Partial Differential Equations

(Semester II; Academic Year 2024-25)

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Assignment - 5

Given Date: April 2, 2025 Submission Date: March 28, 2025
Number of questions: 10 Maximum Marks: 60

1. Use Fourier transform to derive the Poisson formula for the upper half plane in 2-dimension. (5)

2. Fundamental Solution

   (a) Define the fundamental solution: (5)

$$\phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, t > 0, \\ 0, & x \in \mathbb{R}^n, t = 0. \end{cases}$$

   Show that ϕ satisfies $\phi_t - \Delta\phi = 0$ for $x \in \mathbb{R}^n, t > 0$ and that

$$\lim_{(x,t)\to(x_0,0)} \phi(x,t) = 0, \quad \text{for } x_0 \neq 0.$$

   (b) Show that (5)

$$\int_{\mathbb{R}^n} \phi(x,t)dx = 1 \quad \forall t > 0.$$

   (c) For $\delta > 0$, show that (5)

$$\lim_{t\to 0} \int_{|x-y|>\delta} \phi(x-y,t)dy = 0.$$

3. Solve the heat equation: (5)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, t > 0,$$

   with initial condition $u(x,0) = f(x)$ using Fourier transform (assume appropriate conditions on f).

4. Consider the equation: (5)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, t > 0.$$

   Find all solutions of the form $u(x,t) = \frac{1}{\sqrt{t}} v\left(\frac{x}{2\sqrt{t}}\right)$.

5. Let $E(x,t,r)$ be the heat ball and $E(1)=E(0,0,1)$. Show that: (10)

$$\int\int_{E(1)} \frac{|y|^2}{s^2} dyds = 4.$$

Use an appropriate transformation to evaluate:

$$\int\int_{E(r)} \frac{|y|^2}{|s|^2} dyds,$$

where $E(r) = E(0,0,r)$.

6. Define: (5)

$$g(t) = \begin{cases} e^{-\frac{1}{t^\alpha}}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

with $\alpha > 1$, and consider:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Show that this provides infinitely many solutions to the heat equation with zero boundary conditions.

7. Find a sequence of solutions u_n of the one-dimensional heat equation: (5)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in (0, 2\pi), t > 0,$$

with boundary conditions $u(0,t) = u(2\pi, t) = 0$. Using separation of variables, construct a series solution and derive the condition ensuring that $u(x,0) = f(x)$.

8. Show that if u satisfies the heat equation $u_t - \Delta u = 0$ in $\Omega \times (0,T)$, then the following maximum principle holds:

$$\sup_{\Omega\times(0,T)} u(x,t) = \sup_{\Gamma_T} u(x,t).$$

9. Show that if u satisfies: (5)

$$u_t - \Delta u = u, \quad \text{in } \Omega \times (0,T),$$

with $u(x,0) = 0$ for $x \in \Omega$ and $u(x,t) = 0$ on $\partial\Omega \times [0,T]$, then $u(x,t) = 0$ in $\Omega \times (0,T)$.

10. Solve the heat equation: (5)

$$u_t = u_{xx}, \quad x > 0, t > 0,$$

with initial and boundary conditions:

$$u(x,0) = g(x), \quad x > 0, \quad u(0,t) = 0, \quad t > 0.$$

(Hint: Use an odd extension to rewrite the equation in $\mathbb{R} \times (0,\infty)$.)