Induction and Recursion Homework
Theo Shin

1. Recurrence Relations:
   p. 167-168 #3, #9, #13 practice.

p.167

3\. (a) 2^n+1

$a_0 = 2^0+1 = 2$
$a_1 = 2^1+1 = 3$
$a_2 = 2^2+1 = 5$
$a_3 = 2^3+1 = 9$

(b) $(n+1)^{n+1}$

$a_0 = (0+1)^{0+1} = 1^1 = 1$
$a_1 = (1+1)^{1+1} = (2)^2 = 4$
$a_2 = (2+1)^{2+1} = (3)^3 = 27$
$a_3 = (3+1)^{3+1} = 4^4 = 256$

(c) $\lfloor n/2 \rfloor$

$a_0 = \lfloor 0/2 \rfloor = 0$
$a_1 = \lfloor 1/2 \rfloor = 0$
$a_2 = \lfloor 2/2 \rfloor = 1$
$a_3 = \lfloor 3/2 \rfloor = 1$

(d) $\lfloor n/2 \rfloor + \lceil n/2 \rceil$

$a_0 = \lfloor 0/2 \rfloor + \lceil 0/2 \rceil = 0+0 = 0$
$a_1 = \lfloor 1/2 \rfloor + \lceil 1/2 \rceil = 0+1 = 1$
$a_2 = \lfloor 2/2 \rfloor + \lceil 2/2 \rceil = 1+1 = 2$
$a_3 = \lfloor 3/2 \rfloor + \lceil 3/2 \rceil = 1+2 = 3$

Note: $\lfloor n/2 \rfloor + \lceil n/2 \rceil$ equals n

9\. (a) $a_n = 6a_{n-1}$, $a_0 = 2$

$a_1 = 6a_0 = 6 \cdot 2 = 12$
$a_2 = 6a_1 = 6 \cdot 12 = 72$
$a_3 = 6a_2 = 6 \cdot 72 = 432$
$a_4 = 6a_3 = 6 \cdot 432 = 2{,}592$
$a_5 = 6a_4 = 6 \cdot 2592 = 15552$

(b) $a_n = a_{n-1}^2$, $a_1 = 2$

$a_1 = 2$
$a_2 = a_1^2 = 2^2 = 4$
$a_3 = a_2^2 = 4^2 = 16$
$a_4 = a_3^2 = 16^2 = 256$
$a_5 = a_4^2 = 256^2 = 65536$

(c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$

$a_2 = a_1 + 3a_0 = 2 + 3(1) = 5$
$a_3 = a_2 + 3a_1 = 5 + 3(2) = 11$
$a_4 = a_3 + 3a_2 = 11 + 3(5) = 26$
$a_5 = a_4 + 3a_3 = 26 + 3(11) = 59$

(d) $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$

$a_2 = (2)(1) + (2^2)(1) = 6$
$a_3 = (3)(6) + (3^2)(1) = 27$
$a_4 = (4)(27) + (4^2)(6) = 204$

Induction and Recursion Homework
Theo Shin

9. (e) $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1, a_1 = 2, a_2 = 0$

$a_3 = 0 + 1 = 1$

$a_4 = 1 + 2 = 3$

$a_5 = 3 + 0 = 3$

13. (a) $a_n = 8a_{n-1} - 16a_{n-2}$

$a_n = 0$

Since $0 = 0$ when $a_n = 0$, $a_n = 0$ is solution to recurrence relation
TRUE

(b) $a_n = 1$

$1 \neq 8(1) - 16(1)$
FALSE

(c) $a_n = 2^n$

$2^n = 8(2^{n-1}) - 16(2^{n-2})$
$2^n = 2^{n-2}(8 \cdot 2 - 16)$
$2^n \neq 0$ FALSE

(d) $a_n = 4^n$ $\quad 4^n = 8(4^{n-1}) - 16(4^{n-2})$
$4^n = 4^{n-2}(8 \cdot 4 - 16)$
$= 4^{n-2}(16)$
$= 4^{n-2} \cdot 4^2$

$4^n = 4^n$ TRUE

(e) $a_n = n4^n$ $\quad n4^n = 8(n-1)4^{n-1} - 16(n-2)4^{n-2}$
$n' = 4^{n-2}(8(n-1) \cdot 4 - 16(n-2))$
$= 4^{n-2}(32n - 32) - 16n + 32)$
$= 4^{n-2}(16n)$
$= 4^{n-2} \cdot 4^2 n$
$n4^n = n4^n$

Induction and Recursion Homework
Theo Shin

(f) $a_n = 2\cdot 4^n + 3n4^n$

$$2\cdot 4^n + 3n4^n = 8\cdot(2\cdot 4^{n-1} + 3(n-1)4^{(n-1)}) - 16(2\cdot 4^{n-2} + 3(n-2)4^{n-2})$$
$$= 4^{n-2}(8\cdot 2\cdot 4 + 8\cdot 3\cdot (n-1)\cdot 4 - 16\cdot 2 - 16\cdot 3(n-2))$$
$$= 4^{n-2}(64 + 32\cdot(3n-3) - 32 - 16(3n-6))$$
$$= 4^{n-2}(64 + \underbrace{96n - \cancel{96} - 32 - 48n}_{48n} + \cancel{96})$$
$$= 4^{n-2}(48n+32)$$
$$= 4^{n-2}\cdot 4^2(3n+2)$$
$$2\cdot 4^n + 3n4^n = (2+3n)4^n$$

