Apocalypse of the Four Fouriers
and Other Tales of Mathematical Horror

Roger Green

Analysis and Synthesis Equations

Signal x is...	**Aperiodic**	**Periodic**
Continuous	**FT (Analysis):** $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ **IFT (Synthesis):** $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$	**FS (Analysis):** $X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$ **FS (Synthesis):** $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$
Discrete	**DTFT (Analysis):** $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ **IDTFT (Synthesis):** $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}\,d\Omega$	**DFT (Analysis):** $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\Omega_0 kn}$, where $k = \{0 : N-1\}$ and $\Omega_0 = \frac{2\pi}{N}$ **IDFT (Synthesis):** $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\Omega_0 kn}$, where $n = \{0 : N-1\}$ and $\Omega_0 = \frac{2\pi}{N}$

The discrete-time Fourier series (DTFS) differs from the DFT by the scale factor $\frac{1}{N}$, and both use $\Omega_0 = \frac{2\pi}{N}$. The DTFS analysis and synthesis equations are thus

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\Omega_0 kn} \qquad \text{and} \qquad x[n] = \sum_{k=0}^{N-1} X[k]e^{j\Omega_0 kn}.$$

Summary of Properties

Time (or Frequency)	$\longleftrightarrow$	**Frequency (or Time)**
Real	$\longleftrightarrow$	Conjugate symmetric
Imaginary	$\longleftrightarrow$	Conjugate antisymmetric
Even	$\longleftrightarrow$	Even
Odd	$\longleftrightarrow$	Odd
Continuous	$\longleftrightarrow$	Aperiodic
Discrete	$\longleftrightarrow$	Periodic

Linear Vector Spaces

A **Linear Vector Space** $\mathcal{X}$ is a set of elements called vectors together with two operations:

1. **Addition:** $x, y \in \mathcal{X} \Rightarrow x + y \in \mathcal{X}$.
2. **Scalar mult.:** $x \in \mathcal{X}$, $\alpha \in \mathcal{C} \Rightarrow \alpha x \in \mathcal{X}$.

These two operations satisfy the following properties:

1. **Commutative law:** $x + y = y + x$.
2. **Associative law:** $(x + y) + z = x + (y + z)$.
3. **Additive identity:** $\exists \theta \in \mathcal{X}$ s.t. $x + \theta = x$.
4. $\alpha(x + y) = \alpha x + \alpha y$.
5. $(\alpha + \beta)x = \alpha x + \beta x$.
6. $0x = \theta$.
7. $1x = x$.

An **Inner Product** $\langle x, y\rangle \in \mathcal{C}$ defined on $\mathcal{X} \times \mathcal{X}$ (meaning $x \in \mathcal{X}$ and $y \in \mathcal{X}$), satisfies:

1. $\langle x, y\rangle = \langle y, x\rangle^*$.
2. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.
3. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$.
4. $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = \theta$.

Orthogonal (or perpendicular) elements x and y of a linear vector space $\mathcal{X}$ satisfy $\langle x, y\rangle = 0$ ($x \perp y$).

A **Norm** $\|x\|$ on a linear vector space $\mathcal{X}$ satisfies:

1. $\|x\| \geq 0$ for all $x \in \mathcal{X}$, with $\|x\| = 0$ iff $x = \theta$.
2. **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$.
3. $\|\alpha x\| = |\alpha|\|x\|$.

Note: The quantity $\sqrt{\langle x, x\rangle} = \sqrt{\|x\|^2}$ is a valid norm.

Example: The set of all continuous-time energy signals is a linear vector space. For two energy signals $x(\cdot)$ and $y(\cdot)$, define

$$\langle x, y\rangle = \int_{-\infty}^{\infty} x(t)y^*(t)\,dt,$$

and note that $E_x = \|x\|^2 = \langle x, x\rangle$ is a valid norm.

The Orthogonality Principle

Let $\mathcal{X}$ be a linear vector space, $x \in \mathcal{X}$, and let $\{\phi_k\}_{k=1}^K$ be a set of basis vectors in $\mathcal{X}$. We want to estimate x with $\hat{x}_K$, where

$$\hat{x}_K = \sum_{k=1}^{K} c_k\phi_k$$

is a linear combination of the K basis vectors. We want to find the K constants $\{c_k\}_{k=1}^K$ to minimize the norm-squared error between x and $\hat{x}_K$; i.e., find $\{c_k\}_{k=1}^K$ to minimize

$$\mathcal{E}_K = \|x - \hat{x}_K\|^2 = \langle x - \hat{x}_K, x - \hat{x}_K\rangle.$$

