Basic Mathematics and Statistics

CHAPTER 8: MATRICES

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8.1 Definition and Basic Properties

A **matrix** is any rectangular (or square) array of real numbers enclosed within brackets and each number in the array is called an element of the matrix.

In general, a matrix with *m* rows and *n* columns can be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \left[a_{ij}\right]_{m\times n}$$

The matrix **A** is said to have **order (dimension)** *m x n*.

8.1 Definition and Basic Properties

2nd column 4 columns

$$\begin{array}{cc} \text{1st row} & M = \begin{bmatrix} 6 & 5 & 7 & -1 \\ -2 & 12 & -1 & -5 \\ -1 & -3 & 0 & 2 \end{bmatrix} \end{array} \quad \text{3 rows}$$

$5 \to a_{12} \to$ 2nd column

1st row

The array of numbers may be the basis of a table of data that can be organized into a spreadsheet.

8.1 Definition and Basic Properties

The results of 10 matched played by 4 football teams can be shown by a table and a matrix as

Team	Wins	Drawns	Losses
A	5	2	3
B	3	3	4
C	3	6	1
D	2	0	8

Matrix

$$\begin{bmatrix} 5 & 2 & 4 \\ 3 & 3 & 3 \\ 3 & 6 & 1 \\ 2 & 0 & 8 \end{bmatrix}$$

Three points ***P***(1, 2), ***Q***(4, 3) and ***R***(-2, 7) can be represented by a table and a matrix as

Point	P	Q	R
X	1	4	-2
Y	2	3	7

Matrix

$$\begin{bmatrix} 1 & 4 & -2 \\ 2 & 3 & 7 \end{bmatrix}$$

8.2 Types of Matrix

A matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a **column matrix** or column vector of order 3 x 1.

A matrix $\begin{bmatrix} -2 & 3 & 0 & 4 \end{bmatrix}$ is a **row matrix** or row vector of order 1 x 4.

A matrix $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -4 \\ 7 & 0 & 14 \end{bmatrix}$ is a **square matrix** of order 3. (number of rows = number of columns)

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}_{2\times 2} \qquad \begin{bmatrix} x & y & z & 1 \\ a & b & c & 1 \\ p & q & r & 1 \\ m & n & o & 1 \end{bmatrix}_{4\times 4}$$

8.2 Types of Matrix

If the rows and columns of a matrix are interchanged, the matrix is **transposed**;

the **transpose of a matrix A** is written A^T.

$$\text{Let matrix } \mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 4 & -7 \end{bmatrix}, \text{ then } \mathbf{A}^T = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & -7 \end{bmatrix}$$

$3 \times 2 \longrightarrow 2 \times 3$

Remark: $(A^T)^T = A$ and **AA^T** is always a square matrix.

8.2 Types of Matrix

Equal Matrix

Let matrix **A** = $\left[a_{ij}\right]_{m\times n}$ and matrix **B** = $\left[b_{ij}\right]_{m\times n}$, say that **A = B** if $a_{ij} = b_{ij}$ for all *i, j*.

$$\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 2x+1 & y \\ z+2 & w-1 \end{bmatrix}$$

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$$\begin{aligned} 2x+1 &= 5 \\ x &= (5-1)/2 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} z+2 &= 4 \\ z &= 4-2 \\ z &= 2 \end{aligned}$$

$$y = -3$$

$$\begin{aligned} w-1 &= -2 \\ w &= -1 \end{aligned}$$

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Note:

Two matrices can be equal only if they have the same dimensions.

8.2 Types of Matrix

Square matrices are particularly important; they are divided into several types. We consider matrices of order 3 and 4 but the properties apply to all orders.

