Open Channel Flow with Manning's Equation

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Abstract

Equations for analytical and numerical solutions involving Manning's equation are derived for triangular, rectangular, trapezoidal, and parabolic open channels and circular, elliptical, and arch closed conduits.

1 Theory

1.1 Manning's Equation

The Manning's equation for the velocity and discharge of uniform flow in open channels is [1, 2, 3, 4]:

$$v = \frac{K_u}{n} R_h^{\frac{2}{3}} S^{\frac{1}{2}}, \tag{1}$$

$$Q = vA = \frac{K_u}{n} A R_h^{\frac{2}{3}} S^{\frac{1}{2}} = \frac{K_u}{n} A^{\frac{5}{3}} P^{-\frac{2}{3}} S^{\frac{1}{2}}, \tag{2}$$

where

v = Mean velocity, m/s (ft/s),

Q = Discharge, m^3/s (ft^3/s),

n = Manning's coefficient of roughness,

R_h = Hydraulic radius, m (ft). $R_h = P/A$,

P = Wetted perimeter, m (ft),

A = Crossing-section area of flowing water perpendicular to the direction of flow, m^2 (ft^2),

S = Energy slope, m/m (ft/ft). For steady uniform flow $S = S_0$, and

K_u = units conversion factor, 1 for SI, 1.486 for English units.

1.2 Normal Depth

Depending on the geometry of the channel, both A and P are depdent on normal depth y. To calculate y for a given Q, we solve

$$f_d(y_i) = \frac{K_u}{n} A^{\frac{5}{3}} P^{-\frac{2}{3}} S^{\frac{1}{2}} - Q \tag{3}$$

for $f_d(y_i) = 0$. Analytical solutions are available in some special cases, for example, triangular channels. In general, a numerical solution is used to solve the nonlinear equation iteratively. Using Newton's method [5], the iteration starts with an initial guess y_0, and iterates with

$$y_{i+1} = y_i - \frac{f_d(y_i)}{f_d'(y_i)} \tag{4}$$

where

$$f_d'(y_i) = \frac{\partial f_d}{\partial y} = \frac{K_u}{n} S^{\frac{1}{2}} \left(\frac{5}{3} R_h^{\frac{2}{3}} \frac{\partial A}{\partial y} - \frac{2}{3} R_h^{\frac{5}{3}} \frac{\partial P}{\partial y} \right) = \frac{K_u}{3n} S^{\frac{1}{2}} R_h^{\frac{2}{3}} \left(5A' - 2R_h P' \right) \tag{5}$$

until

$$|f_d(y_i)| \leq TOLQ, \tag{6}$$

$$|y_{i+1} - y_i| \leq TOLD, \tag{7}$$

or

$$i \geq MAXI \tag{8}$$

with

$TOLQ =$ discharge tolerance, m^3/s (ft^3/s),

$TOLD =$depth tolerance, m (ft),

$MAXI =$ maximum number of iteration.

When the iteration stops with criteria Eq.(8), an error is shown to notify the user. The user may adjust y_0 or t_0, $TOLQ$, $TOLD$, and/or $MAXI$ to obtain a satisfactory solution. For circular, elliptical, and arch pipes, a replacement of variable y with an alternative variable t (for example) is convient. These equations remain valid. An alternative tolerance $TOLA$ (in lieu of $TOLD$) is specified when t is an angle.

For open channel flow in closed conduits such as circular, elliptical, and arch pipes, the calculated discharge peaks before the pipe is full [1, 2, 4]. The normal depth y_{max} or t_{max} at peak discharge Q_{max} can be calculated by solving

$$\frac{\partial Q}{\partial t} = \frac{K_u}{3n} R_h^{\frac{2}{3}} \left(5 \frac{\partial A}{\partial t} - 2 \frac{A}{P} \frac{\partial P}{\partial t} \right) S^{\frac{1}{2}} = 0. \tag{9}$$

or

$$f(t) = 5P \frac{\partial A}{\partial t} - 2A \frac{\partial P}{\partial t} = 5PA' - 2AP' = 0. \tag{10}$$

In absence of an analytical solution, Newton's method is used with

$$t_{i+1} = t_i - \frac{f(t_i)}{f'(t_i)}, \tag{11}$$

$$f'(t_i) = 5A''P + 3A'P' - 2AP''. \tag{12}$$

If $Q > Q_{max}$, an error is shown with y_{max} returned as normal depth.
In cases where Q is not a monotonically increasing function with y, multiple y values may result in the same Q value. The lowest value is returned as normal depth.
In summary, to calculate normal depth with Newton's method, we need A,P, A' and P'. A'' and P'' are needed when Q_{max} needs to be calculated using Newton's method.

