Geometry of Vector Space R^n and Inner Product Spaces

Dr. Khaing Htun

Vector operations

- Addition (**u** + **v**)
- Subtraction (**u** – **v**)
- Multiplication
  - Scalar (c**u**)
  - **Dot product** (**u**.**v**)
  - Cross product (**u**x**v**)

Dot Product

$\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$

$$\boldsymbol{u} \cdot \boldsymbol{v} = (u_1 v_1 + u_2 v_2 + \ldots + u_n v_n)$$

The dot product is a scalar

Definition: Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in $\mathbf{R}^n$. The **dot product** of **u** and **v** is denoted $\mathbf{u} \bullet \mathbf{v}$ and is defined by

$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

The dot product assigns a real number to each pair of vectors.

Example: Dot Product

Let **u** =(1, -2, 3, -4) and **v** =(3, 0, 1, 4) in R^4

The dot product of vector **u** and **v**,

$$\boldsymbol{u} \cdot \boldsymbol{v} = (1\text{x}3)+(-2\text{x}0)+(3\text{x}1)+(-4\text{x}4)$$

$$= 3 + 0 + 3 - 16 = -10$$

Properties of the Dot Product

Let **u**, **v**, and **w** be vectors in $\mathbf{R}^n$ and let *c* be a scalar. Then

- $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
- $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \mathbf{v} \cdot \boldsymbol{w}$
- $c\boldsymbol{u} \cdot \boldsymbol{v} = c(\boldsymbol{v} \cdot \boldsymbol{u}) = \boldsymbol{u} \cdot c\boldsymbol{v}$
- $\boldsymbol{u} \cdot \boldsymbol{u} = |u|^2$
- $0 \cdot \boldsymbol{u} = 0$
- $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$, and $\boldsymbol{u} \cdot \boldsymbol{u} = 0\ if\ and\ only\ if\ \boldsymbol{u} = 0$

