Partial Differential Equations

(Semester II; Academic Year 2024-25)

Indian Statistical Institute, Bangalore

Midterm Exam

Duration: 3 hrs — Maximum Marks: 30

1. Reduce to canonical form : $u_{xx} - 2u_{xy} - 3u_{yy} + u_y = 0$. (2)

2. Evaluate the integrals:
   (a) $\displaystyle\int_\Omega \frac{2x_1}{1+|x|^2}\, dx$, where $\Omega = \{x \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1\}$ (1)
   (b) $\displaystyle\int_{B(\alpha,1)} \frac{\partial u}{\partial x_1} dx$, where $u = |x|^{-1}$ in $\mathbb{R}^3$ and $\alpha = (2, 0, 0)$. (1)

3. Consider the PDE $xu_x + yu_y + zu_z = 3u$ in $\mathbb{R}^3$.
   (a) Solve the PDE with initial condition $u(x, y, 1) = x^2 + y^2$. (3)
   (b) Is it possible to find unique solution if the initial condition is prescribed on the surface $z = 1 + x^2 + y^2$? (1)

4. Consider the following IVPs:
   A. $u = u_x^2 - 3u_y^2,\ u(x, 0) = x^2,\ x > 0.$
   B. $u = u_x u_y,\ u(x, 0) = x^2,\ x > 0.$
   (a) Discuss the existence and uniqueness of both IVPs. (4)
   (b) Solve any one the above. (3)

5. Let Ω be an open, bounded set in $\mathbb{R}^n$. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $\Delta u = -1$ in Ω, $u = 0$ on $\partial\Omega$. Show that for $x \in \Omega$, $u(x) \geq \frac{1}{2n}(d(x, \partial\Omega))^2$. (3)
   (Hint: For fixed $x_0 \in \Omega$, consider the function $u(x) + \frac{1}{2n}|x - x_0|^2$, $x \in \Omega$.)

6. Suppose u is a harmonic function in $\mathbb{R}^n$ satisfying $|u(x)| \leq C(1 + |x|^m)$, for some non-negative integer m and for all $x \in \mathbb{R}^n$. Show that u is a polynomial of degree at most m. (3)

7. Let Ω is a bounded, open subset of $\mathbb{R}^n$, and $u \in C^1(\Omega)$. If $\int_{\partial B} \frac{\partial u}{\partial \nu} dS = 0$ for every ball B with $\bar{B} \subset \Omega$, show that u is harmonic in Ω. (3)

8. Consider the PDE $xu_x + yu_y = 2u$ on $\mathbb{R}^2$.
   (a) Solve the PDE with the initial condition $u(x, 1) = x$. Determine whether the solution is globally unique? If it is not, find and alternative solution on $\mathbb{R}^2$. (3)
   (b) Find two solutions to the PDE with the initial condition $u(x, e^x) = xe^x$, ensuring that these solutions do not coincide in any neighborhood of the initial curve. (3)

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