Open Channel Flow with Manning's Equation

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December 29, 2020

Abstract

1 Theory

1.1 Manning's Equation

The Manning's equation for the velocity and discharge of uniform flow in open channels is [1, 2, 4, 3]:

$$v = \frac{K_u}{n} R^{\frac{2}{3}} S^{\frac{1}{2}}, \tag{1}$$

$$Q = vA = \frac{K_u}{n} A R^{\frac{2}{3}} S^{\frac{1}{2}} = \frac{K_u}{n} A^{\frac{5}{3}} P^{-\frac{2}{3}} S^{\frac{1}{2}}, \tag{2}$$

where

v = Mean velocity, m/s (ft/s),

Q = Discharge, m^3/s (ft^3/s),

n = Manning's coefficient of roughness,

R = Hydraulic radius, m (ft). $R = P/A$,

P = Wetted perimeter, m (ft),

A = Crossing-section area of flowing water perpendicular to the direction of flow, m^2 (ft^2),

S = Energy slope, m/m (ft/ft). For steady uniform flow $S = S_0$, and

K_u = units conversion factor, 1 for SI, 1.486 for English units.

1.2 Normal Depth

Depending on the geometry of the channel, both A and P are depdent on normal depth y. To calculate y for a given Q, we solve

$$f_d(y_i) = \frac{K_u}{n} A^{\frac{5}{3}} P^{-\frac{2}{3}} S^{\frac{1}{2}} - Q \tag{3}$$

for $f_d(y_i) = 0$. Analytical solutions are available in some special cases, for example, triangular channels. In general, a numerical solution is used to solve the nonlinear equation iteratively. Using Newton's method [5], the iteration starts with an initial guess y_0, and iterates with

$$y_{i+1} = y_i - \frac{f_d(y_i)}{f_d'(y_i)} \tag{4}$$

where

$$f_d'(y_i) = \frac{\partial f_d}{\partial y} = \frac{K_u}{n} S^{\frac{1}{2}} \left(\frac{5}{3} R^{\frac{2}{3}} \frac{\partial A}{\partial y} - \frac{2}{3} R^{\frac{5}{3}} \frac{\partial P}{\partial y} \right) \tag{5}$$

until

$$|f_d(y_i)| \le TOLQ, \tag{6}$$

$$|y_{i+1} - y_i| \le TOLD, \tag{7}$$

or

$$i \ge MAXI \tag{8}$$

with

$TOLQ$ = discharge tolerance, m^3/s (ft^3/s),

$TOLD$ =depth tolerance, m (ft),

$MAXI$ = maximum number of iteration.

When the iteration stops with criteria Eq.(8), an error is shown to notify the user. The user may adjust y_0 or t_0, $TOLQ$, $TOLD$, and/or $MAXI$ to obtain a satisfactory solution. For circular, elliptical, and arch pipes, a replacement of variable y with an alternative variable t (for example) is convient. These equations remain valid. An alternative tolerance $TOLA$ (in lieu of $TOLD$) is specified when t is an angle.

For open channel flow in closed conduits such as circular, elliptical, and arch pipes, the calculated discharge peaks before the pipe is full [1, 2, 4]. The normal depth y_{max} or t_{max} at peak discharge Q_{max} can be calculated by solving

$$\frac{\partial Q}{\partial t} = \frac{K_u}{n} \left(\frac{5}{3} R^{\frac{2}{3}} \frac{\partial A}{\partial t} - \frac{2}{3} R^{\frac{5}{3}} \frac{\partial P}{\partial t} \right) S^{\frac{1}{2}} = 0. \tag{9}$$

or

$$f(t) = 5P \frac{\partial A}{\partial t} - 2A \frac{\partial P}{\partial t} = 5PA' - 2AP' = 0. \tag{10}$$

In absence of an analytical solution, Newton's method is used with

$$t_{i+1} = t_i - \frac{f(t_i)}{f'(t_i)}, \tag{11}$$

$$f'(t_i) = 5A''P + 3A'P' - 2AP''. \tag{12}$$

If $Q > Q_{max}$, an error is shown with y_{max} returned as normal depth.
In cases where Q is not a monotonically increasing function with y, multiple y values may result in the same Q value. The lowest value is returned as normal depth.
In summary, to calculate normal depth with Newton's method, we need A, P, A' and P'. A" and P" are needed when Q_{max} needs to be calculated using Newton's method.

1.3 Critical Depth

Critical flow occurs when the specific energy

$$E = \frac{v^2}{2g} + y = \frac{Q^2}{2gA^2} + y \tag{13}$$

reaches a minimum (g is acceleration of gravity, 32.17 ft/s^2 for US Customery units, 9.81 m/s^2 for SI). Namely,

$$\frac{\partial E}{\partial y} = -\frac{Q^2}{gA^3}\frac{\partial A}{\partial y} + 1 = 0. \tag{14}$$

To solve the equation with Newton's method, critical depth y_c or t_c is calculated by

$$t_{c,i+1} = t_{c,i} - \frac{f(t_{c,i})}{f'(t_{c,i})} \tag{15}$$

where

$$f_c(t) = gA^3 - Q^2\frac{\partial A}{\partial y} \tag{16}$$

$$f_c'(t) = 3gA^2\frac{\partial A}{\partial t} - Q^2\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) \tag{17}$$

The critical velocity v_c is

$$v_c = \frac{Q}{A_c} = \sqrt{g\frac{A}{\frac{\partial A}{\partial y}}} = \sqrt{gD} \tag{18}$$

where $D = A/\frac{\partial A}{\partial y}$ is the hydraulic depth.

The Froude number, $F = v/v_c$. The critical slope S_c is backcalculated from Eq. (1).

