Determinants and Eigenvectors

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Determinants

Determinant of the matrix - assigning a real number to the any square matrix.

The determinant of a matrix *A* is denoted by **|*A*|** or *det A*.

Finding determinants of a 2 × 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -2 \\ 5 & 8 \end{bmatrix} \quad \boxed{= 34}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$|A| = \begin{vmatrix} 3 & -2 \\ 5 & 8 \end{vmatrix} = (3)(8) - (-2)(5)$$

$$= 24 + 10 = 34$$

Determinants

Finding determinants of a 3 × 3 matrix

Third order Determinants

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Determinants

Finding determinants of a $n \times n$ matrix

The determinant of a 3×3 matrix is defined in terms of determinants of 2×2 matrices.

The determinant of a 4×4 matrix is defined in terms of determinants of 3×3 matrices, and so on.

The determinant of a $n \times n$ matrix is defined in terms of determinants of $(n-1) \times (n-1)$ matrices, and so on.

For these definitions we need the following concepts of ***minor*** and ***cofactor***.

Determinants

Minor of the Matrix (M_{ij})

For any square matrix $A=[a_{ij}]_{nxm}$,

minor of an element a_{ij} denoted by M_{ij} can be obtained from the original determinant by deleting the row *i* and column *j.*

$$\text{The minor of } a_{11} \text{ is } M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

$$\text{The minor of } a_{23} \text{ is } M_{23} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{13}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinants

Cofactor of the matrix (C_{ij})

For any square matrix $A=[a_{ij}]_{nxm}$ the cofactor of an element a_{ij} denoted by C_{ij} can be defined **$C_{ij} = (-1)^{i+j} M_{ij}$**.

$$\text{Co-factor of } a_{11} \text{ is } C_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$$

$$\text{Co-factor of } a_{23} \text{ is } C_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -M_{23}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Where C_{ij} is the cofactor of the element a_{ij} is defined by $C_{ij} = (-1)^{i+j} M_{ij}$

Thus, the pattern of signs in front of the number a_{ij} is

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Note that the minor and cofactor differ in at most sign.

Determinants

Cofactor of the matrix (C_{ij})

Definition: The determinant of a square matrix is the sum of the products of the elements of the *first row* and their cofactors.

If **A** is 3 × 3, $|\mathbf{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

If **A** is 4 × 4, $|\mathbf{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$

$\vdots \qquad\qquad\qquad \vdots$

If **A** is $n \times n$, $|\mathbf{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$

These are called the *determinant is being expanded about the first row*. It can be shown that **any row or column can be used to expand a determinant**.

Determinants

For a 3 × 3 matrix A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

determinant is being **expanded about the first row** can be found by multiplying each element of the first row with its corresponding cofactor and then adding the three results.

The determinant can be obtained by the formula

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}\,a_{33} - a_{32}\,a_{23}) - a_{12}(a_{21}\,a_{33} - a_{31}\,a_{23}) + a_{13}(a_{21}\,a_{32} - a_{31}\,a_{22})$$

Determinants

find M_{11} and M_{23}

$$A = \begin{bmatrix} 2 & 1 & 4 \\ -6 & 3 & -2 \\ 4 & 2 & 5 \end{bmatrix}$$

To find M_{11} we first delete row 1 and column 1 of A

$$M_{11} = \begin{vmatrix} 3 & -2 \\ 2 & 5 \end{vmatrix} = (3)(5) - (-2)(2) = 15 + 4 = 19$$

To find M_{23} we first delete row 2 and column 3 of A

$$M_{23} = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = (2)(2) - (1)(4) = 4 - 4 = 0$$

Determinants

find |*A*| by expanded about the first row and the second column

$$A = \begin{bmatrix} 2 & -4 & -5 \\ 1 & 0 & 4 \\ 2 & 3 & -6 \end{bmatrix}$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

a. expanded about the first row

$$|A| = +(2)\begin{vmatrix} 0 & 4 \\ 3 & -6 \end{vmatrix} - (-4)\begin{vmatrix} 1 & 4 \\ 2 & -6 \end{vmatrix} + (-5)\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 2(0-12) + 4(-6-8) - 5(3-0)$$

