Basic Mathematics and Statistics

CHAPTER 5: FUNCTION

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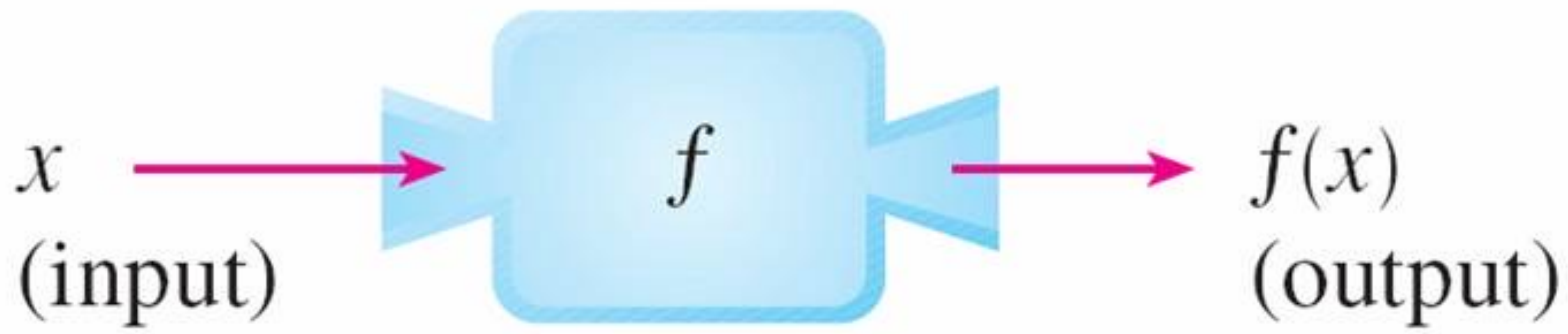
5.1 Function, Domain and Range

Defining a function

A function f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by f (x) (read “f of x”).



Sometimes the output is denoted as single letter

$$y = f(x)$$

5.1 Function, Domain and Range

Defining a function

$$y = f(x)$$

y as a function of x

the variable **x** is called the ***independent variable*** (or argument) of f

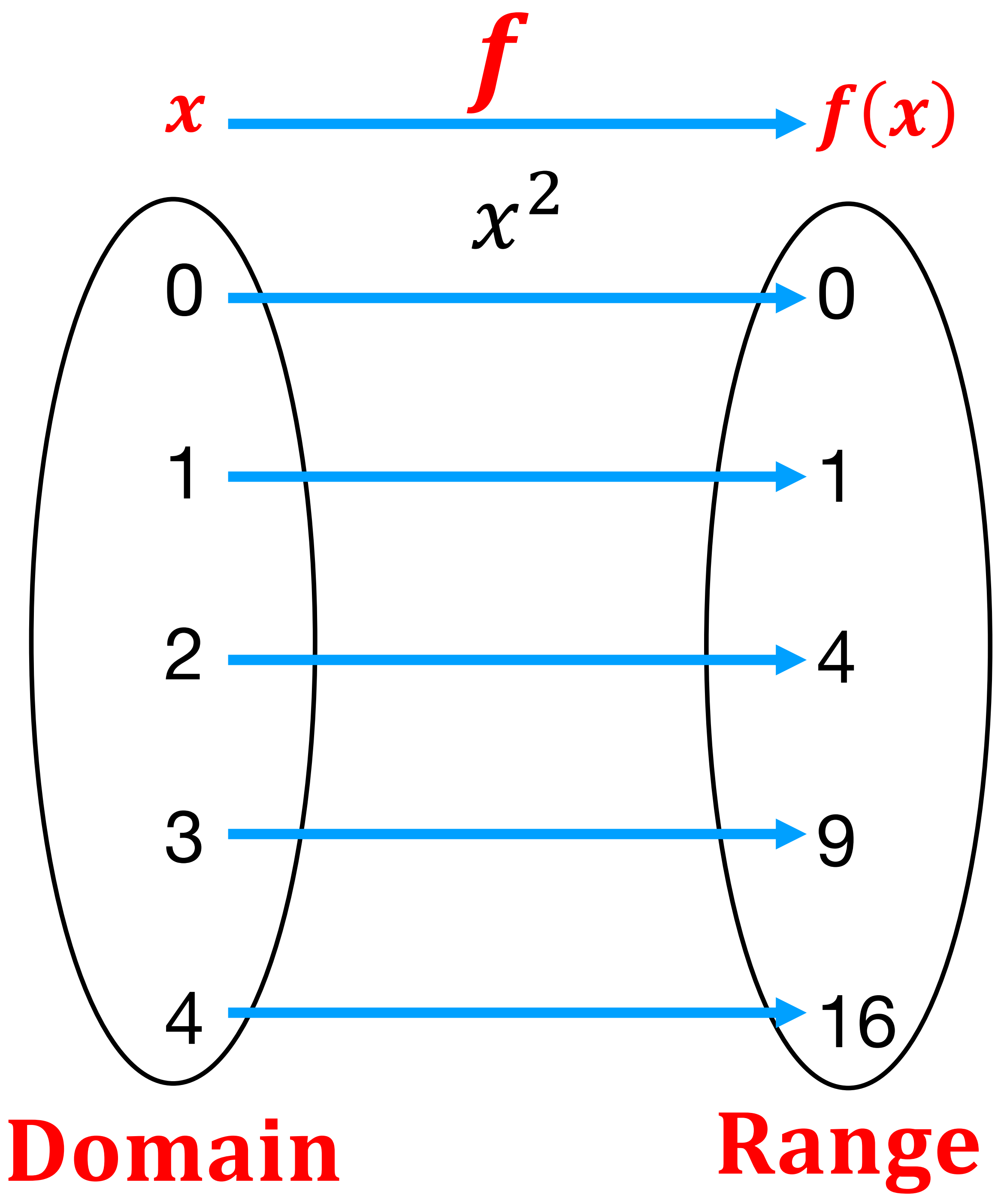
the variable **y** is called the ***dependent variable*** of f

x and y are **real numbers**

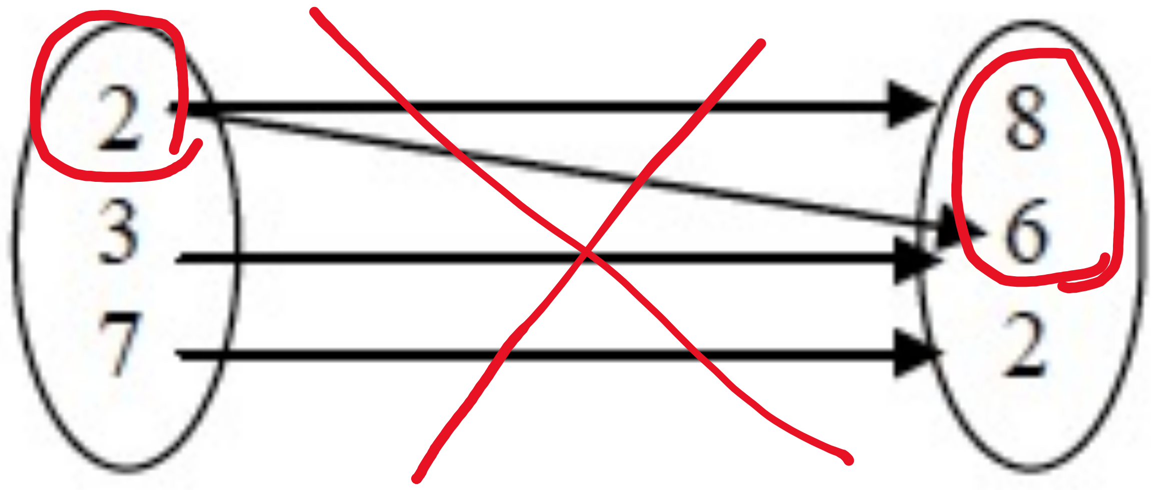
therefore f is a ***real-valued function of a real variable***

5.1 Function, Domain and Range

$$y = \boldsymbol{f}(\boldsymbol{x}) = x^2$$



A **function** is a relation from a set of possible inputs to a set of possible outputs where each input is related to **exactly one** output.



domain **range**

the set of function $= \{(0,0),(1,1),(2,4),(3,9),(4,16)\}$

All the possible values of $\boldsymbol{x}$ (set of real number) is called the **domain**

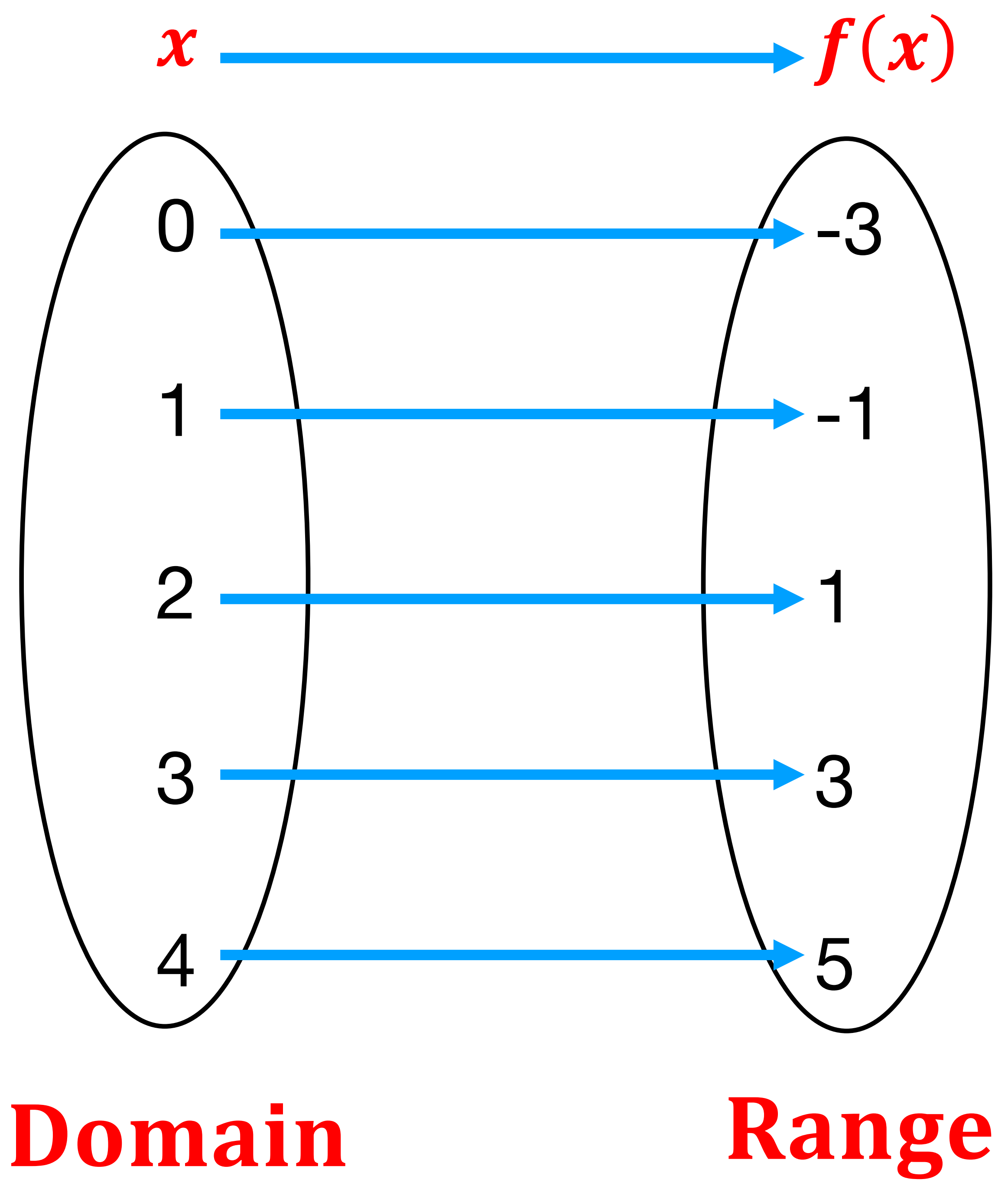
Domain is {0, 1, 2, 3, 4} – ($\boldsymbol{x}$ – ***independent variables***)

All the possible values of $\boldsymbol{y}$ (set of real number) is called the **range**

Range is {0, 1, 4, 9, 16} – ($\boldsymbol{y}$ – ***dependent variables***)

5.1 Function, Domain and Range

$y = \boldsymbol{f}(\boldsymbol{x}) = 2x - 3$



The set of $\{(0,-3),(1,-1),(2,1),(3,3),(4,5)\}$

Domain is {0, 1, 2, 3, 4}

Range is {-3, -1, 1, 3, 5}

5.1 Function, Domain and Range

A **function** is a correspondence between two sets of elements such that to each element in the first set there corresponds one and only one element in the second set. The first set is called the **domain** and the set of all corresponding elements in the second set is called the **range**.

The domain and range of a function will be sets of real numbers. For such a function we often use an equation with two variables to specify both the rule of correspondence and the set of ordered pairs.

5.1 Function, Domain and Range

$y = \boldsymbol{f}(\boldsymbol{x}) = x^2 + 2x$

if $x = 4, then\ y = 4^2 + 2(4) = 16 + 8 = 14$

if $x = -1, then\ y = (-1)^2 + 2(-1) = 1 - 2 = -1$

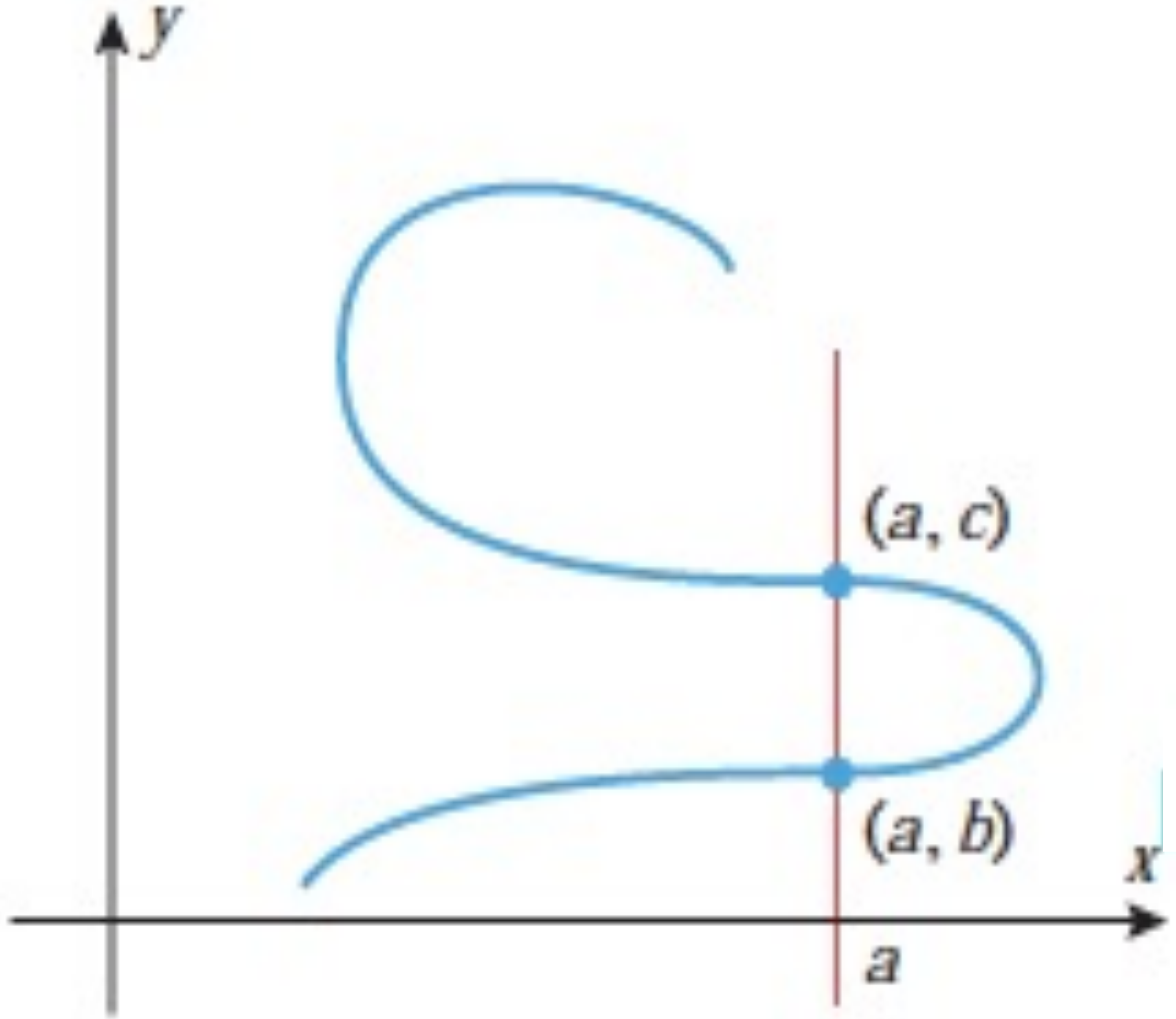
An equation defines a function (how to determine?)