If **A** is a square matrix then

- **symmetric matrix -** $a_{ij} = a_{ji}$, i.e. A = A^T

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

8.2 Types of Matrix

- **skew-symmetric matrix** - $a_{ij} = -a_{ji}$ **, i.e.** the element on the leading diagonal must be zero.

$$\begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix}$$

- **diagonal matrix -** if the only non-zero elements are found on the leading diagonal

$$\begin{bmatrix} \boldsymbol{a} & 0 & 0 & 0 \\ 0 & \boldsymbol{b} & 0 & 0 \\ 0 & 0 & \boldsymbol{c} & 0 \\ 0 & 0 & 0 & \boldsymbol{d} \end{bmatrix}$$

8.2 Types of Matrix

- **unit matrix or identity matrix –** if **A** is diagonal and the diagonal elements are equal to 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ is the unit matrix of order 2.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ is the unit matrix of order 3.

8.3 Matrix Operations

Addition and subtraction of matrices

Two matrices can be added or subtracted if and only if they have the **same order**.
The corresponding numbers or elements in each matrix are added or subtracted.

A + B = $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

A - B = $[a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}]$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 3 & 7 \end{bmatrix}$$

+ + + +

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -1 & -3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

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8.3 Matrix Operations

Multiplication

- **Multiplication by scalar** - In the context of matrices, a scalar is just a number, as in the context of vectors. It means that each element in matrix is multiplied by the scalar.

$$kA = k[a_{ij}] = [ka_{ij}]$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 6 & -1 \end{bmatrix}$$

$$3\mathbf{A} = 3 \times \begin{bmatrix} 2 & 1 & 3 \\ 5 & 6 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 9 \\ 15 & 18 & -3 \end{bmatrix}$$

8.3 Matrix Operations

Multiplication

- **Multiplication of matrices** - The product can be symbolized as **A x B = C** or, more simply, as **AB = C.** If the product exists then **the number of columns of A is equal to the number of rows of *B*.**

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{DC} = \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$$

$3\times2 = 2\times3 \rightarrow 3\times3$

$$= \begin{bmatrix} -3(1)+2(-1) & -3(2)+2(0) & -3(3)+2(4) \\ 1(1)+5(-1) & 1(2)+5(0) & 1(3)+5(4) \\ -1(1)+2(-1) & -1(2)+2(0) & -1(3)+2(4) \end{bmatrix} = \begin{bmatrix} -5 & -6 & -1 \\ -4 & 2 & 23 \\ -3 & -2 & 5 \end{bmatrix}$$

$$\mathbf{CD} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \\ -1 & 2 \end{bmatrix}$$

$2\times3 = 3\times2 \rightarrow 2\times2$

$$= \begin{bmatrix} 1(-3)+2(1)+3(-1) & 1(2)+2(5)+3(2) \\ -1(-3)+0(1)+4(-1) & -1(2)+0(5)+4(2) \end{bmatrix} = \begin{bmatrix} -4 & 18 \\ -1 & 6 \end{bmatrix}$$

$\boxed{\textbf{Note: CD} \neq \textbf{DC}}$

8.3 Matrix Operations

$\longrightarrow$ row $\quad \downarrow$ column

Exercise 8.1

(a) $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ 1×3

row matrix

(b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ 2×1

column matrix

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2×2

square matrix

(d) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 2 & 3 & 5 \end{bmatrix}$ 3×3

square

2. Refer to the following matrices:

$$\mathbf{A}=\begin{bmatrix}1 & -3 & -2\\ 2 & 0 & 6\end{bmatrix};\ \mathbf{B}=\begin{bmatrix}2 & -6 & 5\\ 7 & 3 & 0\end{bmatrix};\ \mathbf{C}=\begin{bmatrix}-6 & -7 & 0\\ 5 & 2 & 1\end{bmatrix};\ \mathbf{D}=\begin{bmatrix}1 & 0 & 2\\ -4 & 5 & 0\\ 2 & -1 & 3\end{bmatrix}$$

Calculate each of the following (where possible):

(a) A + B **(b) A – C** **(c) C – A** **(d) B + C**

(e) –3A **(f) –2A – C** **(g) 2B + C**

(h) Verify A + B = B + A **(i) Verify A + (B – C) = (A + B) – C**

(j) A + D **(k) B – A** **(l) 2A** **(m) AB**

(n) AC **(o) AD** **(p) DB** **(q) $\mathbf{D}^2$**

7. $\begin{bmatrix} x & 0 & a+b \\ -4 & a & y \end{bmatrix} = \begin{bmatrix} 6 & 0 & -5 \\ -4 & 2 & 3 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -3 & -5 & -6 \\ 6 & 1 & 7 & 5 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 6 & 10 \\ 7 & 12 \\ 8 & 11 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

8.4 Inverse of the Square Matrix

Division of matrices cannot be defined but an operation similar in effect to division is that of multiplication of a matrix by its inverse.

