

#### 8.1 Definition and Basic Properties

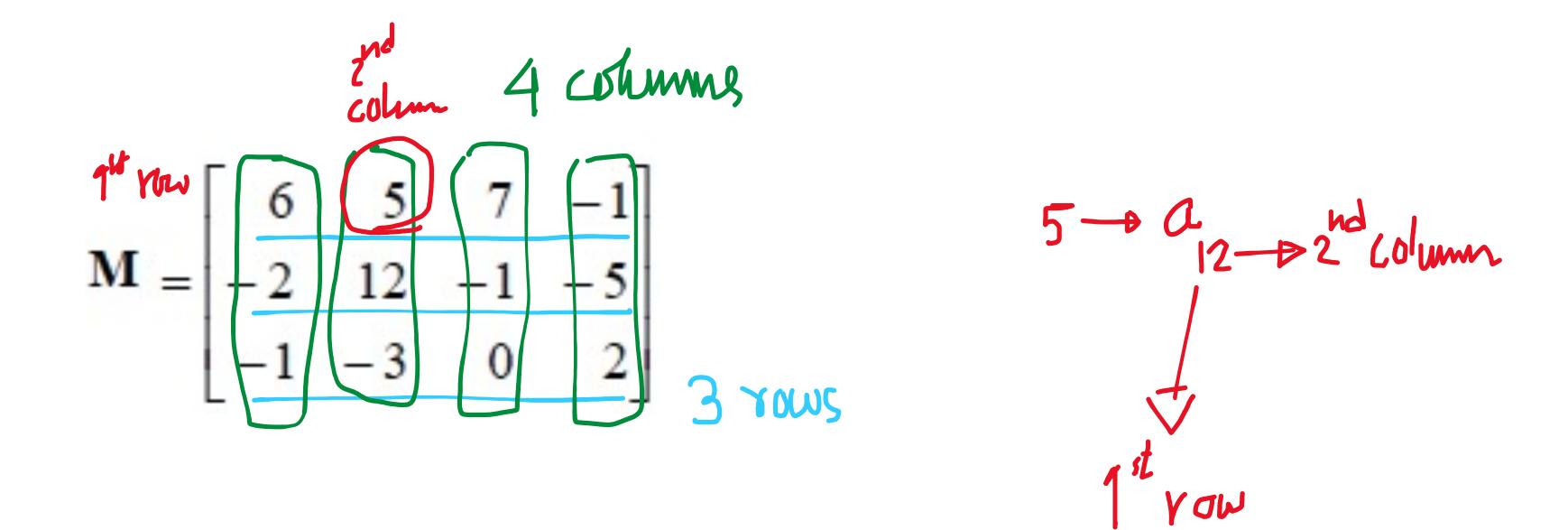
A matrix is any rectangular (or square) array of real numbers enclosed within brackets and each number in the array is called an element of the matrix.

In general, a matrix with *m* rows and *n* columns can be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n}$$

The matrix **A** is said to have **order (dimension)**  $m \times n$ .

### 8.1 Definition and Basic Properties



The array of numbers may be the basis of a table of data that can be organized into a spreadsheet.

### 8.1 Definition and Basic Properties

The results of 10 matched played by 4 football teams can be shown by a table and a matrix as

Team	Wins	Drawns	Losses
A	5	2	3
В	3	3	4
C	3	6	1
D	2	0	8

Three points P(1, 2), Q(4, 3) and R(-2, 7) can be represented by a table and a matrix as

Point	P	Q	R	_
X	1	4	-2	
Y	2	3	7	L

A matrix  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  is a column matrix or column vector of order 3 x 1.

A matrix  $[-2 \ 3 \ 0 \ 4]$  is a row matrix or row vector of order 1 x 4.

A matrix 
$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -4 \\ 7 & 0 & 14 \end{bmatrix}$$
 is a square matrix of order 3. (number of rows = number of columns)

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}_{2 \times 2} \qquad \begin{bmatrix} x & y & z & 1 \\ a & b & c & 1 \\ p & q & r & 1 \\ m & n & o & 1 \end{bmatrix}_{4 \times 2}$$

If the rows and columns of a matrix are interchanged, the matrix is transposed;

