

Open Channel Flow with Manning's Equation

Guoping Tang

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Abstract

Equations for analytical and numerical solutions involving Manning's equation are derived for triangular, rectangular, trapezoidal, and parabolic open channels and circular, elliptical, and arch closed conduits.

1 Theory

1.1 Manning's Equation

The Manning's equation for the velocity and discharge of uniform flow in open channels is [1, 2, 3, 4]:

$$v = \frac{K_u}{n} R^{\frac{2}{3}} S^{\frac{1}{2}}, \quad (1)$$

$$Q = vA = \frac{K_u}{n} AR^{\frac{2}{3}} S^{\frac{1}{2}} = \frac{K_u}{n} A^{\frac{5}{3}} P^{-\frac{2}{3}} S^{\frac{1}{2}}, \quad (2)$$

where

v = Mean velocity, m/s (ft/s),

Q = Discharge, m^3/s (ft^3/s),

n = Manning's coefficient of roughness,

R = Hydraulic radius, m (ft). $R = P/A$,

P = Wetted perimeter, m (ft),

A = Crossing-section area of flowing water perpendicular to the direction of flow, m^2 (ft^2),

S = Energy slope, m/m (ft/ft). For steady uniform flow $S = S_0$, and

K_u = units conversion factor, 1 for SI, 1.486 for English units.

1.2 Normal Depth

Depending on the geometry of the channel, both A and P are dependent on normal depth y . To calculate y for a given Q , we solve

$$f_d(y_i) = \frac{K_u}{n} A^{\frac{5}{3}} P^{-\frac{2}{3}} S^{\frac{1}{2}} - Q \quad (3)$$

for $f_d(y_i) = 0$. Analytical solutions are available in some special cases, for example, triangular channels. In general, a numerical solution is used to solve the nonlinear equation iteratively. Using Newton's method [5], the iteration starts with an initial guess y_0 , and iterates with

$$y_{i+1} = y_i - \frac{f_d(y_i)}{f'_d(y_i)} \quad (4)$$

where

$$f'_d(y_i) = \frac{\partial f_d}{\partial y} = \frac{K_u}{n} S^{\frac{1}{2}} \left(\frac{5}{3} R^{\frac{2}{3}} \frac{\partial A}{\partial y} - \frac{2}{3} R^{\frac{5}{3}} \frac{\partial P}{\partial y} \right) \quad (5)$$

until

$$|f_d(y_i)| \leq TOLQ, \quad (6)$$

$$|y_{i+1} - y_i| \leq TOLD, \quad (7)$$

or

$$i \geq MAXI \quad (8)$$

with

$TOLQ$ = discharge tolerance, m^3/s (ft^3/s),

$TOLD$ = depth tolerance, m (ft),

$MAXI$ = maximum number of iteration.

When the iteration stops with criteria Eq.(8), an error is shown to notify the user. The user may adjust y_0 or t_0 , $TOLQ$, $TOLD$, and/or $MAXI$ to obtain a satisfactory solution. For circular, elliptical, and arch pipes, a replacement of variable y with an alternative variable t (for example) is convenient. These equations remain valid. An alternative tolerance $TOLA$ (in lieu of $TOLD$) is specified when t is an angle.

For open channel flow in closed conduits such as circular, elliptical, and arch pipes, the calculated discharge peaks before the pipe is full [1, 2, 4]. The normal depth y_{max} or t_{max} at peak discharge Q_{max} can be calculated by solving

$$\frac{\partial Q}{\partial t} = \frac{K_u}{n} \left(\frac{5}{3} R^{\frac{2}{3}} \frac{\partial A}{\partial t} - \frac{2}{3} R^{\frac{5}{3}} \frac{\partial P}{\partial t} \right) S^{\frac{1}{2}} = 0. \quad (9)$$

or

$$f(t) = 5P \frac{\partial A}{\partial t} - 2A \frac{\partial P}{\partial t} = 5PA' - 2AP' = 0. \quad (10)$$

In absence of an analytical solution, Newton's method is used with

$$t_{i+1} = t_i - \frac{f(t_i)}{f'(t_i)}, \quad (11)$$

$$f'(t_i) = 5A''P + 3A'P' - 2AP'' \quad (12)$$

If $Q > Q_{max}$, an error is shown with y_{max} returned as normal depth.

In cases where Q is not a monotonically increasing function with y , multiple y values may result in the same Q value. The lowest value is returned as normal depth.

In summary, to calculate normal depth with Newton's method, we need A, P, A' and P' . A'' and P'' are needed when Q_{max} needs to be calculated using Newton's method.

1.3 Critical Depth

Critical flow occurs when the specific energy

$$E = \frac{v^2}{2g} + y = \frac{Q^2}{2gA^2} + y \quad (13)$$

reaches a minimum (g is acceleration of gravity, 32.17 ft/s^2 for US Customary units, 9.81 m/s^2 for SI). Namely,

$$\frac{\partial E}{\partial y} = -\frac{Q^2}{gA^3} \frac{\partial A}{\partial y} + 1 = 0. \quad (14)$$

To solve the equation with Newton's method, critical depth y_c or t_c is calculated by

$$t_{c,i+1} = t_{c,i} - \frac{f(t_{c,i})}{f'(t_{c,i})} \quad (15)$$

where

$$f_c(t) = gA^3 - Q^2 \frac{\partial A}{\partial y} \quad (16)$$

$$f'_c(t) = 3gA^2 \frac{\partial A}{\partial t} - Q^2 \frac{\partial}{\partial t} \left(\frac{\partial A}{\partial y} \right) \quad (17)$$

The critical velocity v_c is

$$v_c = \frac{Q}{A_c} = \sqrt{g \frac{A}{\frac{\partial A}{\partial y}}} = \sqrt{gD_h} \quad (18)$$

where $D_h = A / \frac{\partial A}{\partial y}$ is the hydraulic depth.

