Basic Mathematics and Statistics

CHAPTER 2: BASIC ALGEBRAIC OPERATIONS

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2.1 Operations and Properties of Real Numbers

2.1.1 Subtraction and Division

Note that a and b are ***real numbers***.

a - b = a + (-b)

example:

7 – 5 = 7 + (-5) = 2

0 – 4 = 0 + (-4) = -4

- 2 – 5 = - 2 + (-5) = -7

a ÷ b = $\frac{a}{b} = a \times \frac{1}{b}$ (b $\neq 0$) ***if denominator is "zero", the answer can not be defined!!***

example:

$6 \div 3 = \frac{6}{3} = 6 \times \frac{1}{3} = 2$

$5 \div 2 = \frac{5}{2} = 5 \times \frac{1}{2} = 2.5$

$0 \div 3 = \frac{0}{3} = 0 \times \frac{1}{3} = 0$

$2 \div 0 = \frac{2}{0} = \boldsymbol{undefined!!}$

2.1 Operations and Properties of Real Numbers

2.1.2 Properties of Negatives

$- \times + = -$

$- \times - = +$

$- \div + = -$

$- \div - = +$

1. $-(-a) = a$
2. $(-a)b = -(ab) = a(-b) = -ab$
3. $(-a)(-b) = ab$
4. $(-1)a = -a$
5. $\dfrac{-a}{b} = -\dfrac{a}{b} = \dfrac{a}{-b}, b \neq 0$
6. $\dfrac{-a}{-b} = -\dfrac{-a}{b} = -\dfrac{a}{-b} = \dfrac{a}{b}, b \neq 0$

<u>example</u>: $-(-4) = 4$

$(-2)5 = -(2 \times 5) = 2(-5) = -10$

$-\dfrac{-2}{5} = \dfrac{2}{5}$

$\dfrac{-2}{-5} = -\dfrac{-2}{5} = -\dfrac{2}{-5} = \dfrac{2}{5}$

2.1 Operations and Properties of Real Numbers

2.1.3 Zero Properties

Anything multiply with zero will get zero.

$-4 \times \mathbf{0} = \mathbf{0}$

if $\mathrm{a} \times b = \mathbf{0}$, either a or b is ‘0’ or both can be ‘0’

1. $a \times 0 = 0 \times a = 0$
2. $ab = 0$ if and only if $a = 0$ or $b = 0$ or both

2.1 Operations and Properties of Real Numbers

2.1.4 Fraction Properties

Example:

1. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$ — $\frac{2}{3} = \frac{6}{9} \rightarrow 2\times 9 = 6\times 3$

2. $\frac{ka}{kb} = \frac{a}{b}$ — $\frac{7\times 3}{7\times 5} = \frac{3}{5}$

3. $\frac{a}{b}\times\frac{c}{d} = \frac{ac}{bd}$ — $\frac{3}{5}\times\frac{7}{8} = \frac{3\times 7}{5\times 8} = \frac{21}{40}$

4. $\frac{a}{b}\div\frac{c}{d} = \frac{a}{b}\times\frac{d}{c}$ — $\frac{2}{3}\div\frac{5}{7} = \frac{2}{3}\times\frac{7}{5} = \frac{14}{15}$

5. $\frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$ — $\frac{3}{6}+\frac{4}{6} = \frac{3+4}{6} = \frac{7}{6}$

6. $\frac{a}{b}-\frac{c}{b} = \frac{a-c}{b}$ — $\frac{7}{8}-\frac{2}{8} = \frac{7-2}{8} = \frac{5}{8}$

7. $\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$ — $\frac{2}{3}+\frac{1}{5} = \frac{2(5)+3(1)}{3(5)} = \frac{10+3}{15} = \frac{13}{15}$

2.1 Operations and Properties of Real Numbers

$$\frac{8}{9} - \frac{4}{5}$$

$$= \frac{(8\times5)-(4\times9)}{9\times5}$$

$$= \frac{40-36}{45}$$

$$= \frac{4}{45}$$

$$\left(-\frac{\cancel{2}^{\,1}}{3}\right)\left(\frac{5}{\cancel{6}_{\,3}}\right)$$