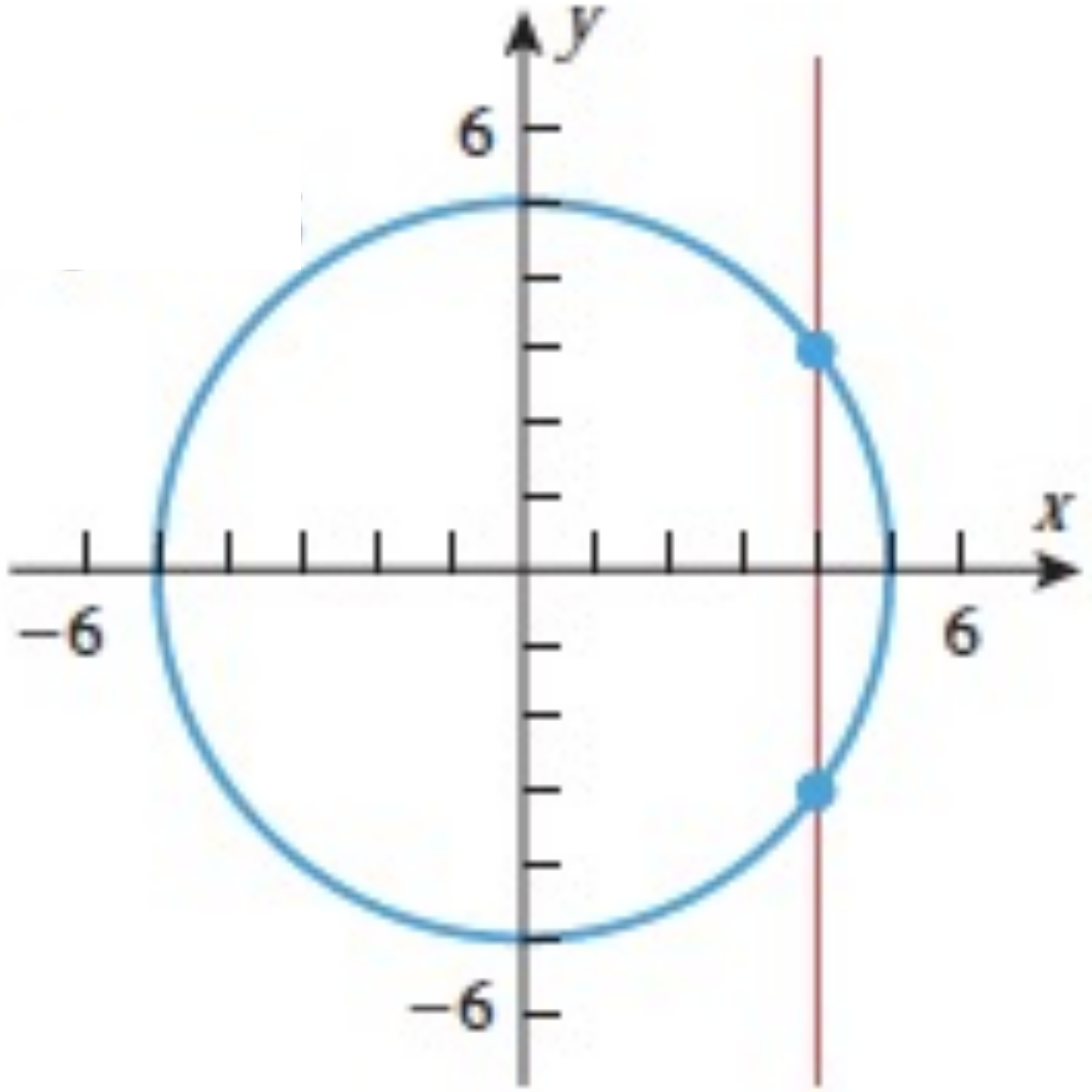
It is very easy to determine whether an equation defines a function if you have the graph of the equation.

An equation defines a function if each vertical line in a rectangular coordinate system passes through at most one point on the graph of the equation. **If any vertical line passes through two or more points on the graph of an equation, then the equation does not define a function.**

5.1 Function, Domain and Range

A curve in the xy-plane is the graph of some function f if and only if no vertical line intersects the curve more than once.





5.1 Function, Domain and Range

example

Determine whether each set specifies a function.

If it does, then state the domain and range.

(a) $S = \{(1, 4), (2, 3), (3, 2), (4, 3), (5, 4)\}$

Domain $\{1, 2, 3, 4, 5\}$ ✓ function

Range $\{4, 3, 2, 3, 4\}$

(b) $T = \{\underline{(1, 4)}, (2, 3), (3, 2), (2, 4), \underline{(1, 5)}\}$

✗ Not a function

Determine if each equation defines a function with independent variable *x*

(a) $y = x^2 - 4$

Any input x will have only one output ✓

(b) $x^2 - y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$ ✗

For any x that provides an output (when $16 - x^2 \geq 0$), there are two choices for y, one positive and one negative. The equation $x^2 - y^2 = 16$ has more than one output for some inputs, so does not define a function.

5.1 Function, Domain and Range

example

Given $f(x) = x^2 + 1$, find

(a) the range when the domain is {-2, 0, 2}

$f(-2) = (-2)^2 + 1 = 4 + 1 = 5$

$f(0) = 0^2 + 1 = 1$

$f(2) = 2^2 + 1 = 5$

Range $= \{1, 5\}$

(b) the domain when the range is {1, 10}

$f(x) = x^2 + 1$

$1 = x^2 + 1$

$0 = x^2$

$0 = x$

$10 = x^2 + 1$

$x^2 = 10 - 1 = 9$

$x = \sqrt{9} = \pm 3$

$D = \{-3, 0, +3\}$

5.1 Function, Domain and Range

Example (Finding domains and range of functions)

Find the domain and range of the following functions. $\frac{1}{0} \to$ undefined

(a) $y = x^2$ $\quad$ (b) $y = \dfrac{1}{x}$ $\quad$ (c) $y = \sqrt{1 - x^2}$ $\quad$ (d) $f(x) = \dfrac{15}{x-3}$

$y = (-10)^2 = 100$

D $\to x \to$ all real number

$(-\infty, +\infty)$

R $\to [0, +\infty)$

$= -1, \frac{1}{5},$

$-1, 5, 0$

D: x all real number except $x = 0$

$(-\infty, 0) \cup (0, +\infty)$

R: $(-\infty, 0) \cup (0, +\infty)$

$\sqrt{\ } = 3$

$1 - 4 = -3$

$1 - x^2 \geq 0$

$-1, 0, 1$

$x^2 \leq 1$

$0 \quad 1$

D: $[-1, 1]$

R: $[0, 1]$

$x - 3 \neq 0$

$x \neq 3$

D: $(-\infty, 3) \cup (3, +\infty)$

R: $(-\infty, 0) \cup (0, +\infty)$

5.1 Function, Domain and Range

Example (Evaluation Functions)

Given $f(x) = 3x + 1$, find

(a) $f(-1) = 3(-1)+1 = -2$

(b) $f(2) = 3(2)+1 = 7$

(c) $f(a) = 3(a)+1 = 3a+1$

(d) $4f(a) = 4(3a+1) = 12a+4$

(e) $f(-3a) = 3(-3a)+1 = -9a+1$

(f) $f(5+a) = 3(5+a)+1 = 15+3a+1 = 16+3a$

5.1 Function, Domain and Range

Example

For $f(x) = x^2 + 4x + 5$, find an simplify $\dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$

$$f(x+h) = (x+h)^2 + 4(x+h) + 5$$

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 + 4(x+h) + 5 - (x^2 + 4x + 5)}{h}$$

$$= \frac{x^2 + 2xh + h^2 + 4x + 4h + 5 - x^2 - 4x - 5}{h}$$

$$= \frac{2xh + h^2 + 4h}{h} = \frac{h(2x + h + 4)}{h} = 2x + h + 4$$

5.1 Function, Domain and Range

Example

piece-wise defined - function

Given $f(x) = \begin{cases} -x\,; & x < 0 \\ x^2\,; & 0 \le x \le 1 \\ 1\,; & x > 1 \end{cases}$. Find the values of $f(-3)$, $f\left(\frac{1}{2}\right)$, and $f(3)$

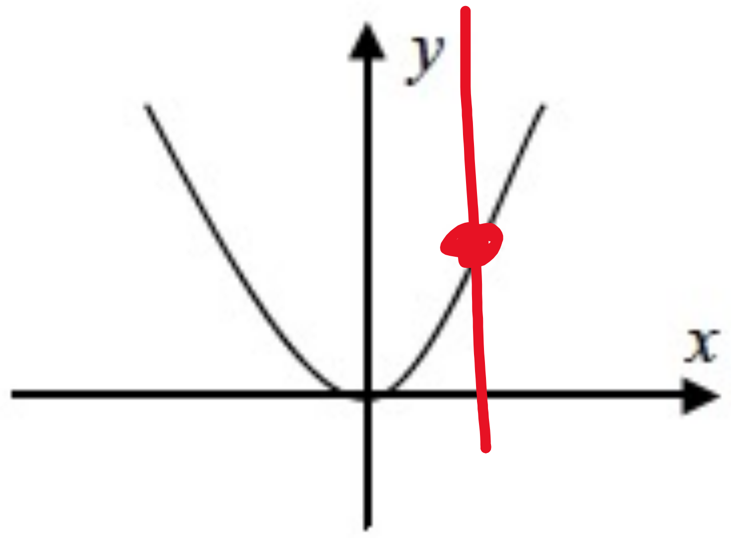
$f(\underset{x}{-3}) = -(-3) = 3$

$f\left(\underset{x}{\frac{1}{2}}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

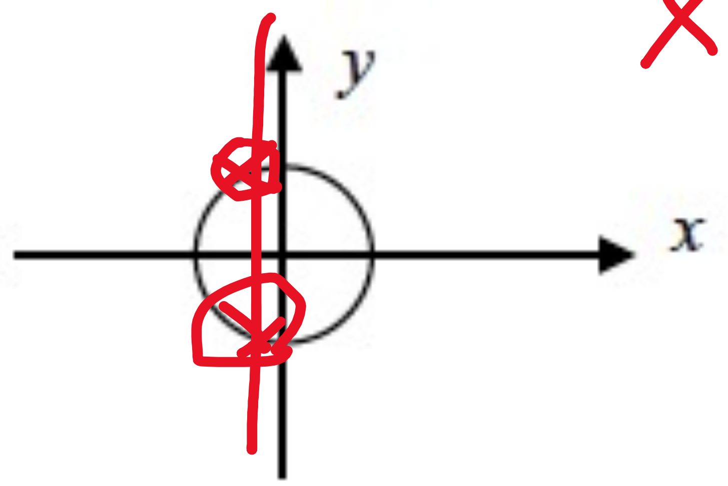
$f(\underset{x}{3}) = 1$

Exercise

- Exercises 5.1

1.


equation

2.


fail the vertical line test

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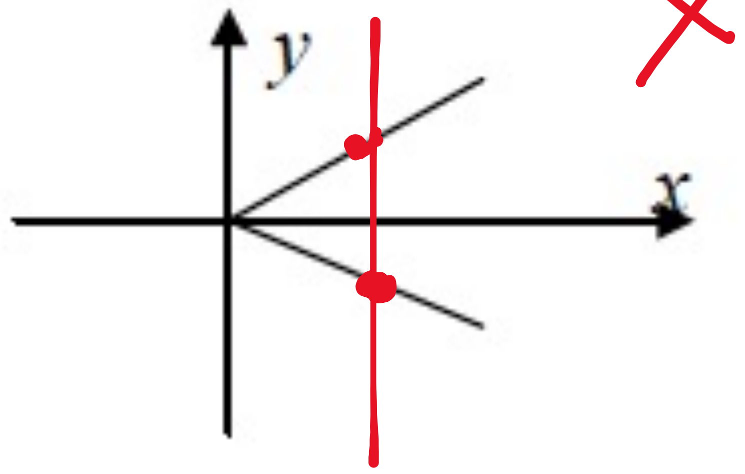

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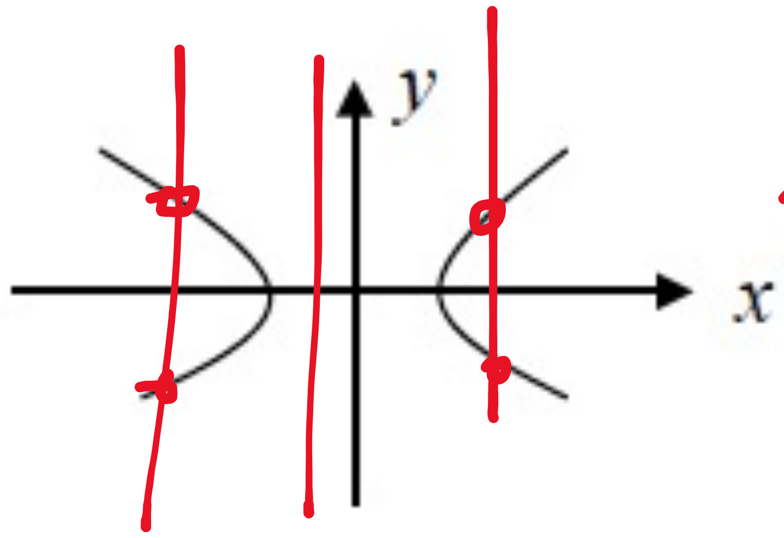