1.3 Critical Depth

Critical flow occurs when the specific energy

$$E = \frac{v^2}{2g} + y = \frac{Q^2}{2gA^2} + y \tag{13}$$

reaches a minimum (g is acceleration of gravity, 32.17 ft/s^2 for US Customery units, 9.81 m/s^2 for SI). Namely,

$$\frac{\partial E}{\partial y} = -\frac{Q^2}{gA^3}\frac{\partial A}{\partial y} + 1 = 0. \tag{14}$$

To solve the equation with Newton's method, critical depth y_c or t_c is calculated by

$$t_{c,i+1} = t_{c,i} - \frac{f(t_{c,i})}{f'(t_{c,i})} \tag{15}$$

where

$$f_c(t) = gA^3 - Q^2\frac{\partial A}{\partial y} \tag{16}$$

$$f_c'(t) = 3gA^2\frac{\partial A}{\partial t} - Q^2\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) \tag{17}$$

The critical velocity v_c is

$$v_c = \frac{Q}{A_c} = \sqrt{g\frac{A}{\frac{\partial A}{\partial y}}} = \sqrt{gD_h} \tag{18}$$

where $D_h = A/\frac{\partial A}{\partial y}$ is the hydraulic depth.
The Froude number, $F = v/v_c$. The critical slope S_c is backcalculated from Eq. (1).
In summary, A, P, A' and P' are needed to calculate normal depth y_n, A, P, A', P', A'' and P'' to calculate t_{max}, y_{max}, Q_{max}, and $\frac{\partial A}{\partial y}$, $\frac{\partial A}{\partial t}$, and $\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right)$ to calculate t_c and y_c using Newton's method.

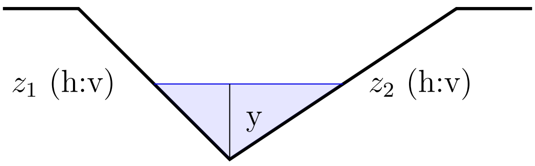


Figure 1: Triangular Section

2 Triangular

$$A = \frac{1}{2}(z_1 + z_2)y^2 \tag{19}$$

$$P = \left(\sqrt{1+z_1^2} + \sqrt{1+z_2^2}\right) y \tag{20}$$

$$R_h = \frac{1}{2}\frac{(z_1 + z_2)}{\sqrt{1+z_1^2} + \sqrt{1+z_2^2}} y \tag{21}$$

For normal flow, Eq. (2) becomes

$$Q = \frac{K_u}{n}\frac{z_1 + z_2}{2}\left(\frac{1}{2}\frac{z_1 + z_2}{\sqrt{1+z_1^2} + \sqrt{1+z_2^2}}\right)^{\frac{2}{3}} y^{\frac{8}{3}} S^{\frac{1}{2}}. \tag{22}$$

Note this equation can be used to backcalculate normal depth y_n from Q.

$$\frac{\partial A}{\partial y} = (z_1 + z_2)y = T_w \tag{23}$$

with T_w as the width of the water surface.
For critical flow, Eq. (16) becomes

$$f_c(y) = \frac{g}{8}(z_1 + z_2)^3 y^6 - Q^2(z_1 + z_2)y = 0. \tag{24}$$

$$y_c^5 = \frac{8Q^2}{g(z_1 + z_2)^2} \tag{25}$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{\frac{1}{2}(z_1 + z_2)y^2}{(z_1 + z_2)y} = \frac{1}{2}y_c \tag{26}$$

$$v_c = \sqrt{\frac{1}{2}g y_c} \tag{27}$$

Analytical solution is derived for normal depth (Eq. 22) and critical depth (Eq. 25).

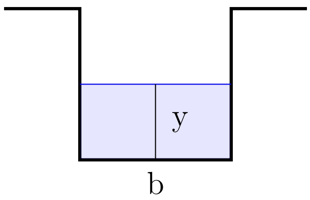


Figure 2: Rectangular Section

3 Rectangular

$$A = by \tag{28}$$

$$P = b + 2y \tag{29}$$

$$R_h = \frac{by}{b + 2y} \tag{30}$$

$$\frac{\partial A}{\partial y} = b = T_w \tag{31}$$

$$f_c(y) = gb^3y^3 - Q^2b = 0 \tag{32}$$

$$y_c^3 = \frac{Q^2}{gb^2} \tag{33}$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{by_c}{b} = y_c \tag{34}$$

$$v_c = \sqrt{gy_c} \tag{35}$$

Analytical solution is derived for critical depth (Eq: 33). Neton's method is used to calculate normal depth y_n.