TRUE

(g) $a_n = (-4)^n$

$$(-4)^n = 8(-4^{n-1}) - 16(-4^{n-2})$$
$$= -4^{n-2}(8\cdot 4 - 16)$$
$$= -4^{n-2}(-48)$$
$$= -4^{n-2}(-4^2)(-3)$$

FALSE

$$(-4)^n \neq -3(-4)^n$$

(h) $a_n = n^2 4^n$

$$n^2 4^n = 8((n-1)^2 4^{(n-1)}) - 16((n-2)^2 4^{(n-2)})$$
$$= 4^{n-2}(8\cdot(n-1)^2\cdot 4 - 16\cdot(n-2)^2)$$
$$= 4^{n-2}(32(n^2-2n+1) - 16(n^2-4n+4))$$
$$= 4^{n-2}(32n^2 - 64n + 32 - 16n^2 + 64n - 64)$$
$$= 4^{n-2}(16n^2 - 32)$$
$$= 4^{n-2}\cdot 4^2(n^2-2)$$
$$n^2 4^n \neq 4^n(n^2-2)$$

FALSE

Induction and Recursion Homework
Theo Shin

p. 168 #12

p.168 #12

Recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$

(a) $a_n = 0$

$a_{n-1} = 0$

$a_{n-2} = 0$

$a_n = -3(0) + 4(0) = 0$

(b) $a_n = 1$

$a_{n-1} = 1$

$a_{n-2} = 1$

$a_n = -3(1) + 4(1) = 1$

(c) $a_n = (-4)^n$

$a_{n-1} = (-4)^{n-1}$

$a_{n-2} = (-4)^{n-2}$

$a_n = -3(-4)^{n-1} + 4(-4)^{n-2}$

$= -3(-4)\cdot(-4)^{n-2} + 4(-4)^{n-2}$

$= (-4)^{n-2}((-3\cdot -4) + 4)$ Factored out -4^{n-2}

$= (-4)^{n-2}(16)$ $16 = -4^2$ in order to get same base

$= (-4)^{n-2}(-4)^2$ • 2's cancel each other (exponents)

$= (-4)^n$ • Left with a_n

Induction and Recursion Homework
Theo Shin

(b) $a_n = 2(-4)^n + 3$

$$a_{n-1} = 2(-4)^{n-1} + 3$$

$$a_{n-2} = 2(-4)^{n-2} + 3$$

$$a_n = -3\left(2(-4)^{n-1} + 3\right) + 4\left(2(-4)^{n-2} + 3\right)$$

distribution

$$= -6(-4)^{n-1} - 9 + 8(-4)^{n-2} + 12$$

$$= -6(-4)^{n-1} + 8(-4)^{n-2} - 9 + 12$$

$$= -6(-4)(-4)^{n-2} + 8(-4)^{n-2} + 3$$

$$= -4^{n-2}(-6\cdot -4 + 8 + 3)$$ factored out -4^{n-2}

$$= -4^{n-2}(32 + 3)$$ → We want to get base -4 and looking at a_n, we are trying to keep $(+3)$

