Basic Mathematics and Statistics

CHAPTER 13: SEQUENCES AND SERIES

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13.1 Sequence and Arithmetic Sequences

Definitions

A **sequence** is a collection of numbers in a certain order.

5, 6, 7

first term

a_1

second term

a_2

last term

finite sequence

1, 6, 11, 16, ...

infinite sequence

a_1 a_2 a_3 a_4 a_n

nth term

$a_2 = 6$, $a_3 = 11$, $a_{10} = ?$

13.1 Sequence and Arithmetic Sequences

Arithmetic Sequences

Arithmetic Sequence

An **arithmetic sequence** (also called an arithmetic progression) is a sequence of numbers In which the difference between successive terms is a constant, called the **common difference.**

$+d \quad +d \quad +d$

a_1 7, 13, 19, 25, … 31

$+6 \quad +6 \quad +6$

common difference, *d*

$a_5 = ?$ $25+6=31$

$a_{20} = ?$

n th Term of an Arithmetic Sequence

The nth term of an arithmetic sequence is given by

$$a_n = a_1 + (n-1)d$$

where a_1 = the first term of the sequence and d = the common difference.

$a_{20} = 7 + (20-1)6 = 7 + 126 = 133$

13.1 Sequence and Arithmetic Sequences

Arithmetic Sequences

Example –

Find the general term expression for the arithmetic sequence 7, 3, -1, -5, …

first term, $a_1 = 7$

common difference, $d = 3 - 7 = -4$

$$a_n = a_1 + (n - 1)d$$

$$a_n = 7 + (n - 1)(-4)$$

$$a_n = 7 - 4n + 4 = 11 + 4n \longleftarrow$$

13.1 Sequence and Arithmetic Sequences

Arithmetic Sequences

Example –

Find the first and sixth term of the arithmetic sequence where the 9th term is 19 and the 21st term is 55.

$a_9 = 19,\ a_{21} = 55$

$a_9 = a_1 + (9 - 1)d$

$19 = a_1 + 8d$

$a_1 = 19 - 8d$

$a_1 = 19 - 8(3)$

$a_1 = -5$

$a_{21} = a_1 + (21 - 1)d$

$55 = a_1 + 20d$

$55 = 19 - 8d + 20d$

$55 = 19 + 12d$

$12d = 36$

$d = 3$

$a_6 = a_1 + (6 - 1)d$

$a_6 = -5 + (6 - 1)3$

$a_6 = 10$

13.1 Sequence and Arithmetic Sequences

Arithmetic Sequences

Example –

Determine how many numbers between 28 and 1,000 are divisible by 3.

a_1 = smallest > 28, a_n = largest < 1000

$a_1 = 30$, $a_n = 999$

$d = 3$

$$a_n = a_1 + (n - 1)d$$

$$999 = 30 + (n - 1)\,3$$

$$n = 324 \leftarrow$$

13.1 Sequence and Arithmetic Sequences

Arithmetic Sequences

Example –

An object moving in a line is given an initial velocity of 5.8 m/s and a constant acceleration of 1.3 m/s². How long will it take the object to reach a velocity of 20.3 m/s?

$a_1 = 5.8$, $d = 1.3$, $a_n = 20.3$, $n=?$

$$a_n = a_1 + (n - 1)d$$

$$20.3 = 5.8 + (n - 1)\ 1.3$$

$$n = 12.2\ \text{s}$$

13.1 Sequence and Arithmetic Sequences

Sum of Arithmetic Sequences

5, 6, 7 → **sum** = 5 + 6 + 7 = 18

+1 +1

$d = 1$

$a_2 = a_1 + d$

5, 6, 7

a_1 $a_3 = a_2 + d = a_1 + 2d$

$= 6 + 1$ $\quad 5 + 2(1)$

11 11 11 11 11

1, 2, 3, 4, 5, 6, 7, 8, 9,10

$1 + 10 = 11$

$\text{sum} = 11 \times 5 = 55$

$(a_1 + a_n)$

$n = 10$ $\quad \frac{n}{2} = \frac{10}{2} = 5$

$\text{sum} = (a_1 + a_n)\frac{n}{2}$

$$
\begin{aligned}
S_n &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
+ \quad S_n &= 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 \\
\hline
2S_n &= 11+11+11+11+11+11+11+11+11+11 \\
2S_n &= 110 \\
S_n &= 55
\end{aligned}
$$