Norm of a Vector in R^n

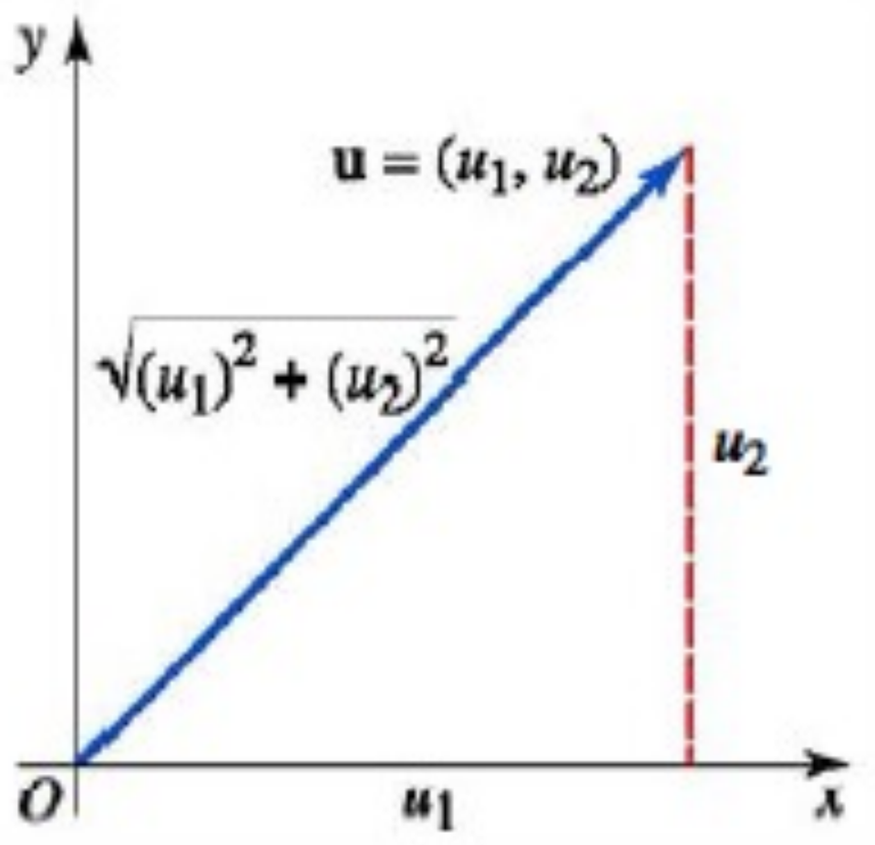
Norm/Length/Magnitude

u =(u_1, u_2) in R^2

Using **Pythagorean Theorem**,

The length of vector **u** is

$$\|\boldsymbol{u}\| = \sqrt{(u_1)^2 + (u_2)^2}$$



Norm of a Vector in R^n

Norm/Length/Magnitude

$\mathbf{u}$ =(u_1, u_2, …, u_n) in R^n

Magnitude of vector $\mathbf{u}$,

$$\|\boldsymbol{u}\| = \sqrt{(u_1)^2 + \cdots + (u_n)^2}$$

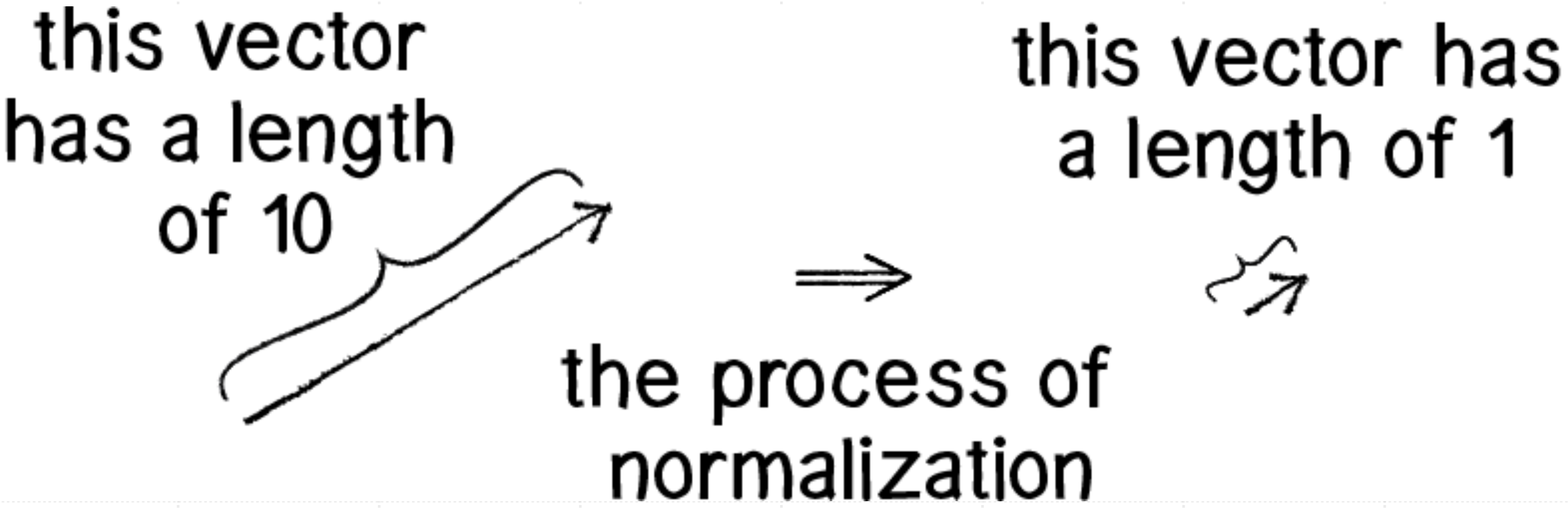
or

$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$$

Norm of a Vector in R^n

Normalizing a vector

$$normalized\ vector, \hat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$$



Norm of a Vector in R^n

Definition: A **unit vector** is a vector whose norm is one. If $\mathbf{v}$ is a nonzero vector, then the vector $\mathbf{u} = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector in the direction of $\mathbf{v}$.

This procedure of constructing a unit vector in the same direction as a given vector is called **normalizing** the vector.