The Froude number, $F = v/v_c$. The critical slope S_c is backcalculated from Eq. (1).

In summary, A, P, A' and P' are needed to calculate normal depth y_n , A, P, A', P', A'' and P'' to calculate t_{max} , y_{max} , Q_{max} , and $\frac{\partial A}{\partial y}$, $\frac{\partial A}{\partial t}$, and $\frac{\partial}{\partial t} \left(\frac{\partial A}{\partial y} \right)$ to calculate t_c and y_c using Newton's method.

2 Triangular

$$A = \frac{1}{2}(z_1 + z_2)y^2 \quad (19)$$

$$P = \left(\sqrt{1 + z_1^2} + \sqrt{1 + z_2^2} \right) y \quad (20)$$

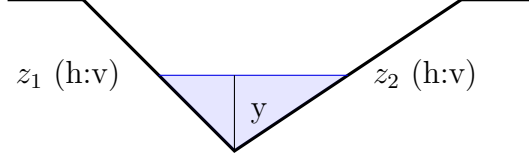


Figure 1: Triangular Section

$$R = \frac{1}{2} \frac{(z_1 + z_2)}{\sqrt{1 + z_1^2} + \sqrt{1 + z_2^2}} y \quad (21)$$

For normal flow, Eq. (2) becomes

$$Q = \frac{K_u}{n} \frac{z_1 + z_2}{2} \left(\frac{1}{2} \frac{z_1 + z_2}{\sqrt{1 + z_1^2} + \sqrt{1 + z_2^2}} \right)^{\frac{2}{3}} y^{\frac{8}{3}} S^{\frac{1}{2}}. \quad (22)$$

Note this equation can be used to backcalculate normal depth y_n from Q .

$$\frac{\partial A}{\partial y} = (z_1 + z_2)y = T_w \quad (23)$$

with T_w as the width of the water surface.

For critical flow, Eq. (16) becomes

$$f_c(y) = \frac{g}{8}(z_1 + z_2)^3 y^6 - Q^2(z_1 + z_2)y = 0. \quad (24)$$

$$y_c^5 = \frac{8Q^2}{g(z_1 + z_2)^2} \quad (25)$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{\frac{1}{2}(z_1 + z_2)y^2}{(z_1 + z_2)y} = \frac{1}{2}(z_1 + z_2)y_c \quad (26)$$

$$v_c = \sqrt{\frac{1}{2}g(z_1 + z_2)y_c} \quad (27)$$

3 Rectangular

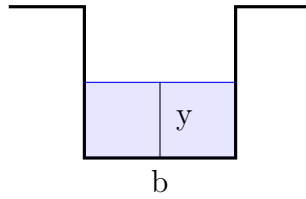


Figure 2: Rectangular Section

$$A = by \quad (28)$$

$$P = b + 2y \quad (29)$$

$$R = \frac{by}{b + 2y} \quad (30)$$

$$\frac{\partial A}{\partial y} = b = T_w \quad (31)$$

$$f_c(y) = gb^3y^3 - Q^2b = 0 \quad (32)$$

$$y_c^3 = \frac{Q^2}{gb^2} \quad (33)$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{by_c}{b} = y_c \quad (34)$$

$$v_c = \sqrt{gy_c} \quad (35)$$

Neton's method is used to calculate normal depth y_n .

4 Trapezoidal

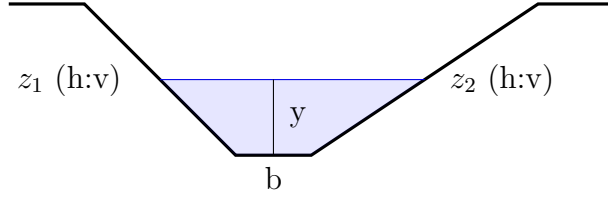


Figure 3: Trapezoidal Section

$$A = \frac{1}{2}(z_1 + z_2)y^2 + by \quad (36)$$

$$P = \left(\sqrt{1 + z_1^2} + \sqrt{1 + z_2^2} \right) y + b \quad (37)$$

$$\frac{\partial A}{\partial y} = (z_1 + z_2)y + b = T_w \quad (38)$$

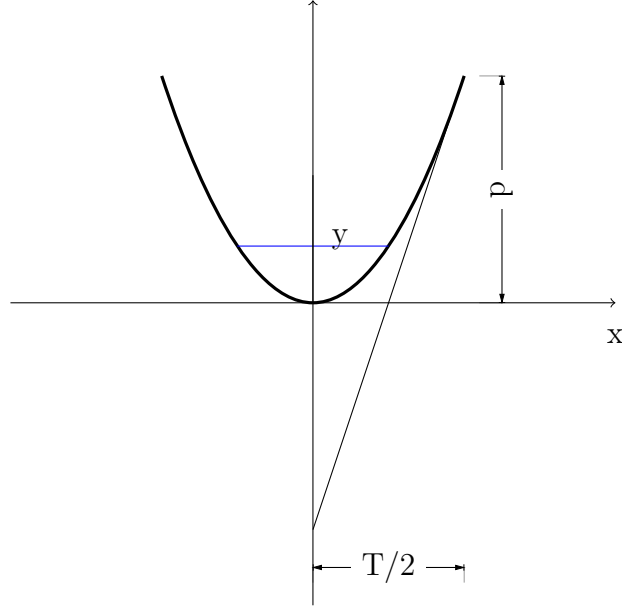
$$\frac{\partial^2 A}{\partial y^2} = z_1 + z_2 \quad (39)$$