The best choice satisfies the orthogonality principle: The error must be orthogonal to the data used in the estimate, or

$$(\text{error} = x - \hat{x}_K) \perp \phi_\ell, \qquad \ell = 1, 2, \ldots, K.$$

Consequently,

$$\langle x - \hat{x}_K, \phi_\ell\rangle = 0, \qquad \ell = 1, 2, \ldots, K,$$

or

$$\langle \hat{x}_K, \phi_\ell\rangle = \langle x, \phi_\ell\rangle, \qquad \ell = 1, 2, \ldots, K.$$

Substituting for $\hat{x}_K$, we find that the c's must satisfy

$$\sum_{k=1}^{K} c_k\langle \phi_k, \phi_\ell\rangle = \langle x, \phi_\ell\rangle, \qquad \ell = 1, 2, \ldots, K.$$

If $\{\phi_k\}_{k=1}^K$ is an orthogonal set of basis vectors (meaning that $\langle \phi_\ell, \phi_k\rangle = 0$ for $\ell \neq k$), then

$$c_k = \frac{\langle x, \phi_k\rangle}{\|\phi_k\|^2}.$$

For mutually orthogonal basis vectors ϕ_k, notice that each c_k posseses the important "finality property". That is, the optimal value of any coefficient c_k is independent of the other coefficients as well as the number of terms used.

The resulting minimum value of $\mathcal{E}_K$ is

$$\mathcal{E}_{K,\min} = \|x\|^2 - \sum_{k=1}^{K} c_k\langle \phi_k, x\rangle.$$

Properties of the Fourier Transform and the Fourier Series

Fourier Transform	Fourier Series
Synthesis: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$	**Synthesis:** $x(t) = \sum_{-\infty}^{\infty} X_k e^{jk\omega_0 t}$
Analysis: $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	**Analysis:** $X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$
Duality: if $x(t) \Longleftrightarrow X(\omega)$, then $X(t) \Longleftrightarrow 2\pi x(-\omega)$	**Duality:**
Linearity: $ax(t) + by(t) \Longleftrightarrow aX(\omega) + bY(\omega)$	**Linearity:** $ax(t) + by(t) \Longleftrightarrow aX_k + bY_k$
Complex Conjugation: $x^*(t) \Longleftrightarrow X^*(-\omega)$	**Complex Conjugation:** $x^*(t) \Longleftrightarrow X^*_{-k}$
Scaling and Reversal: $x(at) \Longleftrightarrow \frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$ $x(-t) \Longleftrightarrow X(-\omega)$	**Scaling and Reversal:** $x(-t) \Longleftrightarrow X_{-k}$
Shifting: $x(t-t_0) \Longleftrightarrow X(\omega)e^{-j\omega t_0}$ $x(t)e^{j\omega_0 t} \Longleftrightarrow X(\omega-\omega_0)$	**Shifting:** $x(t-t_0) \Longleftrightarrow X_k e^{-jk\omega_0 t_0}$ $x(t)e^{jk_0\omega_0 t} \Longleftrightarrow X_{k-k_0}$
Differentiation: $\frac{d}{dt}x(t) \Longleftrightarrow j\omega X(\omega)$ $-jtx(t) \Longleftrightarrow \frac{d}{d\omega}X(\omega)$	**Differentiation:** $\frac{d}{dt}x(t) \Longleftrightarrow jk\omega_0 X_k$
Time Integration: $\int_{-\infty}^{t} x(\tau)d\tau \Longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$	**Time Integration:**
Convolution: $x(t) * y(t) \Longleftrightarrow X(\omega)Y(\omega)$ $x(t)y(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * Y(\omega)$	**Convolution:** $\frac{1}{T_0}x(t)\circledast y(t) \Longleftrightarrow X_k Y_k$ $x(t)y(t) \Longleftrightarrow X_k * Y_k$
Correlation: $\rho_{x,y}(\tau) = x(\tau) * y^*(-\tau) \Longleftrightarrow X(\omega)Y^*(\omega)$	**Correlation:** $\rho_{x,y}(\tau) = \frac{1}{T_0}x(\tau)\circledast y^*(-\tau) \Longleftrightarrow X_k Y_k^*$
Parseval's: $E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\,d\omega$	**Parseval's:** $P_x = \frac{1}{T_0}\int_{T_0} \lvert x(t)\rvert^2\,dt = \sum_{k=-\infty}^{\infty} \lvert X_k\rvert^2$

