Theo Shin
Problem Set 4

1. Prove for all $n \in \aleph$, $n^2 \bmod 4 = 0$ or $n^2 \bmod 4 = 1$.

Cases: n is even, n is odd, (n is zero)
Requirements: Result (remainder) is always between 0 and k-1; The result is 0 when n is divided by k; n mod n = 0; n mod k = r if and only if n = qk + r and $0 \le r \le k-1$, by definition.

Case 1: n is even.

• n = 2k for some integer k	By definition of even
• $= (2k)^2 \bmod 4$	Substitute 2k for n value
• $= 4k^2 \bmod 4$	Divisible by 4

Since n^2 is divisible by 4, we can conclude the remainder or result is zero. This shows that $n^2 \bmod 4 = 0$ when n is even. QED

Case 2: n is odd

• n = 2k +1 for some integer k	By definition of odd
• $= (2k+1)^2 \bmod 4$	Substitute 2k +1 for n value
• $= 4k^2 + 4k + 1 \bmod 4$	Distribution
• $= 4(k^2 + K) + 1$	Factoring
• $= 4\ell + 1$ where $\ell = k^2 + k = k(k+1)$	Assigning ℓ value; Divisible by 4 with 1 for remainder

We know n^2 divided by 4 leaves a remainder of zero. We can conclude that since we have (1) leftover, the conclusion is $n^2 \bmod 4 = 1$ when n is odd. QED

(Case 3: n is zero)

• n = 0k for some integer k	By definition of zero
• $= 0^2 \bmod 4$	Substitute 0k for n value (0 * k = 0)
• $= 0 \bmod 4$	Since 0 * 0 = 0

We didn't really need to check for $n \neq 0$, but to cover all bases I wanted to complete case 3. Dividing 0 by 4 equals 0 without any remainder. We can conclude $n^2 \bmod 4 = 0$ when n is zero. QED

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2. Prove for all $n \in \aleph$, if $3n + 2$ is even, then n is even.

Proof by contradiction:

Supposing $\neg p$, then $3n+2$ is odd:

- We are trying to prove $3n + 2$ is even by showing the contradiction $3n + 2$ is odd is false.
- Starting with $3n + 2$, if we take away or subtract the 2, then $3n$ is still even if that is our assumption.
- To demonstrate $\neg$ p, or that $3n$ is odd we can add 1 to $3n$ by the definition of odd.
- This can be written as $3n + 1 = 2\ell$ for some integer ℓ (2ℓ in this case to try and prove $3n + 2$ is even)
- However, the statement does not hold true that 2ℓ is even since the other side of the equation was written as the definition of an odd number. Therefore, we can conclude $\neg p$, $3n + 2$, is false.

Through proof by contradiction, we have shown through logical steps that when setting up $3n + 2$ to be odd, this is false and therefore when $3n + 2$ is even, then n is even. QED

Proof by contraposition:

Assuming the consequence is false, n is odd ($\neg q \Rightarrow \neg p$). So, $n = 2k+1$ for some integer k

• $= 3(2k+1) + 2$	Plugged in $2k + 1$ for n value
• $= 6k + 3 + 2$	Distribution
• $= 2(3k + 2) + 1$	Factoring
• $= 2\ell + 1$ for some $\ell = 3k + 2$	Assign value for ℓ; By definition of odd

Since we ended with a definition of an odd number, we can conclude through proof by contraposition that since when n is odd ($2k +1$) we obtain an odd result ($2\ell + 1$), then through proof by contraposition we can conclude that if $3n + 2$ is even, then n is even. QED

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3. Prove for all positive integers, A, B, if A is even and B is odd, A + B = C is odd.

• There is d such that A = 2d	By definition of even
• There is h such that B = 2h + 1	By definition of odd
• C = A + B	Premise
• C = 2d + 2h + 1	Substitution
• C = 2(d+h) + 1	Factoring
• Let k = d + h	Defining k for some integer k
• C = 2k + 1	Definition of odd

Since C = 2k + 1 for some integer k, we can conclude C is odd. QED

Prove for all positive integers, A, B, if A is odd and B is odd, A + B = C is even.

• There is d such that A = 2 d + 1	By definition of odd
• There is h such that B = 2 h + 1	By definition of odd
• C = A + B	Premise
• C = 2d + 1 + 2h + 1	Substitution
• C = 2(d+h) + 2	Factoring
• Let k = d + h	Defining k for some integer k
• C = 2k + 2 (-2)	Subtracting 2, an even number
• C = 2k	By definition of even

Since C = 2k for some integer k, we can conclude C is even. QED

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4. Prove for all positive integers, A, B, if A and B are even, A * B = C is even.

