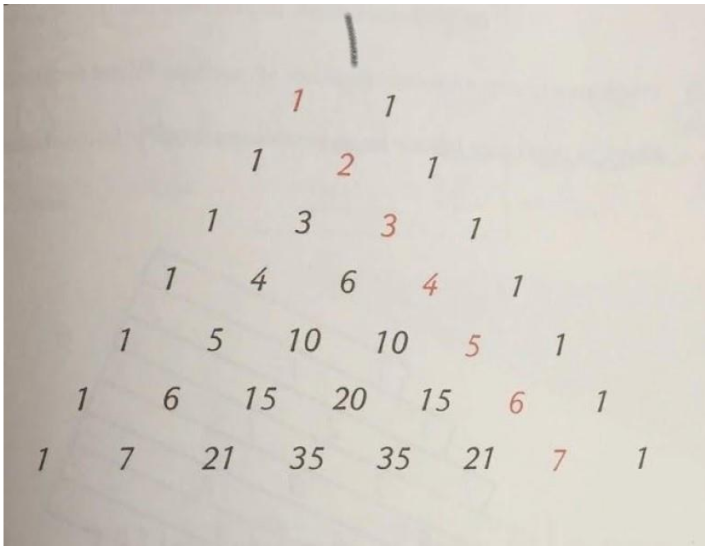
Combinatorics Homework
Theo Shin

1.

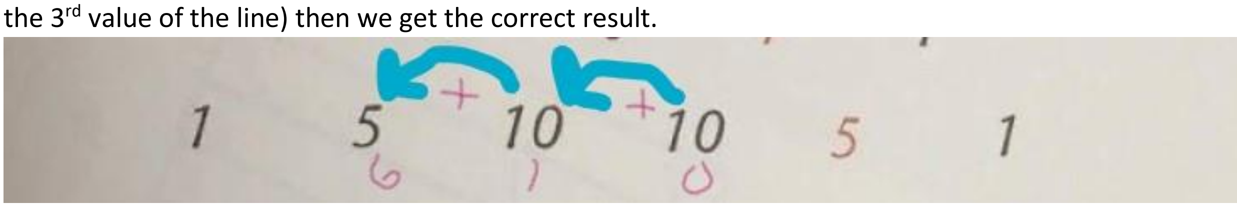


(1)

$11^0 =$	1	sum: 1	2^0
$11^1 =$	11	2	2^1
$11^2 =$	121	4	2^2
$11^3 =$	1331	8	2^3
$11^4 =$	14641	16	2^4

(2) Each line of Pascal's triangle are exponents of 11. Another observance, the sum of each line are exponents of 2 (I added the 1 at the top, it was bothering me the triangle felt incomplete).

(3) $11^5 = 161051 \Rightarrow$ At first glance it may appear the fifth line does not correspond to the value of 11^5, however, if you look at it as grouping the 5,1 and 0,1 (breaking apart the 10, the 3rd value of the line) then we get the correct result.



(4)

(4) The sums of the first six lines are 1, 2, 4, 8, 16, 32, 64 (Taking into account the 1 at the top)

(5) When solving for the first question regarding exponents of 11, I noticed the sum of each line corresponds to exponents of 2. So, we can reasonably assume the pattern will continue with the sum of 7th line equaling 128, 8th line 256, 9th line 512, and so on.

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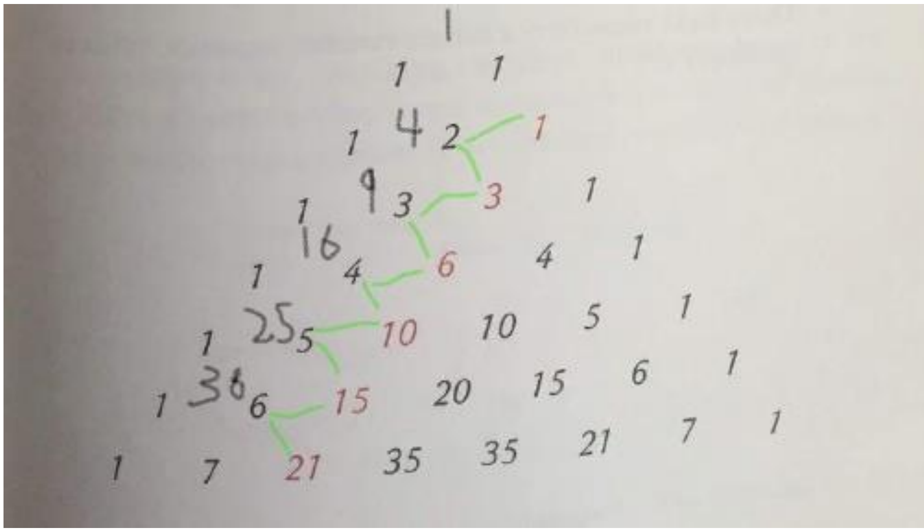
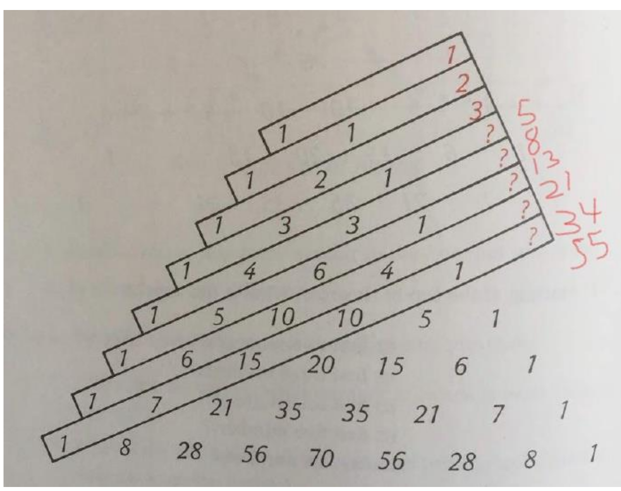
(6) Looking at the sloping row, the sums of 1st and 2nd, 2nd and 3rd, 3rd and 4th, 4th and 5th, 5th and 6th are respectively 4, 9, 16, 25, 36.



