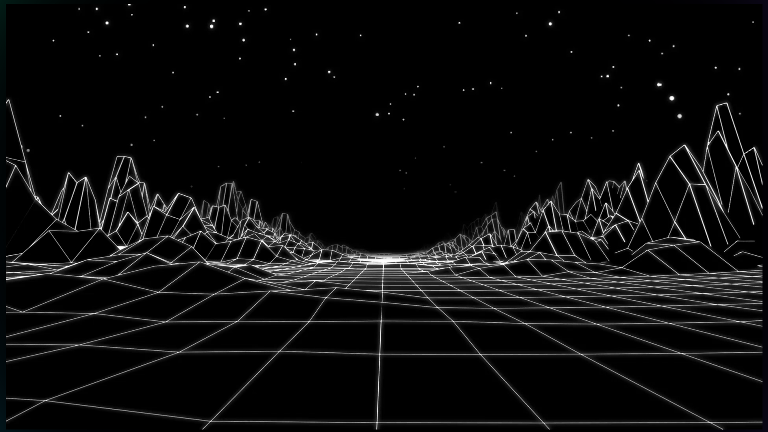
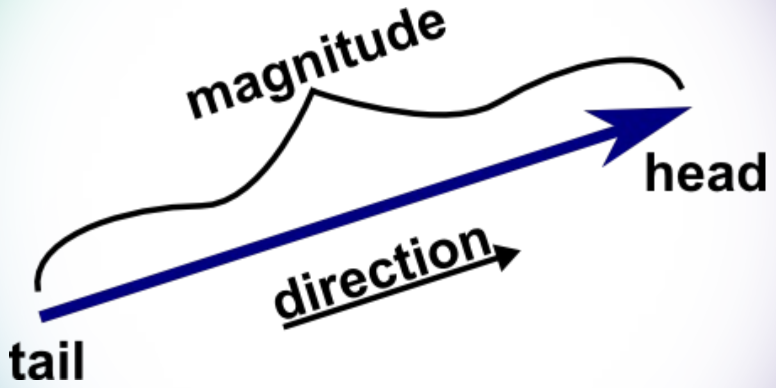
Vector Spaces R^n



DR. KHAING HTUN

What is Vector

- a **magnitude** and a ***direction***
- ***directed line segments or arrows in 2-space or 3-space***
- The tail of the arrow is called the ***initial point*** of the vector, and the head of the arrow is called the ***terminal point***.



How do we write vector

- **Symbolically**, we denote vectors in lowercase **boldface** type (for example: **u**, **v**).

$$\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$$

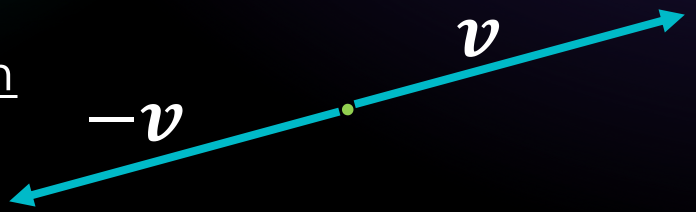
Vector v, $\vec{v} = \overrightarrow{AB}$

- **Scalars (Magnitude, ‖a‖)** is real numbers and denoted in lowercase *italic* type (for example *a*, *c*)

Special vectors

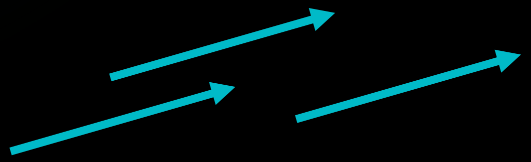
- **Zero vector (0)** - the unique vector having zero length
  - **0**=(0,0) - the zero vector in two dimensions is
  - **0**=(0,0,0) - the zero vector in three dimensions

- **Nagative vector** - the same length but opposite direction



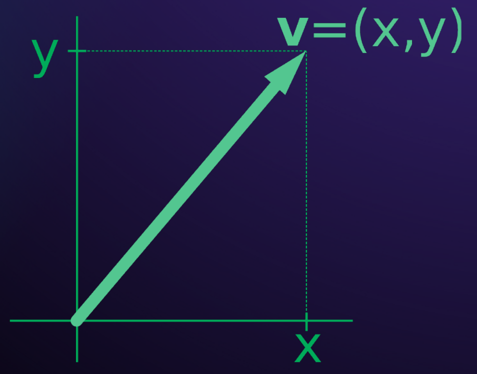
- **Equivalent vector** - the same length and same direction

$$\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w}$$

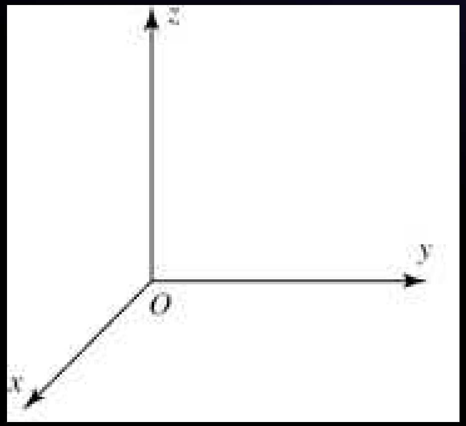


Vector in n space, R^n

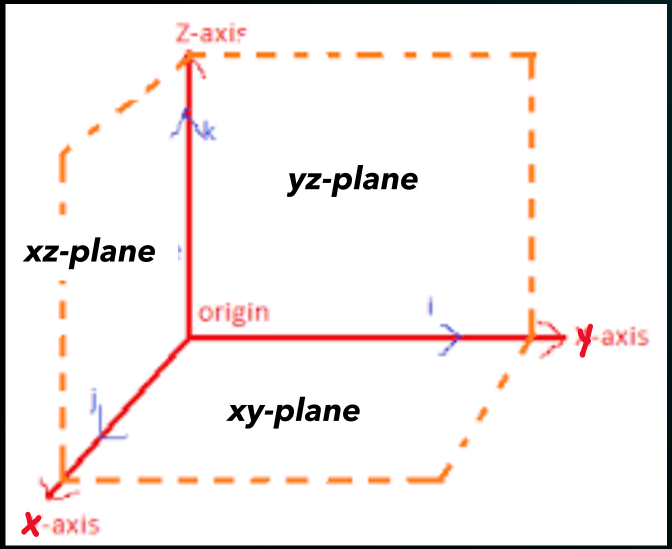
- Vectors in the plane can be described by **ordered pairs** of real numbers R^2.