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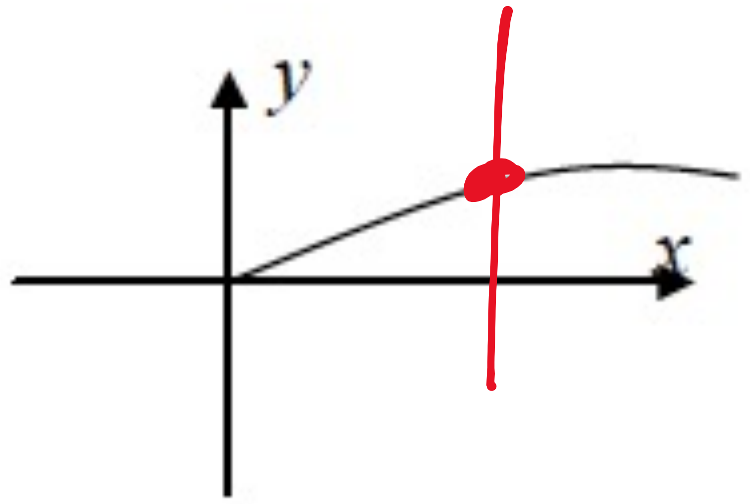

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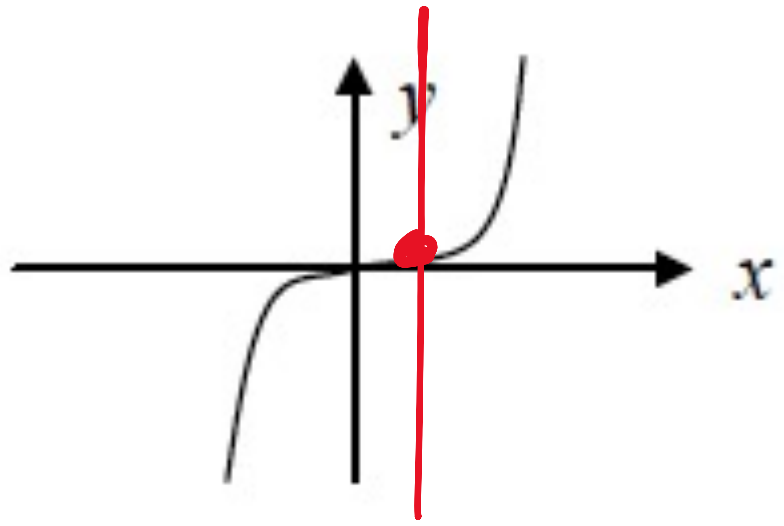


Exercise

- Exercises 5.1

19. Let $F(u) = u^2 - u - 1$. Find $F(10)$, $F(u^2)$, $F(5u)$, $5F(u)$.

$F(10) = 10^2 - 10 - 1 = 89$

$F(u^2) = (u^2)^2 - (u^2) - 1$

$= u^4 - u^2 - 1$

$F(5u) = (5u)^2 - 5u - 1$

$= 25u^2 - 5u - 1$

$5F(u) = 5(u^2 - u - 1)$

$= 5u^2 - 5u - 5$

5.2 Combining Functions

5.2.1 Operations on Functions

If $f(x)$ and $g(x)$ are funtions, then

Sum:	$(f+g)(x) = f(x) + g(x)$
Difference:	$(f-g)(x) = f(x) - g(x)$
Product:	$(f \bullet g)(x) = f(x) \bullet g(x)$
Quotient:	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, \quad g(x) \neq 0$

5.2 Combining Functions

5.2.1 Operations on Functions

If $f(x) = 3x - 1$ and $g(x) = x - 5$. Find

(a) $(f+g)(x)$ $= f(x) + g(x) = 3x-1 + (x-5) = 4x-6 = 2(2x-3)$

(b) $(f-g)(x)$ $= f(x) - g(x) = 3x-1-(x-5) = 3x-1-x+5 = 2x+4 = 2(x+2)$

(c) $(f\cdot g)(x)$ $= f(x)\cdot g(x) = (3x-1)\times(x-5) = 3x^2-5x-x+5 = 3x^2-6x+5$

(d) $\left(\frac{f}{g}\right)(x)$ $= \dfrac{f(x)}{g(x)} = \dfrac{3x-1}{x-5}$

5.2 Combining Functions

5.2.2 Composite Function

The composition of a function *f* with another function *g* is denoted by $f o g$ (read "*f* composed with *g*") and is defined by

$$(f\ o\ g)(x) = f(g(x))$$

Example

$f(x) = x - 7$ and $g(x) = x^2$, then find $(f o g)(x)$ and $(f o g)(2)$ $(g o f)(x) = ?$

$(f o g)(x) = f(g(x)) = f(x^2) = (x^2) - 7 = x^2 - 7$

$(f o g)(2) = (2)^2 - 7 = 4 - 7 = -3$

$= g(f(x))$

$= g(x-7)$

$= (x-7)^2$

5.2 Combining Functions

5.2.2 Composite Function

Example

$f(x) = x^2 - x$ and $g(x) = 3 + 2x$,

find $(fog)(x), (fog)(-1)$ *and* $(gof)(2)$

$= f(g(x))$
$= f(3+2x)$
$= (3+2x)^2 - (3+2x)$
$= 9 + 12x + 4x^2 - 3x - 2x$
$= 4x^2 + 7x + 9$

$(fog)(-1)$
$= 4(-1)^2 + 7(-1) + 9$
$= 4 - 7 + 9 = 6$

$(gog)(x) = g(g(x)) = g(3+2x)$
$= 3 + 2(3+2x)$
$= 3 + 6 + 4x$
$= 9 + 4x$

$(gof)(x)$
$= g(f(x))$
$= g(x^2 - x)$
$= 3 + 2(x^2 - x)$
$= 3 + 2x^2 - 2x$

$(gof)(2) = 3 + 2(2^2) - 2(2)$
$= 7$

Exercise

- Exercises 5.2

5.3 Inverse of a Function

5.3.1 Theorem 1 One – to – One Functions

1. If $f(a) = f(b)$ for at least one pair of domain values *a* and *b*, $a \neq b$, then *f* is not one-to-one.

2. If the assumption $f(a) = f(b)$ always implies that the domain values *a* and *b* are equal, then *f* is one-to-one.

$\boldsymbol{f(x) = x^2}$

$f(a) = f(b) \rightarrow a^2 = b^2$

$a^2 - b^2 = 0$

$(a - b)(a + b) = 0$

$(a - b) = 0 \quad and \quad (a + b) = 0$

$a = b \quad and \quad a = -b$

It means that $\boldsymbol{f(x) = x^2}$ **is not one - to - one.**

$\boldsymbol{g(x) = \sqrt{x}}$

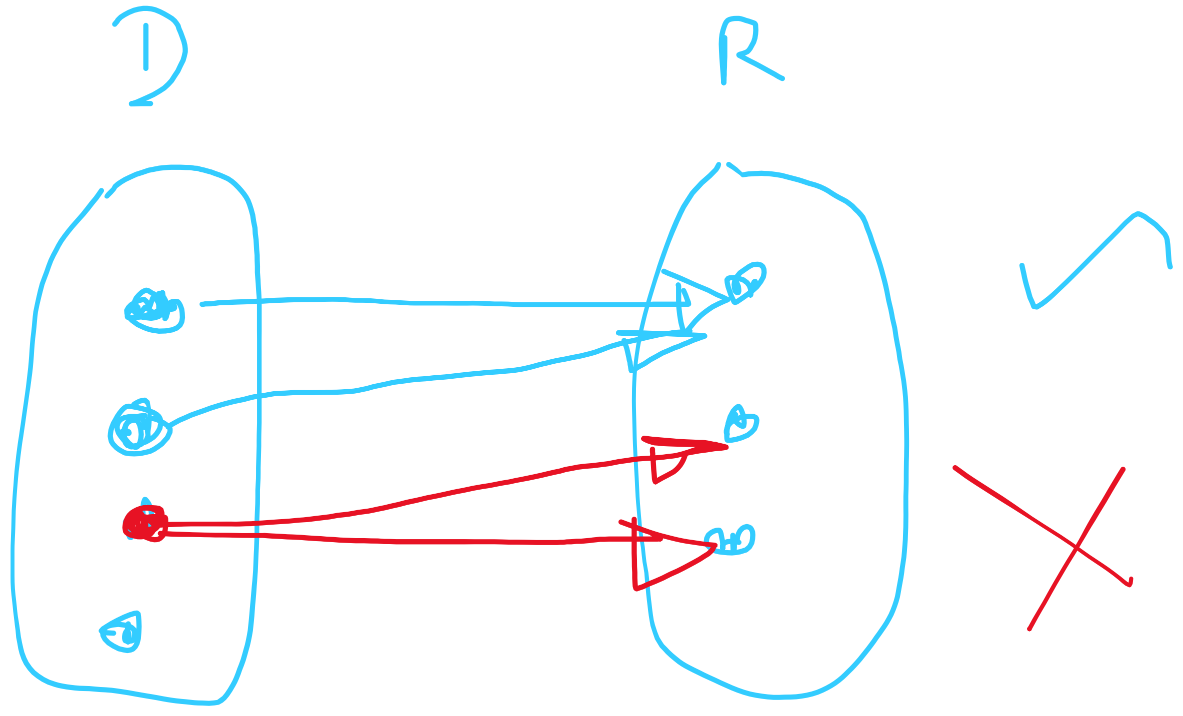
$g(a) = g(b) \rightarrow \sqrt{a} = \sqrt{b}$

$a = b$

It means that is $\boldsymbol{g(x) = \sqrt{x}}$ **is one - to -one**

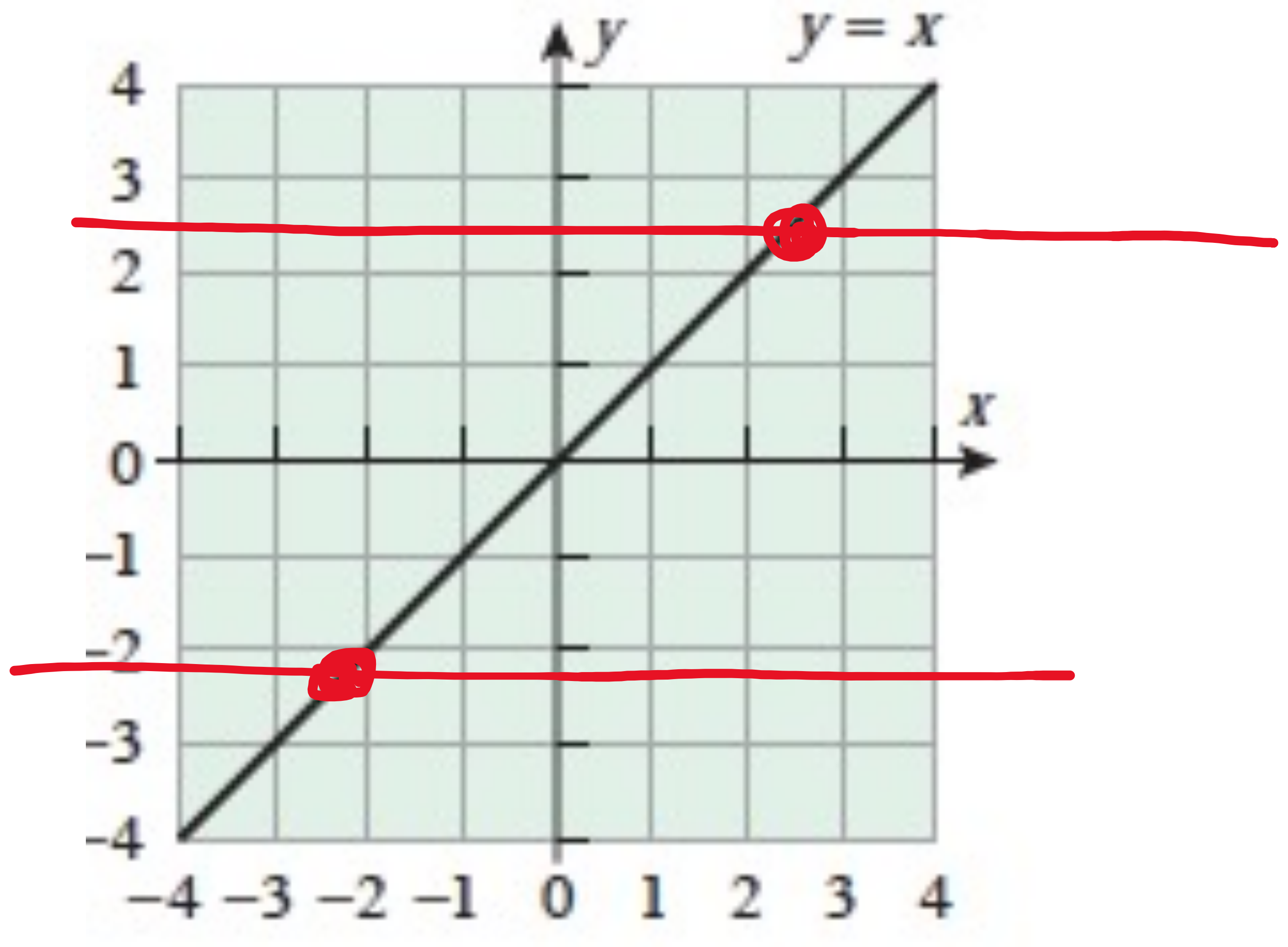
5.3 Inverse of a Function

5.3.2 Theorem 2 Horizontal Line Test

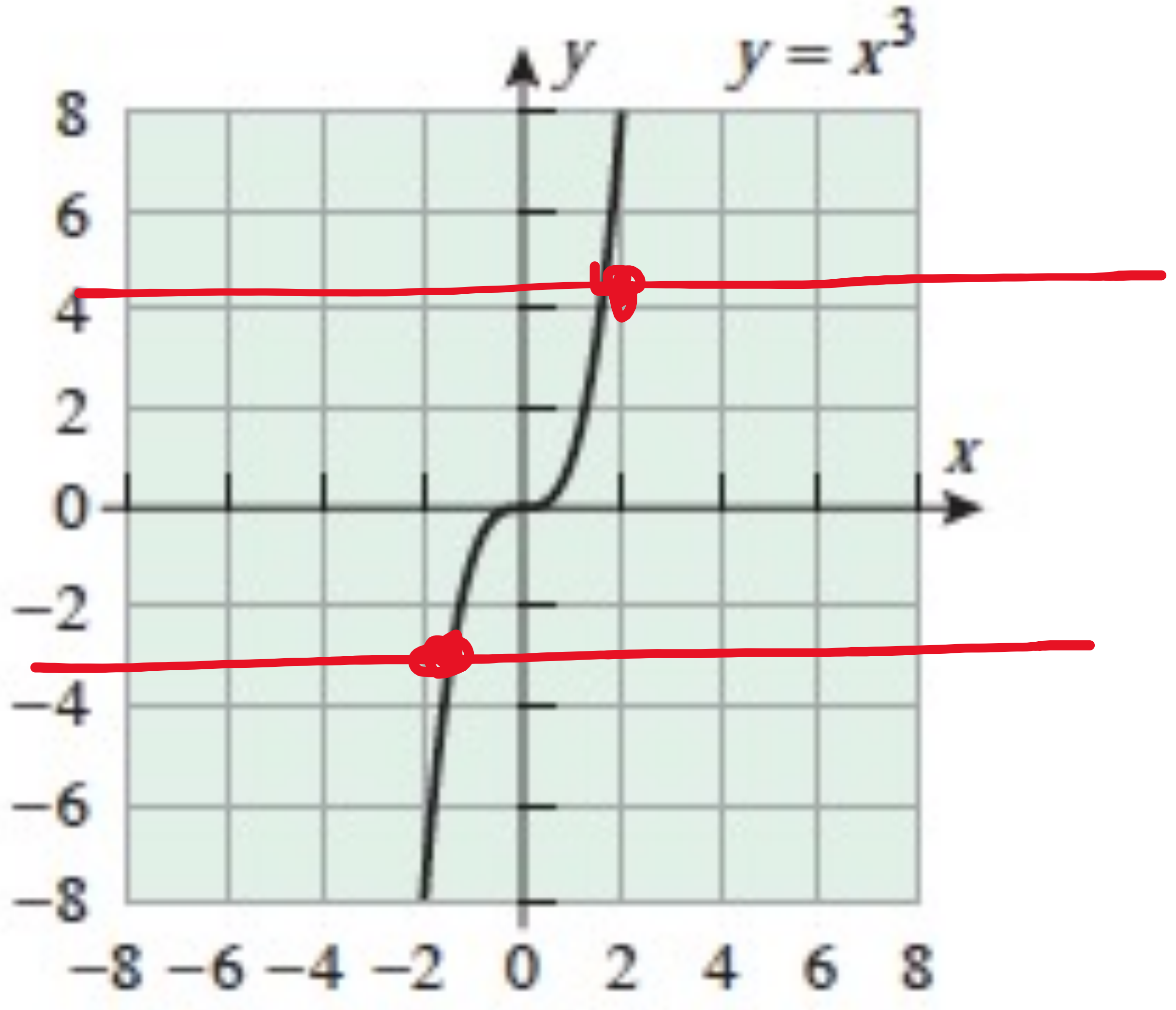


A function is one-to-one if and only if every horizontal line intersects the graph of the function in at most one point.

