Partial Differential Equations

(Semester II; Academic Year 2024-25)

Indian Statistical Institute, Bangalore

Midterm Exam

Duration: 3 hrs Maximum Marks: 30

---

1. Reduce to canonical form : $u_{xx} - 2u_{xy} - 3u_{yy} + u_y = 0$. (2)

> **Solution:** $b^2 - 4ac = 16 > 0$. Hence the PDE is hyperbolic. Roots of the equation $m^2 - 2m - 3 = 0$ are 3 and -1. Choose variables ξ and η such that
>
> $$\xi_x = 3\xi_y \quad \text{and} \quad \eta_x = -\eta_y.$$
>
> Then $\xi = 3x + y$ and $\eta = x - y$ and the canonical form is
>
> $$u_{\xi\eta} + \frac{1}{16}(u_\xi - u_\eta) = 0$$

2. Evaluate the integrals:

   (a) $\displaystyle\int_\Omega \frac{2x_1}{1+|x|^2}\, dx$, where $\Omega = \{x \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1\}$ (1)

   (b) $\displaystyle\int_{B(\alpha,1)} \frac{\partial u}{\partial x_1} dx$, where $u = |x|^{-1}$ in $\mathbb{R}^3$ and $\alpha = (2, 0, 0)$. (1)

> **Solution:**
>
> (a) Here Ω is the region inside the intersection of the planes $\pm x_1 \pm x_2 \pm x_3 = 1$.
>
> $$\int_\Omega \frac{2x_1}{1+|x|^2} dx = \int_\Omega \frac{\partial}{\partial x_1} \log(1+|x|^2) dx = \int_{\partial\Omega} \log(1+|x|^2)\nu^1 dS$$
>
> Hence $\nu_1 = \frac{1}{\sqrt{3}}$ if $x_1 > 0$ and $\nu_1 = -\frac{1}{\sqrt{3}}$ if $x_1 < 0$. Now since the integral is depending only on modulus, we have the value is 0.
>
> (b) Since $|x|^{-1}$ is harmonic in $B(\alpha, 1)$, so is $\frac{\partial u}{\partial x_1}$. Hence, we can use MVP of harmonic functions.
>
> $$\int_{B(\alpha,1)} \frac{\partial u}{\partial x_1} dx = |B(0,1)| \frac{\partial u}{\partial x_1}(\alpha) = -\frac{\pi}{3}$$

3. Consider the PDE $xu_x + yu_y + zu_z = 3u$ in $\mathbb{R}^3$.

   (a) Solve the PDE with initial condition $u(x, y, 1) = x^2 + y^2$. (3)

(b) Is it possible to find unique solution if the initial condition is prescribed on the surface $z = 1 + x^2 + y^2$? (1)

Solution:

(a) Transversality condition:

$$\begin{vmatrix} s_1 & s_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -1.$$

Hence the IVP has a unique solution. Hence

$$\left.\begin{array}{ll} \dot{x} = x; & x(0) = s_1 \\ \dot{y} = y; & y(0) = s_2 \\ \dot{z} = z; & z(0) = 1 \\ \dot{u} = 3u; & p(0) = s_1^2 + s_2^2 \end{array}\right\} \implies \left.\begin{array}{l} x = s_1 e^t \\ y = s_2 e^t \\ z = e^t \\ u = (s_1^2 + s_2^2) e^3 t \end{array}\right\}$$

$$\implies u(x, y) = (x^2 + y^2) z.$$

(b) Transversality Condition:

$$\begin{vmatrix} s_1 & s_2 & 1 + s_1^2 + s_2^2 \\ 1 & 0 & 2s_1 \\ 0 & 1 & 2s_2 \end{vmatrix} = 1 - (s_1^2 + s_2^2).$$

Hence the transversality condition does not hold on the circle $x^2 + y^2 = 1, z = 2$. So we can't guarantee a unique solution in a neighborhood of the initial surface.

4. Consider the following IVPs:

A. $u = u_x^2 - 3u_y^2,\ u(x, 0) = x^2,\ x > 0.$

B. $u = u_x u_y,\ u(x, 0) = x^2,\ x > 0.$

(a) Discuss the existence and uniqueness of both IVPs. (4)

(b) Solve any one the above. (3)

Solution:

(a) A. $p^2 - 3q^2 - z = 0;\ x_0(s) = s,\ y_0(s) = 0,\ z_0(0) = s^2.$

Strip Condition: $p_0(s) x_0'(s) + q_0(s) y_0'(s) = z_0'(s) \implies p_0(s) = 2s$ $\implies q_0(s) = \pm s.$

That is we have 2 choices of initial strips and for each initial data, the transversality condition is

$$\begin{vmatrix} 2p_0(s) & -6q_0(s) \\ 1 & 0 \end{vmatrix} = \pm 6s \neq 0$$

Hence for each initial data, we have unique solution in a neighborhood of initial curve. That is we have 2 solutions.

B. $pq - z = 0;\ x_0(s) = s,\ y_0(s) = 0,\ z_0(0) = s^2$
Strip Condition: $p_0(s)x_0'(s) + q_0(s)y_0'(s) = z_0'(s) \implies p_0(s) = 2s$
$\implies q_0(s) = \frac{s}{2}$.
That is we have only one choices for initial strips and the transversality condition is

$$\begin{vmatrix} q_0(s) & p_0(s) \\ 1 & 0 \end{vmatrix} = -s \neq 0$$

Hence, we have a unique solution in a neighborhood of initial curve.