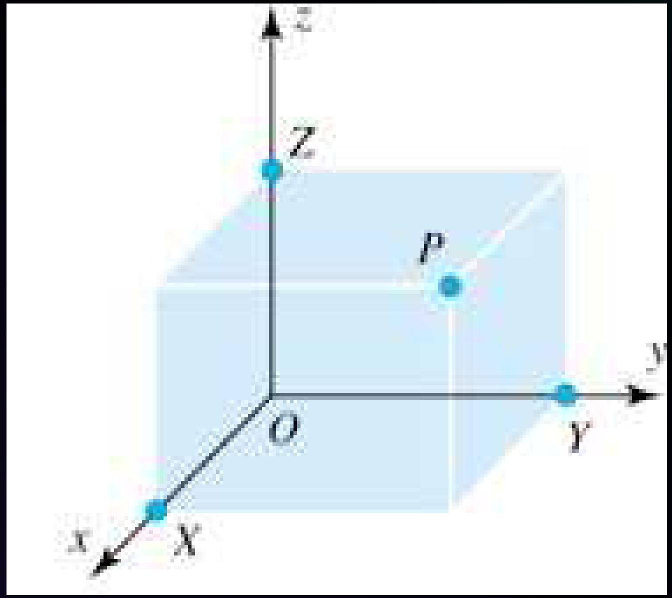


- Vectors in 3-space can be described by triples of real numbers R^3 by introducing a ***rectangular coordinate*** system



Vector in n space, R^n





To construct a ***rectangular coordinate system***,

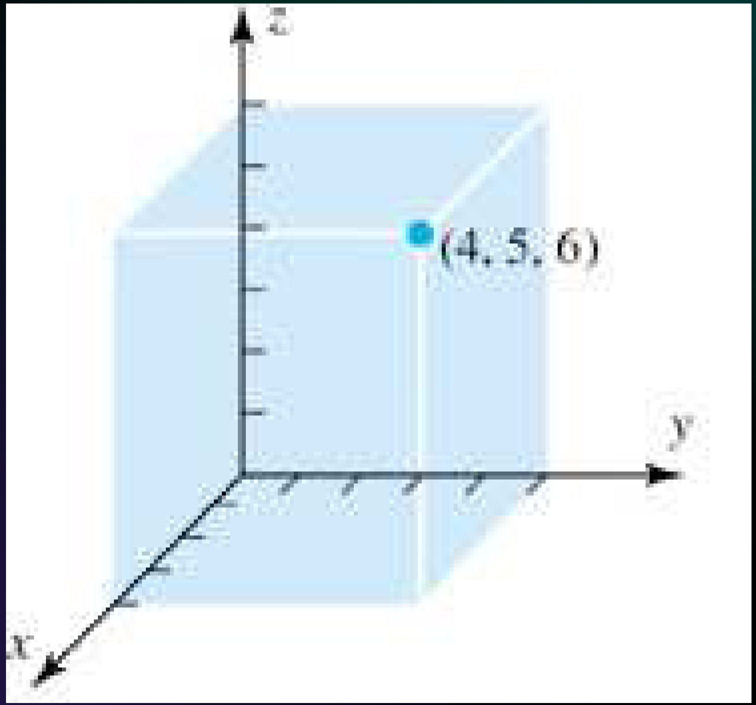
1. Select a point *O*, ***origin***, and three mutually perpendicular lines, ***coordinate axes*** (*X*, *Y*, and *Z*) passing through the origin.
2. Select a positive direction for each coordinate axis as well as a unit of length for measuring distances
3. Each pair of coordinate axes determines a plane called a ***coordinate plane***

   → ***xy-plane***, ***xz-plane***, ***yz-plane***.
4. Assign each point in 3-space with a triple of numbers (*x*, *y*, *z*), called the ***coordinates of*** *P*, **as follows**: Pass three planes through *P* parallel to the coordinate planes, and denote the points of intersection of these planes with the three coordinate axes.

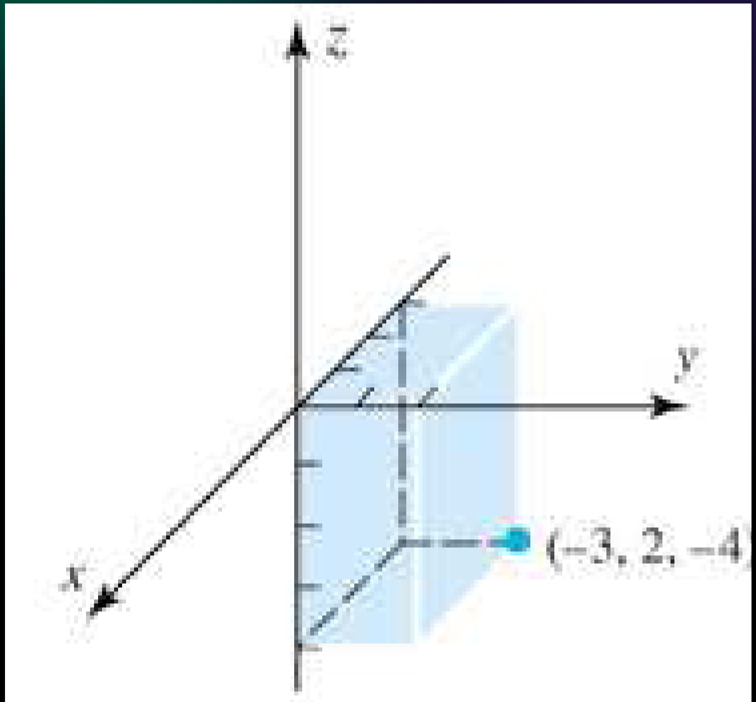
The coordinates of P are defined to be the signed lengths, x=OX, y=OY, and z=OZ