one-to-one function

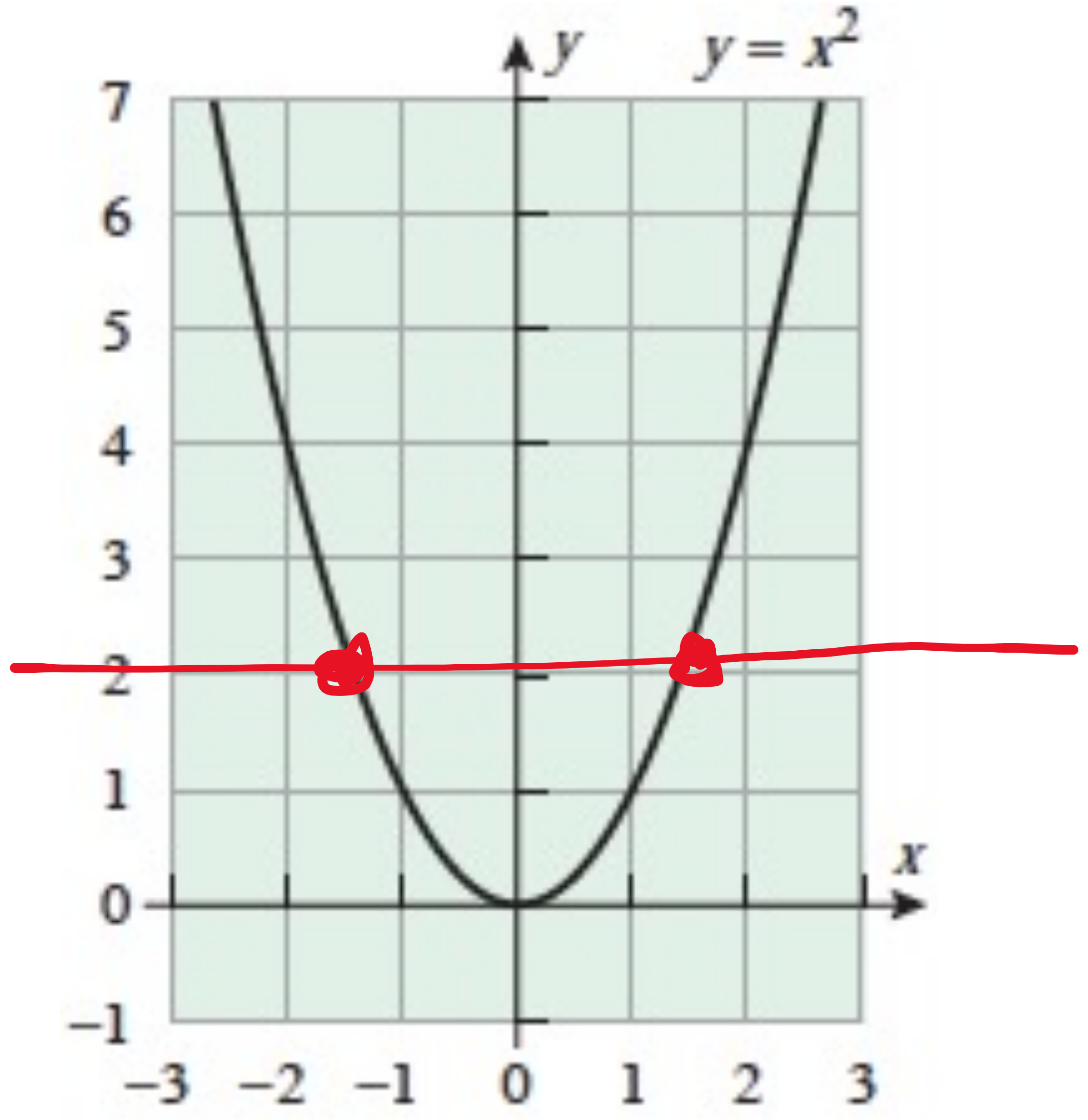


one-to-one function



NOT one-to-one function

many-to-one function



5.3 Inverse of a Function

5.3.3 Theorem 3 Increasing and Decreasing Functions

If a function *f* is increasing throughout its domain or decreasing throughout its domain, then f is a one-to one function

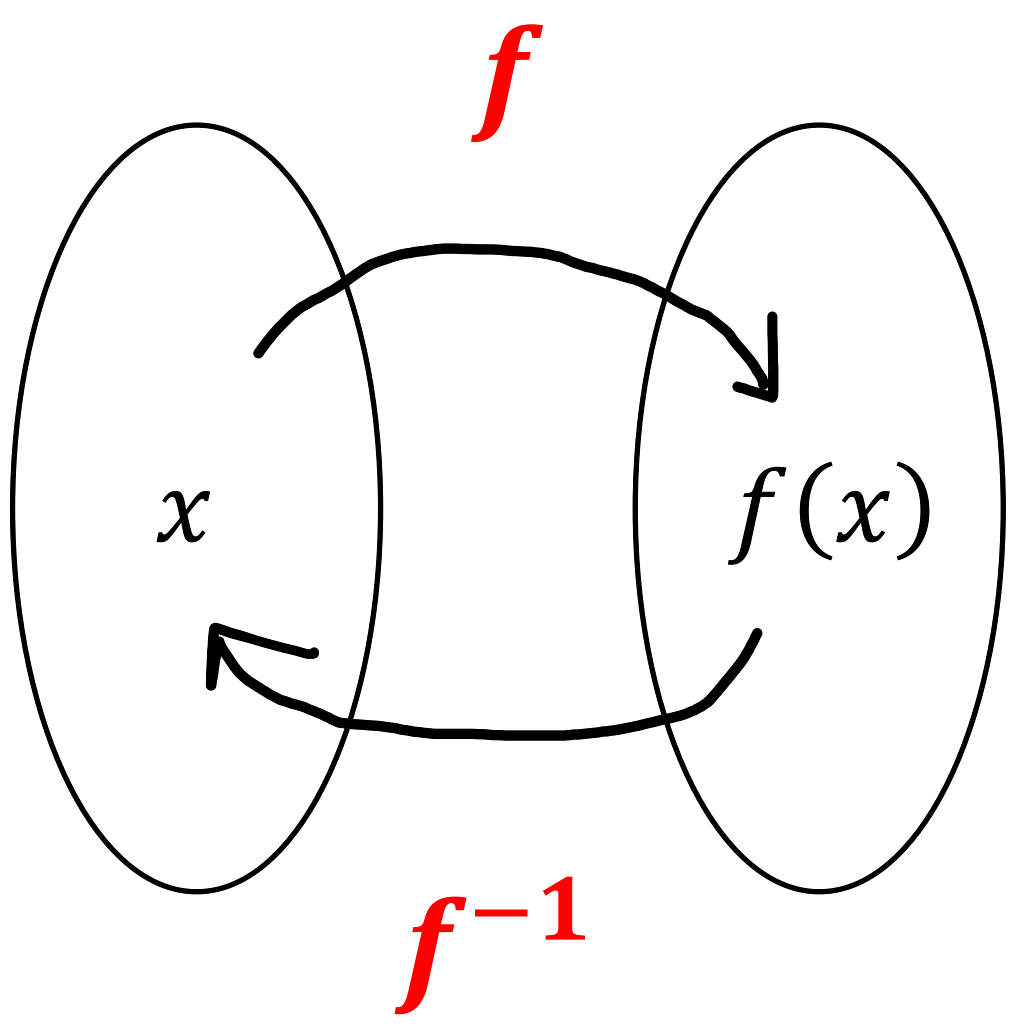
5.3 Inverse of a Function

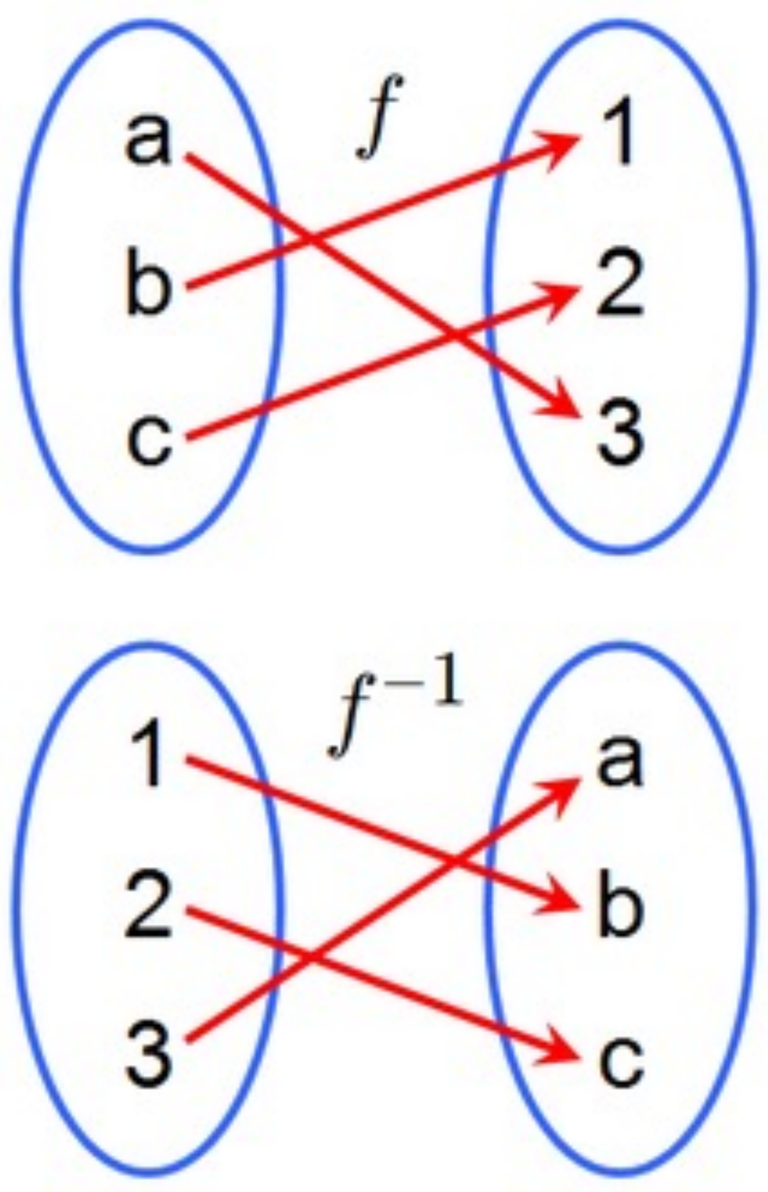
5.3.4 Definition Inverse of a Function

If *f* is a one-to-one function, then the inverse of *f*, denoted $\boldsymbol{f}^{-1}$, is the function formed by reversing all the ordered pairs in *f*. That is,

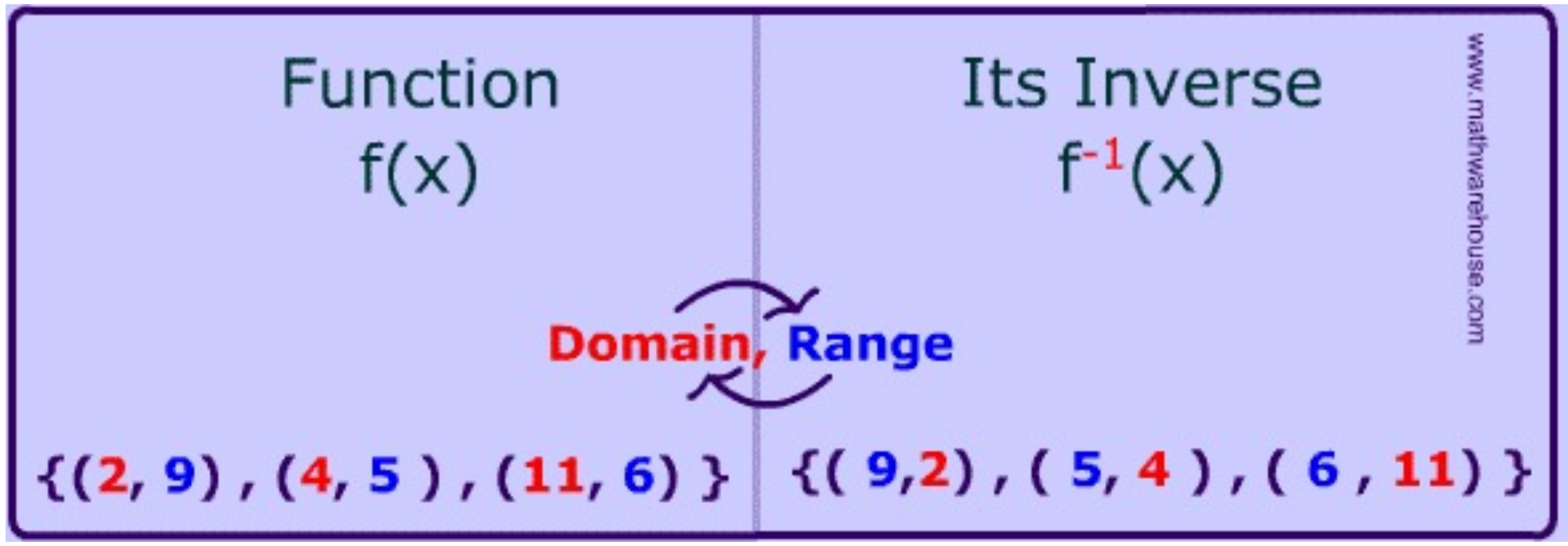
$$f^{-1} = \{(y, x) | (x, y) is\ in\ f\}$$

If *f* is not one-to-one, then *f* **does not have an inverse and $\boldsymbol{f}^{-1}$ does not exist.**





Interchanging the domain and range



5.3 Inverse of a Function

5.3.4 Definition Inverse of a Function

Finding Inverse Function

- **Step 1:** Replace $f(x)$ with y, then interchange $x \text{ and } y$.
- **Step 2:** Solve the resulting equation y . The result is $f^{-1}(x)$.

