Open Channel Flow with Manning's Equation

Guoping Tang

January 1, 2021

Abstract

Equations for analytical and numerical solutions involving Manning's equation are derived for triangular, rectangular, trapezoidal, and parabolic open channels and circular, elliptical, and arch closed conduits.

1 Theory

1.1 Manning's Equation

The Manning's equation for the velocity and discharge of uniform flow in open channels is [1, 2, 3, 4]:

$$v = \frac{K_u}{n} R_h^{\frac{2}{3}} S^{\frac{1}{2}}, \tag{1}$$

$$Q = vA = \frac{K_u}{n} A R_h^{\frac{2}{3}} S^{\frac{1}{2}} = \frac{K_u}{n} A^{\frac{5}{3}} P^{-\frac{2}{3}} S^{\frac{1}{2}}, \tag{2}$$

where

v = Mean velocity, m/s (ft/s),

Q = Discharge, m^3/s (ft^3/s),

n = Manning's coefficient of roughness,

R_h = Hydraulic radius, m (ft). $R_h = P/A$,

P = Wetted perimeter, m (ft),

A = Crossing-section area of flowing water perpendicular to the direction of flow, m^2 (ft^2),

S = Energy slope, m/m (ft/ft). For steady uniform flow $S = S_0$, and

K_u = units conversion factor, 1 for SI, 1.486 for English units.

1.2 Normal Depth

Depending on the geometry of the channel, both A and P are depdent on normal depth y. To calculate y for a given Q, we solve

$$f_d(y_i) = \frac{K_u}{n} A^{\frac{5}{3}} P^{-\frac{2}{3}} S^{\frac{1}{2}} - Q \tag{3}$$

for $f_d(y_i) = 0$. Analytical solutions are available in some special cases, for example, triangular channels. In general, a numerical solution is used to solve the nonlinear equation iteratively. Using Newton's method [5], the iteration starts with an initial guess y_0, and iterates with

$$y_{i+1} = y_i - \frac{f_d(y_i)}{f_d'(y_i)} \tag{4}$$

where

$$f_d'(y_i) = \frac{\partial f_d}{\partial y} = \frac{K_u}{n} S^{\frac{1}{2}} \left(\frac{5}{3} R_h^{\frac{2}{3}} \frac{\partial A}{\partial y} - \frac{2}{3} R_h^{\frac{5}{3}} \frac{\partial P}{\partial y} \right) = \frac{K_u}{3n} S^{\frac{1}{2}} R_h^{\frac{2}{3}} \left(5A' - 2R_h P' \right) \tag{5}$$

until

$$|f_d(y_i)| \leq TOLQ, \tag{6}$$

$$|y_{i+1} - y_i| \leq TOLD, \tag{7}$$

or

$$i \geq MAXI \tag{8}$$

with

$TOLQ$ = discharge tolerance, m^3/s (ft^3/s),

$TOLD$ =depth tolerance, m (ft),

$MAXI$ = maximum number of iteration.

When the iteration stops with criteria Eq.(8), an error is shown to notify the user. The user may adjust y_0 or t_0, $TOLQ$, $TOLD$, and/or $MAXI$ to obtain a satisfactory solution. For circular, elliptical, and arch pipes, a replacement of variable y with an alternative variable t (for example) is convient. These equations remain valid. An alternative tolerance $TOLA$ (in lieu of $TOLD$) is specified when t is an angle.

For open channel flow in closed conduits such as circular, elliptical, and arch pipes, the calculated discharge peaks before the pipe is full [1, 2, 4]. The normal depth y_{max} or t_{max} at peak discharge Q_{max} can be calculated by solving

$$\frac{\partial Q}{\partial t} = \frac{K_u}{3n} R_h^{\frac{2}{3}} \left(5 \frac{\partial A}{\partial t} - 2 \frac{A}{P} \frac{\partial P}{\partial t} \right) S^{\frac{1}{2}} = 0. \tag{9}$$

or

$$f(t) = 5P \frac{\partial A}{\partial t} - 2A \frac{\partial P}{\partial t} = 5PA' - 2AP' = 0. \tag{10}$$

In absence of an analytical solution, Newton's method is used with

$$t_{i+1} = t_i - \frac{f(t_i)}{f'(t_i)}, \tag{11}$$

$$f'(t_i) = 5A''P + 3A'P' - 2AP''. \tag{12}$$

If $Q > Q_{max}$, an error is shown with y_{max} returned as normal depth.
In cases where Q is not a monotonically increasing function with y, multiple y values may result in the same Q value. The lowest value is returned as normal depth.
In summary, to calculate normal depth with Newton's method, we need A,P, A' and P'. A'' and P'' are needed when Q_{max} needs to be calculated using Newton's method.