If **A** and **B** are two square matrices such that **AB = I**, where **I** is a unit matrix, then matrix **B** is called the **inverse matrix** of **A** and is written as **A**$^{-1}$. We could also claim that **A** was the inverse of matrix **B** and write **A** as **B**$^{-1}$.

More correctly **B** is the right-hand inverse of **A** and **A** is the left-hand inverse of **B** so that **AA**$^{-1}$= **A**$^{-1}$**A** = **I**.

$2 \times \frac{1}{2} = 1$

$\frac{2}{2} = 1$

$9 \times \frac{1}{9} = 1$

$\frac{9}{9} = 1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8.4 Inverse of the Square Matrix

$$\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{MI} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{IM} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

MI = IM = M.

If **M** is a square matrix and if there exists a matrix $\mathbf{M}^{-1}$ (read **M** inverse) such that

$\mathbf{M}^{-1}\,\mathbf{M} = \mathbf{M}\,\mathbf{M}^{-1} = \mathbf{I}$ then $\mathbf{M}^{-1}$ is the inverse of **M**.

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A \times B\ (A^{-1}) = I$

$B\ (A^{-1})$ A

$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$.

8.4 Inverse of the Square Matrix

Finding the inverse of a 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \mathrm{ad} - \mathrm{bc} \neq 0$$

If ***ad* – *bc* = 0**, then **A^{-1} does not exist**, and said that **A is singular matrix.**

8.4 Inverse of the Square Matrix

Finding the inverse of a 2 × 2 matrix

$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, **$A^{-1}$** $=?$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{1(5) - 3(2)}\begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= -1\begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

Checking the answer:

$A^{-1} A = A A^{-1} = I$

$$\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -5+6 & -15+15 \\ 2-2 & 6-5 \end{bmatrix} = \begin{bmatrix} -5+6 & 3-3 \\ -10+10 & 6-5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ **= I**

8.4 Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Procedure of Row Operations

Let **A** = $[\, a_{ij} \,]_{n\times n}$ be any square matrix, the inverse of **A** can be perform by the

following algorithm:

1. Write the augmented matrix **[A | I]**.
2. Use the procedure of *row operations* on matrix **[A | I]** in order to transform the coefficients in matrix **[A | I]** into the form of the matrix **[I | B]**.
3. Then $\mathbf{A}^{-1} = \mathbf{B}$.

8.4 Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Procedure of Row Operations

Matrix Inversion by Row Operations: Transform the augmented matrix

[A | I]

$$\left[\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\text{Given matrix } \mathbf{A}} & & & & \\ a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

$\Downarrow$

By successive row operations to

$\Downarrow$

[I | B]

$$\left[\begin{array}{cccc|cccc} & & & & \multicolumn{4}{c}{\text{Inverse matrix } \mathbf{A}} \\ 1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right] \; A^{-1}$$

- ***The procedure of row operations consists of***

1. Interchange any 2 rows *i* and *j* .
2. Multiply any row of the matrix by a nonzero constant.
3. Replace a row by the sum of that row and a constant multiple of some other row.

8.4 Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Procedure of Row Operations

- ***Operation That Produce Row-Equivalent Matrices***

An augmented matrix is transformed into a row-equivalent matrix by performing any of the following **row operations:**

A. Two rows are interchanged ($R_i \leftrightarrow R_j$)

B. A row is multiplied by a nonzero constant. ($kR_i \rightarrow R_j$)

C. A constant multiple of one row is added to another row ($R_i + kR_i \rightarrow R_j$)

[**Note**: The arrow $\rightarrow$ means "replaces."]