$$\frac{dP}{dy} = \sqrt{1 + z_1^2} + \sqrt{1 + z_2^2} \quad (40)$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{\frac{1}{2}(z_1 + z_2)y_c^2 + by_c}{(z_1 + z_2)y_c + b} \quad (41)$$

Neton's method is used to calculate normal depth y_n and critical depth y_c .

5 Parabolic



For a parabola with top width T and depth d ,

$$y = 4\frac{d}{T^2}x^2 \quad (42)$$

$$A = 2xy - \int_{-x}^x 4\frac{d}{T^2}x^2 dx = \frac{16}{3}\frac{d}{T^2}x^3 = \frac{2}{3}\frac{T}{\sqrt{d}}y^{\frac{3}{2}} \quad (43)$$

$$\frac{\partial A}{\partial y} = T\sqrt{\frac{y}{d}} = 2x = T_w \quad (44)$$

$$\frac{\partial^2 A}{\partial y^2} = \frac{T}{2\sqrt{yd}} \quad (45)$$

To calculate P

$$P = 2 \int_0^y \sqrt{1 + (dx/dy)^2} dy \quad (46)$$

$$\frac{\partial y}{\partial x} = 8\frac{d}{T^2}x \quad (47)$$

$$\left(\frac{\partial y}{\partial x}\right)^2 = 16\frac{d}{T^2}y = \frac{y}{a} \quad (48)$$

with $a = T^2/16d$

$$\int \sqrt{1 + \frac{a}{t}} dt = \frac{a}{2} \ln \left(\sqrt{1 + \frac{a}{t}} + 1 \right) - \frac{a}{2} \ln \left(\left| \sqrt{1 + \frac{a}{t}} - 1 \right| \right) + x\sqrt{1 + \frac{a}{t}} + c \quad (49)$$

$$\int \sqrt{1 + \frac{a}{t}} dx = \frac{a}{2} \ln \left| \frac{\sqrt{1 + \frac{a}{t}} + 1}{\sqrt{1 + \frac{a}{t}} - 1} \right| + \sqrt{t^2 + at} + c = \frac{a}{2} \ln \left| \frac{\sqrt{t^2 + at} + t}{\sqrt{t^2 + at} - t} \right| + \sqrt{t^2 + at} + c \quad (50)$$

$$P = 2 \int_0^y \sqrt{1 + (dx/dy)^2} dy = 2 \int_0^y \sqrt{1 + \frac{a}{y}} dy = a \ln \left| \frac{\sqrt{y^2 + ay} + y}{\sqrt{y^2 + ay} - y} \right| + 2\sqrt{y^2 + ay} \quad (51)$$

$$\frac{dP}{dy} = a \frac{\frac{2y+a}{\sqrt{y^2+ay}} + 1}{\sqrt{y^2+ay} + y} - a \frac{\frac{2y+a}{\sqrt{y^2+ay}} - 1}{\sqrt{y^2+ay} - y} + \frac{2y+a}{\sqrt{y^2+ay}} \quad (52)$$

Let $z = \sqrt{y^2 + ay}$,

$$\frac{dP}{dy} = a \frac{\frac{2y+a}{z} + 1}{z + y} - a \frac{\frac{2y+a}{z} - 1}{z - y} + \frac{2y+a}{z} = a \frac{2y+a+z}{z^2 + yz} - a \frac{2y+a-z}{z^2 - yz} + \frac{2y+a}{z} \quad (53)$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{\frac{2}{3} \frac{T}{\sqrt{d}} y^{\frac{3}{2}}}{T \sqrt{\frac{y}{d}}} = \frac{2}{3} y_c \quad (54)$$

$$v_c = \sqrt{\frac{2}{3} g y_c} \quad (55)$$

Neton's method is used to calculate normal depth y_n and critical depth y_c .

6 Circular

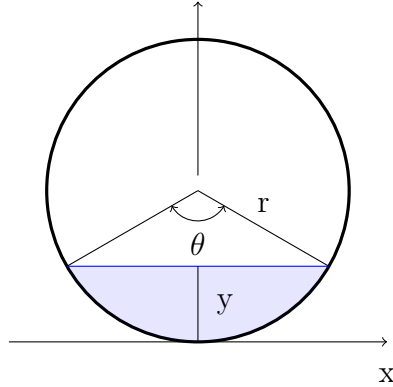


Figure 4: Circular Section

$$y = r(1 - \cos \frac{\theta}{2}) \quad (56)$$

$$A = \frac{1}{2}(\theta - \sin \theta)r^2 \quad (57)$$

$$P = \theta r \quad (58)$$

$$R = \frac{1}{2} \left(1 - \frac{\sin \theta}{\theta} \right) r \quad (59)$$

$$\frac{\partial y}{\partial \theta} = \frac{r}{2} \sin \frac{\theta}{2} \quad (60)$$

