Differential Equations: Computational Practicum

Report:

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Exact Solution:

$$\begin{cases} y' = 3y - xy^{\frac{1}{3}} \\ y(1) = 2 \end{cases}$$

$$y' - 3y = xy^{\frac{1}{3}}$$

This is the Bernoulli equation, let's solve it.

First we should divide both parts by $y^{\frac{2}{3}}$

We get

$$y'y^{-\frac{1}{3}} - 3y^{\frac{2}{3}} = -x$$

then make the following substitution

$$z = y^{\frac{2}{3}}$$

$$z' = \frac{2}{3}y - \frac{1}{3}y'$$

We get

$$\frac{3}{2}z' - 3z = -x \tag{1}$$

Equation(1) is a first-order non-homogeneous linear ordinary differential equation.

First we need to solve the complementary equation

$$\frac{3}{2}z' - 3z = 0$$

$$z' = 2z$$

$$\int \frac{dz}{z} = 2\int dx$$

$$e^{ln|z|} = e^{2x+C_1}$$

$$z = e^{2x}C_2$$

$$z' = 2e^{2x}C_2 + C_2'e^{2x}$$

Substitute to $Equation(1)$

$$3e^{2x}C_2 + \frac{3}{2}C_2'e^{2x} - 3e^{2x}C_2 = -x$$

$$\frac{3}{2}C_2'e^{2x} = -x$$

$$C_2' = -\frac{2}{3}xe^{-2x}$$

$$C_2 = -\frac{2}{3}\int xe^{-2x}dx = \frac{2}{3}.\frac{(2x+1)e^{-2x}}{4} + C_3$$

$$z = \frac{2x+1}{6} + e^{2x}C_3$$

Back substitution

$$y^{\frac{2}{3}} = \frac{x}{3} + \frac{1}{6} + e^{2x}C_3$$

$$y = (\frac{x}{3} + \frac{1}{6} + e^{2x}C_3)^{\frac{3}{2}}$$

So, let's find C_3

$$C_3 = \frac{y^{\frac{2}{3}} - \frac{x}{3} - \frac{1}{6}}{e^{2x}}$$

$$y(1) = 2$$

$$C_3 = (2^{\frac{2}{3}} - \frac{1}{2})e^{-2}$$

$$y = (\frac{x}{3} + \frac{1}{6} + e^{2(x-1)}(2^{\frac{2}{3}} - \frac{1}{2}))^{\frac{3}{2}}$$

Answer:

$$y = (\frac{x}{3} + \frac{1}{6} + e^{2(x-1)}(2^{\frac{2}{3}} - \frac{1}{2}))^{\frac{3}{2}}$$

Results

Solutions charts

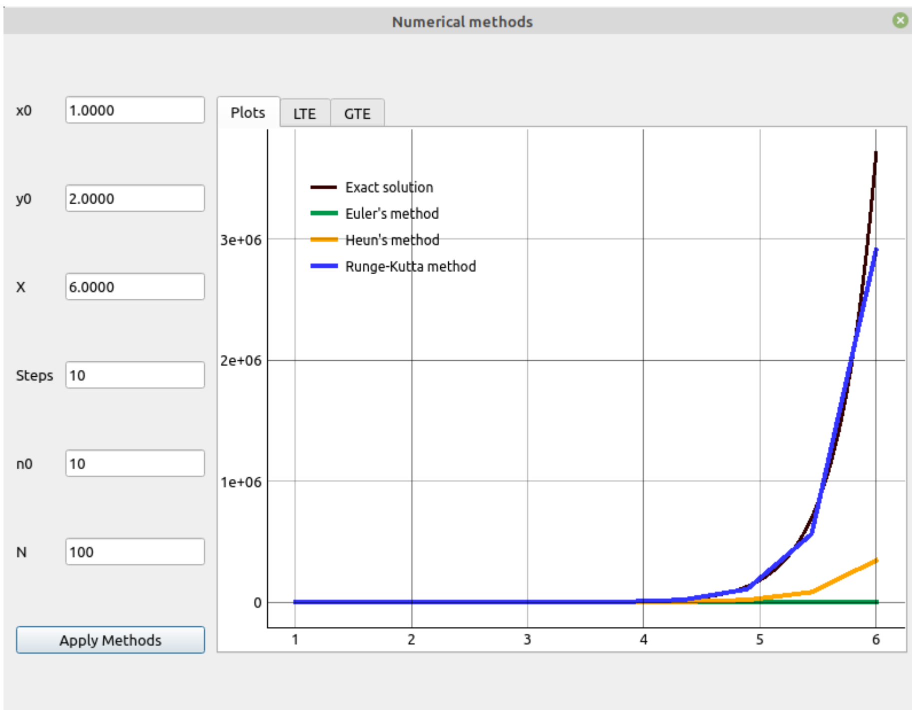

Figure 1: solutions

Chart of solution and approximate values.

We can notice that the Runge-Kutta methods calculates the most approximate values, the worst approximation is done by the Euler method.