2 Triangular

$$A = \frac{1}{2}(z_1 + z_2)y^2 \tag{19}$$

$$P = \left(\sqrt{1+z_1^2} + \sqrt{1+z_2^2}\right)y \tag{20}$$

$$R = \frac{1}{2}\frac{(z_1 + z_2)}{\sqrt{1+z_1^2} + \sqrt{1+z_2^2}}y \tag{21}$$

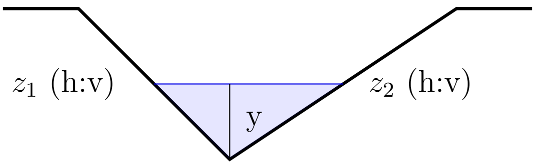


Figure 1: Triangular Section

Eq. (2) becomes

$$Q = \frac{K_u}{n}\frac{z_1+z_2}{2}\left(\frac{1}{2}\frac{z_1+z_2}{\sqrt{1+z_1^2}+\sqrt{1+z_2^2}}\right)^{\frac{2}{3}} y^{\frac{8}{3}} S^{\frac{1}{2}}. \tag{22}$$

Note this equation can be used to backcalculate y from Q.

$$\frac{\partial A}{\partial y} = (z_1+z_2)y \tag{23}$$

Eq. (16) becomes

$$f_c(y) = \frac{g}{8}(z_1+z_2)^3 y^6 - Q^2(z_1+z_2)y = 0. \tag{24}$$

$$y_c^5 = \frac{8Q^2}{g(z_1+z_2)^2} \tag{25}$$

3 Rectangular

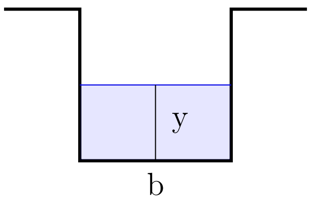


Figure 2: Rectangular Section

$$A = by \tag{26}$$

$$P = b + 2y \tag{27}$$

$$R = \frac{by}{b+2y} \tag{28}$$

$$\frac{\partial A}{\partial y} = b \tag{29}$$

$$f_c(y) = gb^3y^3 - Q^2 b = 0 \tag{30}$$

$$y_c^3 = \frac{Q^2}{gb^2} \tag{31}$$

4 Trapezoidal

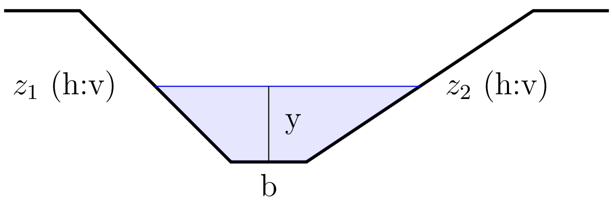


Figure 3: Trapezoidal Section

$$A = \frac{1}{2}(z_1 + z_2)y^2 + by \tag{32}$$

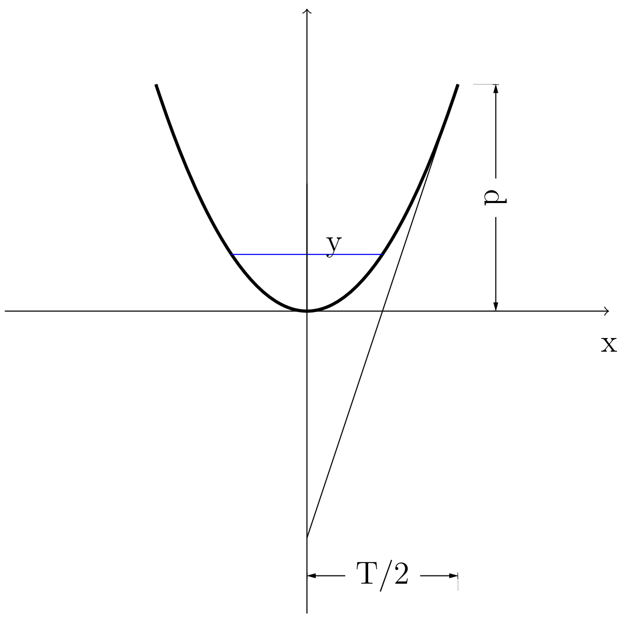
$$P = \left(\sqrt{1+z_1^2} + \sqrt{1+z_2^2}\right) y + b \tag{33}$$

$$\frac{\partial A}{\partial y} = (z_1 + z_2)y + b \tag{34}$$

$$\frac{\partial^2 A}{\partial y^2} = z_1 + z_2 \tag{35}$$

$$\frac{dP}{dy} = \sqrt{1+z_1^2} + \sqrt{1+z_2^2} \tag{36}$$

5 Parabolic



For a parabola with top width T and depth d,

$$y = 4\frac{d}{T^2}x^2 \tag{37}$$

$$A = 2xy - \int_{-x}^{x} 4\frac{d}{T^2}x^2 dx = \frac{16}{3}\frac{d}{T^2}x^3 = \frac{2}{3}\frac{T}{\sqrt{d}}y^{\frac{3}{2}} \tag{38}$$

$$\frac{\partial A}{\partial y} = T\sqrt{\frac{y}{d}} \tag{39}$$

$$\frac{\partial^2 A}{\partial y^2} = \frac{T}{2\sqrt{yd}} \tag{40}$$

To calculate P and P',

$$P = 2\int_0^y \sqrt{1 + (dx/dy)^2} dy \tag{41}$$

$$\frac{\partial y}{\partial x} = 8\frac{d}{T^2}x \tag{42}$$

$$\left(\frac{\partial y}{\partial x}\right)^2 = 16\frac{d}{T^2}y = \frac{y}{a} \tag{43}$$

with $a = T^2/16d$

$$\int \sqrt{1 + \frac{a}{t}}dt = \frac{a}{2}\ln\left(\sqrt{1 + \frac{a}{t}} + 1\right) - \frac{a}{2}\ln\left(|\sqrt{1 + \frac{a}{t}} - 1|\right) + x\sqrt{1 + \frac{a}{t}} + c \tag{44}$$

$$\int \sqrt{1 + \frac{a}{t}}dx = \frac{a}{2}\ln\left|\frac{\sqrt{1 + \frac{a}{t}} + 1}{\sqrt{1 + \frac{a}{t}} - 1}\right| + \sqrt{t^2 + at} + c = \frac{a}{2}\ln\left|\frac{\sqrt{t^2 + at} + t}{\sqrt{t^2 + at} - t}\right| + \sqrt{t^2 + at} + c \tag{45}$$

$$P = 2\int_0^y \sqrt{1 + (dx/dy)^2}dy = 2\int_0^y \sqrt{1 + \frac{a}{y}}dy = a\ln\left|\frac{\sqrt{y^2 + ay} + y}{\sqrt{y^2 + ay} - y}\right| + 2\sqrt{y^2 + ay} \tag{46}$$

$$\frac{dP}{dy} = a\frac{\frac{2y+a}{\sqrt{y^2+ay}} + 1}{\sqrt{y^2 + ay} + y} - a\frac{\frac{2y+a}{\sqrt{y^2+ay}} - 1}{\sqrt{y^2 + ay} - y} + \frac{2y + a}{\sqrt{y^2 + ay}} \tag{47}$$