$$= -24 - 56 - 15 = -95$$

b. expanded about the second column

$$|A| = -(-4)\begin{vmatrix} 1 & 4 \\ 2 & -6 \end{vmatrix} + 0\begin{vmatrix} 2 & -5 \\ 2 & -6 \end{vmatrix} - (3)\begin{vmatrix} 2 & -5 \\ 1 & 4 \end{vmatrix}$$

$$= 4(-6-8) + 0 - 3(8+5) \quad = -\ 56 - 39 = -95$$

Exercise

1. find the minor M_{43} and cofactor C_{43} of **A,** if A = $\begin{bmatrix} 2 & 0 & 1 & -5 \\ 8 & -1 & 2 & 1 \\ 4 & -3 & -5 & 0 \\ 1 & 4 & 8 & 2 \end{bmatrix}$

2. Evaluate the determinants of below matrix in two ways, 2nd column and 3rd column, using the indicated rows and columns. Observe that you get the same answers both ways.

$$\begin{bmatrix} 6 & 3 & 0 \\ -2 & -1 & 5 \\ 4 & 6 & -2 \end{bmatrix}$$

Determinants

$$A = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

$$|A| = a_{13}C_{13} - a_{23}C_{23} + a_{33}C_{33} - a_{43}C_{43}$$

$$= 0(C_{13}) - 0(C_{23}) + 3(C_{33}) - 0(C_{43})$$

$$= +3\begin{vmatrix} 2 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$= +3\,(2)\begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix}$$

$$= 6(3 - 2)$$

$$= 6$$

Determinants

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

$C_2 + 2C_3$

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \longrightarrow \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix}$$

$$(-3)\begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = (-3)(9-2) = -21$$

Exercise

1. Evaluate the determinant of the following matrix.

$$\begin{bmatrix} 1 & 4 & 5 & 9 \\ 2 & 3 & -7 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 8 \end{bmatrix}$$

2. Solve the following equation for the variable *x.*

$$\begin{vmatrix} x-1 & 3 \\ x-2 & x-1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 2 & 0 & 2 \\ 2x & x-1 & 4 \\ -x & x-1 & x+1 \end{vmatrix} = 0$$

Determinants

Properties of Determinants

Let **A** be a $n \times n$ matrix and c be a nonzero scalar.

1. If a matrix **B** is obtained from **A** by multiplying the elements of a row (column) by c then $|\mathbf{B}| = c|\mathbf{A}|$.
2. If a matrix **B** is obtained from **A** by interchanging two rows (columns) then $|\mathbf{B}| = -|\mathbf{A}|$.
3. If a matrix **B** is obtained from **A** by adding a multiple of one row (column) to another row (column), then $|\mathbf{B}| = |\mathbf{A}|$.
4. If any row (or column) of a square matrix **A** contains only 0s, then $|\mathbf{A}| = 0$.
5. $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.
6. $|\mathbf{A}| = |\mathbf{A}^{\mathrm{T}}|$.

Definition: A square matrix **A** is said to be **singular** if $|\mathbf{A}| = 0$. **A** is nonsingular if $|\mathbf{A}| \neq 0$.

Determinants

Determinant of an Upper Triangular Matrix

lower triangular matrix

$$\begin{bmatrix} 2 & -1 & 9 & 4 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -1 & 9 & 4 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 0 & 6 \\ 0 & -5 & 3 \\ 0 & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} -1 & 9 & 4 \\ 0 & -5 & 3 \\ 0 & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} -1 & 9 & 4 \\ 3 & 0 & 6 \\ 0 & 0 & 1 \end{vmatrix} - 0\begin{vmatrix} -1 & 9 & 4 \\ 3 & 0 & 6 \\ 0 & -5 & 3 \end{vmatrix}$$

$$= 2\left(3\begin{vmatrix} -5 & 3 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 6 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 6 \\ -5 & 3 \end{vmatrix}\right) = 2 \times 3 \times (-5) = -30$$

Determinants

Elimination Method for Evaluating a Determinant

Transform the given determinant into upper triangular form using two types of elementary row operations:

1. Add a multiple of one row to another row. This transformation leaves the determinant unchanged.

2. Interchange two rows of the determinant. This transformation multiplies the determinant by –1.

The zeros below the main diagonal are created systematically in the columns, from left to right, according to the Gauss-Jordan pattern.

