Basic Mathematics and Statistics

CHAPTER 7: CONIC SECTION

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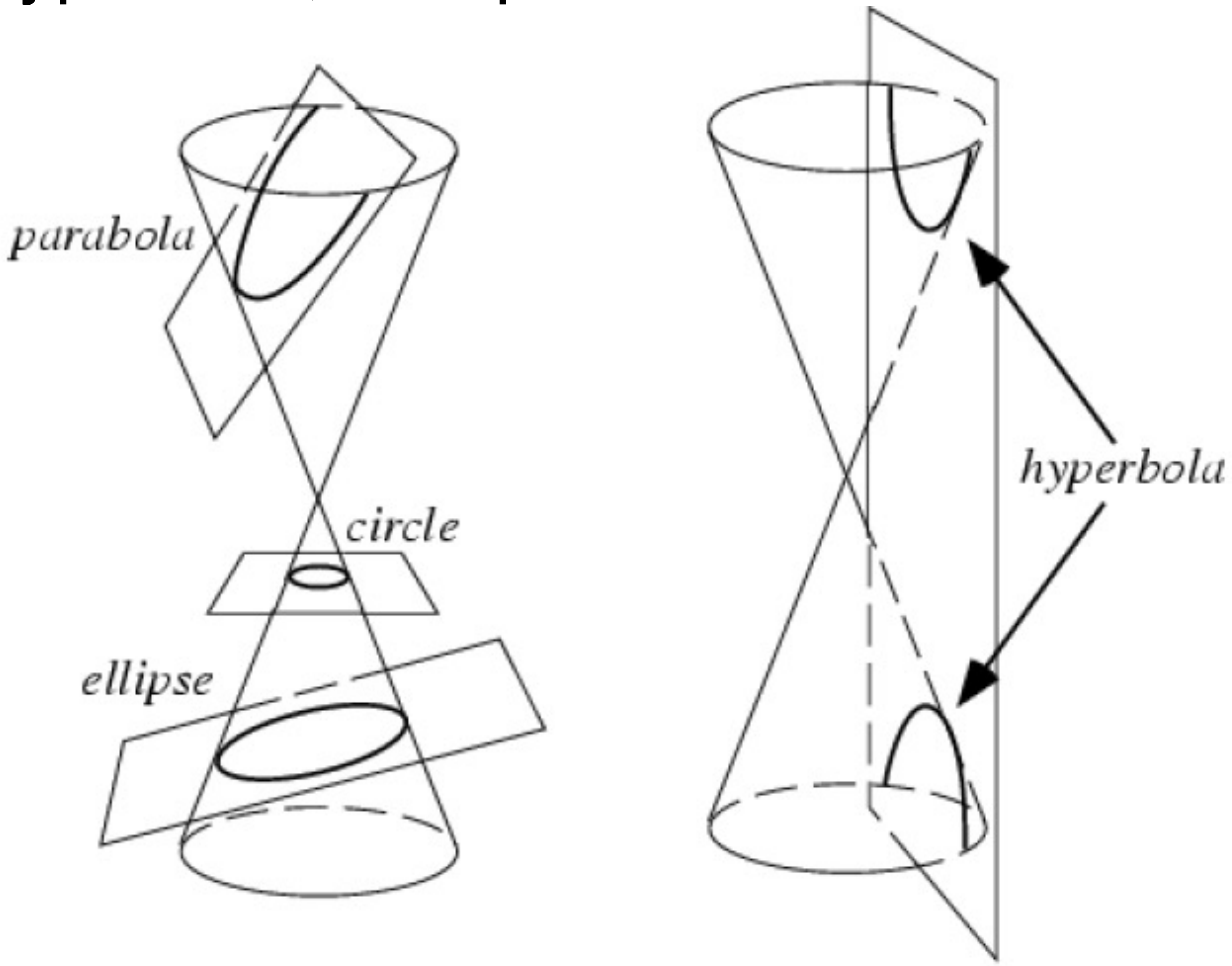
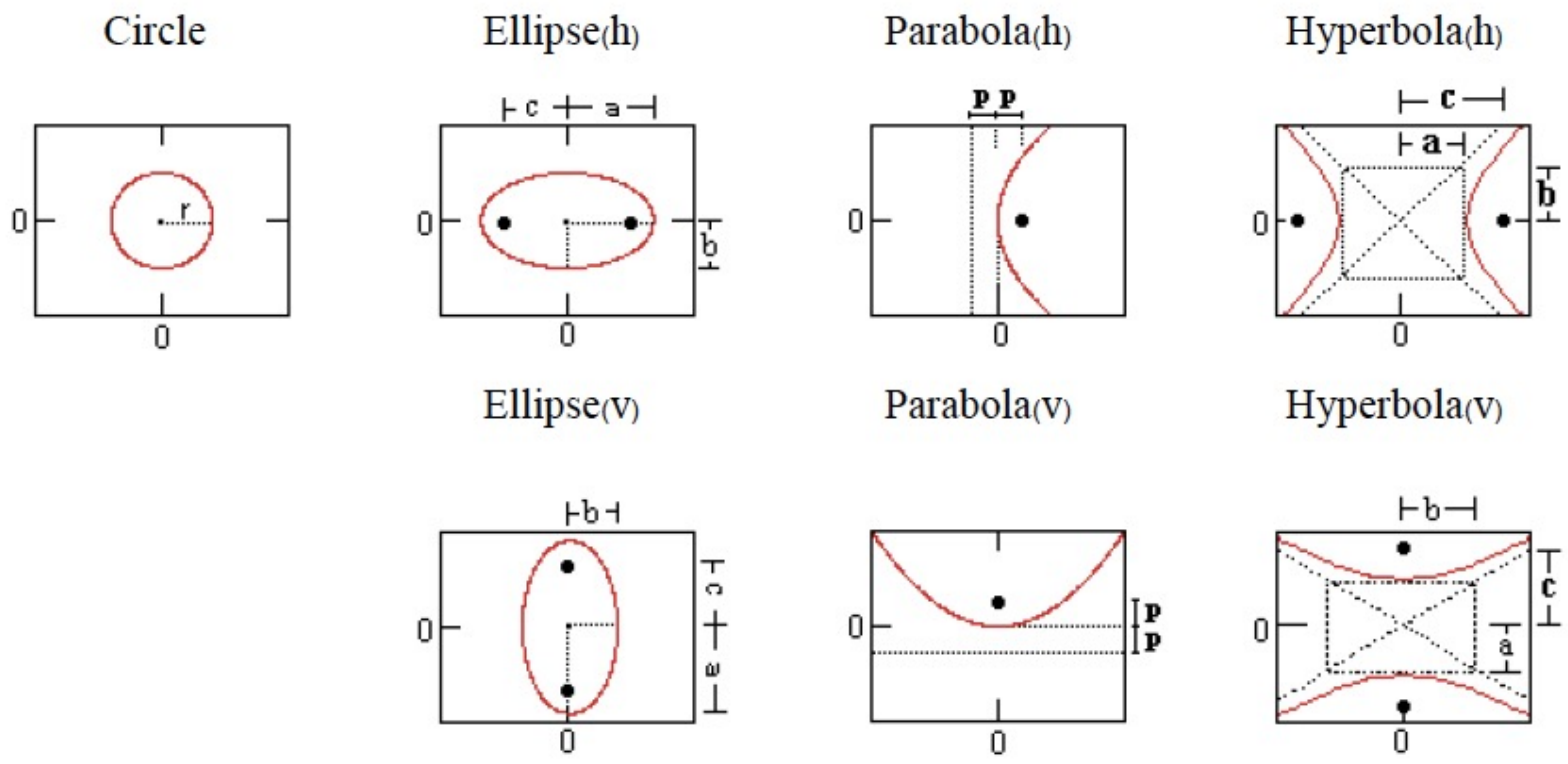
Conics

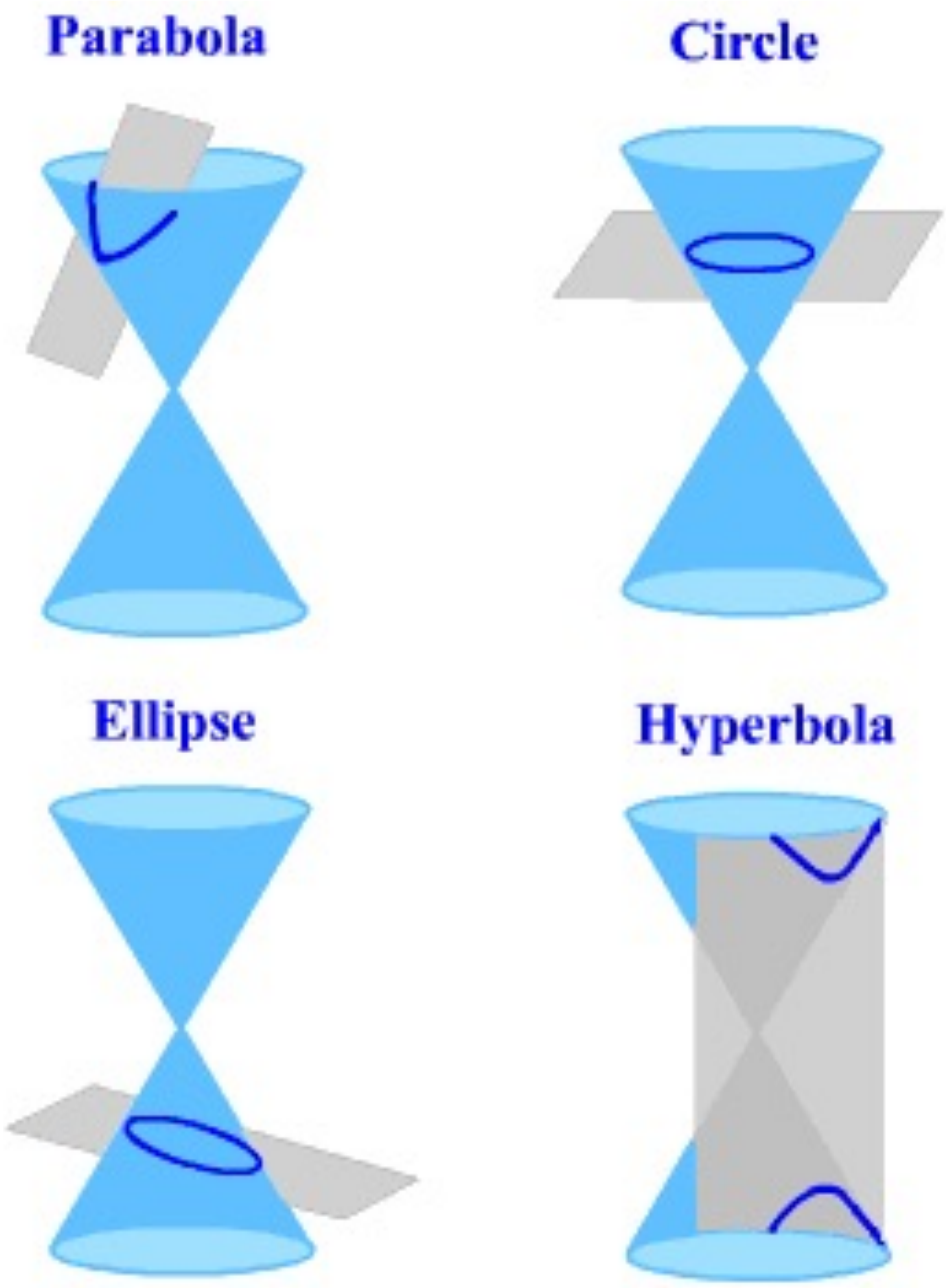
Conics, an abbreviation for **conic sections,** are curves that result from the intersection of a cone and a plane.

Depending on the angle of the plane relative to the cone, the intersection is a circle, an ellipse, a hyperbola, or a parabola





Conics



Conic sections are the cross-sections of a cone.

Eccentricity – a mount a conic section deviates from being perfectly circular.

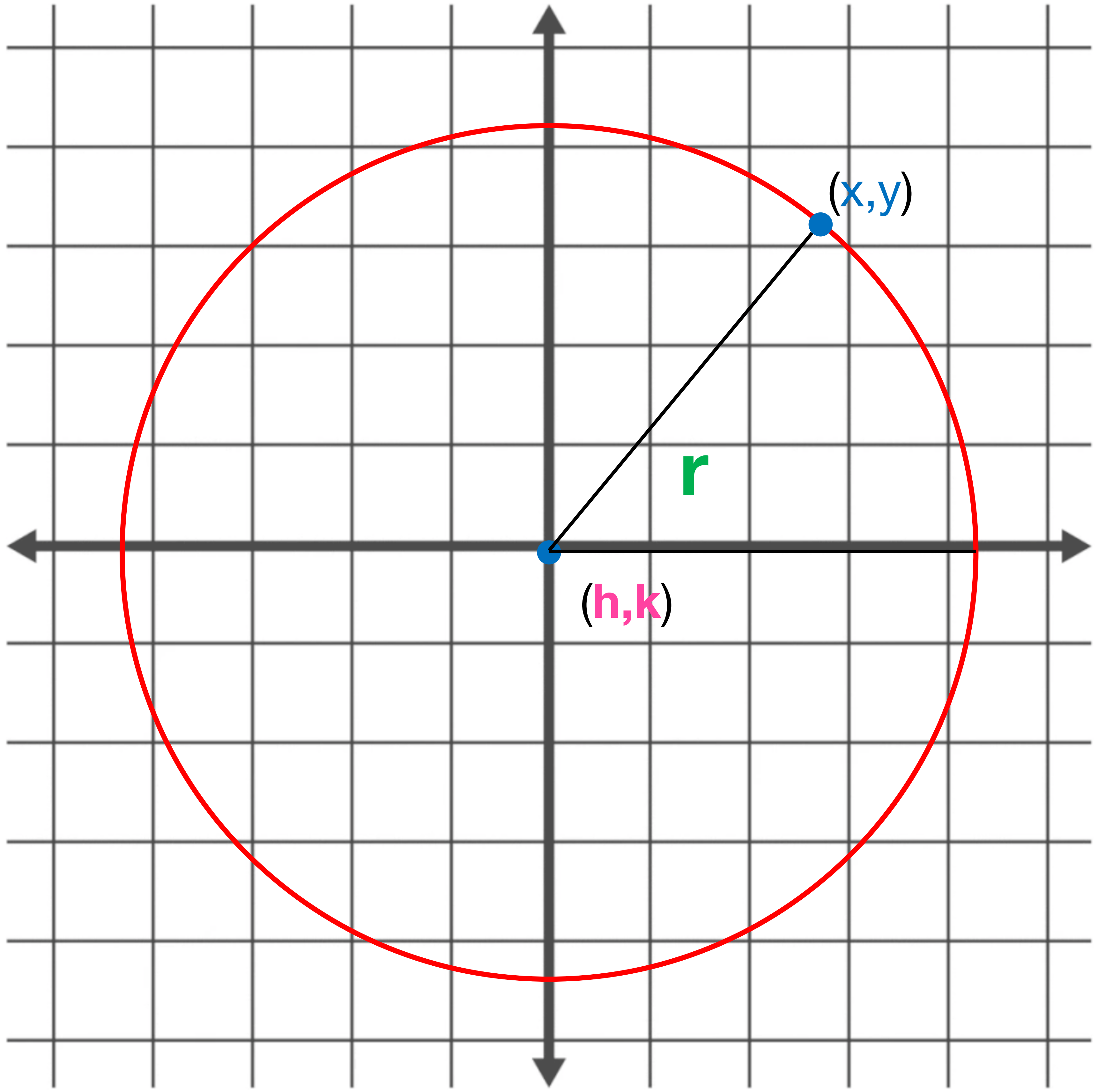
Circle: $e = 0$

Ellipse: $0 < e < 1$

Parabola: $e = 1$

Hyperbola: $e > 1$

7.1 CIRCLE



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The Distance Formula

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

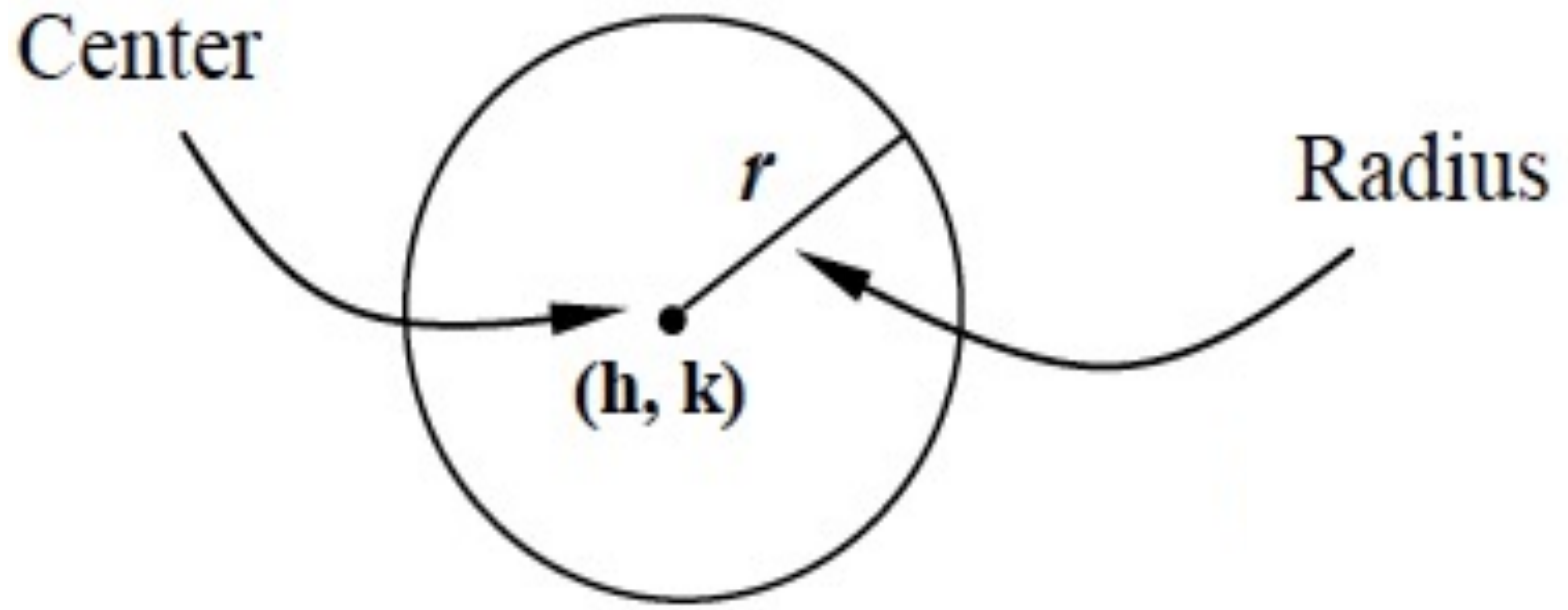
$$r = \sqrt{(x - h)^2 + (y - k)^2}$$

$$r^2 = (x - h)^2 + (y - k)^2$$

7.1 CIRCLE

Definition

The **circle** is defined as the set of points in a plane that are equidistant from a fixed point.



A *fixed point* is called the **center (*h, k*)** of the circle, and the distance from the *center to a point on the circle* is called the **radius, r**.

