0.1 Coupon Bonds with Stochastic Interest Rate

A coupon bond is a financial contract that yields a known amount F, which is usually know as the face value, at a maturity date $t = T$ while also paying out coupons through its lifetime. In this problem, the bond paid out a continuous coupon at the rate of $Ce^{-\alpha t}$ for constants $C = 2.91$ and $\alpha = 0.01$. The risk-neutral process followed by a stochastic interest rate r is given by:

$$dr = \kappa(\theta e^{\mu t} - r)dt + \sigma r^{\beta} dW, \tag{1}$$

where $W(t)$ denotes the Wiener process and the following constants are set at $\kappa = 0.04227$, $\theta = 0.0319$, $\mu = 0.0036$, $\sigma = 0.241$ and $\beta = 0.527$. It then follows that the market value of the coupon bond $B(r, t; T)$ satisfies the following PDE:

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 r^{2\beta} \frac{\partial^2 B}{\partial r^2} + \kappa(\theta e^{\mu t} - r)\frac{\partial B}{\partial r} - rB + Ce^{-\alpha t} = 0; \tag{2}$$

the domain of this problem is $r \in [0, \infty)$ and $0 \leq t < T$. Furthermore, $B(r, t; T)$ then satisfies boundary conditions

$$B(r, t = T; T) = F; \tag{3}$$

$$\frac{\partial B}{\partial t} + \kappa\theta e^{\mu t}\frac{\partial B}{\partial r} + Ce^{-\alpha t} = 0 \text{ at } r = 0; \tag{4}$$

$$B(r, t; T) \to 0 \text{ as } r \to \infty. \tag{5}$$

(5) is known as a Dirichlet condition. The following boundary condition, known as a Neumann condition, is also appropriate:

$$\frac{\partial B}{\partial r} \to \text{ as } r \to \infty. \tag{6}$$

Now consider a call option V to buy the bond at time $T_1 \in [0, T]$ with strike price X. If $V(r, t; T_1, T)$ is the value of the call option to buy then, on the domain $r \in [0, \infty)$, $t < T_1$, it can be shown that $V(r, t; T_1, T)$ satisfies the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r^{2\beta} \frac{\partial^2 V}{\partial r^2} + \kappa(\theta e^{\mu t} - r)\frac{\partial V}{\partial r} - rV = 0. \tag{7}$$

If this is a European call option then $V(r, t; T_1, T)$ satisfies the boundary conditions

$$V(r, t = T_1; T_1, T) = \max\{B(r, T_1; T) - X, 0\}; \tag{8}$$

$$\frac{\partial V}{\partial t} + \kappa\theta e^{\mu t}\frac{\partial V}{\partial r} = 0 \text{ at } r = 0; \tag{9}$$

$$V(r, t; T_1, T) \to 0 \text{ as } r \to \infty. \tag{10}$$

While if this is an American call option the boundary conditions are given by

$$V(r, t; T_1, T) \geq \max\{B(r, t; T) - X, 0\}, \ t \leq T_1; \tag{11}$$

$$V(r, t; T_1, T) = B(r, t; T) - X \text{ at } r = 0; \tag{12}$$

$$V(r, t; T_1, T) \to 0 \text{ as } r \to \infty. \tag{13}$$

0.2 The Crank-Nicolson Method

The Crank-Nicolson scheme is a finite difference method that approximates derivatives halfway between grid points. That is to say if we have a time-step Δt then derivatives are evaluated at $t + \Delta t/2$. Consequently, when using the scheme, all functions, for example the continuous coupon in (2), must also be evaluated at $t + \Delta t/2$ to be consistent. The method says for a general function $f(r, t)$ with a grid such that f_j^i denotes f evaluated at $t = i\Delta t$ and $r = j\Delta r$, where Δr is the step size in r, that

$$f \approx \frac{1}{2}(f_j^i + f_j^{i+1}); \tag{14}$$

$$\frac{\partial f}{\partial t} \approx \frac{f_j^{i+1} - f_j^i}{\Delta t}; \tag{15}$$

$$\frac{\partial f}{\partial r} \approx \frac{1}{4\Delta r}(f_{j+1}^i - f_{j-1}^i + f_{j+1}^{i+1} - f_{j-1}^{i+1}); \tag{16}$$

$$\frac{\partial^2 f}{\partial r^2} \approx \frac{1}{2\Delta r^2}(f_{j+1}^i - 2f_j^i + f_{j-1}^i + f_{j+1}^{i+1} - 2f_j^{i+1} + f_{j-1}^{i+1}), \tag{17}$$

where these are all evaluated at $t + \Delta/2$.

If we apply these approximations to (2) and rearrange to have known values on one side and the unknowns on the other we get

$$\begin{aligned}
&\frac{1}{4}B_{j-1}^i\left(\sigma^2 j^2 r^{2(\beta-1)} - \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} + \kappa j\right) - B_j^i\left(\frac{1}{\Delta t} + \frac{1}{2}\sigma^2 j^2 r^{2(\beta-1)} + \frac{r}{2}\right) \\
&\qquad + \frac{1}{4}B_{j+1}^i\left(\sigma^2 j^2 r^{2(\beta-1)} + \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} - \kappa j\right) \\
&= -\frac{1}{4}B_{j-1}^{i+1}\left(\sigma^2 j^2 r^{2(\beta-1)} - \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} + \kappa j\right) + B_j^{i+1}\left(-\frac{1}{\Delta t} + \frac{1}{2}\sigma^2 j^2 r^{2(\beta-1)} + \frac{r}{2}\right) \\
&\qquad - \frac{1}{4}B_{j+1}^{i+1}\left(\sigma^2 j^2 r^{2(\beta-1)} + \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} - \kappa j\right) - Ce^{-\alpha(i+1/2)\Delta t}.
\end{aligned} \tag{18}$$