Determinants

Elimination Method for Evaluating a Determinant

$$\begin{vmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{vmatrix}$$

$$\left|\begin{matrix} 2 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 1 & 8 \end{matrix}\right| \begin{matrix} \\ R_2 + R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\left|\begin{matrix} 2 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 13 \end{matrix}\right| \begin{matrix} \\ \\ R_3 + R_2 \end{matrix}$$

Then , $\begin{vmatrix} 2 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 13 \end{vmatrix}$

$$2\begin{vmatrix} -1 & 5 \\ 0 & 13 \end{vmatrix} = (2)(-13) = -26$$

Exercise

1. **(Row Operations)** Simplify the determinants of the following matrix by creating zeros in a single row or column, then evaluate the determinant by expanding in terms of that row or column.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2. **(Elimination Method)** Evaluate the following determinant using the elimination method.

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ -2 & 3 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -4 & -2 & 0 & -1 \end{vmatrix}$$

Determinants and Systems of Linear Equations: Cramer’s Rule

Cramer’s Rule (2 × 2 Case)

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The solution for this system is given by

$$x = \frac{D_x}{D}, \qquad \text{and} \qquad y = \frac{D_y}{D}$$

Cramer's Rule

Cramer's Rule (3 × 3 Case)

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_{31}z = d_3$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0 \qquad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, and $z = \dfrac{D_z}{D}$

Remark 1. If $D = 0$ and at least one of D_x , D_y , and D_z is not zero, then the system is non consistent and has no solution.

2. If D , D_x , D_y , and D_z are all zero, then the equations are dependent and there are infinitely many solutions.

Cramer's Rule

$$\frac{x}{2} + \frac{y}{3} = -4$$
$$\frac{x}{4} - \frac{3y}{2} = 20$$

$$\Rightarrow \quad \begin{aligned} 3x + 4y &= -24 \\ x - 6y &= 80 \end{aligned}$$

$$D = \begin{vmatrix} 3 & 4 \\ 1 & -6 \end{vmatrix} = -18 - 4 = -22$$

$$D_x = \begin{vmatrix} -24 & 4 \\ 80 & -6 \end{vmatrix} = 144 - 320 = -176$$

$$D_y = \begin{vmatrix} 3 & -24 \\ 1 & 80 \end{vmatrix} = 240 - (-24) = 264$$

$$x = \frac{D_x}{D} = \frac{-176}{-22} = 8$$

$$y = \frac{D_y}{D} = \frac{264}{-22} = -12$$

The solution set is **{(8, -12)}**.

Cramer's Rule

$$\begin{aligned} x - 2y + z &= -4 \\ 2x + y - z &= 5 \\ 3x + 2y + 4z &= 3 \end{aligned}$$

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29 \qquad D_x = \begin{vmatrix} -4 & -2 & 1 \\ 5 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29 \qquad D_y = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 5 & -1 \\ 3 & 3 & 4 \end{vmatrix} = 58 \qquad D_z = \begin{vmatrix} 1 & -2 & -4 \\ 2 & 1 & 5 \\ 3 & 2 & 3 \end{vmatrix} = -29$$

$$x = \frac{D_x}{D} = \frac{29}{29} = 1 \qquad y = \frac{D_y}{D} = \frac{58}{29} = 2 \qquad z = \frac{D_z}{D} = \frac{-29}{29} = -1$$

The solution set is **{(1, 2, - 1)}**.

Cramer's Rule

$$\begin{aligned} x + 3y - z &= 4 \\ 3x - 2y + z &= 7 \\ 2x + 6y - 2z &= 1 \end{aligned}$$

$$D = \begin{vmatrix} 1 & 3 & -1 \\ 3 & -2 & 1 \\ 2 & 6 & -2 \end{vmatrix} = 0 \qquad D_x = \begin{vmatrix} 4 & 3 & -1 \\ 7 & -2 & 1 \\ 1 & 6 & -2 \end{vmatrix} = -7$$

since $D = 0$ and at least one of D_x , D_y , and D_z is not zero,

then the system is non consistent. The solution set is ∅.

Exercise

Solve the following systems of linear equations using Cramer's Rule.

1.
$$\begin{aligned} 3x_1 - 4x_2 &= -5 \\ -3x_1 + 6x_2 &= 3 \end{aligned}$$

2.
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ x_1 \qquad\quad - x_3 &= -2 \\ 2x_1 - x_2 \qquad\quad &= 0 \end{aligned}$$

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Let A be a square (nxn) matrix.

