Partial Differential Equations

(Semester II; Academic Year 2024-25)

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Assignment - 7

Given Date: April 2, 2025 Submission Date: April 20, 2025
Number of questions: 4 Maximum Marks: 20

1. Is there an f in $L^1(\mathbb{R})$ such that $f * f = f$? What about $L^2(\mathbb{R})$ (5)

2. For $\delta > 0$, let $f_\delta(x) = f(\delta x)$. Compute the Fourier transform of f. Hence or otherwise show the following: (5)

   - If $\|\hat{f}\|_q \leq \|f\|_p$ for all $f \in L^p$, then $\frac{1}{p} + \frac{1}{q} = 1$.
   - If $\|\hat{f}\|_p \leq \|f\|_p$ for all $f \in L^p$, then $p = 2$.

3. Compute the Fourier transform of $\chi_{[-n,n]}$. Let $f_n(x) = \frac{\sin x \sin nx}{x^2}$. Show that $\|f_n\|_1 \to \infty$ as $n \to \infty$. Hence or otherwise prove that the map $f \to \hat{f}$ is **not onto** from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$. Prove that the range of the Fourier transform is dense in $C_0(\mathbb{R})$. (5)

4. If $f, g \in C_c^\infty(\mathbb{R})$ and $f * g = 0$, prove that either f or g is zero. Prove that there exist f and g in $S(\mathbb{R})$ such that $f * g = 0$. (5)