1.3 Critical Depth

Critical flow occurs when the specific energy

$$E = \frac{v^2}{2g} + y = \frac{Q^2}{2gA^2} + y \tag{13}$$

reaches a minimum (g is acceleration of gravity, 32.17 ft/s^2 for US Customery units, 9.81 m/s^2 for SI). Namely,

$$\frac{\partial E}{\partial y} = -\frac{Q^2}{gA^3}\frac{\partial A}{\partial y} + 1 = 0. \tag{14}$$

To solve the equation with Newton's method, critical depth y_c or t_c is calculated by

$$t_{c,i+1} = t_{c,i} - \frac{f(t_{c,i})}{f'(t_{c,i})} \tag{15}$$

where

$$f_c(t) = gA^3 - Q^2\frac{\partial A}{\partial y} \tag{16}$$

$$f_c'(t) = 3gA^2\frac{\partial A}{\partial t} - Q^2\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) \tag{17}$$

The critical velocity v_c is

$$v_c = \frac{Q}{A_c} = \sqrt{g\frac{A}{\frac{\partial A}{\partial y}}} = \sqrt{gD_h} \tag{18}$$

where $D_h = A/\frac{\partial A}{\partial y}$ is the hydraulic depth.
The Froude number, $F = v/v_c$. The critical slope S_c is backcalculated from Eq. (1).
In summary, A, P, A' and P' are needed to calculate normal depth y_n, A, P, A', P', A'' and P'' to calculate t_{max}, y_{max}, Q_{max}, and $\frac{\partial A}{\partial y}$, $\frac{\partial A}{\partial t}$, and $\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right)$ to calculate t_c and y_c using Newton's method.

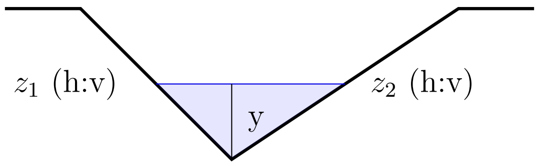


Figure 1: Triangular Section

2 Triangular

$$A = \frac{1}{2}(z_1 + z_2)y^2 \tag{19}$$

$$P = \left(\sqrt{1+z_1^2} + \sqrt{1+z_2^2}\right) y \tag{20}$$

$$R_h = \frac{1}{2}\frac{(z_1 + z_2)}{\sqrt{1+z_1^2} + \sqrt{1+z_2^2}} y \tag{21}$$

For normal flow, Eq. (2) becomes

$$Q = \frac{K_u}{n}\frac{z_1 + z_2}{2}\left(\frac{1}{2}\frac{z_1 + z_2}{\sqrt{1+z_1^2} + \sqrt{1+z_2^2}}\right)^{\frac{2}{3}} y^{\frac{8}{3}} S^{\frac{1}{2}}. \tag{22}$$

Note this equation can be used to backcalculate normal depth y_n from Q.

$$\frac{\partial A}{\partial y} = (z_1 + z_2)y = T_w \tag{23}$$

with T_w as the width of the water surface.
For critical flow, Eq. (16) becomes

$$f_c(y) = \frac{g}{8}(z_1 + z_2)^3 y^6 - Q^2(z_1 + z_2)y = 0. \tag{24}$$

$$y_c^5 = \frac{8Q^2}{g(z_1 + z_2)^2} \tag{25}$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{\frac{1}{2}(z_1 + z_2)y^2}{(z_1 + z_2)y} = \frac{1}{2}(z_1 + z_2)y_c \tag{26}$$

$$v_c = \sqrt{\frac{1}{2}g(z_1 + z_2)y_c} \tag{27}$$

Analytical solution is derived for normal depth (Eq. 22) and critical depth (Eq. 25).

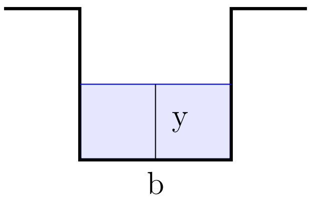


Figure 2: Rectangular Section

3 Rectangular

$$A = by \tag{28}$$

$$P = b + 2y \tag{29}$$

$$R_h = \frac{by}{b + 2y} \tag{30}$$

$$\frac{\partial A}{\partial y} = b = T_w \tag{31}$$

$$f_c(y) = gb^3y^3 - Q^2b = 0 \tag{32}$$

$$y_c^3 = \frac{Q^2}{gb^2} \tag{33}$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{by_c}{b} = y_c \tag{34}$$

$$v_c = \sqrt{gy_c} \tag{35}$$

Analytical solution is derived for critical depth (Eq: 33). Neton's method is used to calculate normal depth y_n.

4 Trapezoidal

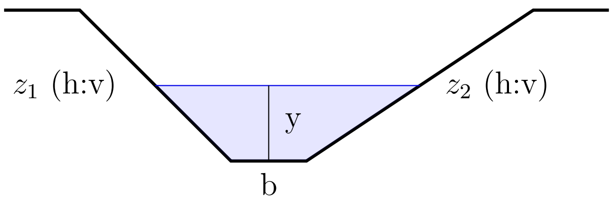


Figure 3: Trapezoidal Section

$$A = \frac{1}{2}(z_1 + z_2)y^2 + by \tag{36}$$

$$P = \left(\sqrt{1+z_1^2} + \sqrt{1+z_2^2}\right) y + b \tag{37}$$

$$\frac{\partial A}{\partial y} = (z_1 + z_2)y + b = T_w \tag{38}$$

$$\frac{\partial^2 A}{\partial y^2} = z_1 + z_2 \tag{39}$$

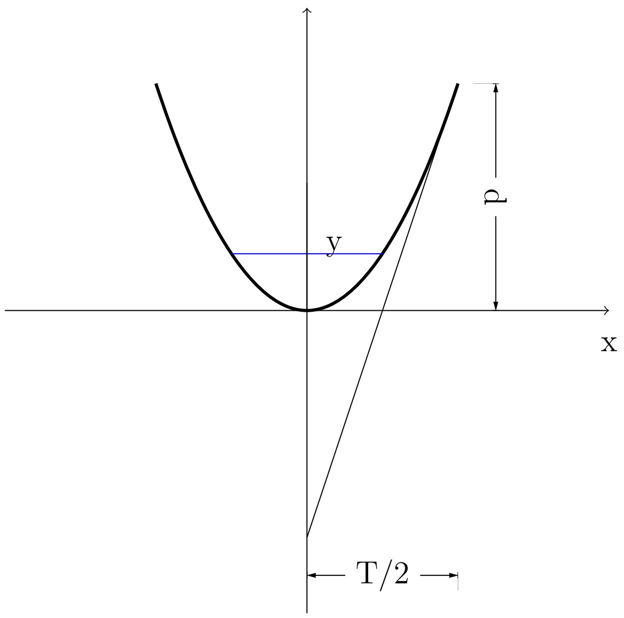
$$\frac{dP}{dy} = \sqrt{1+z_1^2} + \sqrt{1+z_2^2} \tag{40}$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{\frac{1}{2}(z_1 + z_2)y_c^2 + by_c}{(z_1 + z_2)y_c + b} \tag{41}$$

