Matrices and Systems of Linear Equations

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Mathematics and Statistics

for Data Science

Mathematics	Statistics
• Linear Algebra • Calculus	• Descriptive Statistics • Inferential Statistics

Math for Data Science

1. **Linear Algebra**

- Matrices – *solve systems of linear equations*
- Techniques – *Eigenvalues and Eigenvectors*

2. **Calculus**

- Differential
- Integral

Mathematics in Machine Learning

- Basic properties of matrix and vectors: scalar multiplication, linear transformation, transpose, conjugate, rank, determinant
- Inner and outer products, matrix multiplication rule and various algorithms, matrix inverse
- Special matrices: square matrix, identity matrix, triangular matrix, idea about sparse and dense matrix, unit vectors, symmetric matrix, Hermitian, skew-Hermitian and unitary matrices
- Matrix factorization concept/LU decomposition, Gaussian/Gauss-Jordan elimination, solving Ax=b linear system of equation
- Vector space, basis, span, orthogonality, orthonormality, linear least square
- Eigenvalues, eigenvectors, diagonalization, singular value decomposition

Definition and Basic Properties

A **matrix** is any rectangular (or square) array of real numbers enclosed within brackets and each number in the array is called an element of the matrix.

In general, a matrix with m rows and n columns can be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \left[a_{ij}\right]_{m\times n}$$

The matrix **A** is said to have **order (dimension)** m x n.

$$\mathbf{M} = \begin{bmatrix} 6 & 5 & 7 & -1 \\ -2 & 12 & -1 & -5 \\ -1 & -3 & 0 & 2 \end{bmatrix}_{3 \times 4}$$

Definition and Basic Properties

The results of 10 matched played by 4 football teams can be shown by a table and a matrix as

Team	Wins	Drawns	Losses
A	5	2	3
B	3	3	4
C	3	6	1
D	2	0	8

Matrix

$$\begin{bmatrix} 5 & 2 & \boxed{4} \\ 3 & 3 & 3 \\ 3 & 6 & 1 \\ 2 & 0 & 8 \end{bmatrix}$$

Three points ***P*** (1, 2), ***Q*** (4, 3) and ***R*** (-2, 7) can be represented by a table and a matrix as

Point	P	Q	R
X	1	4	-2
Y	2	3	7

Matrix

$$\begin{bmatrix} 1 & 4 & -2 \\ 2 & 3 & 7 \end{bmatrix}$$

Types of Matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \left[a_{ij}\right]_{m\times n}$$

A matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a **column matrix** or column vector of order 3 x 1.

A matrix $\begin{bmatrix} -2 & 3 & 0 & 4 \end{bmatrix}$ is a **row matrix** or row vector of order 1 x 4.

Types of Matrix

A matrix $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -4 \\ 7 & 0 & 14 \end{bmatrix}$ is a **square matrix** of order 3. (number of rows = number of columns)

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}_{2\times 2} \qquad \begin{bmatrix} x & y & z & 1 \\ a & b & c & 1 \\ p & q & r & 1 \\ m & n & o & 1 \end{bmatrix}_{4\times 4}$$

a) **symmetric matrix,** $a_{ij} = a_{ji}$, i.e. A = A^T

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

b) **skew-symmetric matrix,** $a_{ij} = -a_{ji}$ **, i.e.** the element on the leading diagonal must be zero.

$$\begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix}$$

Types of Matrix

c) unit matrix or identity matrix – if **A** is diagonal and the diagonal elements are equal to 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ is the unit matrix of order 2.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ is the unit matrix of order 3.

d) diagonal matrix - if the only non-zero elements are found on the leading diagonal

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

Types of Matrix

Transpose of a matrix

If the rows and columns of a matrix are interchanged, the matrix is **transposed**;

the **transpose of a matrix A** is written **A^T**.