$$= -4^{n-2}((16 \cdot 2) + 3)$$

$$= -4^{n-2}((-4^2) \cdot 2 + 3)$$ exponent property left with -4^n

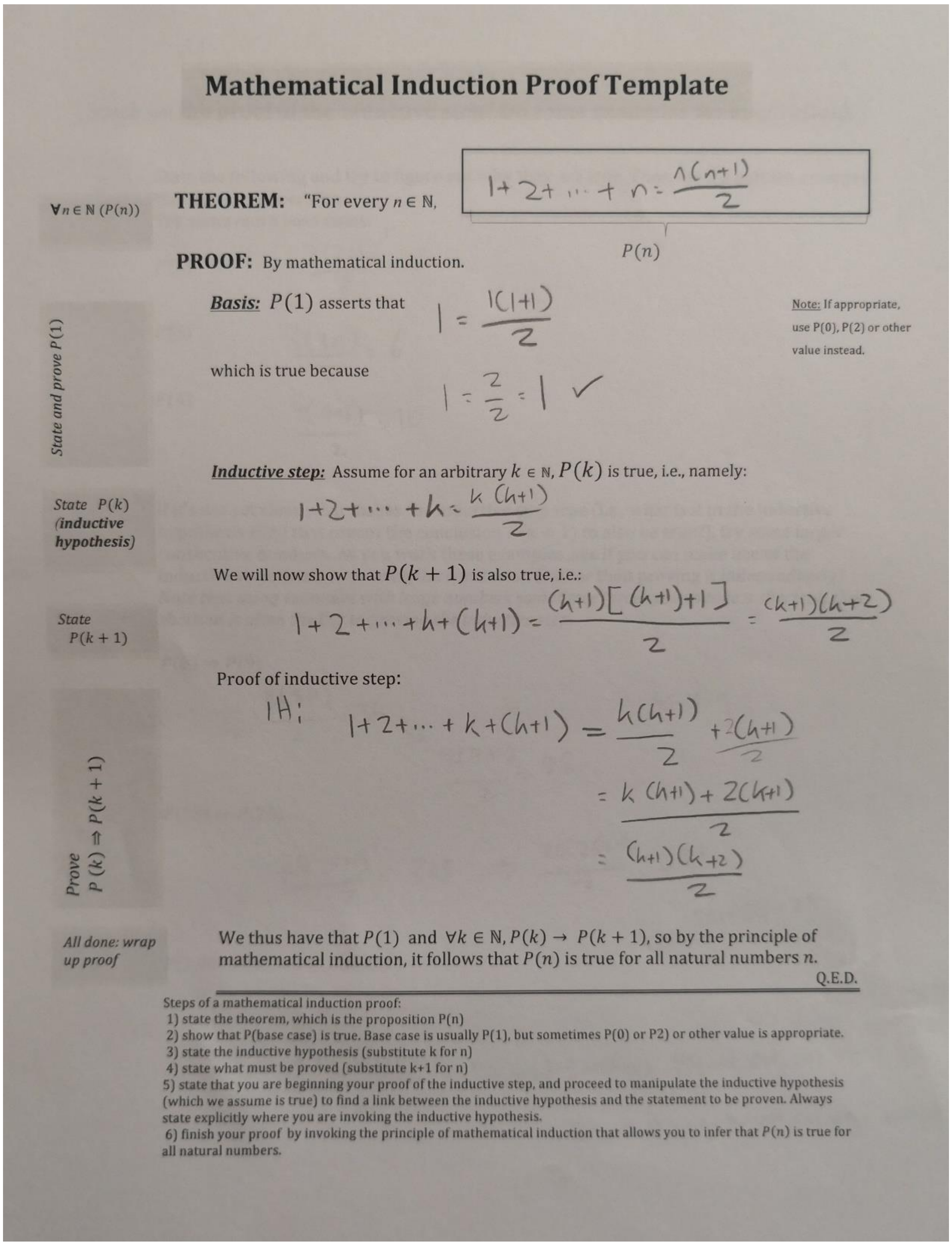
$$= -4^n \cdot 2 + 3$$

$$= \boxed{2(-4^n) + 3} = a_n$$

Induction and Recursion Homework
Theo Shin

2. Induction Workout:
   Rewrite Examples 1,2,3,5,6 from 5.1 using template

   EXAMPLE 1



Mathematical Induction Proof Template

$\forall n \in \mathbb{N}\,(P(n))$

THEOREM: "For every $n \in \mathbb{N}$, $\underbrace{\boxed{1+2+\dots+n=\frac{n(n+1)}{2}}}_{P(n)}$

PROOF: By mathematical induction.

State and prove $P(1)$

Basis: $P(1)$ asserts that $1=\frac{1(1+1)}{2}$

which is true because $1=\frac{2}{2}=1$ ✓

Note: If appropriate, use P(0), P(2) or other value instead.

*State $P(k)$ (**inductive hypothesis**)*

Inductive step: Assume for an arbitrary $k \in \mathbb{N}$, $P(k)$ is true, i.e., namely:

$$1+2+\dots+k=\frac{k(k+1)}{2}$$

State $P(k+1)$

We will now show that $P(k+1)$ is also true, i.e.:

$$1+2+\dots+k+(k+1)=\frac{(k+1)[(k+1)+1]}{2}=\frac{(k+1)(k+2)}{2}$$

Prove $P(k) \Rightarrow P(k+1)$

Proof of inductive step:

IH: $$1+2+\dots+k+(k+1)=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$$
$$=\frac{k(k+1)+2(k+1)}{2}$$
$$=\frac{(k+1)(k+2)}{2}$$

All done: wrap up proof

We thus have that $P(1)$ and $\forall k \in \mathbb{N}, P(k) \rightarrow P(k+1)$, so by the principle of mathematical induction, it follows that $P(n)$ is true for all natural numbers n.

Q.E.D.

Steps of a mathematical induction proof:
1) state the theorem, which is the proposition P(n)
2) show that P(base case) is true. Base case is usually P(1), but sometimes P(0) or P2) or other value is appropriate.
3) state the inductive hypothesis (substitute k for n)
4) state what must be proved (substitute k+1 for n)
5) state that you are beginning your proof of the inductive step, and proceed to manipulate the inductive hypothesis (which we assume is true) to find a link between the inductive hypothesis and the statement to be proven. Always state explicitly where you are invoking the inductive hypothesis.
6) finish your proof by invoking the principle of mathematical induction that allows you to infer that $P(n)$ is true for all natural numbers.

Induction and Recursion Homework
Theo Shin

Stuck on the proof of the inductive step? Do some examples for inspiration!

State the following and try to figure out why they are true. Then see if a pattern emerges that you can generalize.
Try some more base cases:

$P(2)$ $= \frac{2(2+1)}{2} = 3$

$P(3)$ $\frac{3(3+1)}{2} = 6$

$P(4)$ $\frac{4(4+1)}{2} = 10$

If it's not yet clear what makes the inductive step true (i.e., what is it in the inductive hypothesis $P(k)$ that causes the conclusion $P(k+1)$ to also be true?), try some larger consecutive numbers. As you work these examples, see if you can make use of the inductive hypothesis in proving the conclusion (rather than proving it independently). *Note that using examples with large numbers sometimes forces you to take a shortcut; that shortcut is often the key to proving the inductive step.*

$P(8) \Rightarrow P(9)$

$\frac{8(8+1)}{2} = 36 \Rightarrow \frac{9(9+1)}{2} = 45$

$45 - 36 = 9$

$P(25) \Rightarrow P(26)$

$\frac{25(25+1)}{2} = 325 \Rightarrow \frac{26(26+1)}{2} = 351$

$351 - 325 = 26$

$P(1{,}000{,}000) \Rightarrow P(1{,}000{,}001)$

$\frac{1000{,}000(1{,}000{,}000+1)}{2} = 500{,}000{,}500{,}000 \Rightarrow 500{,}001{,}500{,}001$

$500{,}001{,}500{,}001 - 500{,}000{,}500{,}000 = 1000001$

Induction and Recursion Homework
Theo Shin

EXAMPLE 2

Mathematical Induction Proof Template

$\forall n \in \mathbb{N}\ (P(n))$

THEOREM: "For every $n \in \mathbb{N}$, $\underbrace{\boxed{1+3+5+\cdots+(2n-1)=n^2}}_{P(n)}$

PROOF: By mathematical induction.