Neton's method is used to calculate normal depth y_n and critical depth y_c. An initial guess for critical depth [2] is

$$y_{c,0} = 0.81 \left(\frac{Q^2}{gz^{0.75}b^{1.25}}\right)^{0.27} - \frac{b}{30z} \tag{42}$$

5 Parabolic



For a parabola with top width T and depth d,

$$y = 4\frac{d}{T^2}x^2 \tag{43}$$

$$A = 2xy - \int_{-x}^{x} 4\frac{d}{T^2}x^2 dx = \frac{16}{3}\frac{d}{T^2}x^3 = \frac{2}{3}\frac{T}{\sqrt{d}}y^{\frac{3}{2}} \tag{44}$$

$$\frac{\partial A}{\partial y} = T\sqrt{\frac{y}{d}} = 2x = T_w \tag{45}$$

$$\frac{\partial^2 A}{\partial y^2} = \frac{T}{2\sqrt{yd}} \tag{46}$$

To calculate P

$$\frac{\partial y}{\partial x} = 8\frac{d}{T^2}x \tag{47}$$

$$\left(\frac{\partial y}{\partial x}\right)^2 = 8\frac{d}{T^2}8\frac{d}{T^2}x^2 = 16\frac{d}{T^2}y = \frac{y}{a} \tag{48}$$

with $a = T^2/16d$

$$\frac{\partial P}{\partial y} = 2\sqrt{1 + (dx/dy)^2} = 2\sqrt{1 + \frac{a}{y}} \tag{49}$$

$$\int \sqrt{1 + \frac{a}{t}}dx = \frac{a}{2}\ln\frac{\sqrt{1+\frac{a}{t}}+1}{\sqrt{1+\frac{a}{t}}-1} + \sqrt{t^2+at} + c = \frac{a}{2}\ln\frac{\sqrt{t^2+at}+t}{\sqrt{t^2+at}-t} + \sqrt{t^2+at} + c \tag{50}$$

Let $b = \sqrt{y^2 + ay}$,

$$P = 2\int_0^y \sqrt{1 + \frac{a}{y}}dy == a\ln\frac{b+y}{b-y} + 2b \tag{51}$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{\frac{2}{3}\frac{T}{\sqrt{d}}y^{\frac{3}{2}}}{T\sqrt{\frac{y}{d}}} = \frac{2}{3}y_c \tag{52}$$

$$v_c = \sqrt{\frac{2}{3}gy_c} \tag{53}$$

Neton's method is used to calculate normal depth y_n and critical depth y_c. An initial guess for critical depth [2] is

$$y_{c,0} = \left(0.84\frac{4dQ^2}{gT}\right)^{0.25} \tag{54}$$

6 Circular

$$y = r(1 - \cos\frac{\theta}{2}) \tag{55}$$

$$A = \frac{1}{2}(\theta - \sin\theta)r^2 \tag{56}$$

$$P = \theta r \tag{57}$$

$$R = \frac{1}{2}\left(1 - \frac{\sin\theta}{\theta}\right)r \tag{58}$$

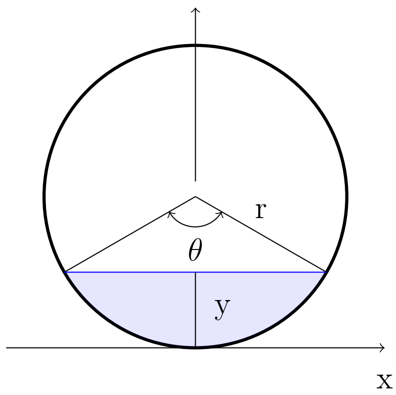


Figure 4: Circular Section

$$\frac{\partial y}{\partial \theta} = \frac{r}{2}\sin\frac{\theta}{2} \tag{59}$$

$$\frac{\partial A}{\partial \theta} = \frac{1}{2}(1-\cos\theta)r^2 \tag{60}$$

$$\frac{\partial A}{\partial y} = \frac{\frac{\partial A}{\partial \theta}}{\frac{\partial y}{\partial \theta}} = \frac{\frac{1}{2}(1-\cos\theta)r^2}{\frac{r}{2}\sin\frac{\theta}{2}} = 2r\sin\frac{\theta}{2} \tag{61}$$

$$\frac{\partial}{\partial \theta}\left(\frac{dA}{dy}\right) = r\cos\frac{\theta}{2} = T_w \tag{62}$$

$$\frac{\partial P}{\partial \theta} = r \tag{63}$$

To calculate θ_{max}, y_{max}, and Q_{max}, Eq. (10)

$$5PA' - 2AP' = 5\theta r\frac{1}{2}(1-\cos\theta)r^2 - 2\frac{1}{2}(\theta-\sin\theta)r^2 r = 3\theta - 5\theta\cos\theta + 2\sin\theta = 0, \tag{64}$$

i.e.,

$$3\theta - 5\theta\cos\theta + 2\sin\theta = 0, \tag{65}$$

The θ when Q peaks,

$$\theta_{max} = 5.27810713, \tag{66}$$

$$Q_{max} = 2.2189\frac{K_u}{n}r^{8/3}S^{1/2}, \tag{67}$$

$$y_{max} = 1.87636243r. \tag{68}$$

Neton's method is used to calculate normal depth y_n and critical depth y_c.