Example: Norm of a Vector in R^n

Let $\mathbf{u} = (1, -2, 3, -4)$ and $\mathbf{v} = (3, 0, 1, 4)$ in R^4

Norm of the vector **u** and normalized vector

$$\| \mathbf{u} \| = \sqrt{(1)^2 + (-2)^2 + (3)^2 + (-4)^2} = \sqrt{1+4+9+16} = \sqrt{30}$$

$$\frac{\mathbf{1}}{\| \mathbf{u} \|}\mathbf{u} = \frac{1}{\sqrt{30}}(1, -2, 3, -4) = \left(\frac{1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{3}{\sqrt{30}}, \frac{-4}{\sqrt{30}} \right)$$

Norm of the vector **v** and normalized vector

$$\| \mathbf{v} \| = \sqrt{(3)^2 + (0)^2 + (1)^2 + (4)^2} = \sqrt{9+0+1+16} = \sqrt{26}$$

$$\frac{\mathbf{1}}{\| \mathbf{v} \|}\mathbf{v} = \frac{1}{\sqrt{26}}(3, 0, 1, 4) = \left(\frac{3}{\sqrt{26}}, 0, \frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}} \right)$$

Angle Between Vectors

Let **u** = (*a*, *b*) and **v** = (*c*, *d*) be position vectors in $\mathbf{R}^2$.

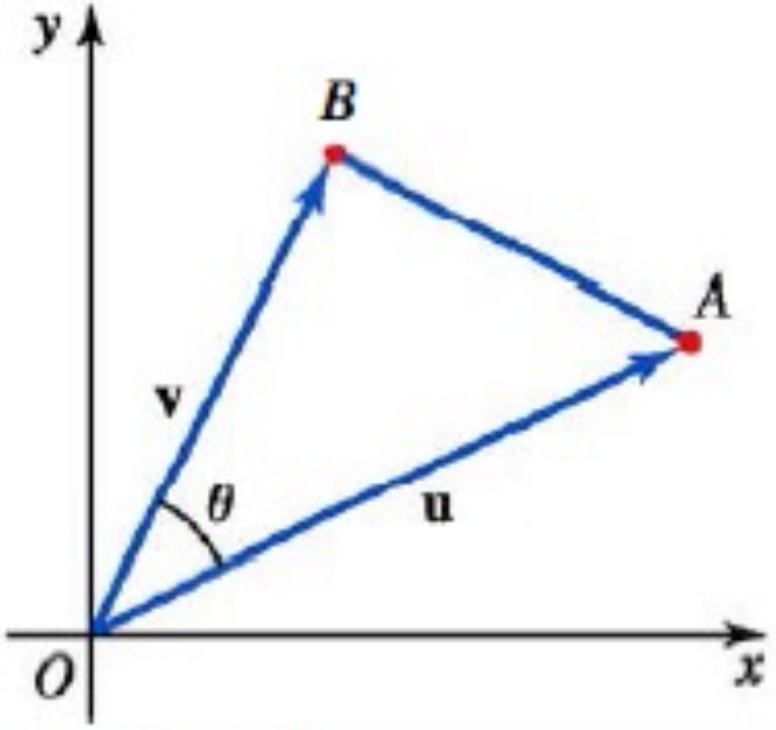
To find the angle between vector **u** and **v**, θ

Law of cosines,

$$(AB)^2 = (OA)^2 + (OB)^2 - 2(OA)(OB)\cos\theta$$

$$\cos\theta = \frac{(OA)^2 + (OB)^2 - (AB)^2}{2(OA)(OB)}$$

$$\boxed{\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}} \qquad 0 \le \theta \le \pi$$



$$(OA)^2 + (OB)^2 - (AB)^2 = \|u\|^2 + \|v\|^2 - \|v-u\|^2$$

$$2(OA)(OB) = 2\|u\|\|v\|$$

Example: Angle Between Vectors

Let **u** =(1, -2, 3, -4) and **v** =(3, 0, 1, 4) in R^4

Angle between the vector **u** and **v**

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

$$\cos\theta = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-10}{\sqrt{30}\times\sqrt{26}} = -0.3581$$

$$\boxed{\theta = 110.98°}$$

$$\boldsymbol{u}\cdot\boldsymbol{v} = \text{(1x3)+(-2x0)+(3x1)+(-4x4)}$$
$$= 3 + 0 + 3 - 16 = -10$$

$$\|\mathbf{u}\| = \sqrt{(1)^2+(-2)^2+(3)^2+(-4)^2} = \sqrt{1+4+9+16} = \sqrt{30}$$

$$\|\mathbf{v}\| = \sqrt{(3)^2+(0)^2+(1)^2+(4)^2} = \sqrt{9+0+1+16} = \sqrt{26}$$