4 Trapezoidal

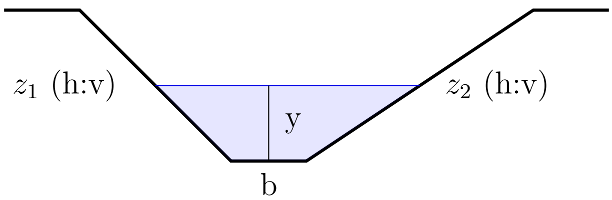


Figure 3: Trapezoidal Section

$$A = \frac{1}{2}(z_1 + z_2)y^2 + by \tag{36}$$

$$P = \left(\sqrt{1+z_1^2} + \sqrt{1+z_2^2}\right) y + b \tag{37}$$

$$\frac{\partial A}{\partial y} = (z_1 + z_2)y + b = T_w \tag{38}$$

$$\frac{\partial^2 A}{\partial y^2} = z_1 + z_2 \tag{39}$$

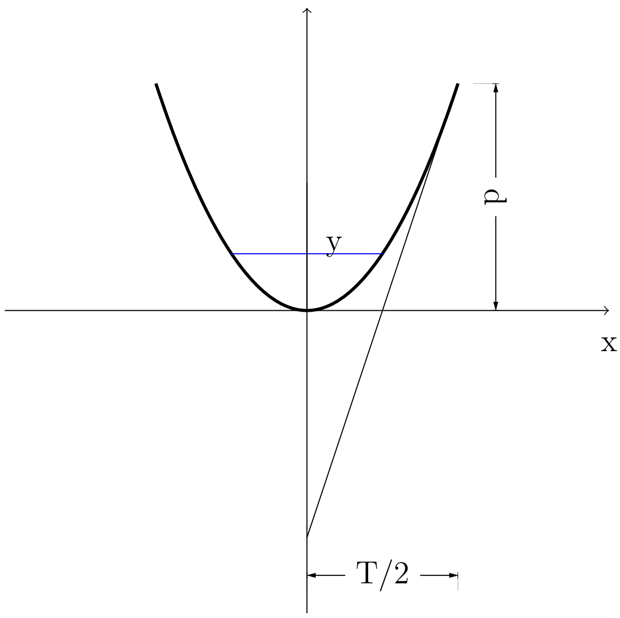
$$\frac{dP}{dy} = \sqrt{1+z_1^2} + \sqrt{1+z_2^2} \tag{40}$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{\frac{1}{2}(z_1+z_2)y_c^2 + by_c}{(z_1+z_2)y_c + b} \tag{41}$$

Neton's method is used to calculate normal depth y_n and critical depth y_c. An initial guess for critical depth [2] is

$$y_{c,0} = 0.81 \left(\frac{Q^2}{gz^{0.75}b^{1.25}}\right)^{0.27} - \frac{b}{30z} \tag{42}$$

5 Parabolic



For a parabola with top width T and depth d,

$$y = 4\frac{d}{T^2}x^2 \tag{43}$$

$$A = 2xy - \int_{-x}^{x} 4\frac{d}{T^2}x^2 dx = \frac{16}{3}\frac{d}{T^2}x^3 = \frac{2}{3}\frac{T}{\sqrt{d}}y^{\frac{3}{2}} \tag{44}$$

$$\frac{\partial A}{\partial y} = T\sqrt{\frac{y}{d}} = 2x = T_w \tag{45}$$

$$\frac{\partial^2 A}{\partial y^2} = \frac{T}{2\sqrt{yd}} \tag{46}$$

To calculate P

$$\frac{\partial y}{\partial x} = 8\frac{d}{T^2}x \tag{47}$$

$$\left(\frac{\partial y}{\partial x}\right)^2 = 8\frac{d}{T^2}8\frac{d}{T^2}x^2 = 16\frac{d}{T^2}y = \frac{y}{a} \tag{48}$$

with $a = T^2/16d$

$$\frac{\partial P}{\partial y} = 2\sqrt{1 + (dx/dy)^2} = 2\sqrt{1 + \frac{a}{y}} \tag{49}$$

$$\int \sqrt{1 + \frac{a}{t}}dx = \frac{a}{2}\ln\frac{\sqrt{1+\frac{a}{t}}+1}{\sqrt{1+\frac{a}{t}}-1} + \sqrt{t^2+at} + c = \frac{a}{2}\ln\frac{\sqrt{t^2+at}+t}{\sqrt{t^2+at}-t} + \sqrt{t^2+at} + c \tag{50}$$

Let $b = \sqrt{y^2 + ay}$,

$$P = 2\int_0^y \sqrt{1 + \frac{a}{y}}dy == a\ln\frac{b+y}{b-y} + 2b \tag{51}$$

$$D_h = \frac{A}{\frac{\partial A}{\partial y}} = \frac{\frac{2}{3}\frac{T}{\sqrt{d}}y^{\frac{3}{2}}}{T\sqrt{\frac{y}{d}}} = \frac{2}{3}y_c \tag{52}$$

$$v_c = \sqrt{\frac{2}{3}gy_c} \tag{53}$$

Neton's method is used to calculate normal depth y_n and critical depth y_c. An initial guess for critical depth [2] is

$$y_{c,0} = \left(0.84\frac{4dQ^2}{gT}\right)^{0.25} \tag{54}$$

6 Circular

$$y = r(1 - \cos\frac{\theta}{2}) \tag{55}$$

$$A = \frac{1}{2}(\theta - \sin\theta)r^2 \tag{56}$$

$$P = \theta r \tag{57}$$

$$R = \frac{1}{2}\left(1 - \frac{\sin\theta}{\theta}\right)r \tag{58}$$

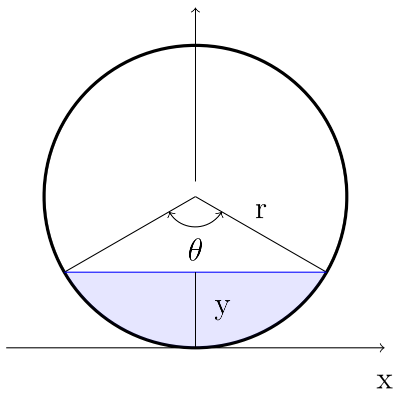


Figure 4: Circular Section

$$\frac{\partial y}{\partial \theta} = \frac{r}{2}\sin\frac{\theta}{2} \tag{59}$$

$$\frac{\partial A}{\partial \theta} = \frac{1}{2}(1-\cos\theta)r^2 \tag{60}$$

$$\frac{\partial A}{\partial y} = \frac{\frac{\partial A}{\partial \theta}}{\frac{\partial y}{\partial \theta}} = \frac{\frac{1}{2}(1-\cos\theta)r^2}{\frac{r}{2}\sin\frac{\theta}{2}} = 2r\sin\frac{\theta}{2} \tag{61}$$

$$\frac{\partial}{\partial \theta}\left(\frac{dA}{dy}\right) = r\cos\frac{\theta}{2} = T_w \tag{62}$$

