Labs
Machine Learning Course
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Problem Set 11, Nov 30, 2021 (Solutions to Theory Questions)

1 Vector Calculus

1. We have $\nabla f(\mathbf{x}) = (\mathbf{A} + \mathbf{A}^\top)\mathbf{x} + \mathbf{b}$. One way to see this is to explicitly expand out the expression. We have

$$f(\mathbf{x}) = \sum_{i,j} A_{i,j} x_i x_j + \sum_i b_i x_i + c.$$

If we now take the derivative with respect to x_k we get

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = \sum_j A_{k,j} x_j + \sum_i A_{i,k} x_i + b_k.$$

2. $\nabla^2 f(\mathbf{x}) = \mathbf{A} + \mathbf{A}^\top$. Taking the derivative of $\frac{\partial f(\mathbf{x})}{\partial x_k}$, as given in the previous expression, with respect to x_l we get

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_k \partial_l} = A_{k,l} + A_{l,k}.$$

2 Maximum Likelihood Principle

1. The likelihood is given by

$$
\begin{aligned}
\mathbb{P}[X_1, ..., X_N|\mu, \sigma^2] &= \prod_{n=1}^{N} \mathbb{P}[X_n|\mu, \sigma^2] \\
&= \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_n-\mu)^2}{2\sigma^2}\right) \\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\left(-\frac{\sum_{n=1}^{N}(X_n-\mu)^2}{2\sigma^2}\right)
\end{aligned}
$$

2. It might be easier to work with the negative log-likelihood, given by

$$
\begin{aligned}
-\log \mathbb{P}[X_1, ..., X_N|\mu, \sigma^2] &= -\log\left[\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \exp\left(-\frac{\sum_{n=1}^{N}(X_n-\mu)^2}{2\sigma^2}\right)\right] \\
&= \frac{N}{2}\log(2\pi\sigma^2) + \frac{1}{2\sigma^2}\sum_{n=1}^{N}(X_n-\mu)^2 \\
&= \frac{N}{2}\log(2\pi) + \frac{N}{2}\log(\sigma^2) + \frac{1}{2\sigma^2}\sum_{n=1}^{N}(X_n-\mu)^2
\end{aligned}
$$

   The derivative with respect to μ is

$$
\begin{aligned}
-\frac{\partial \log \mathbb{P}[X_1, ..., X_N|\mu, \sigma^2]}{\partial\mu} &= \frac{1}{2\sigma^2}\frac{\partial\left(\sum_{n=1}^{N}(X_n^2 - 2X_n\mu + \mu^2)\right)}{\partial\mu} \\
&= \frac{1}{2\sigma^2}\sum_{n=1}^{N}(-2X_n + 2\mu) \\
&= \frac{1}{\sigma^2}\sum_{n=1}^{N}(-X_n + \mu)
\end{aligned}
$$

   Setting this expression to 0, we get $\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} X_n$.
   The derivative with respect to σ^2 is

$$
\begin{aligned}
-\frac{\partial \log \mathbb{P}[X_1, ..., X_N|\mu, \sigma^2]}{\partial\sigma^2} &= \frac{N}{2}\frac{\partial\log(\sigma^2)}{\partial\sigma^2} + \frac{\partial\frac{1}{\sigma^2}}{\partial\sigma^2}\frac{1}{2}\sum_{n=1}^{N}(X_n-\mu)^2 \\
&= \frac{N}{2}\frac{1}{\sigma^2} - \frac{1}{\sigma^4}\frac{1}{2}\sum_{n=1}^{N}(X_n-\mu)^2
\end{aligned}
$$

   Setting this expression to 0, and replacing the unknown quantity μ by the estimate $\hat{\mu}$ we get

$$
\hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(X_n-\hat{\mu})^2 \,.
$$

3. By linearity of expectation, we get $\mathbb{E}[\hat{\mu}] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[X_n] = \mu$. So indeed, this estimate is *unbiased*.

4. We get that

$$
\begin{aligned}
\mathbb{E}[\hat{\sigma}^2] &= \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}(X_n-\hat{\mu})^2\right] \\
&= \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}((X_n-\mu)-(\hat{\mu}-\mu))^2\right] \\
&= \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}\left((X_n-\mu)-\frac{1}{N}\sum_{j=1}^{N}(X_j-\mu)\right)^2\right] \\
&= \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}\left(\frac{N-1}{N}(X_n-\mu)-\frac{1}{N}\sum_{j\neq n}(X_j-\mu)\right)^2\right] \\
&= \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left(\frac{N-1}{N}(X_n-\mu)-\frac{1}{N}\sum_{j\neq n}(X_j-\mu)\right)^2\right].
\end{aligned}
$$

Since the variables $X_i-\mu$ and $X_j-\mu$ for $i\neq j$ are independent and have mean $=0$, we can separate out the expectations as

$$
\begin{aligned}
\mathbb{E}[\hat{\sigma}^2] &= \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left(\frac{N-1}{N}(X_n-\mu)-\frac{1}{N}\sum_{j\neq n}(X_j-\mu)\right)^2\right] \\
&= \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left(\frac{N-1}{N}(X_n-\mu)\right)^2\right]+\frac{1}{N}\sum_{n=1}^{N}\sum_{j\neq n}\mathbb{E}\left[\left(\frac{1}{N}(X_j-\mu)\right)^2\right] \\
&= \frac{(N-1)^2}{N^3}\sum_{n=1}^{N}\mathbb{E}\left[(X_n-\mu)^2\right]+\frac{1}{N^3}\sum_{n=1}^{N}\sum_{j\neq n}\mathbb{E}\left[(X_j-\mu)^2\right] \\
&= \frac{(N-1)^2}{N^3}\sum_{n=1}^{N}\sigma^2+\frac{1}{N^3}\sum_{n=1}^{N}\sum_{j\neq n}\sigma^2 \\
&= \frac{(N-1)^2}{N^2}\sigma^2+\frac{N-1}{N^2}\sigma^2 \\
&= \frac{N^2-2N+1-1+N}{N^2}\sigma^2 \\
&= \frac{N-1}{N}\sigma^2\,.
\end{aligned}
$$

We see that the ML estimate of the variance is biased (but asymptotically as $N\to\infty$ it is unbiased).