$$\frac{\partial A}{\partial \theta} = \frac{1}{2} (1 - \cos \theta) r^2 \quad (61)$$

$$\frac{\partial A}{\partial y} = \frac{\frac{\partial A}{\partial \theta}}{\frac{\partial y}{\partial \theta}} = \frac{\frac{1}{2} (1 - \cos \theta) r^2}{\frac{r}{2} \sin \frac{\theta}{2}} = 2r \sin \frac{\theta}{2} \quad (62)$$

$$\frac{\partial}{\partial \theta} \left(\frac{dA}{dy} \right) = r \cos \frac{\theta}{2} = T_w \quad (63)$$

$$\frac{\partial P}{\partial \theta} = r \quad (64)$$

To calculate θ_{max} , y_{max} , and Q_{max} , Eq. (10)

$$5PA' - 2AP' = 5\theta r \frac{1}{2} (1 - \cos \theta) r^2 - 2 \frac{1}{2} (\theta - \sin \theta) r^2 r = 3\theta - 5\theta \cos \theta + 2 \sin \theta = 0, \quad (65)$$

i.e.,

$$3\theta - 5\theta \cos \theta + 2 \sin \theta = 0, \quad (66)$$

The θ when Q peaks,

$$\theta_{max} = 5.27810713, \quad (67)$$

$$Q_{max} = 2.2189 \frac{K_u}{n} r^{8/3} S^{1/2}, \quad (68)$$

$$y_{max} = 1.87636243r. \quad (69)$$

Neton's method is used to calculate normal depth y_n and critical depth y_c .

7 Elliptical

7.1 Basic Equations

$$\begin{aligned} u &= b \cos \alpha \\ v &= a \sin \alpha \end{aligned} \quad (70)$$

$$\tan \beta = \frac{a}{b} \tan \alpha \quad (71)$$

$$\sin^2 \alpha = \frac{\tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{b^2 \tan^2 \beta}{a^2 + b^2 \tan^2 \beta} \quad (72)$$

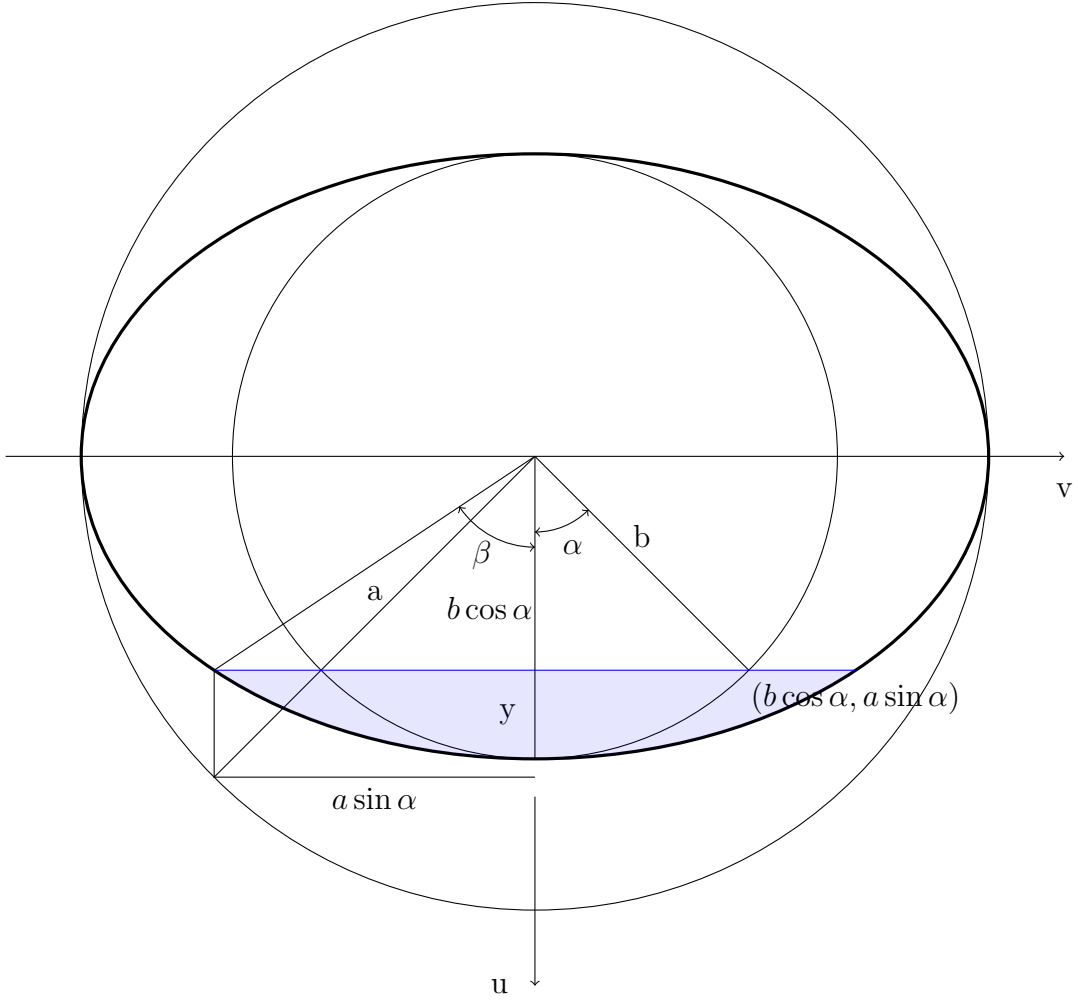


Figure 5: Elliptical Section ($a \geq b$)