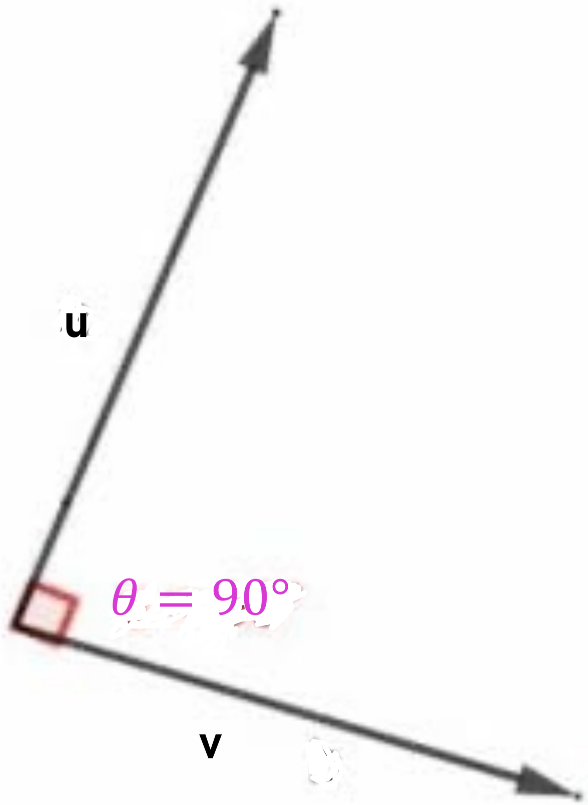
Orthoginal vectors

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

$$\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$$

$$\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos 90°$$

$$\boldsymbol{u}\cdot\boldsymbol{v} = \mathbf{0}$$



Orthoginal vectors

Let **u** =(1, -2, 3, -4) and **v** =(3, 0, 1, 4) in R^4

Angle between the vector **u** and **v**

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-10}{\sqrt{30}\times\sqrt{26}} = -0.3581$$

$$\boldsymbol{\theta = 110.98^\circ} \neq 90^\circ$$

Are the vector **u** and **v** orthogonal?

$\boldsymbol{u \cdot v = 0}??$

No. The dot product is –10, thus the vector **u** and **v** are not orthogonal.

Example

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

a. Dot product of vector **u** and **v.** Are the vector **u** and **v** orthogonal?

b. Norm of the vector **u** and normalized vector.

c. Norm of the vector **v** and normalized vector.

d. Angle between the vector **u** and **v.**

Example

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

a. Dot product of vector **u** and **v.** Are the vector **u** and **v** orthogonal?

$$\mathbf{u} \bullet \mathbf{v} = (1 \times -1) + (3 \times 2) + (-5 \times 1) = -1 + 6 - 5 = 0$$

The dot product is zero, thus the vector **u** and **v** are orthogonal.

Example

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix},$$

b. Norm of the vector **u** and normalized vector.

Norm of the vector **u** = $\|\mathbf{u}\| = \sqrt{(1)^2 + (3)^2 + (-5)^2} = \sqrt{1+9+25} = \sqrt{35}$

Normalized vector = $\dfrac{\mathbf{1}}{\|\mathbf{u}\|}\mathbf{u} = \dfrac{1}{\sqrt{35}}\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{35} \\ 3/\sqrt{35} \\ -5/\sqrt{35} \end{bmatrix}$

This vector is a unit vector in the direction of $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$

Example

$$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

c. Norm of the vector **v** and normalized vector.

Norm of the vector **v** = $\| \mathbf{v} \| = \sqrt{(-1)^2 + (2)^2 + (1)^2} = \sqrt{1+4+1} = \sqrt{6}$

Normalized vector = $\dfrac{\mathbf{1}}{\| \mathbf{v} \|}\mathbf{v} = \dfrac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

This vector is a unit vector in the direction of $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Example

d. Angle between the vector **u** and **v.**

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{0}{\sqrt{35} \times \sqrt{6}} = 0$$

$$\theta = 90^\circ$$ – orthogonal

Euclidean Distance

Distance Between Points

$x = (x_1, x_2)\ and\ y = (y_1, y_2)\ in\ R^2$

$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

Definition: Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbf{R}^n$. The distance between $\mathbf{x}$ and $\mathbf{y}$ is denoted $d(\mathbf{x}, \mathbf{y})$ and is defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_n - y_n)^2}$$