13.1 Sequence and Arithmetic Sequences

Sum of Arithmetic Sequences

5, 6, 7, 8, 9, 10,11, 12, 13, 14, 15, ..., 999, 1000 → **sum** = ?

$$+\begin{cases} S_n = a_1 + (a_1+d) + \dots + (a_n-d) + a_n & \textit{forwards} \\ S_n = a_n + (a_n-d) + \dots + (a_1+d) + a_1 & \textit{backwards} \end{cases}$$

$$2S_n = (a_1+a_n)+(a_1+a_n)+ \dots +(a_1+a_n)+(a_1+a_n)$$

$$2S_n = n(a_1+a_n)$$

$$a_n = a_1 + (n-1)d$$

$$S_n = \frac{n(a_1+a_n)}{2} = \frac{n}{2}\left[2a_1+(n-1)d\right]$$

13.1 Sequence and Arithmetic Sequences

Sum of Arithmetic Sequences

Sum of First n Terms of an Arithmetic Sequence

The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

S_n is called arithmetic series.

Summation Notation, Σ

$$\boxed{S_n = \sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n)}$$

$$a_1 + a_2 + a_3 + a_4 + a_5 \Rightarrow \sum_{i=1}^{5} a_i$$

$$\sum_{i=3}^{5} b_i = b_3 + b_4 + b_5 \qquad \sum_{i=1}^{10} i^2 = 1^2 + 2^2 + 3^2 + \ldots + 10^2 \qquad \sum_{i=2}^{7}\left(\frac{i}{5}\right) = \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \ldots + \frac{7}{5}$$

13.1 Sequence and Arithmetic Sequences

Arithmetic Sequences

Example –

Find the sum of the first one hundred positive even integers.

$2 + 4 + 6 + ... + 100$

$a_1 = 2$, $d = 2$, $a_n = 100$, $n=?$

$a_n = a_1 + (n - 1)d$

$100 = 2 + (n - 1)\,2$

$n = 50$

$$S_n = \frac{n(a_1 + a_n)}{2}$$

$$S_{50} = \frac{50(2+100)}{2}$$

$S_n = 2{,}550$

13.1 Sequence and Arithmetic Sequences

Arithmetic Sequences

Example –

A technician accepts a job to pay off his \$12,000 college loan. If he pays \$457.50 toward the loan the first month and increases his payment by \$15 each month, how long will it take him to pay off the loan?

$a_1 = 457.5$, $d = 15$, $S_n = 12{,}000$, $n = ?$

$a_n = a_1 + (n - 1)d$

$$S_n = \frac{n(a_1 + a_n)}{2} = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$12{,}000 = \frac{n\left[457.5 + (442.5 + 15n)\right]}{2}$$

$$24{,}000 = n\,(900 + 15n)$$

$$24{,}000 = 15n^2 + 900n$$

$$15n^2 + 900n - 24{,}000 = 0$$

$$n^2 + 60n - 1{,}600 = 0$$

$$(n+80)(n-20) = 0$$

$$n = -80 \text{ and } n = 20$$

$n = 20$

13.1 Sequence and Arithmetic Sequences

Summation Notation

$$\boxed{S_n = \sum_{i=1}^{n} a_i = \frac{n}{2}\left(a_1 + a_n\right)}$$

Examples-

$$\sum_{i=1}^{45}(5i+2) = \left[5(1)+2\right]+\left[5(2)+2\right]+\left[5(3)+2\right]+\ldots+\left[5(44)+2\right]+\left[5(45)+2\right]$$

$$= 7+12+17+\ldots+222+227$$

$$= 5,265$$

Since this is a sum of an arithmetic sequence, then

$$S_{45} = \frac{45}{2}(7+227) = 5,265$$

$$\sum_{i=2}^{5} 3i^2 = 3(2)^2+3(3)^2+3(4)^2+3(5)^2 = 12 + 27 + 48 + 75 = 162$$

Note: <u>NOT</u> the indicated sum of an arithmetic sequence

$$2, -1, -4, -7, -10, \ldots\ldots, -55$$

(differences: -3, -3)