$$= -\frac{1\times5}{3\times3}$$

$$= -\frac{5}{9}$$

$$\frac{11}{5} \div \frac{1}{3}$$

$$= \frac{11}{5} \times \frac{3}{1}$$

$$= \frac{11\times3}{5\times1}$$

$$= \frac{33}{5}$$

$$\frac{3}{1} - \frac{5}{2}$$

$$= \frac{(3\times2)-(5\times1)}{1\times2}$$

$$= \frac{6-5}{2}$$

$$= \frac{1}{2}$$

Exercise

- Exercises 2.1 - No 1 to 12

2.2 Exponents and Radicals

Exponent - x^n

exponent

base

x to power of n

x is real number and n =1, 2, 3, …

$$x^n = x \times x \times \cdots \times x \ (n \ times)$$

2 x 2 x 2 x 2 x 2 = 2^5 = 32

-5 x -5 x -5 = $(-5)^3$ = -125

$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

$x^0 = 1$

Radical - $x^{\frac{1}{n}} = \sqrt[n]{x}$

index

radical

radicand

nth root of x

x is real number and n is integer

Note: x cannot be negative when n is even

$9^{\frac{1}{2}} = \sqrt[2]{9} = \sqrt[2]{3^2} = 3$

$8^{\frac{1}{3}} = \sqrt[3]{8} = \sqrt[3]{2^3} = 2$

$9^{-\frac{1}{2}} = \frac{1}{\sqrt[2]{9}} = \frac{1}{\sqrt[2]{3^2}} = \frac{1}{3}$

$2^{\frac{2}{3}} = \sqrt[3]{2^2} = \sqrt[3]{4}$

2.2 Exponents and Radicals

Properties of integer exponents

For m, n integers and $a,\ b \in R$ then

1. $a^m a^n = a^{m+n}$

2. $\left(a^m\right)^n = a^{mn}$

3. $(ab)^m = a^m b^m$

4. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m} \quad ; \quad b \neq 0$

5. $\dfrac{a^m}{a^n} = a^{m-n} \quad ; \quad a \neq 0$

6. $a^0 = 1 \quad , \quad a^{-n} = \dfrac{1}{a^n}$

2.2 Exponents and Radicals

Properties of integer exponents

Example - Simplify using exponent properties, and express answers using positive exponents only.

(a) $(3a^5)(2a^{-3})$

$= (3 \times 2)(a^5 a^{-3})$

$= 6a^2$

(b) $\dfrac{6x^{-2}}{8x^{-5}}$

$= \dfrac{3x^{-2}x^5}{4}$

$= \dfrac{3x^3}{4}$

(c) $-4y^3 - (-4y)^3$

$= -4y^3 - (-4)^3 y^3$

$= -4y^3 - (-64)y^3$

$= -4y^3 + 64y^3$

$= 60y^3$

(d) $(2a^{-3}b^2)^{-2}$

$= 2^{-2} a^{-3(-2)} b^{2(-2)}$

$= \dfrac{a^6 b^{-4}}{2^2}$

$= \dfrac{a^6}{4b^4}$

2.2 Exponents and Radicals

Properties of integer exponents

Example - Simplify

$$\frac{3(2^n)-4(2^{n-2})}{2^n-2^{n-1}} \quad \begin{matrix} \rightarrow \mathit{take\ the\ common\ factor\ out} \rightarrow 2^n \\ \rightarrow \mathit{take\ the\ common\ factor\ out} \rightarrow 2^n \end{matrix}$$

$$=\frac{2^n(3)-2^n(4\times 2^{-2})}{2^n(1-2^{-1})}$$

$$=\frac{2^n(3-4\times\frac{1}{2^2})}{2^n(1-\frac{1}{2})}$$

$$=\frac{3-4\times\frac{1}{4}}{\frac{1}{2}} \qquad =\frac{3-1}{\frac{1}{2}} \qquad =2\times\frac{2}{1} \qquad =4$$

2.2 Exponents and Radicals

Rational Exponents and Radicals

Definition: For m, n integers and $a \in R$ (except a cannot be negative when n is even), then

1. $a^{m/n} = \left(a^{1/n}\right)^m = \left(a^m\right)^{1/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