$$\frac{\partial P}{\partial \theta} = r \tag{63}$$

To calculate θ_{max}, y_{max}, and Q_{max}, Eq. (10)

$$5PA' - 2AP' = 5\theta r\frac{1}{2}(1-\cos\theta)r^2 - 2\frac{1}{2}(\theta-\sin\theta)r^2 r = 3\theta - 5\theta\cos\theta + 2\sin\theta = 0, \tag{64}$$

i.e.,

$$3\theta - 5\theta\cos\theta + 2\sin\theta = 0, \tag{65}$$

The θ when Q peaks,

$$\theta_{max} = 5.27810713, \tag{66}$$

$$Q_{max} = 2.2189\frac{K_u}{n}r^{8/3}S^{1/2}, \tag{67}$$

$$y_{max} = 1.87636243r. \tag{68}$$

Neton's method is used to calculate normal depth y_n and critical depth y_c.

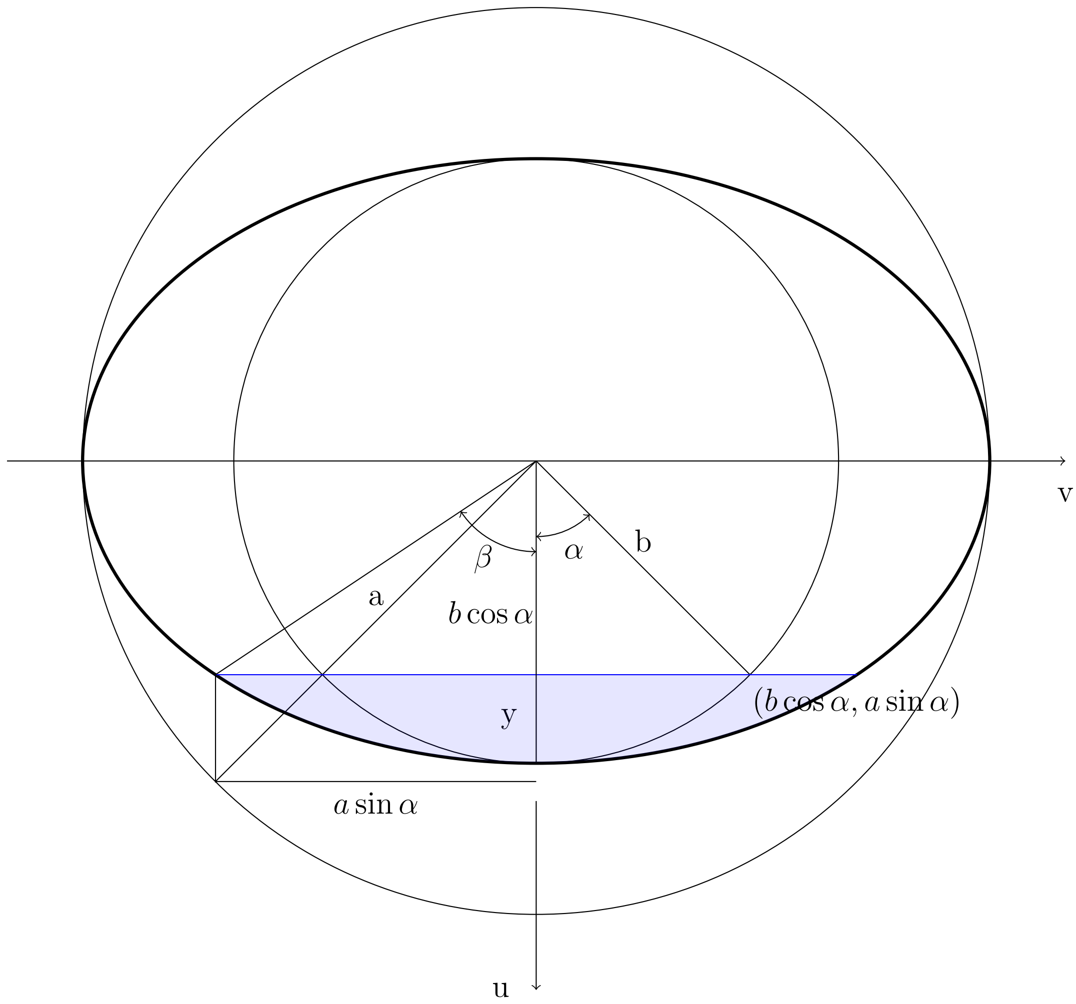


Figure 5: Elliptical Section ($a \geq b$)

7 Elliptical

7.1 Basic Equations

$$\begin{aligned} u &= b\cos\alpha \\ v &= a\sin\alpha \end{aligned} \tag{69}$$

$$\tan\beta = \frac{a}{b}\tan\alpha \tag{70}$$

$$\sin^2\alpha = \frac{\tan^2\alpha}{1+\tan^2\alpha} = \frac{b^2\tan^2\beta}{a^2+b^2\tan^2\beta} \tag{71}$$

$$\cos^2\alpha = \frac{1}{1+\tan^2\alpha} = \frac{a^2}{a^2+b^2\tan^2\beta} \tag{72}$$