State and prove $P(1)$

Basis: $P(1)$ asserts that $1=1^2$ ✓

Note: If appropriate, use P(0), P(2) or other value instead.

which is true because

*State $P(k)$ (**inductive hypothesis**)*

Inductive step: Assume for an arbitrary $k \in \mathbb{N}$, $P(k)$ is true, i.e., namely:

$$1+3+5+\cdots+(2k-1)=k^2$$

State $P(k+1)$

We will now show that $P(k+1)$ is also true, i.e.:

$$1+3+5+\cdots+(2k-1)+(2k+1)=\overset{k^2-2k+1}{(k+1)^2}$$

Prove $P(k) \Rightarrow P(k+1)$

Proof of inductive step:

IH: $$1+3+5+\cdots+(2k-1)+(2k+1)=k^2+(2k+1)$$
$$=k^2+2k+1$$
$$=(k+1)^2$$

All done: wrap up proof

We thus have that $P(1)$ and $\forall k \in \mathbb{N}, P(k) \rightarrow P(k+1)$, so by the principle of mathematical induction, it follows that $P(n)$ is true for all natural numbers n.

Q.E.D.

Steps of a mathematical induction proof:
1) state the theorem, which is the proposition P(n)
2) show that P(base case) is true. Base case is usually P(1), but sometimes P(0) or P2) or other value is appropriate.
3) state the inductive hypothesis (substitute k for n)
4) state what must be proved (substitute k+1 for n)
5) state that you are beginning your proof of the inductive step, and proceed to manipulate the inductive hypothesis (which we assume is true) to find a link between the inductive hypothesis and the statement to be proven. Always state explicitly where you are invoking the inductive hypothesis.
6) finish your proof by invoking the principle of mathematical induction that allows you to infer that $P(n)$ is true for all natural numbers.

Induction and Recursion Homework
Theo Shin

Stuck on the proof of the inductive step? Do some examples for inspiration!

State the following and try to figure out why they are true. Then see if a pattern emerges that you can generalize.
Try some more base cases:

$P(2)$ $1+3=4$

$P(3)$ $1+3+5=9$

$P(4)$ $1+3+5+7=16$

If it's not yet clear what makes the inductive step true (i.e., what is it in the inductive hypothesis $P(k)$ that causes the conclusion $P(k+1)$ to also be true?), try some larger consecutive numbers. As you work these examples, see if you can make use of the inductive hypothesis in proving the conclusion (rather than proving it independently). *Note that using examples with large numbers sometimes forces you to take a shortcut; that shortcut is often the key to proving the inductive step.*

$P(8) \Rightarrow P(9)$

$P(8)=1+3+5+7+9+11+13+15=64$ $[8^2]$

$P(9)=1+3+\dots+15+17=81$ $[9^2]$

$81-64=17$ $(26\cdot2)-1$

$P(25) \Rightarrow P(26)$

$P(25)=625$

$P(26)=676$

$676-625=51=(2n-1)$

$P(1{,}000{,}000) \Rightarrow P(1{,}000{,}001)$

$P(1000000)=1{,}000{,}000{,}000{,}000$

$P(1000001)=1{,}000{,}002{,}000{,}001$

$(-)$ $2{,}000{,}001$

$=(1{,}000{,}001\times2)-1$

Induction and Recursion Homework
Theo Shin

EXAMPLE 3

Mathematical Induction Proof Template

$\forall n \in \mathbb{N}\ (P(n))$

THEOREM: "For every $n \in \mathbb{N}$, $\boxed{1+2+2^2+\cdots+2^n=2^{n+1}-1}$

$\underbrace{\phantom{1+2+2^2+\cdots+2^n=2^{n+1}-1}}_{P(n)}$

PROOF: By mathematical induction.

State and prove $P(1)$

Basis: $P(1)$ asserts that

$$P(0):\ 2^0 = 2^1 - 1$$

which is true because

$$= 1 = 1 \checkmark$$

Note: If appropriate, use P(0), P(2) or other value instead.

*State $P(k)$ (**inductive hypothesis**)*

Inductive step: Assume for an arbitrary $k \in \mathbb{N}$, $P(k)$ is true, i.e., namely:

$$1+2+2^2+\cdots+2^k = 2^{k+1}-1$$

State $P(k+1)$

We will now show that $P(k+1)$ is also true, i.e.:

$$1+2+2^2+\cdots+2^k+2^{k+1} = 2^{(k+1)+1}-1 = 2^{k+2}-1$$

Prove $P(k) \Rightarrow P(k+1)$

Proof of inductive step:

$$\text{IH: } 1+2+2^2+\cdots+2^k+2^{k+1}$$
$$= (2^{k+1}-1)+2^{k+1}$$
$$= 2\cdot 2^{k+1}-1$$
$$= 2^{k+2}-1$$

All done: wrap up proof

We thus have that $P(1)$ and $\forall k \in \mathbb{N}, P(k) \rightarrow P(k+1)$, so by the principle of mathematical induction, it follows that $P(n)$ is true for all natural numbers n.

Q.E.D.