Properties of the DTFT and the DFT

Discrete-Time Fourier Transform	**Discrete Fourier Transform**
Synthesis: $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}\, d\Omega$	**Synthesis:** $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\Omega_0 kn}, \quad \Omega_0 = \frac{2\pi}{N}$
Analysis: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$	**Analysis:** $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\Omega_0 kn}, \quad \Omega_0 = \frac{2\pi}{N}$
Duality:	**Duality:** if $x[n] \longleftrightarrow X[k]$, then $X[n] \longleftrightarrow Nx[\langle -k\rangle_N]$
Linearity: $ax[n] + by[n] \Longleftrightarrow aX(\Omega) + bY(\Omega)$	**Linearity:** $ax[n] + by[n] \longleftrightarrow aX[k] + bY[k]$
Complex Conjugation: $x^*[n] \Longleftrightarrow X^*(-\Omega)$ $x^*[-n] \Longleftrightarrow X^*(\Omega)$	**Complex Conjugation:** $x^*[n] \longleftrightarrow X^*[\langle -k\rangle_N]$ $x^*[\langle -n\rangle_N] \longleftrightarrow X^*[k]$
Reversal: $x[-n] \Longleftrightarrow X(-\Omega)$	**Reversal:** $x[\langle -n\rangle_N] \longleftrightarrow X[\langle -k\rangle_N]$
Shifting: $x[n-m] \Longleftrightarrow X(\Omega)e^{-j\Omega m}$ $x[n]e^{j\Omega_0 n} \Longleftrightarrow X(\Omega - \Omega_0)$	**Shifting:** $x[\langle n-m\rangle_N] \longleftrightarrow X[k]e^{-j\Omega_0 km}, \quad \Omega_0 = \frac{2\pi}{N}$ $x[n]e^{j\Omega_0 mn} \longleftrightarrow X[\langle k-m\rangle_N], \quad \Omega_0 = \frac{2\pi}{N}$
Convolution: $x[n] * y[n] \Longleftrightarrow X(\Omega)Y(\Omega)$ $x[n]y[n] \Longleftrightarrow \frac{1}{2\pi}X(\Omega)\circledast Y(\Omega)$	**Convolution:** $x[n]\circledast y[n] \longleftrightarrow X[k]Y[k]$ $x[n]y[n] \longleftrightarrow \frac{1}{N}X[k]\circledast Y[k]$
Correlation: $\rho_{x,y}[l] = x[l] * y^*[-l] \Longleftrightarrow X(\Omega)Y^*(\Omega)$	**Correlation:** $\rho_{x,y}[l] = x[l]\circledast y^*[\langle -l\rangle_N] \longleftrightarrow X[k]Y^*[k]$
Parseval's: $E_x = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(\Omega)\rvert^2\, d\Omega$	**Parseval's:** $E_x = \sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$

Properties of the Laplace Transform

Bilateral Laplace Transform	Unilateral Laplace Transform
Synthesis: $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$	**Synthesis:** $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$
Analysis: $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$, ROC: R_x	**Analysis:** $X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt$
Linearity: $ax(t)+by(t) \overset{\mathcal{L}}{\Longleftrightarrow} aX(s)+bY(s)$, ROC: At least $R_x \cap R_y$	**Linearity:** $ax(t)+by(t) \overset{\mathcal{L}_u}{\Longleftrightarrow} aX(s)+bY(s)$
Complex Conjugation: $x^*(t) \overset{\mathcal{L}}{\Longleftrightarrow} X^*(s^*)$, ROC: R_x	**Complex Conjugation:** $x^*(t) \overset{\mathcal{L}_u}{\Longleftrightarrow} X^*(s^*)$
Scaling and Reversal: $x(at) \overset{\mathcal{L}}{\Longleftrightarrow} \frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$, ROC: R_x scaled by $1/a$ $x(-t) \overset{\mathcal{L}}{\Longleftrightarrow} X(-s)$, ROC: R_x reflected	**Scaling and Reversal:** If $a>0$: $x(at) \overset{\mathcal{L}_u}{\Longleftrightarrow} \frac{1}{a}X\left(\frac{s}{a}\right)$
Shifting: $x(t-t_0) \overset{\mathcal{L}}{\Longleftrightarrow} X(s)e^{-st_0}$, ROC: R_x $x(t)e^{s_0t} \overset{\mathcal{L}}{\Longleftrightarrow} X(s-s_0)$, ROC: R_x shifted by $\mathrm{Re}\{s_0\}$	**Shifting:** If $t_0>0$: $x(t-t_0) \overset{\mathcal{L}_u}{\Longleftrightarrow} X(s)e^{-st_0}$ $x(t)e^{s_0t} \overset{\mathcal{L}_u}{\Longleftrightarrow} X(s-s_0)$
Differentiation: $\frac{d}{dt}x(t) \overset{\mathcal{L}}{\Longleftrightarrow} sX(s)$, ROC: At least R_x $-tx(t) \overset{\mathcal{L}}{\Longleftrightarrow} \frac{d}{ds}X(s)$, ROC: R_x	**Differentiation:** $\frac{d}{dt}x(t) \overset{\mathcal{L}_u}{\Longleftrightarrow} sX(s)-x(0^-)$ (general case shown below) $-tx(t) \overset{\mathcal{L}_u}{\Longleftrightarrow} \frac{d}{ds}X(s)$
Time Integration: $\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\Longleftrightarrow} \frac{1}{s}X(s)$, ROC: At least $R_x \cap (\mathrm{Re}\{s\}>0)$	**Time Integration:** $\int_{0^-}^{t} x(\tau)d\tau \overset{\mathcal{L}_u}{\Longleftrightarrow} \frac{1}{s}X(s)$
Convolution: $x(t)*y(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s)Y(s)$, ROC: At least $R_x \cap R_y$	**Convolution:** $x(t)*y(t) \overset{\mathcal{L}_u}{\Longleftrightarrow} X(s)Y(s)$