$$\cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha} = \frac{a^2}{a^2 + b^2 \tan^2 \beta} \quad (73)$$

$$r^2 = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha \quad (74)$$

7.2 Wet Perimeter

7.2.1 $a \geq b$

In the case of $a \geq b$ (Figure 5), the wet perimeter is

$$P = 2 \int_0^\alpha \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2a \int_0^\alpha \sqrt{1 - (1 - b^2/a^2) \sin^2 t} dt = 2aE(\alpha, \eta) \quad (75)$$

with $\eta^2 = 1 - b^2/a^2$, $E(\alpha, \zeta)$ as the Legendre elliptical integral of the second kind.

$$E(\alpha, \eta) = \alpha - \frac{1}{2}\eta^2 \int_0^\alpha \sin^2 t dt - \frac{1}{2 \cdot 4}\eta^4 \int_0^\alpha \sin^4 t dt - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}\eta^6 \int_0^\alpha \sin^6 t dt + \dots \quad (76)$$

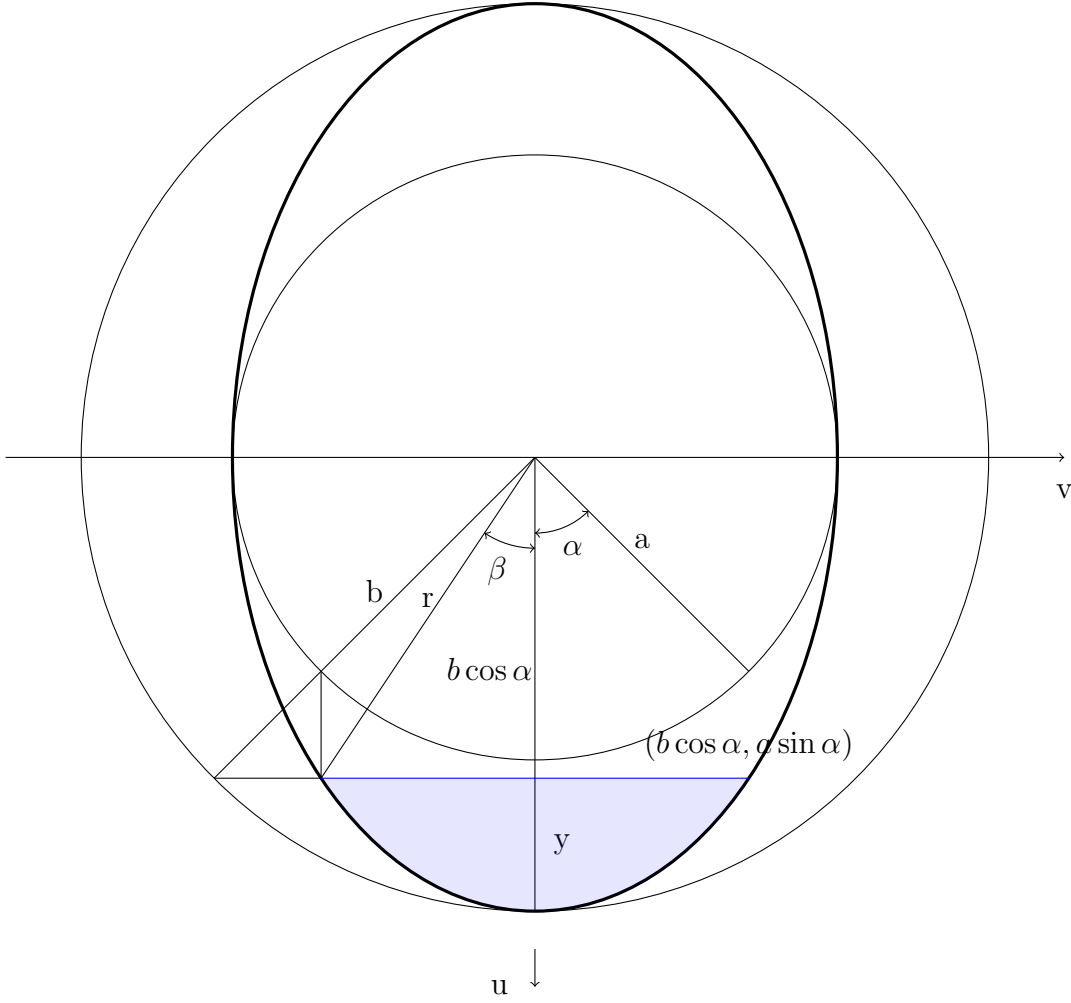


Figure 6: Elliptical Section ($a < b$)

$$\begin{aligned}
\int \sin^n t dt &= -\frac{1}{n} \sin^{n-1} \cos t + \frac{n}{n-1} \int \sin^{n-2} t dt \\
\int_0^\alpha \sin^2 t dt &= -\frac{1}{2} \sin \alpha \cos \alpha + \frac{1}{2} \alpha \\
\int_0^\alpha \sin^4 t dt &= -\frac{1}{4} \sin^3 \alpha \cos \alpha + \frac{3}{4} \int_0^\alpha \sin^2 t dt \\
\int_0^\alpha \sin^6 t dt &= -\frac{1}{6} \sin^5 \alpha \cos \alpha + \frac{5}{6} \int_0^\alpha \sin^4 t dt \\
\int_0^\alpha \sin^{2n} t dt &= -\frac{1}{2n} \sin^{2n-1} \alpha \cos \alpha + \frac{2n-1}{2n} \int_0^\alpha \sin^{2n-2} t dt
\end{aligned} \tag{77}$$