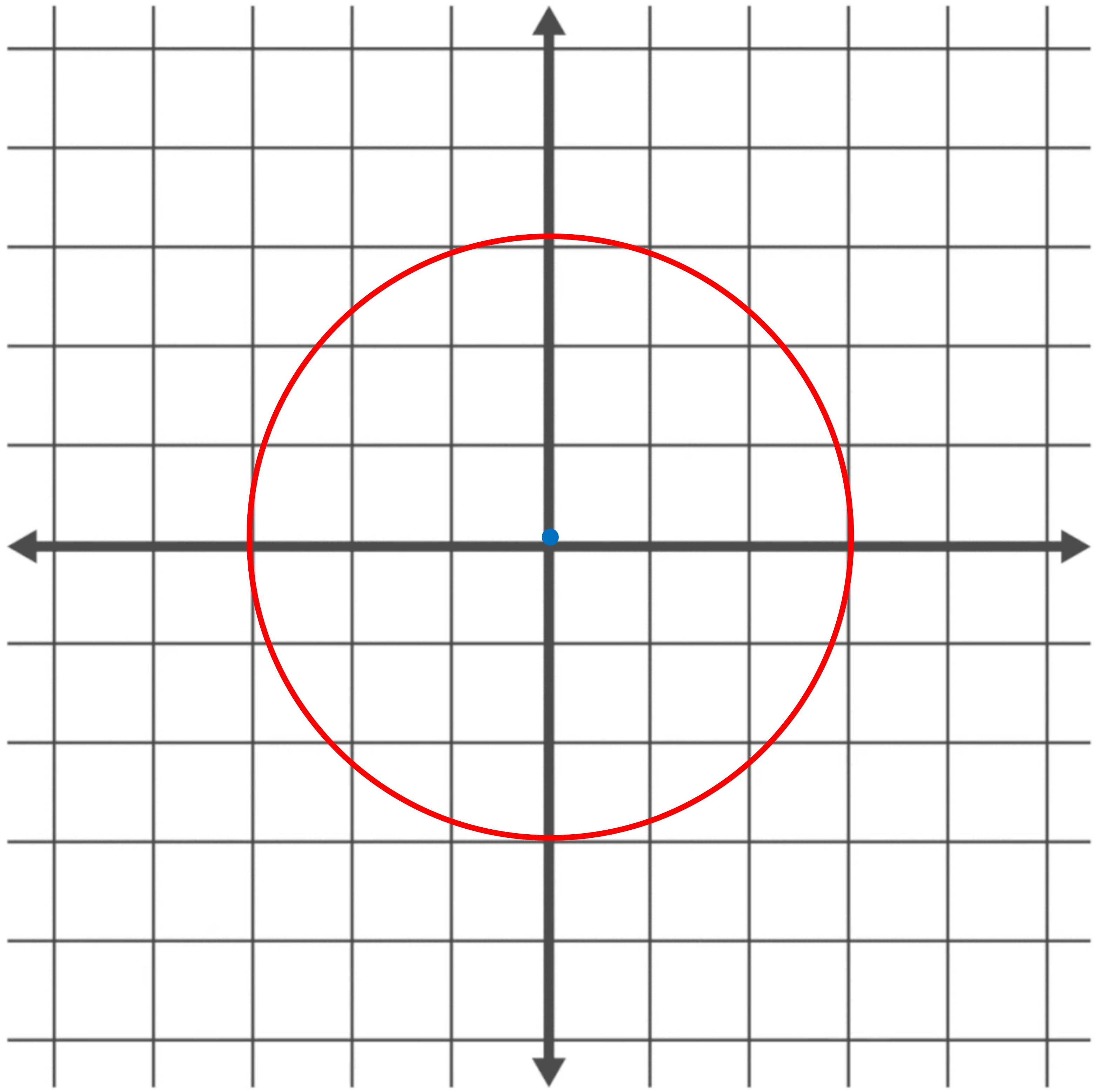
The Standard Form

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

The General Form

$$\boxed{x^2 + y^2 + Ax + By + C = 0}$$

7.1 CIRCLE



$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

$$x^2+y^2=9$$

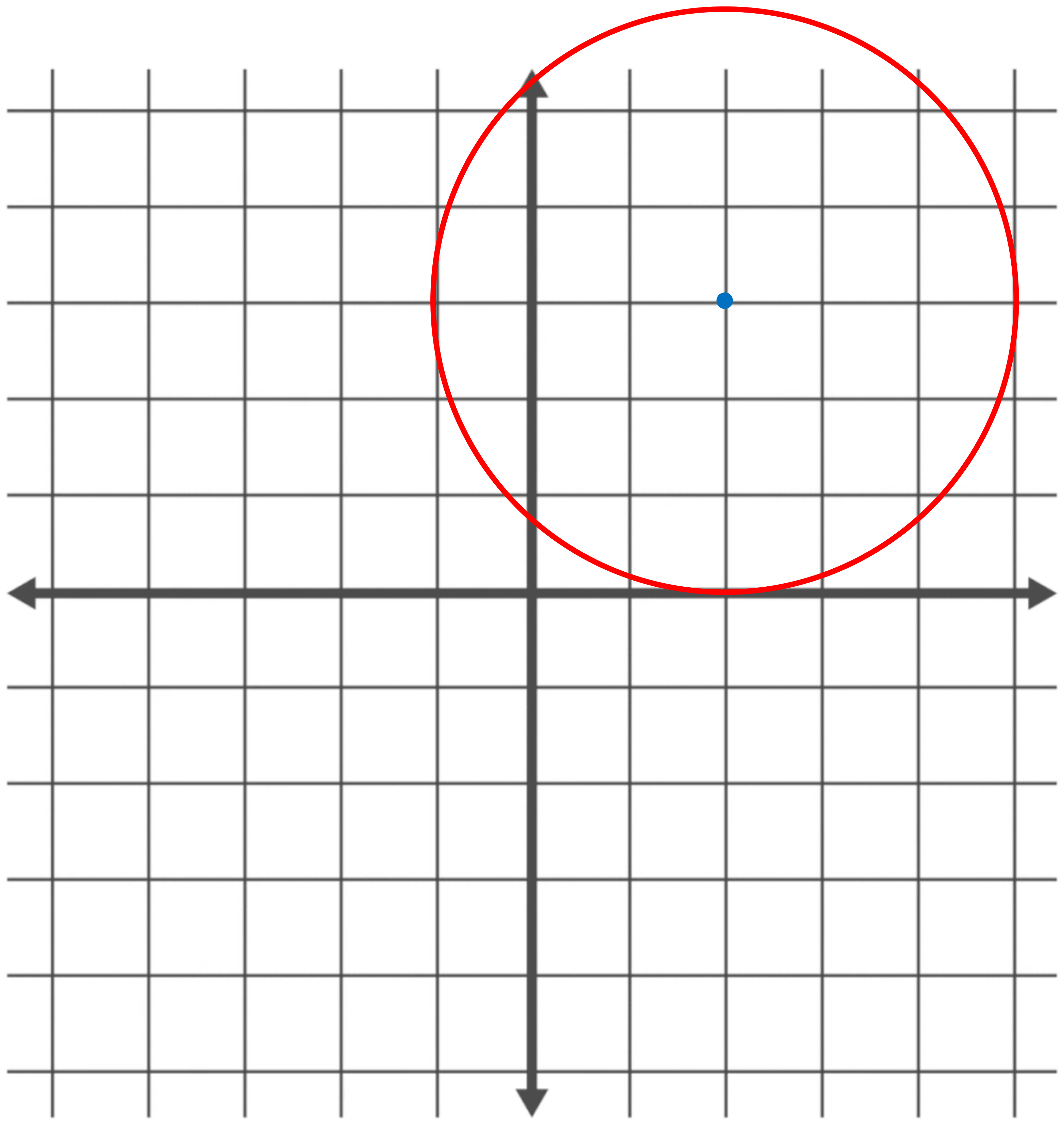
$$h=0 \qquad k=0$$

$$center: (0,0)$$

$$r^2=9$$

$$\mathrm{r}=3$$

7.1 CIRCLE



$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

$$(x-2)^2+(y-3)^2=9$$

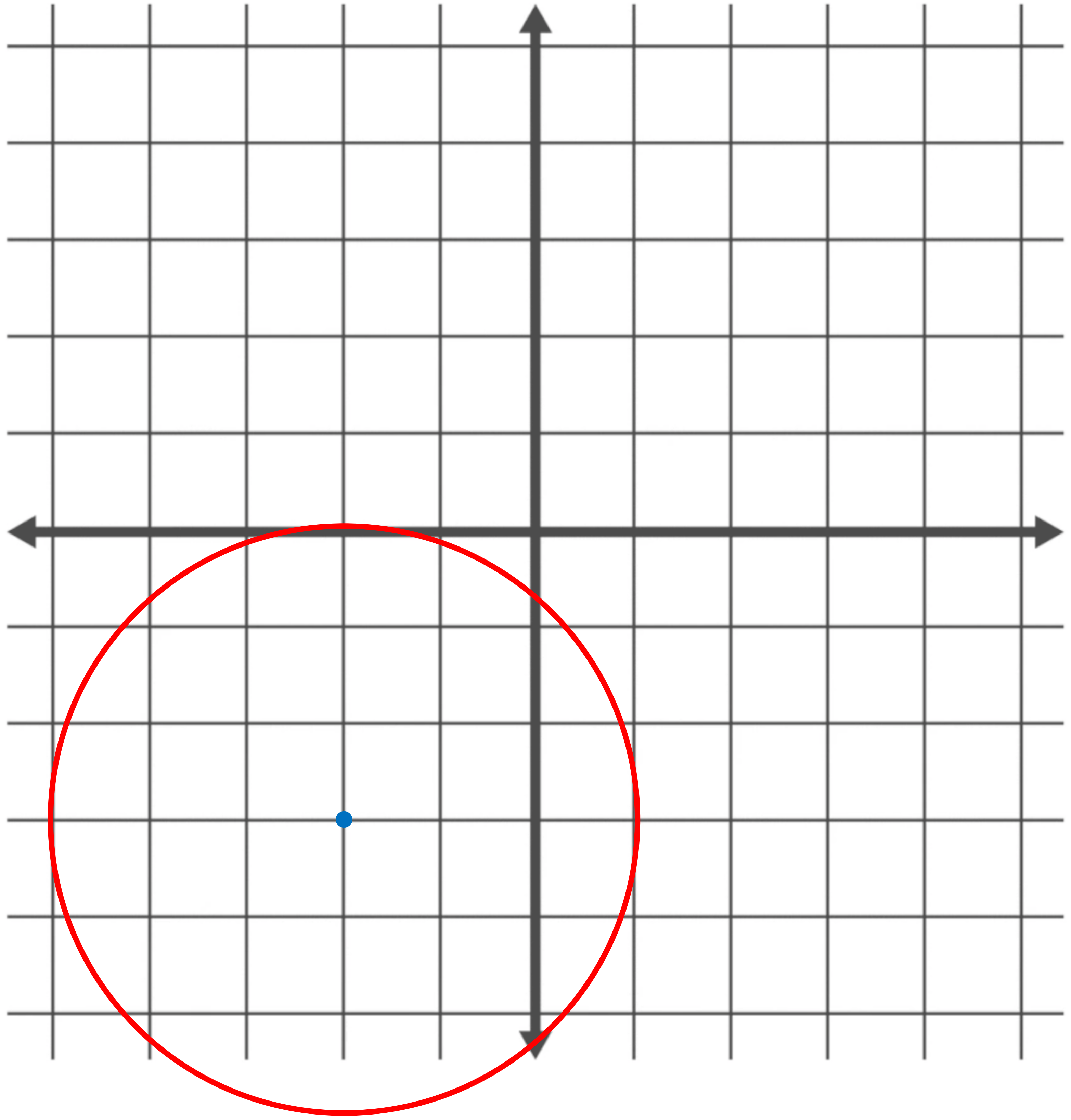
$$h=2 \qquad k=3$$

$$center: (2,3)$$

$$r^2=9$$

$$\mathrm{r}=3$$

7.1 CIRCLE



$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

$$(x+2)^2+(y+3)^2=9$$

$$[x-(-2)]^2+[y-(-3)]^2=9$$

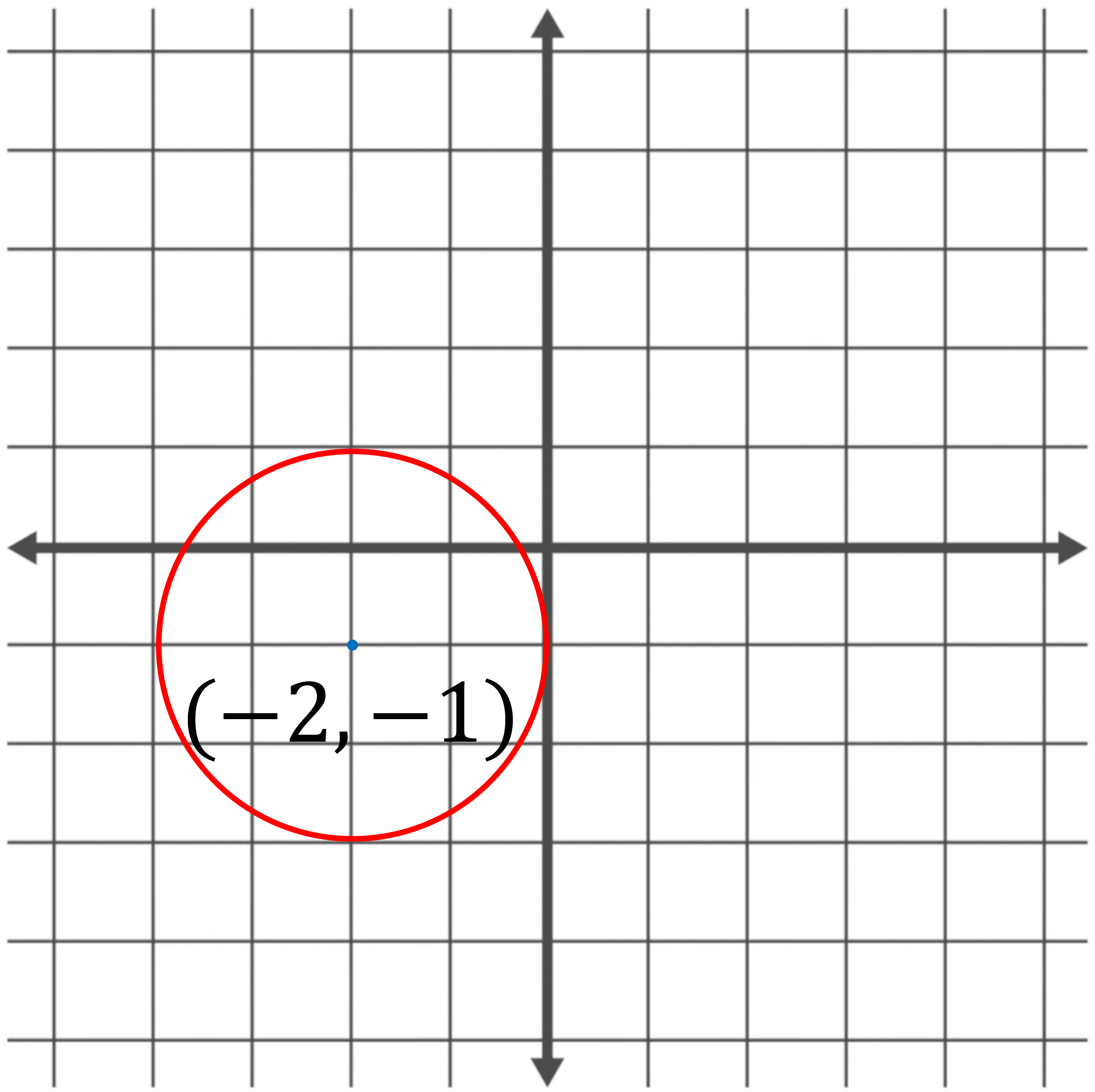
$$h=-2 \qquad k=-3$$

$$center: (-2,-3)$$

$$r^2=9$$

$$\mathrm{r}=3$$

7.1 CIRCLE



$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

$$center: (-2, -1)$$

$$h=-2 \qquad k=-1$$

$$\mathrm{r}=2$$

$$[x-(-2)]^2+[y-(-1)]^2=r^2$$

$$(x+2)^2+(y+1)^2=r^2$$

$$(x+2)^2+(y+1)^2=2^2$$

$$\mathbf{(x+2)^2+(y+1)^2=4}$$

7.1 CIRCLE

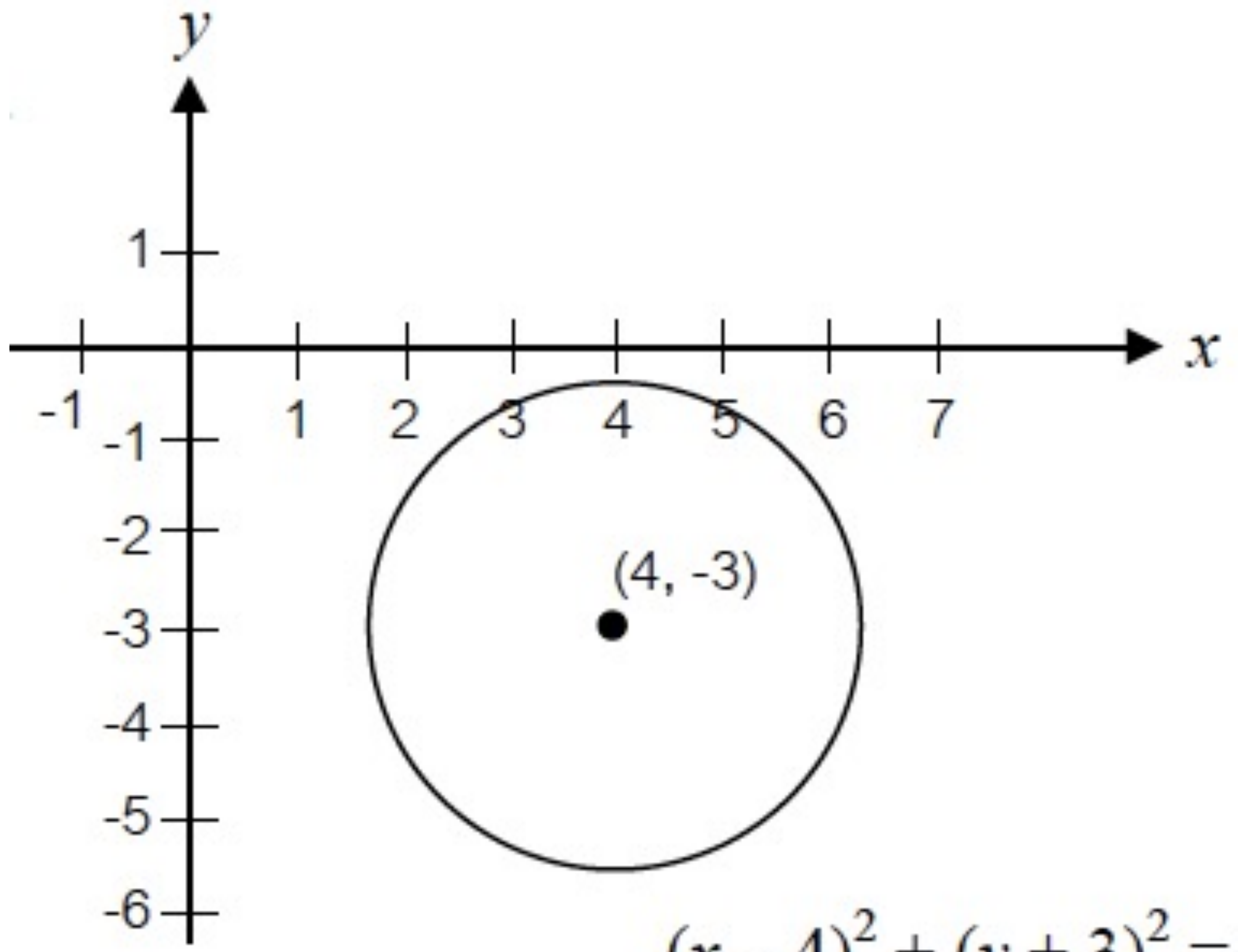
Determine the center, radius, and graph of the circle represented by the equation :

$$\underset{h}{(x \underline{-\ 4})^2} + \underset{k}{(y\ \underline{+\ 3})^2} = \underset{r^2}{\underline{\underline{5}}}$$

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

Standard Form

- center (h, k) = (4, -3)
- radius = $\sqrt{5} \approx 2.2$



$$(x-4)^2 + (y+3)^2 = 5$$

7.1 CIRCLE

Determine the center, radius, and graph for the circle

$$x^2 + y^2 - 2x - 6y + 6 = 0.$$

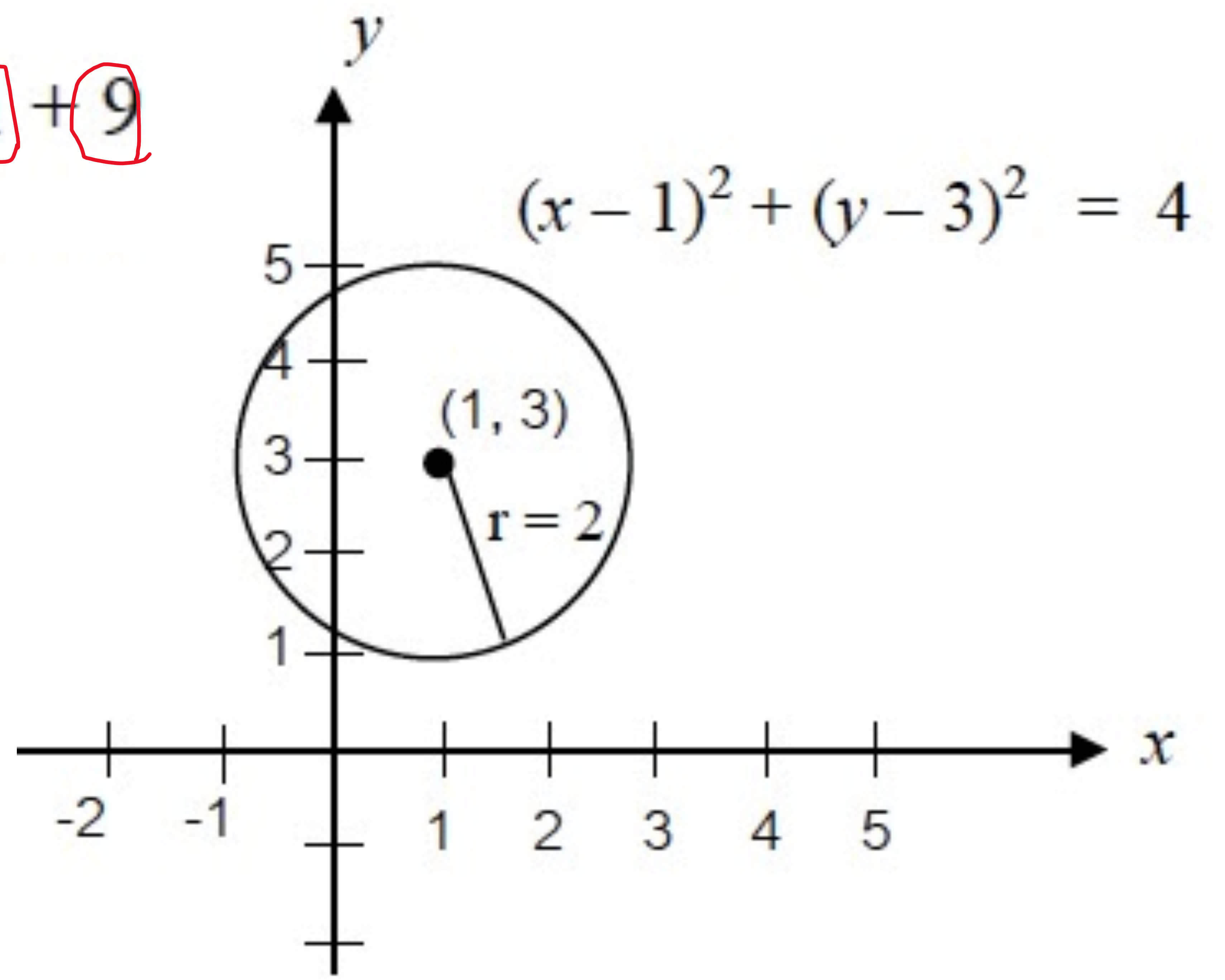
General Form

$$x^2 - 2x + y^2 - 6y = -6$$

$$(x^2 - 2x + 1) + (y^2 - 6y + 9) = -6 + 1 + 9$$

Standard Form

$$(x-1)^2 + (y-3)^2 = 4$$



- center (h, k) = (1, 3)
- radius = $\sqrt{4} = 2$

7.1 CIRCLE

Determine the equation of a circle whose center is at (-3, 1) and radius is 6, in standard form.

h k

r

$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

$$(x-h)^2+(y-k)^2=r^2$$

$$h=-3 \qquad k=1 \qquad r=6$$

$$(x+3)^2+(y-1)^2=36$$

7.1 CIRCLE

Determine the equation of a circle having center (3, -1) and passing through (-1, 2) in standard form.