(b) Since B has a unique solution, we can solve B. By method of characteristics we have

$$\left.\begin{array}{ll} \dot{x} = q; & x(0) = s \\ \dot{y} = p; & y(0) = 0 \\ \dot{z} = 2z; & z(0) = s^2 \\ \dot{p} = p; & p(0) = 2s \\ \dot{q} = q; & q(0) = \dfrac{s}{2} \end{array}\right\} \implies \left.\begin{array}{ll} \dot{x} = \dfrac{s}{2}e^t & x(0) = s \\ \dot{y} = 2se^t & y(0) = 0 \\ \dot{z} = 2z & z(0) = s^2 \end{array}\right\} \implies \left.\begin{array}{l} x = \dfrac{s}{2}e^t + \dfrac{s}{2} \\ y = 2se^t - 2s \\ z = s^2e^{2t} \end{array}\right\}$$

$$\implies z(x,y) = \frac{(4x+y)^2}{16}$$

5. Let Ω be an open, bounded set in $\mathbb{R}^n$. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $\Delta u = -1$ in Ω, $u = 0$ on $\partial\Omega$. Show that for $x \in \Omega$, $u(x) \geq \frac{1}{2n}(d(x, \partial\Omega))^2$. (3)
(Hint: For fixed $x_0 \in \Omega$, consider the function $u(x) + \frac{1}{2n}|x - x_0|^2$, $x \in \Omega$.)

Solution: Since $\Delta u = -1$ in Ω, we get for $x_0 \in \Omega$

$$\Delta\left(u(x) + \frac{1}{2n}|x - x_0|^2\right) = 0$$
$$\implies u(x) + \frac{1}{2n}|x - x_0|^2 \geq \min_{\partial\Omega}\left(u(x) + \frac{1}{2n}|x - x_0|^2\right) = \frac{1}{2n}\mathrm{dist}(x_0, \partial\Omega).$$

Put $x = x_0$ to complete the proof.

6. Suppose u is a harmonic function in $\mathbb{R}^n$ satisfying $|u(x)| \leq C(1 + |x|^m)$, for some non-negative integer m and for all $x \in \mathbb{R}^n$. Show that u is a polynomial of degree at most m. (3)

Solution: By estimates on derivatives we have,

$$\begin{aligned}
|D^\alpha u(x_0)| &\leq \frac{C_k}{r^{n+k}}\|u\|_{L^1(B(x_0,r))} \\
&= \frac{C_k|B(0,1)|}{r^k}\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}|u|dx \\
&\leq \frac{C'}{r^k}\left(1+(|x_0|+r)^m\right)
\end{aligned}$$

For $k>m$, let $r\to\infty$ to get $D^\alpha u(x_0)=0$. That is all the derivatives of order greater than m is 0. Hence u is a polynomial of degree atmost m.

7. Let Ω is a bounded, open subset of $\mathbb{R}^n$, and $u\in C^1(\Omega)$. If $\int_{\partial B}\frac{\partial u}{\partial \nu}dS=0$ for every ball B with $\bar{B}\subset\Omega$, show that u is harmonic in Ω. (3)

Solution:

$$\begin{aligned}
\phi(r) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)}u(y)dS(y) = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)}u(x+rz)dS(z) \\
\phi'(r) &= \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)}\nabla u(x+rz)\cdot z dS(z) \\
&= \frac{1}{|\partial B(0,r)|}\int_{\partial B(x,r)}\nabla u(y)\cdot\frac{y-x}{r}dS(y) \\
&= \frac{1}{|\partial B(0,r)|}\int_{\partial B(x,r)}\frac{\partial u}{\partial\nu}(y)dS(y)=0
\end{aligned}$$

That implies ϕ is a constant. Since u is continuous we have

$$\phi(r)\to u(x) \text{ as } r\to 0 \implies \phi(r)=u(x)\ \forall r>0 \text{ such that } B(x,r)\subset\Omega.$$

That is $u\in C^1(\Omega)$ and satisfies MVP. Hence u is harmonic.

8. Consider the PDE $xu_x+yu_y=2u$ on $\mathbb{R}^2$.
   (a) Solve the PDE with the initial condition $u(x,1)=x$. Determine whether the solution is globally unique? If it is not, find and alternative solution on $\mathbb{R}^2$. (3)
   (b) Find two solutions to the PDE with the initial condition $u(x,e^x)=xe^x$, ensuring that these solutions do not coincide in any neighborhood of the initial curve. (3)

Solution:

(a) Since transversality condition holds on all points on the initial curve, we can get $u(x,y)=xy$ is a unique solution in a neighborhood of initial curve. But it is not globally unique, since the characteristic curves below $x-$axis are not crossing initial curve, we can can get non-uniqueness. Define

$$v(x,y) = \begin{cases} xy, & \text{if } y \geq 0 \\ y^2 + xy, & \text{if } y \leq 0, \end{cases}$$

It is easy to see that $v \in C^1(\mathbb{R}^2)$ and satisfies the IVP. Also $u \neq v$ for $y < 0$.

(b) The transversality condition fails at $(1, e)$. By method of characteristic we can find one solution easily as $u = xy$. Define

$$v(x,y) = \begin{cases} xy, & \text{if } y \geq ex \\ (y - ex)^2 + xy, & \text{if } y \leq ex, \end{cases}$$

It is easy to see that $v \in C^1(\mathbb{R}^2)$ and satisfies the IVP. Also $u \neq v$ in any neighborhood of $(1, e)$,

***