A scalar λ is called and eigenvalue of A if there exists a nonzero vector x in R^n sunh that $\mathbf{Ax = \lambda x}$.

The vector x is called an eigenvector corresponding to λ.

Eigenvalues and Eigenvectors

$x \longrightarrow \boxed{\text{function}} \longrightarrow f(x)$

matrix

Eigenvectors

$\boldsymbol{A}$ x parllel to x

$$\boxed{Ax = \lambda x}$$

Eigenvalues

If A is singular, λ=0 is eigenvalue

Eigenvalues and Eigenvectors

$x^2 - 2x + 1 = 0$
$(x-1)(x-1) = 0$
$x = 1$

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

$$\mathbf{A}\mathbf{x} - \lambda I_n \mathbf{x} = 0$$

$$(\mathbf{A} - \lambda I_n)\mathbf{x} = 0$$

Singluar matrix

Eigenvectors defined to be nonzero vectors

$$|\mathbf{A} - \lambda I| = 0$$

$$\lambda = \checkmark$$

Find λs first

$$(\mathbf{A} - \lambda I_n)\mathbf{x} = 0$$

$$\mathbf{x} = \checkmark$$

Eigenvalues and Eigenvectors

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Fact:

Sum of $\lambda' s$ $(\lambda_1 + \lambda_2 + \ldots + \lambda_n)$ = Sum of down the diagonal entries $(a_{11} + a_{12} + \ldots + a_{nn})$

Eigenvalues and Eigenvectors

Summary To solve the eigenvalue problem for an $n \times n$ matrix, follow these steps:

1. **Compute the determinant** $|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I_n}|$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. **Find the roots of this polynomial,** by solving the equation $|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I_n}| = 0$. The n roots are the n eigenvalues of **A**. They make $\mathbf{A} - \boldsymbol{\lambda}\mathbf{I_n}$ singular.
3. For each eigenvalue λ, **solve** $(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I_n})\mathbf{x} = 0$ **to find an eigenvector** $\boldsymbol{x}$.

These quick checks always work:

- ***The product of the n eigenvalues*** *equals* ***the determinant.***
- ***The sum of the n eigenvalues*** *equals* ***the sum of the n diagonal entries.***

Eigenvalues and Eigenvectors

Example

$$\mathbf{A}\mathbf{x} - \lambda I \mathbf{x} = 0$$

Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$

$\mathbf{A} - \lambda I$

$$= \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -4 - \lambda & -6 \\ 3 & 5 - \lambda \end{bmatrix}$$

$|\mathbf{A} - \lambda I| = 0$

$$\begin{vmatrix} -4 - \lambda & -6 \\ 3 & 5 - \lambda \end{vmatrix} = 0$$

$$(-4 - \lambda)(5 - \lambda) - 18 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\boxed{\lambda = 2 \text{ or } -1}$$

The eigenvalues of **A** are 2 and -1.

Eigenvalues and Eigenvectors

Example

$$\mathbf{A} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$$

The eigenvalues of **A**, λ **= 2 or -1**

The corresponding eigenvectors are found by using these values of λ in the equation **($\mathbf{A}$ - λI_n)x = 0.**

There are many eigenvectors corresponding to each eigenvalue.

Meaning, you will have an eigenvector, x_1, for λ_1 **= 2** and another eigenvector, x_2, for λ_2 **= -1.**

Eigenvalues and Eigenvectors

Example

For $\lambda_1 = 2$,

$(A - \lambda_1 I)x_1 = 0$

$$\begin{bmatrix} -4-2 & -6 \\ 3 & 5-2 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} = 0$$

$$\begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} = 0$$

$-6x_a - 6x_b = 0$ and $3x_a - 3x_b = 0$

$x_a = -x_b$ $\qquad$ $x_a = -x_b$

$$x_1 = \begin{bmatrix} x_a \\ x_b \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ or } 4\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{x_1 = r\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Eigenvalues and Eigenvectors

Example

For $\lambda_2 = -1$,

$(A - \lambda_2 I)x_2 = 0$

$$\begin{bmatrix} -4-(-1) & -6 \\ 3 & 5-(-1) \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix} = 0$$