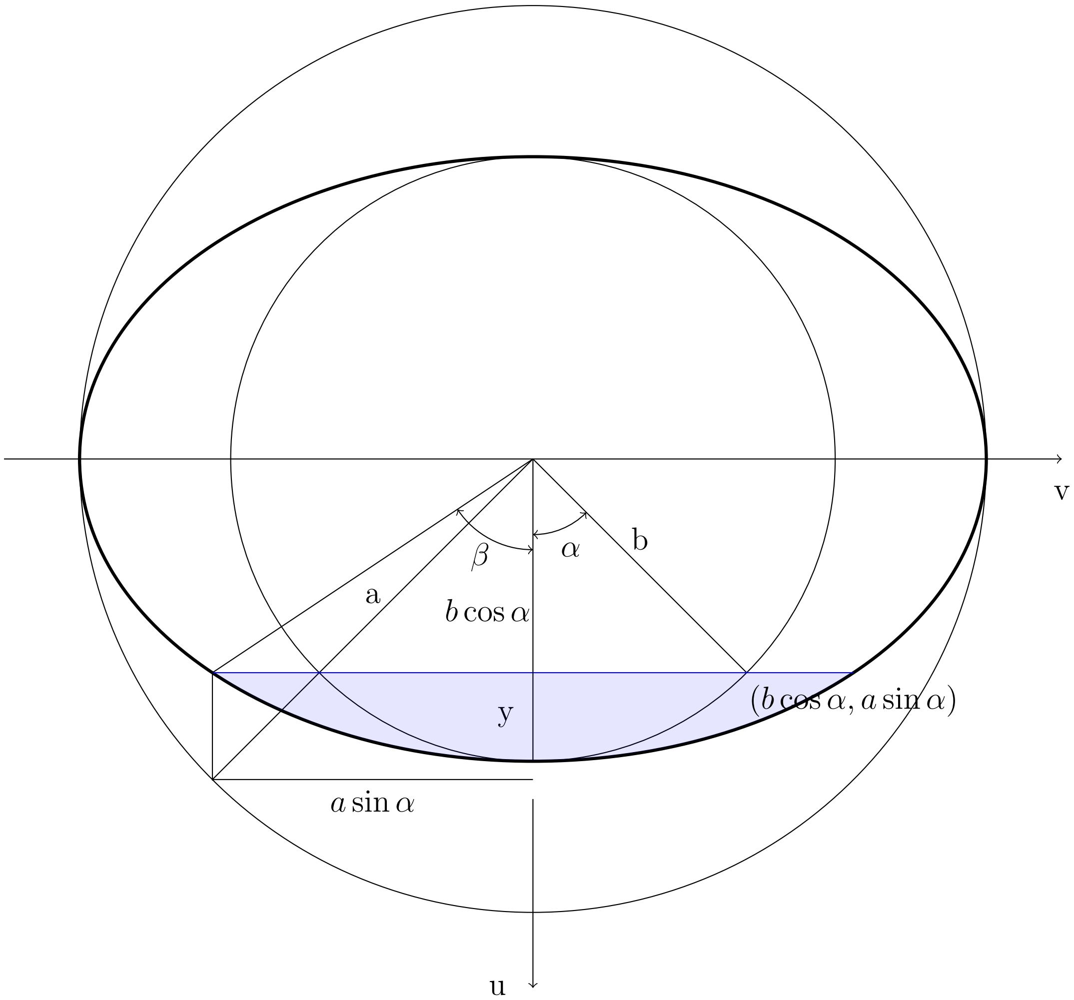


Figure 5: Elliptical Section ($a \geq b$)

7 Elliptical

7.1 Basic Equations

$$\begin{aligned} u &= b\cos\alpha \\ v &= a\sin\alpha \end{aligned} \tag{69}$$

$$\tan\beta = \frac{a}{b}\tan\alpha \tag{70}$$

$$\sin^2\alpha = \frac{\tan^2\alpha}{1+\tan^2\alpha} = \frac{b^2\tan^2\beta}{a^2+b^2\tan^2\beta} \tag{71}$$

$$\cos^2\alpha = \frac{1}{1+\tan^2\alpha} = \frac{a^2}{a^2+b^2\tan^2\beta} \tag{72}$$