Let $z = \sqrt{y^2 + ay}$,

$$\frac{dP}{dy} = a\frac{\frac{2y+a}{z} + 1}{z + y} - a\frac{\frac{2y+a}{z} - 1}{z - y} + \frac{2y + a}{z} = a\frac{2y + a + z}{z^2 + yz} - a\frac{2y + a - z}{z^2 - yz} + \frac{2y + a}{z} \tag{48}$$

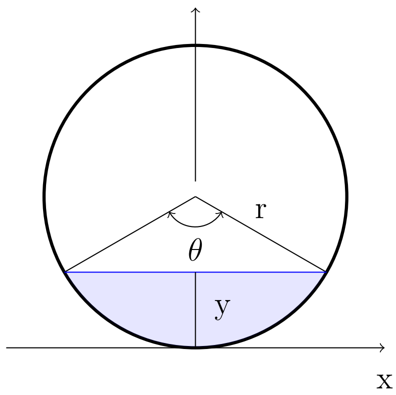


Figure 4: Circular Section

6 Circular

6.1 Basic Equations

$$y = r(1 - \cos\frac{\theta}{2}) \tag{49}$$

$$A = \frac{1}{2}(\theta - \sin\theta)r^2 \tag{50}$$

$$P = \theta r \tag{51}$$

$$R = \frac{1}{2}\left(1 - \frac{\sin\theta}{\theta}\right)r \tag{52}$$

$$T = 2r\sin\frac{\theta}{2} \tag{53}$$

$$\frac{dy}{d\theta} = \frac{r}{2}\sin\frac{\theta}{2} \tag{54}$$

$$\frac{dA}{d\theta} = \frac{1}{2}(1 - \cos\theta)r^2 \tag{55}$$

$$\frac{dP}{d\theta} = r \tag{56}$$

$$\frac{dT}{d\theta} = r\cos\frac{\theta}{2} \tag{57}$$

6.2 Calculate Maximum Discharge Using Manning's Equation

$$Q = \frac{K_u}{n}AR^{2/3}S^{1/2} = \frac{K_u}{n}A^{5/3}P^{-2/3}S^{1/2}, \tag{58}$$

$$\frac{dQ}{d\theta} = \frac{K_u}{n}\left(\frac{5}{3}R^{2/3}\frac{\partial A}{\partial\theta} - \frac{2}{3}R^{5/3}\frac{\partial P}{\partial\theta}\right)S^{1/2} = 0, \tag{59}$$

$$5\frac{\partial A}{\partial \theta} - 2R\frac{\partial P}{\partial \theta} = 0, \tag{60}$$

$$3\theta - 5\theta\cos\theta + 2\sin\theta = 0, \tag{61}$$

$$\theta_{max} = 5.27810713, \tag{62}$$

$$Q_{max} = 2.2189\frac{K_u}{n} r^{8/3} S^{1/2}, \tag{63}$$

$$y_{max} = 1.87636243r, \tag{64}$$

6.3 Calculate Normal Depth

For Q less than Q_{max},

$$\theta_{i+1} = \theta_i - \frac{f_d(\theta_i)}{f'_d(\theta_i)} \tag{65}$$

where

$$f_d(\theta_i) = \frac{K_u}{n} A^{5/3} P^{-2/3} S^{1/2} - Q \tag{66}$$

$$f'_d(\theta_i) = \frac{K_u}{n}\left(\frac{5}{3}R^{2/3}\frac{\partial A}{\partial \theta} - \frac{2}{3}R^{5/3}\frac{\partial P}{\partial \theta}\right) S^{1/2} \tag{67}$$

6.4 Calculate Critical Depth

$$\frac{\partial A}{\partial y}(\theta) = \frac{\frac{\partial A}{\partial \theta}}{\frac{\partial y}{\partial \theta}} = \frac{1-\cos\theta}{\sin\theta/2} r = 2r\sin\frac{\theta}{2} \tag{68}$$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial A}{\partial y}\right) = r\cos\frac{\theta}{2} \tag{69}$$

$$f_c(\theta) = gA^3 - Q^2\frac{\partial A}{\partial y} = gA^3 - 2rQ^2\sin\frac{\theta}{2} \tag{70}$$

$$f'_c(\theta) = 3gA^2\frac{\partial A}{\partial \theta} - Q^2\frac{\partial}{\partial \theta}\left(\frac{\partial A}{\partial y}\right) \tag{71}$$