7.2.2 $a < b$

For $a < b$, the wet perimeter is (Figure 6)

$$P = 2 \int_0^\alpha \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2b \int_0^\alpha \sqrt{1 - (1 - a^2/b^2) \cos^2 t} dt = 2bF(\alpha, \zeta) \tag{78}$$

with $\zeta^2 = 1 - a^2/b^2$.

$$F(\alpha, \eta) = \alpha - \frac{1}{2}\zeta^2 \int_0^\alpha \cos^2 t dt - \frac{1}{2^2 \cdot 2!}\zeta^4 \int_0^\alpha \cos^4 t dt - \frac{1 \cdot 3}{2^3 \cdot 3!}\zeta^6 \int_0^\alpha \cos^6 t dt - \dots \quad (79)$$

$$\begin{aligned} \int \cos^n t dt &= \frac{1}{n} \sin t \cos^{n-1} t + \frac{n}{n-1} \int \cos^{n-2} t dt \\ \int_0^\alpha \cos^2 t dt &= \frac{1}{2} \sin \alpha \cos \alpha + \frac{1}{2} \alpha \\ \int_0^\alpha \cos^4 t dt &= \frac{1}{4} \sin \alpha \cos^3 \alpha + \frac{3}{4} \int_0^\alpha \cos^2 t dt \\ \int_0^\alpha \cos^6 t dt &= \frac{1}{6} \sin \alpha \cos^5 \alpha + \frac{5}{6} \int_0^\alpha \cos^4 t dt \\ \int_0^\alpha \cos^{2n} t dt &= \frac{1}{2n} \sin \alpha \cos^{2n-1} \alpha + \frac{2n-1}{2n} \int_0^\alpha \cos^{2n-2} t dt \end{aligned} \quad (80)$$

For both cases,

$$P' = 2\sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} \quad (81)$$

$$P'' = -\frac{(a^2 - b^2) \sin 2\alpha}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}} \quad (82)$$

7.3 Flow Area

The flow area is

$$A = 2 \int_{b \cos \alpha}^b v du = -2ab \int_\alpha^0 \sin^2 t dt = ab \int_0^\alpha (1 - \cos 2t) dt = ab\left(\alpha - \frac{1}{2} \sin 2\alpha\right) \quad (83)$$

$$A' = ab(1 - \cos 2\alpha) \quad (84)$$

$$A'' = 2ab \sin 2\alpha \quad (85)$$

$$y = b(1 - \cos \alpha) \quad (86)$$

$$y' = \frac{\partial y}{\partial \alpha} = b \sin \alpha \quad (87)$$

$$\frac{\partial A}{\partial y} = \frac{A'}{y'} = 2a \sin \alpha = T_w \quad (88)$$

$$\frac{\partial}{\partial \alpha} \left(\frac{\partial A}{\partial y} \right) = 2a \cos \alpha \quad (89)$$

$$D_h = \frac{A'}{\frac{\partial A}{\partial y}} = \frac{b(\alpha - \frac{1}{2} \sin 2\alpha)}{2 \sin \alpha} \quad (90)$$

Neton's method is used to calculate normal depth y_n , critical depth y_c , α_{max} , y_{max} , and Q_{max} .

8 Arc

The geometry of an arc is determined by r_b , r_t , r_c , and $rise$.

$$\begin{aligned} r_b &= AB = BO = BE \\ r_t &= AC = CG \\ r_c &= DE = DF = DG \\ c &= BC = BO - CO = BO - (OG - CG) = r_b + r_t - rise \end{aligned} \quad (91)$$