2. $a^{-m/n} = \dfrac{1}{a^{m/n}}$

2.2 Exponents and Radicals

Properties of radicals

For n integers number greater than 1, and x, y positive real numbers. Then

1. $\sqrt[n]{x^n} = x$

2. $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$

3. $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$

2.2 Exponents and Radicals

Properties of radicals

example-

$$\left(\frac{4x^{1/3}}{x^{1/2}}\right)^{1/2} = \frac{4^{1/2}x^{1/6}}{x^{1/4}} = \frac{2}{x^{1/4}x^{-1/6}} = \frac{2}{x^{1/4-1/6}} \boxed{= \frac{2}{x^{1/12}}}$$

$$\frac{6}{\sqrt{2x}} = \frac{6}{\sqrt{2x}} \times \frac{\sqrt{2x}}{\sqrt{2x}} = \frac{6\sqrt{2x}}{2x} \boxed{= \frac{3\sqrt{2x}}{x}}$$

$$\sqrt[3]{\frac{8x^4}{y}} = \frac{8^{1/3}x^{4/3}}{y^{1/3}} = \frac{2xx^{1/3}}{y^{1/3}} = 2x\left(\frac{x}{y}\right)^{1/3} \boxed{= 2x\sqrt[3]{\frac{x}{y}}}$$

2.2 Exponents and Radicals

Rationalizing Denominators

$\frac{\sqrt{3}}{\sqrt{2}+1}$ - we need to eliminate the radical in the denominator

$\frac{\sqrt{3}}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$ *Multiply with rationalizing factor*

$\rightarrow (\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$

$= \frac{\sqrt{3}(\sqrt{2}-1)}{2-1}$

$= \frac{\sqrt{3\times2}-\sqrt{3}}{1}$

$=\sqrt{6} - \sqrt{3}$

Exercise

- Exercises 2.2 - No 1 to 4

2.3 Polynomials

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n$$

$$2x - 3, \quad 4x^2 - 3x + 7, \quad x - 2y, \quad x^3 - 3x^2y + xy^2 + 2y^7$$

Polynomials are classified according to their degree –

Monomial: $5x^2y^3$

Binomial: $x^3 + 4$

Trinomial: $x^4 - \sqrt{2}x^2 + 9$

Algeberaic expressions: $\sqrt[3]{x^3+5}$, $5x^4 + 2x^2 - 7$, $\dfrac{x-5}{x^2+2x-5}$.

Polynomial

2.3 Polynomials

$example$-

$$(x^4 - 3x^3 + x^2) + (-x^3 - 2x^2 + 3x) + (3x^2 - 4x - 5)$$

$$x^2 - 8 - (4x^2 - 3x + 5)$$

$$(2x - 3)(3x^2 - 2x + 3)$$

2.4 Factoring

The factors of 12 includes: 1, 2, 3, 4, 6 and 12

6 = 2 x 3 → ***2 and 3 are the factors of 6***

8 = 2 x 2 x 2 (8=2x4=2x2x2)

20 = 2 x 2 x 5 (20=2x10=2x2x5)

24 = 2 x 2 x 2 x 3 (24=2x12=2x2x6=2x2x2x3)

$x^2 - 4 = (\mathrm{x} - 2)(\mathrm{x} + 2)$ → $\boldsymbol{(x-2)}$ ***and*** $\boldsymbol{(x+2)}$ ***are the factors of*** $\boldsymbol{x^2 - 4}$

2.5 Factoring for polynomials

2.5.1 Common Factors

A common factor is a factor that is shared between two different numbers. It can also be referred to as a common divisor.

The factors of 16 include: 1, 2, 4, 8, and 16.

The factors of 12 include: 1, 2, 3, 4, 6, and 12.

Thus, the common factors of 16 and 12 are: 1, 2, and 4.

The Greatest Common Factor (GCF) = 4

Try this:

Common factors of 18 and 24?

GCF of 18 and 24?

2.5 Factoring for polynomials

2.5.1 Common Factors

Finding the Greatest Common Factor (GCF)
Step 1. Factor. Write each number in prime factored form.
Step 2. List common factors. List each prime number that is a factor of every number in the list
Step 3. Choose smallest exponents. Use as exponents on the common prime factors the smallest exponent from the prime factored forms. (If a prime dose not appears in one of the prime factored forms, it cannot appear in the greatest common factor.)
Step 4. Multiply. Multiply the primes from Step 3. If there are no primes left after Step 3., the greatest common factor is 1.

3m + 12

= (3 x m) + (3 x 4) = **3** (m + 4) → ***3 is the GCF***

2.5 Factoring for polynomials

2.5.2 Factoring by groups

Factor by grouping
Step 1. Group terms. Collect the term into two groups so that each group has common factor. **Step 2. Factors with in groups.** Factor out the greatest common factor from each group. **Step 3. Factor the entire polynomial.** Factor the common binomial factor form the result of Step 2. **Step 4. If necessary, rearrange terms.** If step 2 does not result in a common binomial factor, try a different grouping.