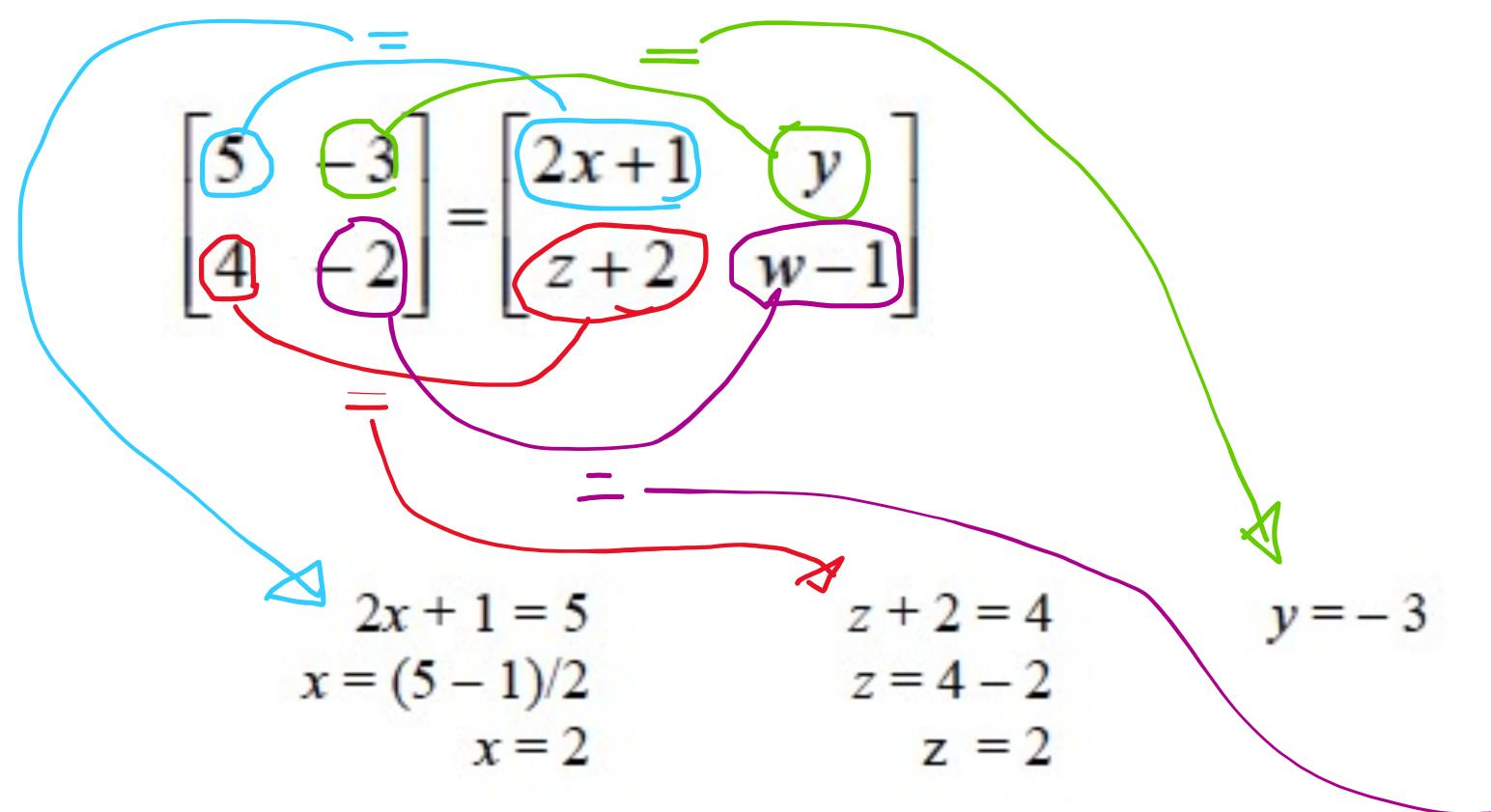
the transpose of a matrix A is written A<sup>T</sup>.

Let matrix 
$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 4 & -7 \end{bmatrix}$$
, then  $\mathbf{A}^{T} = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & -7 \end{bmatrix}$ 

**Remark:**  $(A^T)^T = A$  and  $AA^T$  is always a square matrix.

#### **Equal Matrix**

Let matrix  $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n}$  and matrix  $\mathbf{B} = \begin{bmatrix} b_{ij} \end{bmatrix}_{m \times n}$ , say that  $\mathbf{A} = \mathbf{B}$  if  $a_{ij} = b_{ij}$  for all i, j.



#### Note:

Two matrices can be equal only if they have the same dimensions.

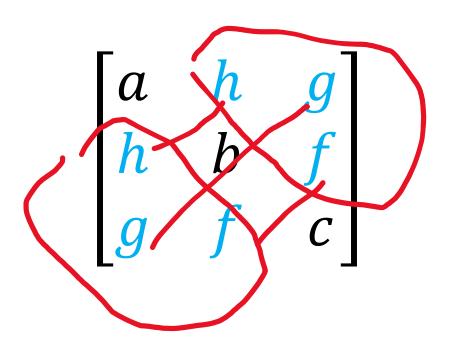
$$w - 1 = -2$$

$$w = -1$$
#

Square matrices are particularly important; they are divided into several types. We consider matrices of order 3 and 4 but the properties apply to all orders.

If A is a square matrix then

• symmetric matrix -  $a_{ij} = a_{ji}$ , i.e.  $A = A^T$ 



• skew-symmetric matrix -  $a_{ij}$  =  $-a_{ji}$ , i.e. the element on the leading diagonal must be zero.

diagonal matrix - if the only non-zero elements are found on the leading diagonal

$$egin{bmatrix} a & 0 & 0 & 0 \ 0 & b & 0 & 0 \ 0 & 0 & c & 0 \ 0 & 0 & 0 & d \end{bmatrix}$$

unit matrix or identity matrix - if A is diagonal and the diagonal elements are equal to 1

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$
 is the unit matrix of order 2.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \text{ is the unit matrix of order 2}$$

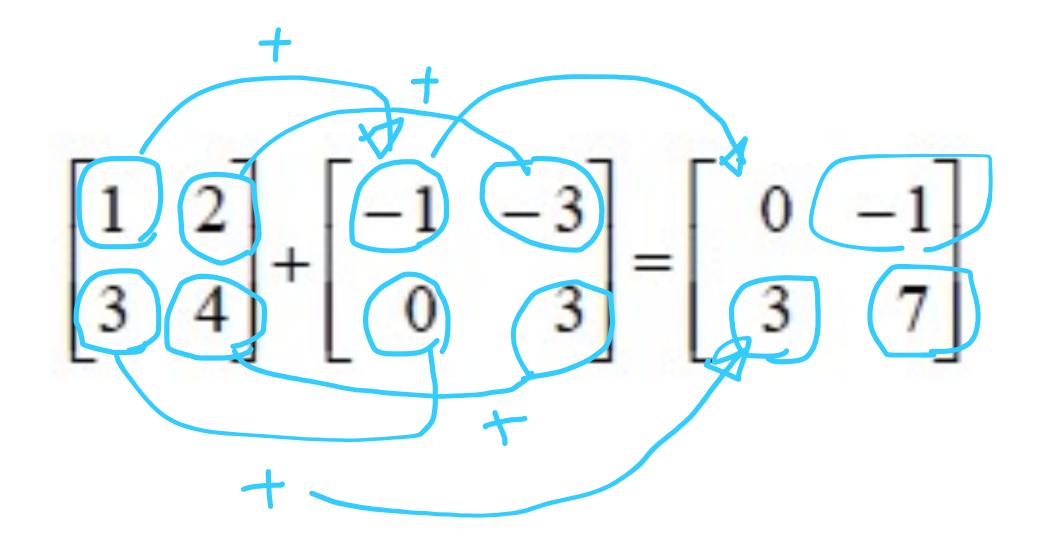
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \text{ is the unit matrix of order 3.}$$

#### Addition and subtraction of matrices

Two matrices can be added or subtracted if and only if they have the **same order**. The corresponding numbers or elements in each matrix are added or subtracted.