A test for Inverse of a function

Functions $f(x)$ and $f^{-1}(x)$ are an inverse pair if and only if

$$(\boldsymbol{f o f^{-1}})(\boldsymbol{x}) = \boldsymbol{x} \quad and \quad (f^{-1} o f)(\boldsymbol{x}) = \boldsymbol{x}$$

NOTE: Be careful that $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$.

5.3 Inverse of a Function

5.3.4 Definition Inverse of a Function

Example: (Deciding if two functions are inverses)

Verify that the functions $f(x) = 3x$ and $g(x) = \frac{x}{3}$ are inverses.

$$\boldsymbol{(fof^{-1})(x) = x \quad and \quad (f^{-1}of)(x) = x}$$

$$(fog)(x) = x \quad and \quad (gof)(x) = x$$

$$(fog)(x) = f\big(g(x)\big) = f\left(\frac{x}{3}\right) = 3\left(\frac{x}{3}\right) = \boldsymbol{x}$$

$$(gof)(x) = g\big(f(x)\big) = g(3x) = \frac{3x}{3} = \boldsymbol{x}$$

Therefore, f and g are inverses.

5.3 Inverse of a Function

5.3.4 Definition Inverse of a Function

Example: Find the inverse of $f(x) = \frac{x}{2} + 1$

Finding Inverse Function

- **Step 1:** Replace $f(x)$ with y, then interchange x and y.

$$y = \frac{x}{2} + 1, \qquad x = \frac{y}{2} + 1$$

- **Step 2:** Solve the resulting equation y . The result is $f^{-1}(x)$.

$$x = \frac{y}{2} + 1$$

$$x - 1 = \frac{y}{2}$$

$$y = 2x - 2 \rightarrow f^{-1}(x)$$

$$\boldsymbol{f^{-1}(x) = 2x - 2}$$

Checking:

$$\boldsymbol{\left(fof^{-1}\right)(x) = x \quad and \quad \left(f^{-1}of\right)(x) = x}$$

$$(fof^{-1})(x) = f\left(f^{-1}(x)\right)$$

$$= f(2x - 2)$$

$$= \frac{(2x-2)}{2} + 1 = \frac{2(x-1)}{2} + 1 = x - 1 + 1 = \boldsymbol{x}$$

$$(f^{-1}of)(x) = f^{-1}(f(x))$$

$$= f^{-1}(\frac{x}{2} + 1)$$

$$= 2\left(\frac{x}{2} + 1\right) - 2 = x + 2 - 2 = \boldsymbol{x}$$

5.3 Inverse of a Function

5.3.4 Definition Inverse of a Function

Example: Find the inverse of $y = x^2; x \geq 0$

Finding Inverse Function

- **Step 1:** Replace $f(x)$ with y, then interchange $x \; and \; y$.

$$y = x^2, \qquad\qquad x = y^2$$

- **Step 2:** Solve the resulting equation y . The result is $f^{-1}(x)$.

$$x = y^2$$

$$y = \sqrt{x} \rightarrow f^{-1}(x)$$

$$f^{-1}(x) = \sqrt{x}$$

Checking:

$$\boldsymbol{(f o f^{-1})(x) = x \;\; and \;\; (f^{-1} o f)(x) = x}$$

$$(f o f^{-1})(x) = f\left(f^{-1}(x)\right)$$

$$= f(\sqrt{x})$$

$$= (\sqrt{x})^2 = \boldsymbol{x}$$

$$(f^{-1} o f)(x) = f^{-1}(f(x))$$

$$= f^{-1}(x^2)$$

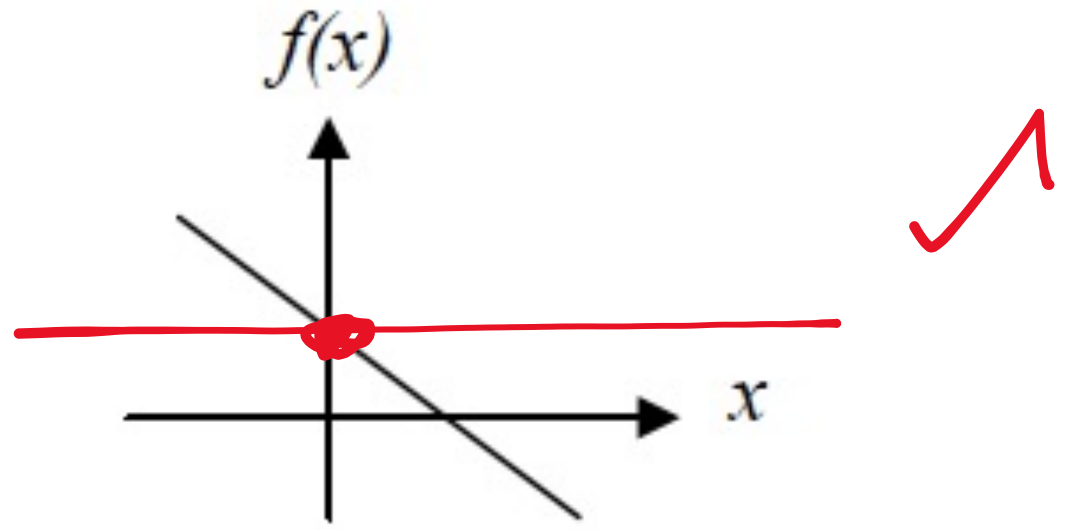
$$= \sqrt{x^2} = \boldsymbol{x}$$

One – to – One Functions?

Exercise

- Exercises 5.3

1.



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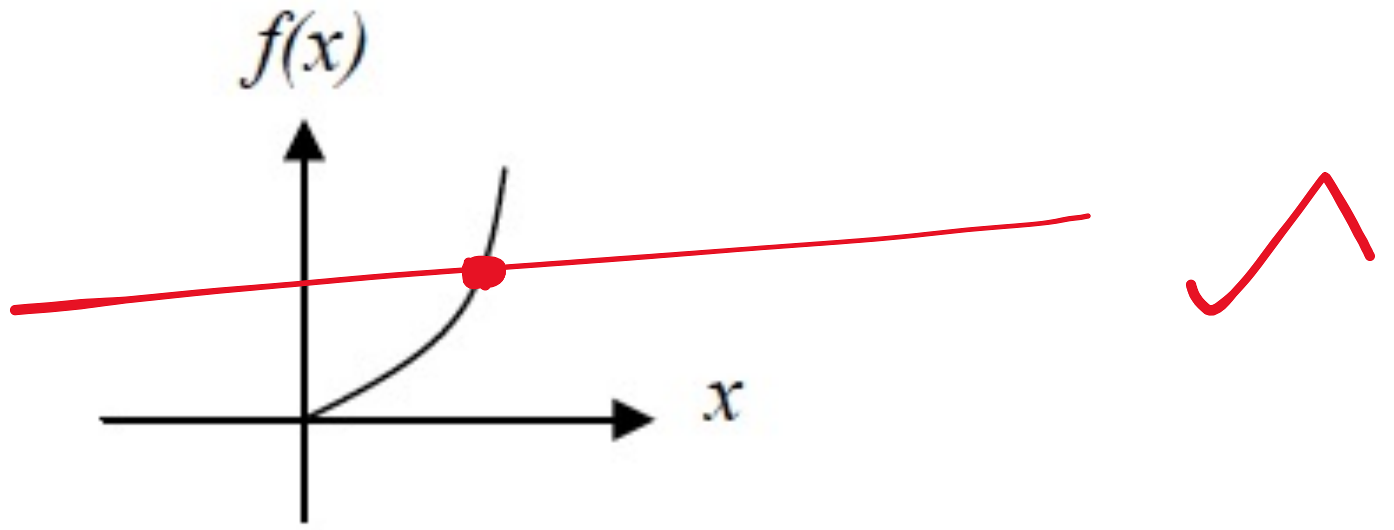

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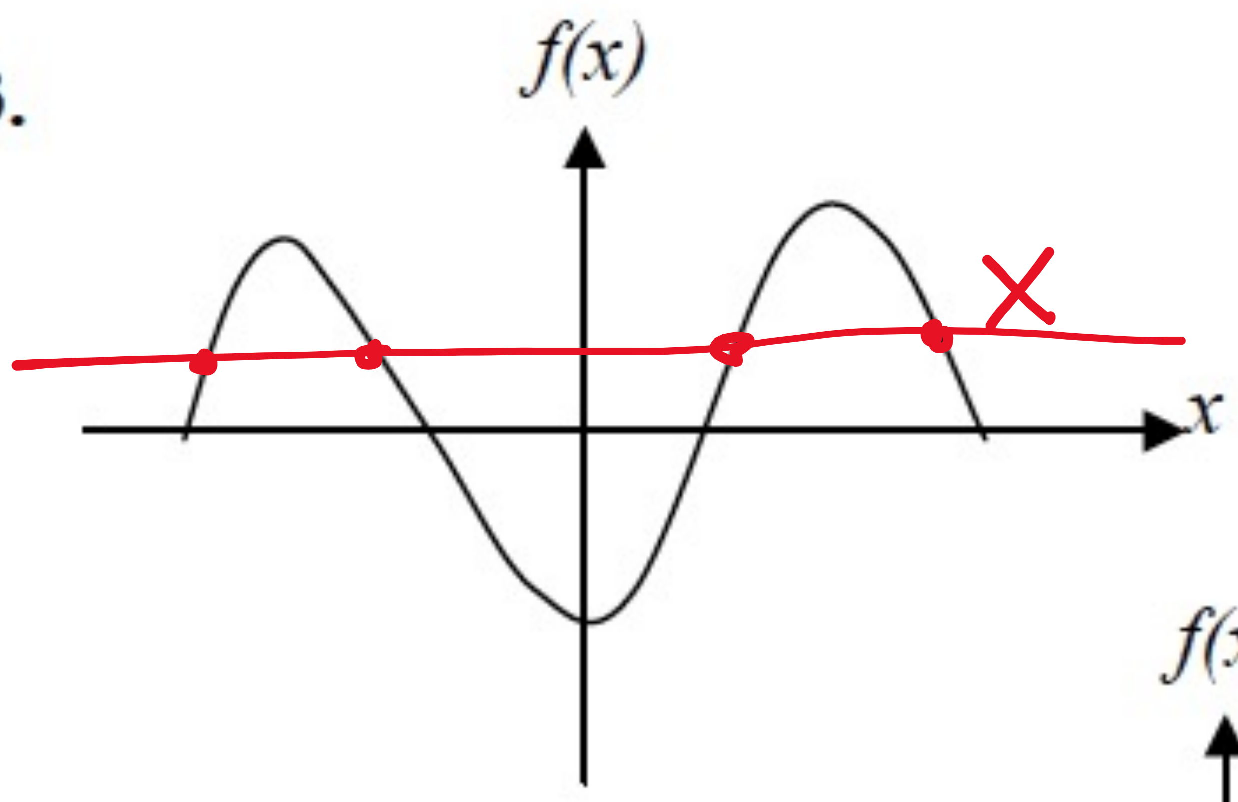

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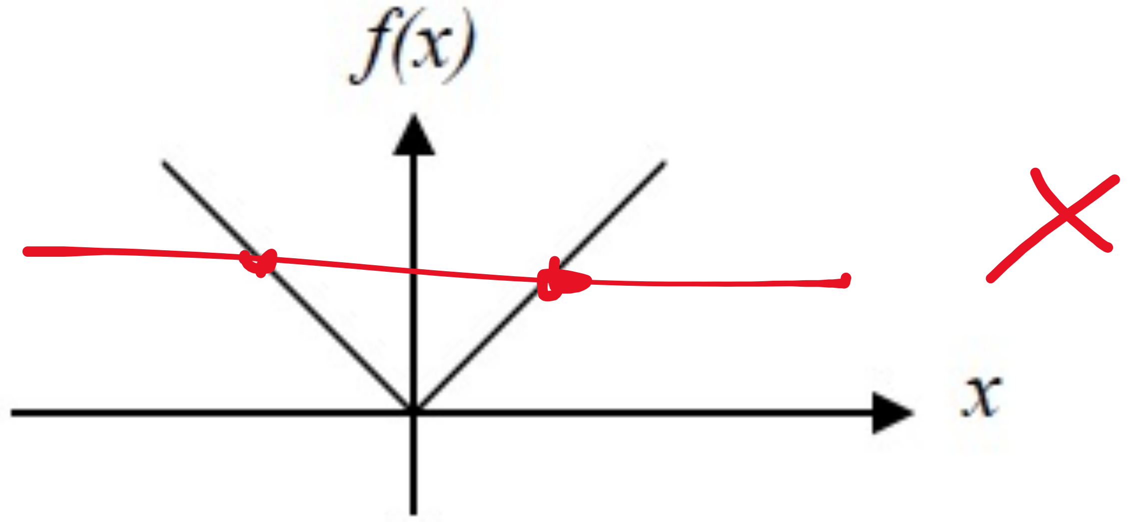

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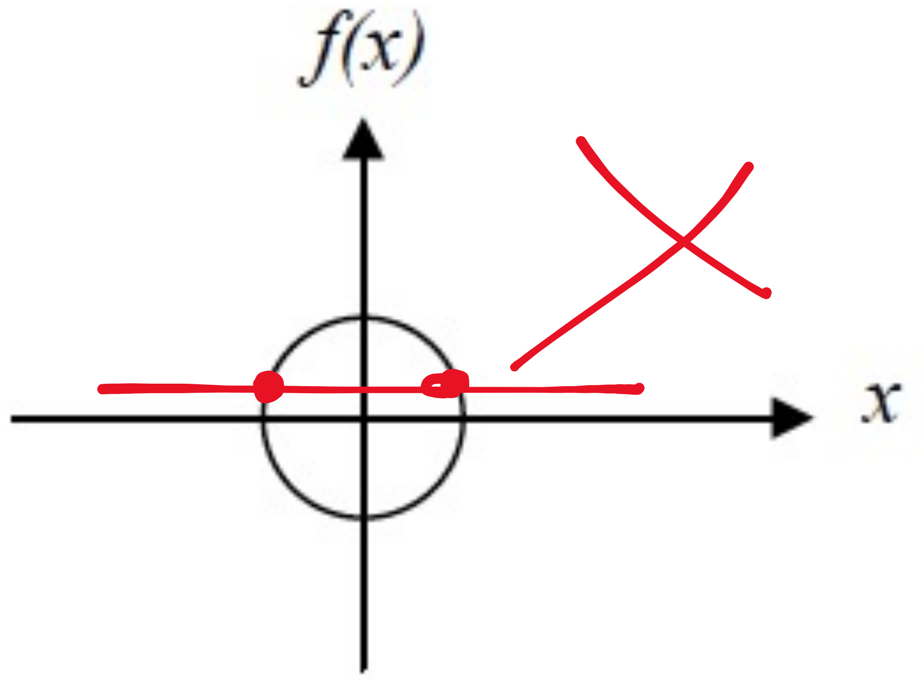