7 Elliptical

7.1 Basic Equations

$$\begin{aligned} u &= b\cos\alpha \\ v &= a\sin\alpha \end{aligned} \tag{72}$$

$$\tan\beta = \frac{a}{b}\tan\alpha \tag{73}$$

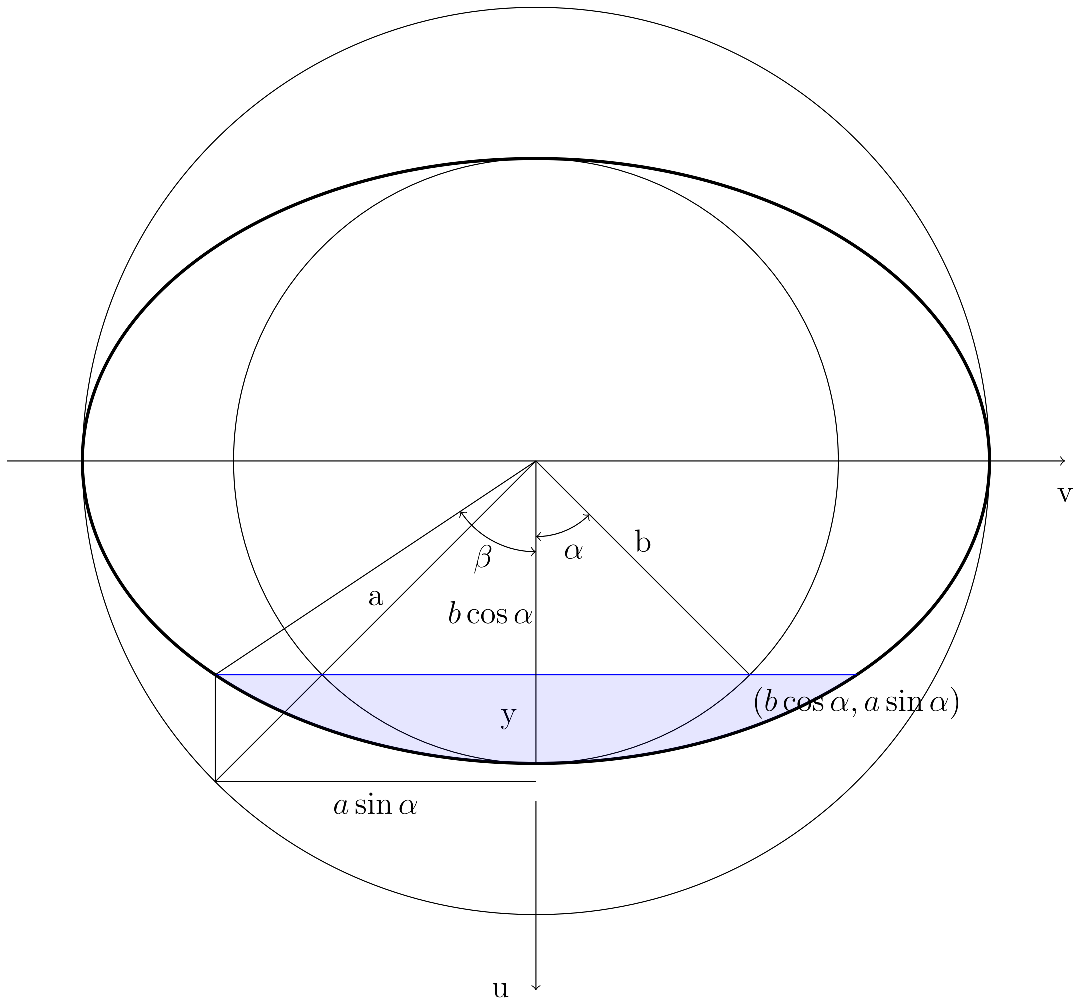


Figure 5: Elliptical Section ($a \geq b$)

$$\sin^2\alpha = \frac{\tan^2\alpha}{1+\tan^2\alpha} = \frac{b^2\tan^2\beta}{a^2+b^2\tan^2\beta} \tag{74}$$

$$\cos^2\alpha = \frac{1}{1+\tan^2\alpha} = \frac{a^2}{a^2+b^2\tan^2\beta} \tag{75}$$

$$r^2 = a^2\sin^2\alpha + b^2\cos^2\alpha \tag{76}$$

7.2 Wet Perimeter

7.2.1 $a \geq b$

In the case of $a \geq b$ (Figure 5), the wet perimeter is

$$P = 2\int_0^\alpha \sqrt{a^2\cos^2 t + b^2\sin^2 t}dt = 2a\int_0^\alpha \sqrt{1-(1-b^2/a^2)\sin^2 t}dt = 2aE(\alpha,\eta) \tag{77}$$

with $\eta^2 = 1 - b^2/a^2$, $E(\alpha,\zeta)$ as the Legendre ellitipcal integral of the second kind.

