

Partial Differential Equations

(Semester II; Academic Year 2024-25)

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Assignment - 5

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Number of questions: 10

Maximum Marks: 60

1. Use Fourier transform to derive the Poisson formula for the upper half plane in 2-dimension. (5)

2. Fundamental Solution

- (a) Define the fundamental solution: (5)

$$\phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, t > 0, \\ 0, & x \in \mathbb{R}^n, t = 0. \end{cases}$$

Show that ϕ satisfies $\phi_t - \Delta\phi = 0$ for $x \in \mathbb{R}^n, t > 0$ and that

$$\lim_{(x,t) \rightarrow (x_0,0)} \phi(x, t) = 0, \quad \text{for } x_0 \neq 0.$$

- (b) Show that (5)

$$\int_{\mathbb{R}^n} \phi(x, t) dx = 1 \quad \forall t > 0.$$

- (c) For $\delta > 0$, show that (5)

$$\lim_{t \rightarrow 0} \int_{|x-y| > \delta} \phi(x-y, t) dy = 0.$$

3. Solve the heat equation: (5)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, t > 0,$$

with initial condition $u(x, 0) = f(x)$ using Fourier transform (assume appropriate conditions on f).

4. Consider the equation: (5)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, t > 0.$$

Find all solutions of the form $u(x, t) = \frac{1}{\sqrt{t}} v\left(\frac{x}{2\sqrt{t}}\right)$.

5. Let $E(x, t, r)$ be the heat ball and $E(1) = E(0, 0, 1)$. Show that: (10)

$$\int \int_{E(1)} \frac{|y|^2}{s^2} dy ds = 4.$$

Use an appropriate transformation to evaluate:

$$\int \int_{E(r)} \frac{|y|^2}{|s|^2} dy ds,$$

where $E(r) = E(0, 0, r)$.

6. Define: (5)

$$g(t) = \begin{cases} e^{-\frac{1}{t^\alpha}}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

with $\alpha > 1$, and consider:

$$u(x, t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Show that this provides infinitely many solutions to the heat equation with zero boundary conditions.

7. Find a sequence of solutions u_n of the one-dimensional heat equation: (5)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in (0, 2\pi), t > 0,$$

with boundary conditions $u(0, t) = u(2\pi, t) = 0$. Using separation of variables, construct a series solution and derive the condition ensuring that $u(x, 0) = f(x)$.

8. Show that if u satisfies the heat equation $u_t - \Delta u = 0$ in $\Omega \times (0, T)$, then the following maximum principle holds:

$$\sup_{\Omega \times (0, T)} u(x, t) = \sup_{\Gamma_T} u(x, t).$$

9. Show that if u satisfies: (5)

$$u_t - \Delta u = u, \quad \text{in } \Omega \times (0, T),$$

with $u(x, 0) = 0$ for $x \in \Omega$ and $u(x, t) = 0$ on $\partial\Omega \times [0, T]$, then $u(x, t) = 0$ in $\Omega \times (0, T)$.

10. Solve the heat equation: (5)

$$u_t = u_{xx}, \quad x > 0, t > 0,$$

with initial and boundary conditions:

$$u(x, 0) = g(x), \quad x > 0, \quad u(0, t) = 0, \quad t > 0.$$

(Hint: Use an odd extension to rewrite the equation in $\mathbb{R} \times (0, \infty)$.)