Let matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 4 & -7 \end{bmatrix}$, then $\mathbf{A}^T = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & -7 \end{bmatrix}$

$3 \times 2 \longrightarrow 2 \times 3$

Remark: $(A^T)^T = A$ and **AA^T** is always a square matrix.

Properties of Transpose

Let A and B be matrices and c be a scalar. Assume that the sizes of the matrices are such that the operations can be performed.

1. $(A + B)^T = A^T + B^T$ Transpose of a sum
2. $(cA)^T = cA^T$ Transpose of a scalar multiple
3. $(AB)^T = B^T A^T$ Transpose of a product
4. $(A^T)^T = A$.

Matrix Operations

Equal Matrix

Two matrices are equal if they are of the same size and if their corresponding elements are equal. Thus **A**= **B** if they are of the same size, and $a_{ij}=b_{ij}$ for all *i* and *j*.

Let matrix **A** = $[a_{ij}]_{m\times n}$ and matrix **B** = $[b_{ij}]_{m\times n}$, say that **A = B** if $a_{ij}= b_{ij}$ for all *i, j*.

$$\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 2x+1 & y \\ z+2 & w-1 \end{bmatrix}$$

$$\begin{aligned} 2x+1 &= 5 \\ x &= (5-1)/2 \\ x &= 2 \end{aligned} \qquad \begin{aligned} z+2 &= 4 \\ z &= 4-2 \\ z &= 2 \end{aligned} \qquad y = -3 \qquad \begin{aligned} w-1 &= -2 \\ w &= -1 \end{aligned}$$

#

Note:

Two matrices can be equal only if they have the same dimensions.

Matrix Operations

Addition and subtraction of matrices

Two matrices can be added or subtracted if and only if they have the **same order**.
The corresponding numbers or elements in each matrix are added or subtracted.

Definition: Let A and B be matrices of the same size. Their sum A + B is the matrix obtained by adding together the corresponding elements of A and B. The matrix A + B will be of the same size as A and B. If A and B are not of the same size they cannot be added, and we say that the sum does not exist. Thus if C = A + B then $c_{ij} = a_{ij} + b_{ij}$

A + B $= [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 3 & 7 \end{bmatrix}$$

A - B $= [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -1 & -3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \end{bmatrix}$$

Matrix Operations

Multiplication by scalar

Definition: Let **A** be a matrix and *c* be a scalar. The scalar multiple of **A** by *c*, denoted *c***A**, is the matrix obtained by multiplying every element of **A** by *c*. The matrix *c***A** will be the same size as **A**. Thus if **B** = *c***A**, $b_{ij}=ca_{ij}$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 6 & -1 \end{bmatrix}$$

$$3\mathbf{A} = 3 \times \begin{bmatrix} 2 & 1 & 3 \\ 5 & 6 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 9 \\ 15 & 18 & -3 \end{bmatrix}$$

$(-1)A$

$$-\mathbf{A} = \begin{bmatrix} -2 & -1 & -3 \\ -5 & -6 & 1 \end{bmatrix}$$

Matrix Operations

Multiplication of matrices

The product can be symbolized as **A x B = C** or, more simply, as **AB = C.** If the product exists then **the number of columns of A is equal to the number of rows of *B*.**

Definition: Let the number of columns in a matrix A be the same as the number of rows in a matrix B. The product AB then exists. The element in row i and column j of AB is obtained by multiplying the corresponding elements of row i of A and column j of B and adding the products. If the number of columns in A does not equal the number of rows in B, the product does not exist.