(7) The number sequence is the squaring of numbers 2 thru 6. We can also see the squared values correspond to the left sloping line (i.e. 2 = (2^2 or 4) = 1+3)

(8) The sums of the next five rows are 5, 8, 13, 21, 34, (55), (89),….



(9) The sums of these rows correspond to the Fibonacci sequence where the next numbers are found by adding up the two previous numbers before it.

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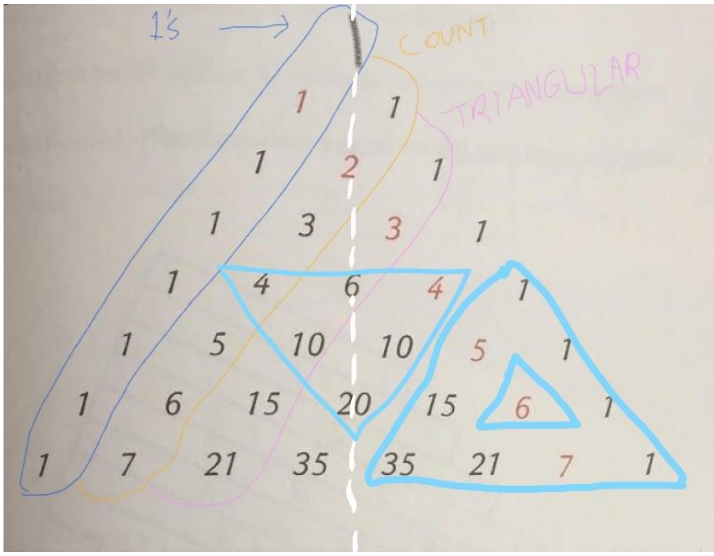
Patterns in Pascal's Triangle :



(10) Aside from the patters we have found thus far (Fibonacci, squares, exponents of 11 and 2), I've highlighted some additional ones. Starting with the left and right edges are all 1's and if you imagine a dotted line down the middle, the left and right halves mirror each other so they're symmetrical. The row next to the 1's (on either side since it's been established they're mirrored) are your counting numbers. The following row are all triangular numbers obtained by summation of natural numbers (1, 1+2=3, 1+2+3=6, 1+2+3+4=10, so on). The last pattern observed (purposely didn't triangle out the entire picture) is if you separate and highlight the even and odd numbers, you are left with a pattern of triangles within triangles.