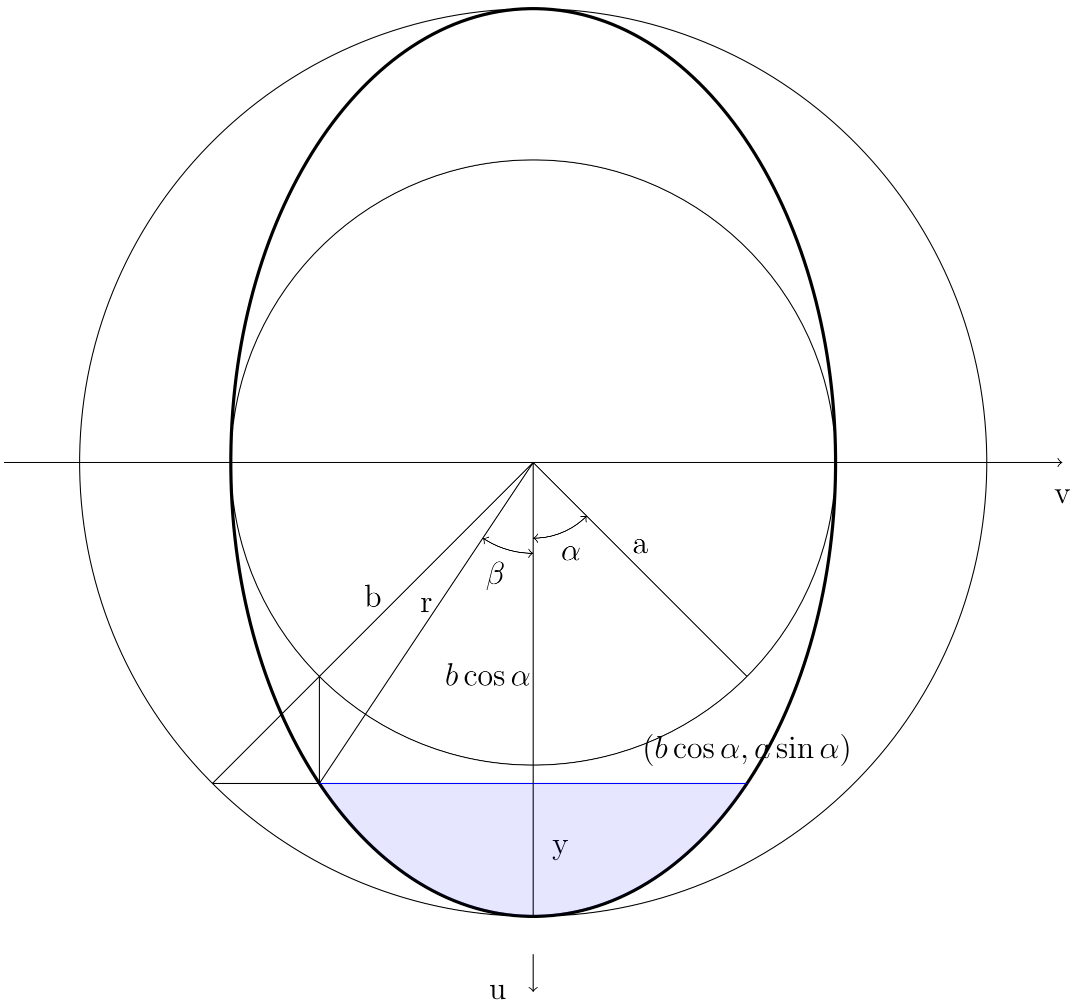


Figure 6: Elliptical Section $(a < b)$

$$E(\alpha,\eta) = \alpha - \frac{1}{2}\eta^2 \int_0^\alpha \sin^2 t dt - \frac{1}{2\cdot 4}\eta^4 \int_0^\alpha \sin^4 t dt - \frac{1\cdot 3}{2\cdot 4\cdot 6}\eta^6 \int_0^\alpha \sin^6 t dt + ... \tag{78}$$

$$\begin{aligned}
\int \sin^n t dt &= -\frac{1}{n}\sin^{n-1}\cos t + \frac{n}{n-1}\int \sin^{n-2} t dt \\
\int_0^\alpha \sin^2 t dt &= -\frac{1}{2}\sin\alpha\cos\alpha + \frac{1}{2}\alpha \\
\int_0^\alpha \sin^4 t dt &= -\frac{1}{4}\sin^3\alpha\cos\alpha + \frac{3}{4}\int_0^\alpha \sin^2 t dt \\
\int_0^\alpha \sin^6 t dt &= -\frac{1}{6}\sin^5\alpha\cos\alpha + \frac{5}{6}\int_0^\alpha \sin^4 t dt \\
\int_0^\alpha \sin^{2n} t dt &= -\frac{1}{2n}\sin^{2n-1}\alpha\cos\alpha + \frac{2n-1}{2n}\int_0^\alpha \sin^{2n-2} t dt
\end{aligned} \tag{79}$$

7.2.2 $a < b$

For $a < b$, the wet perimeter is (Figure 6)

$$P = 2\int_0^\alpha \sqrt{a^2\cos^2 t + b^2\sin^2 t}dt = 2b\int_0^\alpha \sqrt{1-(1-a^2/b^2)\cos^2 t}dt = 2bF(\alpha,\zeta) \quad (80)$$

with $\zeta^2 = 1 - a^2/b^2$.

$$F(\alpha,\eta) = \alpha - \frac{1}{2}\zeta^2\int_0^\alpha \cos^2 tdt - \frac{1}{2^2\cdot 2!}\zeta^4\int_0^\alpha \cos^4 tdt - \frac{1\cdot 3}{2^3\cdot 3!}\zeta^6\int_0^\alpha \cos^6 tdt - \cdots \quad (81)$$

$$\begin{aligned}
\int \cos^n tdt &= \frac{1}{n}\sin t\cos^{n-1} t + \frac{n}{n-1}\int \cos^{n-2} tdt \\
\int_0^\alpha \cos^2 tdt &= \frac{1}{2}\sin\alpha\cos\alpha + \frac{1}{2}\alpha \\
\int_0^\alpha \cos^4 tdt &= \frac{1}{4}\sin\alpha\cos^3\alpha + \frac{3}{4}\int_0^\alpha \cos^2 tdt \\
\int_0^\alpha \cos^6 tdt &= \frac{1}{6}\sin\alpha\cos^5\alpha + \frac{5}{6}\int_0^\alpha \cos^4 tdt \\
\int_0^\alpha \cos^{2n} tdt &= \frac{1}{2n}\sin\alpha\cos^{2n-1}\alpha + \frac{2n-1}{2n}\int_0^\alpha \cos^{2n-2} tdt
\end{aligned} \quad (82)$$

For both cases,

$$P' = 2\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} \quad (83)$$

$$P'' = -\frac{(a^2-b^2)\sin 2\alpha}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}} \quad (84)$$

7.3 Flow Area

The flow area is

$$A = 2\int_{b\cos\alpha}^b vdu = -2ab\int_\alpha^0 \sin^2 tdt = ab\int_0^\alpha (1-\cos 2t)dt = ab(\alpha - \frac{1}{2}\sin 2\alpha) \quad (85)$$

$$A' = ab(1-\cos 2\alpha) \quad (86)$$

$$A'' = 2ab\sin 2\alpha \quad (87)$$

$$T = 2a\sin\alpha \quad (88)$$

$$y = b(1-\cos\alpha) \quad (89)$$

$$\frac{\partial y}{\partial\alpha} = b\sin\alpha \quad (90)$$

$$\cos\alpha = 1 - y/b \quad (91)$$

7.4 Calculate Maximum Discharge Using Manning's Equation

To reach maximum capacity, the α is solved using Newton-Raphson method

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)}{f'(\alpha_i)}, \tag{92}$$

with Eq. 60,

$$f(\alpha) = 5A'P - 2AP' = 0, \tag{93}$$

and

$$f'(\alpha) = 3A'P' + 5A''P - 2AP''. \tag{94}$$

7.5 Calculate Normal Depth

For Q less than Q_{max},

$$\alpha_{i+1} = \alpha_i - \frac{f_d(\alpha_i)}{f'_d(\alpha_i)} \tag{95}$$

where

$$f_d(\alpha_i) = \frac{K_u}{n} A^{5/3} P^{-2/3} S^{1/2} - Q \tag{96}$$

$$f'_d(\alpha_i) = \frac{K_u}{n} \left(\frac{5}{3} R^{2/3} A' - \frac{2}{3} R^{5/3} P' \right) S^{1/2} \tag{97}$$