8.4 Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Procedure of Row Operations

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Answer checking, $\mathbf{AA}^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -5+6 & 3-3 \\ -10+10 & 6-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

[A | I]

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \quad \boxed{2R_1 - R_2 \rightarrow R_2}$$

$2(1) - 2 = 2 - 2 = 0$

$2(3) - 5 = 1$

$\Rightarrow 2(1) - 0 = 2$

$\Rightarrow 2(0) - 1 = 0 - 1 = -1$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad \boxed{R_1 - 3R_2 \rightarrow R_1}$$

$3 - 3(1) = 3 - 3 = 0$

$\Rightarrow 1 - 3(2) = 1 - 6 = -5$

$\Rightarrow 0 - 3(-1) = 0 + 3 = 3$

[I | B]

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

8.4 Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Procedure of Row Operations

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 10 & -1 \\ -2 & -6 & 5 \end{bmatrix}$$

1 0 0

$$[\mathbf{A}\,|\,\mathbf{I}]\quad \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 3 & 10 & -1 & 0 & 1 & 0 \\ -2 & -6 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 5 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \to R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -17 & 10 & -3 & 0 \\ 0 & 1 & 5 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + 17R_3 \to R_1 \\ R_2 - 5R_3 \to R_2 \\ \\ \end{array}$$

$$[\mathbf{I}\,|\,\mathbf{B}]\quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 44 & -3 & 17 \\ 0 & 1 & 0 & -13 & 1 & -5 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 44 & -3 & 17 \\ -13 & 1 & -5 \\ 2 & 0 & 1 \end{bmatrix}$$

8.4 Inverse of the Square Matrix

Finding Inverse of a $n \times n$ Matrix Using Procedure of Row Operations

Answer checking: $\mathbf{AA}^{-1} = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 10 & -1 \\ -2 & -6 & 5 \end{bmatrix} \begin{bmatrix} 44 & -3 & 17 \\ -13 & 1 & -5 \\ 2 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 44-39-4 & -3+3-0 & 17-15-2 \\ 132-130-2 & -9+10-0 & 51-50-1 \\ -88+78+10 & 6-6+0 & -34+30+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

8.5 Solving System of Linear Equations Using Gauss-Jordan Elimination

One of the most important applications of matrices is to find the solution of the system linear equations. Consider the system of linear equations with *m* equations and *n* variable:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad \text{or} \quad \mathbf{AX} = \mathbf{B}$$

$X = \frac{B}{A} = A^{-1}B$

$X = A^{-1}B$

Where $\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ is called coefficient matrix,

$\mathbf{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is called variable matrix, and

$\mathbf{B} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ is called constant matrix.

If **A** is nonsingualr matrix, that is $\mathbf{A}^{-1}$ exist. The solution **X** can be solved by the matrix equation $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

8.5 Solving System of Linear Equations Using Gauss-Jordan Elimination

$$x + 3y = 7$$
$$2x + 5y = 12$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$ or $\mathbf{AX} = \mathbf{B}$ where $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{5-6} \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore the solutions are $x = 1$ and $y = 2$

8.5 Solving System of Linear Equations Using Gauss-Jordan Elimination

Gauss-Jordan elimination

- a step-by-step for solving system of linear equations
- works for any system of linear equations that was in large-scale
- easily implemented on a computer
- systematically transforms **an augmented matrix into a reduced form**

The system corresponding to a reduced augmented coefficient matrix is called a **reduced system**.

8.5 Solving System of Linear Equations Using Gauss-Jordan Elimination

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

GAUSS-JORDAN ELIMINATION	
Step 1.	Choose the leftmost nonzero column and use appropriate row operations to get a 1 at the top.
Step 2.	Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.
Step 3.	Repeat step 1 with the **submatrix** formed by (mentally) deleting the row used in step 2 and all rows above this row.
Step 4.	Repeat step 2 with the **entire matrix**, including the mentally deleted rows. Continue this process until the entire matrix is in reduced form.