**A** + **B** = 
$$[a_{ij}]$$
 +  $[b_{ij}]$  =  $[a_{ij} + b_{ij}]$ 

**A** - **B** = 
$$[a_{ij}]$$
 -  $[b_{ij}]$  =  $[a_{ij} - b_{ij}]$ 



$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -1 & -3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

#### Multiplication

 Multiplication by scalar - In the context of matrices, a scalar is just a number, as in the context of vectors. It means that each element in matrix is multiplied by the scalar.

$$kA = k[a_{ij}] = [ka_{ij}]$$

$$\begin{bmatrix}
 2 & 1 & 3 \\
 4 & 6 & -1
 \end{bmatrix}$$

$$3\mathbf{A} = 3 \times \begin{bmatrix} 2 & 1 & 3 \\ 5 & 6 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 9 \\ 15 & 18 & -3 \end{bmatrix}$$

#### Multiplication

Multiplication of matrices - The product can be symbolized as A x B = C or, more simply, as AB = C. If the product exists then the number of columns of A is equal to the number of rows of B.

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{CD} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-3) + 2(1) + 3(-1) & 1(2) + 2(5) + 3(2) \\ -1(-3) + 0(1) + 4(-1) & -1(2) + 0(5) + 4(2) \end{bmatrix} = \begin{bmatrix} -1(1) + 2(1) \\ -4 & 18 \\ -1 & 6 \end{bmatrix}$$

$$DC = \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -1 & 0 & 4 \end{bmatrix}$$

$$3 \times 2 = 2 \times 3$$

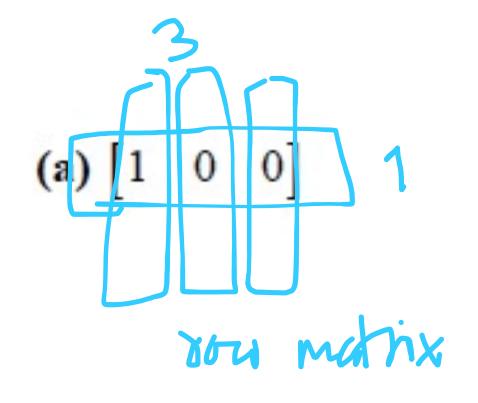
$$3 \times 3 \times 3$$

$$= \begin{bmatrix} -3(1) + 2(-1) & -3(2) + 2(0) & -3(3) + 2(4) \\ 1(1) + 5(-1) & 1(2) + 5(0) & 1(3) + 5(4) \\ -1(1) + 2(-1) & -1(2) + 2(0) & -1(3) + 2(4) \end{bmatrix} = \begin{bmatrix} -5 & -6 & -1 \\ -4 & 2 & 23 \\ -3 & -2 & 5 \end{bmatrix}$$

**Note:** CD ≠ DC



Exercise 8.1



(b) 
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(c) 
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$
Square

(d) 
$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

2. Refer to the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 0 & 6 \end{bmatrix}; \ \mathbf{B} = \begin{bmatrix} 2 & -6 & 5 \\ 7 & 3 & 0 \end{bmatrix}; \ \mathbf{C} = \begin{bmatrix} -6 & -7 & 0 \\ 5 & 2 & 1 \end{bmatrix}; \ \mathbf{D} = \begin{bmatrix} 1 & 0 & 2 \\ -4 & 5 & 0 \\ 2 & -1 & 3 \end{bmatrix}$$

Calculate each of the following (where possible):

(a) A + B

(b) A - C

(c) C - A

(d) B + C

(e) -3A

(f) -2A - C

(g) 2B + C

(h) Verify A + B = B + A

(i) Verify A + (B - C) = (A + B) - C

(j) A + D

(k) B - A

(l) 2A

(m) AB

(n) AC

(o) AD

(p) DB

(q) D<sup>2</sup>

7. 
$$\begin{bmatrix} x & 0 & a+b \\ -4 & a & y \end{bmatrix} = \begin{bmatrix} 6 & 0 & -5 \\ -4 & 2 & 3 \end{bmatrix}$$

23. 
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -3 & -5 & -6 \\ 6 & 1 & 7 & 5 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 6 & 10 \\ 7 & 12 \\ 8 & 11 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Division of matrices cannot be defined but an operation similar in effect to division is that of multiplication of a matrix by its inverse.

If **A** and **B** are two square matrices such that  $\mathbf{AB} = \mathbf{I}$ , where **I** is a unit matrix, then matrix **B** is called the **inverse matrix** of **A** and is written as  $\mathbf{A}^{-1}$ . We could also claim that **A** was the inverse of matrix **B** and write **A** as  $\mathbf{B}^{-1}$ .

More correctly **B** is the right-hand inverse of **A** and **A** is the left-hand inverse of **B** so that  $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}\mathbf{A} = \mathbf{I}$ .

$$\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$MI = IM = M$$
.

$$\mathbf{MI} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{IM} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

If M is a square matrix and if there exists a matrix M<sup>-1</sup> (read M inverse) such that

 $\mathbf{M}^{-1} \mathbf{M} = \mathbf{M} \mathbf{M}^{-1} = \mathbf{I}$  then  $\mathbf{M}^{-1}$  is the inverse of  $\mathbf{M}$ .