$$r^2 = a^2\sin^2\alpha + b^2\cos^2\alpha \tag{73}$$

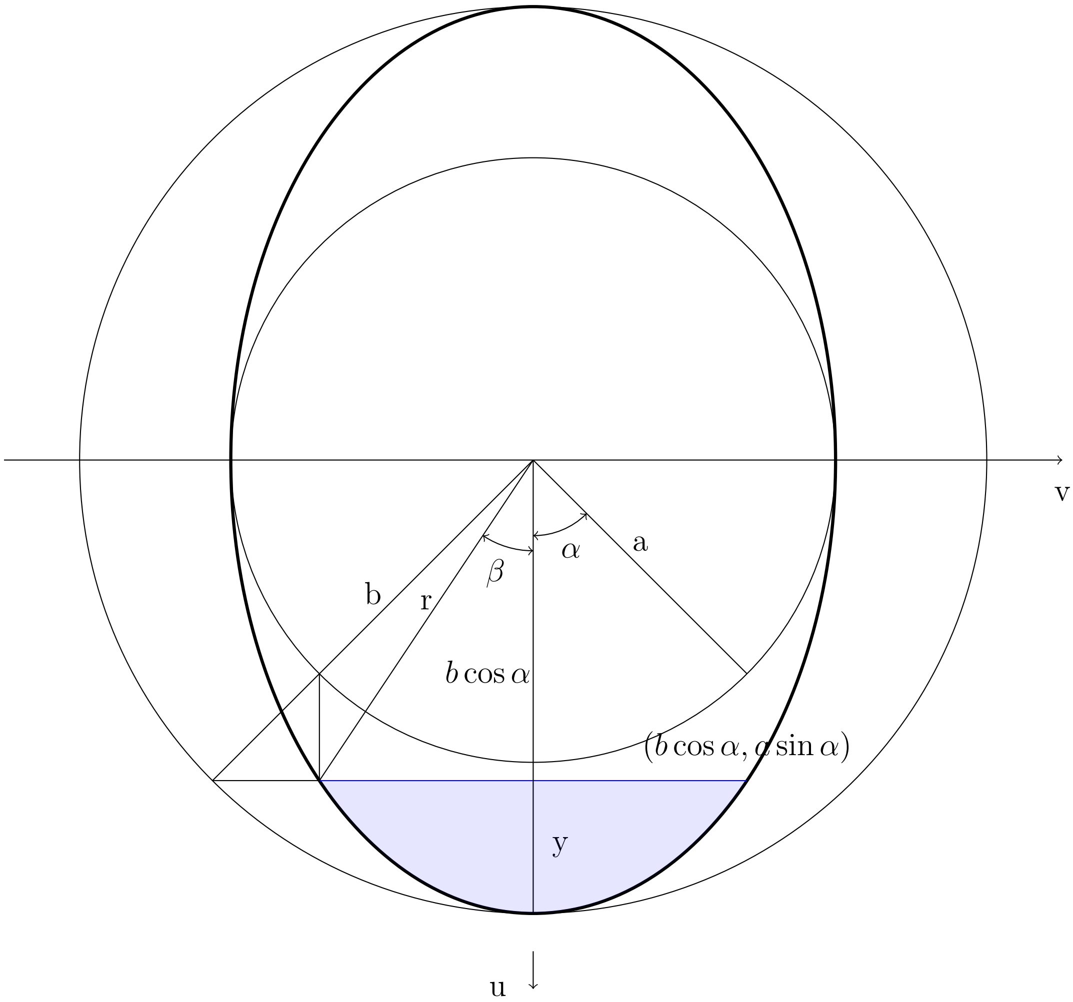


Figure 6: Elliptical Section ($a < b$)

$$y = b(1 - \cos\alpha) \tag{74}$$

$$\cos\alpha = 1 - \frac{y}{b} \tag{75}$$

$$y' = \frac{\partial y}{\partial \alpha} = b\sin\alpha \tag{76}$$

7.2 Flow Area

The flow area is

$$A = 2\int_{b\cos\alpha}^{b} v du = -2ab\int_{\alpha}^{0} \sin^2 t dt = ab\int_{0}^{\alpha}(1-\cos 2t)dt = ab(\alpha - \frac{1}{2}\sin 2\alpha) \tag{77}$$

$$A' = ab(1 - \cos 2\alpha) \tag{78}$$

$$A'' = 2ab \sin 2\alpha \tag{79}$$

$$\frac{\partial A}{\partial y} = \frac{A'}{y'} = 2a \sin \alpha = T_w \tag{80}$$

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial A}{\partial y}\right) = 2a \cos \alpha \tag{81}$$

$$D_h = \frac{A'}{\frac{\partial A}{\partial y}} = \frac{b(\alpha - \frac{1}{2}\sin 2\alpha)}{2 \sin \alpha} \tag{82}$$

7.3 Wet Perimeter

7.3.1 $a \geq b$

In the case of $a \geq b$ (Figure 5), the wet perimeter is

$$P = 2\int_0^\alpha \sqrt{a^2 \cos^2 t + b^2 \sin^2 t}dt = 2a\int_0^\alpha \sqrt{1 - (1 - b^2/a^2)\sin^2 t}dt = 2aE(\alpha, \eta) \tag{83}$$

with $\eta^2 = 1 - b^2/a^2$, $E(\alpha, \zeta)$ as the Legendre ellitipcal integral of the second kind.

$$E(\alpha, \eta) = \alpha - \frac{1}{2}\eta^2 \int_0^\alpha \sin^2 tdt - \frac{1}{2 \cdot 4}\eta^4 \int_0^\alpha \sin^4 tdt - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\eta^6 \int_0^\alpha \sin^6 tdt + ... \tag{84}$$

$$\begin{aligned}
\int \sin^n tdt &= -\frac{1}{n}\sin^{n-1}\cos t + \frac{n-1}{n}\int \sin^{n-2} tdt \\
\int_0^\alpha \sin^2 tdt &= -\frac{1}{2}\sin\alpha\cos\alpha + \frac{1}{2}\alpha \\
\int_0^\alpha \sin^4 tdt &= -\frac{1}{4}\sin^3\alpha\cos\alpha + \frac{3}{4}\int_0^\alpha \sin^2 tdt \\
\int_0^\alpha \sin^6 tdt &= -\frac{1}{6}\sin^5\alpha\cos\alpha + \frac{5}{6}\int_0^\alpha \sin^4 tdt \\
\int_0^\alpha \sin^{2n} tdt &= -\frac{1}{2n}\sin^{2n-1}\alpha\cos\alpha + \frac{2n-1}{2n}\int_0^\alpha \sin^{2n-2} tdt
\end{aligned} \tag{85}$$