Unilateral Laplace Transform Time Differentiation, General Case
$x^{(k)}(t) = \frac{d^k}{dt^k}x(t) \overset{\mathcal{L}_u}{\Longleftrightarrow} s^kX(s) - \sum_{i=0}^{k-1} s^{k-1-i}x^{(i)}(0^-)$

Properties of the z-Transform

Bilateral z-Transform	Unilateral z-Transform
Synthesis: $x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$	**Synthesis:** $x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$
Analysis: $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$, ROC: R_x	**Analysis:** $X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$
Linearity: $ax[n]+by[n] \overset{\mathcal{Z}}{\Longleftrightarrow} aX(z)+bY(z)$, ROC: At least $R_x \cap R_y$	**Linearity:** $ax[n]+by[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} aX(z)+bY(z)$
Complex Conjugation: $x^*[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X^*(z^*)$, ROC: R_x	**Complex Conjugation:** $x^*[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} X^*(z^*)$
Time Reversal: $x[-n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(1/z)$, ROC: $1/R_x$	**Time Reversal:**
Time Shifting: $x[n-m] \overset{\mathcal{Z}}{\Longleftrightarrow} z^{-m}X(z)$, ROC: Almost R_x	**Time Shifting:** If $m>0$: $x[n-m]u[n-m] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z)$ (general case given below)
z-Domain Scaling: $\gamma^n x[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z/\gamma)$, ROC: $\lvert\gamma\rvert R_x$	**z-Domain Scaling:** $\gamma^n x[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} X(z/\gamma)$
z-Domain Differentiation: $nx[n] \overset{\mathcal{Z}}{\Longleftrightarrow} -z\frac{d}{dz}X(z)$, ROC: R_x	**z-Domain Differentiation:** $nx[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} -z\frac{d}{dz}X(z)$
Time Convolution: $x[n] * y[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z)Y(z)$, ROC: At least $R_x \cap R_y$	**Time Convolution:** $x[n] * y[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} X(z)Y(z)$

Unilateral z-Transform Time Shifting, General Case

If $m > 0$: $x[n-m]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z) + z^{-m}\sum_{n=1}^{m} x[-n]z^n$

If $m < 0$: $x[n-m]u[n] \overset{\mathcal{Z}_u}{\Longleftrightarrow} z^{-m}X(z) - z^{-m}\sum_{n=0}^{-m-1} x[n]z^{-n}$

A Selection of Useful Sums

1.	$\sum_{m=p}^{n} r^m = \frac{r^p - r^{n+1}}{1-r}$	$r \neq 1$
2.	$\sum_{m=0}^{n} m = \frac{n(n+1)}{2}$	
3.	$\sum_{m=0}^{n} m^2 = \frac{n(n+1)(2n+1)}{6}$	
4.	$\sum_{m=0}^{n} mr^m = \frac{r+[n(r-1)-1]r^{n+1}}{(r-1)^2}$	$r \neq 1$
5.	$\sum_{m=0}^{n} m^2 r^m = \frac{r[(1+r)(1-r^n)-2n(1-r)r^n-n^2(1-r)^2r^n]}{(r-1)^3}$	$r \neq 1$