Vector in n space, R^n

(4, 5, 6)

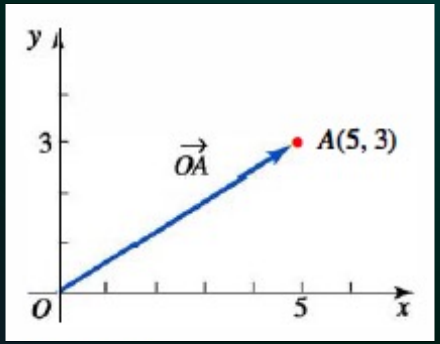


(-3, 2, - 4)



Vector in R^2

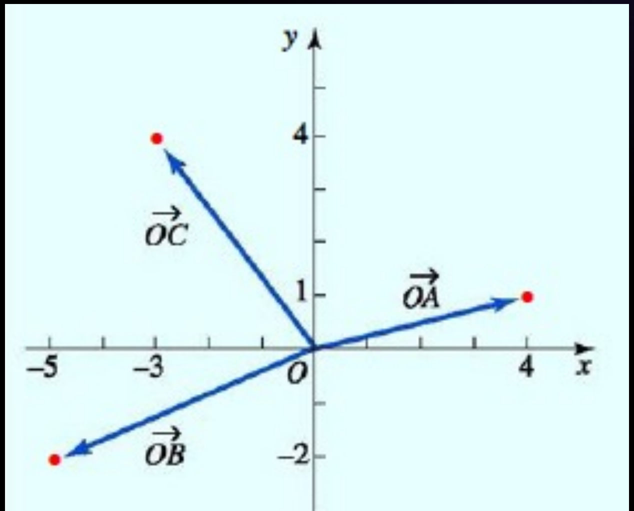
(**R** stands for real number and 2 stand for the number of entries; it is pronounced "r-two.")



the significance of "ordered" here; for example, the point (5, 3) is not the same point as (3, 5).

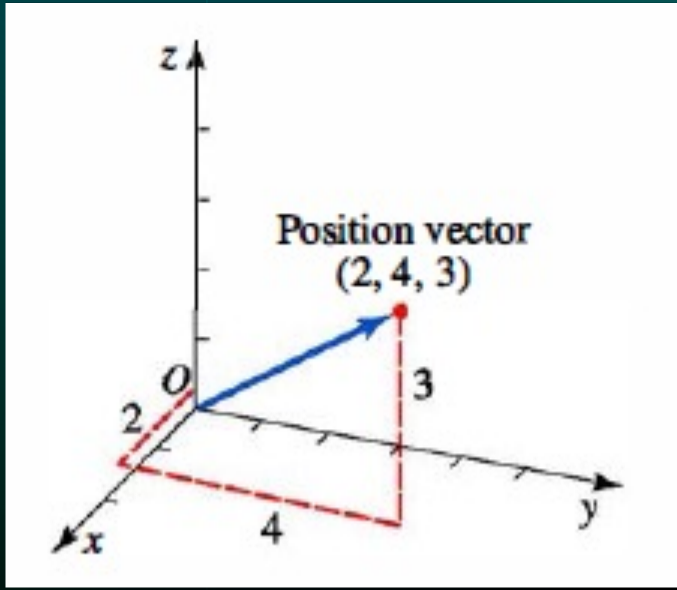
The order is important.

Sketch the position vectors $\overrightarrow{\boldsymbol{OA}}$ = (4,1), $\overrightarrow{\boldsymbol{OB}}$ =(-5,-2) and $\overrightarrow{\boldsymbol{OC}}$ =(-3,4)



Vector in R^3, ... R^n

- R^3 : (2, 4, 3)



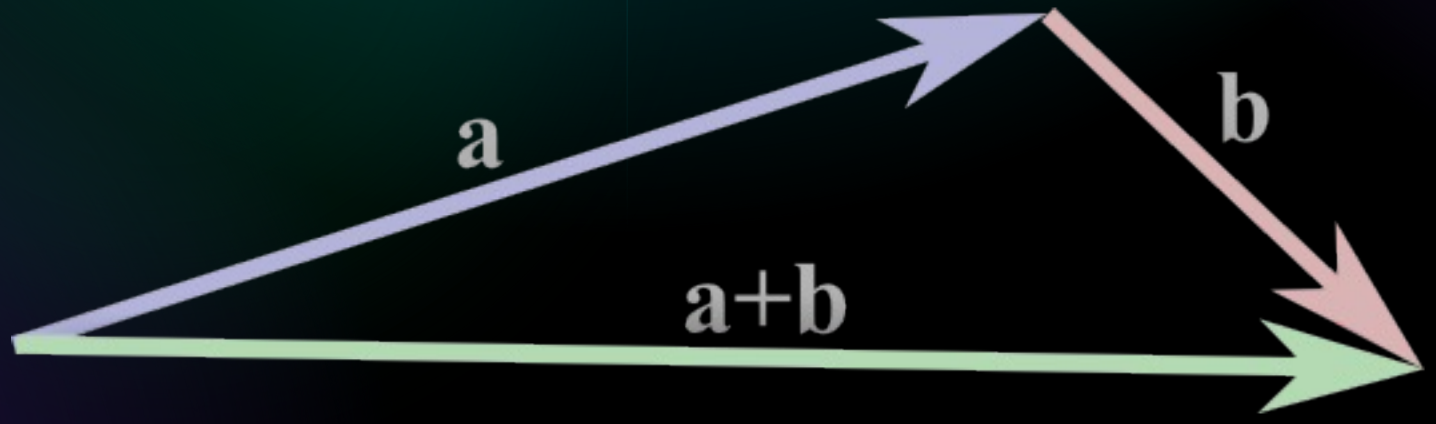
$\mathbf{R}^4$ is the set of sequences of four real numbers; (1, 2, 3, 4) and (–1, 3, 5.2, 0) are in $\mathbf{R}^4$.
$\mathbf{R}^5$ is the set of sequences of five real numbers; (–1, 2, 0, 3, 9) is in this set.
$\mathbf{R}^n$ - $(u_1,u_2,u_3,\ldots,u_n)$

Operations on vectors

Addition of vectors

Given two vectors **a** and **b**,

a+**b**



Operations on vectors

Addition of vectors

Let **u** and **v** be vectors in $\mathbf{R}^n$

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \quad \mathbf{v} = (v_1, v_2, \ldots, v_n)$$

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n)$$

Operations on vectors

Addition of vectors *(example)*

Let $\mathbf{u} = (-1, 4, 3, 7)$ and $\mathbf{v} = (-2, -3, 1, 0)$ be vectors in R^4. Find $\mathbf{u} + \mathbf{v}$.