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding the inverse of a 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{ad} - bc \neq 0$$

If ad - bc = 0, then A-1 does not exist, and said that A is singular matrix.

#### Finding the inverse of a 2 × 2 matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$
,  $A^{-1} = ?$ 

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{1(5) - 3(2)} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= -1 \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

#### **Checking the answer:**

$$A^{-1}A = AA^{-1} = I$$

$$\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -5+6 & -15+15 \\ 2-2 & 6-5 \end{bmatrix} = \begin{bmatrix} -5+6 & 3-3 \\ -10+10 & 6-5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

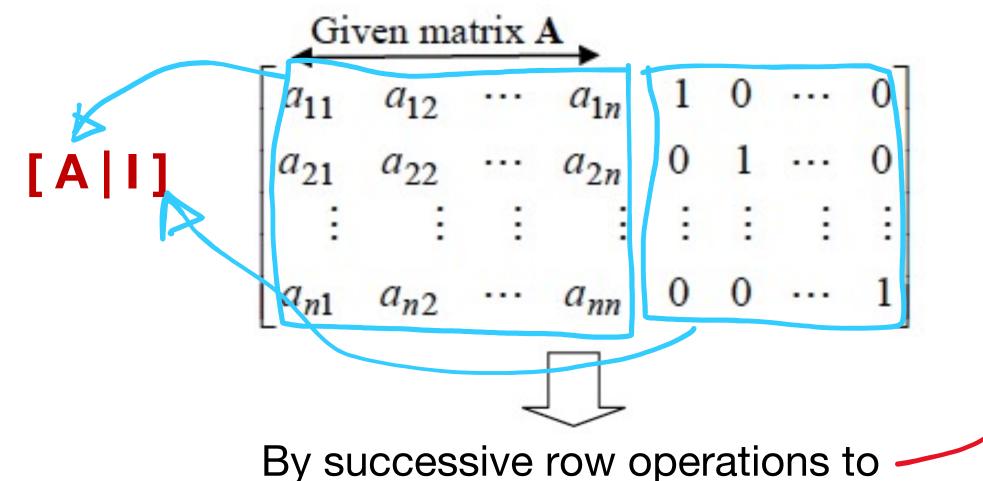
Finding Inverse of a  $n \times n$  Matrix Using Procedure of Row Operations

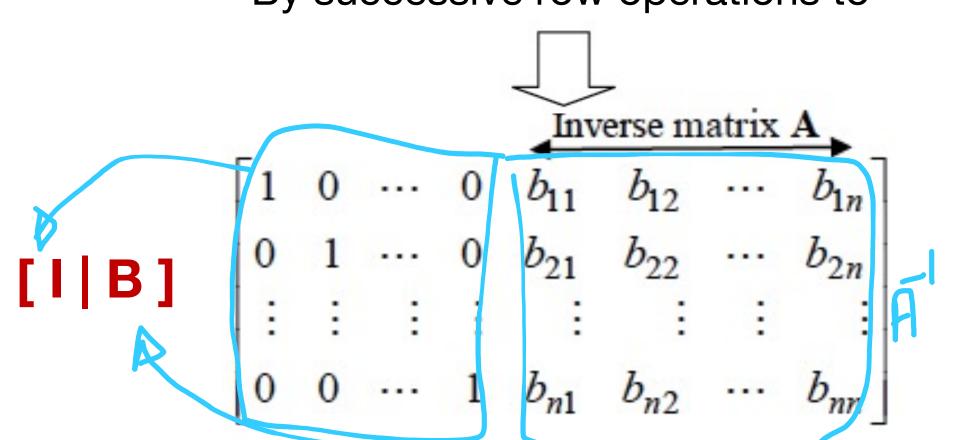
Let  $\mathbf{A} = [a_{ij}]_{n \times n}$  be any square matrix, the inverse of  $\mathbf{A}$  can be perform by the following algorithm:

- 1. Write the augmented matrix [A I].
- 2. Use the procedure of *row operations* on matrix [A | I ] in order to transform the coefficients in matrix [A | I ] into the form of the matrix [I | B].
- 3. Then  $A^{-1} = B$ .

Finding Inverse of a  $n \times n$  Matrix Using Procedure of Row Operations

Matrix Inversion by Row Operations: Transform the augmented matrix





- The procedure of row operations consists of
- 1. Interchange any 2 rows i and j.
- 2. Multiply any row of the matrix by a nonzero constant.
- 3. Replace a row by the sum of that row and a constant multiple of some other row.

#### Finding Inverse of a $n \times n$ Matrix Using Procedure of Row Operations

Operation That Produce Row-Equivalent Matrices

An augmented matrix is transformed into a row-equivalent matrix by performing any of the following row operations:

- A. Two rows are interchanged ( $R_i \leftrightarrow R_i$ )
- B. A row is multiplied by a nonzero constant. ( $kR_i \rightarrow R_i$ )
- C. A constant multiple of one row is added to another row ( $R_i + kR_i \rightarrow R_j$ )

[Note: The arrow → means "replaces."]