We can then rewrite this valuation problem in terms of matricies as

$$\begin{pmatrix}
b_0 & c_0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
a_1 & b_1 & c_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & a_2 & b_2 & c_2 & 0 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & a_3 & b_3 & c_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & a_j & b_j & c_j & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{jmax} & b_{jmax}
\end{pmatrix}
\begin{pmatrix}
B_0^i \\ B_1^i \\ B_2^i \\ B_3^i \\ \cdot \\ \cdot \\ B_{jmax-1}^i \\ B_{jmax}^i
\end{pmatrix}
=
\begin{pmatrix}
d_0^i \\ d_1^i \\ d_2^i \\ d_3^i \\ \cdot \\ \cdot \\ d_{jmax-1}^i \\ d_{jmax}^i
\end{pmatrix}, \tag{19}$$

where, for $1 \leq j < jmax$, we have:

$$\begin{aligned}
a_j &= \frac{1}{4}\left(\sigma^2 j^2 r^{2(\beta-1)} - \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} + \kappa j\right) \\
b_j &= -\left(\frac{1}{\Delta t} + \frac{1}{2}\sigma^2 j^2 r^{2(\beta-1)} + \frac{r}{2}\right) \\
c_j &= \frac{1}{4}\left(\sigma^2 j^2 r^{2(\beta-1)} + \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} - \kappa j\right) \\
d_j &= -\frac{1}{4}B_{j-1}^{i+1}\left(\sigma^2 j^2 r^{2(\beta-1)} - \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} + \kappa j\right) + B_j^{i+1}\left(-\frac{1}{\Delta t} + \frac{1}{2}\sigma^2 j^2 r^{2(\beta-1)} + \frac{r}{2}\right) \\
&\quad - \frac{1}{4}B_{j+1}^{i+1}\left(\sigma^2 j^2 r^{2(\beta-1)} + \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} - \kappa j\right) - Ce^{-\alpha(i+1/2)\Delta t}.
\end{aligned} \tag{20}$$

It is useful to keep in mind we can then say

$$a_j B_{j-1}^i + b_j B_j^i + c_j B_{j+1}^i = d_j. \tag{21}$$

For $j = 0$ and $j = jmax$, we will need to use the boundary conditions of the problem. (4) can similarly be rearranged with appropriate approximations for derivatives to obtain

$$B_0^i\left(\frac{1}{\Delta t} + \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t/2}\right) - B_1^i\frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t/2} = B_0^{i+1}\frac{1}{\Delta t} + Ce^{-\alpha(i+1/2)\Delta t}, \tag{22}$$

which implies

$$\begin{aligned}
b_0^i &= \frac{1}{\Delta t} + \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t/2} \\
c_0^i &= -\frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t/2} \\
d_0^i &= \frac{1}{\Delta t}B_0^{i+1} + Ce^{-\alpha(i+1/2)\Delta t}.
\end{aligned} \tag{23}$$

It is easy to see (5) then states that $B^i_{jmax} = 0$. This implies

$$a^i_{jmax} = 0,\ b^i_{jmax} = 1,\ d^i_{jmax} = 0. \tag{24}$$

Finally, with a simple approximation for the derivative, (6) implies that

$$a^i_{jmax} = -\frac{1}{\Delta r},\ b^i_{jmax} = \frac{1}{\Delta r},\ d^i_{jmax} = 0. \tag{25}$$

The situation is exceedingly similar for the call option V. V follows the same PDE as B except for the continuous coupon and so we get

$$\begin{aligned}
a_j &= \frac{1}{4}\left(\sigma^2 j^2 r^{2(\beta-1)} - \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} + \kappa j\right) \\
b_j &= -\left(\frac{1}{\Delta t} + \frac{1}{2}\sigma^2 j^2 r^{2(\beta-1)} + \frac{r}{2}\right) \\
c_j &= \frac{1}{4}\left(\sigma^2 j^2 r^{2(\beta-1)} + \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t} - \kappa j\right) \\
d_j &= -a_j V^{i+1}_{j-1} - \left(b_j + \frac{2}{\Delta t}\right) V^{i+1}_j - c_j V^{i+1}_{j+1}
\end{aligned} \tag{26}$$

for $1 \leq j < jmax$. Note that we have rewritten d_j for brevity; this can also be found in the code implementation. For a European call, again we have the same as B for the boundary condition at $r = 0$ except for the continuous coupon so

$$\begin{aligned}
b^i_0 &= \frac{1}{\Delta t} + \frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t/2} \\
c^i_0 &= -\frac{1}{\Delta r}\kappa\theta e^{\mu(i+1/2)\Delta t/2} \\
d^i_0 &= \frac{1}{\Delta t} V^{i+1}_0.
\end{aligned} \tag{27}$$

Similarly, (10) implies that

$$a^i_{jmax} = 0,\ b^i_{jmax} = 1,\ d^i_{jmax} = 0. \tag{28}$$

This is also true of the American call option. The only difference is the boundary condition at $r = 0$. (12) implies that

$$b^i_0 = 1,\ c^i_0 = 0,\ d^i_0 = B(0, i\Delta t; T) - X. \tag{29}$$