$xy + ax + 2by + 2ab$

$= x(y+a) + 2b(y+a)$

$= (x+2b)\,(y+a)$

Exercise

- Exercises 2.3 - No 1 to 35

2.5 Factoring for polynomials

2.5.3 Factoring Second-Degree Polynomials

- **Special Factoring Formulas**

1. Perfect Square	$(a+b)^2 = a^2 + 2ab + b^2$
2. Perfect Square	$(a-b)^2 = a^2 - 2ab + b^2$
3. Difference of Squares	$a^2 - b^2 = (a-b)(a+b)$
4. Difference of Cubes	$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$
5. Sum of Cubes	$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

Examples -

$x^2 - 4 = x^2 - 2^2 = (x-2)(x+2)$

$16x^2 - 9 = (4x)^2 - 3^2 = (4x-3)(4x+3)$

$(x+3y)^2 = x^2 + 2(x)(3y) + (3y)^2 = x^2 + 6xy + 9y^2$

$(3x^2 - 4y)^2 = (3x^2)^2 - 2(3x^2)(4y) + (4y)^2 = 9x^4 - 24x^2y + 16y^2$

2.6 Product of two Binomials (FOIL Method)

$$(ax + by)\,(cx + dy) = acx^2 + (ad + bc)\,xy + bdy^2$$

First

Outer

$$(ax + by)(cx + dy) = acx^2 + \underbrace{adxy + bcxy}_{\Downarrow} + bdy^2$$

Inner

Last

$$= acx^2 + (ad + bc)\,xy + bdy^2$$

2.6 Product of two Binomials (FOIL Method)

Example -

$$(3x - 2a)(2x - 6a) = (3x)(2x) + (3x)(-6a) + (-2a)(2x) + (-2a)(-6a)$$

$$= 6x^2 - 18ax - 4ax + 12a^2$$

$$= \mathbf{6x^2 - 22ax + 12a^2}$$

Exercise

- Exercises 2.4 - No 1 to 45

2.7 Factoring Trinomials

Trinomial is produced when two binomials of the 1st degree are multiplied.

$$x^2 + 8x + 15 = (x + .?.)(x + .?.)$$

$$= (x + 3)(x + 5)$$

factors of **+15** = $1 \times 15 = 16$

$-1 \times -15 = -16$

$3 \times 5 = 8$

$-3 \times -5 = -8$

Which pair?

- Check middle term: **+8x**
- Pick the pair that gives +8

2.7 Factoring Trinomials

Guidelines for Factoring Trinomials:

1. If the last term is *positive*, then the factor will look like $(+)(+)$ or $(-)(-)$
   The + or – sign is determined by the coefficient of middle term.

2. If the last term is *negative*, then the factor will look like $(+)(-)$ or $(-)(+)$

Examples -

$2x^2 + 7x + 6 = (2x + 3)(x + 2)$

$x^2 + 4x - 5 = (x + 5)(x - 1)$

$5x^2 - 16x + 3 = (5x - 1)(x - 3)$

$6x^2 - x - 12 = (3x + 4)(2x - 3)$

2.7 Factoring Trinomials

2.7.1 Perfect-Square Trinomials

A trinomial is a perfect square if it can be written as the square of binomial.

Then Perfect-Square Trinomials can be obtained as follow:

Perfect-Square Trinomials:
$a^2 + 2ab + b^2 = (a+b)^2$ $a^2 - 2ab + b^2 = (a-b)^2$

2.7 Factoring Trinomials

2.7.1 Perfect-Square Trinomials

Example -

$x^2 + 4xy + 4y^2$

The first term x^2 and the last term $4y^2 = (2y)^2$ are both perfect squares.

The middle term is $2(x)(2y) = 4xy$.

Therefore, $x^2 + 4xy + 4y^2 = \mathbf{(x + 2y)^2}$

2.7 Factoring Trinomials

2.7.1 Perfect-Square Trinomials

Example -

$9x^2 - 30xy + 25y^2$

The first term: $9x^2 = (3x)^2$

the last term: $25y^2 = (5y)^2$

The middle term: $-2(3x)(5y) = -30xy$.