$-3x_a - 6x_b = 0$ and $3x_a + 6x_b = 0$

$x_a = -2x_b$ $x_a = -2x_b$

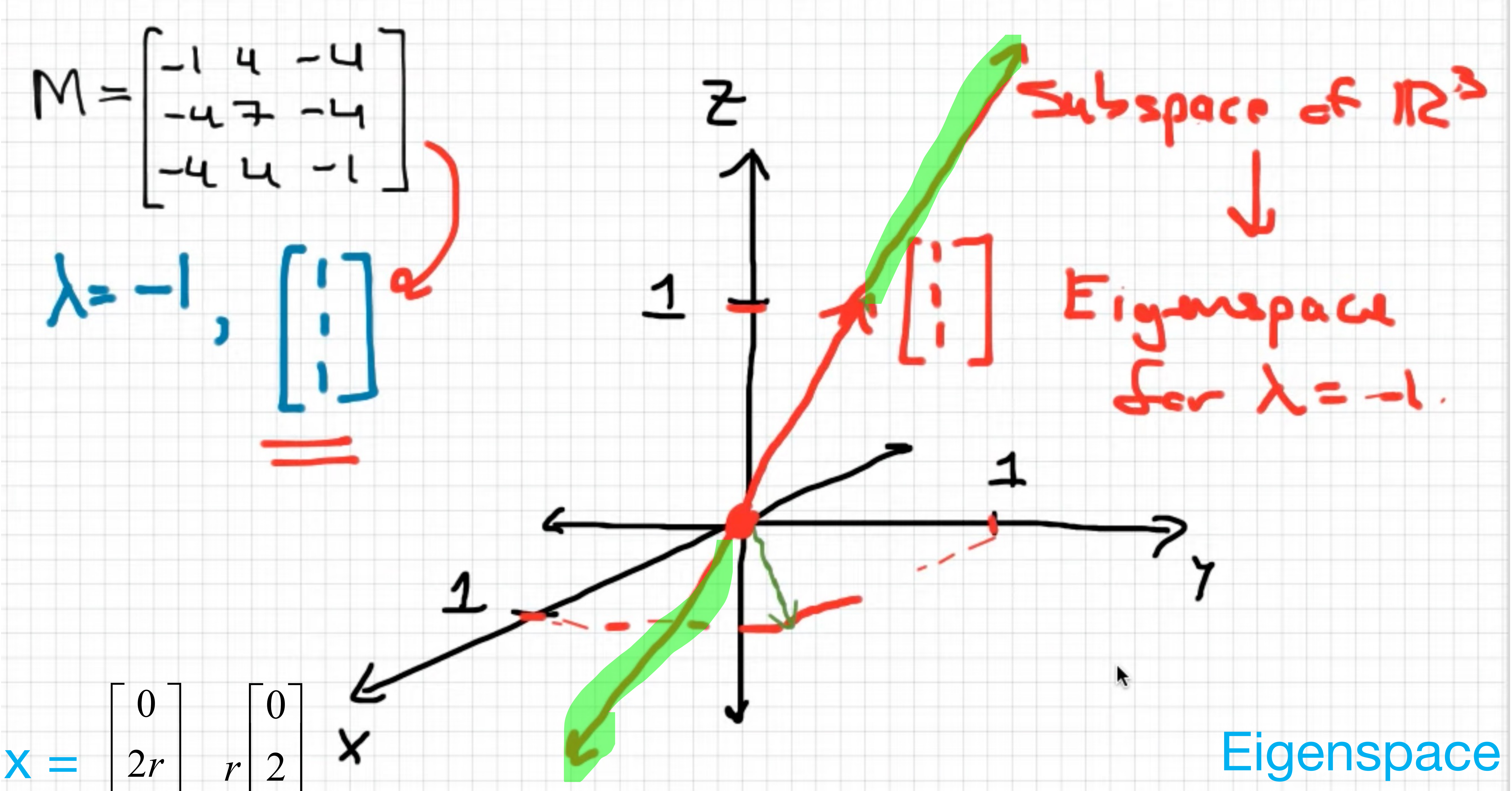
$$x_2 = \begin{bmatrix} x_a \\ x_b \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\boxed{x_2 = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Eigenspace

Eigenspace of an eigenvlaue is the subspace of R^n that contains all the eigenvectors that correspond to that eigenvlaues.