Finding Inverse of a  $n \times n$  Matrix Using Procedure of Row Operations

Answer checking, 
$$AA^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -5 + 6 & 3 - 3 \\ -10 + 10 & 6 - 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\begin{vmatrix} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 2R_1 - R_2 \rightarrow R_2 \\ 2(1) - 2 = 2 - 2 = 0 \end{vmatrix}$$

$$\begin{vmatrix} 2(1) - 2 = 2 - 2 = 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 5 & 3 \\ 2 & -1 \end{vmatrix}$$

Finding Inverse of a  $n \times n$  Matrix Using Procedure of Row Operations

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 10 & -1 \\ -2 & -6 & 5 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 10 & -1 \\ -2 & -6 & 5 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{A} \mid \mathbf{I} \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \mid 1 & 0 & 0 \\ 3 & 10 & 1 & 0 & 1 & 0 \\ -2 & -6 & 5 \mid 0 & 0 & 1 \end{bmatrix} R_2 - 3R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 \mid 1 & 0 & 0 \\ 0 & 1 & 5 \mid -3 & 1 & 0 \\ 0 & 0 & 1 \mid 2 & 0 & 1 \end{bmatrix} R_1 - 3R_2 \to R_1$$

$$\begin{bmatrix} 1 & 0 & -17 \mid 10 & -3 & 0 \\ 0 & 1 & 5 \mid -3 & 1 & 0 \\ 0 & 0 & 1 \mid 2 & 0 & 1 \end{bmatrix} R_1 + 17R_3 \to R_1$$

$$\begin{bmatrix} 1 & 0 & -17 \mid 10 & -3 & 0 \\ 0 & 1 & 5 \mid -3 & 1 & 0 \\ 0 & 0 & 1 \mid 2 & 0 & 1 \end{bmatrix} R_1 + 17R_3 \to R_1$$

$$\begin{bmatrix} \mathbf{I} \mid \mathbf{B} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 44 & -3 & 17 \\ 0 & 1 & 0 & -13 & 1 & -5 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 44 & -3 & 17 \\ -13 & 1 & -5 \\ 2 & 0 & 1 \end{bmatrix}$$

Finding Inverse of a  $n \times n$  Matrix Using Procedure of Row Operations

Answer checking: 
$$AA^{-1} = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 10 & -1 \\ -2 & -6 & 5 \end{bmatrix} \begin{bmatrix} 44 & -3 & 17 \\ -13 & 1 & -5 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 44 - 39 - 4 & -3 + 3 - 0 & 17 - 15 - 2 \\ 132 - 130 - 2 & -9 + 10 - 0 & 51 - 50 - 1 \\ -88 + 78 + 10 & 6 - 6 + 0 & -34 + 30 + 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

One of the most important applications of matrices is to find the solution of the system linear equations. Consider the system of linear equations with *m* equations and *n* variable:

If **A** is nonsingualr matrix, that is **A**<sup>-1</sup> exist. The solution **X** can be solved by the matrix equation  $X = A^{-1}B$ 

$$x + 3y = 7$$

$$2x + 5y = 12$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix} \quad \text{or } \mathbf{A}\mathbf{X} = \mathbf{B} \qquad \text{where } \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{5 - 6} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore the solutions are x = 1 and y = 2

#### **Gauss-Jordan elimination**

- a step-by-step for solving system of linear equations
- works for any system of linear equations that was in large-scale
- easily implemented on a computer
- systematically transforms an augmented matrix into a reduced form

The system corresponding to a reduced augmented coefficient matrix is called a reduced system.

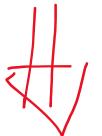
$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
  
 $a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$   
 $\vdots$   
 $a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$ 

CATICC	TODD	A DI LUI	ATTON
GAUSS-	JUKD	AIN L	ATION

- Step 1. Choose the leftmost nonzero column and use appropriate row operations to get a 1 at the top.
- Step 2. Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.
- Step 3. Repeat step 1 with the submatrix formed by (mentally) deleting the row used in step 2 and all rows above this row.
- Step 4. Repeat step 2 with the entire matrix, including the mentally deleted rows.

  Continue this process until the entire matrix is in reduced form.

Note: If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop, since we will have a contradiction: 0 = n,  $n \neq 0$ . We can then conclude that the system has no solution.



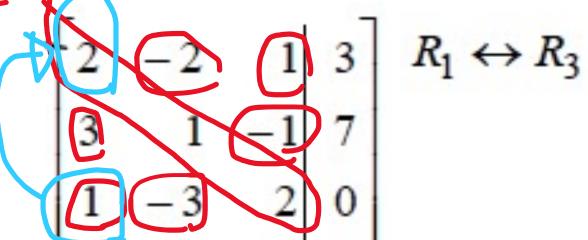
Pultion

$$\begin{pmatrix}
1 & 0 & 0 & \chi_1 & \chi_1 \\
0 & 1 & 0 & \chi_2 & \chi_2 \\
0 & 0 & 1 & \chi_3 & \chi_3
\end{pmatrix}$$

Elimination

Solve by Gauss-Jordan elimination:

$$2x_1 - 2x_2 + x_3 = 3$$
$$3x_1 + x_2 - x_3 = 7$$
$$x_1 - 3x_2 + 2x_3 = 0$$



$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{bmatrix} 0.1R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & -0.2 & 0.2 \end{bmatrix} -5R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{array}{c} 0.1R_3 + R_1 \rightarrow R_1 \\ 0.7R_3 + R_2 \rightarrow R_2 \\ 0.7R_3 + R_2 \rightarrow R_2 \end{array}$$

$$2x_{1} - 4x_{2} + x_{3} = -4 
4x_{1} - 8x_{2} + 7x_{3} = 2 
-2x_{1} + 4x_{2} - 3x_{3} = 5$$

$$\begin{bmatrix}
2 & -4 & 1 & -4 \\
4 & -8 & 7 & 2 \\
-2 & 4 & -3 & 5
\end{bmatrix}$$

$$\begin{bmatrix}
1 & -2 & 0.5 & -2 \\
0 & 0 & 1 & 2 \\
0 & 0 & -2 & 1
\end{bmatrix}$$