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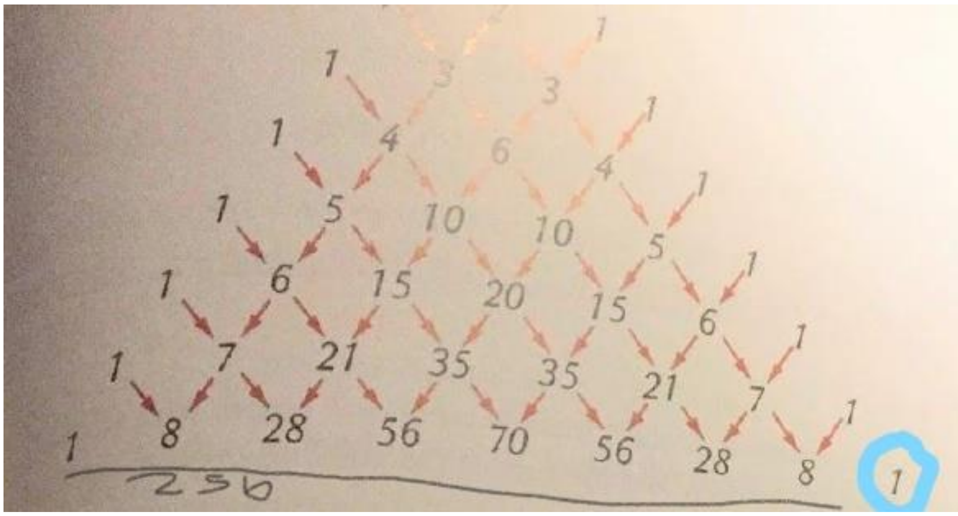
2. From Pascal's triangle and related pages from Jacob's book, the 8th line of Pascal's shows a total of 256 total possibilities (you can sum the 8th line or calculate $\frac{1}{2}$ ^ 8 for the probability) of boys/girls for 8 children. Since there's only one possible way (denoted by the rightmost circled 1) for all 8 children to be girls, the probability is calculated as
1 / 256 = .0039 or .39% (less than 1% chance).



- To determine the probability for exactly 5 girls, we see from the 8th line that 5 girls corresponds to a total of 56 possibilities (GGGGGBBB, BBBGGGGG, GBGGGGBB, GGBGGBGG, etc). The probability is then calculated as 56 / 256 = 0.21875 or 21.875%

Number of girls	0	1	2	3	4	5	6	7	8
Number of ways	1	8	28	56	70	56	28	8	1

- Calculating the probability of having exactly five boys can be reworded as having exactly three girls. We see again that the number of possibilities for having three girls is 56, so the calculation and result is the same from the previous answer (56 / 256 = 0.21875 or 21.875%).

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3.

$$(x+y)^4$$
$$= (x+y)(x+y)(x+y)(x+y)$$
$$= (x^2+2xy+y^2)(x+y)(x+y)$$
$$= x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3 \ (x+y) \quad \text{simplify}$$
$$= (x^3 + 3x^2y + 3xy^2 + y^3)(x+y)$$
$$= x^4 + x^3y + 3x^3y + 3x^2y^2 + 3x^2y^2 + 3xy^3 + xy^3 + y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

(For explanation purposes, I distributed the (x+y)(x+y) and got (x^2+2xy+y^2). Then I distributed the (x^2+2xy+y^2) * (x+y) and purposely added a simplification process to show how it was distributed. I continued the process until all of the (x+y)'s were distributed)

- Using Binomial Theorem:

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$${}_4C_0 = \frac{4!}{0!(4-0)!} = 1 \qquad {}_4C_1 = \frac{4!}{1!(4-1)!} = 4 \qquad {}_4C_2 = \frac{4!}{2!(4-2)!} = 6 \qquad {}_4C_3 = \frac{4!}{3!(4-3)!} = 4 \qquad {}_4C_4 = \frac{4!}{4!(4-4)!} = 1$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

(Used combination formula n!/r!(n-r!) to establish coefficients)

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4. For permutations, order matters. So, if for instance we were determining how many different ways 50 students can place 1st, 2nd, and 3rd for a hack-a-thon, we would choose to use the permutation calculation over combinations because certainly the top three qualifiers care whether they are given gold, silver, bronze or maybe the 1st place finisher has first choice for which company they want to intern for. The easy way to see this is there are 50 students who can get 1st place, then 49 students left who can get 2nd place, and then 48 students who are going for 3rd place. We get 117,600 possible scenarios following the computation.

$$(50)(49)(48) = 117{,}600$$

$$P(50,3) = \frac{50!}{(50-3)!} = \frac{50 \cdot 49 \cdot 48 \cdot \cancel{47 \cdot 46 \cdot 45 \cdots 1}}{\cancel{47 \cdot 46 \cdot 45 \cdot 44 \cdots}}$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

- For combinations, the order does not matter. Using the same example, if we had to determine the number of different ways 50 students can compete in a hack-a-thon and the top three finishers win a round-trip stay at Disney World in Florida, then it does not matter if a student places 1st, 2nd, or 3rd so long as they are one of the top 3 finishers. So, to compute this, we would use a combination formula which is similar to the permutation formula EXCEPT since order not matter, the formula involves dividing my r! (in this 3!), which omits the duplicate scenarios. Following the computation, we get a result of 19,600.

$$C(50,3) = \frac{50!}{47!\,3!} = \frac{50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!}\,\underbrace{3 \cdot 2 \cdot 1}_{\text{duplicate possibilities}}} = 19{,}600$$

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

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5. Section 6.2 #4 (Any Pigeonhole Principle Problem)
   A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

   A) How many balls must she select to be sure of having at least three balls of the same color?