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{CD} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1(-3)+2(1)+3(-1) & 1(2)+2(5)+3(2) \\ -1(-3)+0(1)+4(-1) & -1(2)+0(5)+4(2) \end{bmatrix} = \begin{bmatrix} -4 & 18 \\ -1 & 6 \end{bmatrix}$$

$$\mathbf{DC} = \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3(1)+2(-1) & -3(2)+2(0) & -3(3)+2(4) \\ 1(1)+5(-1) & 1(2)+5(0) & 1(3)+5(4) \\ -1(1)+2(-1) & -1(2)+2(0) & -1(3)+2(4) \end{bmatrix} = \begin{bmatrix} -5 & -6 & -1 \\ -4 & 2 & 23 \\ -3 & -2 & 5 \end{bmatrix}$$

$\boxed{\textbf{Note: } \mathbf{CD \neq DC}}$

Matrix Operations

Properties of Matrix Operations

Let **A**, **B**, and **C** be matrices and r and s be scalars. Assume that the sizes of the matrices are such that the operations can be performed

Properties of Matrix Addition and Scalar Multiplication

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
3. $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$ (*where **0** is the appropriate zero matrix*)
4. $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$
5. $(r + s)\mathbf{C} = r\mathbf{C} + s\mathbf{C}$
6. $r(s\mathbf{C}) = (rs)\mathbf{C}$

Properties of Matrix Multiplication

1. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
2. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
3. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
4. $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ (*where **I** is the appropriate identity matrix*)
5. $r(\mathbf{AB}) = (r\mathbf{A})\mathbf{B} = \mathbf{A}(r\mathbf{B})$

Note: $\mathbf{AB} \neq \mathbf{BA}$ in general. *Multiplication of matrices is not commutative.*

Matrix Operations

Properties of Matrix Operations

Caution

In algebra we know that the following cancellation laws apply.

- If $ab = ac$ and $a \neq 0$ then $b = c$.
- If $pq = 0$ then $p = 0$ or $q = 0$.

However, the corresponding **results are not true for matrices**.

- $\mathbf{AB} = \mathbf{AC}$ does not imply that $\mathbf{B} = \mathbf{C}$.
- $\mathbf{PQ} = \mathbf{0}$ does not imply that $\mathbf{P} = \mathbf{0}$ or $\mathbf{Q} = \mathbf{0}$.

Class Assignment

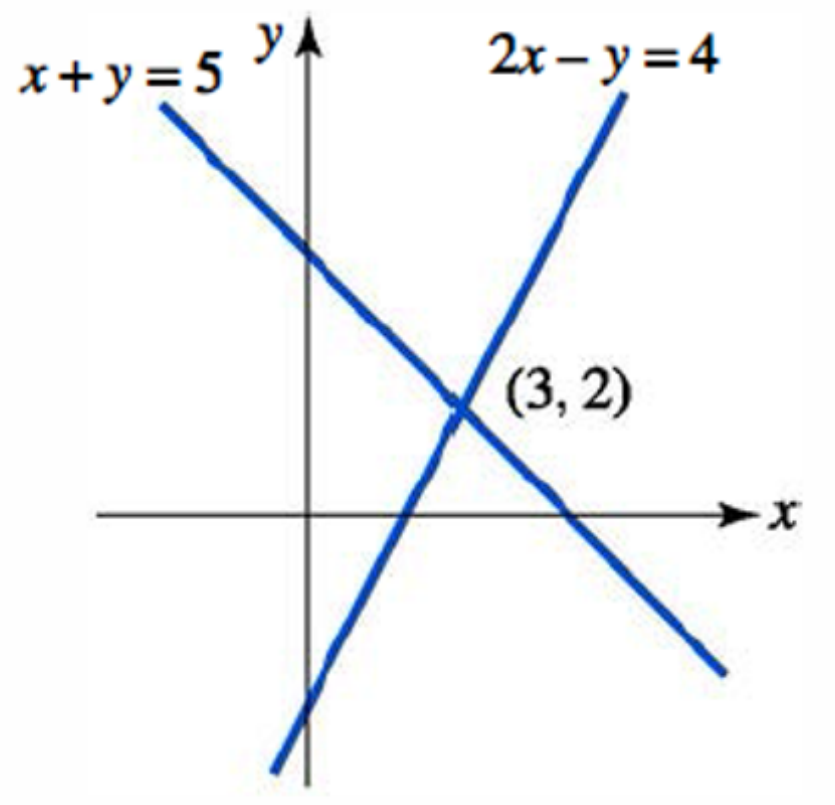
Matrices and Solving System of Linear Equations Using Gauss-Jordan Elimination