(x, y)

$$(x - 3)^2 + (y + 1)^2 = r^2$$

The circle passes through (-1, 2), we have a point (x, y) is (-1, 2). Substitute x = -1 and y = 2 into equation

$$(-1 - 3)^2 + (2 + 1)^2 = r^2$$

$$16 + 9 = r^2$$

$$25 = r^2$$

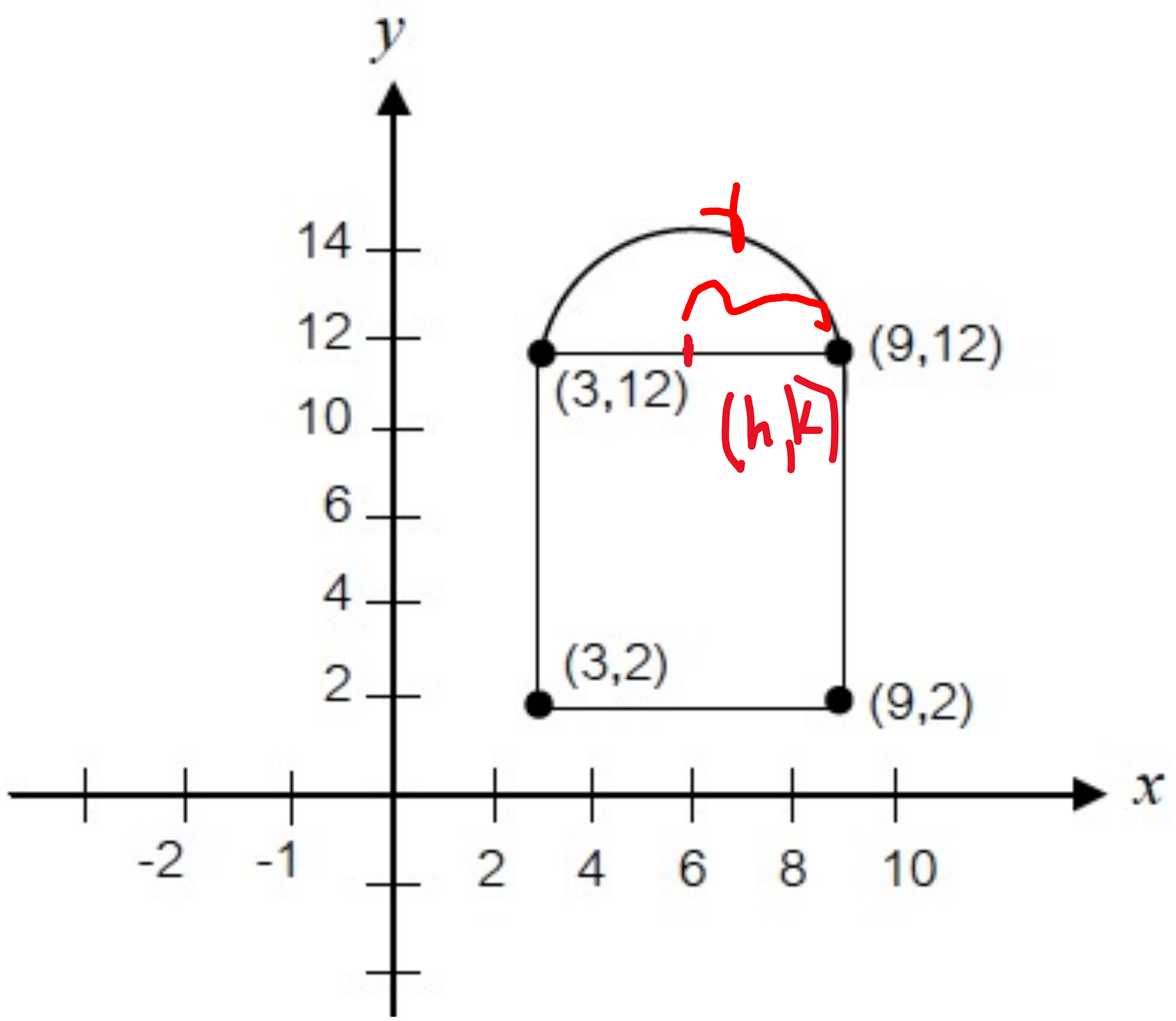
$$r = 5$$

$$(x - 3)^2 + (y + 1)^2 = 5^2$$

$$(x - 3)^2 + (y + 1)^2 = 25$$

7.1 CIRCLE

(Application) An archway is composed of a semicircle atop a rectangle. If the rectangular portion is drawn to scale on the rectangular coordinate system, the four vertices of the rectangle are (3, 2), (9, 2), (3, 12) and (9, 12). Find the equation of the circle containing the semicircle top. Use standard form for your answer.



h, $x_{mid} = \frac{3+9}{2} = \frac{12}{2} = 6$

k, $y_{mid} = \frac{12+12}{2} = \frac{24}{2} = 12$

center (h, k) = (6, 12)

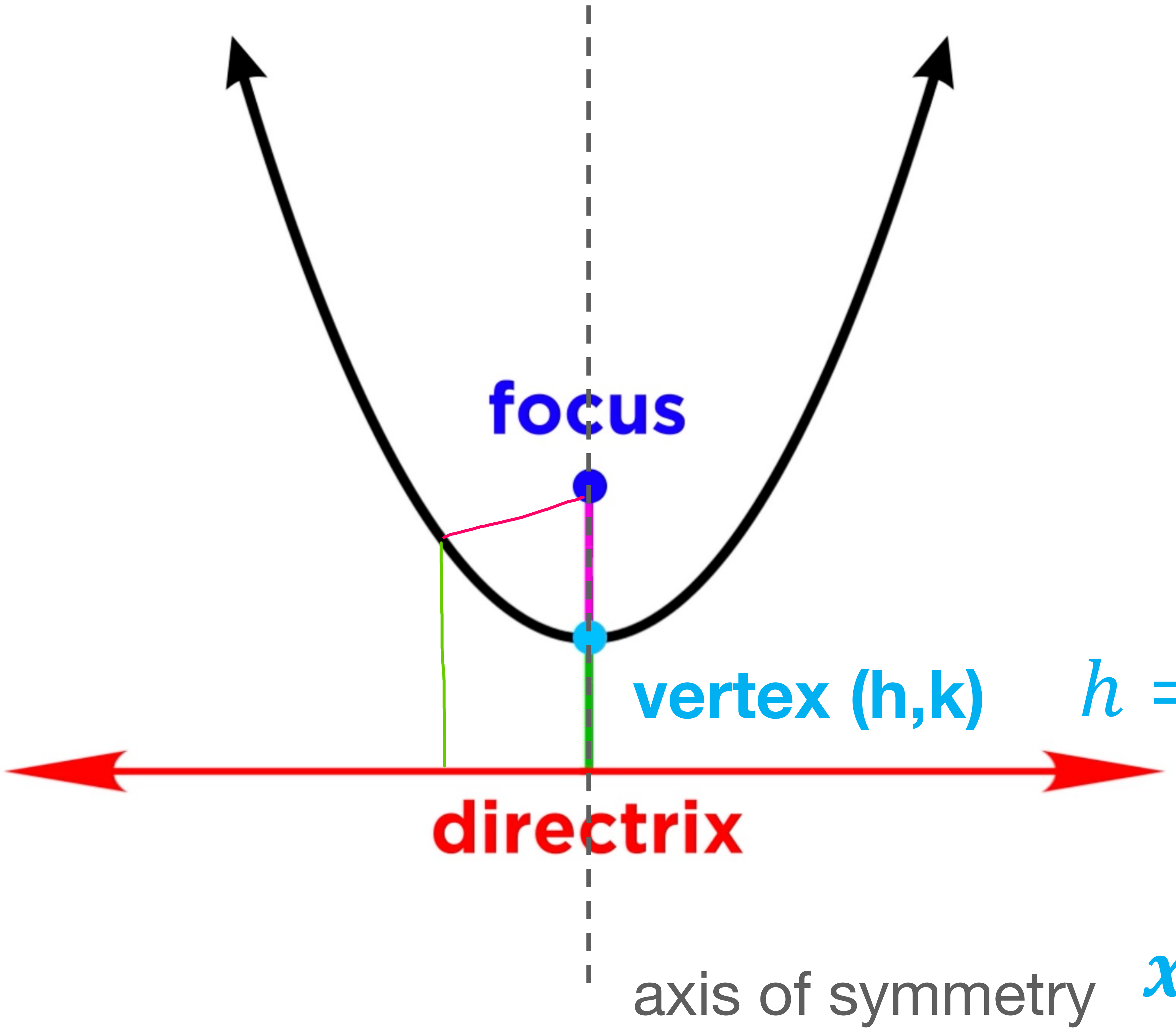
$$r = \frac{9-3}{2} = \frac{6}{2} = 3$$

$$(x-6)^2 + (y-12)^2 = 9$$

7.2 PARABOLA

$$f(x) = ax^2 + bx + c$$

distance to focus = distance to directrix



$$h = -\frac{b}{2a}, k = f(h)$$

$$x = -\frac{b}{2a}$$

7.2 PARABOLA

$$f(x) = -x^2 - 2x + 1$$

Strategy 1:

Convert equation into vertex form

we must simply complete the square

Vertex form:

$$f(x) = a(x - h)^2 + k$$

$$h = \frac{-b}{2a}, \qquad k = c - \frac{b^2}{4a}$$

$$f(x) = ax^2 + bx + c$$

$$f(x) = a(x^2 + \frac{b}{a}x) + c$$

half: $\frac{b}{2a}$, square: $\frac{b^2}{4a^2}$

$$f(x) = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + c - \frac{b^2}{4a}$$

$$f(x) = a(x - \underbrace{\frac{-b}{2a}}_{h})^2 + \underbrace{c - \frac{b^2}{4a}}_{k}$$

7.2 PARABOLA

$a = -1$ (parabola opens downwards)

$$f(x) = -x^2 - 2x + 1$$

Vertex form:

$$f(x) = a(x - \boldsymbol{h})^2 + \boldsymbol{k}$$

$$h = \frac{-b}{2a}$$

$$= -\frac{-2}{2(-1)} = -1$$

$$k = c - \frac{b^2}{4a}$$

$$= 1 - \frac{(-2)^2}{4(-1)} = 2$$

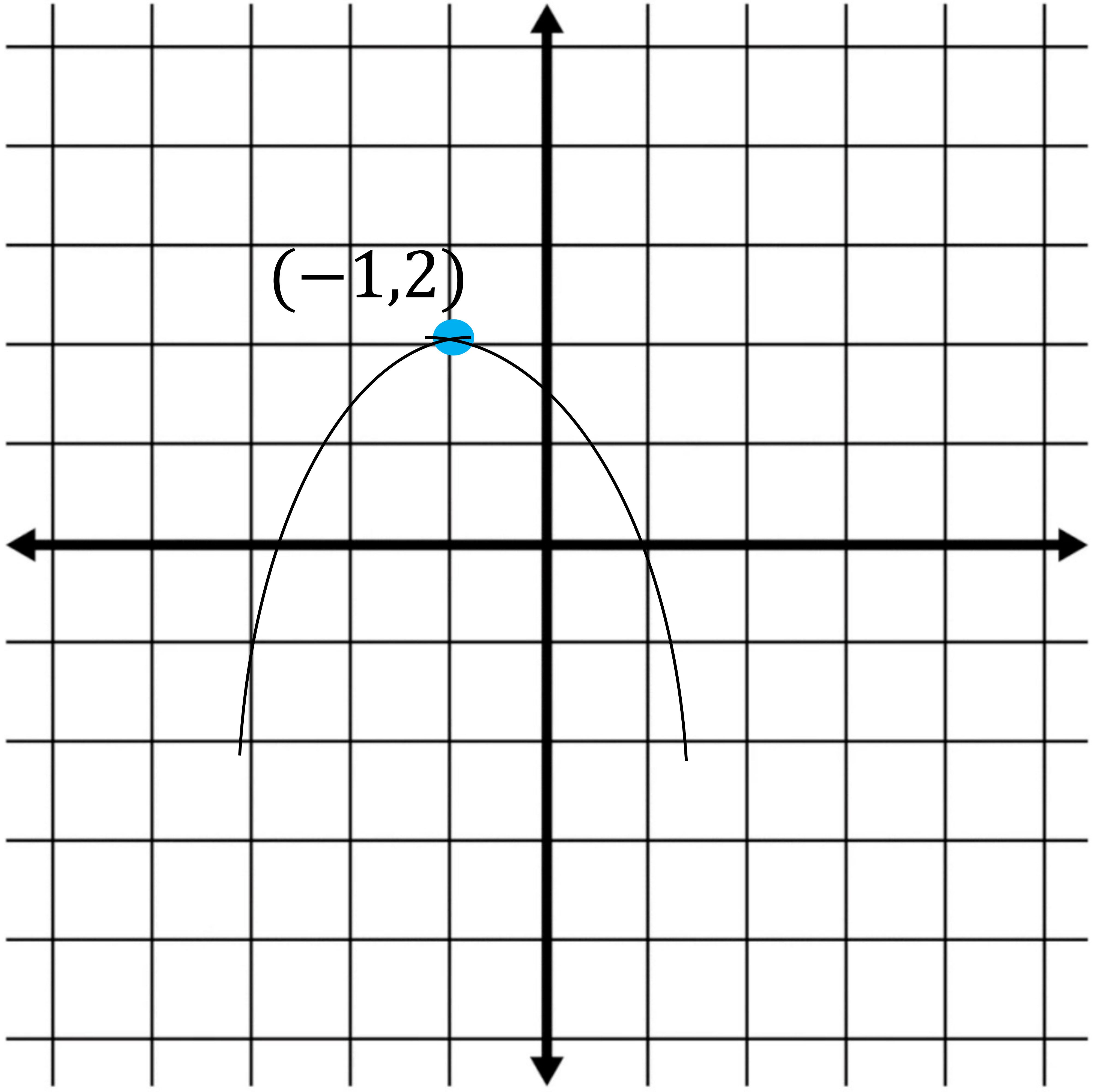
$$\boxed{f(x) = -x^2 - 2x + 1}$$

$$f(x) = -(x^2 + 2x) + 1$$

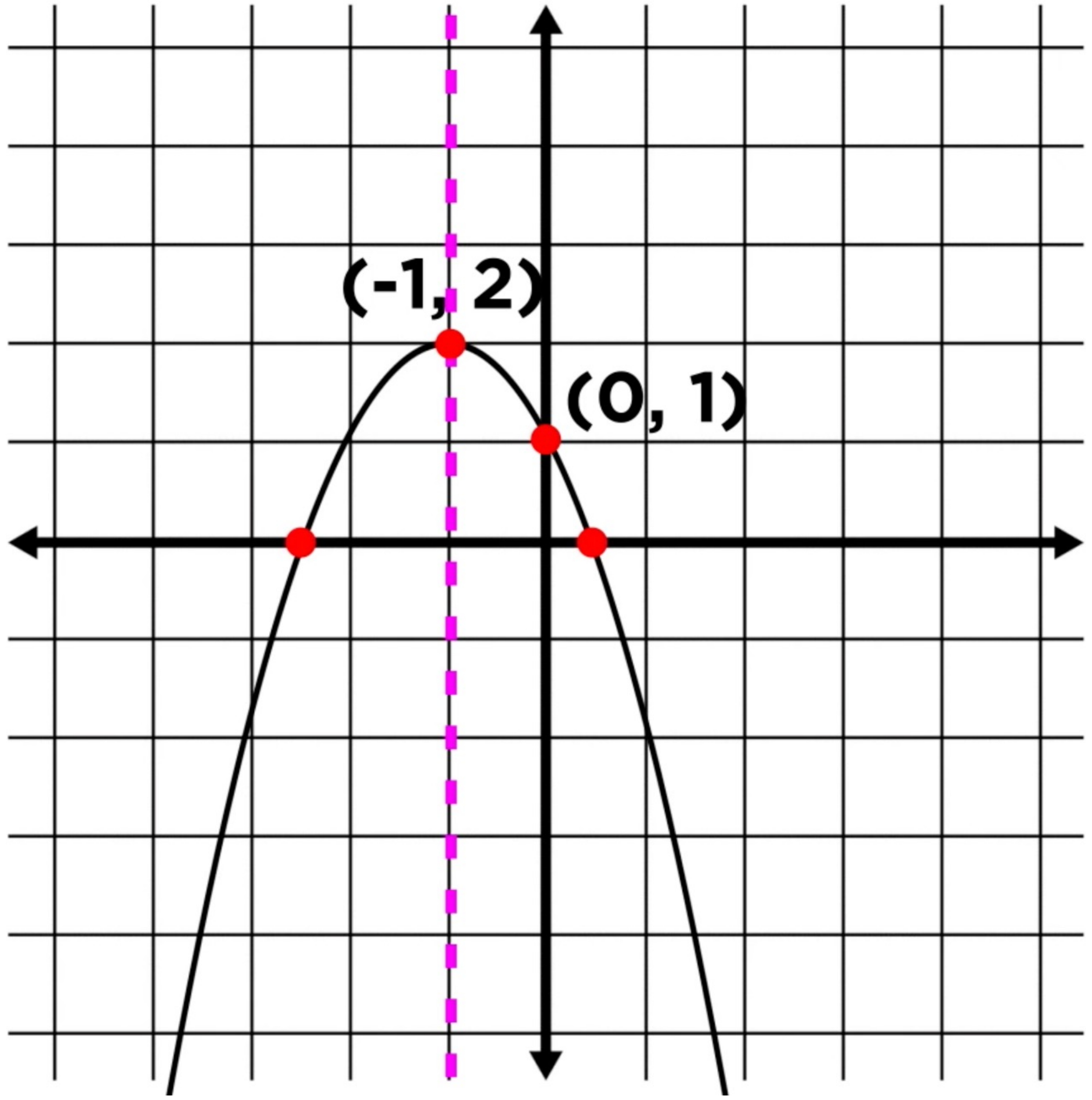
$$f(x) = -(x^2 + 2x + 1) + 2$$

$$f(x) = -(x + 1)^2 + 2$$

± h k



7.2 PARABOLA



$$f(x) = -x^2 - 2x + 1$$

Strategy 2:

Identify vertex and other key features

1. Identify the direction of the parabola opens

   positive: upwards

   negative: downwards $(a = -1)$

2. Find the vertex (h,k)

$$h = -\frac{b}{2a} = -\frac{-2}{2(-1)} = -1$$

$$k = f(h) = f(-1) = -(-1)^2 - 2(-1) + 1 = 2$$

3. Find x and y intercepts

   x-intercept: f=0 $-x^2 - 2x + 1 = 0$

   y-intercepts: x=0 $f(0) = -0^2 - 2(0) + 1 = 1$

4. axis of symmetry

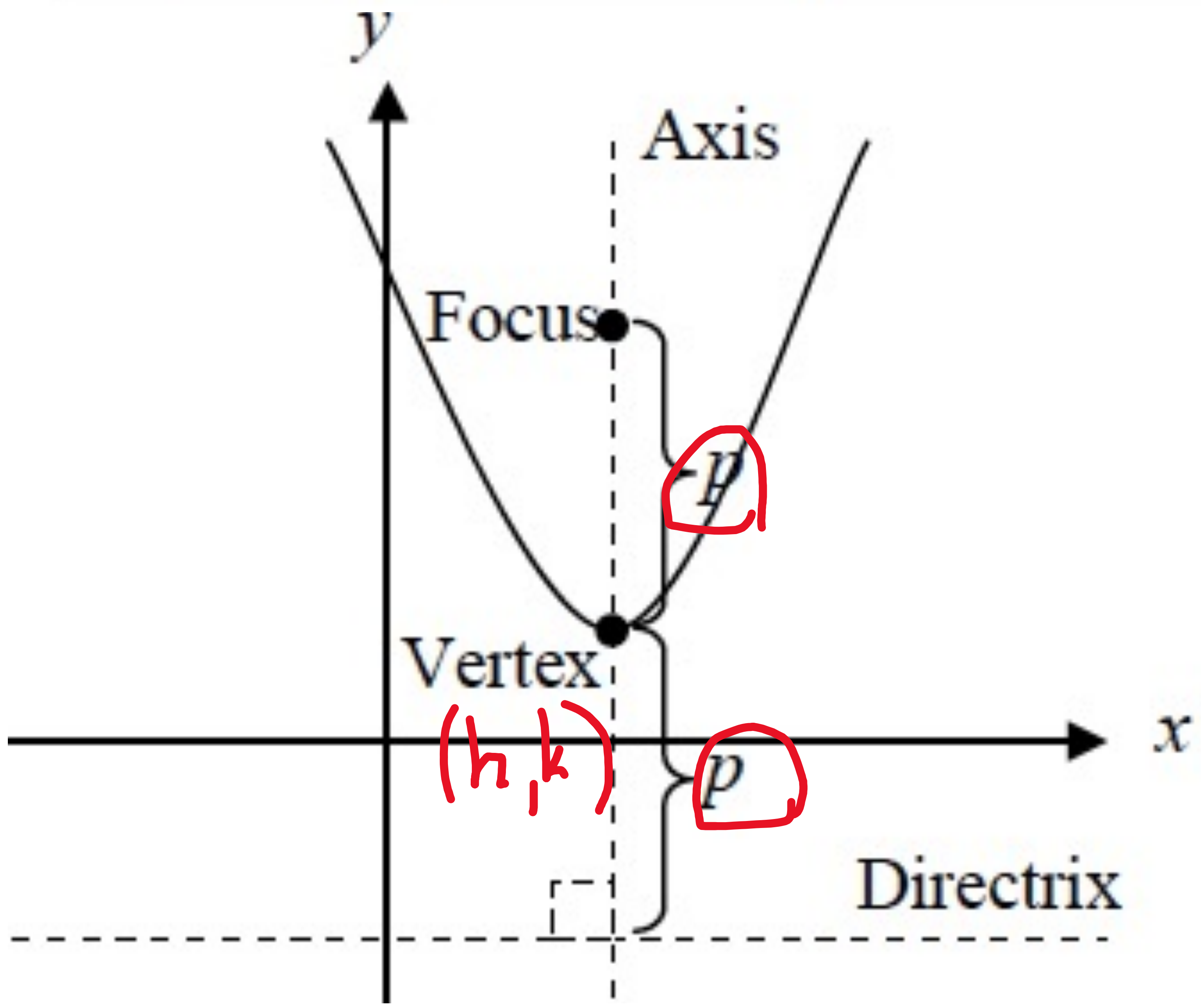
$$\boldsymbol{x} = -\frac{b}{2a} = -\frac{-2}{2(-1)} = -1$$

5. Sketch the function

7.2 PARABOLA

Definition

The **parabola** is defined as the set of points in a plane that are equidistant from a given point and a given line.



The Standard Form

$$\boxed{(x-h)^2 = 4p(y-k)}$$

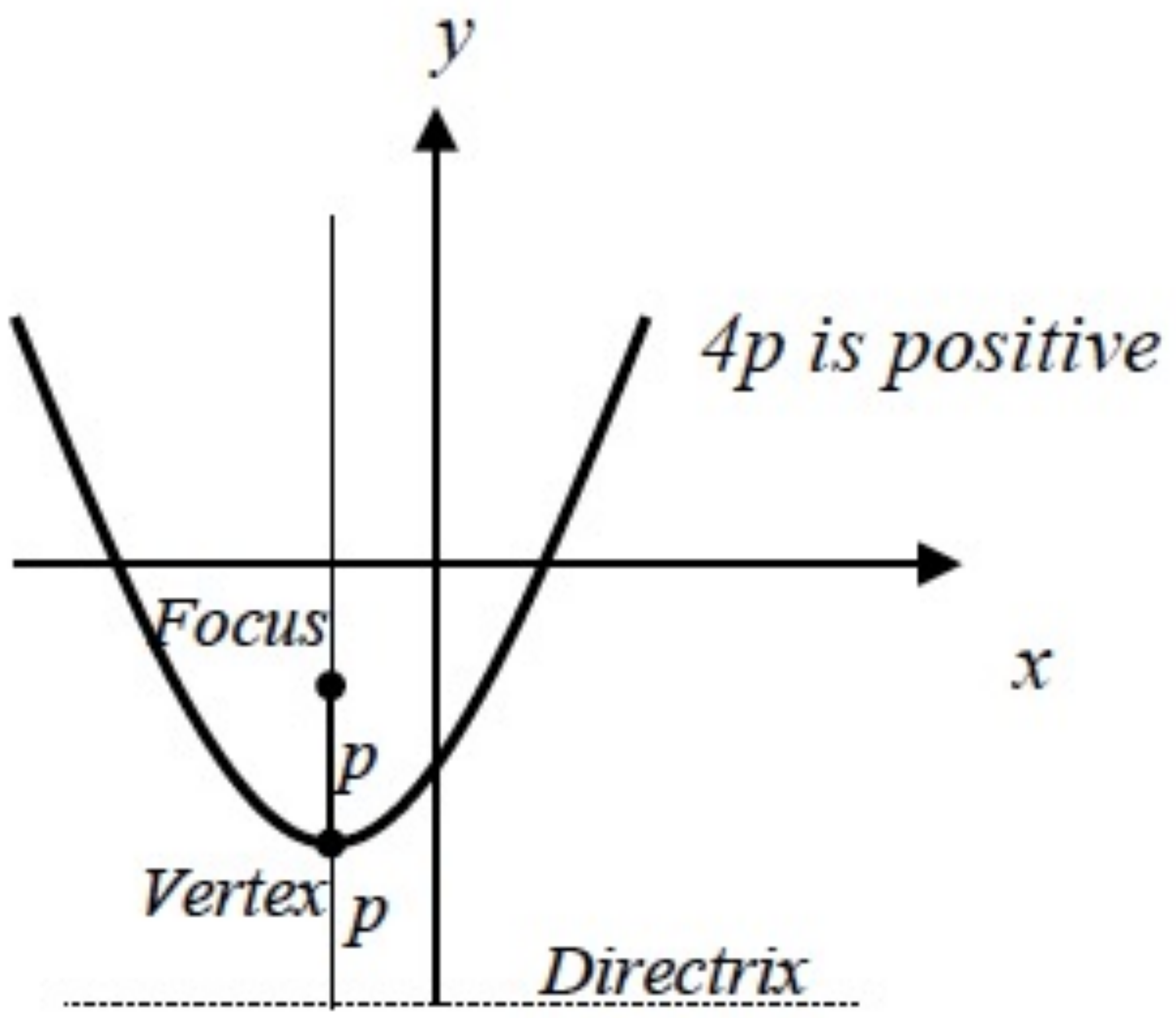
where (h, k) represents the coordinate of the vertex, and p represents the directed distance along the parabola's axis from the vertex to the focus or directrix

7.2 PARABOLA

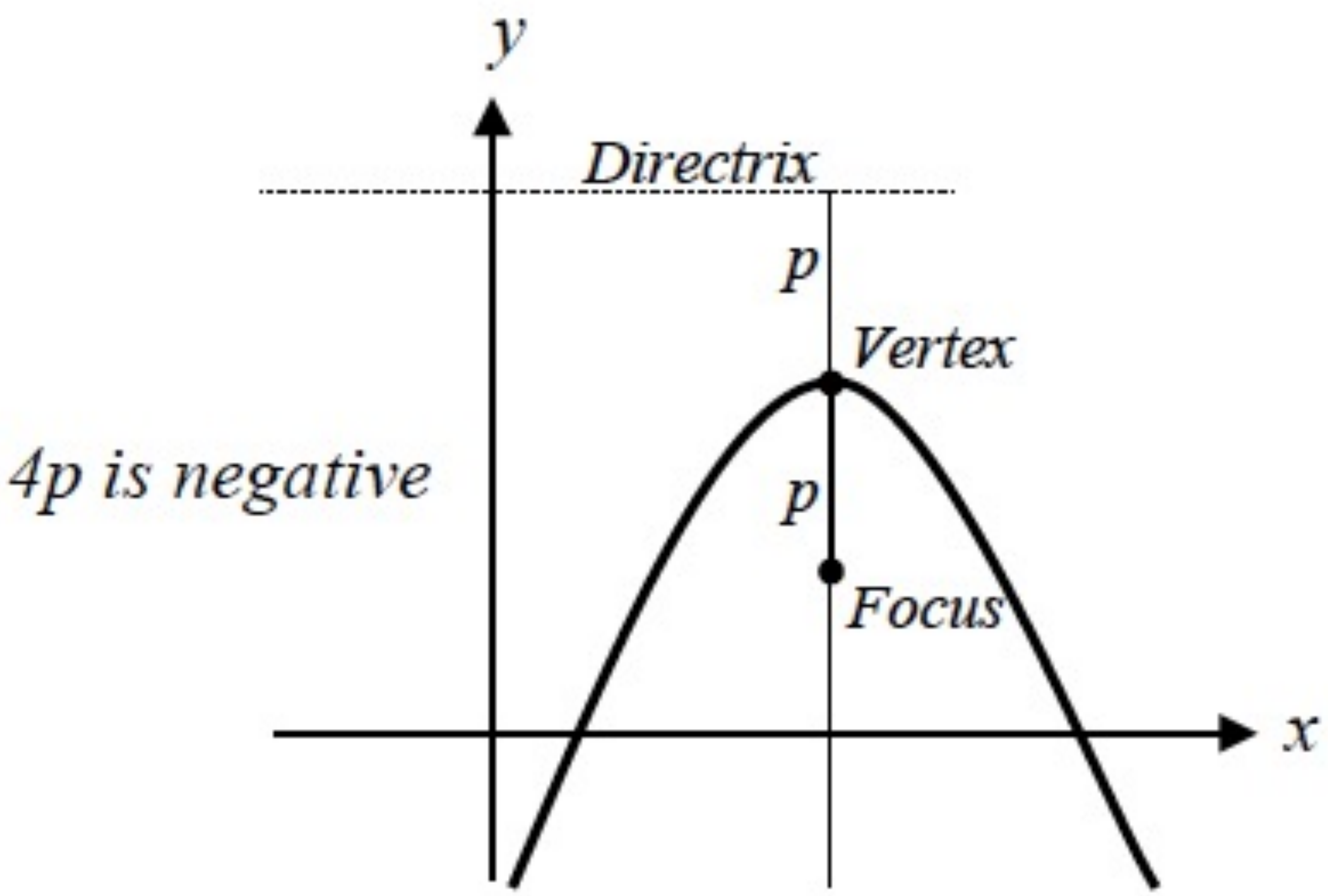
axis of parabola is a vertical line

$$\boxed{(x-h)^2 = 4p(y-k)}$$

Case 1 :
p is positive number,
the parabola opens upward.



Case 2 :
p is negative number,
the parabola opens downward.

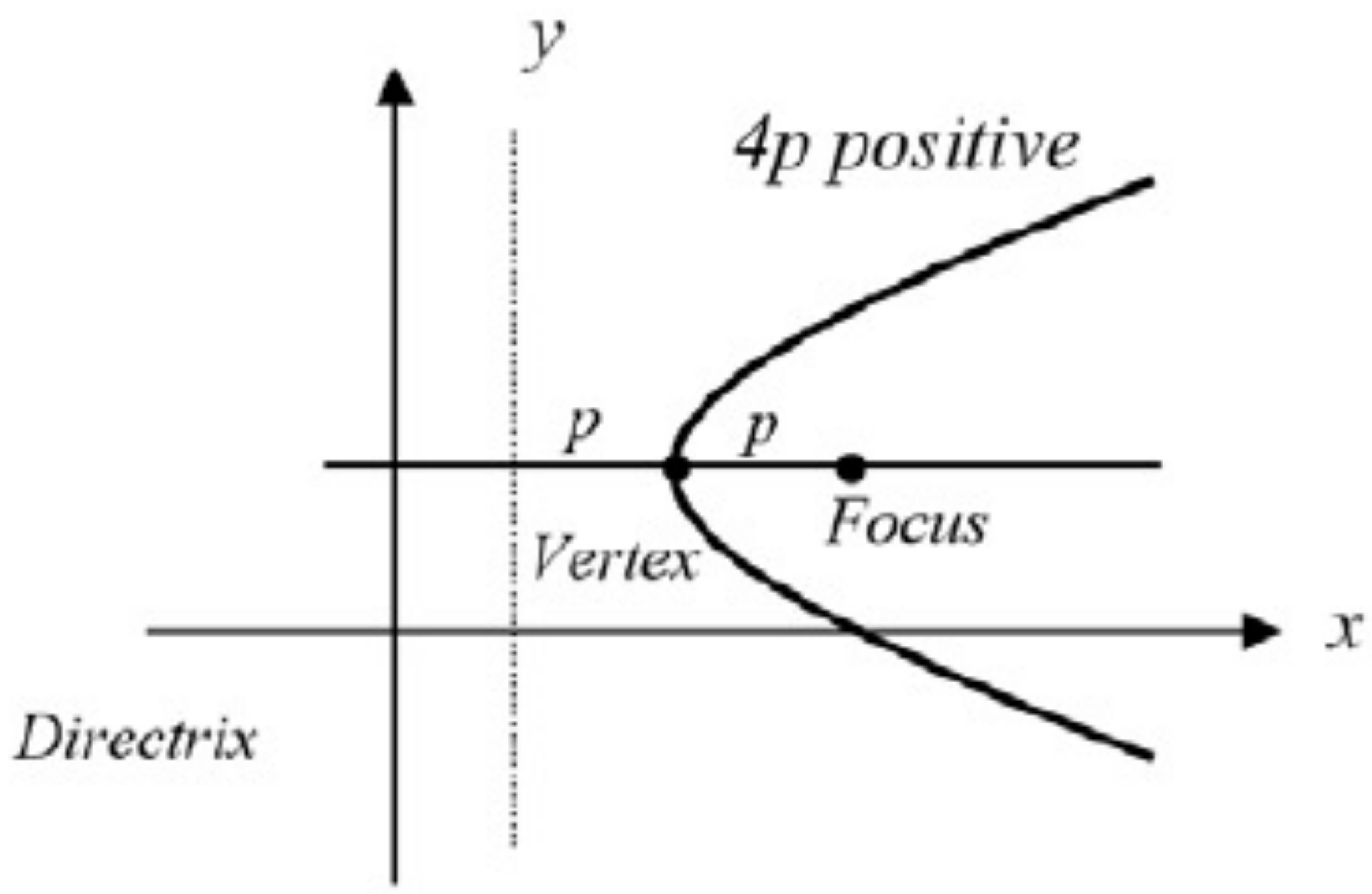


7.2 PARABOLA

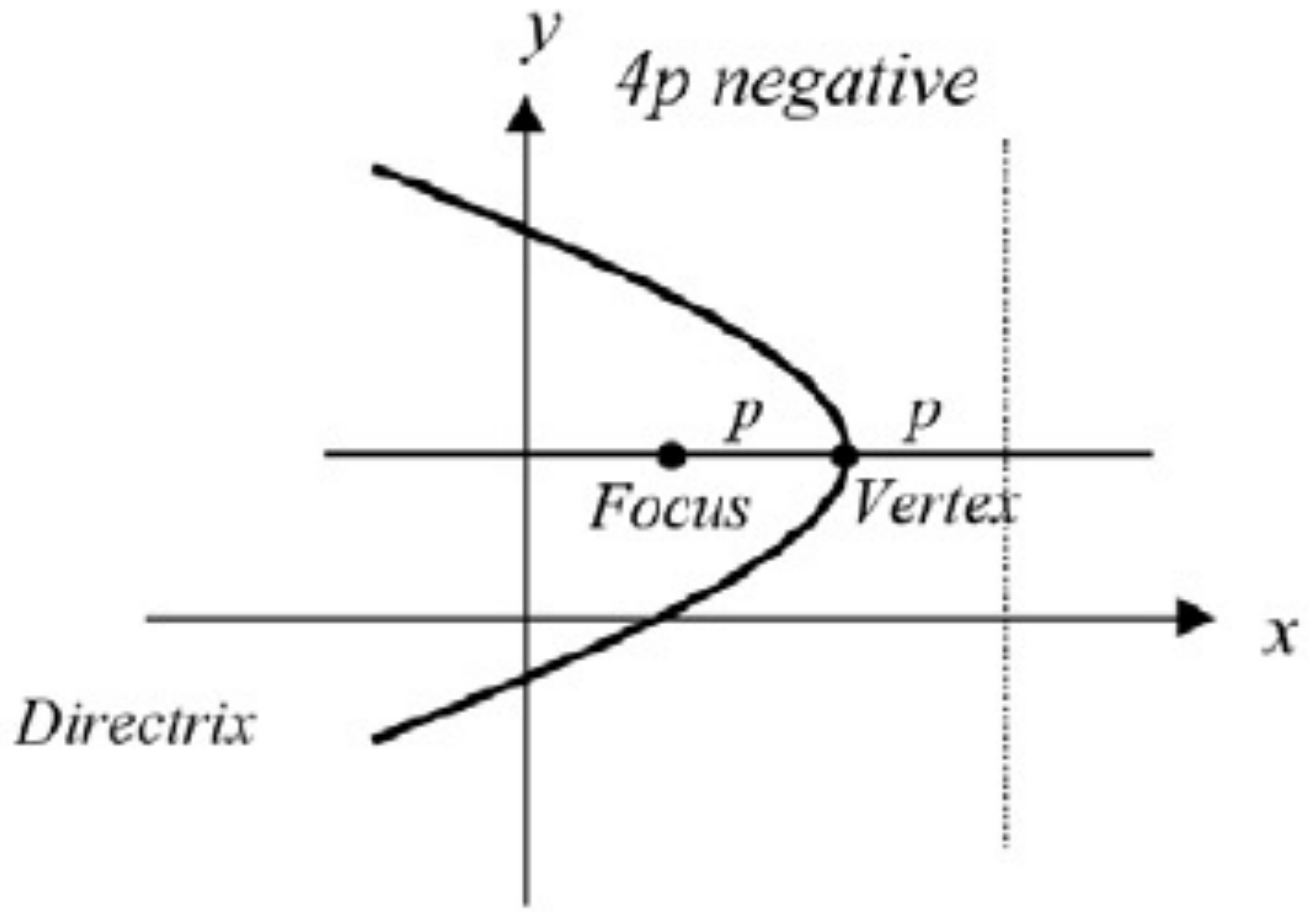
axis is a horizontal line

$$(y-k)^2 = 4p(x-h)$$

Case 1 :
p is positive number,
the parabola opens right.



Case 2 :
p is negative number,
the parabola opens left.