Eigenvector, x = $\begin{bmatrix} 0 \\ 2r \\ 3r \end{bmatrix}$ $r\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

Eigenspace = $\left\{ r\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$

Eigenspace

Theorem:

Let **A** be an *n x n* matrix and λ an eigenvalue of **A.** The set of all eigenvectors corresponding to λ, together with the zero vector, is a subspace of $\mathbf{R}^n$. This subspace is called the **eigenspace** of .

Eigenvalues and Eigenvectors

Exercise

1. Find the eigenvalues and eigenvectors of the matrix **A** = $\begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} \end{bmatrix}$

2. Find the eigenvalues and eigenvectors of the matrix **B** = $\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{4} & \mathbf{1} \end{bmatrix}$

3. Find the eigenvalues and eigenvectors of the matrix **C =** $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{-2} & \mathbf{0} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 3 & 0 \\ 6 & -6 & 7 \end{bmatrix}$$

$$|A - \lambda I_3| = 0$$

$$\begin{vmatrix} 2-\lambda & 3 & -2 \\ 2 & 3-\lambda & 0 \\ 6 & -6 & 7-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ -6 & 7-\lambda \end{vmatrix} = 3\begin{vmatrix} 2 & 0 \\ 6 & 7-\lambda \end{vmatrix} + (-2)\begin{vmatrix} 2 & 3-\lambda \\ 6 & -6 \end{vmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(7-\lambda)] - 3(2)(7-\lambda) - 2[(-12) - 6(3-\lambda)] = 0$$

$$(2-\lambda)(3-\lambda)(7-\lambda) - 6(7-\lambda) - 2(-12-18+6\lambda) = 0$$

$$(2-\lambda)(3-\lambda)(7-\lambda) - 6(7-\lambda) - 2\underbrace{(-30+6\lambda)}_{6(-5+\lambda)} = 0$$

$$(2-\lambda)(3-\lambda)(7-\lambda) - 6(7-\lambda) - 2(6)(-5+\lambda) = 0$$

$$(2-\lambda)(3-\lambda)(7-\lambda) - 6(7-\lambda) - 12(\lambda-5) = 0$$

$$(7-\lambda)[(2-\lambda)(3-\lambda) - 6] - 12(\lambda-5) = 0$$

$$(7-\lambda)[\cancel{6} - 2\lambda - 3\lambda + \lambda^2 - \cancel{6}] - 12(\lambda-5) = 0$$

$$(7-\lambda)(-5\lambda + \lambda^2) - 12(\lambda-5) = 0$$

$$(7-\lambda)\lambda(-5+\lambda) - 12(\lambda-5) = 0$$

$$(7-\lambda)\lambda(\lambda-5) - 12(\lambda-5) = 0$$

$$(\lambda-5)[(7-\lambda)\lambda - 12] = 0$$

$$(\lambda-5)(-\lambda^2 + 7\lambda - 12) = 0$$

$$-(\lambda-5)(\lambda^2 - 7\lambda + 12) = 0$$

$$-(\lambda-5)(\lambda-4)(\lambda-3) = 0$$

$$\boxed{\begin{array}{l} \lambda = 5 \\ \lambda = 4 \\ \lambda = 3 \end{array}}$$

$\lambda = 3, 4 \ \& \ 5$

$$\boxed{(A - \lambda I)x = 0}$$

$$\begin{pmatrix} 2-\lambda & 3 & -2 \\ 2 & 3-\lambda & 0 \\ 6 & -6 & 7-\lambda \end{pmatrix}$$

$\boxed{\lambda = 3}$

$$\begin{bmatrix} 2-3 & 3 & -2 \\ 2 & 3-3 & 0 \\ 6 & -6 & 7-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 3 & -2 \\ 2 & 0 & 0 \\ 6 & -6 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -x_1 + 3x_2 - 2x_3 \\ 2x_1 + 0 + 0 \\ 6x_1 - 6x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_2 - 2x_3 = 0$ — (1)

$2x_1 = 0 \Rightarrow x_1 = 0$ ✓

$-6x_2 + 4x_3 = 0$ — (2)

$3x_2 - 2x_3 = 0$ — (3)

$[A|B]$

$$\left[\begin{array}{ccc|c} -1 & 3 & -2 & 0 \\ 2 & 0 & 0 & 0 \\ 6 & -6 & 4 & 0 \end{array}\right]$$