$$-0.5R_{2} + R_{1} \rightarrow R_{1}$$

$$\begin{bmatrix}
1 & -2 & 0.5 & -2 \\
0 & 0 & 1 & 2 \\
0 & 0 & -2 & 1
\end{bmatrix}$$

$$2R_{2} + R_{3} \rightarrow R_{3}$$

$$\begin{bmatrix} 1 & -2 & 0.5 & | & -2 \\ 4 & -8 & 7 & | & 2 \\ -2 & 4 & -3 & | & 5 \end{bmatrix} \xrightarrow{-4R_1 + R_2 \to R_2} R_2$$

$$\begin{bmatrix} 1 & -2 & 0 & | & -3 \\ 0 & 0 & 1 & | & 2 \\ 2R_1 + R_3 \to R_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -2 & 1 \end{bmatrix} \quad 0.2R_2 \to R_2$$

$$\begin{bmatrix} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \quad -0.5R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Now the last matrix is not in reduced form, since the last row produces a contradiction.

The system is inconsistent and has no solution.

$$3x_{1} + 6x_{2} + 9x_{3} = 15$$

$$2x_{1} + 4x_{2} - 6x_{3} = 10$$

$$-2x_{1} - 3x_{2} + 4x_{3} = -6$$

$$\begin{bmatrix} 3 & 6 & -9 & 15 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \end{bmatrix}$$

$$R_{2} \leftrightarrow R_{3}$$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} -2R_2 + R_1 \rightarrow R_1$$

Now the last matrix is in reduced form, since the last row produces all 0. Write the corresponding reduced system 0 1 -2 4 2 and solve. We discard the equation corresponding to the third (all 0) row in the reduced form, since it is satisfied by All values of  $x_1$ ,  $x_2$ , and  $x_3$ .

For the first row:  $x_1 + x_3 = -3$  or  $x_1 = -x_3 - 3$ 

For the second row:  $x_2 - 2x_3 = 4$  or  $x_2 = 2x_3 + 4$ 

This dependent system has an infinite number of solutions. We will use a parameter to represent all the solutions. If we let  $x_3 = t$ , then for all real number t,

$$\begin{cases} x_1 = -t - 3 \\ x_2 = 2t + 4 \\ x_3 = t \end{cases}$$
 is a solution.

Some particular solutions as: (1) when t = 0, then  $x_1 = -3$ ,  $x_2 = 4$ ,  $x_3 = 0$ (2) when t = -2, then  $x_1 = -1$ ,  $x_2 = 0$ ,  $x_3 = -2$ 

Exercise 8.2

#### 8.6 Determinants

Determinant of the matrix - assigning a real number to the any square matrix.

The determinant of a matrix A is denoted by |A| or det A.

#### 8.6.1 Finding determinants of a 2 × 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{vmatrix} A \\ A \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} 3 & -2 \\ 5 & 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -2 \\ 5 & 8 \end{vmatrix}$$