$\mathbf{u} + \mathbf{v}$

$= (-1, 4, 3, 7) + (-2, -3, 1, 0)$

$= (-3, 1, 4, 7)$

Operations on vectors

Subtraction of vectors

$$\mathbf{b} - \mathbf{a} = \mathbf{b} + (-\mathbf{a})$$

$$\mathbf{a} + \mathbf{x} = \mathbf{b}$$

Operations on vectors

Subtraction of vectors

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^n$

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \qquad \mathbf{v} = (v_1, v_2, \ldots, v_n)$$

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, \ldots, u_n - v_n)$$

Operations on vectors

Subtraction of vectors *(example)*

Let $\mathbf{u} = (5, 3, -6)$ and $\mathbf{v} = (2, 1, 3)$ be vectors in R^3. Find $\mathbf{u} - \mathbf{v}$.

$\mathbf{u} - \mathbf{v}$

$= (5, 3, -6) - (2, 1, 3)$

$= (3, 2, -9)$

$= (5, 3, -6) + \mathbf{(-1)}(2, 1, 3)$

$= (3, 2, -9)$

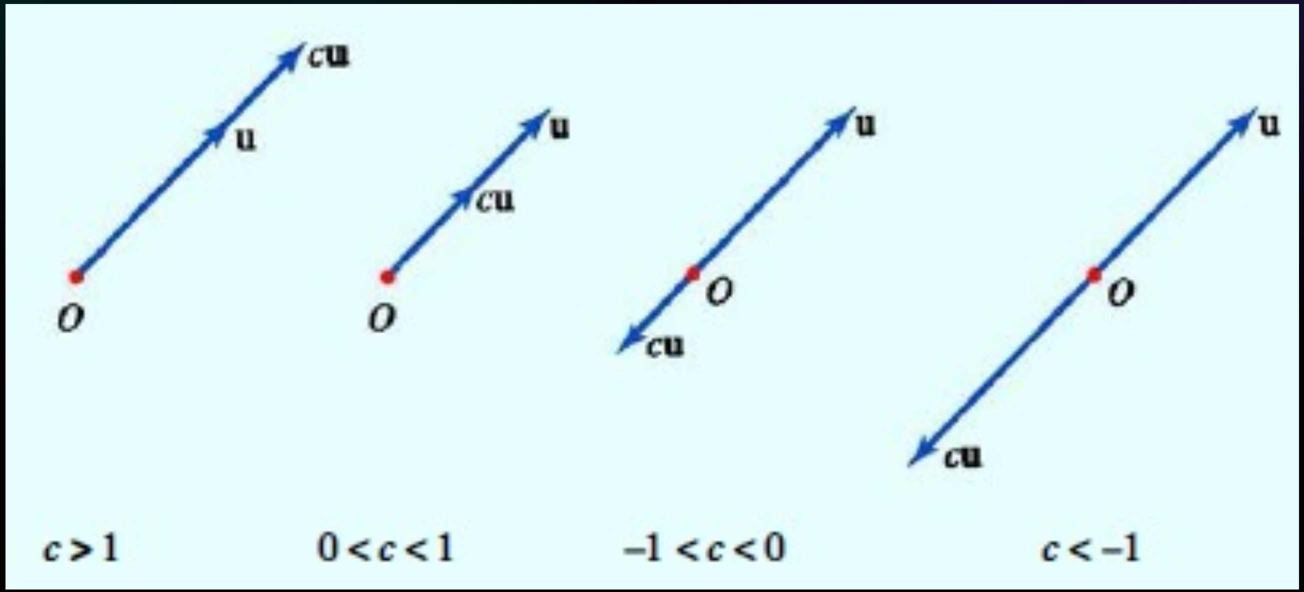
Operations on vectors

Scalar Multiplication

Let **u** be vector in R^n and let c be a scalar.

$$\mathbf{u} = (u_1, u_2, \ldots, u_n)$$

$$c\mathbf{u} = (cu_1, cu_2, \ldots, cu_n)$$



Operations on vectors

Scalar Multiplication *(example)*

Let **u** = (–1, 4, 3, 7) be vectors in R^4. Find 3**u**.

3**u**

= 3(–1, 4, 3, 7)

= (–3, 12, 9, 21)

Operations on vectors

Properties of Vector Addition and Scalar Multiplication

Let **u**, **v**, and **w** be vectors in $\mathbf{R}^n$ and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative property
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$	Property of the zero vector
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Property of the negative vector
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive properties
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$	
8. $1\mathbf{u} = \mathbf{u}$	Scalar multiplication by 1

Column vectors

Addition and scalar multiplication of column vectors in $\mathbf{R}^n$ in a component-wise manner

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

$$c \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$$

Column vectors

For example, in $\mathbf{R}^3$

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 3 \end{bmatrix}$$

$$4\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -12 \end{bmatrix}$$

Exercises

1. If **u** = (2, 1, 3), **v** = (–1, 3, 2), and **w** = (2, 4, –2) are the row vectors in $\mathbf{R}^3$. Compute the following linear combinations.

   a. **u** + **w** **b.** 2**u** – 3**v** – 4**w**

2. If **u** = $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, **v** = $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, and **w** = $\begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix}$ are the row vectors in $\mathbf{R}^3$. Compute the following linear combination

   a. – 4**v** + 3**w** **b.** 2**u** + 3**v** – 8**w**

Linear Combinations of vectors

$a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$ - a **linear combination** of the vectors **u**, **v**, and **w**

Definition: Let $v_1, v_2, \ldots, v_m$ be vectors in $\mathbf{R}^n$. The vector **v** in $\mathbf{R}^n$ is a *linear combination* of $v_1, v_2, \ldots, v_m$ if there exist scalars $c_1, c_2, \ldots, c_m$ such that **v** can be written

$$\mathbf{v} = c_1 v_1, c_2 v_2, \ldots, c_m v_m$$

If the system had no solutions, then it would not have been possible to express it as a linear combination of the other vectors.