n	a_n	b_n	c_n	d_n	e_n	f_n
1	$-\frac{1}{2}\eta^2$	$-\frac{1}{2}$	$\sin\alpha\cos\alpha$	$\frac{1}{2}$	$b_1c_1 + d_1\alpha$	a_1e_1
2	$a_1\frac{\eta^2}{4}$	$-\frac{1}{4}$	$c_1\sin^2\alpha$	$\frac{3}{4}$	$b_2c_2 + d_2e_1$	a_2e_2
3	$a_2\frac{\eta^2}{6}$	$-\frac{1}{6}$	$c_2\sin^2\alpha$	$\frac{5}{6}$	$b_3c_3 + d_3e_2$	a_3e_3
n	$a_{n-1}\frac{\eta^2}{2n}$	$-\frac{1}{2n}$	$c_{n-1}\sin^2\alpha$	$\frac{2n-1}{2n}$	$b_nc_n + d_ne_{n-1}$	a_ne_n

7.3.2 $a < b$

For $a < b$, the wet perimeter is (Figure 6)

$$P = 2\int_0^\alpha \sqrt{a^2\cos^2 t + b^2\sin^2 t}dt = 2b\int_0^\alpha \sqrt{1-(1-a^2/b^2)\cos^2 t}dt = 2bF(\alpha,\zeta) \quad (86)$$

with $\zeta^2 = 1 - a^2/b^2$.

$$F(\alpha,\eta) = \alpha - \frac{1}{2}\zeta^2\int_0^\alpha \cos^2 tdt - \frac{1}{2^2\cdot 2!}\zeta^4\int_0^\alpha \cos^4 tdt - \frac{1\cdot 3}{2^3\cdot 3!}\zeta^6\int_0^\alpha \cos^6 tdt - \cdots \quad (87)$$

$$\begin{aligned}
\int \cos^n tdt &= \frac{1}{n}\sin t\cos^{n-1} t + \frac{n}{n-1}\int \cos^{n-2} tdt \\
\int_0^\alpha \cos^2 tdt &= \frac{1}{2}\sin\alpha\cos\alpha + \frac{1}{2}\alpha \\
\int_0^\alpha \cos^4 tdt &= \frac{1}{4}\sin\alpha\cos^3\alpha + \frac{3}{4}\int_0^\alpha \cos^2 tdt \\
\int_0^\alpha \cos^6 tdt &= \frac{1}{6}\sin\alpha\cos^5\alpha + \frac{5}{6}\int_0^\alpha \cos^4 tdt \\
\int_0^\alpha \cos^{2n} tdt &= \frac{1}{2n}\sin\alpha\cos^{2n-1}\alpha + \frac{2n-1}{2n}\int_0^\alpha \cos^{2n-2} tdt
\end{aligned} \quad (88)$$

For both cases,

$$P' = 2\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \quad (89)$$

$$P'' = -\frac{(a^2-b^2)\sin 2\alpha}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}} \quad (90)$$

Neton's method is used to calculate normal depth y_n, critical depth y_c, α_{max}, y_{max}, and Q_{max}.

8 Arc

The geometry of an arc is determined by r_b, r_t, r_c, and $rise$.

$$\begin{aligned}
r_b &= AB = BO = BE \\
r_t &= AC = CG \\
r_c &= DE = DF = DG \\
c &= BC = BO - CO = BO - (OG - CG) = r_b + r_t - rise
\end{aligned} \quad (91)$$