$$r^2 = a^2\sin^2\alpha + b^2\cos^2\alpha \tag{73}$$

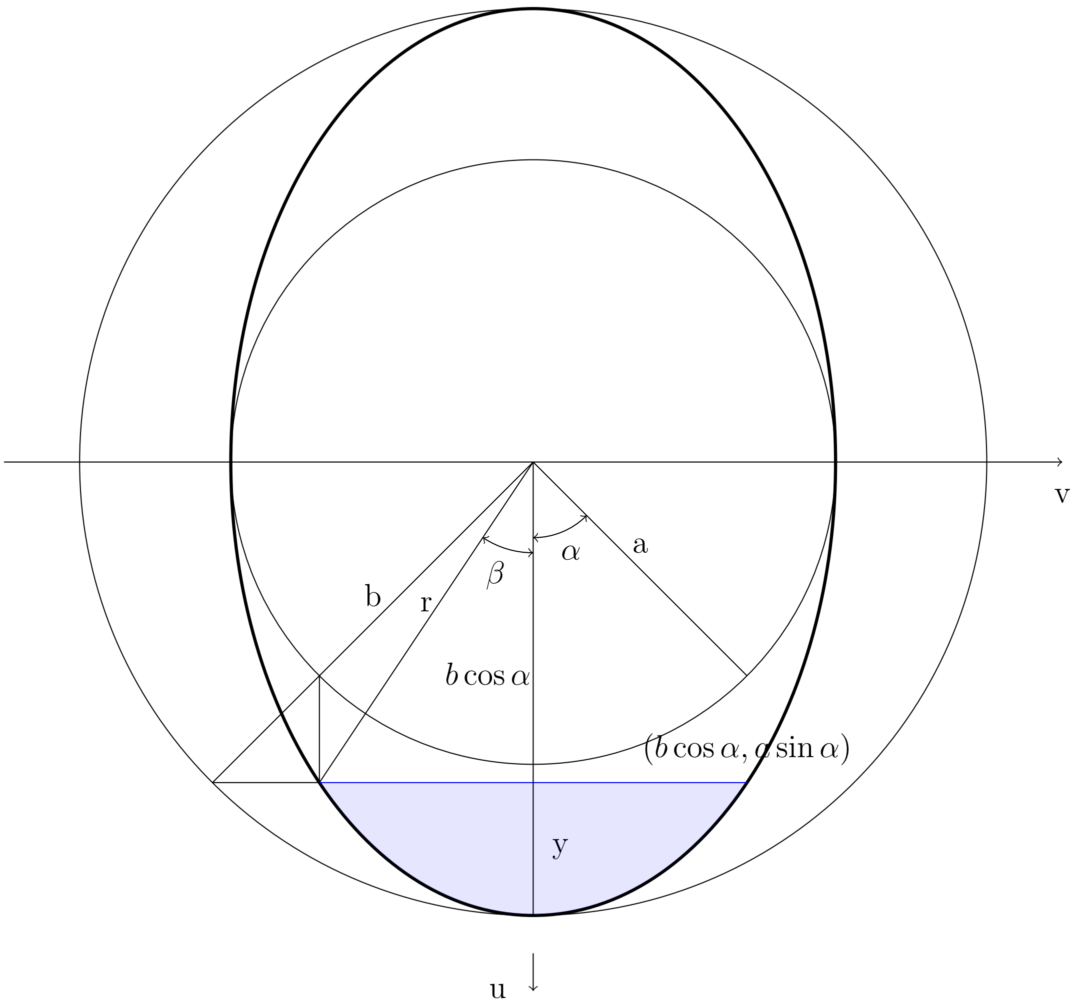


Figure 6: Elliptical Section ($a < b$)

$$y = b(1 - \cos\alpha) \tag{74}$$

$$\cos\alpha = 1 - \frac{y}{b} \tag{75}$$

$$y' = \frac{\partial y}{\partial \alpha} = b\sin\alpha \tag{76}$$

7.2 Flow Area

The flow area is

$$A = 2\int_{b\cos\alpha}^{b} v du = -2ab\int_{\alpha}^{0}\sin^2 t dt = ab\int_{0}^{\alpha}(1-\cos 2t)dt = ab(\alpha - \frac{1}{2}\sin 2\alpha) \tag{77}$$

$$A' = ab(1 - \cos 2\alpha) \tag{78}$$

$$A'' = 2ab\sin 2\alpha \tag{79}$$

$$\frac{\partial A}{\partial y} = \frac{A'}{y'} = 2a\sin\alpha = T_w \tag{80}$$

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial A}{\partial y}\right) = 2a\cos\alpha \tag{81}$$

$$D_h = \frac{A'}{\frac{\partial A}{\partial y}} = \frac{b(\alpha - \frac{1}{2}\sin 2\alpha)}{2\sin\alpha} \tag{82}$$

7.3 Wet Perimeter

7.3.1 $a \geq b$

In the case of $a \geq b$ (Figure 5), the wet perimeter is

$$P = 2\int_0^\alpha \sqrt{a^2\cos^2 t + b^2\sin^2 t}dt = 2a\int_0^\alpha \sqrt{1-(1-b^2/a^2)\sin^2 t}dt = 2aE(\alpha,\eta) \tag{83}$$

with $\eta^2 = 1 - b^2/a^2$, $E(\alpha,\zeta)$ as the Legendre ellitipcal integral of the second kind.

