李导数

$\mathcal{L}_X f = X^i\partial_i f$

$(\mathcal{L}_X V)^j = X^i\partial_i V^j - V^i\partial_i X^j$

$(\mathcal{L}_X \omega)_j = X^i\partial_i \omega_j + \omega_i\partial_j X^i$

n维环面：$T^n = \underbrace{S^1\times S^1\times\cdots\times S^1}_{n}$

push-forward

$(f_*V)^\alpha = V^\mu \frac{\partial y^\alpha(x)}{\partial x^\mu}$ $(f(x)=y(x))$

$(f_*T)^{\alpha_1\cdots\alpha_q} = T^{\mu_1\cdots\mu_q}\frac{\partial y^{\alpha_1}(x)}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\alpha_q}(x)}{\partial x^{\mu_q}}$

$(g\circ f)_* = g_*\circ f_*$

pull-back

$(f^*\omega)_\mu = \omega_\alpha \frac{\partial y^\alpha(x)}{\partial x^\mu}$

$(f^*T)_{\mu_1\cdots\mu_r} = T_{\alpha_1\cdots\alpha_r}\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}}$

$(g\circ f)^* = f^*\circ g^*$ (反序)

李括号

$[X,Y]^\mu = X^\mu\partial_\mu Y^\nu - Y^\mu\partial_\mu X^\nu$

微分形式楔积

$dx^\mu\wedge dx^\nu = dx^\mu\otimes dx^\nu - dx^\nu\otimes dx^\mu$

微分形式外积 $\wedge: \Omega^q_p(M)\times\Omega^r_p(M)\to\Omega^{q+r}_p(M)$

$(\omega\wedge\xi)(V_1,\cdots,V_{q+r}) = \frac{1}{q!r!}\sum_{P\in S_{q+r}}\mathrm{sgn}(P)\times\omega(V_{P(1)},\cdots,V_{P(q)})\cdot\xi(V_{P(q+1)},\cdots,V_{P(q+r)})$

- 结合律 $(\xi\wedge\eta)\wedge\omega = \xi\wedge(\eta\wedge\omega)$, $\forall\xi,\eta,\omega\in\Omega^*_p(M)$
- q-r律 $\xi\wedge\eta = (-1)^{qr}\eta\wedge\xi$, $\forall\xi\in\Omega^q_p(M),\eta\in\Omega^r_p(M)$

外导数 $d: \Omega^r(M)\to\Omega^{r+1}(M)$

$d\omega = \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1\cdots\mu_r}\right)dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}$

- Leibniz $d(\xi\wedge\eta) = d\xi\wedge\eta + (-1)^q\xi\wedge d\eta$ $(\xi\in\Omega^q(M), \eta\in\Omega^p(M))$
- 幂零性 $d^2=0$

与李括号的关系：$d\omega(X,Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X,Y])$

de Rham 复型

$0\xrightarrow{i}\Omega^0(M)\xrightarrow{d_0}\Omega^1(M)\xrightarrow{d_1}\Omega^2(M)\to\cdots\to\Omega^m(M)\xrightarrow{d_m}0$

$\mathrm{Im}\, d_r\subseteq\mathrm{Ker}\, d_{r+1}$

恰当形式 ⇒ 闭形式 $(\{\omega\in\Omega^r(M), d\omega=0\})$

$(\{\omega\in\Omega^r(M), \exists\omega'\in\Omega^{r-1}(M), d\omega'=\omega\})$

内积

$i_X\omega = \frac{1}{r!}\sum_{s=1}^r X^{\mu_s}\omega_{\mu_1\cdots\mu_s\cdots\mu_r}(-1)^{s-1}dx^{\mu_1}\wedge\cdots\wedge\widehat{dx^{\mu_s}}\wedge\cdots\wedge dx^{\mu_r}$

与李导数关系：

$\mathcal{L}_X\omega = (di_X + i_X d)\omega$ $(\omega\in\Omega^r(M))$

Hodge star

$\cancel{*(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_q}) = \frac{1}{(m-q)!}}$

$*(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_q}) = \frac{1}{(m-q)!}\epsilon^{\mu_1\cdots\mu_q}{}_{\mu_{q+1}\cdots\mu_m}dx^{\mu_{q+1}}\wedge\cdots\wedge dx^{\mu_m}$

Levi-Civita 张量

$\epsilon_{\mu_1\cdots\mu_m} = \begin{cases}+1 & \text{下标偶排}\\ -1 & \text{奇}\\ 0 & \text{其他}\end{cases}$