7.2 PARABOLA

Determine the vertex, directrix, focus, and graph of the parabola given by $x^2 = 16y$.

standard equation:

$$(x - h)^2 = 4p(y - k)$$

$$(x - 0)^2 = 16(y - 0)$$

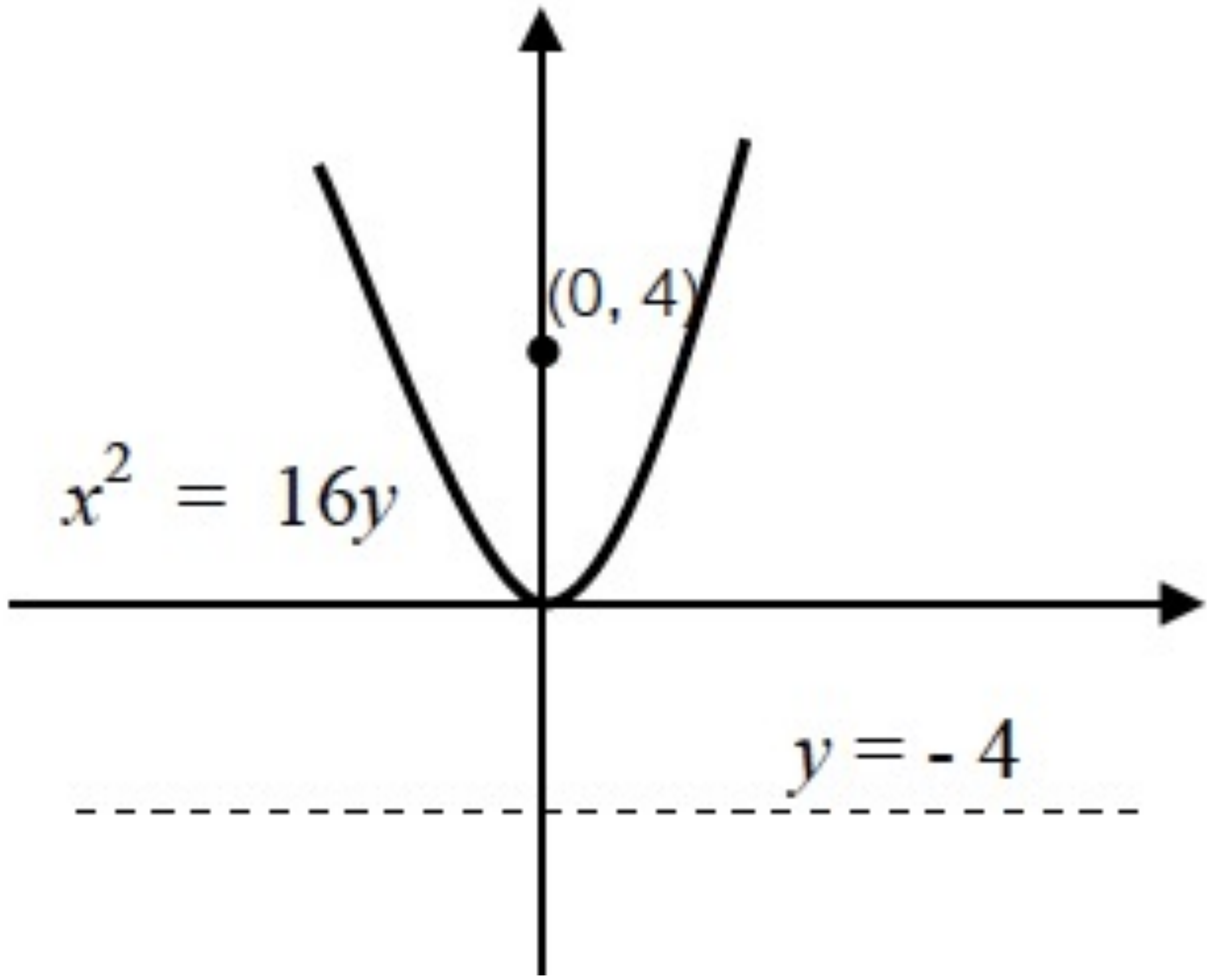
$$(x - 0)^2 = 4(4)(y - 0)$$

- vertex (h, k): (0, 0)
- p = 4. Since the value of $4p$ is positive, the graph opens upward. The focus is 4 units from the vertex inside the parabola.

focus : (0, 4).

- The directrix is 4 units from the vertex outside the parabola.

Directrix : y = -4.



7.2 PARABOLA

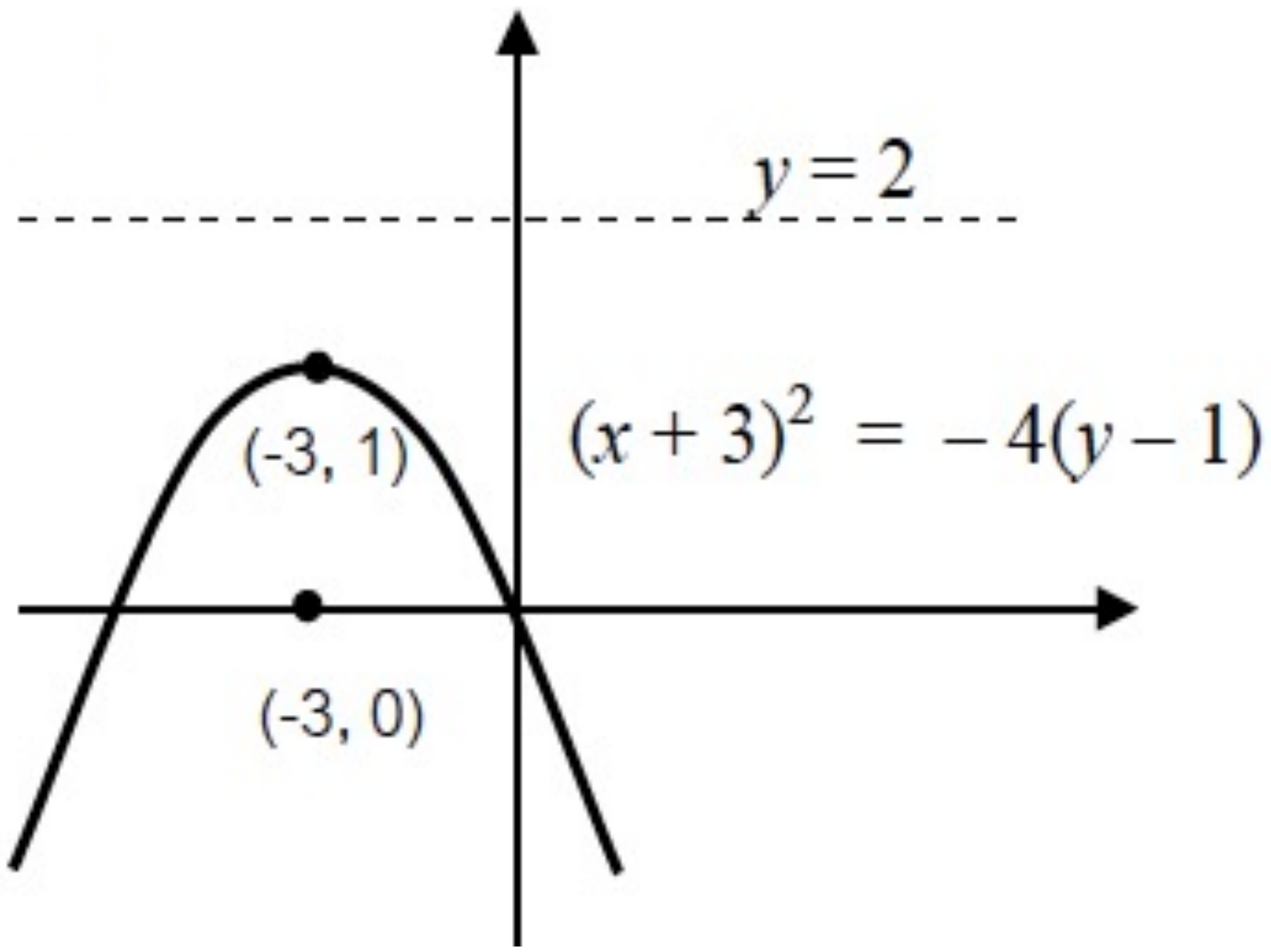
Determine the vertex, directrix, focus, and graph of the parabola given by $(x + 3)^2 = -4(y - 1)$

$\boldsymbol{(x - h)^2 = 4p(y - k)}$

$(x - (-3))^2 = -4(y - 1)$

$(x - (-3))^2 = 4(-1)(y - 1)$

- vertex (h, k): (-3, 1)
- $p = -1$. The graph opens downward.
- focus : (-3, 0)
- directrix : $y = 2$

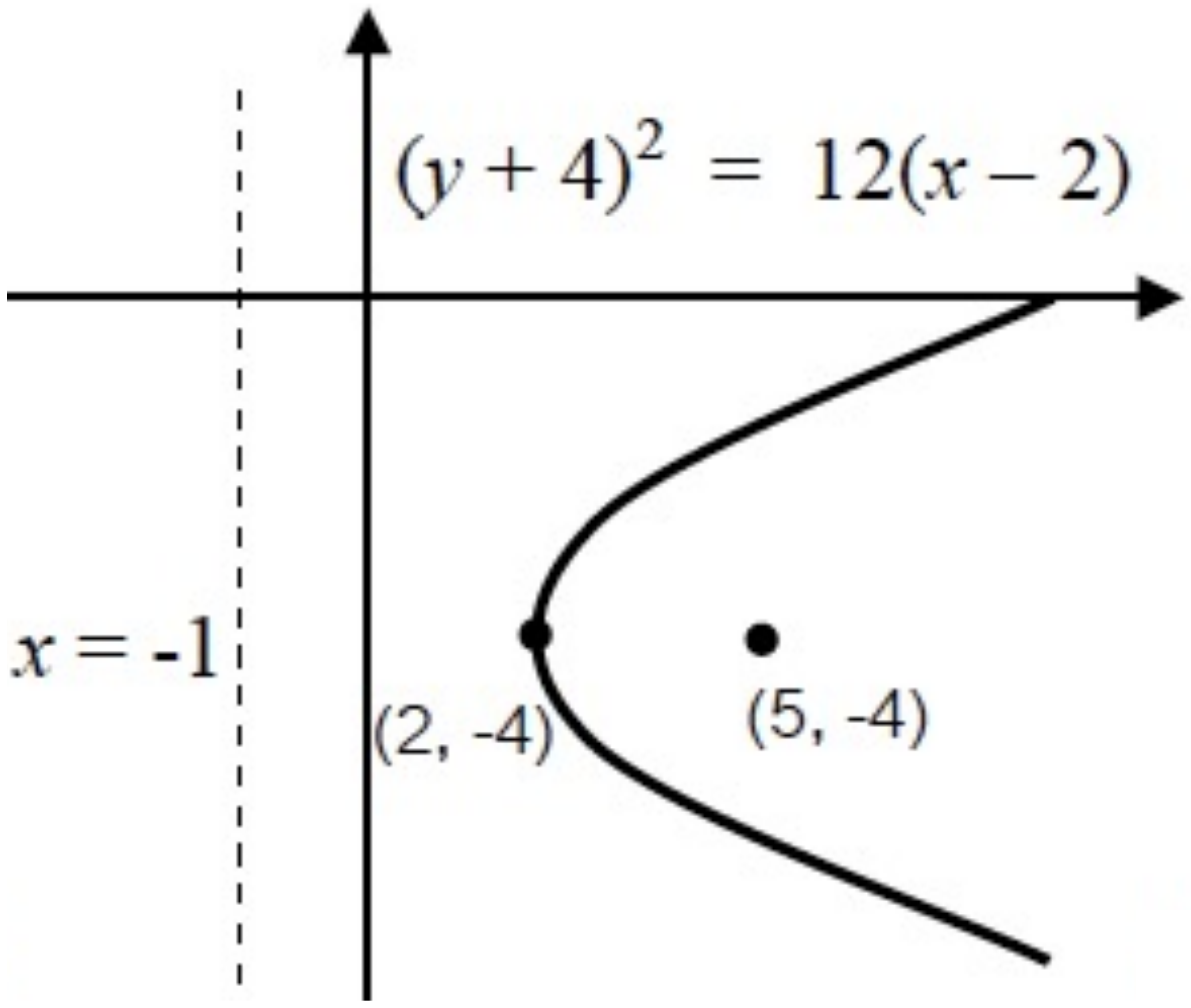


7.2 PARABOLA

Determine the vertex, directrix, focus, and graph of the parabola given by $y^2 + 8y = 12x - 40$

$$\begin{aligned} y^2 + 8y &= 12x - 40 \\ (y^2 + 8y + 16) &= 12x - 40 + 16 \\ (y+4)^2 &= 12x - 24 \\ (y+4)^2 &= 12(x-2) \end{aligned}$$

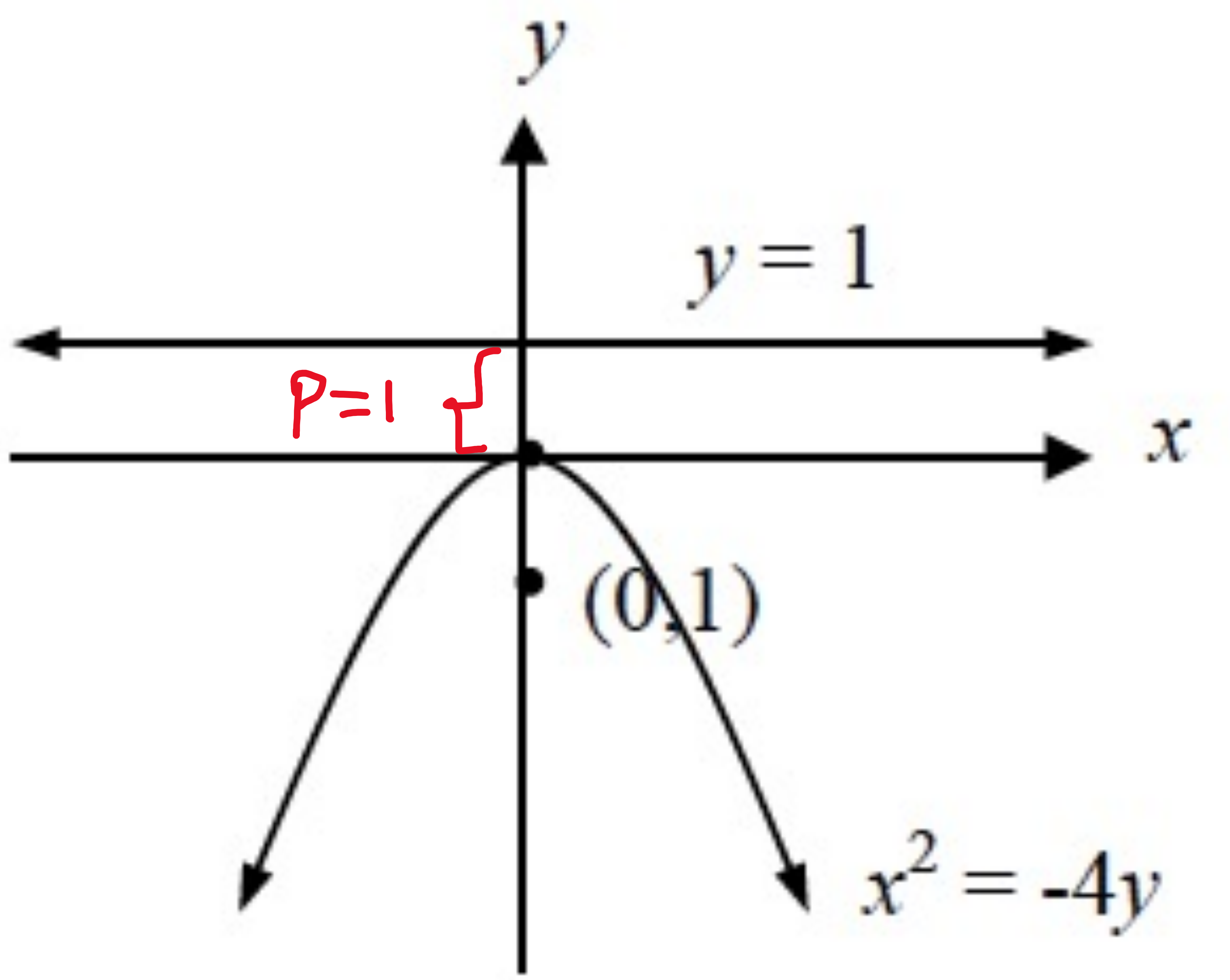
1. vertex : (2, -4)
2. $p = 3$. The graph opens right.
3. focus : (5, -4)
4. directrix : $x = -1$



7.2 PARABOLA

Find the equation of a parabola whose directrix is the line $y = 1$ and whose vertex is the point (0, 0)

(h, k)



p = -1 (from vertex to directrix)

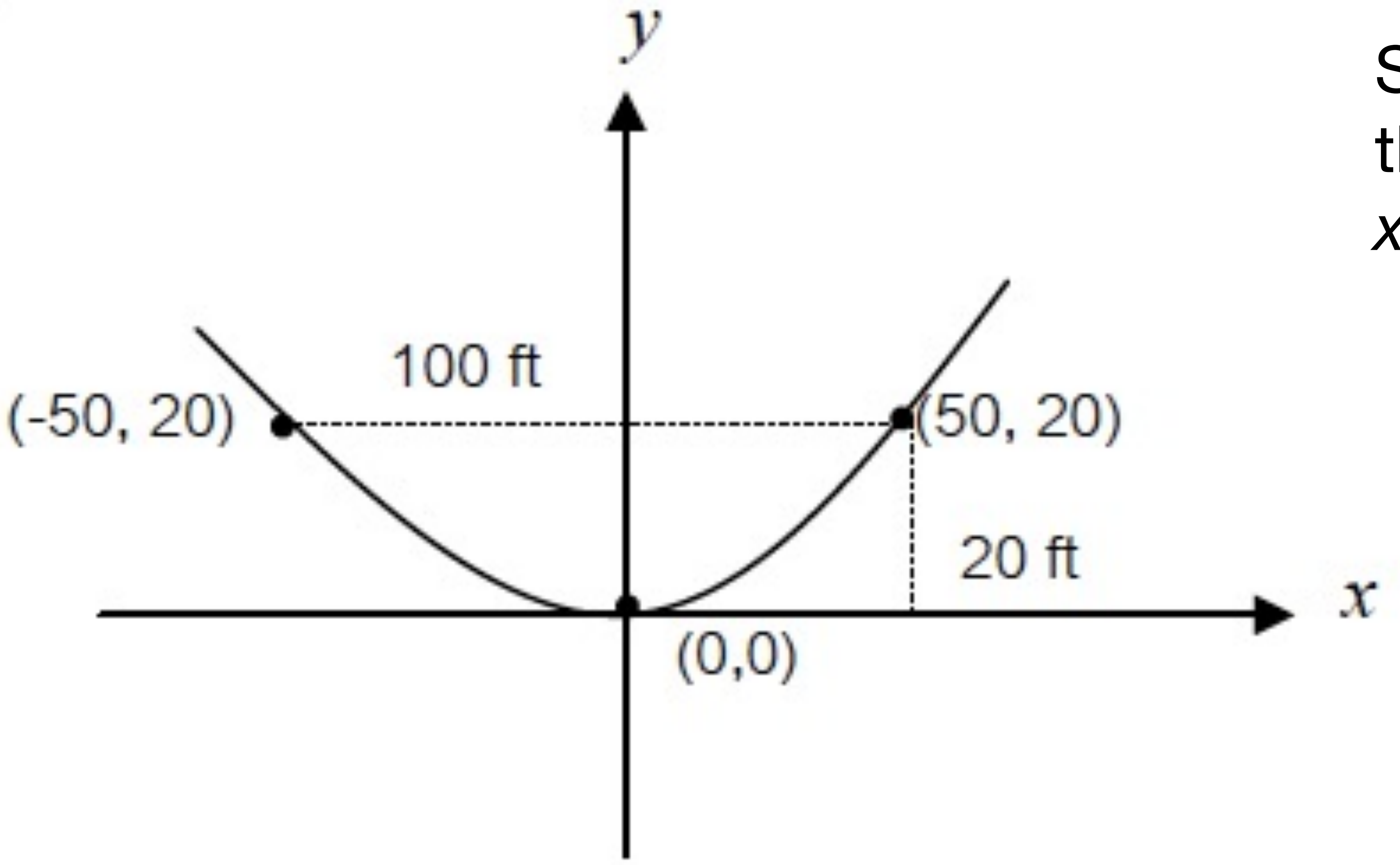
$(x - h)^2 = 4p(y - k)$

$(x - 0)^2 = 4(1)(y - 0)$

$x^2 = -4y$

7.2 PARABOLA

(Application). A cable used to support a swinging bridge approximates the shape of a parabola. Determine the equation of the parabola if the length of the bridge is 100 ft and the vertical distance from where the cable is attached to the bridge to the lowest point of the cable is 20 ft. Assume that the origin is at the lowest point of the cable.