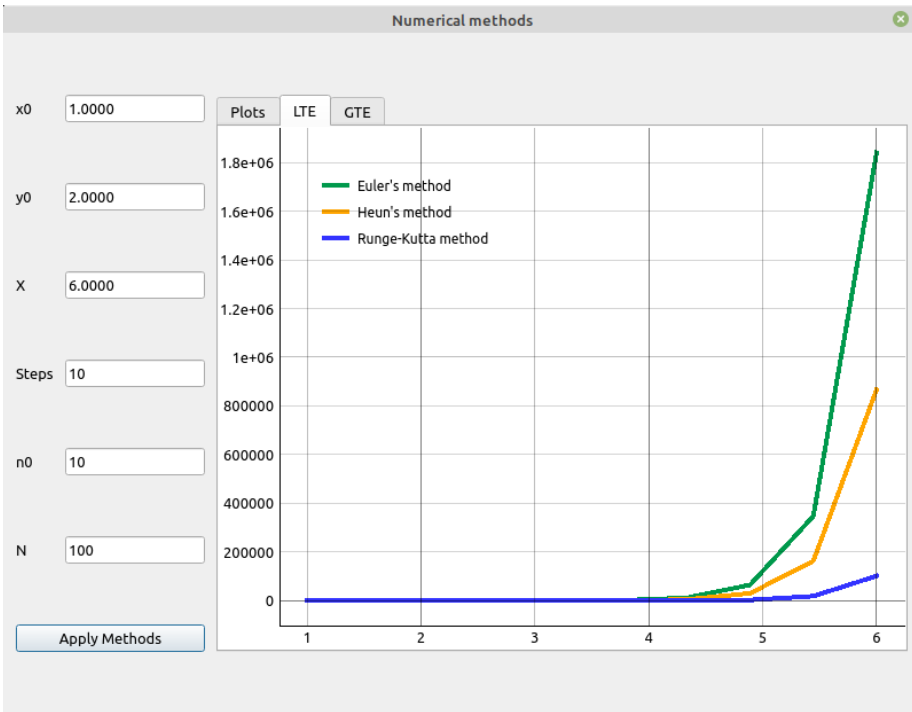
Numerical methods
x0
1.0000
y0
2.0000
X
6.0000
Steps
10
n0
10
N
100
Apply Methods
Plots
LTE
GTE
Euler's method
Heun's method
Runge-Kutta method
1.8e+06
1.6e+06
1.4e+06
1.2e+06
1e+06
800000
600000
400000
200000
0
1
2
3
4
5
6

Figure 2: LTE

LTE charts

Also we can see that the Runge-Kutta methods has the smallest error and the Euler has larger errors.

Global Errors chart

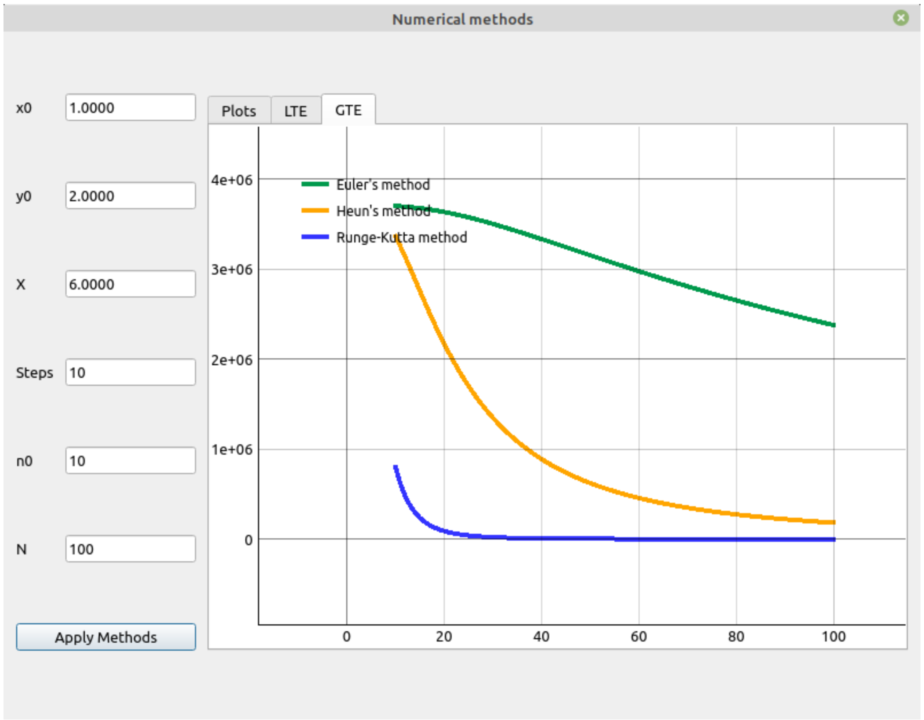

Figure 3: GTE

We can see here that if we increase the steps, the value of GTE decreases.

UML Diagram

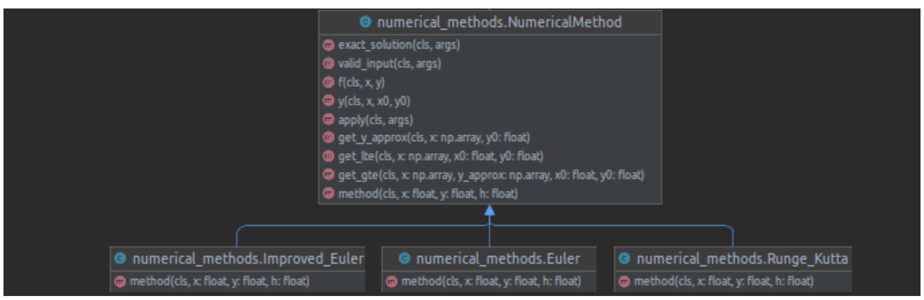
numerical_methods.NumericalMethod
exact_solution(cls, args)
valid_input(cls, args)
f(cls, x, y)
y(cls, x, x0, y0)
apply(cls, args)
get_y_approx(cls, x: np.array, y0: float)
get_lte(cls, x: np.array, x0: float, y0: float)
get_gte(cls, x: np.array, y_approx: np.array, x0: float, y0: float)
method(cls, x: float, y: float, h: float)
numerical_methods.Improved_Euler
method(cls, x: float, y: float, h: float)
numerical_methods.Euler
method(cls, x: float, y: float, h: float)
numerical_methods.Runge_Kutta
method(cls, x: float, y: float, h: float)

Figure 4: UML