Steps of a mathematical induction proof:
1) state the theorem, which is the proposition P(n)
2) show that P(base case) is true. Base case is usually P(1), but sometimes P(0) or P2) or other value is appropriate.
3) state the inductive hypothesis (substitute k for n)
4) state what must be proved (substitute k+1 for n)
5) state that you are beginning your proof of the inductive step, and proceed to manipulate the inductive hypothesis (which we assume is true) to find a link between the inductive hypothesis and the statement to be proven. Always state explicitly where you are invoking the inductive hypothesis.
6) finish your proof by invoking the principle of mathematical induction that allows you to infer that $P(n)$ is true for all natural numbers.

Induction and Recursion Homework
Theo Shin

Stuck on the proof of the inductive step? Do some examples for inspiration!

State the following and try to figure out why they are true. Then see if a pattern emerges that you can generalize.
Try some more base cases:

$P(2)$

$2^2 = 2^3 - 1$

$1 + 2 + 2^2 = 7$

$P(3)$

$1 + 2 + 2^2 + 2^3 = 15 = 2^4 - 1$

$P(4)$

$1 + 2 + 2^2 + 2^3 + 2^4 = 2^5 - 1$

$= 31$

If it's not yet clear what makes the inductive step true (i.e., what is it in the inductive hypothesis $P(k)$ that causes the conclusion $P(k+1)$ to also be true?), try some larger consecutive numbers. As you work these examples, see if you can make use of the inductive hypothesis in proving the conclusion (rather than proving it independently). *Note that using examples with large numbers sometimes forces you to take a shortcut; that shortcut is often the key to proving the inductive step.*

$P(8) \Rightarrow P(9)$

$P(8) = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 = 2^9 - 1 = 511$

$P(9) = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 = 2^{10} - 1 = 1023$

1023
$(-)\ 511$
512

$P(25) \Rightarrow P(26)$

$P(25) = 2^{26} - 1 = 67108863$ $\quad (-)\ 67108864 = 2^{26}$

$P(26) = 2^{27} - 1 = 134217727$

$P(1{,}000{,}000) \Rightarrow P(1{,}000{,}001)$

Induction and Recursion Homework
Theo Shin

EXAMPLE 5

Mathematical Induction Proof Template

$\forall n \in \mathbb{N}\,(P(n))$

THEOREM: "For every $n \in \mathbb{N}$, $\boxed{n < 2^n}$

$\underbrace{\qquad\qquad}_{P(n)}$

PROOF: By mathematical induction.

State and prove $P(1)$

Basis: $P(1)$ asserts that

$$P(1) = 1 < 2^1$$

which is true because

$$1 < 2 \checkmark$$

Note: If appropriate, use P(0), P(2) or other value instead.

State $P(k)$ (inductive hypothesis)

Inductive step: Assume for an arbitrary $k \in \mathbb{N}$, $P(k)$ is true, i.e., namely:

$$k < 2^k$$

State $P(k+1)$

We will now show that $P(k+1)$ is also true, i.e.:

$$k+1 < 2^{k+1}$$

Prove $P(k) \Rightarrow P(k+1)$

Proof of inductive step:

$$\begin{aligned} \text{IH: } k+1 &< 2^k + 1 \\ &\leq 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

All done: wrap up proof

We thus have that $P(1)$ and $\forall k \in \mathbb{N}, P(k) \rightarrow P(k+1)$, so by the principle of mathematical induction, it follows that $P(n)$ is true for all natural numbers n.

Q.E.D.

Steps of a mathematical induction proof:
1) state the theorem, which is the proposition P(n)
2) show that P(base case) is true. Base case is usually P(1), but sometimes P(0) or P2) or other value is appropriate.
3) state the inductive hypothesis (substitute k for n)
4) state what must be proved (substitute k+1 for n)
5) state that you are beginning your proof of the inductive step, and proceed to manipulate the inductive hypothesis (which we assume is true) to find a link between the inductive hypothesis and the statement to be proven. Always state explicitly where you are invoking the inductive hypothesis.
6) finish your proof by invoking the principle of mathematical induction that allows you to infer that $P(n)$ is true for all natural numbers.

Induction and Recursion Homework
Theo Shin

Stuck on the proof of the inductive step? Do some examples for inspiration!

State the following and try to figure out why they are true. Then see if a pattern emerges that you can generalize.
Try some more base cases:

$P(2)$ $2 < 2^2$ $2 < 4$

$P(3)$ $3 < 2^3$ $3 < 8$

$P(4)$ $4 < 2^4$ $4 < 16$

If it's not yet clear what makes the inductive step true (i.e., what is it in the inductive hypothesis $P(k)$ that causes the conclusion $P(k+1)$ to also be true?), try some larger consecutive numbers. As you work these examples, see if you can make use of the inductive hypothesis in proving the conclusion (rather than proving it independently). *Note that using examples with large numbers sometimes forces you to take a shortcut; that shortcut is often the key to proving the inductive step.*

$P(8) \Rightarrow P(9)$

$8 < 2^8$ $9 < 2^9$
$8 < 256$ $9 < 512$

$P(25) \Rightarrow P(26)$

$P(1{,}000{,}000) \Rightarrow P(1{,}000{,}001)$

Induction and Recursion Homework
Theo Shin

EXAMPLE 6

Mathematical Induction Proof Template

$\forall n \in \mathbb{N}\,(P(n))$

THEOREM: "For every $n \in \mathbb{N}$, $\underbrace{\boxed{2^n < n! \text{ with } n \geq 4}}_{P(n)}$

PROOF: By mathematical induction.

State and prove $P(1)$

Basis: $P(1)$ asserts that

$$P(4) = 2^4 < 4!$$

which is true because

$$16 < 24$$

Note: If appropriate, use P(0), P(2) or other value instead.