$y = mx + b$ - linear equation

<u>Unique solution</u>

$$x + y = 5$$
$$2x - y = 4$$

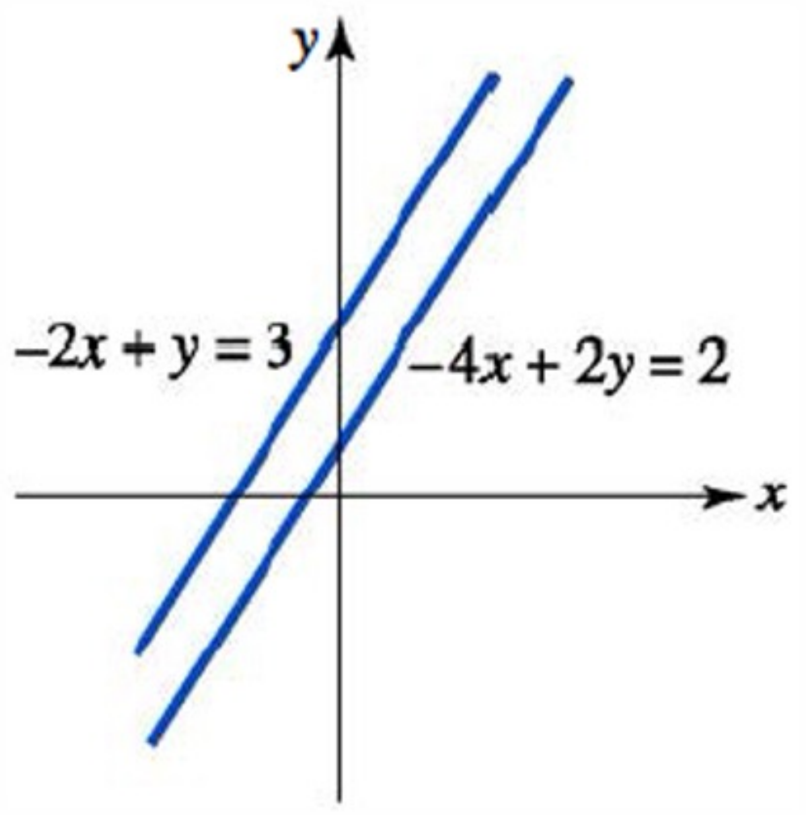
Write as $y = -x + 5$ and $y = 2x - 4$. The lines have slopes – 1 and 2, and y-intercepts 5 and – 4. They intersect at a point, the solution. There is a unique solution, $x = 3,\ y = 2$.



<u>No solution</u>

$$-2x + y = 3$$
$$-4x + 2y = 2$$

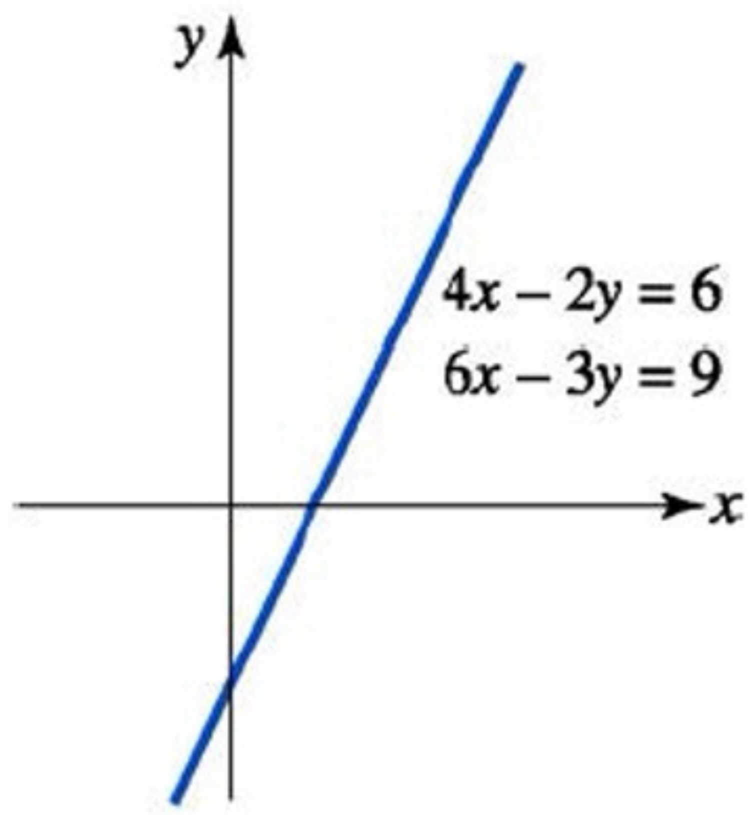
Write as $y = 2x + 3$ and $y = 2x + 1$. The lines have slopes 2, and y-intercepts 3 and 1. They are parallel. There is no point of intersection. No solution.