Note: If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop, since we will have a contradiction: $0 = n$, $n \neq 0$. We can then conclude that the system has no solution.

$$
\left[\begin{array}{ccc|c}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a_{22} & a_{23} & b_2 \\
a_{31} & a_{32} & a_{33} & b_3
\end{array}\right]
$$

$$\Downarrow$$

solution

$$
\left[\begin{array}{ccc|c}
1 & 0 & 0 & x_1 \\
0 & 1 & 0 & x_2 \\
0 & 0 & 1 & x_3
\end{array}\right]
\begin{array}{l}
\to x_1 \\
\to x_2 \\
\to x_3
\end{array}
$$

8.5 Solving System of Linear Equations Using Gauss-Jordan Elimination

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{array}\right]$$

Solve by Gauss-Jordan elimination:

$$2x_1 - 2x_2 + x_3 = 3$$
$$3x_1 + x_2 - x_3 = 7$$
$$x_1 - 3x_2 + 2x_3 = 0$$

$\Rightarrow$

1

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

0

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right] \quad \begin{array}{l} \\ -3R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array}$$

1

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 0 & 4 & -3 & 3 \end{array}\right] \quad 0.1R_2 \to R_2$$

0

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 4 & -3 & 3 \end{array}\right] \quad \begin{array}{l} 3R_2 + R_1 \to R_1 \\ \\ -4R_2 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & -0.2 & 0.2 \end{array}\right] \quad -5R_3 \to R_3$$

1

0

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.1 & 2.1 \\ 0 & 1 & -0.7 & 0.7 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \begin{array}{l} 0.1R_3 + R_1 \to R_1 \\ 0.7R_3 + R_2 \to R_2 \\ \\ \end{array}$$

$$\begin{array}{cc} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} & \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] & \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \end{array}$$

$$\boxed{\begin{array}{l} x_1 = 2 \\ x_2 = 0 \\ x_3 = -1 \end{array}}$$

8.5 Solving System of Linear Equations Using Gauss-Jordan Elimination

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &= -4 \\ 4x_1 - 8x_2 + 7x_3 &= 2 \\ -2x_1 + 4x_2 - 3x_3 &= 5 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 2 & -4 & 1 & -4 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \quad \frac{1}{2}R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 4 & -8 & 7 & 2 \\ -2 & 4 & -3 & 5 \end{array}\right] \quad \begin{array}{l} -4R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & -2 & 1 \end{array}\right] \quad 0.2R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0.5 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right] \quad \begin{array}{l} -0.5R_2 + R_1 \to R_1 \\ \\ 2R_2 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

Now the last matrix is not in reduced form, since the last row produces a contradiction.

The system is inconsistent and has no solution.

8.5 Solving System of Linear Equations Using Gauss-Jordan Elimination

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 15 \\ 2x_1 + 4x_2 - 6x_3 &= 10 \\ -2x_1 - 3x_2 + 4x_3 &= -6 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 6 & -9 & 15 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \quad \frac{1}{3}R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -2 & -3 & 4 & -6 \end{array}\right] \quad \begin{array}{l} -2R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad -2R_2 + R_1 \to R_1$$

$$\begin{array}{cccccc} & x_1 & x_2 & x_3 & & \\ \left[\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right. & \begin{array}{c} 1 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \end{array} & \begin{array}{c} 1 \\ -2 \\ 0 \end{array} & \left|\begin{array}{c} -3 \\ 4 \\ 0 \end{array}\right] & \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \end{array}$$

Now the last matrix is in reduced form, since the last row produces all 0. Write the corresponding reduced system and solve. We discard the equation corresponding to the third (all 0) row in the reduced form, since it is satisfied by all values of x_1, x_2, and x_3.

For the first row: $1x_1 + 1x_3 = -3$ or $x_1 = -x_3 - 3$

For the second row: $1x_2 - 2x_3 = 4$ or $x_2 = 2x_3 + 4$

This dependent system has an infinite number of solutions. We will use a parameter to represent all the solutions. If we let $x_3 = t$, then for all real number t,

$$\left.\begin{aligned} x_1 &= -t - 3 \\ x_2 &= 2t + 4 \\ x_3 &= t \end{aligned}\right\} \text{ is a solution.}$$

Some particular solutions as: (1) when $t = 0$, then $x_1 = -3,\ x_2 = 4,\ x_3 = 0$

(2) when $t = -2$, then $x_1 = -1,\ x_2 = 0,\ x_3 = -2$

8.5 Solving System of Linear Equations Using Gauss-Jordan Elimination

Exercise 8.2

8.6 Determinants

Determinant of the matrix - assigning a real number to the any square matrix.