7.6 Calculate Critical Depth

$$\frac{\partial A}{\partial y}(\alpha) = \frac{\frac{\partial A}{\partial \alpha}}{\frac{\partial y}{\partial \alpha}} = \frac{ab(1 - \cos 2\alpha)}{b \sin \alpha} = 2a \sin \alpha \tag{98}$$

$$\frac{\partial}{\partial \alpha} \left(\frac{\partial A}{\partial y} \right) = 2a \cos \alpha \tag{99}$$

$$f_c(\alpha) = gA^3 - Q^2 \frac{\partial A}{\partial y} = gA^3 - 2aQ^2 \sin \alpha \tag{100}$$

$$f'_c(\alpha) = 3gA^2 \frac{\partial A}{\partial \alpha} - Q^2 \frac{\partial}{\partial \alpha} \left(\frac{\partial A}{\partial y} \right) \tag{101}$$

8 Arc

8.1 Geometry

$$\begin{aligned} r_b &= AB = BO = BE \\ r_t &= AC = CG \\ r_c &= DE = DF = DG \\ c &= BC = BO - CO = BO - (OG - CG) = r_b + r_t - rise \end{aligned} \tag{102}$$

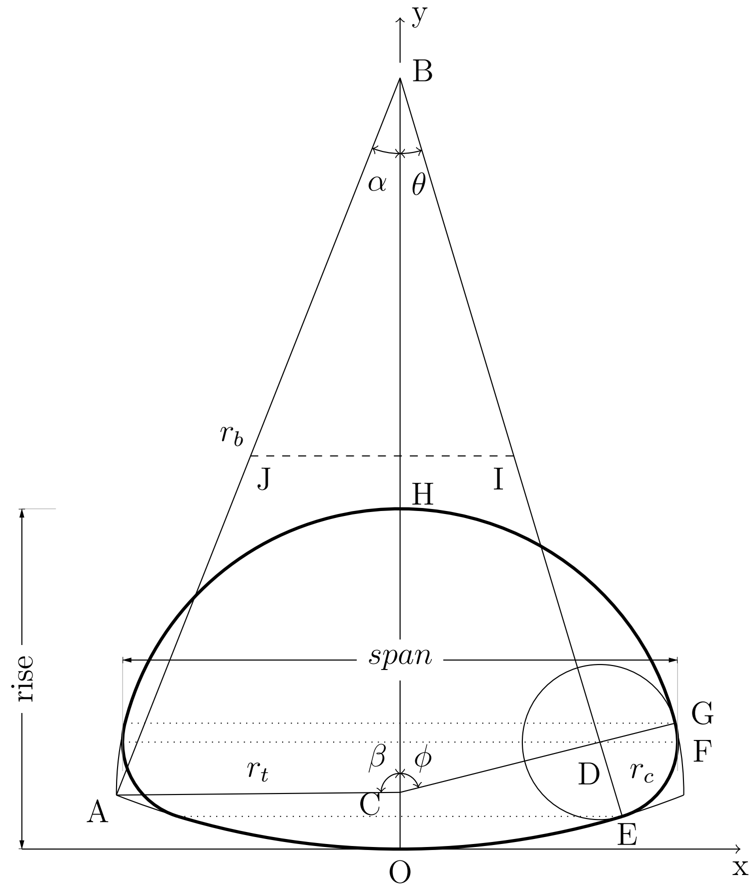


Figure 7: Arc Section

$$
\begin{aligned}
\cos\alpha &= \frac{r_b^2 + c^2 - r_t^2}{2r_b c} \\
\cos\beta &= \frac{r_t^2 + c^2 - r_b^2}{2r_t c} \\
\cos\theta &= \frac{(r_b - r_c)^2 + c^2 - (r_t - r_c)^2}{2(r_b - r_c)c} \\
\cos\phi &= \frac{(r_t - r_c)^2 + c^2 - (r_b - r_c)^2}{2(r_t - r_c)c}
\end{aligned}
\tag{103}
$$

$$
\begin{aligned}
x_A &= r_b \sin\alpha \\
y_A &= r_b(1-\cos\alpha) \\
y_B &= r_b \\
y_C &= rise - r_t = OG - CG \\
x_D &= (r_b - r_c)\sin\theta = (r_t - r_c)\sin\phi \\
y_D &= r_b - (r_b - r_c)\cos\theta = y_F \\
x_E &= r_b \sin\theta \\
y_E &= r_b(1-\cos\theta) \\
x_F &= x_D + r_c \\
x_G &= r_t \sin\phi \\
y_G &= y_C + r_t\cos\phi = rise - r_t(1-\cos\phi) \\
y_H &= rise
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
A_E &= r_b^2(\theta - \sin\theta\cos\theta) \\
P_E &= 2r_b\theta \\
T_E &= 2r_b\sin\theta \\
A_F &= A_E + r_c^2(\pi/2-\theta) + (x_E + x_D)(y_D - y_E) \\
P_F &= P_E + 2r_c(\pi/2-\theta) \\
T_F &= 2(x_D + r_c) \\
A_G &= A_F + r_c^2(\pi/2-\phi) + (x_D + x_G)(y_G - y_D) \\
P_G &= P_F + 2r_c(\pi/2-\phi) \\
T_G &= 2r_t\sin\phi \\
A_T &= A_G + r_t^2(\phi - \sin\phi\cos\phi) \\
P_T &= P_G + 2r_t\phi
\end{aligned}
\tag{105}
$$

For $0 \le y \le y_E$, $0 \le t \le \theta$

$$
\begin{aligned}
x &= r_b\sin t \\
y &= r_b(1-\cos t) \\
y' &= r_b\sin t = \frac{\partial y}{\partial t} \\
A &= r_b^2(t - \sin t\cos t) \\
A' &= r_b^2(1-\cos 2t) = \frac{\partial A}{\partial t} \\
\frac{\partial A}{\partial y}(t) &= 2r_b\sin t \\
\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) &= 2r_b\cos t \\
P &= 2r_b t \\
T &= 2r_b\sin t
\end{aligned}
\tag{106}
$$