6. Determine whether the following function is one-to-one.

(a) $f(x) = 5x + 13$ ✓

(b) $f(x) = -|x|$ ✗

(c) $f(x) = 5 - x^3$ → ✓

(d) $f(x) = \sqrt{x}$ → ✓

a) $f(-2) = 5(-2) + 13$
$= -10 + 13 = \boxed{3}$

$f(0) = 13$

$f(+10) = 5(10) + 13 = 63$

(b) $f(-10) = -|-10| = \boxed{-10}$

$f(5) = -|5| = -5$

$f(10) = -10$

$f(-1) = 5-(-1)^3$
$= 5 + 1 = 6$

$f(0) = 5$

$f(1) = 5 - 1$
$= 4$

5.4 Exponential Functions

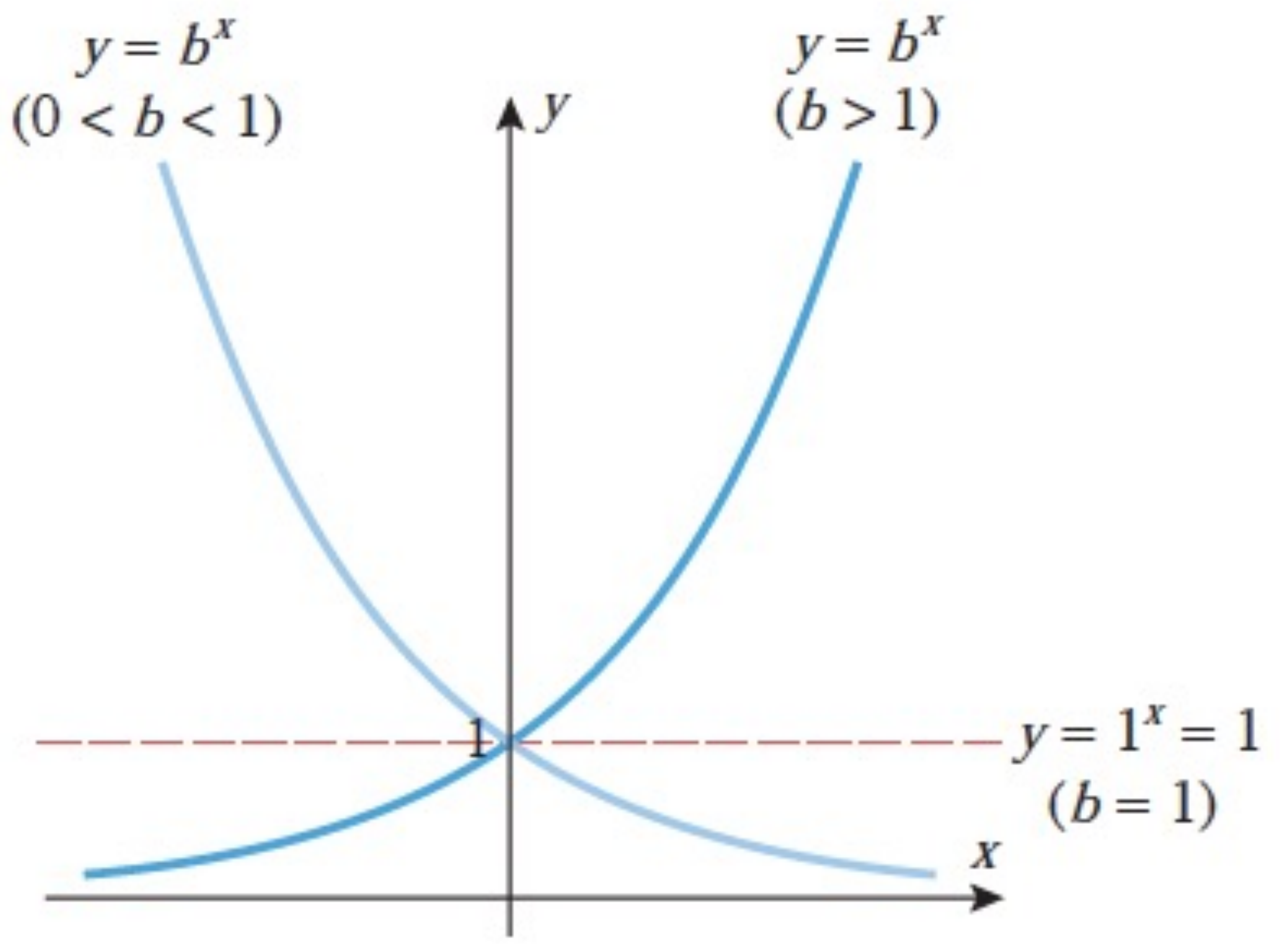
An **exponential function** is a function of the form with base b

$$y = f(x) = b^{x}$$

x → must be a variable

b → base

where
b = any positive real number other than 1
x = any real number
y = any positive real number



Note:
1. The domain of $y = b^{x}$ is $(-\infty, +\infty)$ and the range if $(0, +\infty)$
2. If $b > 1$, $y = b^{x}$ is increasing function.
   If $0 < b < 1$, $y = b^{x}$ is decreasing function
3. From one-to-one function $b^{x} = b^{y}$ if and only if $x = y$.

5.4 Exponential Functions

Note:

a constant base and variable exponent

$y = 3^x$, $y = 6^{2x-1}$, $y = e^x$, $y = (\frac{1}{2})^x$

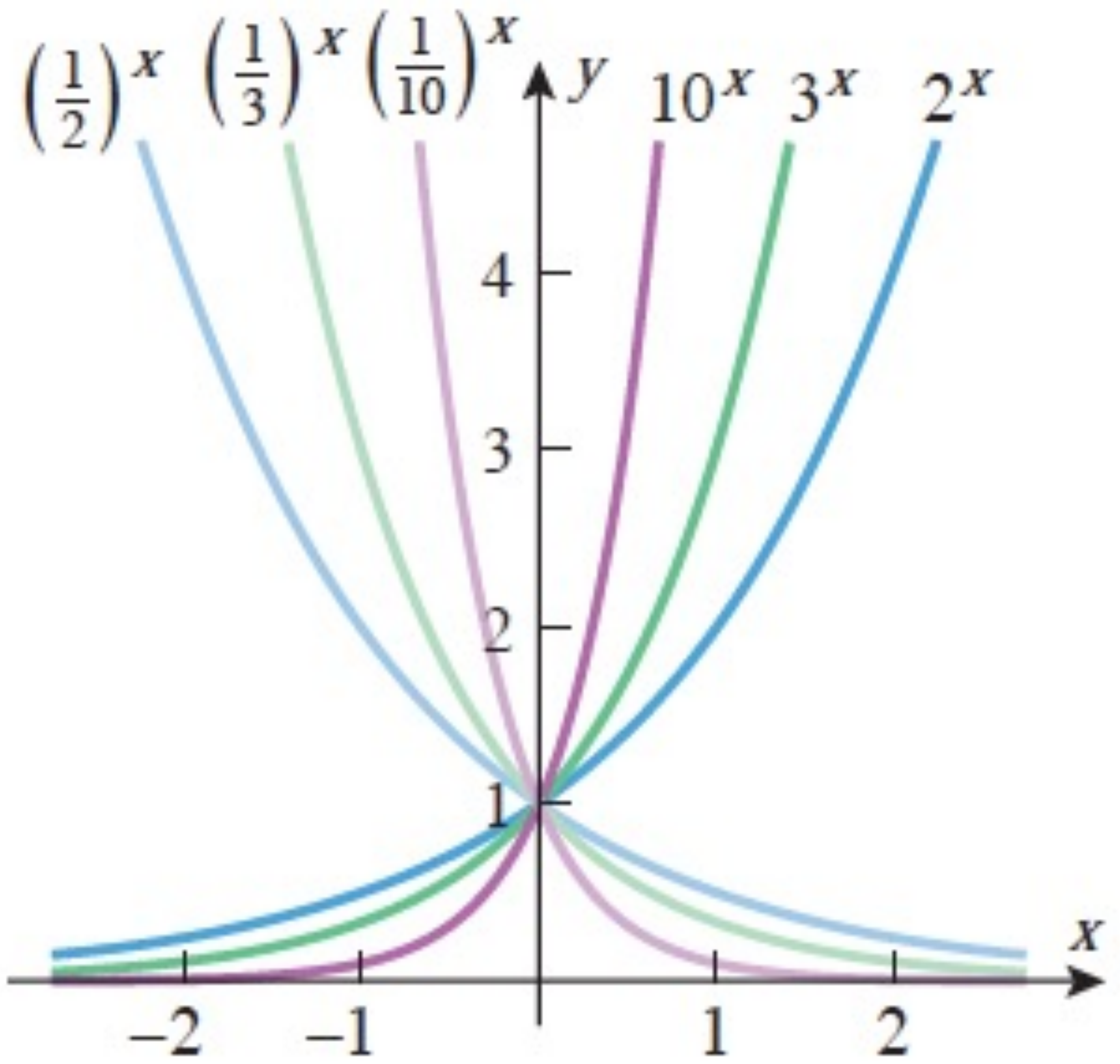
exponential function

not a variable base and a constant exponent

These functions are <u>not exponential functions</u>

$y = x^2, y = x^\pi$

NOT exponential function



5.4 Exponential Functions

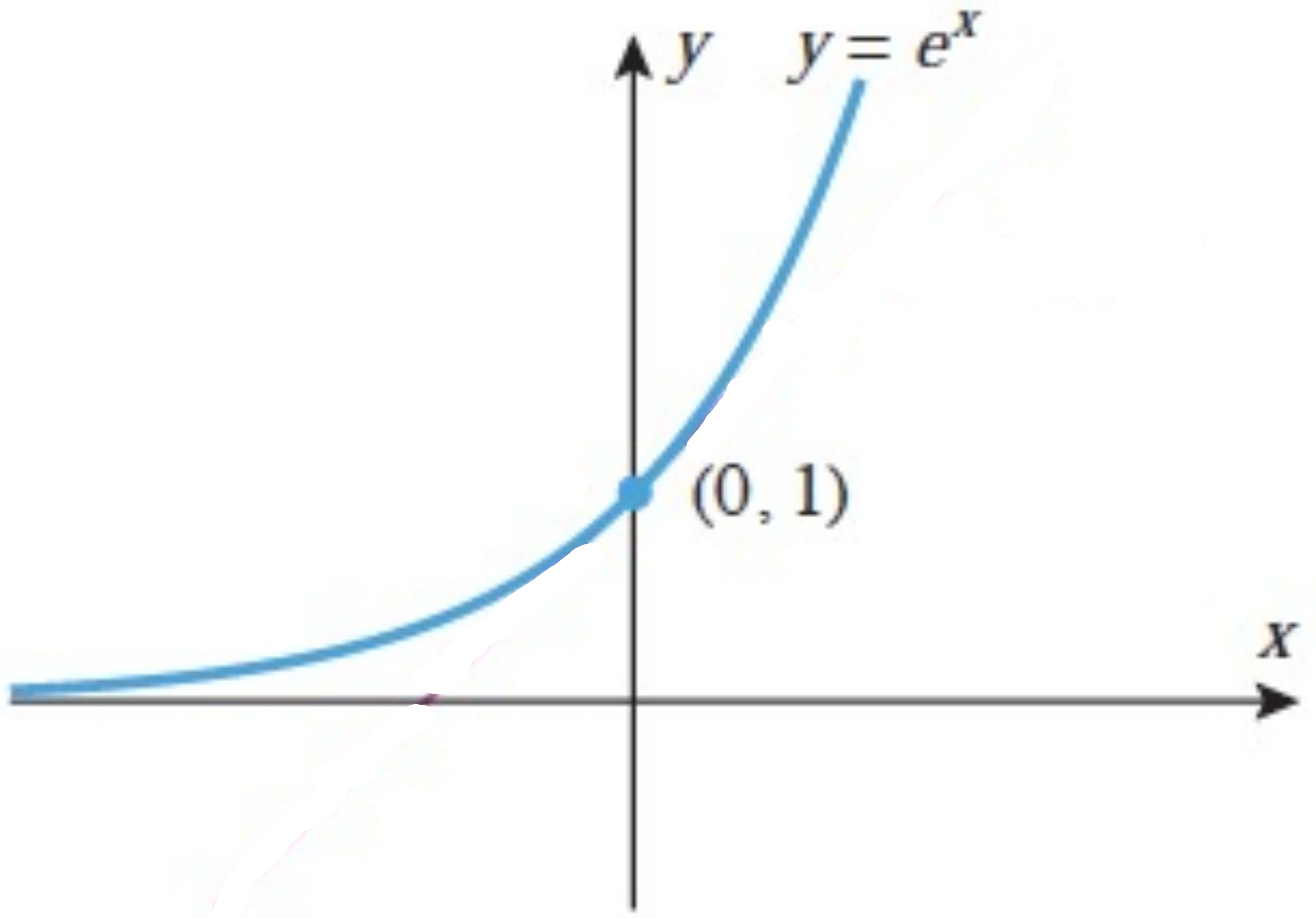
The Natural Exponential Function e^x

The most important exponential function for modeling natural, physical, and Economic phenomena is the *natural exponential function* , whose base is the number $e \approx 2.718281828\ldots$

$$y = f(x) = e^x$$

Properties

1. y-intercept is 1
2. No x-intercept.
3. Domain : $\{x \mid x \in R\}$
4. Range : $\{y \mid y \in R,\ y > 0\}$



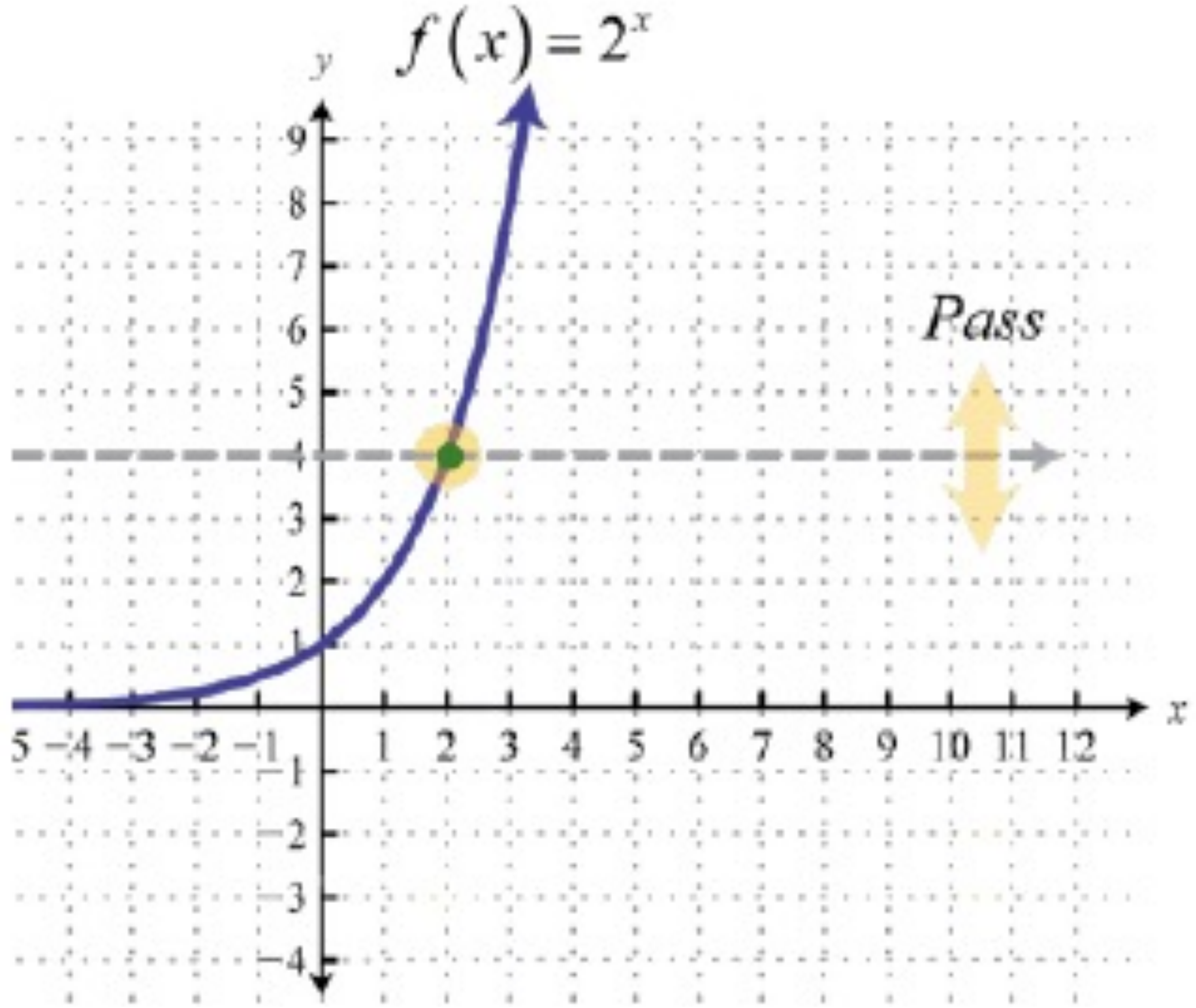
5.4 Exponential Functions

Draw the graph of $y = 2^x$

x	y
-3	1/8
-2	1/4
-1	1/2
0	1
1	2
2	4
3	8

Properties

1. The graph is rising from left to right
2. The *y*-intercept is 1 and there is no *x*-intercept.
3. The domain of the function is the set of real numbers.
4. The range of the function is the set of positive real numbers