The determinant of a matrix A is denoted by $|A|$ or *det A*.

8.6.1 Finding determinants of a 2 × 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} 3 & -2 \\ 5 & 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -2 \\ 5 & 8 \end{vmatrix}$$

$$= (3)(8) - (-2)(5)$$

$$= 24 + 10 = 34$$

8.6 Determinants

8.6.2 Finding determinants of a 3 × 3 matrix

Third order Determinants

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

− +

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

8.6 Determinants

Minor of the Matrix (M_{ij})

For any square matrix $A=[a_{ij}]_{nxm}$ minor of an element a_{ij} denoted by M_{ij} can be obtained from the original determinant by deleting the row *i* and column *j*.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

The minor of a_{23} is $M_{23} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{13}$

8.6 Determinants

Cofactor of the matrix (C_{ij})

For any square matrix $A=[a_{ij}]_{nxm}$ the cofactor of an element a_{ij} denoted by C_{ij} can be defined **$C_{ij} = (-1)^{i+j} M_{ij}$**.

$$\text{Co-factor of } a_{11} \text{ is } C_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$$

$$\text{Co-factor of } a_{23} \text{ is } C_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -M_{23}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

8.6 Determinants

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If A is a 3 × 3 matrix, then the minor of the a_{ij} element (denoted by M_{ij}) is the determinant of the 2 × 2 matrix obtained by deleting row i and column j of A, and can be use these meaning to find the determinant of the matrix for any dimension $n \times n$.

This can be obtained by the following *expansion method*.

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a1 & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}\,a_{33} - a_{32}\,a_{23}) + a_{12}(a_{21}\,a_{33} - a_{31}\,a_{23}) + a_{13}(a_{21}\,a_{32} - a_{31}\,a_{22})$$

Where C_{ij} is the cofactor of the element a_{ij} is defined by $C_{ij} = (-1)^{i+j} M_{ij}$

Thus, the pattern of signs in front of the number a_{ij} is

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

We found that the determinant of a 3 × 3 matrix can be found by multiplying each element of the first column by its corresponding cofactor and then adding the three results. This is called the *determinant is being expanded about the first column*. It can be shown that any row or column can be used to expand a determinant.

8.6 Determinants

find M_{11} and M_{23}

$$A = \begin{bmatrix} 2 & 1 & 4 \\ -6 & 3 & -2 \\ 4 & 2 & 5 \end{bmatrix}$$

To find M_{11} we first delete row 1 and column 1 of A

$$M_{11} = \begin{vmatrix} 3 & -2 \\ 2 & 5 \end{vmatrix} = (3)(5) - (-2)(2) = 15 + 4 = 19$$

To find M_{23} we first delete row 2 and column 3 of A

$$M_{23} = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = (2)(2) - (1)(4) = 4 - 4 = 0$$

8.6 Determinants

find |A| by expanded about the first row and the second column

$$A = \begin{bmatrix} 2 & -4 & -5 \\ 1 & 0 & 4 \\ 2 & 3 & -6 \end{bmatrix}$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

a. expanded about the first row

$$|A| = +(2)\begin{vmatrix} 0 & 4 \\ 3 & -6 \end{vmatrix} - (-4)\begin{vmatrix} 1 & 4 \\ 2 & -6 \end{vmatrix} + (-5)\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 2(0-12) + 4(-6-8) - 5(3-0)$$

$$= -24 - 56 - 15 = -95$$

b. expanded about the second column

$$|A| = -(-4)\begin{vmatrix} 1 & 4 \\ 2 & -6 \end{vmatrix} + 0\begin{vmatrix} 2 & -5 \\ 2 & -6 \end{vmatrix} - (3)\begin{vmatrix} 2 & -5 \\ 1 & 4 \end{vmatrix}$$

$$= 4(-6-8) + 0 - 3(8+5) \quad = -56 - 39 = -95$$

8.6 Determinants

- **Properties of Determinants**
  1. If any row (or column) of a square matrix A contains only 0s , then $|A| = 0$.