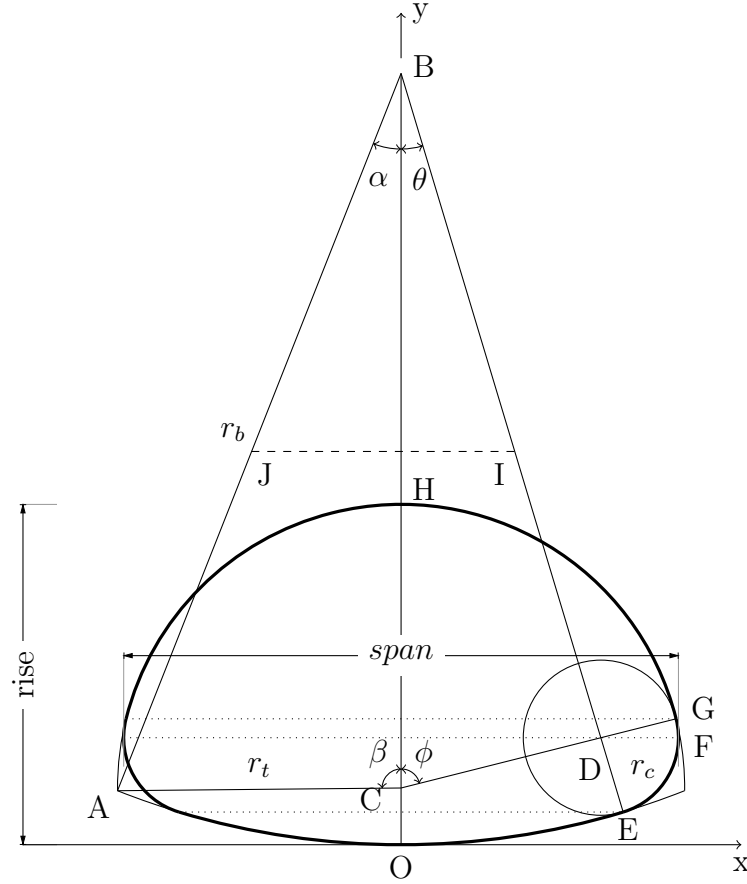


Figure 7: Arc Section

$$\begin{aligned}
\cos \alpha &= \frac{r_b^2 + c^2 - r_t^2}{2r_b c} \\
\cos \beta &= \frac{r_t^2 + c^2 - r_b^2}{2r_t c} \\
\cos \theta &= \frac{(r_b - r_c)^2 + c^2 - (r_t - r_c)^2}{2(r_b - r_c)c} \\
\cos \phi &= \frac{(r_t - r_c)^2 + c^2 - (r_b - r_c)^2}{2(r_t - r_c)c}
\end{aligned} \tag{92}$$

$$\begin{aligned}
x_A &= r_b \sin \alpha \\
y_A &= r_b(1 - \cos \alpha) \\
y_B &= r_b \\
y_C &= rise - r_t = OG - CG \\
x_D &= (r_b - r_c) \sin \theta = (r_t - r_c) \sin \phi \\
y_D &= r_b - (r_b - r_c) \cos \theta = y_F \\
x_E &= r_b \sin \theta \\
y_E &= r_b(1 - \cos \theta) \\
x_F &= x_D + r_c = span/2 \\
x_G &= r_t \sin \phi \\
y_G &= y_C + r_t \cos \phi = rise - r_t(1 - \cos \phi) \\
y_H &= rise
\end{aligned} \tag{93}$$

$$\begin{aligned}
A_E &= r_b^2(\theta - \sin \theta \cos \theta) \\
P_E &= 2r_b \theta \\
T_E &= 2r_b \sin \theta \\
A_F &= A_E + r_c^2(\pi/2 - \theta) + (x_E + x_D)(y_D - y_E) \\
P_F &= P_E + 2r_c(\pi/2 - \theta) \\
T_F &= 2(x_D + r_c) \\
A_G &= A_F + r_c^2(\pi/2 - \phi) + (x_D + x_G)(y_G - y_D) \\
P_G &= P_F + 2r_c(\pi/2 - \phi) \\
T_G &= 2r_t \sin \phi \\
A_T &= A_G + r_t^2(\phi - \sin \phi \cos \phi) \\
P_T &= P_G + 2r_t \phi
\end{aligned} \tag{94}$$

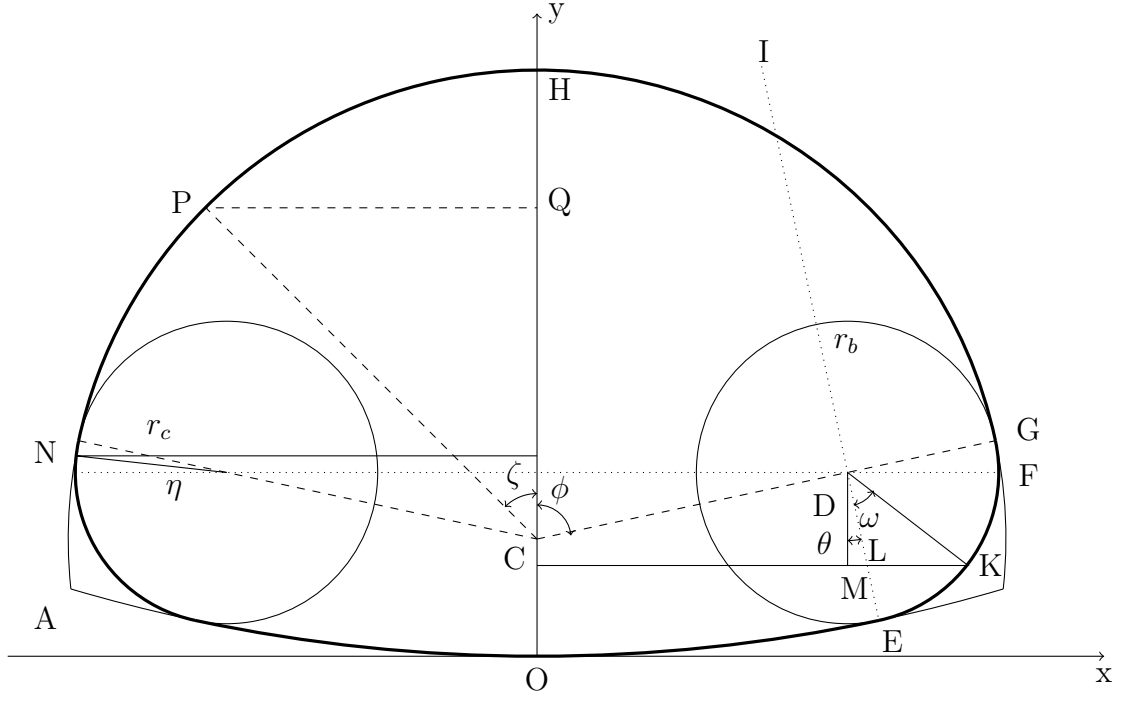


Figure 8: Arc Section

For $0 \leq y \leq y_E$, $0 \leq t \leq \theta$

$$\begin{aligned}
 x &= r_b \sin t \\
 y &= r_b(1 - \cos t) \\
 y' &= r_b \sin t = \frac{\partial y}{\partial t} \\
 A &= r_b^2(t - \sin t \cos t) \\
 A' &= r_b^2(1 - \cos 2t) = \frac{\partial A}{\partial t} \\
 \frac{\partial A}{\partial y} &= 2r_b \sin t = T_w \\
 \frac{\partial}{\partial t} \left(\frac{\partial A}{\partial y} \right) &= 2r_b \cos t \\
 P &= 2r_b t
 \end{aligned} \tag{95}$$

