Strong Stability Preserving Integrating Factor Runge Kutta Methods

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Overview

The Setting

- We are numerically solving hyperbolic conservation laws:

$$u_t + f(u)_x = 0$$

- For a problem with a smooth solution, looking at the L_2 stability of the linearized problem is enough to ensure convergence of a consistent numerical method.
- We wish to consider problems with a discontinuous solution, where this is not enough.
- One choice is to have a method be total variation diminishing (TVD), in other words satisfying the strong stability property:

$$||u^{n+1}||_{\mathrm{TV}} \leq ||u^n||_{\mathrm{TV}}$$

- A method of lines approach is popular where the spatial discretization satisfies this property when paired with forward Euler.

How does TVD help?

- One specific difficulty in getting numerical approximations on discontinuous solutions are the artificial oscillations that tend to appear near the discontinuities.
- It has been shown that if a method is TVD, it is monotonicity preserving.
- For a given numerical method this means that if $u_j^0 \geq u_{j+1}^0$ for all j then $u_j^n \geq u_{j+1}^n$ for all n, j.
- In particular this will not allow oscillations near an isolated propagating discontinuity since the initial data would be monotone.

Strong Stability Preserving

- Assume we have a high order spatial discretization that satisfies the strong stability property when paired with forward Euler.
- Applying this spatial discretization our conservation law can be written in semi-discrete form:

$$u_t = F(u)$$

- We want to evolve this with a time discretization that is higher order than forward Euler, but allows the spatial discretization to retain the strong stability property.
- By using a convex combination of forward Euler steps we can create such discretizations which are called strong stability preserving (SSP).

Review on Explicit Runge-Kutta

- For the equation $u_t = F(u)$, an explicit Runge-Kutta method can be written in Shu-Osher form:

$$u^{(0)} = u^n$$

$$u^{(i)} = \sum_{j=0}^{i-1} \left(\alpha_{i,j} u^{(j)} + \Delta t \beta_{i,j} F\left(u^{(j)}\right)\right), \quad i = 1, \dots, s$$

$$u^{n+1} = u^{(s)}$$

- For consistency we need $\sum_{j=0}^{i-1} \alpha_{i,j} = 1$.
- If all $\alpha_{i,j}$ and $\beta_{i,j}$ are non-negative and a corresponding $\alpha_{i,j}$ is zero iff $\beta_{i,j}$ is zero, it is possible to rearrange each stage into a convex combination of forward Euler steps.

SSP Runge Kutta Methods

- Because the spatial discretization satisfies the strong stability property when paired with forward Euler we have:

$$\left\|u^{n+1}\right\| = \|u^n + \Delta t F\left(u^n\right)\| \leq \|u^n\|$$

under the step size restriction $0 \leq \Delta t \leq \Delta t_{FE}$.

SSP Runge Kutta Methods

- When we rearrange each stage and write the Runge-Kutta scheme as a convex combination of forward Euler steps:

$$\left\| u^{(i)} \right\| = \left\| \sum_{j=0}^{i-1} \left(\alpha_{i,j} u^{(j)} + \Delta t \beta_{i,j} F\left(u^{(j)} \right) \right) \right\|$$
$$\leq \sum_{j=0}^{i-1} \alpha_{i,j} \left\| u^{(j)} + \Delta t \frac{\beta_{i,j}}{\alpha_{i,j}} F\left(u^{(j)} \right) \right\| \leq \| u^n \|$$

  under the time step restriction $\Delta t \leq \min_{i,j} \frac{\alpha_{i,j}}{\beta_{i,j}} \Delta t_{FE}$ (by using the previous inequality).

- We call this method strong stability preserving so long as $C = \min_{i,j} \frac{\alpha_{i,j}}{\beta_{i,j}} > 0$

Integrating Factor Methods

- We now get even more specific to conservation laws of the form $u_t = Lu + N(u)$, where L is a stiff linear operator and N is a non-stiff nonlinear operator.
- The stiff linear operator causes the need for a very small time step when using an explicit time discretization, we will solve this part **exactly** to alleviate this restriction.
- We do this by using the familiar integrating factor technique.

Integrating Factor Methods

Starting with $u_t = Lu + N(u)$, multiply by the matrix exponential e^{-Lt}:

$$e^{-Lt}u_t - e^{-Lt}Lu = e^{-Lt}N(u)$$
$$\left(e^{-Lt}u\right)_t = e^{-Lt}N(u)$$

Finally through a change of variables we have:

$$w_t = e^{-Lt}N\left(e^{Lt}w\right)$$

Since the stiff linear portion is being solved exactly through the matrix exponential, we are free to evolve this with an explicit time discretization.

Combining Methods

- Each stage of an explicit SSP Runge Kutta method written in Shu-Osher form, $u^{(i)}$ corresponds to the solution at time $t_i = t^n + c_i \Delta t$.
- For an explicit SSP Runge Kutta method, if these c_i values are non-decreasing, we can combine it with the integrating factor method $w_t = e^{-Lt} N\left(e^{Lt} w\right)$, and the resulting method will also be SSP.