<u>Many solutions</u>

$$4x - 2y = 6$$
$$6x - 3y = 9$$

Each equation can be written as $y = 2x - 3$. The graph of each equation is a line with slope 2 and y-intercept –3. Any point on the line is a solution. Many solutions.



Matrices and Solving System of Linear Equations Using Gauss-Jordan Elimination

Consider the system of linear equations with *m* equations and *n* variable:

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{gathered}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad \text{or} \quad \mathbf{AX} = \mathbf{B}.$$

Where $\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ is called coefficient matrix, $\mathbf{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ is called constant matrix. $\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is called variable matrix, and

Matrices and Solving System of Linear Equations Using Gauss-Jordan Elimination

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

matrix of coefficients

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

augmented matrix

Matrices and Solving System of Linear Equations Using Gauss-Jordan Elimination

GAUSS-JORDAN ELIMINATION

Step 1.	Choose the leftmost nonzero column and use appropriate row operations to get a 1 at the top.
Step 2.	Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.
Step 3.	Repeat step 1 with the **submatrix** formed by (mentally) deleting the row used in step 2 and all rows above this row.
Step 4.	Repeat step 2 with the **entire matrix**, including the mentally deleted rows. Continue this process until the entire matrix is in reduced form.

Note: If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop, since we will have a contradiction: $0 = n$, $n \neq 0$. We can then conclude that the system has no solution.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

solution

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{array}\right] \begin{array}{l} \to x_1 \\ \to x_2 \\ \to x_3 \end{array}$$

Matrices and Solving System of Linear Equations Using Gauss-Jordan Elimination

- ***The procedure of elementary row operations consist of***
  1. Interchange any 2 rows i and j .
  2. Multiply any row of the matrix by a nonzero constant.
  3. Replace a row by the sum of that row and a constant multiple of some other row.

- ***Operation That Produce Row-Equivalent Matrices***

  An augmented matrix is transformed into a row-equivalent matrix by performing any of the following row operations:
  1. Two rows are interchanged $(R_i \leftrightarrow R_j)$.
  2. A row is multiplied by a nonzero constant $(kR_i \rightarrow R_i)$.
  3. A constant multiple of one row is added to another row $(R_i + kR_i \rightarrow R_i)$.

[**Note:** The arrow $\rightarrow$ means "replaces."]

Matrix Inversion by Row Operations: Transform the augmented matrix

Given matrix A

$$[\mathbf{A} \mid \mathbf{B}], \quad \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

By successive elementary row operations to

solution of the system of linear equations

$$[\mathbf{I} \mid \mathbf{X}], \quad \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & x_1 \\ 0 & 1 & \cdots & 0 & x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_n \end{array}\right]$$

Homogeneous Systems of Linear Equations

A system of linear equations is said to be **homogeneous** if all the constants are zero.

$$\begin{aligned} x_1 + x_2 \qquad\quad - x_4 &= 0 \\ x_1 + 2x_2 - x_3 + 2x_4 &= 0 \\ -x_1 - 2x_2 + 2x_3 - 3x_4 &= 0 \end{aligned}$$

Theorem:
A homogeneous system of linear equations in n variables always has the solution $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. This solution is called the **trivial solution**.