Linear Combinations of vectors

Let **u** = (2, 5, -3), **v** = (-4, 1, 9), **w** = (4, 0, 2). Determine the linear combination 2**u** – 3**v** + **w**.

2**u** – 3**v** + **w**

= 2(2, 5, -3) – 3(-4, 1, 9) + (4, 0, 2)

= (4, 10, -6) – (-12, 3, 27) + (4, 0, 2)

= (4 + 12 + 4, 10 – 3 + 0, -6 – 27 + 2)

= (20, 7, -31)

vector (20, 7, -31) is a linear combination of the three vectors (2, 5, -3), (-4, 1, 9), and (4, 0, 2).

(20, 7, -31) = 2(2, 5, -3) – 3(-4, 1, 9) + (4, 0, 2)

Linear Combinations of vectors

Example: Determine whether the vector (–1, 1, 5) is a linear combination of the vectors (1, 2, 3), (0, 1, 4) and (2, 3, 6).

$$c_1(1, 2, 3) + c_2(0, 1, 4) + c_3(2, 3, 6) = (-1, 1, 5)$$

$$(c_1, 2c_1, 3c_1) + (0, c_2, 4c_2) + (2c_3, 3c_3, 6c_3) = (-1, 1, 5)$$

$$(c_1 + 2c_3,\ \ 2c_1 + c_2 + 3c_3,\ \ 3c_1 + 4c_2 + 6c_3) = (-1, 1, 5)$$

$$\begin{array}{rcrcrcr} c_1 & & & + & 2c_3 & = & -1 \\ 2c_1 & + & c_2 & + & 3c_3 & = & 1 \\ 3c_1 & + & 4c_2 & + & 6c_3 & = & 5 \end{array}$$

Linear Combinations of vectors

$$\begin{array}{ccccccc} c_1 & & & + & 2c_3 & = & -1 \\ 2c_1 & + & c_2 & + & 3c_3 & = & 1 \\ 3c_1 & + & 4c_2 & + & 6c_3 & = & 5 \end{array}$$

Hence **[A | B]** is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 2 & 1 & 3 & 1 \\ 3 & 4 & 6 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & -1 & 1 & -3 \\ 0 & -4 & 0 & -8 \end{array}\right] \begin{array}{c} \\ 2R_1 - R_2 \\ 3R_1 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & -4 & 0 & -8 \end{array}\right] \begin{array}{c} \\ -R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -4 & 4 \end{array}\right] \begin{array}{c} \\ \\ 4R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} \\ \\ R_1/(-4) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 - 2R_3 \\ R_2 + R_3 \\ \\ \end{array}$$

$c_1 = 1$
$c_2 = 2$
$c_3 = -1$

Thus the vector (–1, 1, 5) can be written in one way as a linear combination

(-1, 1, 5) = 1(1, 2, 3) + 2(0, 1, 4) –1(2, 3, 6)

Linear Combinations of vectors

Example: Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space M_{22} of *2x2* matrices.

$$c_1\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + c_2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + c_3\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 2c_2 & -3c_2 + c_3 \\ 2c_1 + 2c_3 & c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$c_1 + 2c_2 = -1$ $\qquad$ $-3c_2 + c_3 = 7$

$2c_1 + 2c_3 = 8$ $\qquad$ $c_1 + 2c_2 = -1$

$c_1 = 3$, $c_2 = -2$, and $c_3 = 1$

Linear Combinations of vectors

Example: Determine whether the function f(x)=2x²+6x+7 is a linear combination of g(x)=x²-1 and h(x)=2x+3

$$c_1 g(x) + c_2 h(x) = f(x)$$

$$c_1(x^2 - 1) + c_2(2x+3) = 2x^2 + 6x + 7$$

$$c_1 x^2 + 2c_2 x - c_1 + 3c_2 = 2x^2 + 6x + 7$$

$$c_1 = 2$$

$$2c_2 = 6$$

$$-c_1 + 3c_2 = 7$$

$c_1 = 2$ and $c_2 = -3$

Thus f(x) can be written in one way as a linear combination

$$f(x) = 2g(x) + 3h(x)$$

$$2x^2 + 6x + 7 = 2(x^2 - 1) + 3(2x+3)$$

Exercise

Determine whether the first vector/matrix/function is a linear combination of the other vectors/matrices/functions.

1. (6, 13, 9); (1, 1, 1), (1, 2, 4), (0, 1, –3)
2. (1, 2, -1); (1, 2, 0), (–1, –1, 2), (1, 3, 2)
3. $\begin{bmatrix} 7 & 6 \\ -5 & -3 \end{bmatrix}; \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
4. $\begin{bmatrix} 4 & 1 \\ 7 & 10 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$
5. $f(x)=2x^2+x-3$; $g(x)= x^2-x+1$; $h(x)= x^2+2x-2$
6. $f(x)=x^2+4x+5$; $g(x)= x^2+x-1$; $h(x)= x^2+2x+1$

Linear Independence of Vectors

Definition: (a) The set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ in a vector space V is said to be *linearly dependent* if there exist scalars $c_1, \ldots, c_m$ not all zero, such that

$$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

(b) The set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is *linearly independent* if

$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$ can only be satisfied when $c_1 = 0, \ldots, c_m = 0$

Linear Independence of Vectors

According to this definition,

the set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is seen to be

linearly independent in $\mathbf{R}^3$:

- $c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$
- can only be satisfied if $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$.