$a_{20} = ?$

$a_1 = 2$

$d = -3$

$$a_{20} = a_1 + (n-1)d$$

$$a_{20} = 2 + (20-1)(-3)$$

$$= 2 + 19(-3)$$

$$= 2 - 57$$

$$= -55$$

$S_{20} = ?$

$$2 + (-1) + (-4) + (-7) + \ldots + (-55)$$

$$S_{20} = \frac{n}{2}(a_1 + a_{20}) = \frac{20}{2}(2 + (-55)) = 10 \times (-53)$$

$$= -530$$

13.2 Geometric Sequences

$a_2 = a_1 r$ $\quad a_3 = a_1 r^2$ $\quad a_4 = a_1 r^3$

a_1 2, 14, 98, 686, ... 4802

$\times 7 \quad \times 7 \quad \times 7$

common ratio, *r*

$a_5 = a_1 r^4$
$= 2(7)^4 = 2(2401) = 4802$

$a_{20} = ?$

A **geometric sequence** is a sequence in which each successive term is a *constant multiple*, called the **common ratio** *r*, of the previous term.

$$\begin{array}{cccccc} a_1 & a_1 r & a_1 r^2 & a_1 r^3 & \cdots & a_1 r^{n-1} \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_n \end{array}$$

The *n*th term of a geometric sequence is given by the following formula.

$16, 8, 4, 2, 1, \frac{1}{2}, \ldots$

$r = \frac{8}{16} = \frac{1}{2}$

13.2 Geometric Sequences

n th Term of a Geometric Sequence

The nth term of a geometric sequence is given by

$$a_n = a_1 r^{n-1}$$

where a_1 = the first term of the sequence and r = the common ratio.

$a_{20} = 2(7)^{20-1} = 2 \times 7^{19} = 2.28 \times 10^{16}$ A

13.1 Sequence and Arithmetic Sequences

Geometric Sequences

Example –

Find the general term of the geometric sequence 16, 8, 4, 2, 1, ...

$a_1 = 16$, $\quad r = \frac{8}{16} = \frac{1}{2}$

$a_n = a_1 r^{n-1}$

$$= (16)\left(\frac{1}{2}\right)^{n-1} = 2^4 \times 2^{-(n-1)} = 2^4 \times 2^{-n+1} = 2^{-n+5}$$

13.1 Sequence and Arithmetic Sequences

Geometric Sequences

Example –

A radioactive product has a half-life of 5 years. If the radioactivity level is 56.0 microcuries after 23 years, determine the original level of radioactivity.

r= 0.5,

a_n = 56,

n = 23/5 = 4.6

$$a_n = a_1 r^{n-1}$$

$$56.0 = a_1 (0.5)^{\cancel{5.6}\,4.6-1}$$

$$a_1 \cancel{= 1,360}\ 679$$

13.2 Geometric Sequences

Sum of Geometric Sequences

2, 14, 98, 686 → **sum** = 2 + 14 + 98 + 686 = 800

2, 14, 98, 686, 4802, …, 80707214 → **sum** = ?

2, 6, 18, 54, 162

$r = \frac{6}{2} = 3$

$S_n = 2 + 6 + 18 + 54 + 162$

$rS_n = (2\times3) + (6\times3) + (18\times3) + (54\times3) + (162\times3)$

$S_n - rS_n = 2 - 0 - 0 - 0 - 0 - 486$

$S_n (1-r) = -484$

$S_n = -484/(1-r) = -484/(1-3) = -484/-2 = \mathbf{242}$

13.2 Geometric Sequences

Sum of Geometric Sequences

$$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-2} + a_1 r^{n-1} \qquad \text{sum of } n \text{ terms}$$

$$-rS_n = \quad -\left(a_1 r + a_1 r^2 + \ldots + a_1 r^{n-2} + a_1 r^{n-1} + a_1 r^n\right) \qquad \text{multiply by } \boldsymbol{-r}$$

$$S_n - rS_n = a_1 + 0 + 0 + 0 + \ldots + 0 + 0 - a_1 r^n$$

$$S_n(1-r) = a_1 - a_1 r^n$$

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

Sum of First *n* Terms of a Geometric Sequence

The sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

13.1 Sequence and Arithmetic Sequences

Geometric Sequences

Example –

Find the sum 1 + 3 + 9 +27 +…+ 6561.

$a_1 = 1$, $a_n = 6561$, $r = 3$

$$a_n = a_1 r^{n-1}$$

$$6561 = 1\left(3^{n-1}\right)$$

$$3^8 = 3^{n-1}$$

$$8 = n - 1$$

$$n = 9$$

$$S_n = \frac{a_1\left(1 - r^n\right)}{1 - r}$$

$$S_9 = \frac{1\left(1 - 3^9\right)}{1 - 3} = 9{,}841$$

13.1 Sequence and Arithmetic Sequences

Geometric Sequences

5% simple interest

500 ⇒ 1st: 500 + 25 = 525; 2nd: 25; 3rd: 25

Example –

Determine the total worth of a yearly \$500 investment after 15 years if the interest rate is 5.5% compounded annually.