$$E(\alpha,\eta) = \alpha - \frac{1}{2}\eta^2\int_0^\alpha \sin^2 tdt - \frac{1}{2\cdot 4}\eta^4\int_0^\alpha \sin^4 tdt - \frac{1\cdot 3}{2\cdot 4\cdot 6}\eta^6\int_0^\alpha \sin^6 tdt + ... \tag{84}$$

$$\begin{aligned}
\int \sin^n tdt &= -\frac{1}{n}\sin^{n-1}\cos t + \frac{n-1}{n}\int \sin^{n-2} tdt \\
\int_0^\alpha \sin^2 tdt &= -\frac{1}{2}\sin\alpha\cos\alpha + \frac{1}{2}\alpha \\
\int_0^\alpha \sin^4 tdt &= -\frac{1}{4}\sin^3\alpha\cos\alpha + \frac{3}{4}\int_0^\alpha \sin^2 tdt \\
\int_0^\alpha \sin^6 tdt &= -\frac{1}{6}\sin^5\alpha\cos\alpha + \frac{5}{6}\int_0^\alpha \sin^4 tdt \\
\int_0^\alpha \sin^{2n} tdt &= -\frac{1}{2n}\sin^{2n-1}\alpha\cos\alpha + \frac{2n-1}{2n}\int_0^\alpha \sin^{2n-2} tdt
\end{aligned} \tag{85}$$

n	a_n	b_n	c_n	d_n	e_n	f_n
1	$-\frac{1}{2}\eta^2$	$-\frac{1}{2}$	$\sin\alpha\cos\alpha$	$\frac{1}{2}$	$b_1c_1 + d_1\alpha$	a_1e_1
2	$a_1\frac{\eta^2}{4}$	$-\frac{1}{4}$	$c_1\sin^2\alpha$	$\frac{3}{4}$	$b_2c_2 + d_2e_1$	a_2e_2
3	$a_2\frac{\eta^2}{6}$	$-\frac{1}{6}$	$c_2\sin^2\alpha$	$\frac{5}{6}$	$b_3c_3 + d_3e_2$	a_3e_3
n	$a_{n-1}\frac{\eta^2}{2n}$	$-\frac{1}{2n}$	$c_{n-1}\sin^2\alpha$	$\frac{2n-1}{2n}$	$b_nc_n + d_ne_{n-1}$	a_ne_n

7.3.2 $a < b$

For $a < b$, the wet perimeter is (Figure 6)

$$P = 2\int_0^\alpha \sqrt{a^2\cos^2 t + b^2\sin^2 t}dt = 2b\int_0^\alpha \sqrt{1-(1-a^2/b^2)\cos^2 t}dt = 2bF(\alpha,\zeta) \quad (86)$$

with $\zeta^2 = 1 - a^2/b^2$.

$$F(\alpha,\eta) = \alpha - \frac{1}{2}\zeta^2\int_0^\alpha \cos^2 tdt - \frac{1}{2^2\cdot 2!}\zeta^4\int_0^\alpha \cos^4 tdt - \frac{1\cdot 3}{2^3\cdot 3!}\zeta^6\int_0^\alpha \cos^6 tdt - \cdots \quad (87)$$

$$\begin{aligned}
\int \cos^n tdt &= \frac{1}{n}\sin t\cos^{n-1} t + \frac{n}{n-1}\int \cos^{n-2} tdt \\
\int_0^\alpha \cos^2 tdt &= \frac{1}{2}\sin\alpha\cos\alpha + \frac{1}{2}\alpha \\
\int_0^\alpha \cos^4 tdt &= \frac{1}{4}\sin\alpha\cos^3\alpha + \frac{3}{4}\int_0^\alpha \cos^2 tdt \\
\int_0^\alpha \cos^6 tdt &= \frac{1}{6}\sin\alpha\cos^5\alpha + \frac{5}{6}\int_0^\alpha \cos^4 tdt \\
\int_0^\alpha \cos^{2n} tdt &= \frac{1}{2n}\sin\alpha\cos^{2n-1}\alpha + \frac{2n-1}{2n}\int_0^\alpha \cos^{2n-2} tdt
\end{aligned} \quad (88)$$

For both cases,

$$P' = 2\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \quad (89)$$

$$P'' = -\frac{(a^2-b^2)\sin 2\alpha}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}} \quad (90)$$

Neton's method is used to calculate normal depth y_n, critical depth y_c, α_{max}, y_{max}, and Q_{max}.

8 Arc

The geometry of an arc is determined by r_b, r_t, r_c, and $rise$.

$$\begin{aligned}
r_b &= AB = BO = BE \\
r_t &= AC = CG \\
r_c &= DE = DF = DG \\
c &= BC = BO - CO = BO - (OG - CG) = r_b + r_t - rise
\end{aligned} \quad (91)$$