5.4 Exponential Functions

Draw the graph of $y = 2^{-x}$ or $\boldsymbol{y} = (\frac{\mathbf{1}}{\mathbf{2}})^{\boldsymbol{x}}$

x	y
-3	8
-2	4
-1	2
0	1
1	1/2
2	1/4
3	1/8

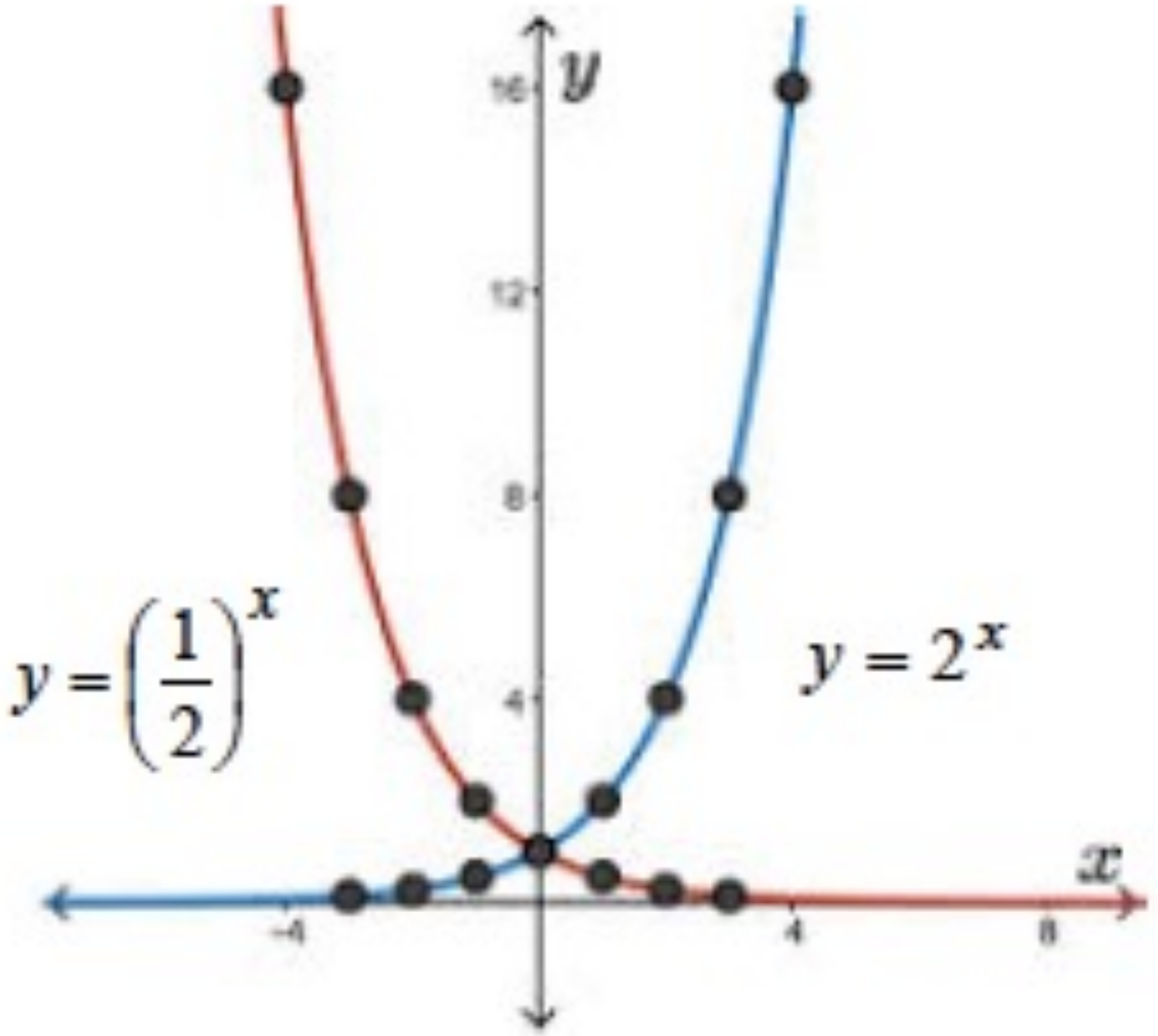
Properties

1. The graph is falling from left to right

2. *y*-intercept is 1 and there is no *x*-intercept.

3. Domain is the set of real numbers.

4. The range is the set of positive real numbers.



5.4 Exponential Functions

(a) $5^x = \frac{1}{125}$

$= 5^{-3}$

then $x = -3$

(b) $2^{x+6} = 32$

$2^{x+6} = 2^5$

then $x + 6 = 5$

$x = -1$

(c) $4^x = 2^{2x-1} \times 8^{4-2x}$

$2^{2x} = 2^{2x-1} \times (2^3)^{4-2x}$

$2^{2x} = 2^{2x-1} \times 2^{12-6x}$

$2^{2x} = 2^{11-4x}$

then $2x = 11 - 4x$

$6x = 11$

$x = \frac{11}{6}$

2. Solve the equation

(a) $16^x = \frac{1}{128}$ $\Rightarrow$ $2^{4x} = \frac{1}{2^7}$ $\Rightarrow$ $2^{4x} = 2^{-7}$

$4x = -7 \Rightarrow x = \frac{-7}{4}$

(b) $5^{-2x} = 125$

(c) $9^{2x-5} = 27$

(d) $8^{2x-3} = \left(\frac{1}{16}\right)^{x-2}$ $\Rightarrow$ $(2^3)^{2x-3} = \left(\frac{1}{2^4}\right)^{x-2}$

$2^{6x-9} = \left(2^{-4}\right)^{x-2} = 2^{-4x+8}$ $\Rightarrow$ $6x-9 = -4x+8$

$10x = 17$

$x = 1.7$

(e) $4^{3x+2} = 32^{5-x}$

(f) $49^{2x+3} = 343^{1-2x}$

(g) $\left(\frac{2}{3}\right)^{3x+2} = \left(\frac{9}{4}\right)^{x-3}$

(h) $\left(\frac{9}{25}\right)^{3x+4} = \left(\frac{125}{27}\right)^{x-5}$

(i) $a^{3/5} = \frac{1}{216}$

(j) $a^{7/3} = 128$

$B = 100e^{0.693t}$

(a) $B = 100e^0 = 100 \times 1 = 100$

(b) $B = 100e^{0.693(6)} = 100 \times e^{4.158} = 100 \times 63.94 = 6394$

5.5 Logarithm Functions

When working with the exponential function $y = b^x$, often we must solve for x.

To do so, we must define a new type of function called a logarithm.

$$y = b^x \quad = \quad x = \log_b y \qquad b > 1, b \neq 1$$

$2^{x+6} = 32$

$2^{x+6} = 2^5 \Rightarrow x+6=5$

$2^{x+6} = 20$

?

5.5 Logarithm Functions

Logarithmic functions is defined as

$$y = \log_b x \qquad b > 1, b \neq 1$$

$$\log_{10} 100 = 2, \quad \log_{10}(1/1000) = -3, \quad \log_2 16 = 4, \quad \log_b 1 = 0, \quad \log_b b = 1$$

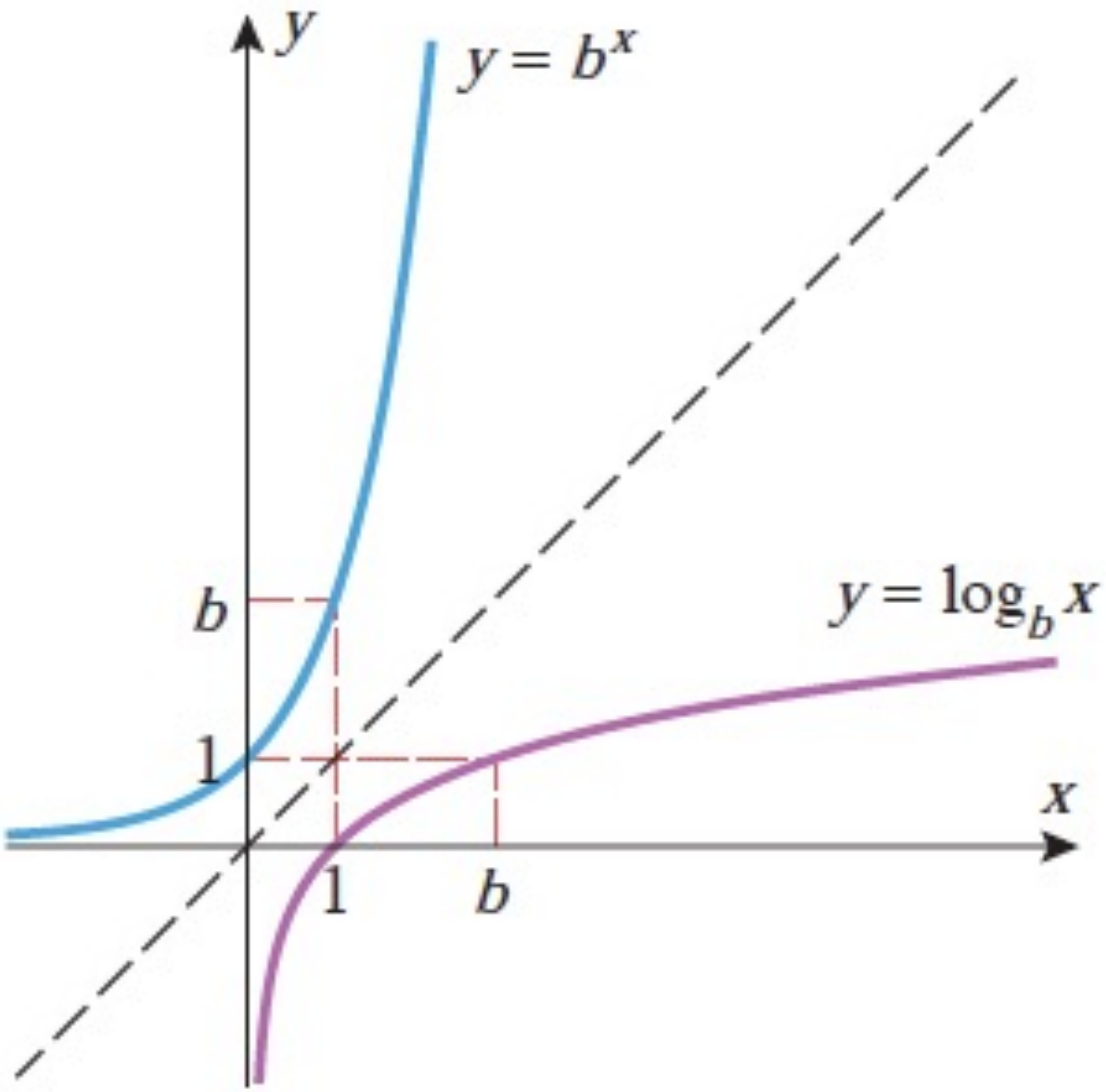
$$\boxed{10^2 = 100} \quad \boxed{10^{-3} = 1/1000} \quad \boxed{2^4 = 16} \quad \boxed{b^0 = 1} \quad \boxed{b^1 = b}$$

Logarithmic functions can also be viewed as inverses of exponential functions.

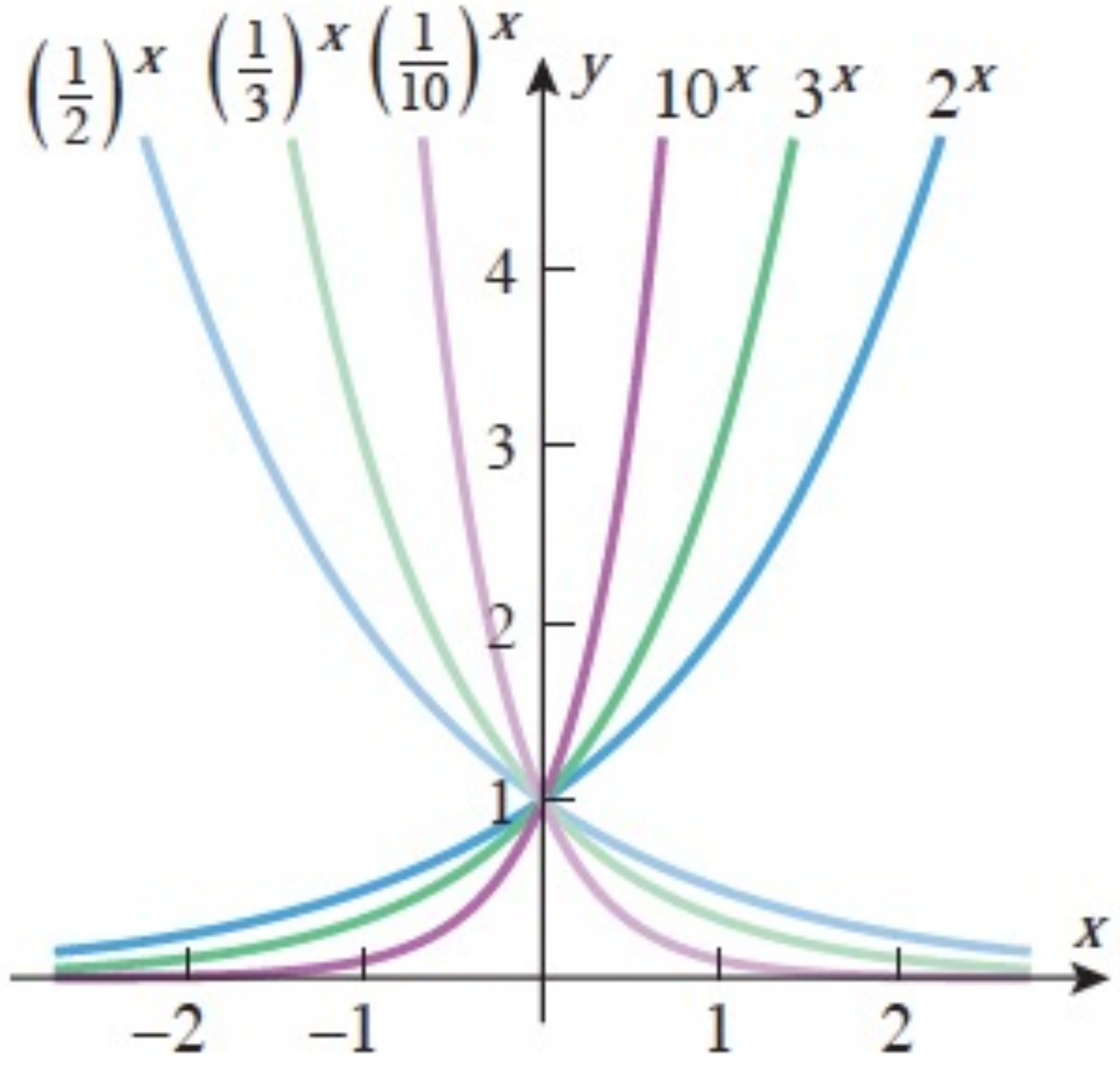
$$y = \log_b x \quad \rightarrow \quad x = b^y$$