      To find the least number of balls selected being three balls of the same color, we can use the "generalized pigeonhole principle" where k is equal to the number of colors (2 – red and blue), the result is three, and we are looking to find N.

      $\lceil N/2 \rceil = 3$
      $\lceil 5/2 \rceil = 3$

      In this case, the lowest integer to satisfy the ceiling function would be 3.

   B) How many balls must she select to be sure of having at least three blue balls?

      In order to choose three blue balls with 100% certainty, the "worst" possible outcome is having the first ten choices be red, which we would need to account for. So, she must select 13 (10 red $\Rightarrow$ 3 blue) to ensure at least 3 of them end up being blue balls.

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6. Section 6.4 #32

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

a. Set up induction base case

Base Case: $n=1$

$\frac{1!}{(1-0)!\cdot 0!} = \frac{1}{(1)(1)}$ $\qquad$ $\frac{1!}{(1-1)!1!} = \frac{1}{(1)(1)}$

$$(x+y)^1 = \sum_{j=0}^{1} \binom{1}{0} x^{1-0} y^0 + \binom{1}{1} x^{1-1} y^1$$

$$x^1 y^0 + x^0 y^1$$

$\checkmark = x+y$

b. Set up IH

Inductive Step: Assume for an arbitrary $k \in \mathbb{N}$, $P(k)$ is true

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j$$

Show $P(k+1)$ is also true

$$(x+y)^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} x^{k+1-j} y^j$$

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c. Begin with IH – Multiply both sides by (x+y)

IH: ① Multiply both sides by (x+y)

$$(x+y)^k(x+y) = \sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^j\,[(x+y)]$$

$$= x\sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^j + y\sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^j$$

$$= x\sum_{j=0}^{k}\binom{k}{j}x^{k+1-j}y^j + \sum_{j=0}^{k}\binom{k}{j}x^{k-j}y^{j+1}$$

$$= \left[\binom{k}{0}x^{k+1-0}y^0 + \binom{k}{1}x^{k+1-1}y^1 + \binom{k}{2}x^{k+1-2}y^2 + \binom{k}{3}x^{k+1-3}y^3 + \dots + \binom{k}{m}x^{k+1-m}y^m + \dots + \binom{k}{k}x^{0+1}y^k\right]$$

$$+ \left[\binom{k}{0}x^{k-0}y^{0+1} + \binom{k}{1}x^{k-1}y^{1+1} + \binom{k}{2}x^{k-2}y^{2+1} + \binom{k}{3}x^{k-3}y^{3+1} + \dots + \binom{k}{m}x^{k-m}y^{m+1} + \dots \binom{k}{k}x^0y^{k+1}\right]$$

d. Combine like terms

$$= \binom{k}{0}x^{k+1}y^0 + \binom{k}{1}x^{k}y^1 + \binom{k}{2}x^{k-1}y^2 + \binom{k}{3}x^{k-2}y^3 + \dots + \binom{k}{m}x^{k+1-m}y^m + \dots + \binom{k}{k}x^1y^k]$$

$$+ \left[\binom{k}{0}x^{k}y^1 + \binom{k}{1}x^{k-1}y^2 + \binom{k}{2}x^{k-2}y^3 + \binom{k}{3}x^{k-3}y^4 + \dots + \binom{k}{m}x^{k-m}y^{m+1} + \dots \binom{k}{k}x^0y^{k+1}\right]$$

$$= \binom{k}{0}x^{k+1}y^0 + \left[\binom{k}{1}+\binom{k}{0}\right]x^ky^1 + \left[\binom{k}{2}+\binom{k}{1}\right]x^{k-1}y^2 + \left[\binom{k}{3}+\binom{k}{2}\right]x^{k-2}y^3 + \dots + \left[\binom{k}{m}+\binom{k}{m-1}\right]x^{k+1-m}y^m$$

$$+ \dots + \left[\binom{k}{k-1}+\binom{k}{k}\right]x^1y^k + \binom{k}{k}x^0y^{k+1}$$

$$= \binom{k+1}{0}x^{k+1}y^0 + \binom{k+1}{1}x^ky^1 + \binom{k+2}{2}x^{k-1}y^2 + \binom{k+3}{3}x^{k-2}y^3 + \dots + \binom{k+1}{m}x^{k+1-m}y^m$$

$$+ \dots + \binom{k+1}{k+1}x^0y^{k+1}$$

e. Finish with Pascal's Identity

$$\binom{k}{m}+\binom{k}{m-1}=\binom{k+1}{m}$$

$$= \sum_{j=0}^{k+1}\binom{k+1}{j}x^{k+1-j}y^j$$

We thus have that $P(1)$ and $\forall k \in \mathbb{N}, P(k) \to P(k+1)$, so by principle of mathematical induction it follows that $P(n)$ is true for all natural numbers n. △