Linear Independence of Vectors

linearly independent

the set $\{(1, 2, 3), (5, 1, 0), (2, 0, 0)\}$

- $c_1(1, 2, 3) + c_2(5, 1, 0) + c_3(2, 0, 0) = (0, 0, 0)$
- $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$
- This set is linearly independent in $\mathbf{R}^3$.

Linear Independence of Vectors

linearly dependent

the set $\{(4, -1, 0), (2, 1, 3), (0, 1, 2)\}$

- $c_1(4, -1, 0) + c_2(2, 1, 3) + c_3(0, 1, 2) = (0, 0, 0)$
- $1(4, -1, 0) - 2(2, 1, 3) + 3(0, 1, 2) = (0, 0, 0)$
- $c_1 = 1$, $c_2 = -2$, and $c_3 = 3$
- This set is linearly dependent in $\mathbf{R}^3$.

Linear Independence of Vectors

Determine whether the set {(1, 2, 0), (0, 1, –1), (1, 1, 2)} is linearly independent in $\mathbf{R}^3$.

- Examine the identity $c_1(1, 2, 0) + c_2(0, 1, -1) + c_3(1, 1, 2) = (0, 0, 0)$

$$(c_1+2\,c_1+0)+(0+c_2-c_2)+(c_3+c_3+2c_3)=(0,0,0)$$

$$\mathbf{(c_1+c_3\;,\;2\,c_1+c_2+c_3\;,\;-\,c_2+2c_3)=(0,0,0)}$$

$$\begin{array}{rcrcrcr} c_1 & & & + & c_3 & = & 0 \\ 2c_1 & + & c_2 & + & c_3 & = & 0 \\ & - & c_2 & + & 2c_3 & = & 0 \end{array}$$

Linear Independence of Vectors

$$\begin{array}{rcrcrcl} c_1 & & & + & c_3 & = & 0 \\ 2c_1 & + & c_2 & + & c_3 & = & 0 \\ & - & c_2 & + & 2c_3 & = & 0 \end{array}$$

$[A \mid B]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \begin{array}{l} \\ 2R_1 - R_2 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ -R_2 \\ R_3 - R_2 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ \\ \end{array}$$

$c_1 = 0$, $c_2 = 0$, and $c_3 = 0$

Thus the set is linearly independent.

If the system had other solutions, the vectors would have been linearly dependent.

Linear Independence of Vectors

a) Show that the set $\{x^2+1, 3x-1, -4x+1\}$ is linearly independent in $\mathbf{P}_2$.

(*Note:* $\mathbf{P}_n$ denote the set of real polynomial functions of degree $\leq n$.)

$$c_1(x^2+1)+c_2(3x-1)+c_3(-4x+1)=0$$

$$c_1x^2+(3c_2-4c_3)x+c_1-c_2+c_3=0$$

$$c_1=0,\ 3c_2-4c_3=0,\ c_1-c_2+c_3=0$$

$c_1 = 0$, $c_2 = 0$, and $c_3 = 0$

The functions are linearly independent.

Linear Independence of Vectors

b) Show that the set {x+1, x-1, -x+5} is linearly dependent in $\mathbf{P}_1$.

(*Note:* $\mathbf{P}_n$ denote the set of real polynomial functions of degree $\leq n$.)

$$c_1(x+1)+c_2(x-1)+c_3(-x+5)=0$$

$$(c_1+c_2-c_3)x+(c_1-c_2+5c_3)=0$$

$$c_1+c_2-c_3=0,\ c_1-c_2+5c_3=0$$

$c_1 = -2r$, $c_2 = 3r$, and $c_3 = r$

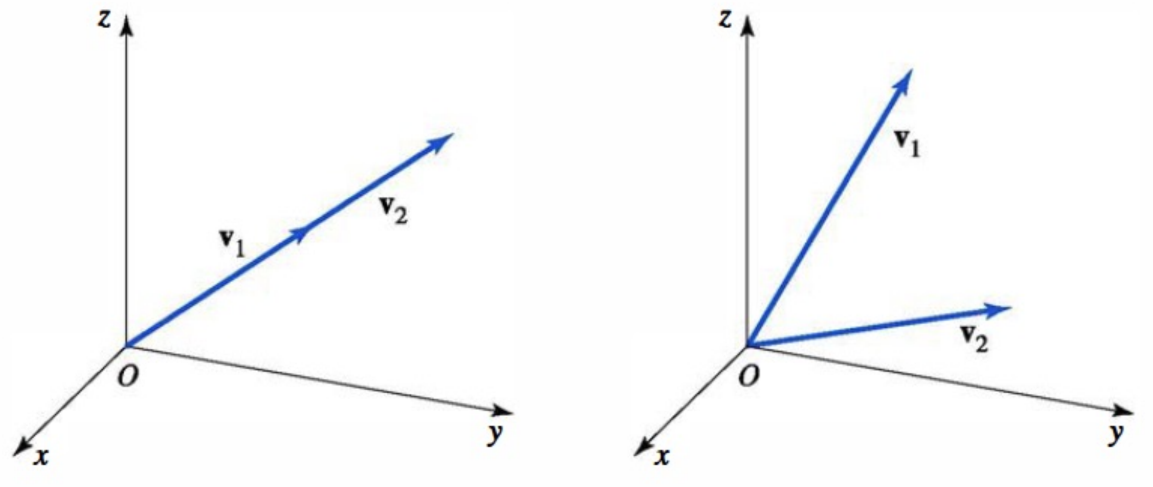
$$-2r(x+1)+3r(x-1)+r(-x+5)=0$$

If $r = 1$ for example $-2(x+1)+3(x-1)+1(-x+5)=0$

The functions are linearly dependent.