  2. If the elements of one row (or column) of a determinant are multiplied by constant k, the value of the determinant is multiplied by constant k. That is $|B| = k|A|$.

  3. $|AB| = |A|\,|B|$.

  4. $|A| = |A^T|$.

8.6 Determinants

Exercises 8.3

8.7 Cramer's Rule

8.7.1 Cramer's Rule (2 × 2 Case)

Determinant provide the basis for another method of solving linear systems

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}$$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The solution for this system is given by

$$x = \frac{D_x}{D}, \qquad \text{and} \qquad y = \frac{D_y}{D}$$

8.7 Cramer's Rule

8.7.2 Cramer's Rule (3 × 3 Case)

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_{31}z &= d_3 \end{aligned}$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0 \qquad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, and $z = \dfrac{D_z}{D}$

Remark 1. If $D = 0$ and at least one of D_x , D_y , and D_z is not zero, then the system is non consistent and has no solution.

2. If D , D_x , D_y , and D_z are all zero, then the equations are dependent and there are infinitely many solutions.

8.7 Cramer's Rule

$$\begin{aligned} \frac{x}{2} + \frac{y}{3} &= -4 \\ \frac{x}{4} - \frac{3y}{2} &= 20 \end{aligned} \Rightarrow \begin{aligned} 3x + 4y &= -24 \\ x - 6y &= 80 \end{aligned}$$

$$D = \begin{vmatrix} 3 & 4 \\ 1 & -6 \end{vmatrix} = -18 - 4 = -22$$

$$D_x = \begin{vmatrix} -24 & 4 \\ 80 & -6 \end{vmatrix} = 144 - 320 = -176$$

$$D_y = \begin{vmatrix} 3 & -24 \\ 1 & 80 \end{vmatrix} = 240 - (-24) = 264$$

$$x = \frac{D_x}{D} = \frac{-176}{-22} = 8$$

$$y = \frac{D_y}{D} = \frac{264}{-22} = -12$$

The solution set is **{(8, -12)}**.

8.7 Cramer's Rule

$$\begin{aligned} x - 2y + z &= -4 \\ 2x + y - z &= 5 \\ 3x + 2y + 4z &= 3 \end{aligned}$$

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29 \qquad D_x = \begin{vmatrix} -4 & -2 & 1 \\ 5 & 1 & -1 \\ 3 & 2 & 4 \end{vmatrix} = 29 \qquad D_y = \begin{vmatrix} 1 & -4 & 1 \\ 2 & 5 & -1 \\ 3 & 3 & 4 \end{vmatrix} = 58 \qquad D_z = \begin{vmatrix} 1 & -2 & -4 \\ 2 & 1 & 5 \\ 3 & 2 & 3 \end{vmatrix} = -29$$

$$x = \frac{D_x}{D} = \frac{29}{29} = 1 \qquad y = \frac{D_y}{D} = \frac{58}{29} = 2 \qquad z = \frac{D_z}{D} = \frac{-29}{29} = -1$$

The solution set is **{(1, 2, - 1)}**.

8.7 Cramer's Rule

$$\begin{aligned} x+3y-z &= 4 \\ 3x-2y+z &= 7 \\ 2x+6y-2z &= 1 \end{aligned}$$

$$D=\begin{vmatrix} 1 & 3 & -1 \\ 3 & -2 & 1 \\ 2 & 6 & -2 \end{vmatrix}=0 \qquad D_x=\begin{vmatrix} 4 & 3 & -1 \\ 7 & -2 & 1 \\ 1 & 6 & -2 \end{vmatrix}=-7$$

since $D = 0$ and at least one of D_x , D_y , and D_z is not zero,

then the system is consistent. The solution set is ∅.

Assignment

Deadline for submission: Monday August, 2021

- Exercises 8.1
- Exercises 8.2 – 9, 17, 29
- Exercises 8.3 – 6, 29
- Exercises 8.4 - 15
- Review exercise