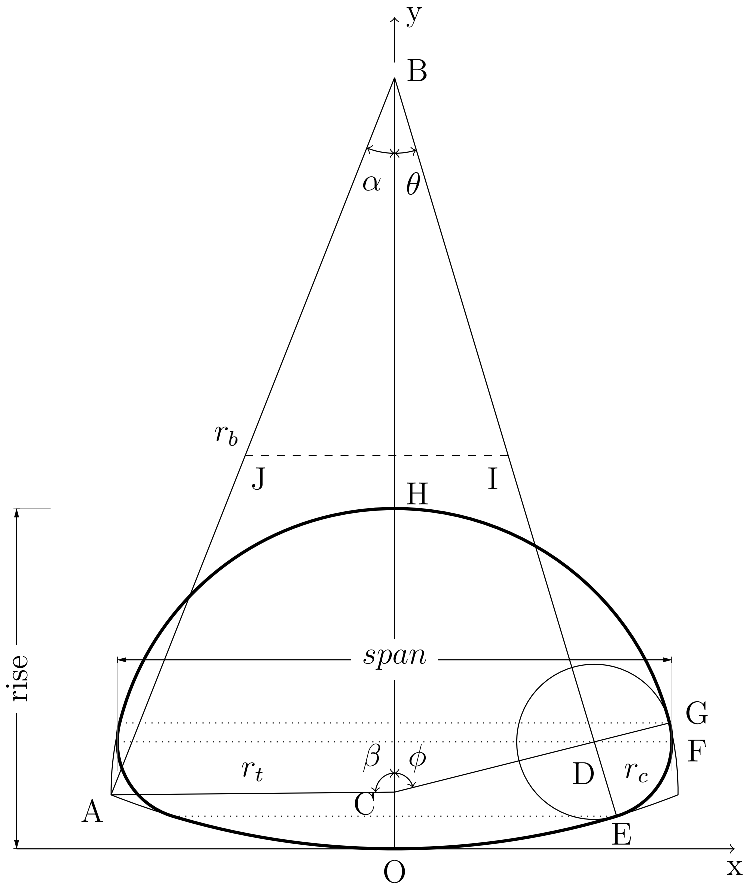


Figure 7: Arc Section

$$
\begin{aligned}
\cos\alpha &= \frac{r_b^2 + c^2 - r_t^2}{2r_b c} \\
\cos\beta &= \frac{r_t^2 + c^2 - r_b^2}{2r_t c} \\
\cos\theta &= \frac{(r_b - r_c)^2 + c^2 - (r_t - r_c)^2}{2(r_b - r_c)c} \\
\cos\phi &= \frac{(r_t - r_c)^2 + c^2 - (r_b - r_c)^2}{2(r_t - r_c)c}
\end{aligned} \tag{92}
$$

$$
\begin{aligned}
x_A &= r_b \sin\alpha \\
y_A &= r_b(1-\cos\alpha) \\
y_B &= r_b \\
y_C &= rise - r_t = OG - CG \\
x_D &= (r_b - r_c)\sin\theta = (r_t - r_c)\sin\phi \\
y_D &= r_b - (r_b - r_c)\cos\theta = y_F \\
x_E &= r_b \sin\theta \\
y_E &= r_b(1-\cos\theta) \\
x_F &= x_D + r_c = span/2 \\
x_G &= r_t \sin\phi \\
y_G &= y_C + r_t\cos\phi = rise - r_t(1-\cos\phi) \\
y_H &= rise
\end{aligned} \tag{93}
$$

$$
\begin{aligned}
A_E &= r_b^2(\theta - \sin\theta\cos\theta) \\
P_E &= 2r_b\theta \\
T_E &= 2r_b\sin\theta \\
A_F &= A_E + r_c^2(\pi/2-\theta) + (x_E + x_D)(y_D - y_E) \\
P_F &= P_E + 2r_c(\pi/2-\theta) \\
T_F &= 2(x_D + r_c) \\
A_G &= A_F + r_c^2(\pi/2-\phi) + (x_D + x_G)(y_G - y_D) \\
P_G &= P_F + 2r_c(\pi/2-\phi) \\
T_G &= 2r_t\sin\phi \\
A_T &= A_G + r_t^2(\phi - \sin\phi\cos\phi) \\
P_T &= P_G + 2r_t\phi
\end{aligned} \tag{94}
$$

For $0 \le y \le y_E$, $0 \le t \le \theta$

$$
\begin{aligned}
x &= r_b \sin t \\
y &= r_b(1-\cos t) \\
y' &= r_b \sin t = \frac{\partial y}{\partial t} \\
A &= r_b^2(t - \sin t\cos t) \\
A' &= r_b^2(1-\cos 2t) = \frac{\partial A}{\partial t} \\
\frac{\partial A}{\partial y} &= 2r_b\sin t = T_w \\
\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) &= 2r_b\cos t \\
P &= 2r_b t
\end{aligned} \tag{95}
$$