$9x^2 - 30xy + 25y^2 = (3x)^2 - 2(3x)(5y) + (5y)^2 = \mathbf{(3x - 5y)^2}$

2.7 Factoring Trinomials

2.7.2 General Trinomials

General Trinomials:
$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$
$acx^2 + (ad + bc)xy + bdy^2 = (ax + by)(cx + dy)$

Example- $2x^2 - 5x + 3 = (2x - 3)\,(x - 1)$

$2x^2 = 2x$ and x

Next, look for a product of the coefficient of last term = 3

and have the coefficient of middle term = $-5x$.

$$-3x$$

$$(2x - 3)\,(x - 1)$$

$$-2x$$

2.7 Factoring Trinomials

2.7.2 General Trinomials

Example- $9x^2 - 219x + 72$

Take common factor out

$3\,(3x^2 - 73x + 24)$

$$-x$$

$$(3x - 1)\ (x - 24)$$

$$-72x$$

$$9x^2 - 219x + 72 = 3(3x - 1)\,(x - 24)$$

Exercise

- Exercises 2.5 - No 1 to 50

2.8 Rational Expressions: Basic Operations

2.8.1 Multiplication and Division

If a, b, c and d are real numbers with b, *d ≠ 0, then*

1. $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$

e.g. $\dfrac{2}{3} \times \dfrac{x}{x-1} \quad \boxed{= \dfrac{2x}{3(x-1)}}$

2. $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c}, \quad c \neq 0$

e.g. $\dfrac{2}{3} \div \dfrac{x}{x-1} \quad = \dfrac{2}{3} \times \dfrac{x-1}{x}$

$$\boxed{= \dfrac{2(x-1)}{3x}}$$

2.8 Rational Expressions: Basic Operations

2.8.2 Adding and Subtracting with the same denominators

If a, b and c are real numbers with b $\neq$ *0, then*

1. $\frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$ e.g. $\frac{x}{x-3}+\frac{2}{x-3} \boxed{=\frac{x+2}{x-3}}$

2. $\frac{a}{b}-\frac{c}{b} = \frac{a-c}{b}$ e.g. $\frac{x}{x-3}-\frac{2}{x-3} \boxed{=\frac{x-2}{x-3}}$

2.8 Rational Expressions: Basic Operations

2.8.2 Adding and Subtracting with the different denominators

Different denominators → use the **least common denominator (LCD)**

↓

find **prime factorizations**

Example-

$$\frac{4\times x}{4\times 3} - \frac{x\times 3}{4\times 3} + \frac{5x\times 2}{6\times 2} = \frac{4x-3x+10x}{12} \boxed{= \frac{11x}{12}}$$

LCD of 3, 4 and 6 → 3 x 2 x 2 = 12

Prime factorizations for 3, 4 and 6:

3 = 1 x 3

4 = 2 x 2

6 = 2 x 3

$$\begin{array}{r|l} 3 & 3, 4, 6 \\ \hline 2 & 1, 4, 2 \\ \hline & 1, 2, 1 \end{array}$$

2.8 Rational Expressions: Basic Operations

2.8.2 Adding and Subtracting with the different denominators

Example-

$$\frac{1}{x}-\frac{1}{x+1}+\frac{1}{x+2}$$

$$\boxed{\text{LCD of } x,(x+1),(x+2) \text{ is } x(x+1)(x+2)}$$

$$=\frac{(x+1)(x+2)-x(x+2)+x(x+1)}{x(x+1)(x+2)}$$

$$=\frac{x^2+3x+2-x^2-2x+x^2+x}{x(x+1)(x+2)}$$

$$=\frac{x^2+2x+2}{x(x+1)(x+2)}$$

$$\frac{y}{y^2-3y+2}-\frac{y+3}{y^2-1}=\frac{y}{(y-1)(y-2)}-\frac{y+3}{(y-1)(y+1)}$$

$$y^2-3y+2=(y-1)(y-2)$$

$$y^2-1=(y-1)(y+1)$$

$$\boxed{\text{LCD}=(y-1)(y-2)(y+1)}$$

$$=\frac{y(y+1)-(y+3)(y-2)}{(y-1)(y-2)(y+1)}$$

$$=\frac{y^2+y-(y^2+y-6)}{(y-1)(y-2)(y+1)}$$

$$=\frac{y^2+y-y^2-y+6}{(y-1)(y-2)(y+1)}$$

$$=\frac{6}{(y-1)(y-2)(y+1)}$$

Exercise

- Exercises 2.6 - No 1 to 10

Assignment

Deadline for submission: next week ***Monday***

- Exercises 2.2 -
- Exercises 2.3 –
- Exercises 2.4 –
- Exercises 2.5 –
- Exercises 2.6 –