• There is d such that A = 2d	By definition of even
• There is h such that B = 2h	By definition of even
• C = A * B	Premise
• C = 2d * 2h	Substitution
• C = 2(dh)	Factoring
• Let k = dh	Defining k for some integer k
• C = 2k	By definition of even

Since C = 2k for some integer k, we can conclude C is even. QED

Prove for all positive integers, A, B, if A is even and B is odd, A * B = C is even.

• There is d such that A = 2d	By definition of even
• There is h such that B = 2h + 1	By definition of odd
• C = A * B	Premise
• C = 2d * (2h + 1)	Substitution
• C = 4dh + 2d	Distribution
• C = 2(dh + d)	Factoring
• Let k = dh + d	Defining k for some integer k
• C = 2k	Definition of even

Since C = 2k for some integer k, we can conclude C is even. QED

Prove for all positive integers, A, B, if A and B are odd, A * B = C is odd.

• There is d such that A = 2 d + 1	By definition of odd
• There is h such that B = 2 h + 1	By definition of odd
• C = A * B	Premise
• C = (2d + 1) * (2h + 1)	Substitution
• C = 4dh + 2d + 2h + 1	Distribution
• C = 2(2dh + d + h) + 1	Factoring
• Let k = 2dh + d + h	Defining k for some integer k
• C = 2k + 1	Definition of odd

Since C = 2k + 1 for some integer k, we can conclude C is odd. QED

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5. Prove for all $n \in \aleph$, if $3n + 2$ is even, then n is even (using direct approach from solved proofs).

   Assuming n is an even integer, then $n = 2k$ for some integer k.

   - $= 3(2k) + 2$ Plug 2k for n by definition of even
   - $= 6k + 2$ Distribution
   - $= 2(3k + 1)$ Factoring
   - $= 2\ell$ where $\ell = 3k + 1$ for some integer ℓ Assign value for ℓ

   Since 2ℓ is an even integer, we can conclude $3n + 2$ is even when n is even. QED

6. Prove or disprove for all $n \in \aleph$, if n^2 is even, then n is even.

   Assuming n is an even integer, then $n = 2k$ for some integer k.

   - $= (2k)^2$ Plug 2k for n by definition of even
   - $= 4k^2$ Distribution
   - $= 2(2k2)$ Factoring
   - $= 2\ell$ where $\ell = 2k2$ Assign value for ℓ

   Since 2ℓ is an even integer, we can conclude n^2 is even when n is even. QED

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7. Prove for any integers m, n, and k > 0, (m+n) mod k = ((m mod k) + (n mod k)) mod k.

*Rewritten as (A1 + A2) mod B = (A1 mod B + A2 mod B) mod B
Had a difficult time understanding modular arithmetic and researched other tutorials. Switching variables from (m,n,k) to (A1,A2,B) was easier to manage.

REQUIREMENTS:

$\frac{A}{B}$ = Q remainder R

A = B * Q + R where 0 ≤ R < B (quotient remainder theorem) ⇒ A mod B = R

A mod B = (A + K * B) mod B for any integer K; # "Increase A by a multiple of B won't change mod calculations"

- A1 = B * Q1 + R1 where 0 ≤ R1 < B and Q1 is some integer Q.
- A1 mod B = R1
- A2 = B * Q2 + R2 where 0 ≤ R2 < B and Q2 is some integer Q.
- A2 mod B = R2
  # From stated equation, divide by B and eliminate Q1 since does not affect mod calculations

(A1 + A2) mod B:	Solving for LHS (Left-hand side)
= ((B * Q1 + R1) + (B * Q2 + R2)) mod B	Plug values for A1 and A2 from above
= B * Q1 + B * Q2 + R1 + R2 mod B	Commutative Property
= B (Q1 + Q2) + R1 + R2 mod B	Factoring
= (R1 + R2) mod B	Multiplying B by Q1
	and/or Q2 does not affect mod calculations,
	therefore B(Q1 + Q2) mod B = 0.

(A1 mod B) + (A2 mod B) mod B:	Solving for RHS (Right-hand side)
= (R1 + R2) mod B	Plugged in R1 and R2 values from above
	#(Again, Q1 does affect our mod calculations based on A mod B = (A + K * B) mod B

Both LHS and RHS are equivalent. QED

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8. Prove that $\sqrt{2}$ is irrational.