State $P(k)$ (inductive hypothesis)

Inductive step: Assume for an arbitrary $k \in \mathbb{N}$, $P(k)$ is true, i.e., namely:

$2^k < k!$ for positive integer k with $k \geq 4$

State $P(k+1)$

We will now show that $P(k+1)$ is also true, i.e.:

$$2^{k+1} < (k+1)!$$

Prove $P(k) \Rightarrow P(k+1)$

Proof of inductive step:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k && \text{• definition of exponent} \\ &< 2 \cdot k! && \text{• inductive hypothesis} \\ &< (k+1)k! && \text{, Since } 2 < k+1 \\ &= (k+1)! && \text{• definition of factorial} \end{aligned}$$

All done: wrap up proof

We thus have that $P(1)$ and $\forall k \in \mathbb{N}, P(k) \rightarrow P(k+1)$, so by the principle of mathematical induction, it follows that $P(n)$ is true for all natural numbers n.

Q.E.D.

Steps of a mathematical induction proof:
1) state the theorem, which is the proposition P(n)
2) show that P(base case) is true. Base case is usually P(1), but sometimes P(0) or P2) or other value is appropriate.
3) state the inductive hypothesis (substitute k for n)
4) state what must be proved (substitute k+1 for n)
5) state that you are beginning your proof of the inductive step, and proceed to manipulate the inductive hypothesis (which we assume is true) to find a link between the inductive hypothesis and the statement to be proven. Always state explicitly where you are invoking the inductive hypothesis.
6) finish your proof by invoking the principle of mathematical induction that allows you to infer that $P(n)$ is true for all natural numbers.

Induction and Recursion Homework
Theo Shin

Stuck on the proof of the inductive step? Do some examples for inspiration!

State the following and try to figure out why they are true. Then see if a pattern emerges that you can generalize.
Try some more base cases:

$P(2)$

$n \geq 4$

$P(3)$

$n \geq 4$

$P(4)$

$2^4 < 4!$

$16 < 24$

If it's not yet clear what makes the inductive step true (i.e., what is it in the inductive hypothesis $P(k)$ that causes the conclusion $P(k+1)$ to also be true?), try some larger consecutive numbers. As you work these examples, see if you can make use of the inductive hypothesis in proving the conclusion (rather than proving it independently). *Note that using examples with large numbers sometimes forces you to take a shortcut; that shortcut is often the key to proving the inductive step.*

$P(8) \Rightarrow P(9)$

$2^8 < 8!$ $\quad$ $2^9 < 9!$

$256 < 40320$ $\quad$ $512 < 362880$

$P(25) \Rightarrow P(26)$

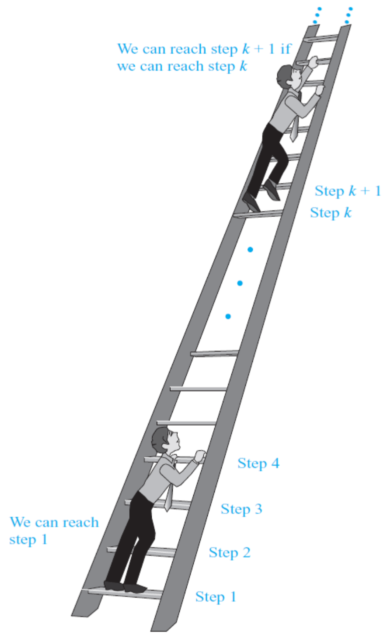
$2^{25} < 25!$ $\quad$ $2^{26} < 26!$

$33554432 < 1.551121\text{e}25$ $\quad$ $67108864 < 4.03291461\text{e}26$

$P(1{,}000{,}000) \Rightarrow P(1{,}000{,}001)$

Induction and Recursion Homework
Theo Shin

3.



Going off the Figure 1 from Rosen p.512, I view the induction process as starting from the bottom rung of the ladder by proving the basis step. In most cases, that is proving P(1) is true or the lowest element in the set. Once it's been verified the basis step is true, we can imagine stepping up to the kth step. With the Inductive hypothesis, we are going to assume P(k) is true meaning we are going to assume we can get to the kth rung of the ladder. Then, by verifying we can get to the P(k+1)st rung of the ladder (verifying if P(k) is true then P(k+1) is true) then we can prove the original statement P(n) (i.e. summation, squared numbers, etc.) is true for all k's within that domain (i.e. all natural numbers). We can test the bottom rungs of the ladder by verifying P(2) is true due to inferring P(1) being true and subsequently, P(3) is true because P(2) is true, and so on. By proving and verifying while going up the ladder each rung holds true, then essentially we can "believe" that the induction process holds true no matter how high we want to climb (infinite). As stated in the lectures, we need to remember to state the inductive hypothesis where IF we assume P(k) is true for an arbitrary positive integer k and show under this assumption P(k+1) is true, we can show P(n) is then true for all positive integers. Also, as stated in the textbook, mathematical induction is valid due to the well-ordering property where every nonempty subset of the set of positive integers has a least element and its proof through contradiction.

Induction and Recursion Homework
Theo Shin

Induction Proofs:

4.

