Partial Differential Equations

(Semester II; Academic Year 2024-25)

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Assignment - 6

Given Date: April 12, 2025 Submission Date: April 20, 2025
Number of questions: 8 Maximum Marks: 40

1. Prove the characteristic parallelogram property for wave equations. (5)

2. Solve the problem with two different characteristic speeds c_1 and c_2: (5)

$$\left(\frac{\partial}{\partial t} - c_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c_2\frac{\partial}{\partial x}\right)u = 0, \quad \text{in } \mathbb{R}\times(0,\infty).$$

Analyze the cases $c_1 \neq c_2$ and $c_1 = c_2$, and derive D'Alembert's formula as a special case. Discuss any potential loss of regularity in the one-dimensional case.

3. Integrate the wave equation $u_{tt} - u_{xx} = f(x,t)$ in the characteristic triangle $P(x,t)$, $Q(x-ct,0)$, $R(x+ct,0)$ to derive a formula for the solution. (Hint: Use the identity $u_{tt} - u_{xx} = (u_t)_t - (u_x)_x$.) (5)

4. Solve the wave equation in the first quadrant with non-homogeneous Dirichlet boundary condition: (5)

$$u_{tt} - u_{xx} = 0, \quad \text{in } (0,\infty)\times(0,\infty)$$

with initial and boundary conditions:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad u(0,t) = h(t), \quad t > 0.$$

Derive the formula for $x < ct$ using the parallelogram identity.

5. Solve the above equation with the Neumann non-homogeneous boundary condition where $u(0,t) = h(t)$ is replaced by $u_x(0,t) = h(t)$. (5)

6. Let $c_1, \ldots, c_k$ be distinct positive real numbers. Show that the solution of the equation: (5)

$$(\partial_t^2 - c_1^2\partial_x^2)\ldots(\partial_t^2 - c_k^2\partial_x^2)u = 0,$$

can be written as:

$$u(x,t) = \sum_{j=0}^{k} u_j(x,t),$$

where each u_j satisfies $\partial_t^2 u_j - c_j^2\partial_x^2 u_j = 0$. This result also holds in higher dimensions.

7. Consider the case $n = 3$ for: (5)

$$(\partial_t^2 - c^2\partial_x^2)(\partial_t^2 - c^2\partial_x^2)u = 0, \quad c > 0.$$

Given smooth initial data $\partial_t^j u(x,0) = f_j(x)$ for $j = 0, 1, 2, 3$, explicitly determine the solution.

8. Consider the wave equation: (5)

$$u_{tt} - \Delta u = 0, \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

with initial data:

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).$$

Verify that the solution is given by:

$$u(x,t) = u_\phi(x,t) + \int_0^t u_\psi(x,s)ds,$$

and also by:

$$u(x,t) = v_\psi(x,t) + \frac{\partial}{\partial t} v_\phi(x,t).$$