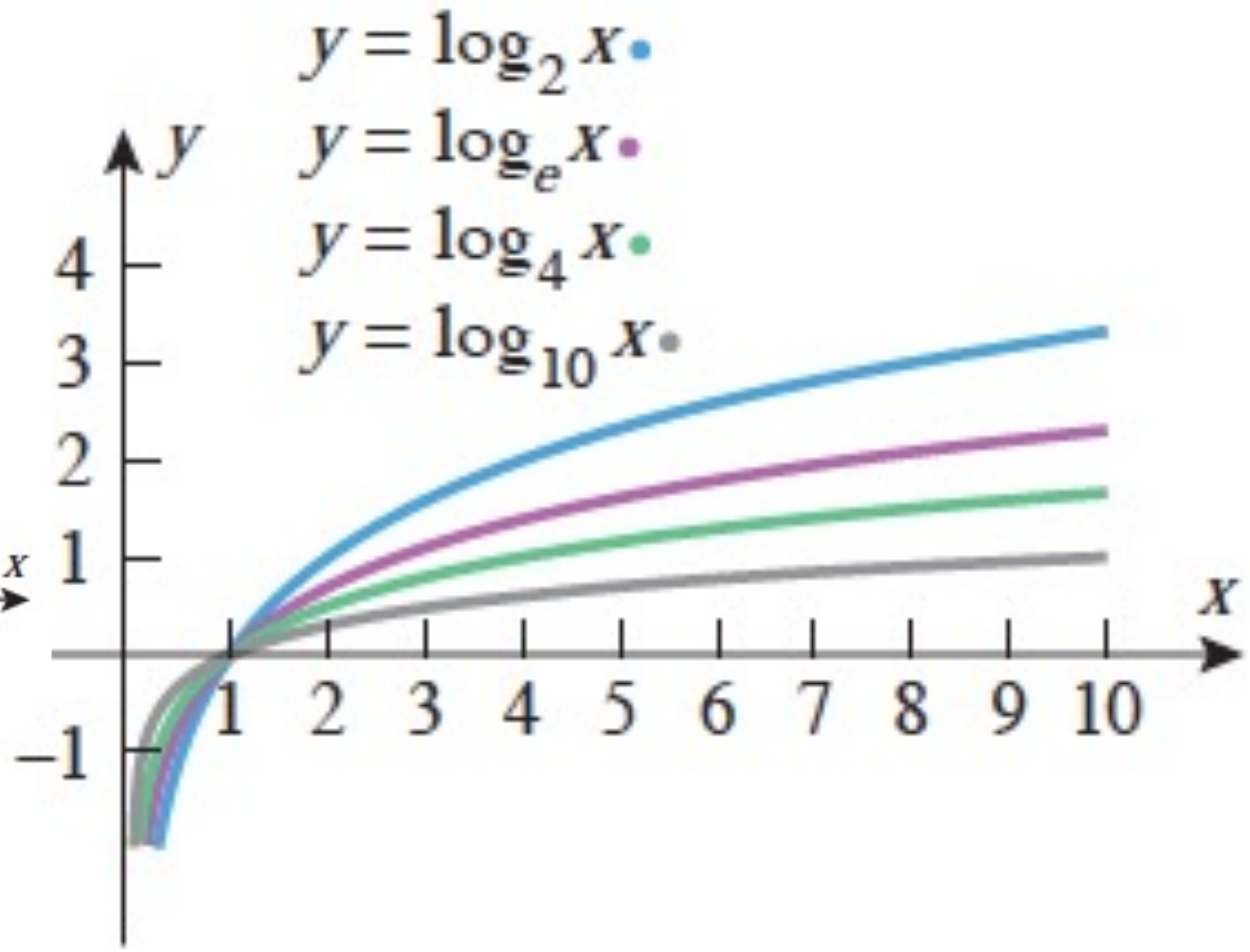
$$y = b^x \quad \rightarrow \quad x = \log_b y$$

5.5 Logarithm Functions



the graphs of y = b^x and y = $\log_b$ x are **reflections** of one another about the **line y = x**





5.5 Logarithm Functions

Note

1. The domain of $y = \log_b x$ is ~~R^+~~ D $= (0, \infty)$. The range of $y = \log_b x$ is $R = (-\infty, \infty)$.
2. If $b > 1$, $y = \log_b x$ is increasing function

   If $0 < b < 1$, $y = \log_b x$ is decreasing function
3. From one - to - one function, $\log_a x = \log_b y$ if and only if $x = y$.
4. $y = \log_e x$ is written as $y = \mathit{ln}\, x$ called *natural logarithm function*

   $y = \log_{10} x$ is written as $y = \mathit{log}\, x$ called *common logarithm function*

$5^2 = 25 \Rightarrow 2 = \log_5 25$

$= \log_5 5^2 = 2 \log_5 5 = 2 \times 1 = 2$

$\log_2 32 = 5 \Rightarrow 2^5 = 32$

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$

$\log_2 \frac{1}{8} = -3 \Rightarrow 2^{-3} = \frac{1}{8}$

$\frac{1}{2^3} = \frac{1}{8}$

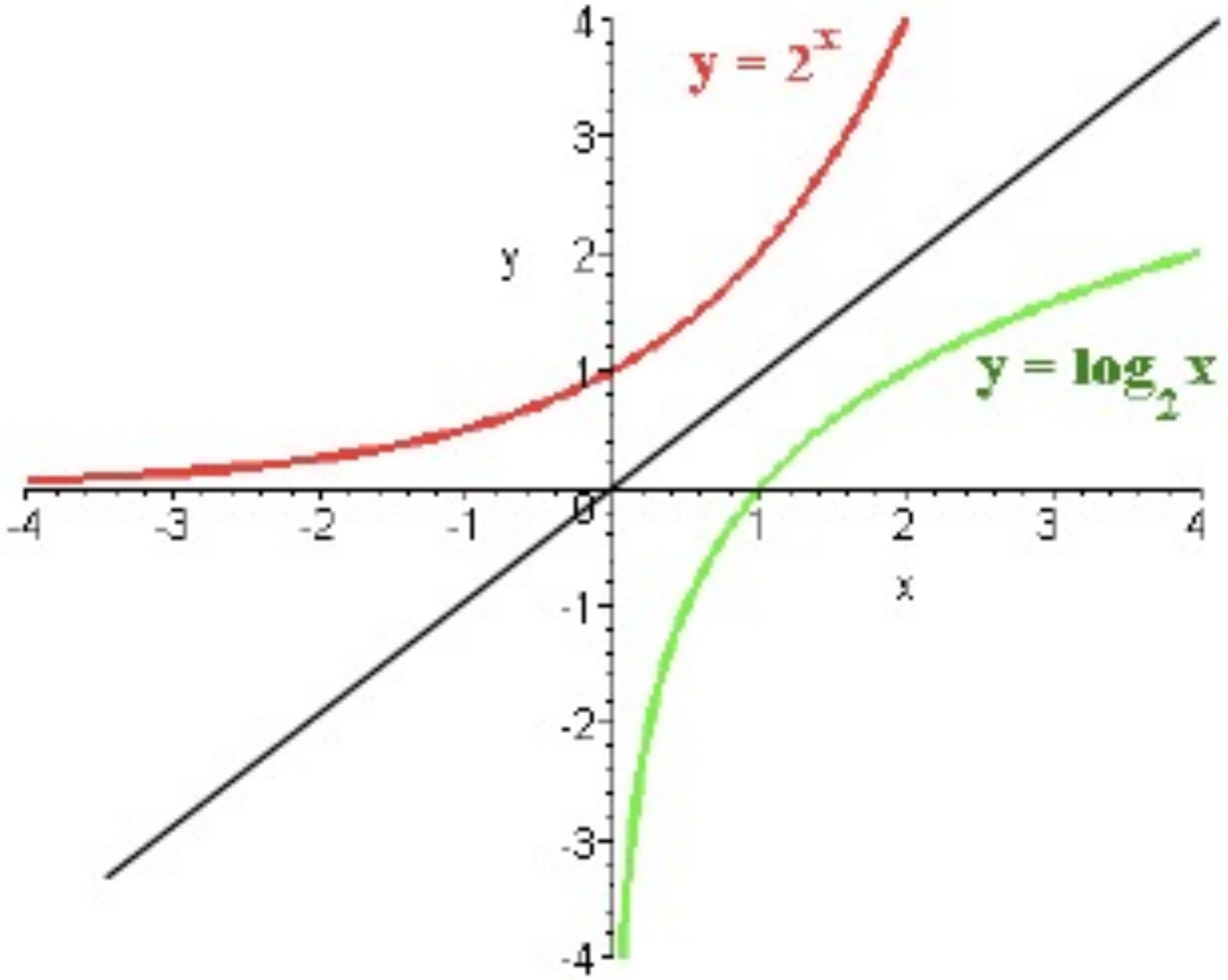
$= \frac{1}{8}$

5.5 Logarithm Functions

The graph of the function $y = 2^x$ and its inverse $x = b^y \rightarrow y = \log_2 x$

x	$y = 2^x$
-3	1/8
-2	1/4
-1	1/2
0	1
1	2
2	4
3	8

$x = 2^y$	$y = \log_2 x$
1/8	-3
1/4	-2
1/2	-1
1	0
2	1
4	2
8	3



Because b^x and $log_b\, x$ are inverse,

composing them in either order gives the identity function.

5.5 Logarithm Functions

Natural logarithms

$\log_e x$

$y = \ln x$

$\ln 1 = 0,\quad \ln e = 1,\quad \ln 1/e = -1,\quad \ln(e^2) = 2$

$\boxed{\text{Since } e^0 = 1}\quad \boxed{\text{Since } e^1 = e}\quad \boxed{\text{Since } e^{-1} = 1/e}\quad \boxed{\text{Since } e^2 = e^2}$

Inverse Properties for b^x and $log_b\, x$

1. Base b : $\quad b^{\log_b x} = x \quad$ and $\quad \log_b b^x = x$
2. Base e : $\quad e^{ln\, x} = x \quad$ and $\quad ln\, e^x = x$

5.5 Logarithm Functions

(a) $\ln x = 3t + 5$

$$e^{\ln x} = e^{3t+5}$$ → *Change to the natural exponential function of each side.*

$$x = e^{3t+5}$$

(b) $e^{2x} = 10$

$$\ln e^{2x} = \ln 10$$ → *Take the natural logarithm of each side.*

$$2x = \ln 10$$ → *Apply the property of* $\ln e^{x} = x$

$$x = \frac{\ln 10}{2} \approx 1.15$$

(c) $\log_{27} 9 = x$

$$27^{x} = 9$$

$$(3^{3})^{x} = 3^{2}$$

$$3^{3x} = 3^{2}$$

Then $3x = 2$,

$$x = \frac{2}{3}$$

5.5 Logarithm Functions

Properties of Logarithms:

For any real number $x > 0$ *and* $y > 0$, *then*

1. **Product Rule:** $\log_b(xy) = \log_b x + \log_b y$

2. **Quotient Rule:** $\log_b \frac{x}{y} = \log_b x - \log_b y$

3. **Power Rule:** $\log_b x^y = y \log_b x$

5.5 Logarithm Functions

Solve the equation $3^{\log_3 7} - 4^{\log_4 2} = 5^{(\log_5 x - \log_5 x^2)}$

$$3^{\log_3 7} - 4^{\log_4 2} = 5^{\log_5\left(\frac{x}{x^2}\right)}$$

$$7 - 2 = \frac{x}{x^2}$$

$$5 = \frac{1}{x}$$

$$x = \frac{1}{5}$$

5.5 Logarithm Functions

Change of Base Formula:

If $a, b, x > 0$ with $a, b \neq 1$, then $\log_a x = \dfrac{\log_b x}{\log_b a}$

$\log_{10} = \log$

$\ln = \log_e$

Note 1. If $b = 10$, then $\log_a x = \dfrac{\log x}{\log a}$.

2. If $b = e$, then $\log_a x = \dfrac{\ln x}{\ln a}$.

$\log_7 5 = \dfrac{\log 5}{\log 7} = \dfrac{\ln 5}{\ln 7}$

$\boxed{\log_2 7 = \dfrac{\log_{10} 7}{\log_{10} 2}}$

5.5 Logarithm Functions

if not given

Given that $\log_5 123 = 2.9898$, find $\log_{125} 123$ $\Rightarrow$ $\dfrac{\log_{10} 123}{\log_{10} 125} = \dfrac{\quad}{\quad}$

$$= \frac{\log_5 123}{\log_5 125} = \frac{\log_5 123}{\log_5 5^3} = \frac{\log_5 123}{3} = \frac{2.9898}{3} = 0.9966$$

Sarah invests \$1,000 in an account that earns 5.25 % interest compounded annually. How long will it take the account to reach \$ 2,500?

\$ 100 ← 5%

\$ 105 ← 5%

The amount in the account at any time t in years is

$$y = 1,000(1.0525)^t$$

$$2,500 = 1,000(1.0525)^t$$

$$(1.0525)^t = 2.5$$

$$t \ln 1.0525 = \ln 2.5 \quad \rightarrow \quad t = \frac{\ln 2.5}{\ln 1.0525} \approx 17.9 \text{ years}$$

5.6 Solving Exponential Equations with the Different Bases

$5^{2x-3} = 18$ use the properties of logarithms

→ $\log(5^{2x-3}) = \log(18)$ →*Take the common logarithm or natural logarithm of each side.*

→ $(2x-3)\log 5 = \log 18$ →*Use the properties of logarithms to rewrite the problem. Move the exponent out front which turns this into a multiplication problem.*

→ $(2x-3) = \dfrac{\log 18}{\log 5}$ →*Divide each side by log 5.*

→ $(2x-3) = 1.795889$ →*Use a calculator to find log 18 divided by log 5. Round the answer as appropriate, these answers will use 6 decimal places.*

→ $x = 2.3979$ →*Finish solving the problem by adding 3 to each side and then dividing each side by 2.*

Therefore, the solution to the problem $5^{2x-3} = 18$ is $x = 2.3979$.

Exercise and Assignment

Deadline for submission: next week Monday

- Exercises 5.4 – 2(d)
- Exercises 5.5 – 1(f), 2(d), 4(a), 9(d)