边缘算子 ∂_r

$\partial_r(p_0p_1\cdots p_r) = \sum_{i=0}^r(-1)^i(p_0\cdots\hat{p}_i\cdots p_r)$

→有幂零性 $\partial_r\partial_{r+1}\sigma^{r+1}=0$

闭链 $Z_r(K)$

$\{c\in C_r(K)\mid\partial_r c=0\}$

边缘链 $B_r(K)$

$\{c\in C_r(K)\mid\exists c'\in C_{r+1}(K)\ s.t.\ \partial c'=c\}$

同调群 $H_r(K)=Z_r(K)/B_r(K)$

$H_0(K)=\mathbb{Z}$ (K是连通复形)

$H_0(K)=\mathbb{Z}^n$ (K有n个连通分支)

计算同调群的技巧：利用同调等价关系 $c\sim c+\partial\Sigma$ $(c\in C_r(K), \Sigma\in C_{r+1}(K))$

挤到边上

$H_1(T^2)=\mathbb{Z}\oplus\mathbb{Z}$

n个洞则为 $\underbrace{\mathbb{Z}\oplus\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{2n个}$

Betti数 $b_r=\dim H_r(M,\mathbb{Z})$ →有多少个Z直和部分

Euler示性数：$\chi(M) = b_0-b_1+b_2-\cdots+(-1)^{\dim M}b_{\dim M}$

多连通分支：

$H_r(K)=H_r(K_1)\oplus\cdots\oplus H_r(K_N)$

同调群结构

$H_r(K,\mathbb{Z})\cong\underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{\text{自由部分}}\oplus\underbrace{\mathbb{Z}_{k_1}\oplus\cdots\oplus\mathbb{Z}_{k_p}}_{\text{挠子群}}$

仅自由部分由 b_r （循环群）

→系数群为R时：$H_r(K,\mathbb{Z})\cong\underbrace{\mathbb{R}\oplus\cdots\oplus\mathbb{R}}_{\ell个}=\mathbb{R}^\ell$ (无挠)

Thm (Euler-Poincaré) n维复形K有 I_r 个r维单形

则：$\chi(K)\equiv\sum_{r=0}^{n}(-1)^r I_r=\sum_{r=0}^{n}(-1)^r b_r$

上链群 $C^r(K)=\mathrm{Hom}(C_r(K),\mathbb{Z})$

上闭链群 $Z^r(K)=\{c^r\in C^r(K)\mid \delta c^r=0\}$

上边缘链群 $B^r(K)=\{c^r\in C^r(K)\mid \exists c^{r-1}\in C^{r-1}(K),\ \text{s.t. } c^r=\delta c^{r-1}\}$

上同调群 $H^r(K,\mathbb{Z})=Z^r(K,\mathbb{Z})/B^r(K,\mathbb{Z})$

$\delta: C^{r-1}(K)\to C^r(K)$ $\quad\langle\delta c^{r-1},c_r'\rangle=\langle c^{r-1},\partial c_r'\rangle$

$\forall c_r'\in C_r(K)$

上同调群结构：自由⊕挠 $\quad(\langle\omega,c\rangle=\int_c\omega)$

$$H^r(K,\mathbb{Z})=\underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{b^r}\oplus T^r(K)$$

$$\begin{cases} b^r=b_r \\ T^r(K)\cong T_{r-1}(K)\end{cases}$$

Thm (Stokes) $\forall\omega\in\Omega^{r-1}(M),\ c\in C_r(M)$

$$\int_c d\omega=\int_{\partial c}\omega$$

de-Rham上同调群：

$$H^r_{dR}(M,\mathbb{R})=\frac{\mathrm{Ker}\{d:\Omega^r(M)\to\Omega^{r+1}(M)\}}{\mathrm{Im}\{d:\Omega^r(M)\to\Omega^{r+1}(M)\}}$$

M有n个连通分支：

$$H^0_{dR}(M,\mathbb{R})\cong\underbrace{\mathbb{R}\oplus\cdots\oplus\mathbb{R}}_{n}$$

Thm (de Rham)

M紧，$H_r(M)$、$H^r_{dR}(M)$有限维，$\langle[\omega],[c]\rangle$为非退化

$H^r_{dR}(M)$对偶$H_r(M)$

e.g. $H^1_{dR}(S^1)\cong H_1(S^1)\cong\mathbb{R}$

Thm (Poincaré引理)