$x_1 = 0$

$3x_2 - 2x_3 = 0$

$3x_2 = 2x_3$

$x_2 = \frac{2}{3}x_3$

if $x_3 = 3$, $x_2 = \frac{2}{\cancel{3}}(\cancel{3}) = 2$

$x_3 = 6$, $x_2 = 4$

$$\therefore \gamma \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \begin{matrix} 0 & 0 \\ 4 & 6 \\ 6 & 9 \end{matrix}$$

$\gamma = 1$ $(0, 2, 3)$

$\gamma = 2$ $(0, 4, 6)$

$\underline{\underline{\lambda=4}}$

$$\begin{vmatrix} 2-\lambda & 3 & -2 \\ 2 & 3-\lambda & 0 \\ 6 & -6 & 7-\lambda \end{vmatrix} = 0$$

$$x \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 & -2 \\ 2 & -1 & 0 \\ 6 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$-2x_1 + 3x_2 - 2x_3 = 0$ —— (1)

$2x_1 - x_2 = 0$ —— (2)

$6x_1 - 6x_2 + 3x_3 = 0$

$2x_1 - 2x_2 + x_3 = 0$ —— (3)

(2) $2x_1 = x_2$

if $x_1 = 1$, $x_2 = 2$ | $x_3 = -2x_1 + 2x_2 = -2 + 4$

$= 2$

$\lambda = 5$

$$\begin{bmatrix} -3 & 3 & -2 \\ 2 & -2 & 0 \\ 6 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \begin{bmatrix} 2-\lambda & 3 & -2 \\ 2 & 3-\lambda & 0 \\ 6 & -6 & 7-\lambda \end{bmatrix} \gamma$$

$-3x_1 + 3x_2 - 2x_3 = 0$

$2x_1 - 2x_2 = 0 \Rightarrow \underline{x_1 = x_2}$ if $x_1 = 1, x_2 = 1, x_3 = 0$

$6x_1 - 6x_2 + 2x_3 = 0$

$3x_1 - 3x_2 + x_3 = 0 \Rightarrow \cancel{3} - \cancel{3} + x_3 = 0$

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\boxed{(A - \lambda I)x = 0}$$

$$\begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0 \xrightarrow{R_1 - R_2} \begin{vmatrix} 1-\lambda & -1+\lambda & 0 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} \xrightarrow{C_1 + C_2} \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 9-\lambda & 2 \\ 2 & 4 & 2-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 9-\lambda & 2 \\ 2 & 4 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\begin{vmatrix} 9-\lambda & 2 \\ 4 & 2-\lambda \end{vmatrix} - 0 + 0 = 0$$

$$(1-\lambda)\left[(9-\lambda)(2-\lambda) - 8\right] = 0$$

$$(1-\lambda)\left[18 + \lambda^2 - 2\lambda - 9\lambda - 8\right] = 0$$

$$(1-\lambda)(\lambda^2 - 11\lambda + 10) = 0$$

$$(1-\lambda)(\lambda - 1)(\lambda - 10) = 0$$

$$\lambda = 1, 1, 10$$

$\lambda = 1$ (repeated)

$$\begin{bmatrix} 5-1 & 4 & 2 \\ 4 & 5-1 & 2 \\ 2 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right]$$

$\cancel{R_1 \leftrightarrow R_2 \to R_1}$, $R_1 - 2R_3 \to R_1$

$R_2 - 2R_3 \to R_2$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \frac{R_3}{2}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$$\boxed{x_1 + x_2 + \frac{1}{2} x_3 = 0}$$

$$x_1 + x_2 = -\frac{1}{2} x_3$$

if $x_3 = 2$, $x_1 + x_2 = -1$

if $x_1 = -1$, $x_2 = 0$ $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$\lambda = 10$

$$\begin{bmatrix} 5-10 & 4 & 2 \\ 4 & 5-10 & 2 \\ 2 & 2 & 2-10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array}\right]$$

$x_1 = 1$

$x_2 = 1$

$x_3 = \frac{1}{2}$

$$\begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix}$$

✓

4

?

Assignment 5

Determine the eigenvalues and corresponding eigenspaces of the given matrices.

1. $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

7. $\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 5 & -2 \\ -2 & 4 & -1 \end{bmatrix}$

10. $\begin{bmatrix} 15 & 7 & -7 \\ -1 & 1 & 1 \\ 13 & 7 & -5 \end{bmatrix}$