Since we can assume that the origin is at the lowest point, the vertex is (0, 0), and the general form of the equation is $x^2 = 4py$. Find the value of *p*.

$$x^2 = 4py$$

$$(50)^2 = 4p(20)$$

$$2500 = 80p$$

$$31.25 = p$$

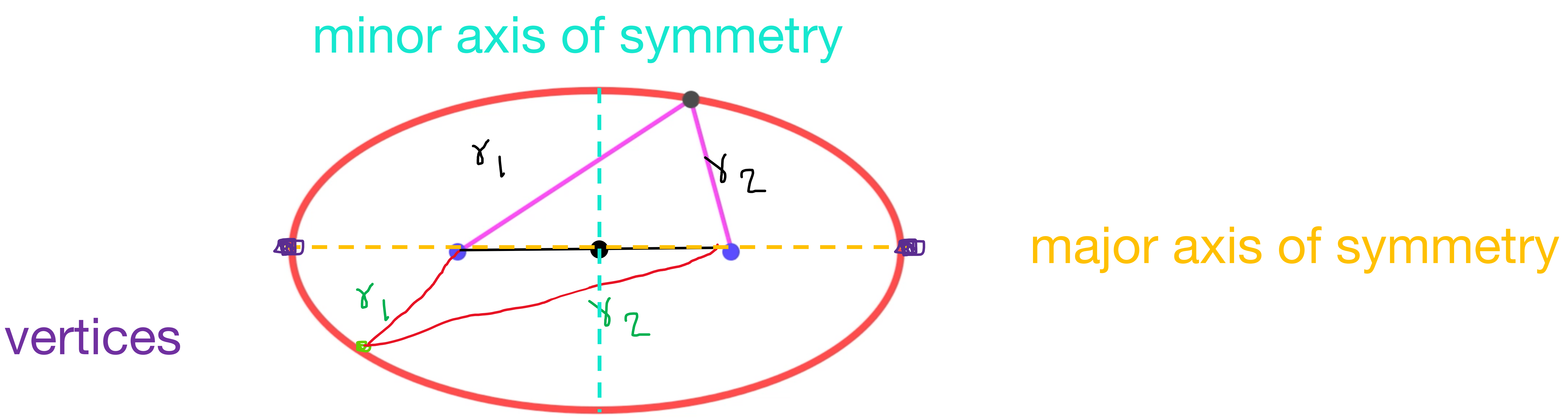
$$x^2 = 4py$$

$$\boxed{x^2 = 4(31.25)y = 125y}$$

7.2 PARABOLA

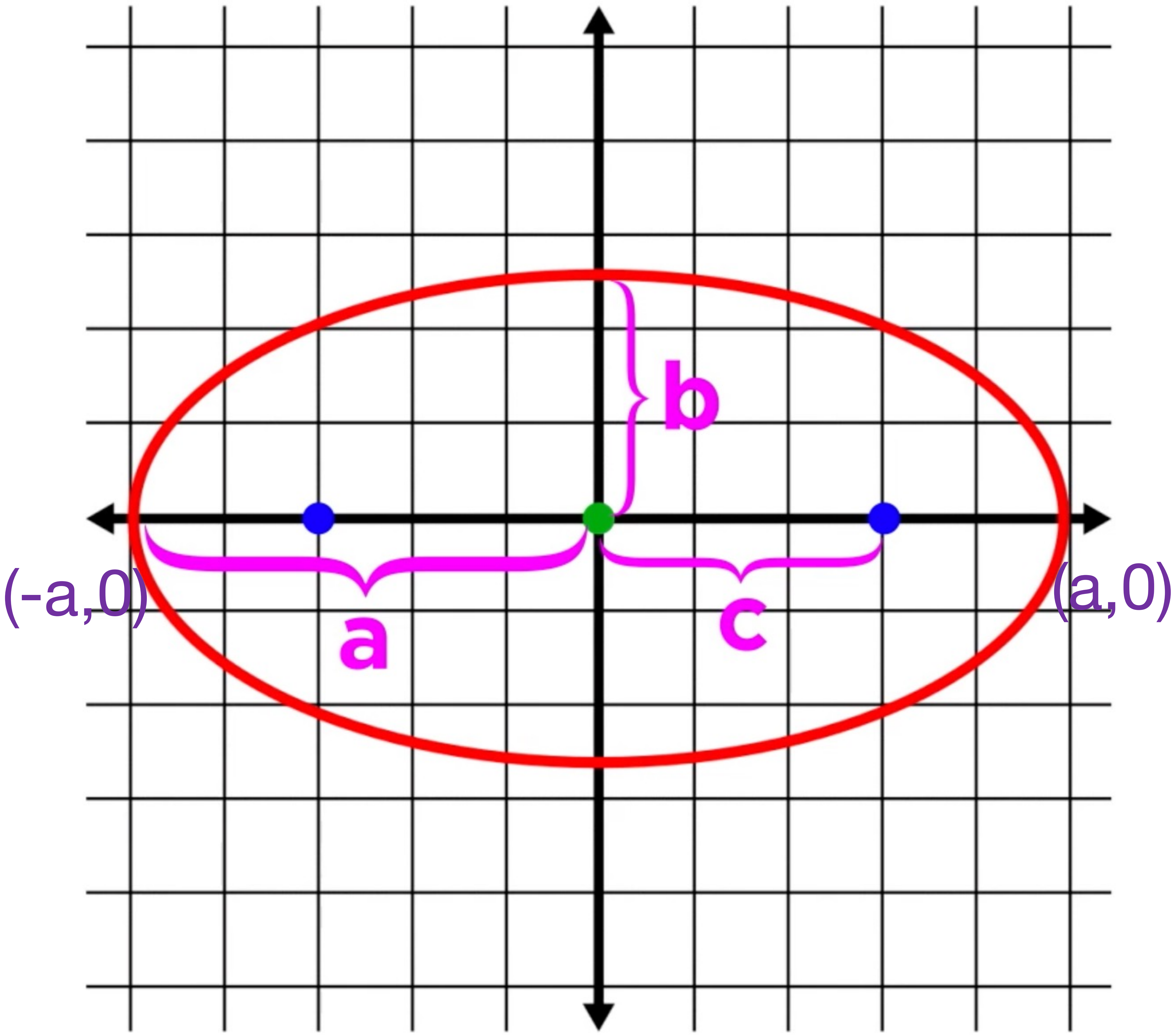
Exercise 7.2

7.3 ELLIPSE



sum of distances
to foci is constant

7.3 ELLIPSE



standard form of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Length of major axis = 2a

Length of minor axis = 2b

(a = distance from center to vertex)

Vertices are at:

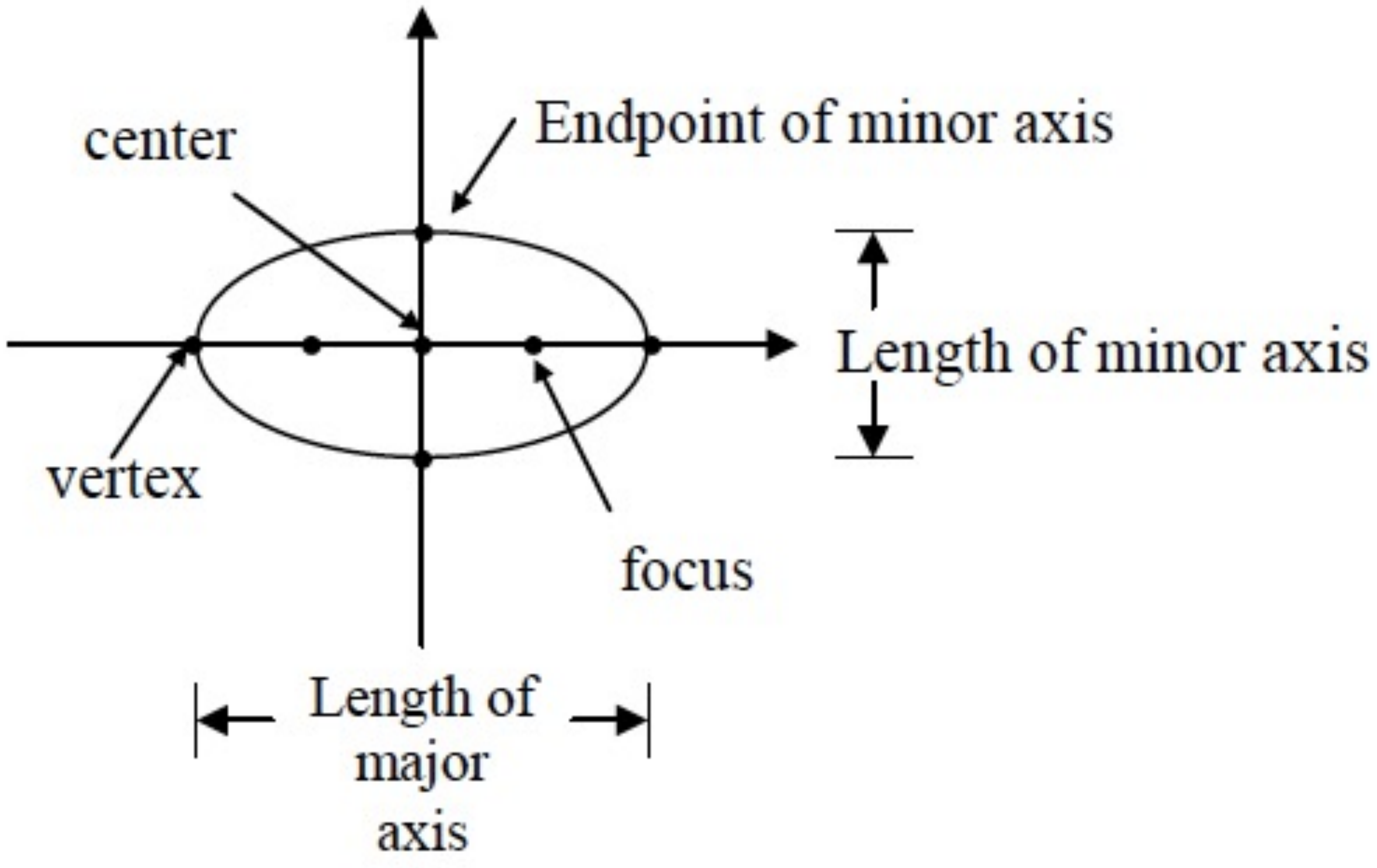
(-a,0) and (a,0)

$$c^2 = a^2 - b^2$$

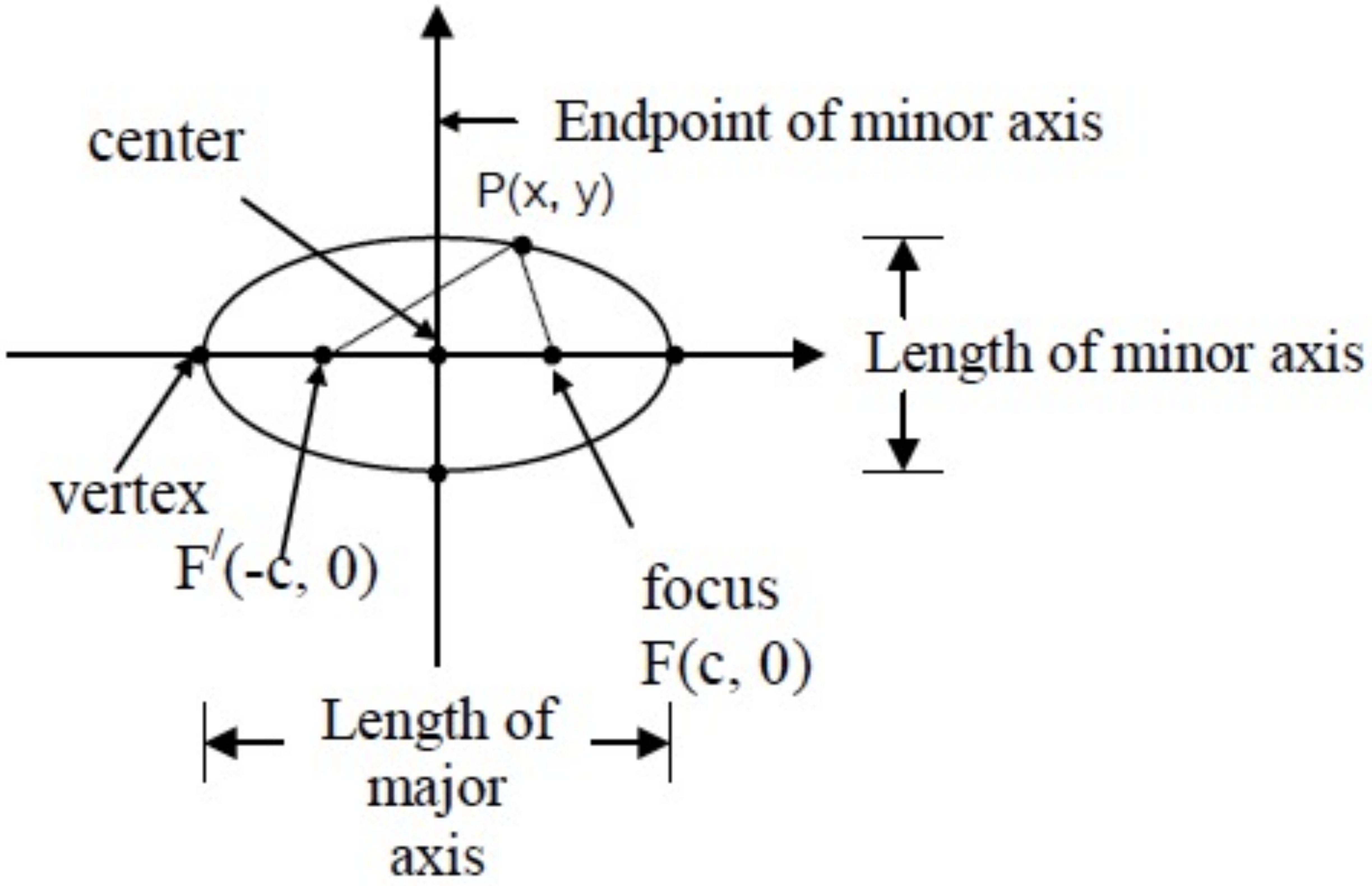
7.3 ELLIPSE

Definition

The **ellipse** is the set of all points in a plane such that the sum of its distances from two fixed points F and F' in the plane (the **foci**) is a constant.



7.3 ELLIPSE

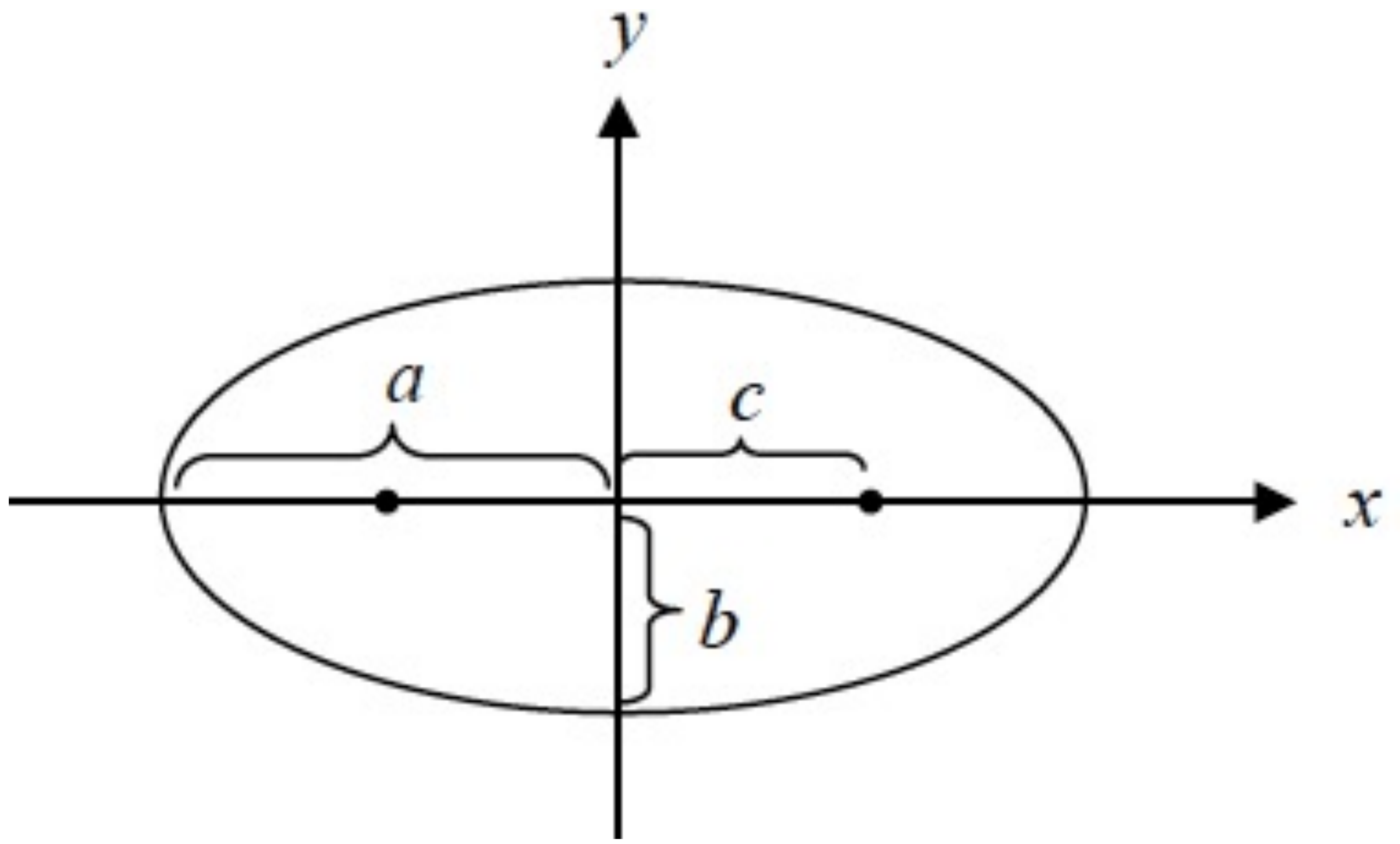


The distance between $F^{/}$ and F is $2c$ units. We will let $2a$ represent the constant sum of $FP + F^{/}P$. Note that $2a > 2c$ and therefore $a > c$. For any point P on the ellipse,

$$FP + F^{/}P = 2a$$

7.3 ELLIPSE

an ellipse elongated horizontally



The Standard Form

$$\boxed{\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1\,, \qquad a^2>b^2}$$

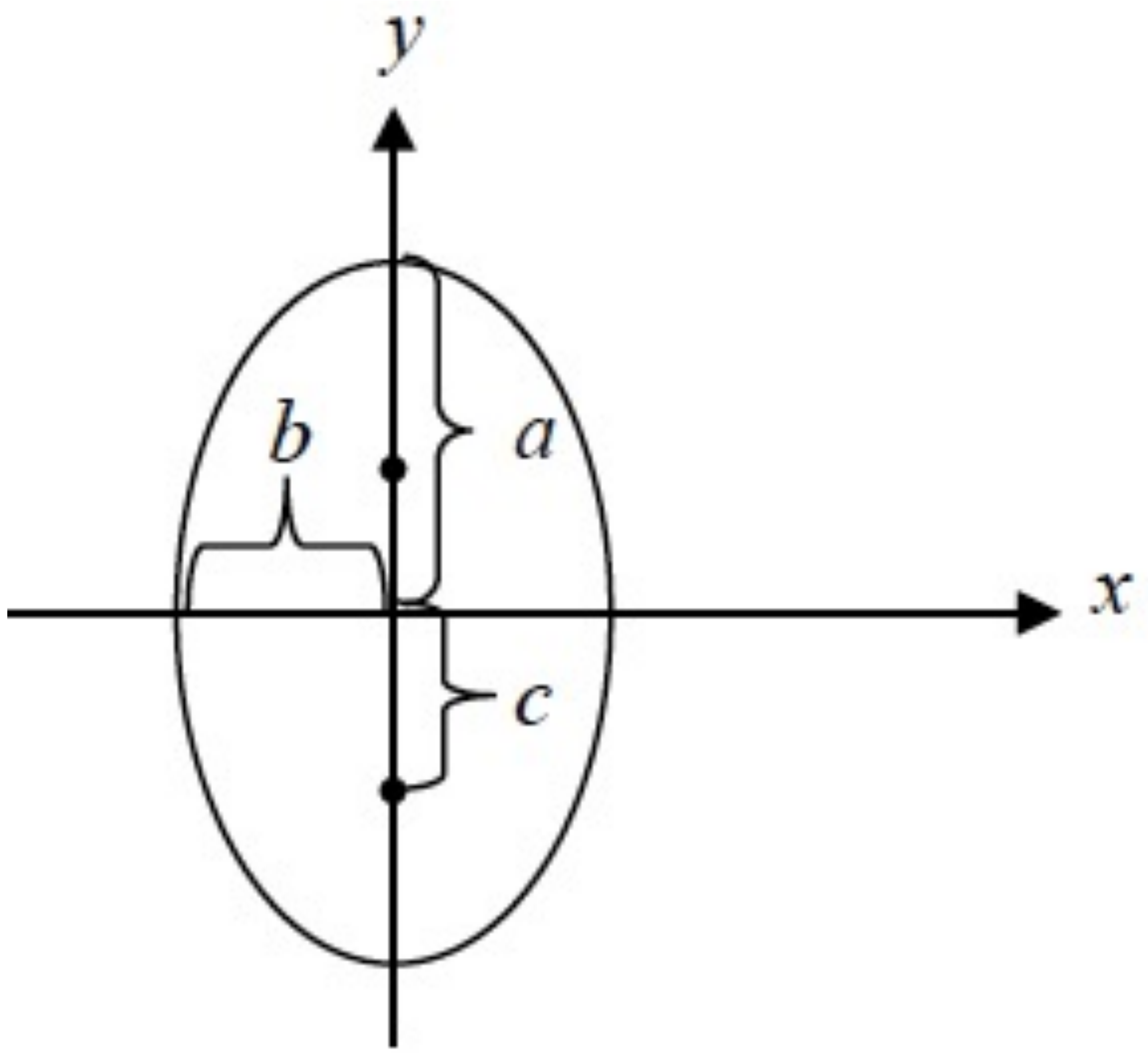
where

- *(h, k)* - the coordinates of the center
- *a* - distance from center to the vertices
- *b* - distance from the center to the endpoints of the minor axis.
- *c* - distance from the center to either focus
  $\boldsymbol{c=\sqrt{a^2-b^2}}$

Notice that the **foci are always located on the major axis**.