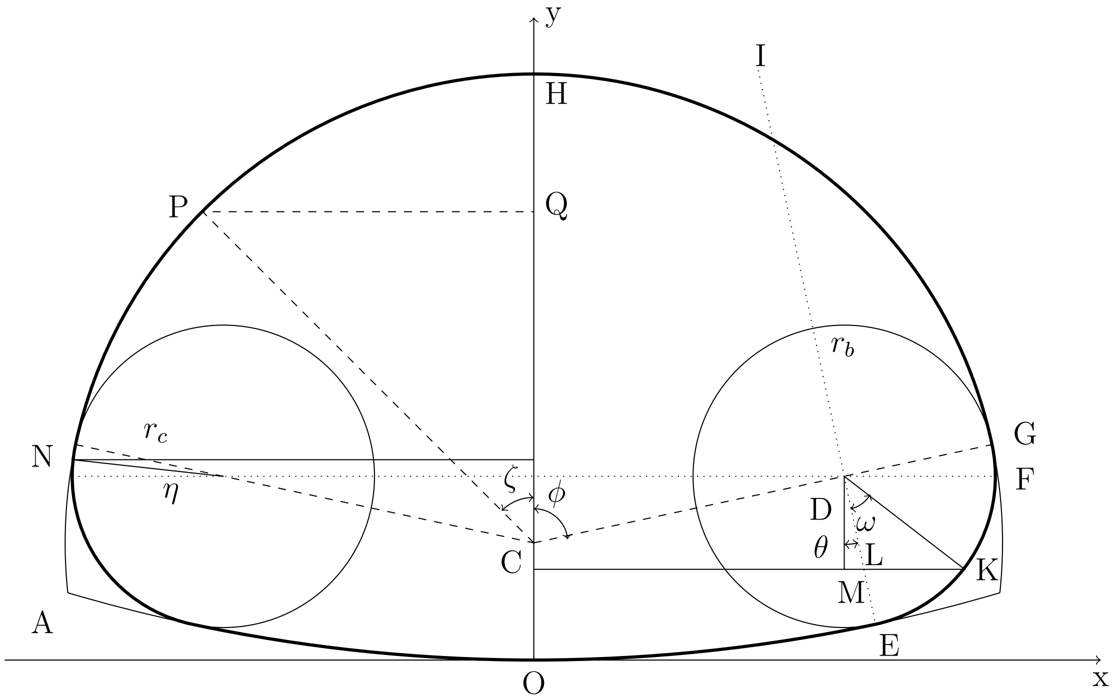


Figure 8: Arc Section

For $y_E \leq y \leq y_F$, $0 \leq w \leq \pi/2 - \theta$

$$
\begin{aligned}
x &= x_D + r_c \sin(\omega + \theta) \\
x_L &= x_D + r_c \cos(\omega + \theta) \tan\theta \\
x'_L &= -r_c \sin(\omega + \theta) \tan\theta \\
x''_L &= -r_c \cos(\omega + \theta) \tan\theta \\
y &= y_D - r_c \cos(\omega + \theta) \\
y' &= r_c \sin(\omega + \theta) \\
y'' &= r_c \cos(\omega + \theta) \\
A &= A_E + r_c^2 \omega - r_c^2 \sin(\omega + \theta) \cos(\omega + \theta) + r_c^2 \cos^2(\omega + \theta) \tan\theta + (x_E + x_L)(y - y_E) \\
A' &= r_c^2 - r_c^2 \cos\left[2(\omega + \theta)\right] - r_c^2 \sin[2(\omega + \theta)] \tan\theta + (x_E + x_L) y' + x'_L (y - y_E) \\
A'' &= 2r_c^2 \sin\left[2(\omega + \theta)\right] - 2r_c^2 \cos[2(\omega + \theta)] \tan\theta + (x_E + x_L) y'' + x'_L y' + x''_L (y - y_E) + x'_L y' \\
\frac{\partial A}{\partial y}(t) &= \frac{A'}{y'} \\
\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) &= \frac{A'' y' - A' y''}{y'^2} \\
P &= P_E + 2r_c \omega \\
T &= 2x_D + 2r_c \sin(\omega + \theta)
\end{aligned}
\tag{107}
$$

For $y_F \le y \le y_G$, $0 \le \eta \le \pi/2 - \phi$

$$\begin{aligned}
x &= x_D + r_c \cos\eta \\
y &= y_D + r_c \sin\eta \\
A &= A_F + r_c^2\eta + (2x_D + r_c\cos\eta) r_c \sin\eta \\
P &= P_F + 2r_c\eta \\
T &= 2x_D + 2r_c\cos\eta \\
y' &= r_c\cos\eta \\
y'' &= -r_c\sin\eta \\
A' &= r_c^2 + 2x_D r_c\cos\eta + r_c^2\cos(2\eta) \\
A'' &= -2x_D r_c\sin\eta - 2r_c^2\sin(2\eta) \\
\frac{\partial A}{\partial y}(t) &= \frac{A'}{y'} \\
\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) &= \frac{A''y' - A'y''}{y'^2}
\end{aligned} \tag{108}$$

For $y_G \le y \le y_H$, $0 \le \zeta \le \phi$

$$\begin{aligned}
y &= y_C + r_t\cos\zeta \\
A &= A_T - r_t^2(\zeta - \sin\zeta\cos\zeta) \\
P &= P_T - 2r_t\zeta = P_G + 2r_t(\phi - \zeta) \\
T &= 2r_t\sin\zeta \\
y' &= -r_t\sin\zeta \\
y'' &= -r_t\cos\zeta \\
A' &= -r_t^2[1 - \cos(2\zeta)] \\
A'' &= -2r_t^2\sin(2\zeta) \\
P'(\zeta) &= -2r_t \\
P''(\zeta) &= 0 \\
\frac{\partial A}{\partial y}(t) &= \frac{A'}{y'} = 2r_t\sin\zeta \\
\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial y}\right) &= 2r_t\cos\zeta
\end{aligned} \tag{109}$$