M上U可缩 ⇒ U上闭形式必恰当

Thm (Poincaré对偶) M(m维紧，无边)

$\langle\omega,\eta\rangle\equiv\int_M\omega\wedge\eta\quad([\omega]\in H^r_{dR}(M),\ [\eta]\in H^{m-r}_{dR}(M))$

给出向量空间对偶

$$H^r_{dR}(M)\cong H^{m-r}_{dR}(M)$$

Künneth公式 $\quad M=M_1\times M_2$

$$H^r(M)=\bigoplus_{p+q=r}H^p(M_1)\otimes H^q(M_2)$$

(注：$\mathbb{R}\otimes\mathbb{R}\cong\mathbb{R}$)

初值：$H^0_{dR}(S^1)\cong\mathbb{R}$，$H^1_{dR}(S^1)\cong\mathbb{R}$

$$\begin{cases} b_r(M)=\sum_{p+q=r}b_p(M_1)b_q(M_2) \\ \chi(M)=\chi(M_1)\chi(M_2)\end{cases}$$

Thm (Whitehead) $\pi_n(Y_0)\cong\pi_n(Y_1),\ \forall n\geq0$

$$\Rightarrow Y_0\simeq Y_1$$

$\pi_n(X\times Y)\cong\pi_n(X)\times\pi_n(Y)$

同伦群计算：纤维丛 $F\xrightarrow{i}E\xrightarrow{p}B$

诱导长正合列 $\cdots\to\pi_n(F)\to\pi_n(E)\to\pi_n(B)\to\cdots\to\pi_0(E)\to\underset{=0}{\pi_0(B)}$

纤维丛举例：

$\mathbb{Z}^m\xrightarrow{i}\mathbb{R}^m\xrightarrow{p}T^m\quad(T^m=\mathbb{R}^m/\mathbb{Z}^m)$

$S^1\xrightarrow{i}S^3\xrightarrow{p}S^2\quad$(Hopf纤维丛)

$SO(N-1)\xrightarrow{i}SO(N)\xrightarrow{p}S^{N-1}\quad(SO(N)/SO(N-1)\cong S^{N-1})$

$\mathbb{Z}_2\xrightarrow{i}S^3\xrightarrow{p}SO(3)\quad(SO(3)\cong SU(2)/\mathbb{Z}_2\cong S^3/\mathbb{Z}_2)$

$SU(N-1)\xrightarrow{i}SU(N)\xrightarrow{p}S^{2N-1}\quad(SU(N)/SU(N-1)\cong S^{2N-1})$

$\mathbb{Z}_2\xrightarrow{i}U(N)\xrightarrow{p}SU(N)\quad(N=1$不满足$)$

常用结果 $\pi_n(S^n)\cong\mathbb{Z}\ (n=1,2,\cdots)$

$\pi_0(S^0)\cong\mathbb{Z}_2$

$\pi_0(\text{离散集}\,\Gamma)\cong\Gamma,\ \pi_{n>0}(\Gamma)=0$

短正合列 $0\to\pi_\alpha(A)\to\pi_\beta(B)\to0$ 给出 $\pi_\alpha(A)\cong\pi_\beta(B)$

诱导度规 $\overset{m}{g}_{\mu\nu}=\overset{N}{g}_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}$

协变导数 $\nabla_\mu V_\nu=\partial_\mu V_\nu-\Gamma^\rho_{\mu\nu}V_\rho$ (对偶矢 V_ν)

$\nabla_\mu V^\nu=\partial_\mu V^\nu+\Gamma^\nu_{\mu\rho}V^\rho$ (矢 V^ν)

Levi-Civita联络：

$$\Gamma^\mu_{\nu\rho}=\frac{1}{2}g^{\mu\sigma}(g_{\sigma\nu,\rho}+g_{\sigma\rho,\nu}-g_{\nu\rho,\sigma})$$

Riemann曲率张量

$$R^\kappa_{\ \lambda\mu\nu}=\partial_\mu\Gamma^\kappa_{\nu\lambda}-\partial_\nu\Gamma^\kappa_{\mu\lambda}+\Gamma^\kappa_{\mu\rho}\Gamma^\rho_{\nu\lambda}-\Gamma^\kappa_{\nu\rho}\Gamma^\rho_{\mu\lambda}$$

测地线方程：(要先计算 $\Gamma^\mu_{\nu\rho}$)

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds}=0$$

Killing矢量场 (要先计算 $\Gamma^\mu_{\nu\rho}$)

$$\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu=0$$