Note: We can also write this distance formula as $d(\mathbf{x}, \mathbf{y}) = \| \mathbf{x} - \mathbf{y} \|$

Example: Distance Between Points

$x = (1, -2, 3, 0)\ and\ y = (4, 0, -3, 5)\ in\ R^4$

the distance between the points x and y,

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + (x_4 - y_4)^2}$$

$$= \sqrt{(1-4)^2 + (-2-0)^2 + (3-(-3))^2 + (0-5)^2}$$

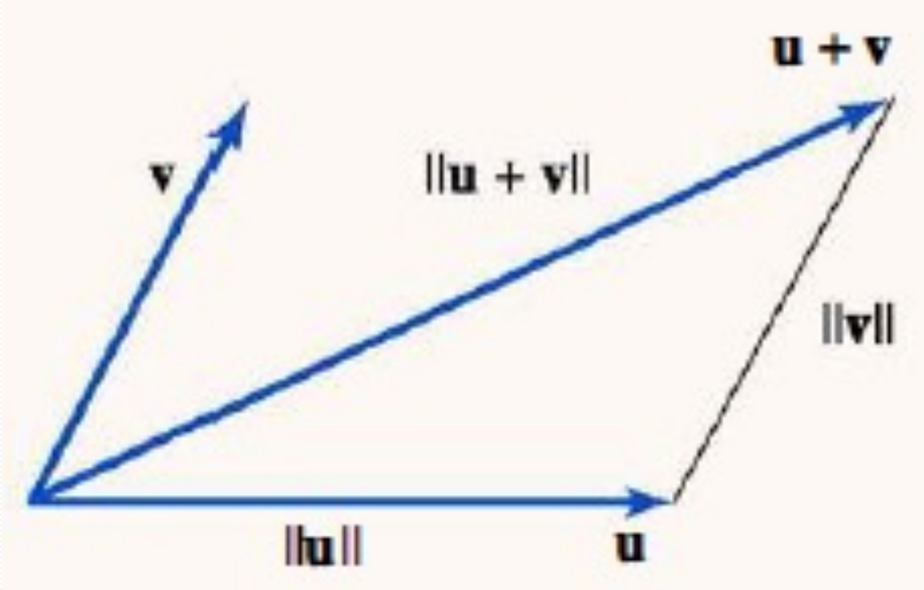
$$= \sqrt{9+4+36+25}$$

$$= \sqrt{74}$$

Distance Between Points

Triangle inequality:

$\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\|+\|\boldsymbol{v}\|$



length of one side of a triangle **cannot exceed** the sum of the lengths of the other two side

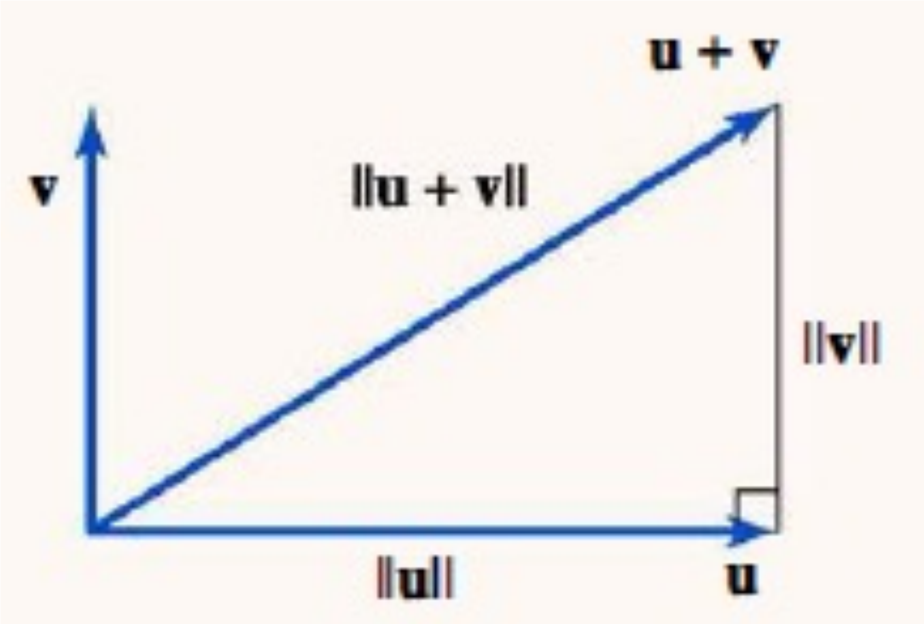
Distance Between Points

Pythagorean Theorem:

If $\boldsymbol{u} \cdot \boldsymbol{v} = \mathbf{0}$,

then $||\boldsymbol{u} + \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$

Orthogonal vectors



Example: Distance Between Points

Let **u** = (1, −2, 3, −4) and **v** = (3, 0, 1, 4) in $\mathbf{R}^4$, **using Pythagorean Theorem** to prove that vector **u** and **v** are not orthogonal vectors.

$$\boxed{\|\mathbf{u}+\mathbf{v}\|^{\mathbf{2}} \neq (\|\mathbf{u}\|^{\mathbf{2}} + \|\mathbf{v}\|^{\mathbf{2}})}$$

vector **u** and **v** are not orthogonal vectors

Pythagorean Theorem:

If $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$, then $\boldsymbol{u}\cdot\boldsymbol{v} = \mathbf{0}$ (**u** and **v** are orthogonal vectors)

$$\|\mathbf{u}\| = \sqrt{(1)^2 + (-2)^2 + (3)^2 + (-4)^2} = \sqrt{1+4+9+16} = \sqrt{30}$$

$$\|\mathbf{v}\| = \sqrt{(3)^2 + (0)^2 + (1)^2 + (4)^2} = \sqrt{9+0+1+16} = \sqrt{26}$$

$$\|\mathbf{u}\|^2 = 30$$

$$\|\mathbf{v}\|^2 = 26$$

$\|\mathbf{u}\|^{\mathbf{2}} + \|\mathbf{v}\|^{\mathbf{2}}$= 30+26 = 56

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{(1+3)^2 + (-2+0)^2 + (3+1)^2 + (-4+4)^2} = \sqrt{16+4+16+0} = \sqrt{36}$$

$$\|\mathbf{u}+\mathbf{v}\|^2 = 36$$

Inner Product Spaces

- A vector space – any set that satisfied **axioms**-based on the properties of $\mathbf{R}^n$
- Extention of the concepts of R^n to **general vector space**
  - dot product of two vectors
  - norm of a vector
  - angle between vectors
  - distance between points
- This concepts are used to approximate functions by polynomials – a technique that is used to implement functions on calculators and computers.

Dot Product

Definition:

An inner product on a real vector space V is a function that associates a number, denoted, $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ of V. This function must satisfy the following conditions for vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and scalar c.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (Symmetry axiom)
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ (Additive axiom)
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ (Homogeneity axiom)
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$ (positive definite axiom)

Norm of a Vector

Definition:

Let V be an inner product space. The norm of a vector $\mathbf{v}$ is denoted $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Angle Between Two Vectors

Definition:

Let V be a real inner product space The angle θ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in V is given by $\cos\theta = \dfrac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$.

Orthogonal Vectors

Let V be an inner product space.