7.3 ELLIPSE

an ellipse elongated vertically



$$\boxed{\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1\ , \qquad a^2>b^2}$$

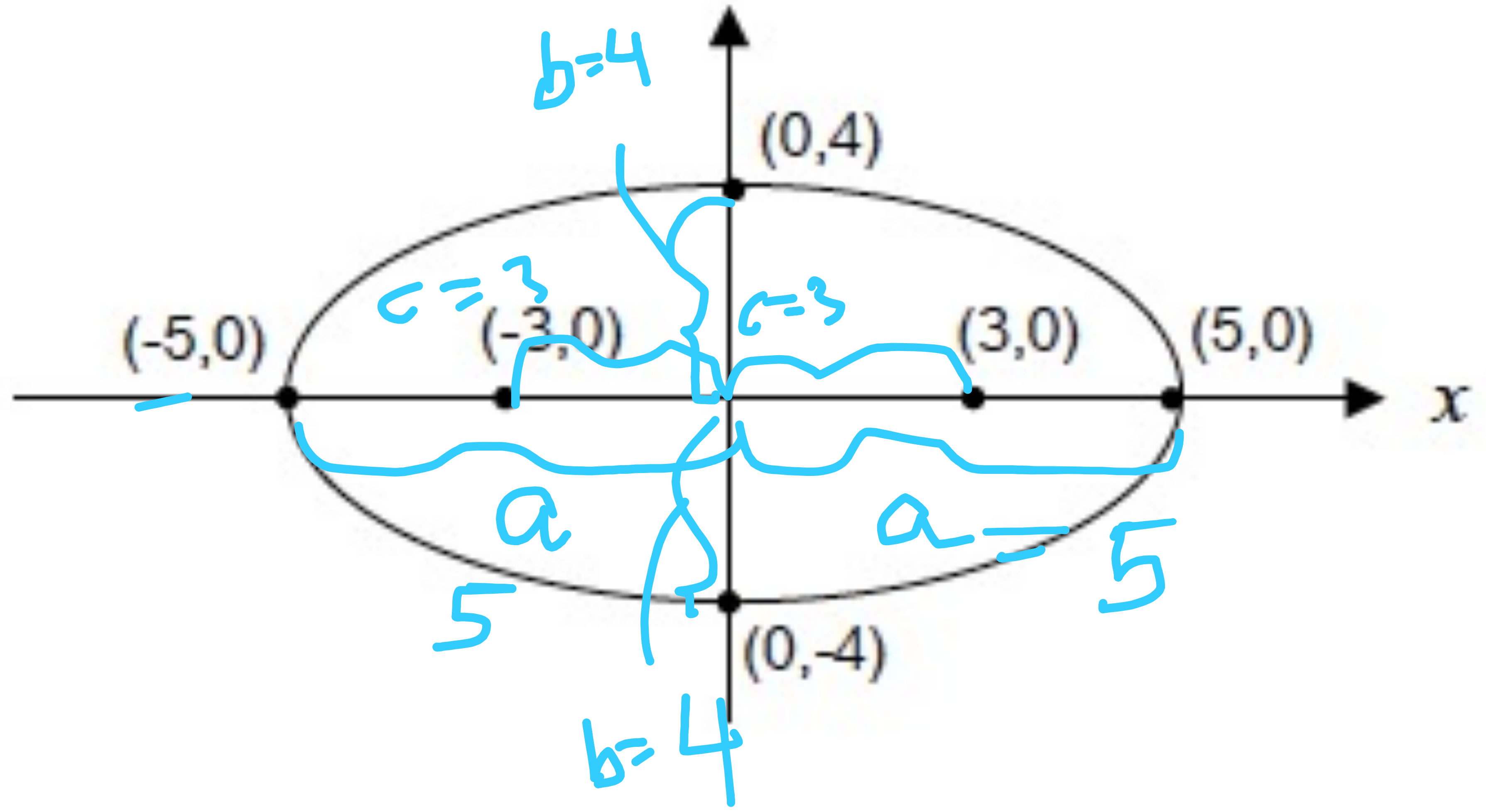
The major axis, vertices, and foci are located on a vertical line

7.3 ELLIPSE

$$\boxed{\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1, \qquad a^2>b^2}$$

Find the coordinates for the center, vertices, endpoints of the minor axis, and foci of the ellipse represented by

$16x^2 + 25y^2 = 400$



$$\frac{x^2}{25}+\frac{y^2}{16}=1 \quad or \quad \frac{(x-0)^2}{25}+\frac{(y-0)^2}{16}=1$$

$a^2 = 5^2$ $\quad 4^2 = b^2$

1. The center is at the origin (0, 0).
2. The major axis is horizontal, because the larger denominator is under x^2.
3. The value of a is $\sqrt{25}=5$, so the vertices move 5 units horizontally from the center.
4. The value of b is $\sqrt{16}=4$, so the endpoints of the minor axis, move 4 units vertically from the center.
5. The coordinates of the foci, move $\sqrt{25-16}=3$ units horizontally from the center.

$c=\sqrt{a^2-b^2}=\sqrt{25-16}=\sqrt{9}=3$

Center : (0, 0)
Vertices : (– 5, 0) and (5, 0)
Endpoints of minor axis : (0, 4) and (0, – 4)
Foci : (3, 0) and (-3, 0)

Example 7.13 Find the coordinates for the center, vertices, endpoints of the minor axis, and foci of the ellipse represented by

$$\frac{(x-3)^2}{4}+\frac{(y+1)^2}{49}=1$$

$2^2 \qquad 7^2$

$a \qquad b$

$$\boxed{\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1, \qquad a^2>b^2}$$

$$\boxed{\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1, \qquad a^2>b^2}$$

$C(h,k)=(3,-1)$

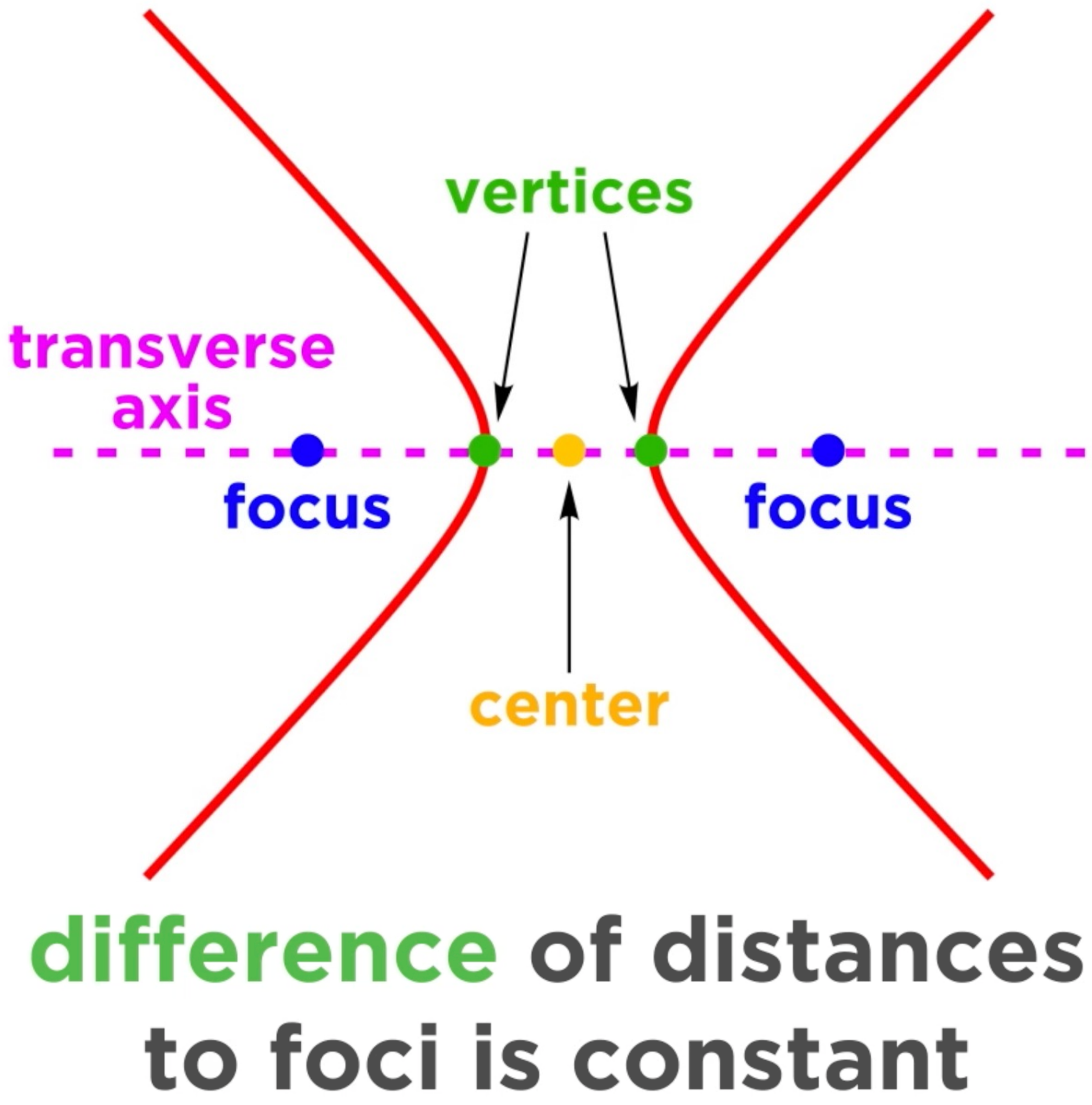
$C=\sqrt{a^2-b^2}=\sqrt{49-4}=\sqrt{45}$

$=6.7$

7.3 ELLIPSE

Exercise 7.3

7.4 HYPERBOLA



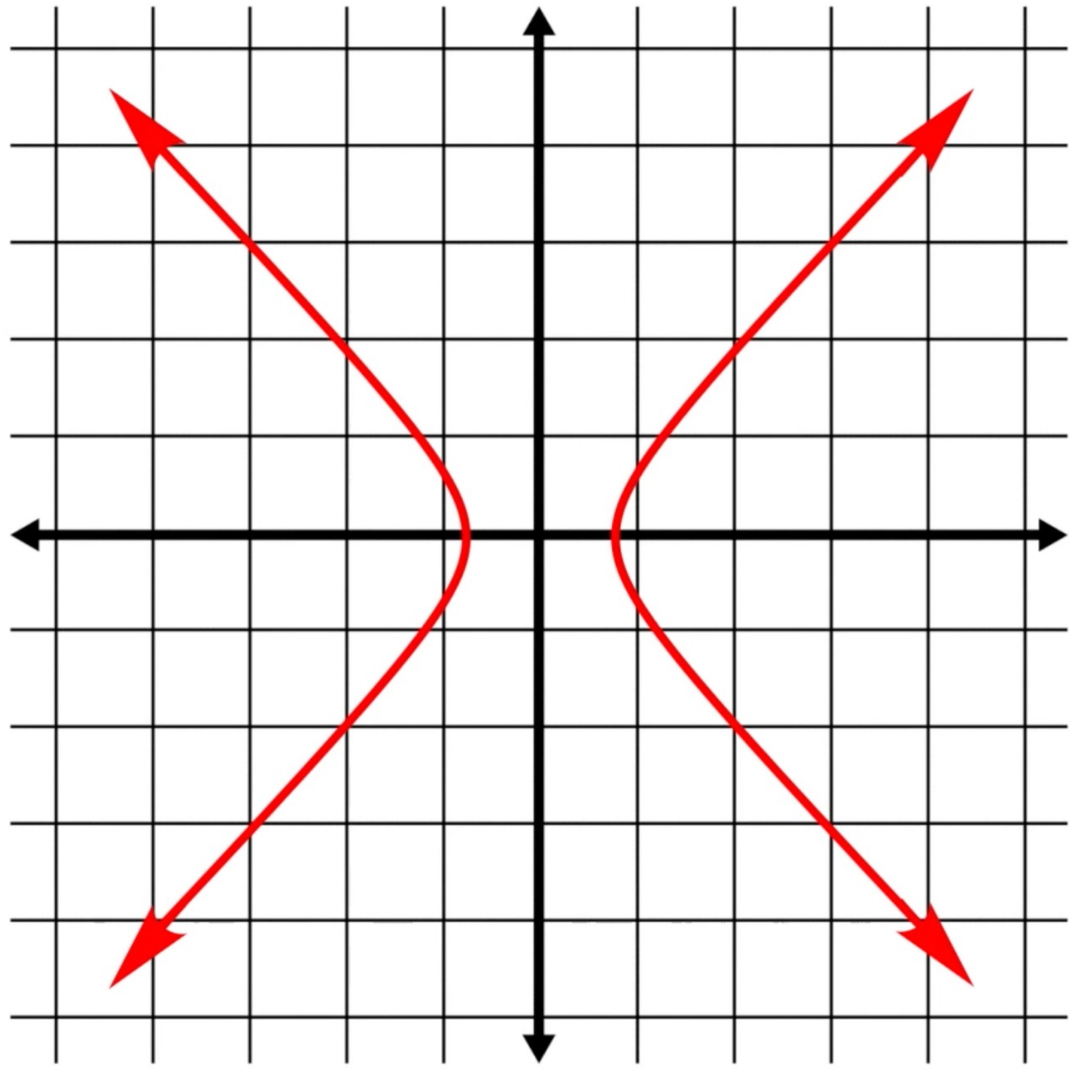
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

7.4 HYPERBOLA

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

x term is first:

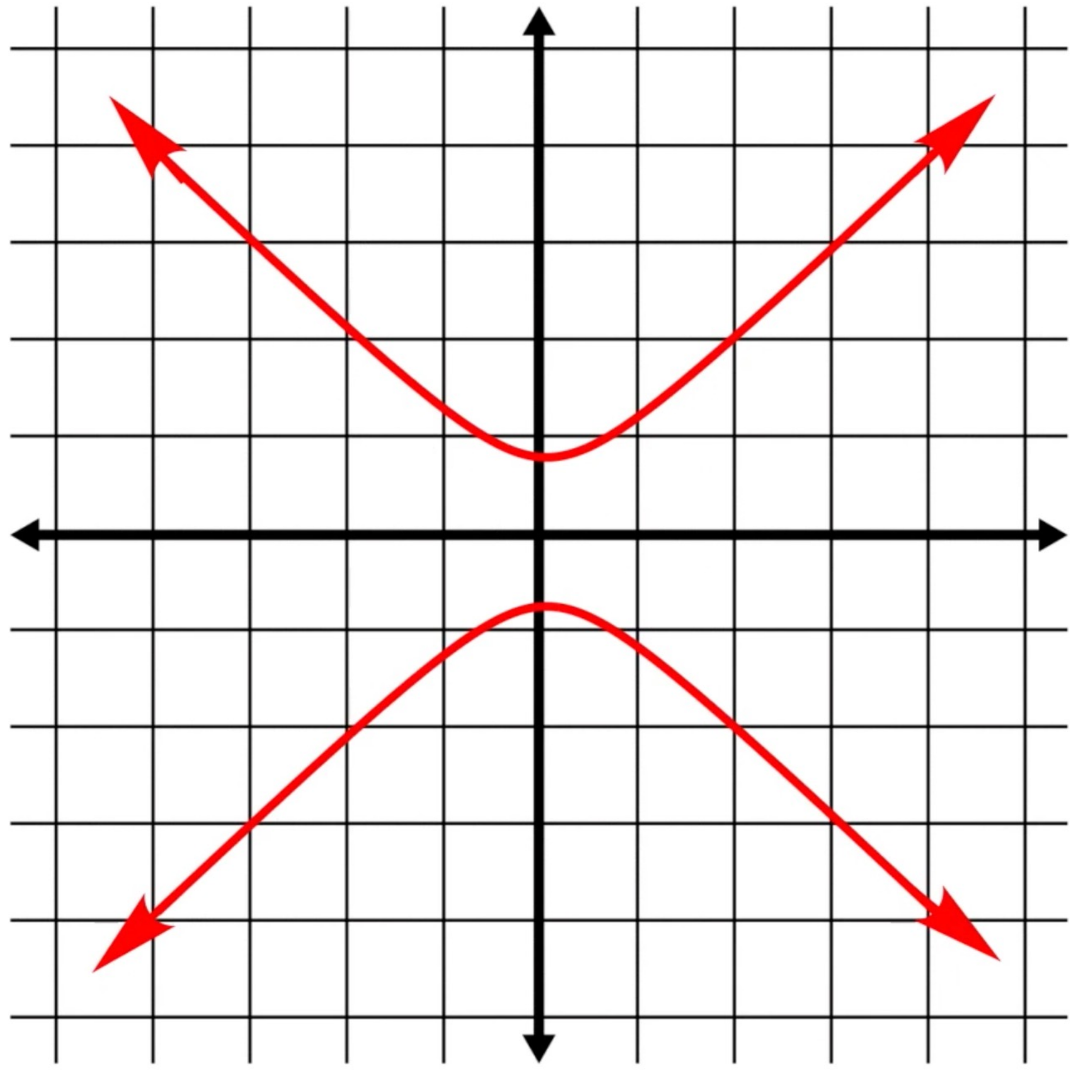
opens around x-axis



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

y term is first:

opens around y-axis



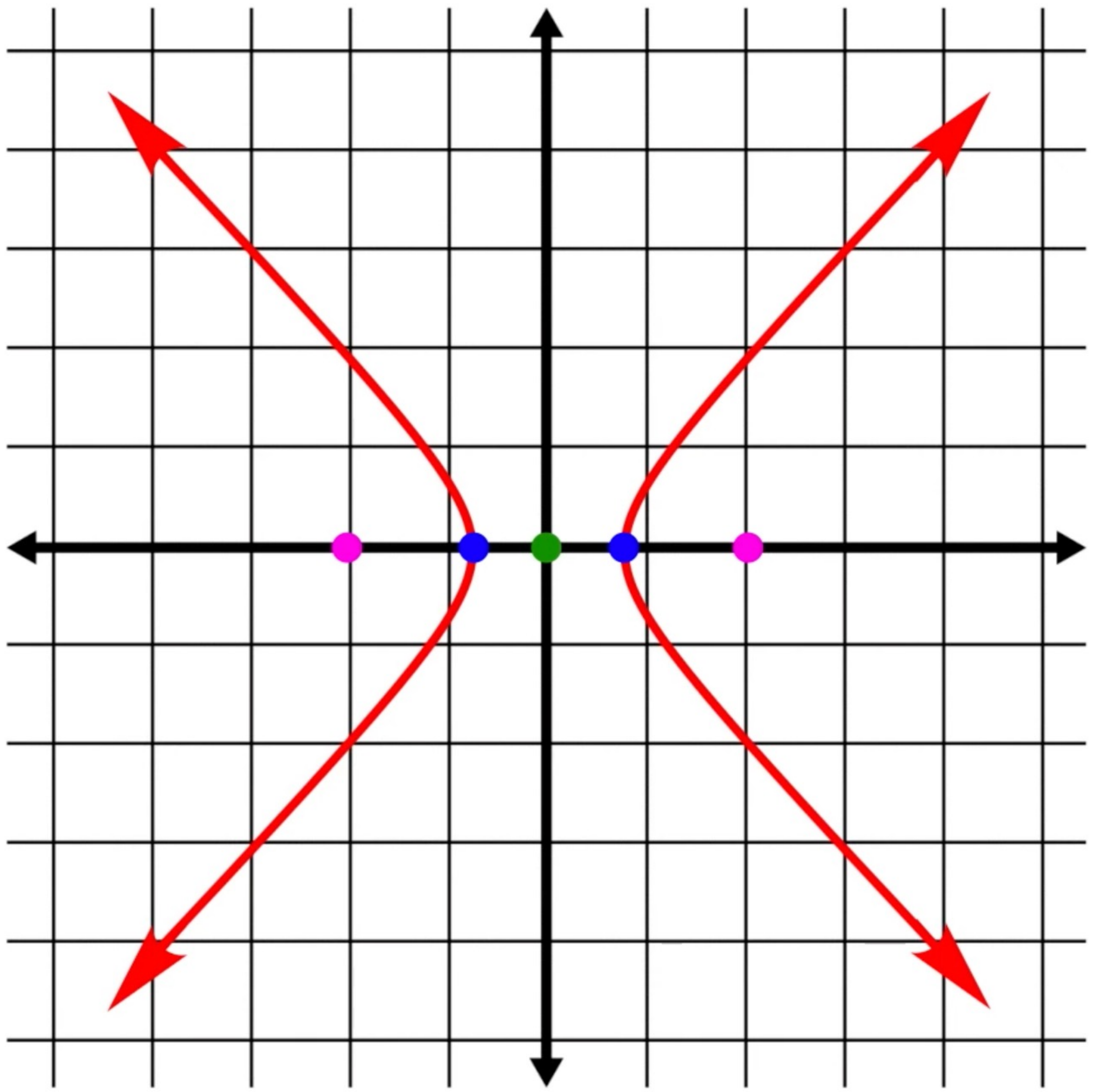
7.4 HYPERBOLA

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

a = distance from center to vertex

c = distance from center to focus

$$c^2 = a^2 + b^2$$



7.4 HYPERBOLA

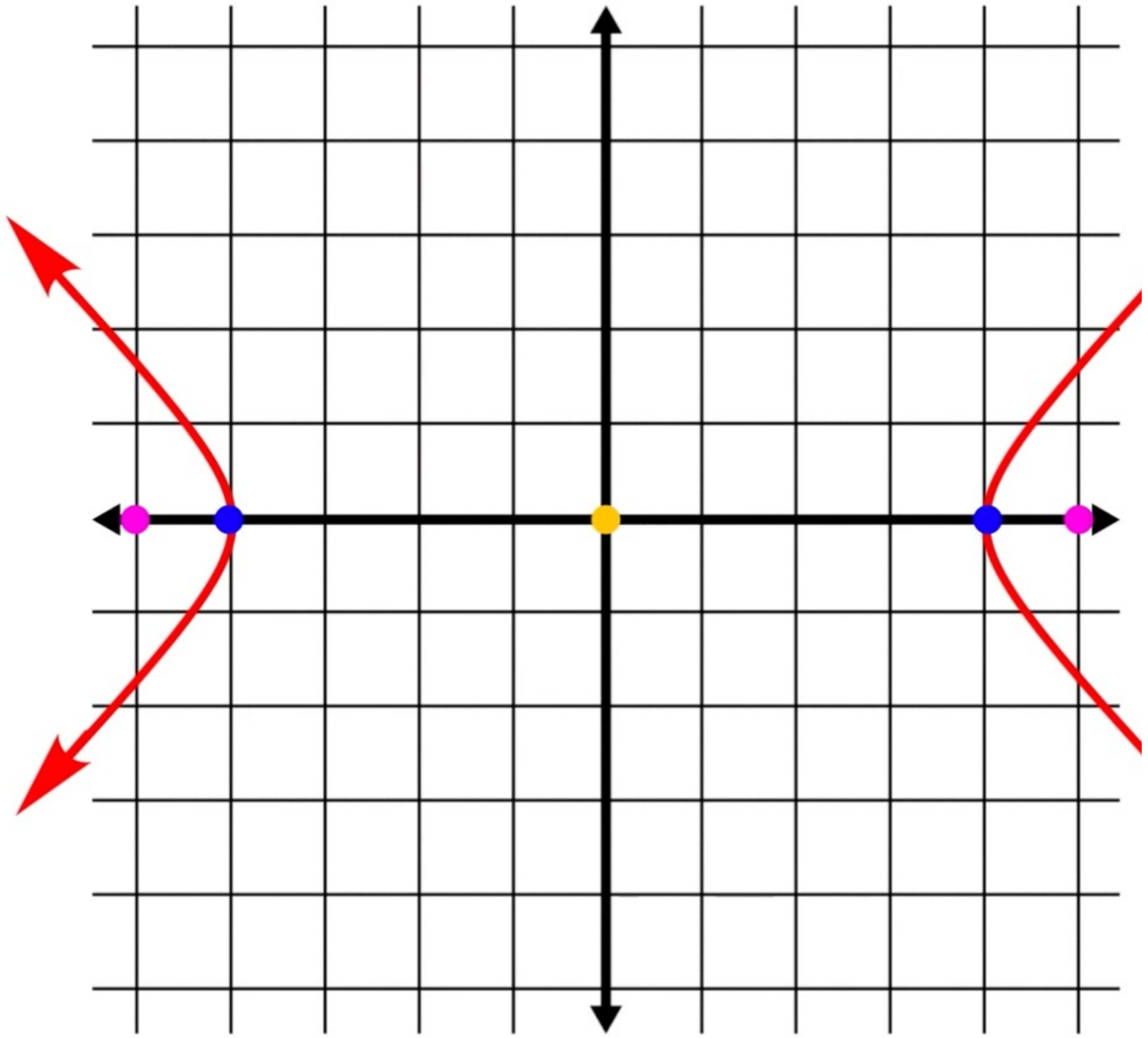
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$a^2 = 16$$
$$b^2 = 9$$

$$a = 4 \qquad b = 3$$

$$c^2 = a^2 + b^2 = 25$$

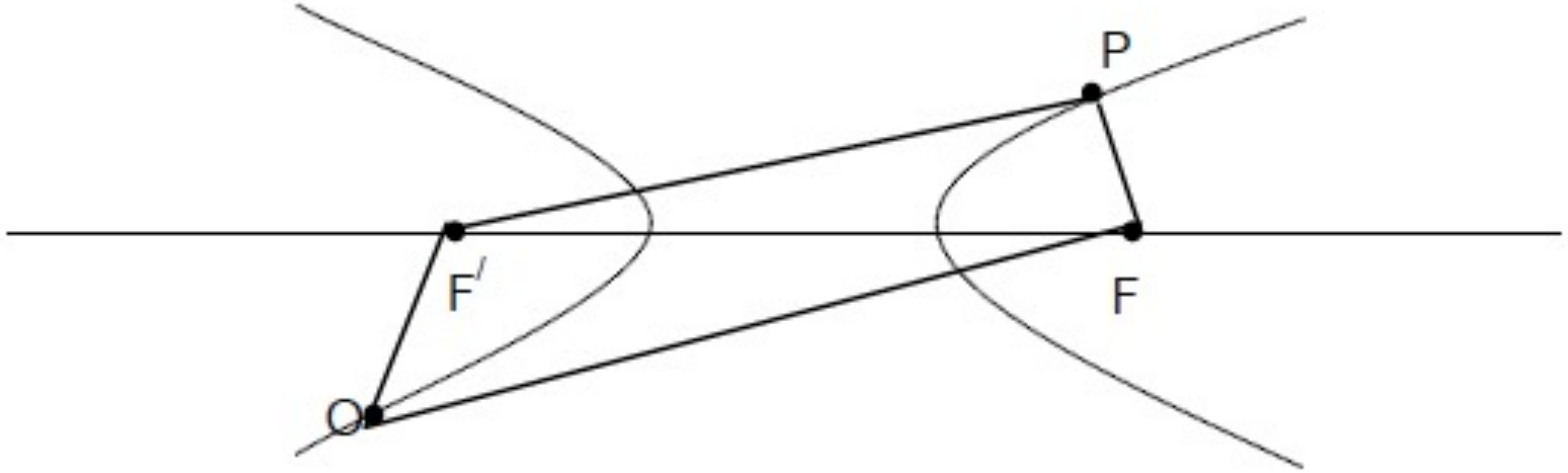
$$c = 5$$



7.4 HYPERBOLA

Definition

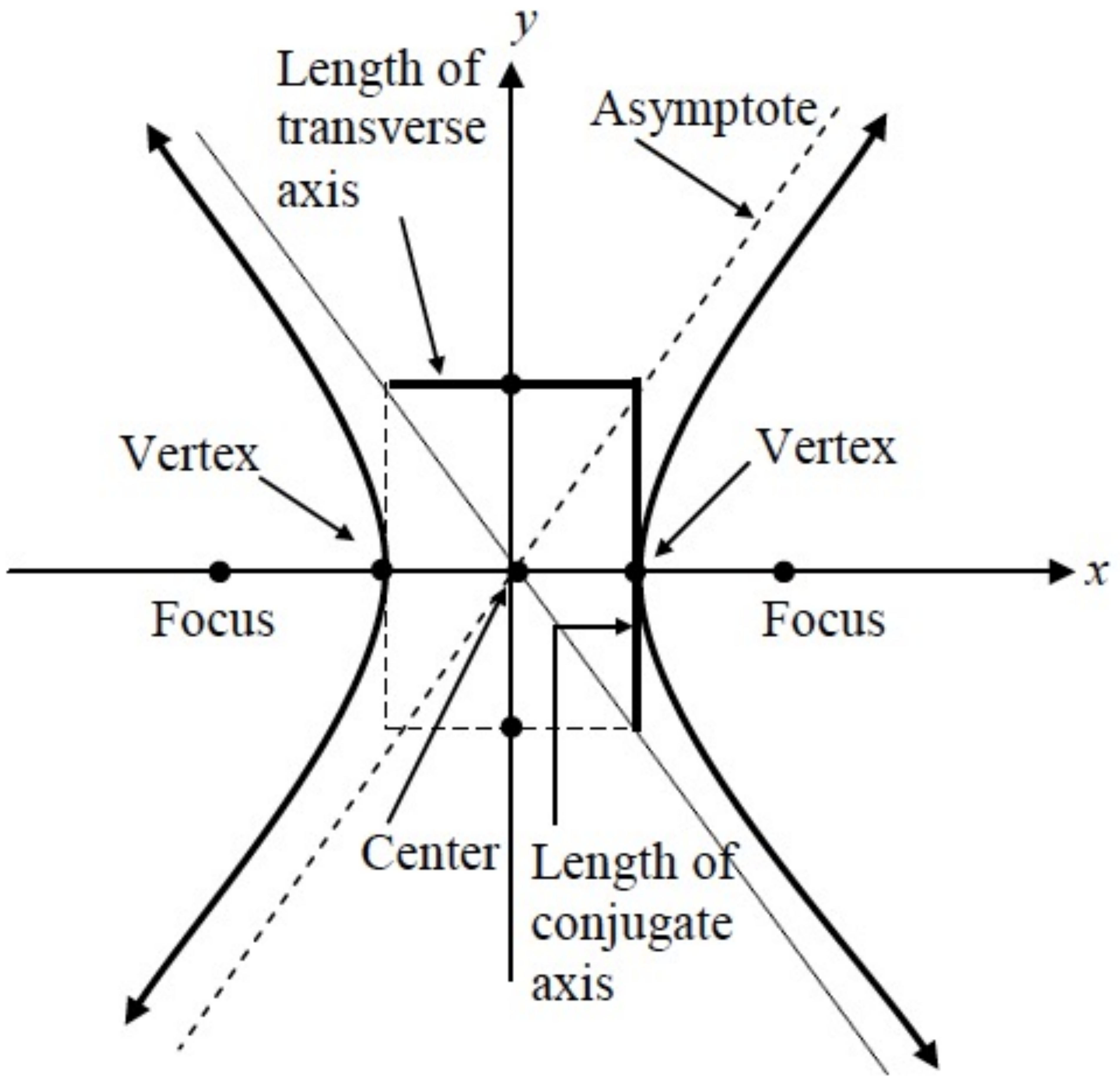
The **hyperbola** is the set of all points in a plane such that the difference of its distances from two fixed points F and F' in the plane (the **foci**) is a positive constant.



For all points P, $PF^{/}$ - PF is a positive constant.

Likewise, all points O, OF – $OF^{/}$ is the same positive constant.

7.4 HYPERBOLA

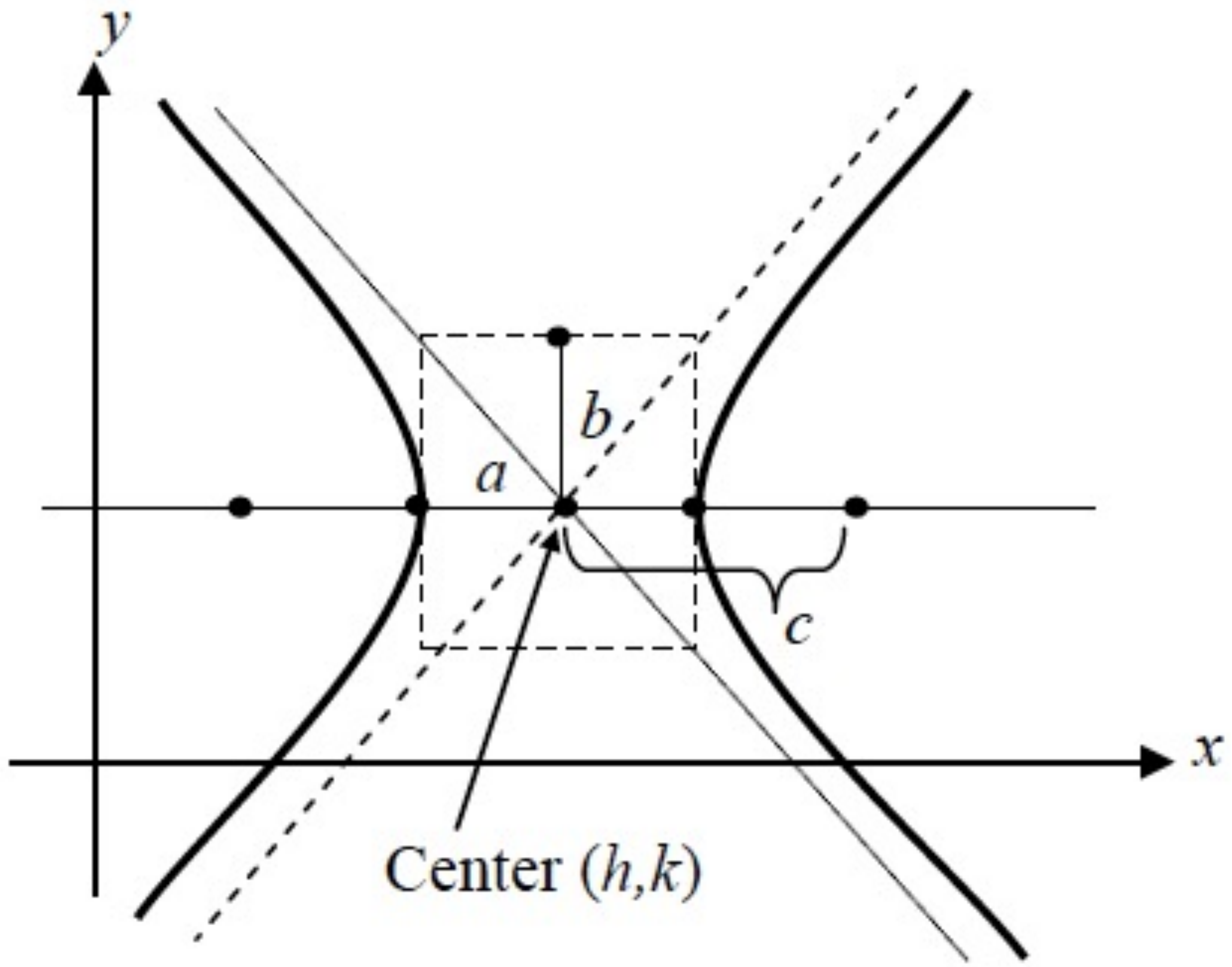


The Standard Form

$$\boxed{\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1}$$

7.4 HYPERBOLA

hyperbola with a horizontal transverse axis



$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

where the point ***(h, k)*** is the center, ***a*** represents the distance from the center to a vertex, and ***b*** represent the distance from the center to an endpoint of the conjugate axis.

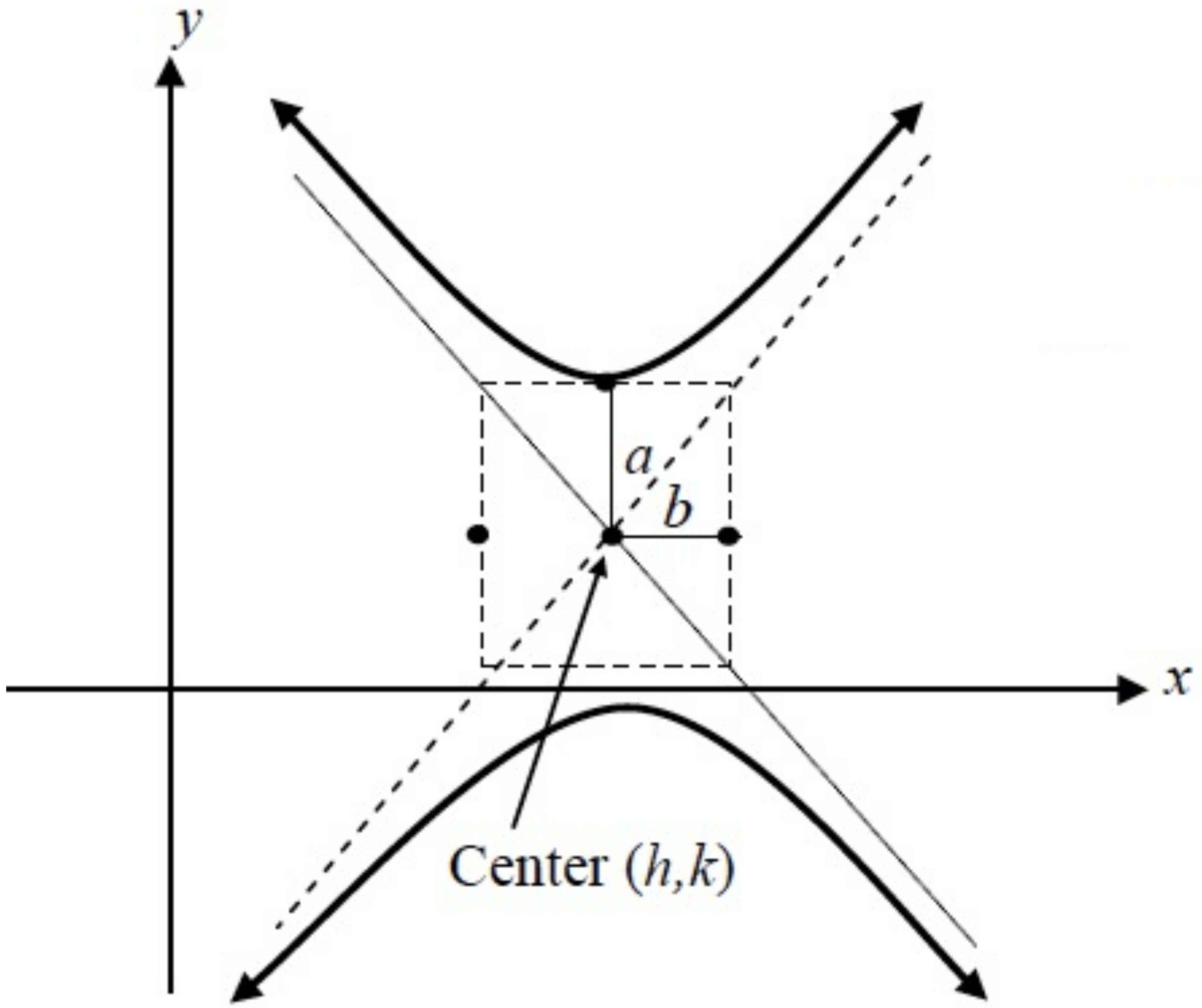
The **slopes** of the asymptotes are given by

$$m=\pm\frac{b}{a}$$

The foci are a distance c from the center where $c=\sqrt{a^2+b^2}$ along the transverse axis.

7.4 HYPERBOLA

hyperbola whose transverse axis is vertical



$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

where h, k, a, and b represent the same quantities as previous.

The slopes of the asymptotes are given by

$$m=\pm\frac{a}{b}$$

7.4 HYPERBOLA

Note **:**

If the equation is in standard form, the direction of opening of the hyperbola is determined by which term is positive.

If the y term is positive, the hyperbola opens upward and downward.

If the x term is positive, the hyperbola opens left and right.

The letter a^2 always appears in the denominator of the positive term (and a^2 need *not* be larger than b^2)

7.4 HYPERBOLA

Hyperbolas of the form $xy = k$

An equation of the form $xy = k$ where k is some nonzero constant produces a hyperbola. The x and y axes serve as the asymptotes.

If k is positive, the hyperbola lies in quadrants I and III, and if k is negative, the hyperbola lies in quadrants II and IV.

The vertices are given by ($\sqrt{|k|} = \sqrt{|k|}$) placed in the appropriate quadrant with appropriate signs.

The foci are given by ($\sqrt{|2k|} = \sqrt{|2k|}$) placed in the appropriate quadrant with appropriate signs. The transverse axis is $y = x$ if $k < 0$.

7.4 HYPERBOLA

Exercise 7.4

Assignment

Deadline for submission: Monday August 16, 2021

- Exercises 7.1 - 9
- Exercises 7.2 - 9
- Exercises 7.3 - 5
- Exercises 7.4 - 3