8.2 Calculate Maximum Discharge Using Manning's Equation

Maximum discharge occurs close to the top of arch, the ζ is solved using Newton-Raphson method

$$\zeta_{i+1} = \zeta_i - \frac{f(\zeta_i)}{f'(\zeta_i)}, \tag{110}$$

with Eq. 60,

$$f(\zeta) = 5A'P - 2AP', \tag{111}$$

and

$$f'(\zeta) = 3A'P' + 5A''P - 2AP''. \tag{112}$$

8.3 Calculate Normal Depth

For Q less than Q_{max},

$$\alpha_{i+1} = \alpha_i - \frac{f_d(\alpha_i)}{f_d'(\alpha_i)} \tag{113}$$

where

$$f_d(\alpha_i) = \frac{K_u}{n} A^{5/3} P^{-2/3} S^{1/2} - Q \tag{114}$$

$$f_d'(\alpha_i) = \frac{K_u}{n} \left(\frac{5}{3} R^{2/3} A' - \frac{2}{3} R^{5/3} P' \right) S^{1/2} \tag{115}$$

8.4 Critical Flow

Critical flow occurs when the specific energy of the cross-section

$$E = \frac{v^2}{2g} + y = \frac{Q^2}{2gA^2} + y \tag{116}$$

reaches a minimum. Namely,

$$\frac{\partial E}{\partial y} = -\frac{Q^2}{gA^3} \frac{\partial A}{\partial y} + 1 = 0 \tag{117}$$

Newton-Raphson method is used to calculate critical depth as

$$y_{c,i+1} = y_{c,i} - \frac{f(y_{c,i})}{f'(y_{c,i})} \tag{118}$$

where

$$f_c(y) = gA^3 - Q^2 \frac{\partial A}{\partial y} \tag{119}$$

$$f_c'(y) = 3gA^2 \frac{\partial A}{\partial y} - Q^2 \frac{\partial^2 A}{\partial y^2} \tag{120}$$

for A and y are parameterized as a function of t with $A = A(t)$, $A' = \frac{\partial A(t)}{\partial t}$, $y = y(t)$, $y' = \frac{\partial y(t)}{\partial t}$,

$$t_{c,i+1} = t_{c,i} - \frac{f(t_{c,i})}{f'(t_{c,i})} \tag{121}$$

$$f_c(t) = gA^3(t) - Q^2 \frac{\partial A}{\partial y}(t) \tag{122}$$

$$f_c'(t) = 3gA^2 \frac{\partial A}{\partial t} - Q^2 \frac{\partial}{\partial t} \left(\frac{\partial A}{\partial y} \right) \tag{123}$$

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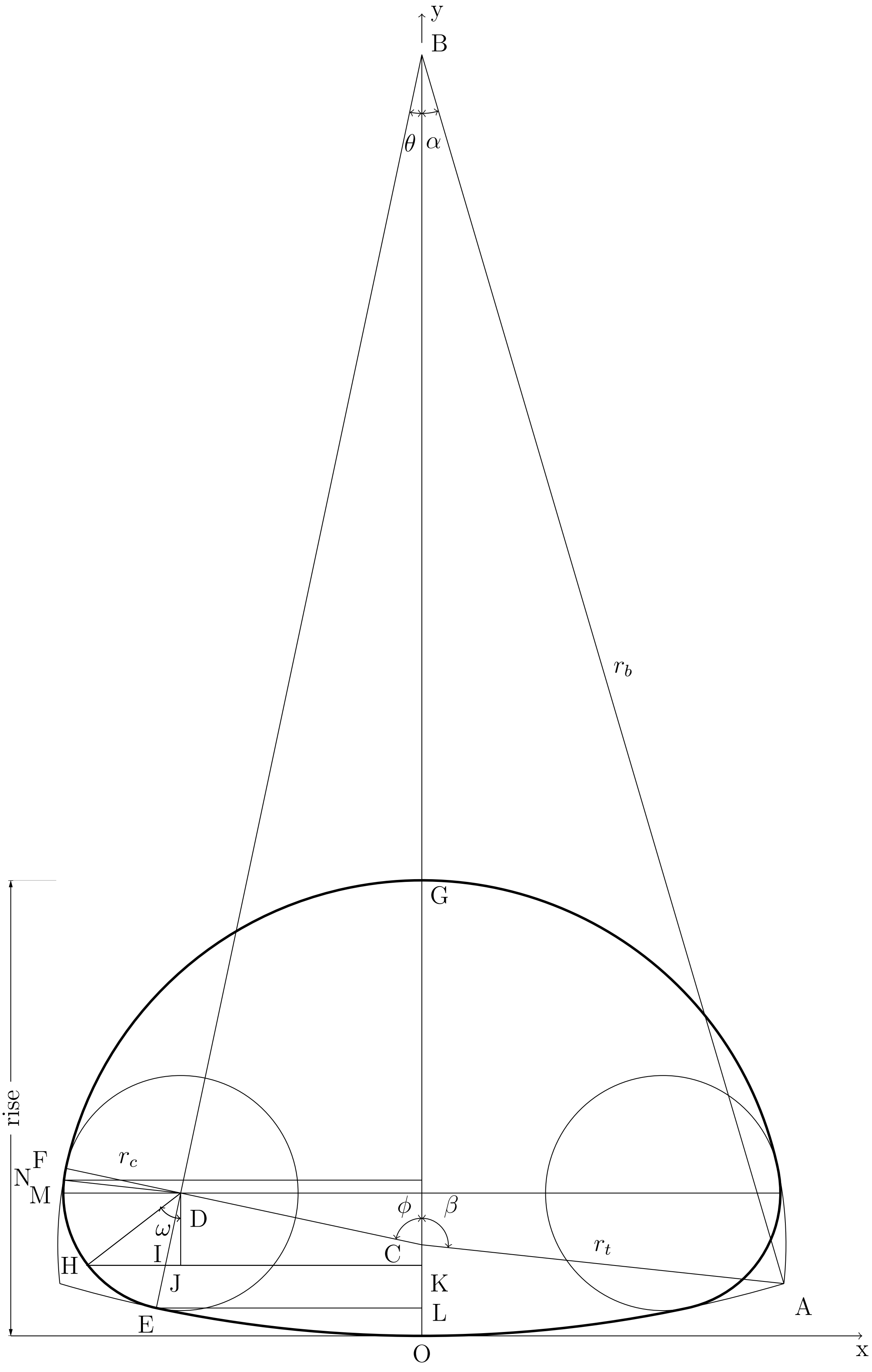

Figure 9: Arc Section