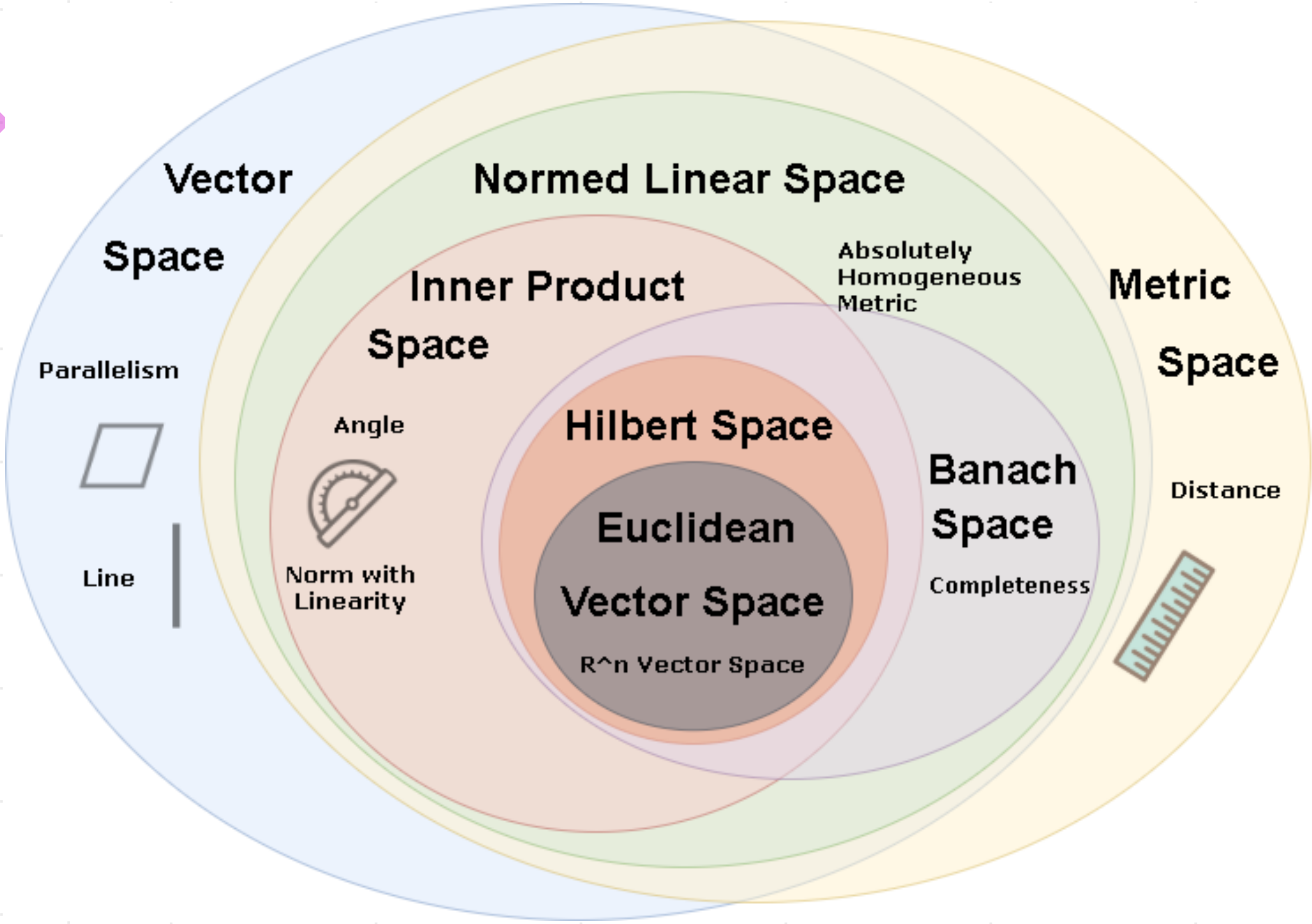
Two nonzero vectors in *V* are said to be orthogonal if the angle between them is a right angle.