Example Method

The method SSPIFRK(3,3)$^{+}$ is given:

$$
\begin{aligned}
u^{(1)} &= \frac{1}{2} e^{\frac{2}{3}\Delta t L} u^n + \frac{1}{2} e^{\frac{2}{3}\Delta t L} \left(u^n + \frac{4}{3} \Delta t N\left(u^n\right) \right) \\
u^{(2)} &= \frac{2}{3} e^{\frac{2}{3}\Delta t L} u^n + \frac{1}{3} \left(u^{(1)} + \frac{4}{3} \Delta t N\left(u^{(1)}\right) \right) \\
u^{n+1} &= \frac{59}{128} e^{\Delta t L} u^n + \frac{15}{128} e^{\Delta t L} \left(u^n + \frac{4}{3} \Delta t N\left(u^n\right) \right) \\
&\quad + \frac{27}{64} e^{\frac{1}{3}\Delta t L} \left(u^{(2)} + \frac{4}{3} \Delta t N\left(u^{(2)}\right) \right)
\end{aligned}
$$

Notation: SSPIFRK(s,p)$^{+}$ has s stages and is order p. The $+$ denotes the nondecreasing c_i values.

Our Application

- We plan on using an SSPIFRK method as a component of an asymptotic preserving (AP) scheme to solve the compressible Euler equations for any Mach number.
- A quick review on AP schemes:

Asymptotic Preserving (AP) Schemes

- Asymptotic preserving schemes are used on multi-scale problems, phenomena that can be viewed at microscopic and macroscopic levels by different models.
- Often these models are connected by a scaling parameter ε, such that when $\varepsilon \to 0$, solutions of the microscopic model converge to solutions of the macroscopic model.

Solving Multi-Scale Problems

Why not just use any numerical method to solve the microscopic model and take ε small to get the macroscopic solution too?

- In many cases using an explicit scheme yields a convergence criterion where grid size is dependant on ε.
- We can try to use an implicit scheme to avoid this dependence, but even so we are not guaranteed a numerical solution consistent with the macroscopic model's solution as $\varepsilon \to 0$.

AP Scheme

- $\mathcal{F}^0$: Macroscopic model
- $\mathcal{F}^\varepsilon$: Microscopic model
- $\mathcal{F}^\varepsilon_\delta$: Numerical discretization of microscopic model
- $\mathcal{F}^0_\delta$: Numerical discretization of macroscopic model

$$
\begin{tikzcd}
\mathcal{F}^\varepsilon_\delta \arrow[r, "\varepsilon \to 0"] \arrow[d, "\delta \to 0"] & \mathcal{F}^0_\delta \arrow[d, "\delta \to 0"] \\
\mathcal{F}^\varepsilon \arrow[r, "\varepsilon \to 0"] & \mathcal{F}^0
\end{tikzcd}
$$

Figure:

AP Scheme

- $\mathcal{F}^0$: Incompressible Euler
- $\mathcal{F}^\varepsilon$: Compressible Euler
- $\mathcal{F}^\varepsilon_\delta$: Numerical discretization of Compressible Euler
- $\mathcal{F}^0_\delta$: Numerical discretization of Incompressible Euler
- ε in this case is the Mach number.

$$
\begin{tikzcd}
\mathcal{F}^\varepsilon_\delta \arrow[r, "\varepsilon \to 0"] \arrow[d, "\delta \to 0"] & \mathcal{F}^0_\delta \arrow[d, "\delta \to 0"] \\
\mathcal{F}^\varepsilon \arrow[r, "\varepsilon \to 0"] & \mathcal{F}^0
\end{tikzcd}
$$

Figure:

Compressible Euler

We can write the non-dimensionalized compressible Euler equations in divergence form as:

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0$$

where:

$$U = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho E \end{pmatrix}, F(U) = \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u} \otimes \mathbf{u} + \frac{p}{\varepsilon^2} \mathbf{Id} \\ (\rho E + p) \mathbf{u} \end{pmatrix}$$

Here ρ is density, $\mathbf{u}$ is velocity, $\rho \mathbf{u}$ is momentum, E is total specific energy, ρE is total energy, and p is pressure.
Also $\varepsilon = \frac{u_{\mathrm{ref}}}{\sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}}$ is the mach number.

Compressible Euler

The system is closed by an equation of state:

$$p = (\gamma - 1)\left(\rho E - \frac{\varepsilon^2}{2}\rho\|\mathbf{u}\|^2\right)$$

where $\gamma > 1$ is the ratio of specific heats.

Applying SSPIFRK

- First split the flux term F into a non-stiff nonlinear term $\hat{F}$ and a stiff linear term $\tilde{F}$.

$$\hat{F}(U) = \begin{pmatrix} \rho\mathbf{u} \\ \rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{Id} \\ (\rho E + \Pi)\mathbf{u} \end{pmatrix}, \tilde{F}(U) = \begin{pmatrix} 0 \\ \frac{1-\varepsilon^2}{\varepsilon^2} p\mathbf{Id} \\ (p - \Pi)\mathbf{u} \end{pmatrix}$$

  So $F = \tilde{F} + \hat{F}$ and Π is an auxiliary variable depending on pressure.

- We now have

$$\frac{\partial U}{\partial t} + \nabla \cdot \hat{F}(U) + \nabla \cdot \tilde{F}(U) = 0$$

Applying SSPIFRK

- Next apply an appropriate spatial discretization to the flux term to get the problem into a semi-discrete form.
- Then we can apply the SSPIFRK method to the problem. Hopefully by solving the linear part exactly, we remove the stability constraint due to epsilon completely.

References

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