Partial Differential Equations

(Semester II; Academic Year 2024-25)

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Assignment - 3

Given Date: January 22, 2025 Submission Date: January 31, 2025
Number of questions: 3 Maximum Marks: 50

1. Determine the types of the following equations, and reduce them to canonical form. (20)
   (a) $u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$.
   (b) $u_{xx} + 2u_{xy} + 4u_{xz} + 5u_{zz} + u_x + 2u_y = 0$ (in $\mathbb{R}^3$).
   (c) $u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0$.
   (d) $\sin^4(2x)u_{xx} + 4\sin^4(2x)u = \frac{\partial^2 u}{\partial t^2}$.

2. Use Gauss-Divergence Theorem to drive the following integration by parts formula (10)
   (a) $\displaystyle\int_\Omega v\frac{\partial u}{\partial x_i} = -\int_\Omega u\frac{\partial v}{\partial x_i} + \int_{\partial\Omega} uv\nu^i$
   (b) $\displaystyle\int_\Omega v\Delta u = -\int_\Omega \nabla u\nabla v + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}v$

3. Let Ω be a open bounded subset of $\mathbb{R}^n$ with ν the outward unit normal vector at $\partial\Omega$, A be an $n \times n$ matrix and $L = \operatorname{div}(A\nabla u)$ be an elliptic operator on Ω. Discuss the uniqueness of solution (if it exists) for the following PDEs: (20)
   (a)
   $$\begin{cases} Lu = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega. \end{cases}$$
   (b)
   $$\begin{cases} Lu = f \text{ in } \Omega \\ A\nabla u \cdot \nu = g \text{ on } \partial\Omega. \end{cases}$$
   (c)
   $$\begin{cases} Lu - u = f \text{ in } \Omega \\ A\nabla u \cdot \nu = g \text{ on } \partial\Omega. \end{cases}$$