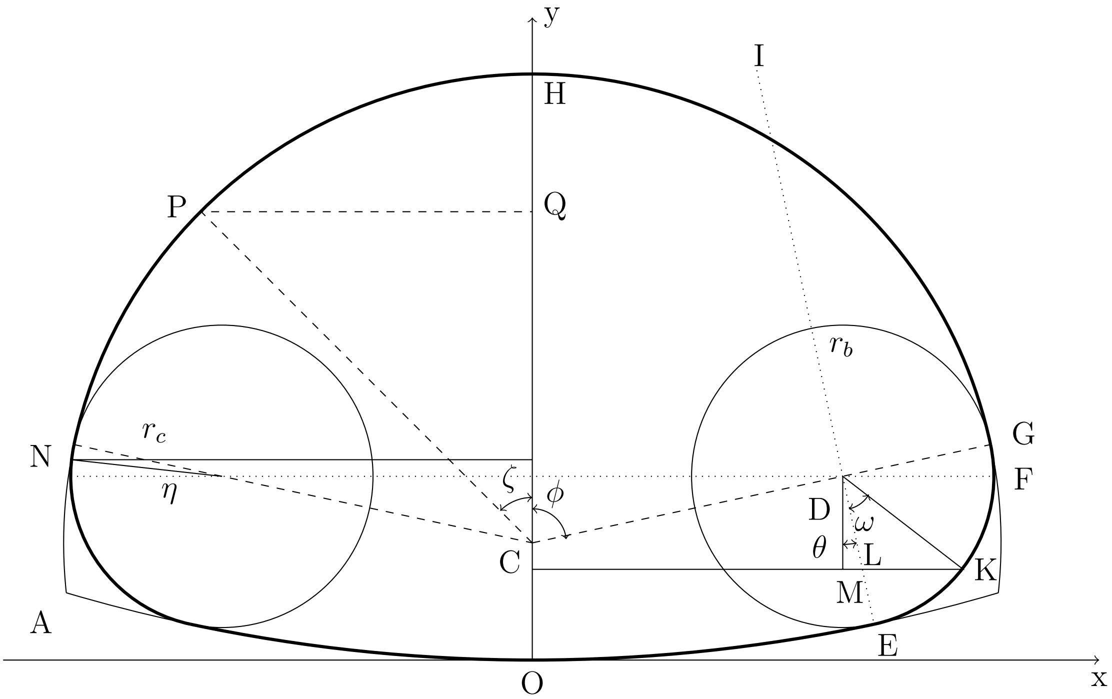


Figure 8: Arc Section

For $y_E \le y \le y_F$, $0 \le w \le \pi/2 - \theta$

$$
\begin{aligned}
x &= x_D + r_c \sin(\omega+\theta) \\
x_L &= x_D + r_c \cos(\omega+\theta)\tan\theta \\
x'_L &= -r_c \sin(\omega+\theta)\tan\theta \\
x''_L &= -r_c \cos(\omega+\theta)\tan\theta \\
y &= y_D - r_c \cos(\omega+\theta) \\
y' &= r_c \sin(\omega+\theta) \\
y'' &= r_c \cos(\omega+\theta) \\
A &= A_E + r_c^2\omega - r_c^2 \sin(\omega+\theta)\cos(\omega+\theta) + r_c^2\cos^2(\omega+\theta)\tan\theta + (x_E + x_L)(y - y_E) \\
A' &= r_c^2 - r_c^2 \cos 2(\omega+\theta) - r_c^2 \sin 2(\omega+\theta)\tan\theta + (x_E + x_L)y' + x'_L(y - y_E) \\
&= 2r_c^2 \sin^2(\omega+\theta) - 2r_c^2 \sin(\omega+\theta)\cos(\omega+\theta)\tan\theta + (x_E + x_L)y' + x'_L(y - y_E) \\
\frac{\partial A}{\partial y} &= \frac{A'}{y'} = \frac{2r_c^2 \sin^2(\omega+\theta) - 2r_c^2 \sin(\omega+\theta)\cos(\omega+\theta)\tan\theta}{r_c \sin(\omega+\theta)} + x_E + x_L - (y - y_E)\tan\theta \\
&= 2r_c \sin(\omega+\theta) - 2r_c \cos(\omega+\theta)\tan\theta + x_E + x_D + r_c\cos(\omega+\theta)\tan\theta \\
&\quad - [y_D - r_c\cos(\omega+\theta) - y_E]\tan\theta \\
&= 2r_c \sin(\omega+\theta) + x_E + x_D - (y_D - y_E)\tan\theta = 2x_D + 2r_c\sin(\omega+\theta) = T_w \\
\frac{\partial}{\partial\omega}\left(\frac{\partial A}{\partial y}\right) &= 2r_c\cos(\omega+\theta) \\
P &= P_E + 2r_c\omega
\end{aligned}
\tag{96}
$$

For $y_F \le y \le y_G$, $0 \le \eta \le \pi/2 - \phi$

$$\begin{aligned}
x &= x_D + r_c \cos\eta \\
y &= y_D + r_c \sin\eta \\
A &= A_F + r_c^2\eta + (2x_D + r_c\cos\eta)r_c\sin\eta \\
P &= P_F + 2r_c\eta \\
y' &= r_c\cos\eta \\
A' &= r_c^2 + 2x_D r_c\cos\eta + r_c^2\cos(2\eta) \\
\frac{\partial A}{\partial y} &= \frac{A'}{y'} = \frac{r_c^2 + 2x_D r_c\cos\eta + r_c^2\cos(2\eta)}{r_c\cos\eta} = 2x_D + 2r_c\cos\eta = T_w \\
\frac{\partial}{\partial\eta}\left(\frac{\partial A}{\partial y}\right) &= -2r_c\sin\eta
\end{aligned} \tag{97}$$

For $y_G \le y \le y_H$, $0 \le \zeta \le \phi$

$$\begin{aligned}
y &= y_C + r_t\cos\zeta \\
A &= A_T - r_t^2(\zeta - \sin\zeta\cos\zeta) \\
P &= P_T - 2r_t\zeta = P_G + 2r_t(\phi - \zeta) \\
y' &= -r_t\sin\zeta \\
A' &= -r_t^2[1 - \cos(2\zeta)] \\
A'' &= -2r_t^2\sin(2\zeta) \\
P'(\zeta) &= -2r_t \\
P''(\zeta) &= 0 \\
\frac{\partial A}{\partial y} &= \frac{A'}{y'} = 2r_t\sin\zeta = T_w \\
\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) &= 2r_t\cos\zeta
\end{aligned} \tag{98}$$

Maximum discharge occurs close to the top of arch, the ζ_{max} is solved using Newton's method

$$\zeta_{i+1} = \zeta_i - \frac{f(\zeta_i)}{f'(\zeta_i)}, \tag{99}$$

with

$$f(\zeta) = 5A'P - 2AP', \tag{100}$$

and

$$f'(\zeta) = 3A'P' + 5A''P - 2AP''. \tag{101}$$

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