Inverse of the Square Matrix

Division of matrices cannot be defined but an operation similar in effect to division is that of multiplication of a matrix by its inverse.

If **A** and **B** are two square matrices such that **AB = I** , where **I** is a unit matrix, then matrix **B** is called the **inverse matrix** of **A** and is written as $\mathbf{A}^{-1}$. We could also claim that **A** was the inverse of matrix **B** and write **A** as $\mathbf{B}^{-1}$.

More correctly **B** is the right-hand inverse of **A** and **A** is the left-hand inverse of **B** so that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Definition: Let **A** be an $n \times n$ matrix. If a matrix **B** can be found such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, then **A** is said to be invertible and **B** is called an inverse of matrix **A,** written as $\mathbf{A}^{-1}$. If such a matrix **B** does not exist, then **A** has no inverse.

Inverse of the Square Matrix

$$\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{MI} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{IM} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

MI = IM = M.

If **M** is a square matrix and if there exists a matrix $\mathbf{M}^{-1}$ (read **M** inverse) such that

$\mathbf{M}^{-1}\mathbf{M} = \mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$ then $\mathbf{M}^{-1}$ is the inverse of **M**.

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}} \times \underset{\substack{B \\ (A^{-1})}}{\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$\overset{B\,(A^{-1})}{\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}}$ is the inverse of $\overset{A}{\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}}$.

Inverse of the Square Matrix

Finding the inverse of a 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boldsymbol{A^{-1}} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \mathrm{ad} - \mathrm{bc} \neq 0$$

If ***ad* – *bc* = 0**, then **A^{-1} does not exist**, and said that **A is singular matrix.**

Inverse of the Square Matrix

Finding the inverse of a 2 × 2 matrix

$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $A^{-1} = ?$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{1(5) - 3(2)}\begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= -1\begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

Checking the answer:

$A^{-1} A = A A^{-1} = I$

$$\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -5+6 & -15+15 \\ 2-2 & 6-5 \end{bmatrix} = \begin{bmatrix} -5+6 & 3-3 \\ -10+10 & 6-5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Gauss- Jordan elimination

Let $\mathbf{A} = [\, a_{ij} \,]_{n\times n}$ be any square matrix, the inverse of **A** can be perform by the following algorithm:

1. Write the augmented matrix **[A | I]**.
2. Use the procedure of *row operations* on matrix **[A | I]** in order to transform the coefficients in matrix **[A | I]** into the form of the matrix **[I | B]**.
3. Then $\mathbf{A}^{-1} = \mathbf{B}$.

Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Gauss- Jordan elimination

Matrix Inversion by Row Operations: Transform the augmented matrix

[A | I]

$$
\left[\begin{array}{cccc|cccc}
\multicolumn{4}{c|}{\overleftrightarrow{\text{Given matrix } \mathbf{A}}} & & & & \\
a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1
\end{array}\right]
$$

By successive row operations to

[I | B]

$$
\left[\begin{array}{cccc|cccc}
 & & & & \multicolumn{4}{c}{\overleftrightarrow{\text{Inverse matrix } \mathbf{A}}} \\
1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\
0 & 1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & b_{n1} & b_{n2} & \cdots & b_{nn}
\end{array}\right]
$$

Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Gauss- Jordan elimination

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 10 & -1 \\ -2 & -6 & 5 \end{bmatrix}$$

$$[\mathbf{A} \mid \mathbf{I}] \quad \begin{array}{cc} & \begin{array}{cccccc} 1 & & 0 & & & \end{array} \\ \begin{array}{c} \\ \\ 0 \end{array} & \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 3 & 10 & -1 & 0 & 1 & 0 \\ -2 & -6 & 5 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ R_2 - 3R_1 \rightarrow R_2 \\ R_3 + 2R_1 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & \underline{1} & 5 & -3 & 1 & 0 \\ 0 & 0 & \underline{1} & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \rightarrow R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -17 & 10 & -3 & 0 \\ 0 & 1 & 5 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + 17R_3 \rightarrow R_1 \\ R_2 - 5R_3 \rightarrow R_2 \\ \\ \end{array}$$

$$[\mathbf{I} \mid \mathbf{B}]. \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 44 & -3 & 17 \\ 0 & 1 & 0 & -13 & 1 & -5 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 44 & -3 & 17 \\ -13 & 1 & -5 \\ 2 & 0 & 1 \end{bmatrix}$$

Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Procedure of Row Operations

$$\textbf{Answer checking: } \mathbf{AA}^{-1} = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 10 & -1 \\ -2 & -6 & 5 \end{bmatrix} \begin{bmatrix} 44 & -3 & 17 \\ -13 & 1 & -5 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 44-39-4 & -3+3-0 & 17-15-2 \\ 132-130-2 & -9+10-0 & 51-50-1 \\ -88+78+10 & 6-6+0 & -34+30+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Inverse Matrix and Solving System of Linear Equations

Theorem:
Let $\mathbf{AX} = \mathbf{Y}$ be a system of n linear equations in n variables. If $\mathbf{A}$ is nonsingular matrix, that is $\mathbf{A}^{-1}$ exist. The solution $\mathbf{X}$ can be solve by the matrix equation $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$.

$$\begin{aligned} x + 3y &= 7 \\ 2x + 5y &= 12 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix} \quad \text{or } \mathbf{AX} = \mathbf{B} \quad \text{where } \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{5-6}\begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore, the solutions are $x = 1$ and $y = 2$

Inverse Matrix and Solving System of Linear Equations

$$\begin{aligned} x_1 - x_2 - 2x_3 &= 1 \\ 2x_1 - 3x_2 - 5x_3 &= 3 \\ -x_1 + 3x_2 + 5x_3 &= -2 \end{aligned}$$

[A | I]

$$\left[\begin{array}{rrr|rrr} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & -3 & -5 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ -R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 2R_2 \end{array}$$

[I | B]

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 5 & -3 & -1 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$$

This system can be written in the following matrix form

$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

So the solution to this system is

$x_1=1$, $x_2=-2$, $x_3=1$

Class Assignment

1. Solve the following systems of three equations in three variables using the method of Gauss- Jordan elimination.

$$\begin{aligned} -2x_1 - 2x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 - 6x_3 &= 2 \\ -2x_1 + 3x_2 + 2x_3 &= 6 \end{aligned}$$

2. Solve the following systems of three equations in three variables by determining the inverse of the matrix of coefficients and then using matrix multiplication.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ x_1 \qquad\quad - x_3 &= -2 \\ 2x_1 - x_2 \qquad\quad &= 0 \end{aligned}$$

$$\begin{aligned} 3x_1 - 2x_2 + x_3 &= 14 \\ -x_1 + x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 - x_3 &= 2 \end{aligned}$$

Assignment 4

Due: 20 August

Determine the inverse of each of the following matrices, if it exits, using the method of Gauss-Jordan elimination.

$$\begin{bmatrix} 3 & 2 \\ -6 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Solve the following systems of three equations in three variables by determining the inverse of the matrix of coefficients and then using matrix multiplication.

$$\begin{aligned} 3x_1 + 5x_2 - x_3 &= -7 \\ x_1 + x_2 + x_3 &= -1 \\ 2x_1 \quad + 11x_3 &= 7 \end{aligned} \qquad \begin{aligned} 3x_1 - 4x_2 + x_3 &= 2 \\ 2x_1 - 3x_2 + x_3 &= 1 \\ x_1 - 2x_2 + 3x_3 &= 2 \end{aligned}$$