For $y_E \leq y \leq y_F$, $0 \leq w \leq \pi/2 - \theta$

$$\begin{aligned}
x &= x_D + r_c \sin(\omega + \theta) \\
x_L &= x_D + r_c \cos(\omega + \theta) \tan \theta \\
x'_L &= -r_c \sin(\omega + \theta) \tan \theta \\
x''_L &= -r_c \cos(\omega + \theta) \tan \theta \\
y &= y_D - r_c \cos(\omega + \theta) \\
y' &= r_c \sin(\omega + \theta) \\
y'' &= r_c \cos(\omega + \theta) \\
A &= A_E + r_c^2 \omega - r_c^2 \sin(\omega + \theta) \cos(\omega + \theta) + r_c^2 \cos^2(\omega + \theta) \tan \theta + (x_E + x_L)(y - y_E) \\
A' &= r_c^2 - r_c^2 \cos 2(\omega + \theta) - r_c^2 \sin 2(\omega + \theta) \tan \theta + (x_E + x_L)y' + x'_L(y - y_E) \\
&= 2r_c^2 \sin^2(\omega + \theta) - 2r_c^2 \sin(\omega + \theta) \cos(\omega + \theta) \tan \theta + (x_E + x_L)y' + x'_L(y - y_E) \\
\frac{\partial A}{\partial y} &= \frac{A'}{y'} = \frac{2r_c^2 \sin^2(\omega + \theta) - 2r_c^2 \sin(\omega + \theta) \cos(\omega + \theta) \tan \theta}{r_c \sin(\omega + \theta)} + x_E + x_L - (y - y_E) \tan \theta \\
&= 2r_c \sin(\omega + \theta) - 2r_c \cos(\omega + \theta) \tan \theta + x_E + x_D + r_c \cos(\omega + \theta) \tan \theta \\
&\quad - [y_D - r_c \cos(\omega + \theta) - y_E] \tan \theta \\
&= 2r_c \sin(\omega + \theta) + x_E + x_D - (y_D - y_E) \tan \theta = 2x_D + 2r_c \sin(\omega + \theta) = T_w \\
\frac{\partial}{\partial \omega} \left(\frac{\partial A}{\partial y} \right) &= 2r_c \cos(\omega + \theta) \\
P &= P_E + 2r_c \omega
\end{aligned} \tag{96}$$

For $y_F \leq y \leq y_G$, $0 \leq \eta \leq \pi/2 - \phi$

$$\begin{aligned}
x &= x_D + r_c \cos \eta \\
y &= y_D + r_c \sin \eta \\
A &= A_F + r_c^2 \eta + (2x_D + r_c \cos \eta) r_c \sin \eta \\
P &= P_F + 2r_c \eta \\
y' &= r_c \cos \eta \\
A' &= r_c^2 + 2x_D r_c \cos \eta + r_c^2 \cos(2\eta) \\
\frac{\partial A}{\partial y} &= \frac{A'}{y'} = \frac{r_c^2 + 2x_D r_c \cos \eta + r_c^2 \cos(2\eta)}{r_c \cos \eta} = 2x_D + 2r_c \cos \eta = T_w \\
\frac{\partial}{\partial \eta} \left(\frac{\partial A}{\partial y} \right) &= -2r_c \sin \eta
\end{aligned} \tag{97}$$

For $y_G \leq y \leq y_H$, $0 \leq \zeta \leq \phi$

$$\begin{aligned}
y &= y_C + r_t \cos \zeta \\
A &= A_T - r_t^2 (\zeta - \sin \zeta \cos \zeta) \\
P &= P_T - 2r_t \zeta = P_G + 2r_t (\phi - \zeta) \\
y' &= -r_t \sin \zeta \\
A' &= -r_t^2 [1 - \cos(2\zeta)] \\
A'' &= -2r_t^2 \sin(2\zeta) \\
P'(\zeta) &= -2r_t \\
P''(\zeta) &= 0 \\
\frac{\partial A}{\partial y} &= \frac{A'}{y'} = 2r_t \sin \zeta = T_w \\
\frac{\partial}{\partial t} \left(\frac{\partial A}{\partial y} \right) &= 2r_t \cos \zeta
\end{aligned} \tag{98}$$

Maximum discharge occurs close to the top of arch, the ζ_{max} is solved using Newton's method

$$\zeta_{i+1} = \zeta_i - \frac{f(\zeta_i)}{f'(\zeta_i)}, \tag{99}$$

with

$$f(\zeta) = 5A'P - 2AP', \tag{100}$$

and

$$f'(\zeta) = 3A'P' + 5A''P - 2AP''. \tag{101}$$

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