For all natural numbers n, $a^n \times b^n = (ab)^n$

$P(n)$ is proposition $a^n \times b^n = (ab)^n$

Step 1: BASIS STEP

$a^1 \times b^1 = (ab)^1$

$a^1 \times b^1 = ab$ ✓ $P(1)$ is true

Step 2: INDUCTIVE STEP

IH: $a^k \cdot b^k = (ab)^k$

[Showing $P(k+1)$ is also true: $a^{k+1} \cdot b^{k+1} = (ab)^{k+1}$]

$= (a^k) \cdot (a^1) \cdot (b^k)(b^1)$

$= a^k \cdot b^k (ab)$ • Commutativity of multiplication

$= (ab)^k (ab)$ • Using IH: substituted $a^k \cdot b^k$ with $(ab)^k$

$= \boxed{(ab)^{k+1}}$ • Exponent product property

△ Thus, we have $P(1)$ and $\forall k \in \mathbb{N}, P(k) \to P(k+1)$, so by principle of mathematical induction, it follows $P(n)$ is true for all natural numbers n.

Induction and Recursion Homework
Theo Shin

5.

For all natural numbers n, $10^n \bmod 3 = 1$

$P(n)$ is proposition $10^n \bmod 3 = 1$

① BASIS STEP

$10^1 \bmod 3 = 1 = 1$ ✓
$P(1)$ holds true

② INDUCTIVE STEP

IH: $10^k \bmod 3 = 1$

[Showing $P(k+1)$ is also true, $10^{k+1} \bmod 3 = 1$]

$10^{k+1} \bmod 3 = (10^k \bmod 3) \times (10 \bmod 3)$

$= 1 \times (10 \bmod 3)$ Using IH for substitution

$= 1 \times (1)$ Calculating mod equation

$= \boxed{1}$

∆ Thus, we have $P(1)$ and $\forall k \in \mathbb{N}, P(k) \to P(k+1)$, so by principle of mathematical induction, it follows that $P(n)$ is true for all natural numbers n.

Induction and Recursion Homework
Theo Shin

6.

For all natural numbers >1, $1^3+2^3+3^3+\cdots+n^3 = n^2(n+1)^2/4$

P(n) is proposition $1^3+2^3+3^3+\cdots+n^3 = n^2(n+1)^2/4$

① BASIS STEP

$$P(2): \quad 1^3+2^3 = 2^2(2+1)^2/4$$
$$9 = 4(3)^2/4$$
$$= 4\cdot 9/4$$
$$= 36/4$$
$$= 9$$

P(2) holds true

② INDUCTIVE STEP

$$\text{IH: } 1^3+2^3+3^3+\cdots+k^3 = k^2(k+1)^2/4$$

[Showing P(k+1) is also true, $1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = (k+1)^2(k+2)^2/4$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 \rightarrow \frac{4(k+1)^3}{4}$$

• By assumption of IH, substitution
• Multiplicative identity

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

Distributive Property
Factored out $(k+1)^2$

$$= \frac{(k+1)^2\cdot(k^2+4(k+1))}{4}$$

property of exponents

$$= \frac{(k+1)^2\cdot(k^2+4k+4)}{4}$$

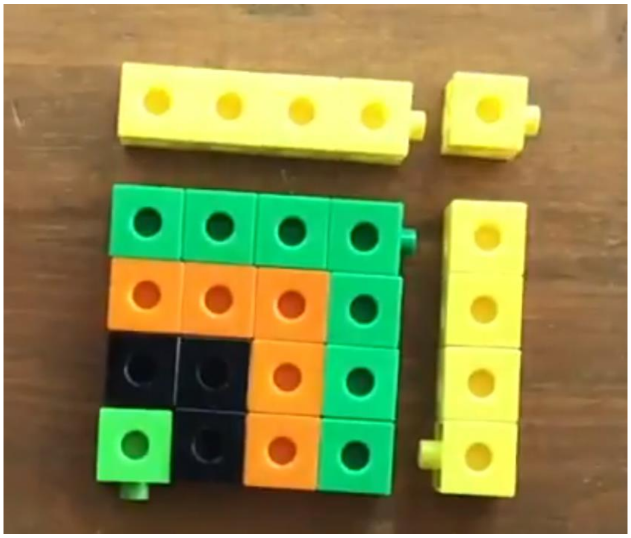
Distribution

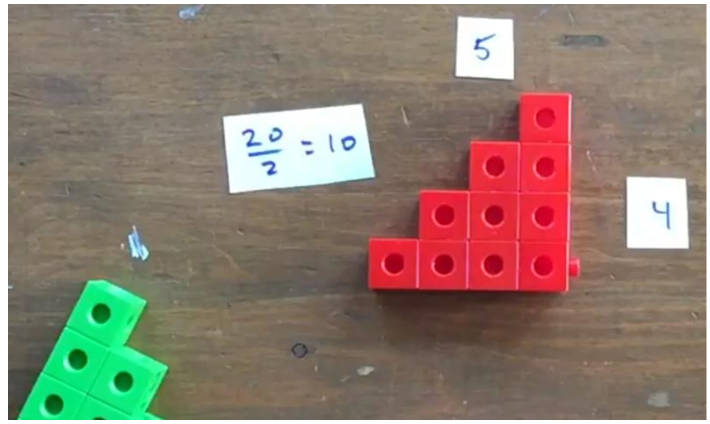
$$= \boxed{\frac{(k+1)^2\cdot(k+2)^2}{4}}$$

∴ Thus, we have P(1) and $\forall k \in \mathbb{N}$, $P(k) \rightarrow P(k+1)$, so by principle of mathematical induction, it follows that P(n) is true for all natural numbers n.

Induction and Recursion Homework
Theo Shin

7.