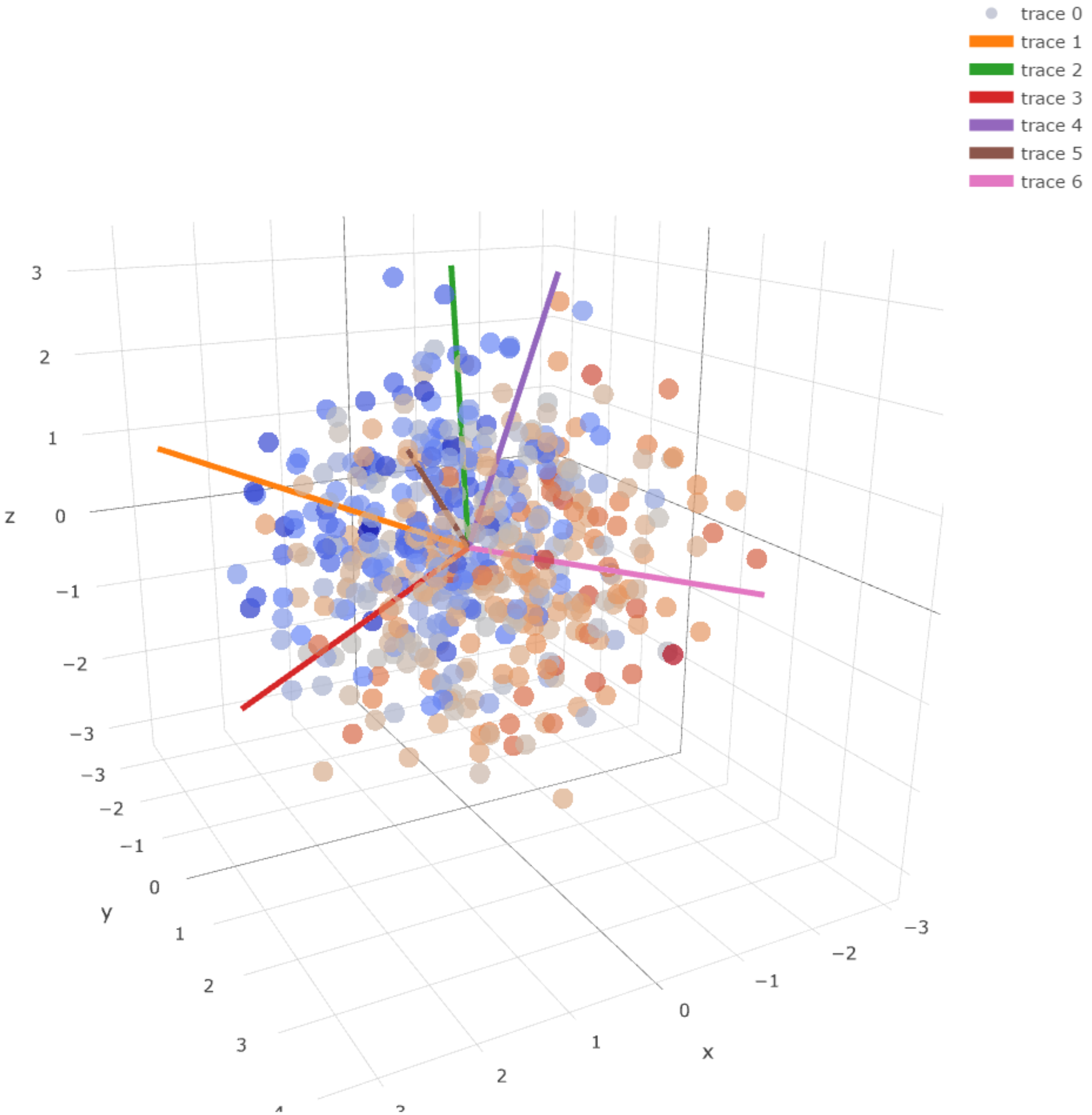
Thus two nonzero vectors **u** and **v** in *V* are orthogonal if

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

Distance

Definition: Let V be an inner product space with vector norm defined by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. The **distance** between two vectors (points) $\mathbf{u}$ and $\mathbf{v}$ is given by $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$


Vector Space
Normed Linear Space
Inner Product Space
Absolutely Homogeneous Metric
Metric Space
Parallelism
Angle
Hilbert Space
Banach Space
Euclidean Vector Space
Line
Norm with Linearity
Completeness
Distance
R^n Vector Space


trace 0
trace 1
trace 2
trace 3
trace 4
trace 5
trace 6
z
y
x

Class Assignment

1. For each pairs of vector **u** and **v** determine the following values

   a. Dot product of vector **u** and **v.**

   b. Norm of the vector **u** and normalized vector.

   c. Norm of the vector **v** and normalized vector.

   d. Angle between the vector **u** and **v.**

   e. Are the vector **u** and **v** orthogonal? Using Pythagorean Theorem.

a) **u** = (1, 2, 3), **v** = (4, 1, 0)

b) **u** = (1, −2, 3, −4), **v** = (−9, 8, −7, 6)

c) **u** = $\begin{bmatrix} 7 \\ 1 \\ -2 \end{bmatrix}$, **v** = $\begin{bmatrix} 3 \\ -5 \\ 8 \end{bmatrix}$

2. Find the distances between the following pairs of point **x** and **y**.

a) **x** = (1, 2, 3), **y** = (4, 1, 0)

b) **x** = (1, 2, −1, 3, 1), **y** = (2, 0, 1, 0, 4)