$a_1 + a_2 + a_3 + \ldots + a_{15} \Rightarrow 500(1.055) + 500(1.055)^2 + \ldots + 500(1.055)^{15} \Rightarrow 500[(1.055) + (1.055)^2 + \ldots + (1.055)^{15}]$

(a_1, $a_2 = a_1 r$)

1st year

principal → \$500 + \$500(0.055) ← interest

500(1 + 0.055)

500(1.055)

2nd year

total worth of \$500 after 1 year + total worth of \$500 after 2 years

$\$500(1.055) + \$500(1.055)^2$

$a_1 = 1.055$, $r = 1.055$, $n = 15$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{15} = \frac{1.055\left[1-(1.055)^{15}\right]}{1-1.055} \approx 23.64$$

The total value of the investment is \$500(23.64) = \$11,820.57

13.2 Geometric Sequences

Sum of Infinite Geometric Sequences

$\times 7$ $\times 7$

2, 14, 98, 686, 4802, …

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

10000, 5000, 2500, 1250, ...

→ **sum** = ???

infinite sequence

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1 - a_1 r^n}{1-r} = \frac{a_1}{1-r} - \frac{a_1 r^n}{1-r}$$

0 → "0"

bigger

$\boxed{|r| < 1}$

–1 < r < 1

$$S_n = \frac{a_1}{1-r}, \quad |r| < 1$$

13.2 Geometric Sequences

Sum of Infinite Geometric Sequences

$$S = \frac{a_1}{1-r}, \qquad |r| < 1$$

Example-

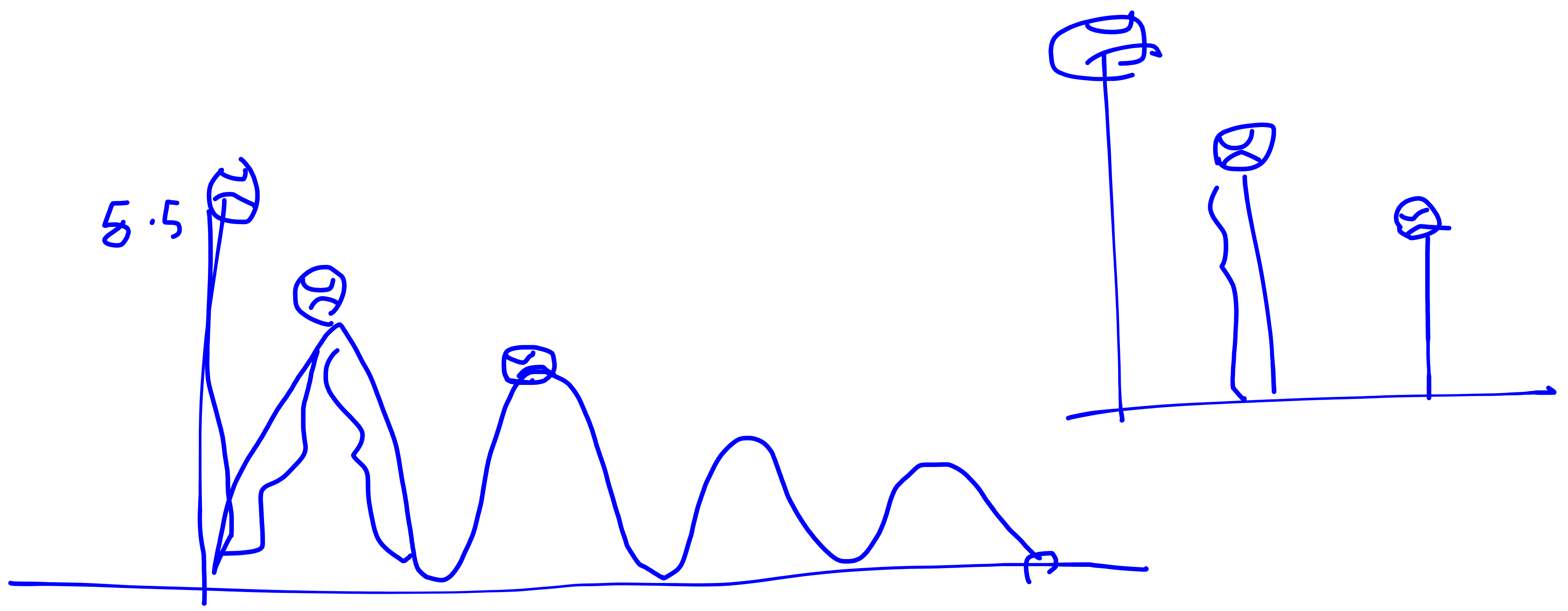
63 + 21 + 7 + ...