Linear Independence of $\{v_1, v_2\}$

(2, -1, 3) and (4, -2, 6) are linearly dependent in R^3
(1, -2, 4) and (3, -6, 8) are linearly independent in R^3



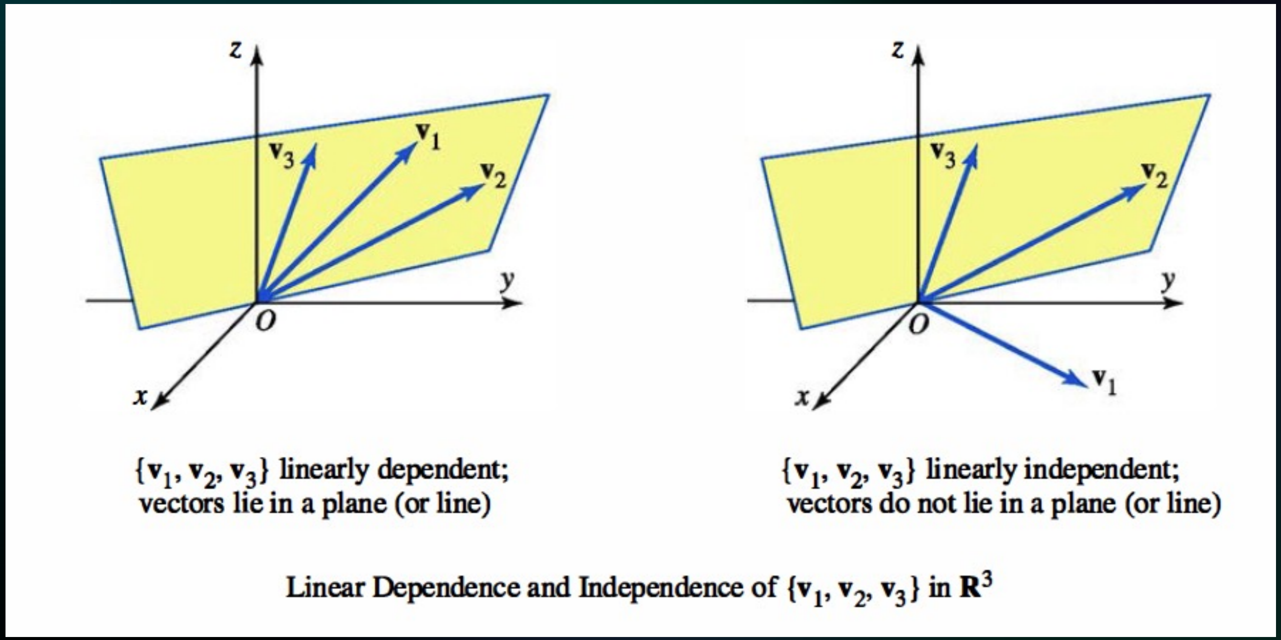
$\{v_1, v_2\}$ linearly dependent; vectors lie on a line

$\{v_1, v_2\}$ linearly independent; vectors do not lie on a line

Linear Dependence and Independence of $\{v_1, v_2\}$ in $\mathbf{R}^3$

$x - 1$ and $x^2 + 2$?
$3x - 2$ and $9x - 6$?

Linear Independence of $\{v_1, v_2, v_3\}$



Exercise

Determine whether the following sets of functions are linearly dependent in $\mathbf{P}_2$.

1. $\{ f, g, h \}$ where $f(x)=2x^2+1$; $g(x)= x^2+4x$; $h(x)= x^2-4x+1$
2. $\{ f, g, h \}$ where $f(x)=x^2+3$; $g(x)= x+1$; $h(x)= 2x^2-3x+3$

Diagonalization of Matrices

1. **The Eigenvector Problem**

   Given a matrix **A**, does there exist a basis for consisting of eigenvectors of **A**?

2. **The Diagonalization Problem (Matrix Form)**

   Given an *nxn* matrix **A**, does there exist an invertible matrix **X** such that $\mathbf{B}=\mathbf{C}^{-1}\mathbf{A}\mathbf{C}$ is a diagonal matrix?

Diagonalization of Matrices

Definition:

A square matrix $\mathbf{A}$ is called **diagonalizable** if there is an invertible matrix $\mathbf{C}$ such that $\mathbf{C}^{-1}\mathbf{AC}$ is a diagonal matrix; the matrix $\mathbf{P}$ is said to **diagonalize** $\mathbf{A}$.

Definition:

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same size. $\mathbf{B}$ is said to be ***similar*** to $\mathbf{A}$ if there exists an invertible matrix $\mathbf{C}$ such that $\mathbf{B} = \mathbf{C}^{-1}\mathbf{AC}$. The transformation of the matrix $\mathbf{A}$ into the matrix $\mathbf{B}$ in this manner is called a **similarity transformation**.

Diagonalization of Matrices

Theorem: If **A** be an $n \times n$ matrix, then the following are equivalent.

(a) If **A** has n linearly independent eigenvectors $\lambda_1, \lambda_2, \ldots, \lambda_n$, then **A** is diagonalizable.

The matrix **C** whose columns consist of n linearly independent eigenvectors can be used in a similarity transformation $\mathbf{C}^{-1}\mathbf{AC}$ to give a diagonal matrix **D**. The diagonal elements of **D** will be the eigenvalues of A.

$$C^{-1}AC = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

(b) If **A** is diagonalizable, then **A** has n linearly independent eigenvectors $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Diagonalization of Matrices

$$A = XDX^{-1}$$

$$X^{-1} A X = X^{-1}(XDX^{-1}) X$$

$$X^{-1}AX = D$$

Diagonalization of Matrices

$$X^{-1}AX = D$$

The matrix, X, is said to be diagonalize A

That give The matrix, D, diagonal matrix

The matrix, **A** has unique eigenvalues (λ)

If duplicate eigenvalues (λ), then they must have **linearly independent** eigenvectors.

