0.1 Stock Options

A stock price can sometimes follow a risk-neutral distribution at time t as

$$S_t \sim N(f(S_0,t), v^2(S_0,t)t), \tag{1}$$

where S_0 is the current stock price, S_t is the stock price at t; f and v are calibrated functions. In this problem,

$$f(S_0,t) = S_0(\alpha T + \tan(\beta T)) + \theta\cos(\alpha T + \beta T), \tag{2}$$

$$v(S_0,t) = \frac{1}{2}\sigma(1+\alpha T)(S_0+\theta)^\gamma, \tag{3}$$

where we have parameters α=0.02, $\beta = 0.02$, $\theta = 70$, $\gamma = 1.03$, $T = 1$, the time at option maturity in years, $\sigma = 0.19$, the volatility.

It can be shown that a European put option then has analytic solution

$$P(S_0,t=0) = \left[XN(z) + v(S_0,T)\sqrt{\frac{T}{2\pi}}e^{-z^2/2} - f(S_0,T)N(z)\right]e^{-rT}, \tag{4}$$

where X is the strike price and r is the risk-free interest rate. $N(z)$ is the standard cumulative normal distribution for variable

$$z = \frac{X - f(S_0,t)}{v(S_0,t)\sqrt{T}}. \tag{5}$$

To carry out a Monte Carlo valuation of this option we use samples from a standard normal distribution, ϕ, to write the i-th stock path as

$$S_T^i = f(S_0,T) + v(S_0,T)\sqrt{T}\phi_i. \tag{6}$$

We can then average over n approximations of the discounted payoff to get the value of the put option:

$$P(S_0,t=0) \approx e^{-rT}\frac{1}{n}\sum_{i=1}^{n}\max\left(X - S_T^i, 0\right). \tag{7}$$

0.2 Asian Option

Stock prices can also be viewed as following a risk neutral stochastic process with SDE

$$dS = f'(S,t)dt + v'(S,t)dW, \tag{8}$$

where W is a Wiener process and f' and v' are calibrated functions. In this problem,

$$f'(S,t) = \alpha\theta - \beta S, \tag{9}$$

$$v'(S,t) = \sigma(|S|)^\gamma dW. \tag{10}$$

These options are path dependent and so the pricing depends on the path taken. We can approximate this be taking K steps to maturity time T with a respective time-step of Δt. To go from one point to the next we can use an Euler type scheme to write

$$S^i(t_k) = S^i(t_{k-1}) + f(S^i(t_{k-1}), t_{k-1})\Delta t + v(S^i(t_{k-1}), t_{k-1})\sqrt{\Delta t}\phi_{i,k-1} \tag{11}$$

for t_k the k-th time-step. In this assignment a value of 70 was used for K.

For a discrete floating-strike Asian put option, the payoff function is

$$G(S,A) = \max(A - S(t_K), 0), \tag{12}$$

$$A = \frac{1}{K}\sum_{k=1}^{K}S(t_k). \tag{13}$$

As before, to then carry out a Monte Carlo valuation we average over n approximations of the discounted payoff:

$$P(S_0,t=0) \approx e^{-rT}\frac{1}{n}\sum_{i=1}^{n}G(S^i,A). \tag{14}$$

0.3 Error

The Central Limit Theorem states that if $V(S_i^T)$ is a sequence of independent and identically distributed random variables with mean $E_t^Q[V(S_T)]$ and variance η^2 then we can say that

$$\lim_{n\to\infty} \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^{n} V(S_i^T) - E_t^Q[V(S_T)] \right) = N(0, \eta^2). \tag{15}$$

Therefore, each estimate of a solution V_N with a Monte-Carlo simulation is such that

$$V_N \sim N(V^*, \eta^2), \tag{16}$$

where V^* is the true solution. Then if we choose to take a sample mean from a distribution $\bar{V}$ of M samples of V_N we have that

$$\bar{V} \sim N(V^*, \frac{\eta^2}{M}). \tag{17}$$

To then obtain an estimate of the error on our approximation we find the variance of our M samples, divide this by M and then square root. This was taken to be the error on all succeeding results unless otherwise specified. It is also equivalent to the standard deviation here.