The use of blocks to demonstrate the statements “sum of n” and “sum of the odds” is a good demonstration of going about proofs using mathematical induction. In both instances, you should start mathematical induction by verifying the least element [P(1)] is true. Using the colored blocks, we can visually see and count to verify P(1), P(2), P(3), P(4)….. holds true. In the left image (sum of the odds), we can see the pattern of how it literally forms a square as you continue adding odd number of blocks to the top and right sides. On the right, I liked how the stairs were similarly visually representing the rungs of a ladder, but instead forming a staircase. Connecting the green and red staircases, we can visually see how the sum of n positive integers is equal to n(n+1)/2, since it forms a rectangle with one side having one more block than the other. By dividing by 2 or separating the colored staircases, we can count and verify the statement “sum of n” holds true.

Induction and Recursion Homework
Theo Shin

8. P. 370 #3 and #4 with explanations regarding RSA

p.370 #3

$\gcd(\overset{a}{8}, \overset{b}{13})$	$13 = 8(1) + 5$
$\gcd(5, 8)$	$8 = 5(1) + 3$
$\gcd(3, 5)$	$5 = 3(1) + 2$
$\gcd(2, 3)$	$3 = 2(1) + 1$
$\gcd(1, 2)$	$2 = 1(2) + 0$
$\gcd(0, \overset{b}{1})$	

4.

$\gcd(\overset{a}{12}, \overset{b}{17})$	$17 = 12(1) + 5$
$\gcd(5, 12)$	$12 = 5(2) + 2$
$\gcd(2, 5)$	$5 = 2(2) + 1$
$\gcd(1, 2)$	$2 = 1(2) + 0$
$\gcd(0, \overset{b}{1})$	

The “Euclidean_Alg(a,b)” function used in my RSA project carried over the basic principles and many similarities from Algorithm 3 in the textbook. Essentially, I used a while loop to continue tracking the remainder (%) until it reached 0, returning the “b” coefficient. I was able to plug in the values of a and b from problems #3 and #4 in RSA algorithm I used and in both cases, it returned a 1 for the value of b.

Induction and Recursion Homework
Theo Shin

```python
def Euclidean_Alg(a, b):
    """
    - Calculate the Greatest Common Divisor of a and b.
    - Unless b==0, the result will have the same sign as b (so that when b is divided
      by it, the result comes out positive).
    - The function must return a single integer 'x' which is the gcd of a and b.
    - Implement Algorithm 1 as Euclidean_Alg from page 269. Ultimately you will input
      the binary expansion as a string (see 6) so keep that in mind. You can practice
      it however you like though.)

    Args:
        a (int): An integer
        b (int): An integer

    Returns:
        int: Greatest Common Divisor of a and b
    """
    # SIMPLIFIED VERSION:
    #if b == 0:
    #     return a
    #else:
    #    return Euclidean_Alg(b,a%b)

    if a > b:            #stating the lower of a,b
        result = b
    result = a

    if result == 1:      #if a,b is equal to 1, return 1
        return 1

    while result > 0:    #while a,b are positive
        if a % result == 0 and b % result == 0:     #if a mod b= 0 and b mod a =0
            return result
        result = result - 1

print(Euclidean_Alg(12,17))
```

1

Induction and Recursion Homework
Theo Shin

9. P. 370 #5 and #5 with RSA explanations.

5. $m=5, n=11, b=3$

$3^{11} \bmod 5$

Since $n=11$ is not even

$mpower(3, \lfloor 11/2 \rfloor, 5)^2 \bmod 5 \cdot 3 \bmod 5) \bmod 5$

$mpower(3, 5, 5)^2 \bmod 5 \cdot 3 \bmod 5) \bmod 5$

Since $n=5$ is not even

$mpower(3, \underbrace{\lfloor 5/2 \rfloor}_{=2}, 5)^2 \bmod 5 \cdot 3 \bmod 5) \bmod 5$

Since $n=2$ is even

$mpower(3, \underbrace{2/2}_{=1}, 5)^2 \bmod 5$

Since $n=1$ is not even

$mpower(3, \underbrace{\lfloor 1/2 \rfloor}_{=0}, 5)^2 \bmod 5 \cdot 3 \bmod 5) \bmod 5$

Since $n=0$

output is $3^0 \bmod 5 = 1 = mpower(3, 0, 5)$

Induction and Recursion Homework
Theo Shin

6. $m=7, n=10, b=2$

$2^{10} \bmod 7$

Since $n=10$ is even

$\text{mpower}(2, \underbrace{10/2}_{=5}, 7)^2 \bmod 7$

Since $n=5$ is not even

$\text{mpower}(2, \underbrace{\lfloor 5/2 \rfloor}_{=2}, 7)^2 \bmod 7 \cdot 2 \bmod 7) \bmod 7$

Since $n=2$ is even

$\text{mpower}(2, \underbrace{2/2}_{=1}, 7)^2 \bmod 7$

Since $n=1$ is not even

$\text{mpower}(2, \underbrace{\lfloor 1/2 \rfloor}_{=0}, 7)^2 \bmod 7 \cdot 2 \bmod 7) \bmod 7$

Since $n=0$, output is $2^0 \bmod 7 = 1 = \text{mpower}(2,0,7)$

There are similarities between Algorithm 4 and the RSA - FME(b,n.m) function. The majority of the algorithm was similar with the main difference being the use of a binary string conversion (Convert_Binary_String(n)) and incorporating the string into the function. As written in the RSA code, I used similar parameters where an if loop was nested inside a while loop and the conditions were set depending on whether n % 2 == 1 (if n value is odd) or else statement (n%2==0). I will keep in mind using the Python pow() function as I hope to rebuild the RSA project in the near future.