Diagonalization of Matrices

$$X^{-1}AX = D$$

D is made of the eigenvalues of A

X is made if the eigenvectors of A

Diagonalization of Matrices

nxn matrix with eigenvalues λ_1, λ_2, to λ_n

and eigenvectors $\vec{x_1}, \vec{x_2}, \ldots \vec{x_n}$

$$X^{-1}AX = D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$$X = [\vec{x_1}, \vec{x_2}, \ldots \vec{x_n}]$$

Diagonalization of Matrices

A= $\begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$

Fine the eigenvalues |A $-\lambda$I|=0

$$\begin{vmatrix} -3-\lambda & -4 \\ 5 & 6-\lambda \end{vmatrix} = 0$$

(-3 $-\lambda$)(6 $-\lambda$)-(4)(5)=0

-18 + λ+3-6λ+ λ^2+20=0

λ^2-3λ+2=0

(λ-1)(λ-2)=0

λ=1, and λ=2

$\vec{x_1}=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad$ $\vec{x_2}=\begin{pmatrix} -4/5 \\ 1 \end{pmatrix}$

Fine the eigenvectors (A $-\lambda$I) $\vec{x}$ =0

A- (1)I = $\begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$ - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ = $\begin{bmatrix} -4 & -4 \\ 5 & 5 \end{bmatrix}$

x_1=-1, x_2=1; $\vec{x_1}=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

A- (2)I = $\begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$ - $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ = $\begin{bmatrix} -5 & -4 \\ 5 & 4 \end{bmatrix}$

x_1=-4/5, x_2=1; $\vec{x_2}=\begin{pmatrix} -4/5 \\ 1 \end{pmatrix}$

Diagonalization of Matrices

A= $\begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$ with $\lambda=1$ $\overrightarrow{\boldsymbol{x_1}}=\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $\lambda=2$ $\overrightarrow{\boldsymbol{x_2}}=\begin{pmatrix} -4/5 \\ 1 \end{pmatrix}$

$$X^{-1}AX = D$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix};\ X = \begin{bmatrix} -1 & -\,^{4}/_{5} \\ 1 & 1 \end{bmatrix};\ X^{-1} = \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix}$$

$$A = X\,D\,X^{-1}$$

Diagonalization of Matrices

$$A = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \quad \mathbf{X} = \begin{bmatrix} -1 & -\,^{4}/_{5} \\ 1 & 1 \end{bmatrix}; \quad \mathbf{X^{-1}} = \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix}$$

$$\mathbf{A = X\, D\, X^{-1}}$$

$$\mathbf{A} = \begin{bmatrix} -1 & -\,^{4}/_{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -5 & -4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$$

Diagonalization of Matrices

Consider the following matrices A= $\begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix}$ and C= $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, **C** is invertible. Use the similarity transformation **B** = **C^{-1}AC** to transform **A** into a matrix **B**.

$$\mathbf{B} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & -10 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & -10 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C}^{-1} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1}$$
$$= \frac{1}{(2)(3)-(5)(1)} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Note: **A** is transformed into a diagonal matrix **B**. Not every square matrix can be "diagonalized" in this manner.

Diagonalization of Matrices

Given A= $\begin{bmatrix} -4 & -6 \\ 3 & -5 \end{bmatrix}$

a) Show that the matrix **A** is diagonalizable.

b) Find a diagonal matrix **D** that is similar to **A**.

c) Determine the similarity transformation that diagonalizes **A**.

Diagonalization of Matrices

Given A= $\begin{bmatrix} -4 & -6 \\ 3 & -5 \end{bmatrix}$

a) Show that the matrix **A** is diagonalizable.

The eigenvalues and corresponding eigenvectors of this matrix were found that

$$\lambda_1 = 2, \quad v_1 = r\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda_2 = -1, \quad v_2 = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Since **A** be 2x2 matrix and has two linearly independent eigenvectors, **A** is diagonalizable.

Diagonalization of Matrices

Given A= $\begin{bmatrix} -4 & -6 \\ 3 & -5 \end{bmatrix}$

a) Show that the matrix **A** is diagonalizable. ✅

b) Find a diagonal matrix **D** that is similar to **A**.

D also has diagonal elements $\boxed{\lambda_1 = 2}$ $\boxed{\lambda_2 = -1}$

Thus, diagonal matrix D = $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

Diagonalization of Matrices

Given A= $\begin{bmatrix} -4 & -6 \\ 3 & -5 \end{bmatrix}$

a) Show that the matrix **A** is diagonalizable. ✅

b) Find a diagonal matrix **D** that is similar to **A**. $\boxed{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}$

c) Determine the similarity transformation that diagonalizes **A**.

linearly independent eigenvectors $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ the column vectors of the diagonalizing matrix $\mathbf{C} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$$\mathbf{C}^{-1}\mathbf{AC} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}$$

Diagonalization of Matrices

Given A= $\begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$,show that the matrix **A** is NOT diagonalizable.

Step 1: find eigenvalues of given matrix A

$$A - \lambda I_2 = \begin{bmatrix} 5-\lambda & -3 \\ 3 & -1-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I_2| &= 0 \\ (5-\lambda)(-1-\lambda) + 9 &= 0 \\ \lambda^2 - 4\lambda + 4 &= 0 \\ (\lambda - 2)(\lambda - 2) &= 0 \end{aligned}$$

The eigenvalues of A, $\lambda = 2\ (repeated)$

Step 2: find corresponding eigenvectors of matrix A

(A - 2I)x = 0

$$\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

$3x_1 - 3x_2 = 0$

$x_1 = x_2$

The eigenvectors of A, $x = r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The eigenspace is a one-dimensional space.

A is a 2 X 2 matrix but it does **not** have two linearly independent eigenvectors. Thus A is **not** diagonalizable.

Assignment 6

1. Transform the matrix **A** into a matrix **B** using the similarity transformation **$C^{-1}AC$**, with the given matrix **C**.

$$A=\begin{bmatrix}0 & 4\\ 3 & 2\end{bmatrix}, C=\begin{bmatrix}2 & 1\\ 7 & 4\end{bmatrix}$$

2. Diagonalize (if possible) each of the following matrices. Give the similarity transformation.

   a) $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

   b) $\begin{bmatrix}4 & -1\\ 2 & 1\end{bmatrix}$

   c) $\begin{bmatrix}15 & 7 & -7\\ -1 & 1 & 1\\ 13 & 7 & -5\end{bmatrix}$