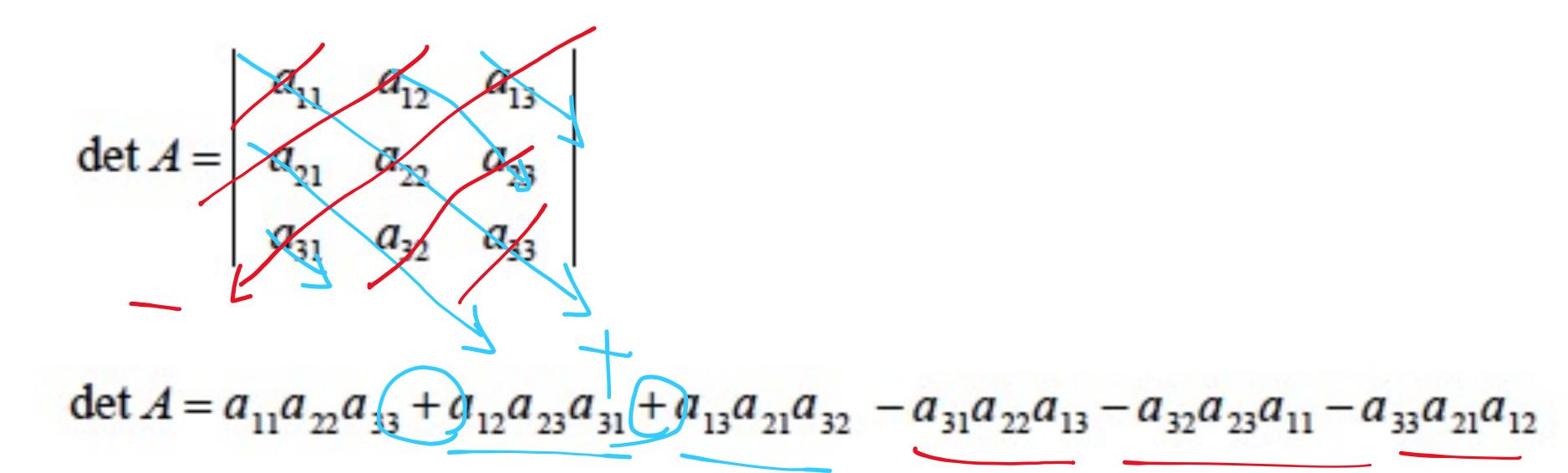
$$= (3)(8) - (-2)(5)$$

$$= 24 + 10 = 34$$

#### 8.6 Determinants

#### 8.6.2 Finding determinants of a 3 × 3 matrix

#### **Third order Determinants**

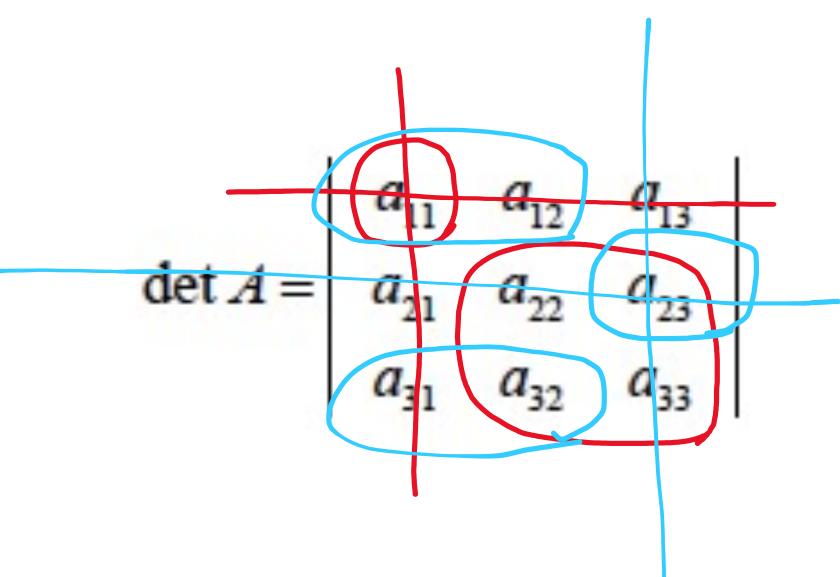


### Minor of the Matrix (M<sub>ij</sub>)

For any square matrix  $A=[a_{ij}]_{n\times m}$  minor of an element  $a_{ij}$  denoted by  $M_{ij}$  can be obtained from the original determinant by deleting the row *i* and column *j*.

The minor of 
$$a_{11}$$
 is  $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$ 

The minor of 
$$a_{23}$$
 is  $M_{23} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{13}$ 

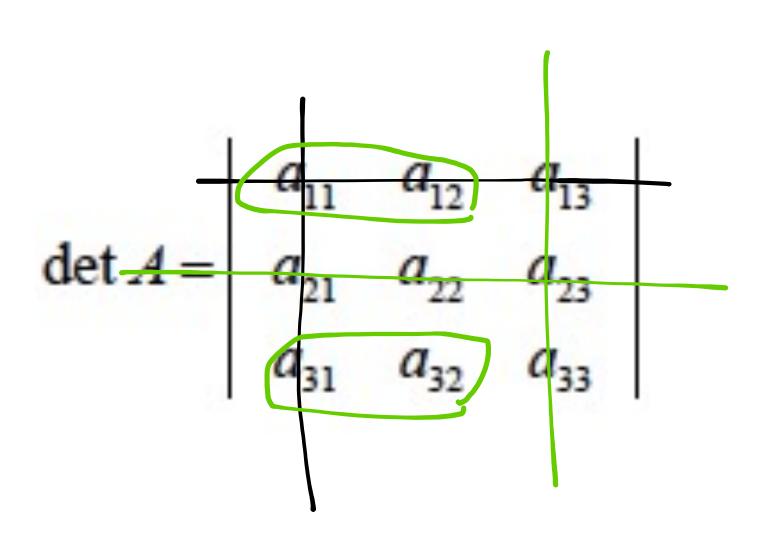


#### Cofactor of the matrix (C<sub>ii</sub>)

For any square matrix  $A=[a_{ij}]_{nxm}$  the cofactor of an element  $a_{ij}$  denoted by  $C_{ij}$  can be defined  $C_{ij}=(-1)^{i+j}M_{ij}$ .

Co-factor of 
$$a_{11}$$
 is  $C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$ 

Co-factor of 
$$a_{23}$$
 is  $C_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{32} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{32} \end{vmatrix} = -M_{23}$ 



$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If A is a 3 × 3 matrix, then the minor of the  $a_{ij}$  element (denoted by  $M_{ij}$ ) is the determinant of the 2 × 2 matrix obtained by deleting row i and column j of A, and can be use these meaning to find the determinant of the matrix for any dimension  $n \times n$ .

This can be obtained by the following expansion method.

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_1 & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

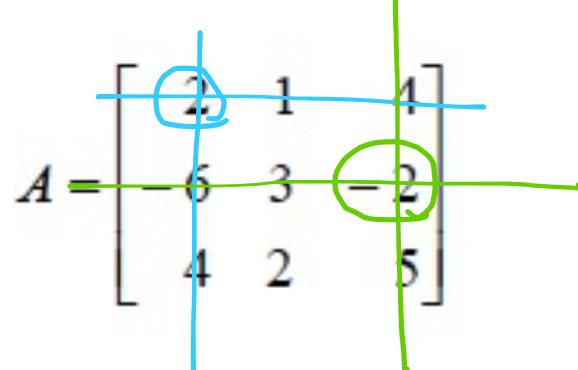
$$= a_{11}(a_{22} a_{33} - a_{32} a_{23}) + a_{12}(a_{21} a_{33} - a_{31} a_{23}) + a_{13}(a_{21} a_{32} - a_{31} a_{22})$$

Where  $C_{ij}$  is the cofactor of the element  $a_{ij}$  is defined by  $C_{ij} = (-1)^{i+j} M_{ij}$ 

Thus, the pattern of signs in front of the number  $a_{ij}$  is

We found that the determinant of a  $3 \times 3$  matrix can be found by multiplying each element of the first column by its corresponding cofactor and then adding the three results. This is called the determinant is being expanded about the first column. It can be shown that any row or column can be used to expand a determinant.

find  $M_{11}$  and  $M_{23}$ 



To find  $M_{11}$  we first delete row 1 and column 1 of A

$$M_{11} = \begin{vmatrix} 3 & -2 \\ 2 & 5 \end{vmatrix} = (3)(5) - (-2)(2) = 15 + 4 = 19$$