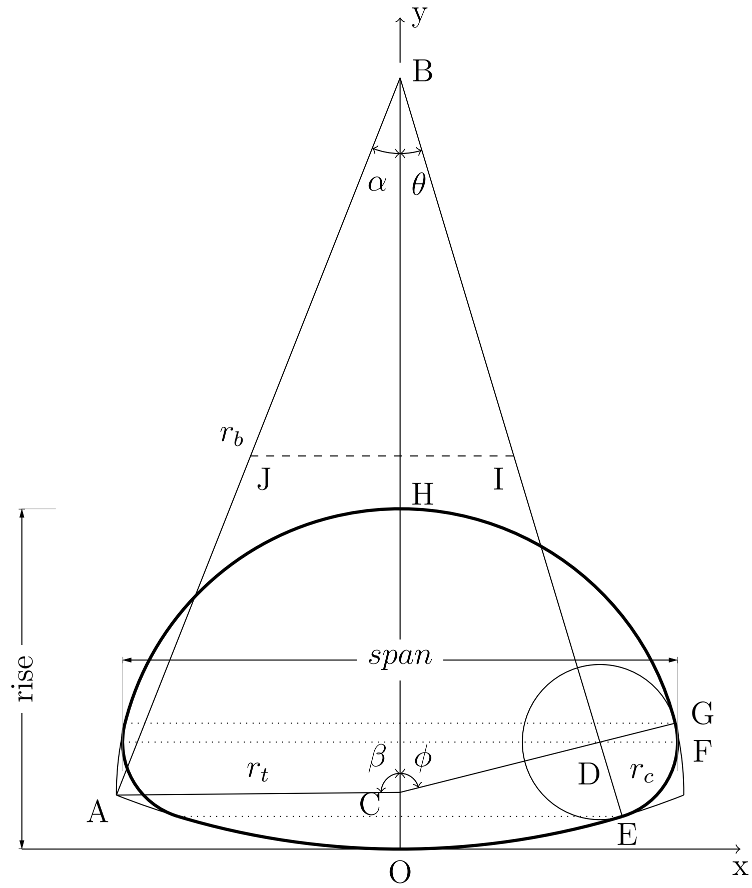


Figure 7: Arc Section

$$
\begin{aligned}
\cos\alpha &= \frac{r_b^2 + c^2 - r_t^2}{2r_b c} \\
\cos\beta &= \frac{r_t^2 + c^2 - r_b^2}{2r_t c} \\
\cos\theta &= \frac{(r_b - r_c)^2 + c^2 - (r_t - r_c)^2}{2(r_b - r_c)c} \\
\cos\phi &= \frac{(r_t - r_c)^2 + c^2 - (r_b - r_c)^2}{2(r_t - r_c)c}
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
x_A &= r_b \sin\alpha \\
y_A &= r_b(1-\cos\alpha) \\
y_B &= r_b \\
y_C &= rise - r_t = OG - CG \\
x_D &= (r_b - r_c)\sin\theta = (r_t - r_c)\sin\phi \\
y_D &= r_b - (r_b - r_c)\cos\theta = y_F \\
x_E &= r_b \sin\theta \\
y_E &= r_b(1-\cos\theta) \\
x_F &= x_D + r_c = span/2 \\
x_G &= r_t \sin\phi \\
y_G &= y_C + r_t\cos\phi = rise - r_t(1-\cos\phi) \\
y_H &= rise
\end{aligned}
\tag{93}
$$

$$
\begin{aligned}
A_E &= r_b^2(\theta - \sin\theta\cos\theta) \\
P_E &= 2r_b\theta \\
T_E &= 2r_b\sin\theta \\
A_F &= A_E + r_c^2(\pi/2-\theta) + (x_E + x_D)(y_D - y_E) \\
P_F &= P_E + 2r_c(\pi/2-\theta) \\
T_F &= 2(x_D + r_c) \\
A_G &= A_F + r_c^2(\pi/2-\phi) + (x_D + x_G)(y_G - y_D) \\
P_G &= P_F + 2r_c(\pi/2-\phi) \\
T_G &= 2r_t\sin\phi \\
A_T &= A_G + r_t^2(\phi - \sin\phi\cos\phi) \\
P_T &= P_G + 2r_t\phi
\end{aligned}
\tag{94}
$$

For $0 \le y \le y_E$, $0 \le t \le \theta$

$$
\begin{aligned}
x &= r_b \sin t \\
y &= r_b(1-\cos t) \\
y' &= r_b \sin t = \frac{\partial y}{\partial t} \\
A &= r_b^2(t - \sin t\cos t) \\
A' &= r_b^2(1-\cos 2t) = \frac{\partial A}{\partial t} \\
\frac{\partial A}{\partial y} &= 2r_b\sin t = T_w \\
\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) &= 2r_b\cos t \\
P &= 2r_b t
\end{aligned}
\tag{95}
$$