$r = \frac{21}{63} = \frac{1}{3}$

$S = \frac{63}{1-\frac{1}{3}} = \frac{63}{\frac{2}{3}} = 94\frac{1}{2}$

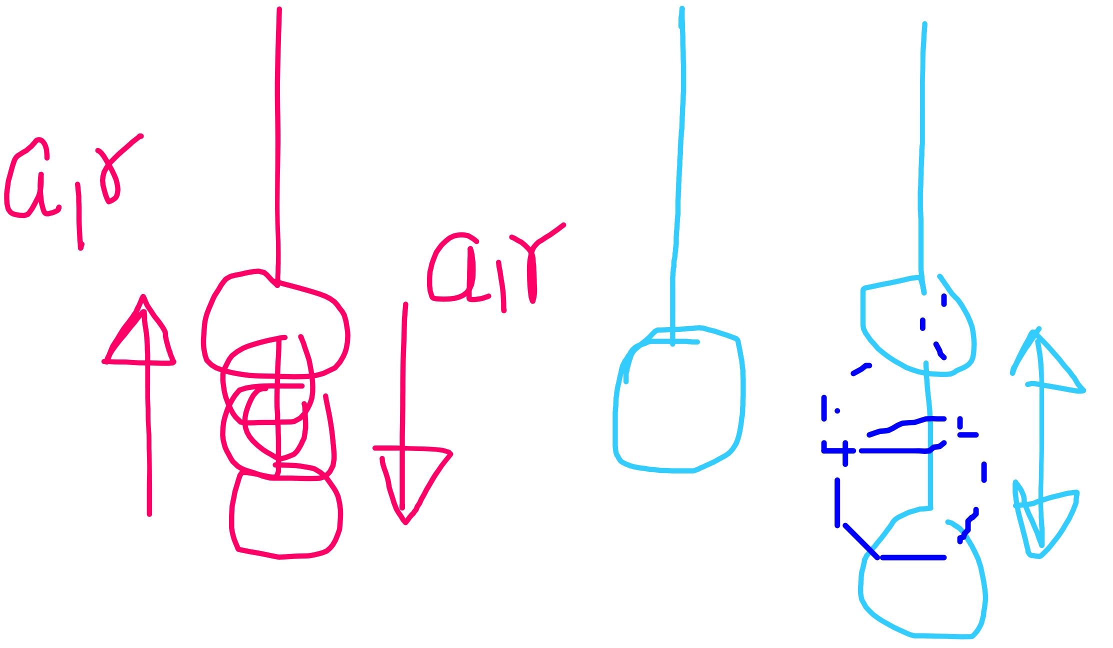
13.2 Geometric Sequences

Sum of Infinite Geometric Sequences



Example-

A ball attached to the end of an elastic band oscillates up and down. The total distance traveled during its initial oscillation is 8.5 cm, and the distance traveled during each successive oscillation is 72% of the previous distance. Find the total vertical distance the ball traveled before coming to rest.



$r = 72\% = 0.72$

a_1

8.5 cm

$S = \frac{a_1}{1-r}$

$a_1 r = ?$

$a_1 + a_2 + a_3 + \ldots$

$a_1 r + a_1 r$

$a_1 r^2 + a_1 r^2$

up down up down

$$S = a_1 + (a_1 r + a_1 r) + (a_1 r^2 + a_1 r^2) + (a_1 r^3 + a_1 r^3) + \ldots$$

$$= a_1 + 2(a_1 r) + 2(a_1 r^2) + 2(a_1 r^3) + \ldots$$

$$= a_1 + 2[a_1 r + a_1 r^2 + a_1 r^3 + \ldots]$$

$$= a_1 + 2\left[\frac{a_1 r}{1-r}\right] = 8.5 + 2\left[\frac{8.5(0.72)}{1-0.72}\right]$$

$$S = 8.5 + 43.71 \text{ cm}$$

$$S = 52.21$$

Assignment

Deadline for submission: Monday, September 6, 2021

- Exercises 13.1 – 15, 17, 23 ,27, 34
- Exercises 13.2 – 7, 11, 13, 16, 24