To find  $M_{23}$  we first delete row 2 and column 3 of A

$$M_{23} = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = (2)(2) - (1)(4) = 4 - 4 = 0$$

find |A| by expanded about the first row and the second column

$$A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

a. expanded about the first row

$$A = \begin{bmatrix} 2 & -4 & -5 \\ 1 & 0 & 4 \\ 2 & 3 & -6 \end{bmatrix}$$

$$\begin{vmatrix} A \end{vmatrix} = +(2) \begin{vmatrix} 0 & 4 \\ 3 & -6 \end{vmatrix} - (-4) \begin{vmatrix} 1 & 4 \\ 2 & -6 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 2(0 - 12) + 4(-6 - 8) - 5(3 - 0)$$
$$= -24 - 56 - 15 = -95$$

b. expanded about the second column

$$\begin{vmatrix} A \end{vmatrix} = -(-4) \begin{vmatrix} 1 & 4 \\ 2 & -6 \end{vmatrix} + 0 \begin{vmatrix} 2 & -5 \\ 2 & -6 \end{vmatrix} - (3) \begin{vmatrix} 2 & -5 \\ 1 & 4 \end{vmatrix}$$
$$= 4(-6-8) + 0 - 3(8+5) = -56 - 39 = -95$$

#### Properties of Determinants

- 1. If any row (or column) of a square matrix A contains only 0s, then |A| = 0.
- 2. If the elements of one row (or column) of a determinant are multiplied by constant k, the value of the determinant is multiplied by constant k. That is |B| = k|A|.
- $3. \quad |AB| = |A| |B|.$
- $4. \quad |A| = |A^T|.$

Exercises 8.3

### 8.7.1 Cramer's Rule (2 × 2 Case)

Determinant provide the basis for another method of solving linear systems

$$a_1 x + b_1 y = c_1$$
  
 $a_2 x + b_2 y = c_2$ 

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The solution for this system is given by

$$x = \frac{D_x}{D}$$
, and  $y = \frac{D_y}{D}$ 

### 8.7.2 Cramer's Rule (3 × 3 Case)

$$a_1x + b_1y + c_1z = d_1$$
  
 $a_2x + b_2y + c_2z = d_2$   
 $a_3x + b_3y + c_{31}z = d_3$ 

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0 \qquad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$D_X = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then 
$$x = \frac{D_x}{D}$$
,  $y = \frac{D_y}{D}$ , and  $z = \frac{D_z}{D}$ 

$$y=\frac{D_y}{D}$$
,

$$z = \frac{D_z}{D}$$

- **Remark** 1. If D = 0 and at least one of  $D_x$ ,  $D_v$ , and  $D_z$  is not zero, then the system is non consistent and has no solution.
  - 2. If D, D<sub>x</sub>,  $D_v$ , and  $D_z$  are all zero, then the equations are dependent and there are infinitely many solutions.

$$\frac{x}{2} + \frac{y}{3} = -4$$

$$\frac{x}{4} - \frac{3y}{2} = 20$$

$$3x + 4y = -24$$

$$x - 6y = 80$$

$$3x + 4y = -24$$
  
 $x - 6y = 80$ 

$$D = \begin{vmatrix} 3 & 4 \\ 1 & -6 \end{vmatrix} = -18 - 4 = -22$$

$$D_X = \begin{vmatrix} -24 & 4 \\ 80 & -6 \end{vmatrix} = 144 - 320 = -176$$

$$D_y = \begin{vmatrix} 3 & -24 \\ 1 & 80 \end{vmatrix} = 240 - (-24) = 264$$

$$X = \frac{D_X}{D} = \frac{-176}{-22} = 8$$

$$y = \frac{D_y}{D} = \frac{264}{-22} = -12$$

The solution set is  $\{(8, -12)\}$ .

$$x-2y+z = -4$$
  
 $2x + y - z = 5$   
 $3x + 2y + 4z = 3$ 

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29$$

$$D_X = \begin{vmatrix} -4 & -2 & 1 \\ 5 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29$$

$$D_y = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 5 & -1 \\ 3 & 3 & 4 \end{vmatrix} = 58$$

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29$$

$$D_X = \begin{vmatrix} -4 & -2 & 1 \\ 5 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29$$

$$D_Y = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 5 & -1 \\ 3 & 3 & 4 \end{vmatrix} = 58$$

$$D_Z = \begin{vmatrix} 1 & -2 & -4 \\ 2 & 1 & 5 \\ 3 & 2 & 3 \end{vmatrix} = -29$$

$$x = \frac{D_X}{D} = \frac{29}{29} = 1$$
  $y = \frac{D_y}{D} = \frac{58}{29} = 2$   $z = \frac{D_z}{D} = \frac{-29}{29} = 1$ 

$$y = \frac{D_y}{D} = \frac{58}{29} = 2$$

$$z = \frac{D_z}{D} = \frac{-29}{29} = 1$$

The solution set is  $\{(1, 2, -1)\}$ .

$$x + 3y - z = 4$$
  
 $3x - 2y + z = 7$   
 $2x + 6y - 2z = 1$ 

$$D = \begin{vmatrix} 1 & 3 & -1 \\ 3 & -2 & 1 \\ 2 & 6 & -2 \end{vmatrix} = 0$$

$$D = \begin{vmatrix} 1 & 3 & -1 \\ 3 & -2 & 1 \\ 2 & 6 & -2 \end{vmatrix} = 0$$

$$D_X = \begin{vmatrix} 4 & 3 & -1 \\ 7 & -2 & 1 \\ 1 & 6 & -2 \end{vmatrix} = -7$$

since D = 0 and at least one of  $D_x$ ,  $D_y$ , and  $D_z$  is not zero,

then the system is consistent. The solution set is Ø.

# Assignment

Deadline for submission: Monday August, 2021

- Exercises 8.1
- Exercises 8.2 9, 17, 29
- Exercises 8.3 6, 29
- Exercises 8.4 15
- Review exercise