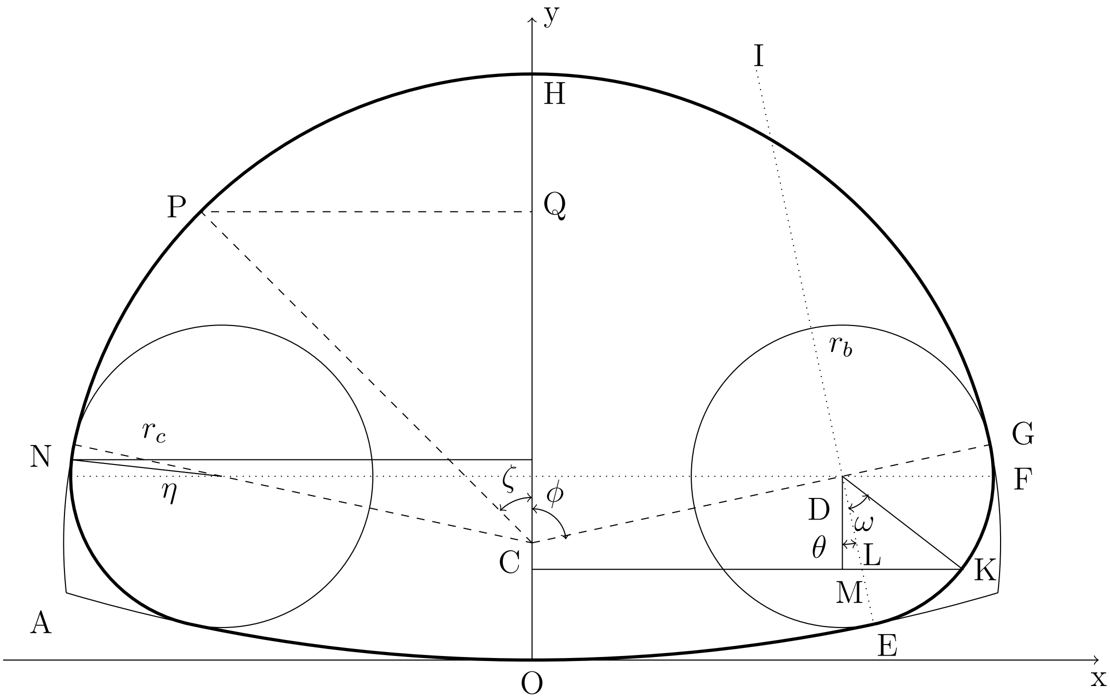


Figure 8: Arc Section

For $y_E \leq y \leq y_F$, $0 \leq w \leq \pi/2 - \theta$

$$
\begin{aligned}
x &= x_D + r_c \sin(\omega+\theta) \\
x_L &= x_D + r_c \cos(\omega+\theta)\tan\theta \\
x'_L &= -r_c \sin(\omega+\theta)\tan\theta \\
x''_L &= -r_c \cos(\omega+\theta)\tan\theta \\
y &= y_D - r_c \cos(\omega+\theta) \\
y' &= r_c \sin(\omega+\theta) \\
y'' &= r_c \cos(\omega+\theta) \\
A &= A_E + r_c^2\omega - r_c^2 \sin(\omega+\theta)\cos(\omega+\theta) + r_c^2\cos^2(\omega+\theta)\tan\theta + (x_E + x_L)(y - y_E) \\
A' &= r_c^2 - r_c^2\cos 2(\omega+\theta) - r_c^2 \sin 2(\omega+\theta)\tan\theta + (x_E+x_L)y' + x'_L(y-y_E) \\
&= 2r_c^2\sin^2(\omega+\theta) - 2r_c^2\sin(\omega+\theta)\cos(\omega+\theta)\tan\theta + (x_E+x_L)y' + x'_L(y-y_E) \\
\frac{\partial A}{\partial y} &= \frac{A'}{y'} = \frac{2r_c^2\sin^2(\omega+\theta) - 2r_c^2\sin(\omega+\theta)\cos(\omega+\theta)\tan\theta}{r_c\sin(\omega+\theta)} + x_E + x_L - (y-y_E)\tan\theta \\
&= 2r_c\sin(\omega+\theta) - 2r_c\cos(\omega+\theta)\tan\theta + x_E + x_D + r_c\cos(\omega+\theta)\tan\theta \\
&\quad - [y_D - r_c\cos(\omega+\theta) - y_E]\tan\theta \\
&= 2r_c\sin(\omega+\theta) + x_E + x_D - (y_D - y_E)\tan\theta = 2x_D + 2r_c\sin(\omega+\theta) = T_w \\
\frac{\partial}{\partial\omega}\left(\frac{\partial A}{\partial y}\right) &= 2r_c\cos(\omega+\theta) \\
P &= P_E + 2r_c\omega
\end{aligned}
\tag{96}
$$

For $y_F \leq y \leq y_G$, $0 \leq \eta \leq \pi/2 - \phi$

$$\begin{aligned}
x &= x_D + r_c \cos\eta \\
y &= y_D + r_c \sin\eta \\
A &= A_F + r_c^2\eta + (2x_D + r_c\cos\eta) r_c \sin\eta \\
P &= P_F + 2r_c\eta \\
y' &= r_c \cos\eta \\
A' &= r_c^2 + 2x_D r_c \cos\eta + r_c^2\cos(2\eta) \\
\frac{\partial A}{\partial y} = \frac{A'}{y'} &= \frac{r_c^2 + 2x_D r_c\cos\eta + r_c^2\cos(2\eta)}{r_c\cos\eta} = 2x_D + 2r_c\cos\eta = T_w \\
\frac{\partial}{\partial\eta}\left(\frac{\partial A}{\partial y}\right) &= -2r_c\sin\eta
\end{aligned} \tag{97}$$

For $y_G \leq y \leq y_H$, $0 \leq \zeta \leq \phi$

$$\begin{aligned}
y &= y_C + r_t\cos\zeta \\
A &= A_T - r_t^2(\zeta - \sin\zeta\cos\zeta) \\
P &= P_T - 2r_t\zeta = P_G + 2r_t(\phi - \zeta) \\
y' &= -r_t\sin\zeta \\
A' &= -r_t^2[1 - \cos(2\zeta)] \\
A'' &= -2r_t^2\sin(2\zeta) \\
P'(\zeta) &= -2r_t \\
P''(\zeta) &= 0 \\
\frac{\partial A}{\partial y} = \frac{A'}{y'} &= 2r_t\sin\zeta = T_w \\
\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) &= 2r_t\cos\zeta
\end{aligned} \tag{98}$$

Maximum discharge occurs close to the top of arch, the ζ_{max} is solved using Newton's method

$$\zeta_{i+1} = \zeta_i - \frac{f(\zeta_i)}{f'(\zeta_i)}, \tag{99}$$

with

$$f(\zeta) = 5A'P - 2AP', \tag{100}$$

and

$$f'(\zeta) = 3A'P' + 5A''P - 2AP''. \tag{101}$$

References

[1] Ven Te Chow. *Open-Channel Hydraulics.* McGraw-Hill, New York, NY, 1985.

[2] Richard H. French. *Open-Channel Hydraulics.* McGraw-Hill, New York, NY, 1985.

[3] Everett V. Richardson James D. Schall and Johnny L. Morris. Introduction to hightway hydraulics, 4th ed. Technical Report FHWA-NHI-08-090(HDS-4), FHWA, 2008.

[4] Bruce R. Munson, Theodore H. Okiishi, Wade W. Huebsch, and Alric P. Rothmayer. *Fundamentals of Fluid Mechanics.* Wiley, 7 edition, 2012.

[5] Gilbert Strang. *Calculus.* Wellesley-Cambridge Press, Wellesley, MA, 1985.