Attempting to prove using proof by contradiction, we can show $\neg p \Rightarrow q$. $\neg p$ states $\sqrt{2}$ is rational. From the definition of rational numbers, we can write $\sqrt{2} = \frac{a}{b}$; $b \neq 0$

#When a rational number is in its lowest form $\frac{a}{b}$ and its denominator is a positive integer, the numerator and denominator have no common factor other than 1 by definition.

• $\sqrt{2} = \frac{a}{b}$	By definition of rational number
• $2 = \left(\frac{a}{b}\right)^2$	Square both sides
• $2 = \left(\frac{a^2}{b^2}\right)$	Distribution
• $a^2 = 2b^2$	# Multiplying both sides by b^2, we determine a^2 or simply a is even by the definition of even. We can say
• a = 2k for some integer k	Assigning k value to denote a is even
• $2b^2 = (2k)^2$	Plug 2k for a value
• $2b^2 = 4k^2$	Distribution
• $b^2 = 2k^2$	# Dividing both sides by 2, we determine b^2 or simply b even due to definition of even.

Since we have concluded that both a and b are even, we have proved $\frac{a}{b}$ has a common factor other than 1, in this case 2. Through proof by contradiction we have shown that $\sqrt{2}$ is an irrational number. QED

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9. Prove that $(a^5)^3 = a^{(5*3)} = a^{15}$

- $(a^5)^3 = a^5 * a^5 * a^5$ — By definition of exponent
- $a^5 * a^5 * a^5$ = aaaaa * aaaaa * aaaaa — By definition of exponent
- =aaaaaaaaaaaaaaa — Multiplication
- $= a^{15}$ — By definition of exponent
- $= a^{(5*3)}$ — By definition of multiplication (5 * 3 == 15)

Prove that $(ab)^5 = a^5 b^5$

- $(ab)^5$ = (ab) * (ab) * (ab) * (ab) * (ab) — By definition of exponent
- = a * a * a * a *a * b * b * b * b * b — Commutative property of multiplication
- aaaaa * bbbbb — Multiplication
- $= a^5 * b^5$ — By definition of exponent

Adding definitions of exponents are a consequence of the existing structure since proving $(a^5)^3 = a^{(5*3)}$ and $(ab)^5 = a^5 b^5$ demonstrated by breaking apart the stated exponents into their smaller equivalent counterparts, we could translate the logical equivalences using just algebraic principles. QED

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10. Prove the Quadratic formula: $ax^2 + bx + c = 0$, $a \neq 0$

- $x^2 + \frac{b}{a}x + \frac{c}{a} = \frac{0}{a}$ — Divide by the coefficient (a) of x^2; $0 \div a = 0$
- $x^2 + \frac{b}{a}x = -\frac{c}{a}$ — Move the constant $(\frac{c}{a})$ to the other side
- $x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$ — Take half the coefficient of x, square it and add it to both sides (from algebra notes)

LHS (Left-hand side)

$x^2 + \frac{b}{a}x + \frac{b}{2a}^2$

$= (x + \frac{b}{2a})^2$ — (algebra notes)

$= x^2 + \frac{b}{2a}x + \frac{b}{2a}x + (\frac{b}{2a})^2$ — Distribution

$= x^2 + \frac{2b}{2a}x + (\frac{b}{2a})^2$ — Added fractions $\frac{b}{2a} + \frac{b}{2a} = \frac{2b}{2a}$

$= x^2 + \frac{b}{a}x + \frac{b}{2a}^2$ — Cancel out 2's from num./den. in coefficient x

$= (x + \frac{b}{2a})^2$ — Back to factored form

RHS (Right-hand side)

$= -\frac{c}{a} + \frac{b}{2a}^2$

$= -\frac{c}{a} + \frac{b^2}{4a^2}$ — $\frac{b}{2a}^2 = \frac{b^2}{4a^2}$

$= \frac{b^2}{4a^2} - \frac{c}{a}(\frac{4a}{4a})$ — Multiply by $(\frac{4a}{4a})$

$= \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$ — Distribution

$= \frac{b^2-4ac}{4a^2}$ — Common denominator

- $(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$ — LHS = RHS
- $(x + \frac{b}{2a}) = \pm\frac{\sqrt{b^2-4ac}}{2a}$ — Square root property ($4a^2 \Rightarrow 2a$)
- $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$ — Subtracted $\frac{b}{2a}$ from both sides
- $x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$ — Common denominator

We can conclude $x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$ where the values of x can be plugged into the quadratic formula $ax^2 + bx + c$ with a result of 0. QED