Relations Problem Set
Theo Shin

1. (a)

Reflexive iff $(a,a) \in R$ for every $a \in A$

Symmetric iff $(b,a) \in R$ whenever $(a,b) \in R$

Anti symmetric if $\forall a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $\boxed{a=b}$

Transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ $\forall a,b,c \in A$

(a) A = set of all CU students

Reflexive ✓

Since "a" shares atleast one class with "a" itself

Symmetric ✓

iff $(a,b) \in R$ wherever $(a,b) \in R$

Since "a" shares at least one class with "b" and "b" shares at least one class with "a"

Transitive ✗

whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ $\forall a,b,c \in A$

"a" shares at least one class with "b" and "b" shares at least one class with "c" however "a" does not necessarily share a class with "c"

Equivalence ✗

Because $(a,c) \in R$ is not transitive, therefore set R is not an equivalence relation

"a" shares at least one class with "b"
"b" shares at least one class with "c" $\not\Rightarrow$ "a" shares at least one class with "c"

1. (b)

(b) Reflexive ✗

"a" cannot have a higher gpa than with "a" itself

Symmetric ✗

Since $(a,b) \in R$ iff a has higher GPA than b, GPA of "a" is greater than GPA of "b," but not vice versa (GPA b $\not>$ GPA a)

Transitive ✓

Since GPA of "a" is greater than GPA of "b" and GPA of "b" is greater than GPA of "c" then GPA of "a" is greater than GPA of "c"

GPA "a" > GPA "b" ; GPA "b" > GPA "c" ⇒ GPA "a" > GPA "c"

Since it did not hold set R is either reflexive or symmetric, it is not an equivalence relation

1. (c)

(c) Reflexive ✓

"a" is roomate with "a"

Symmetric ✓

Since "a" is roommates with "b" then "b" must be roommates with "a"

Transitive ✓

If "a" is roomates with "b" and "b" is roommates with "c" then "a" is roommates with "c"

"a" and "b" are roommates; "b" and "c" are roommates ⇒ "a" and "c" are roommates

Equivalence ✓

Yes, because $(a,b) \in R$ are reflexive, transitive, and symmetric, an equivalence relationship is satisfied

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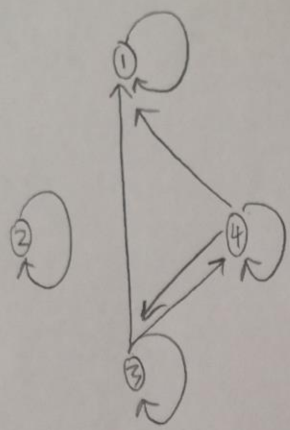
2. (a,b,c)

(a)



(b) Reflexive ✓

Looking at graph of R, every element of $A = \{1,2,3,4\}$ have arrows leading back to same number (i.e. (1,1), (2,2), (3,3), (4,4))

Symmetric ✗

Even though (3,4) and (4,3) are symmetric not every element is such as $(3,1) \in R \Rightarrow (1,3) \notin R$ and $(4,1) \in R \Rightarrow (1,4) \notin R$

Transitive ✓

Since $(3,1), (1,1) \Rightarrow (3,1) \in R$ and $(4,3), (3,1) \Rightarrow (4,1) \in R$ and $(3,4), (4,3) \Rightarrow (3,3) \in R$ and $(3,4), (4,1) \Rightarrow (3,1) \in R$ ....

Equivalence ✗

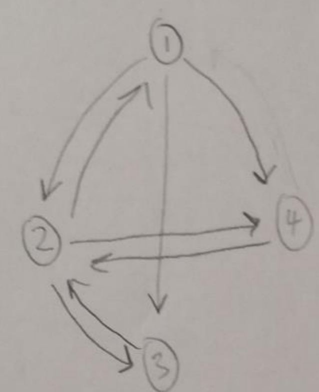
Because it is not necessarily symmetric, it is not an equivalence relation

(c) (i) $\bar{R} = (A \times A) - R$

$= \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,2), (4,2)\}$

2. (c – con't)

(ii)



(iii) $\overline{R}$ is not symmetric since $(1,3) \in R \Rightarrow (3,1) \notin R$. True, relation R is symmetric iff its complement $\overline{R}$ is symmetric. We notice in both cases, neither R or $\overline{R}$ were symmetric and in fact, in R we have $(3,1) \in R \Rightarrow (1,3) \notin R$ and $(4,1) \in R \Rightarrow (1,4) \notin R$. In $\overline{R}$, we have exactly $(1,3) \in R \Rightarrow (3,1) \notin R$ and $(1,4) \in R \Rightarrow (4,1) \notin R$. Also, by definition of complement, we can conclude $\overline{R} \cap R = \{\}$, so if R is symmetric we have $(a,b) \in R \Rightarrow (b,a) \in R$ then $\overline{R}$ we have $(c,d) \in \overline{R} \Rightarrow (d,c) \in \overline{R}$. There would be a contradiction if $\overline{R}$ was not symmetric and R was symmetric.

3.

EXAMPLE 3 **Congruence Modulo m** Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

THEOREM 3 Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \textbf{ mod } m = b \textbf{ mod } m$.

The proof of Theorem 3 is left as Exercises 15 and 16. Recall that $a \textbf{ mod } m$ and $b \textbf{ mod } m$ are the remainders when a and b are divided by m, respectively. Consequently, Theorem 3 also says that $a \equiv b \pmod{m}$ if and only if a and b have the same remainder when divided by m.

Reflexive ✓

$a \equiv b$ (mod m) iff $m / a-b$

Since $a-a = 0$ and $m \mid 0$ (since $0 = 0m$) due to definition of modulus

$a \equiv a$ (mod m)

Symmetric ✓

Using theorems 3 and 4

$a-b = km$ for integer k

$b-a = -mk = (-k)m$ due to associative property

$b-a = (-k)m$ therefore $b-a / m$ and $m / b-a$ thus

$b \equiv a$ (mod m)

THEOREM 4 Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

Proof: If $a \equiv b \pmod{m}$, by the definition of congruence (Definition 3), we know that $m \mid (a - b)$. This means that there is an integer k such that $a - b = km$, so that $a = b + km$. Conversely, if there is an integer k such that $a = b + km$, then $km = a - b$. Hence, m divides $a - b$, so that $a \equiv b \pmod{m}$. ◁

3. Continued

Transitive ✓

Showing $(a,b) \in R$ and $(b,c) \in R$ then $a \equiv b \pmod m$ and $b \equiv c \pmod m$. From what's been already shown through symmetric and reflexive calculations/proofs, we have $a-b$ is divisible by m and $b-c$ is divisible by m from modulus properties. Therefore, $a-b = (h)m$ and $b-c = (\ell)m$, where ℓ and h are some integers.

$$+$$

$$(a-b) + (b-c) = (h)m + (\ell)m$$

$$a-c = (h)m + (\ell)m \quad \text{associative property}$$

$$a-c = m(h+\ell) \quad \text{distribution}$$

Since $a-c$ is also divisible by m, $a \equiv c \pmod m$ and thus set R is transitive.

In order to demonstrate the relation R = {(a,b) | (a ≡ b (mod m)}, in needed to be proven that the relationship was reflexive, symmetric, and transitive. Since we have demonstrated those properties, it is established set R is an equivalence relation.