Selected Fourier Transform Pairs

	$x(t)$	$X(\omega)$	
1.	$e^{\lambda t}u(t)$	$\frac{1}{j\omega-\lambda}$	$\mathrm{Re}\{\lambda\}<0$
2.	$e^{\lambda t}u(-t)$	$-\frac{1}{j\omega-\lambda}$	$\mathrm{Re}\{\lambda\}>0$
3.	$e^{\lambda\|t\|}$	$\frac{-2\lambda}{\omega^2+\lambda^2}$	$\mathrm{Re}\{\lambda\}<0$
4.	$t^k e^{\lambda t}u(t)$	$\frac{k!}{(j\omega-\lambda)^{k+1}}$	$\mathrm{Re}\{\lambda\}<0$
5.	$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
6.	$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
7.	$\Pi\left(\frac{t}{\tau}\right)$	$\tau\mathrm{sinc}\left(\frac{\tau\omega}{2\pi}\right)$	$\tau>0$
8.	$\frac{B}{\pi}\mathrm{sinc}\left(\frac{Bt}{\pi}\right)$	$\Pi\left(\frac{\omega}{2B}\right)$	$B>0$
9.	$\Lambda\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\mathrm{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$	$\tau>0$
10.	$\frac{B}{2\pi}\mathrm{sinc}^2\left(\frac{Bt}{2\pi}\right)$	$\Lambda\left(\frac{\omega}{2B}\right)$	$B>0$
11.	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	$\sigma>0$
12.	$\delta(t)$	1	
13.	1	$2\pi\delta(\omega)$	
14.	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
15.	$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$	
16.	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
17.	$\cos(\omega_0 t)$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$	
18.	$\sin(\omega_0 t)$	$\frac{\pi}{j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$	
19.	$\cos(\omega_0 t)u(t)$	$\frac{\pi}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]+\frac{j\omega}{\omega_0^2-\omega^2}$	
20.	$\sin(\omega_0 t)u(t)$	$\frac{\pi}{2j}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
21.	$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$	$\omega_0=\frac{2\pi}{T}$

Selected Discrete-Time Fourier Transform Pairs

	$x[n]$	$X(\Omega)$	
1.	$\delta[n-k]$	$e^{-jk\Omega}$	integer k
2.	$\gamma^n u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\lvert\gamma\rvert < 1$
3.	$-\gamma^n u[-n-1]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\lvert\gamma\rvert > 1$
4.	$\gamma^{\lvert n\rvert}$	$\frac{1-\gamma^2}{1-2\gamma\cos(\Omega)+\gamma^2}$	$\lvert\gamma\rvert < 1$
5.	$n\gamma^n u[n]$	$\frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}$	$\lvert\gamma\rvert < 1$
6.	$\lvert\gamma\rvert^n \cos(\Omega_0 n+\theta)u[n]$	$\frac{e^{j\Omega}[e^{j\Omega}\cos(\theta)-\lvert\gamma\rvert\cos(\Omega_0-\theta)]}{e^{j2\Omega}-2\lvert\gamma\rvert\cos(\Omega_0)e^{j\Omega}+\lvert\gamma\rvert^2}$	$\lvert\gamma\rvert < 1$
7.	$u[n]-u[n-L_x]$	$\frac{\sin(L_x\Omega/2)}{\sin(\Omega/2)}\, e^{-j\Omega(L_x-1)/2}$	
8.	$\frac{B}{\pi}\operatorname{sinc}\left(\frac{Bn}{\pi}\right)$	$\sum_{k=-\infty}^{\infty}\Pi\left(\frac{\Omega-2\pi k}{2B}\right)$	
9.	$\frac{B}{2\pi}\operatorname{sinc}^2\left(\frac{Bn}{2\pi}\right)$	$\sum_{k=-\infty}^{\infty}\Lambda\left(\frac{\Omega-2\pi k}{2B}\right)$	
10.	1	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$	
11.	$u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-1}+\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$	
12.	$e^{j\Omega_0 n}$	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)$	
13.	$\cos(\Omega_0 n)$	$\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)$	
14.	$\sin(\Omega_0 n)$	$\frac{\pi}{j}\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)-\delta(\Omega+\Omega_0-2\pi k)$	
15.	$\cos(\Omega_0 n)\,u[n]$	$\frac{e^{j2\Omega}-e^{j\Omega}\cos(\Omega_0)}{e^{j2\Omega}-2\cos(\Omega_0)e^{j\Omega}+1}+\frac{\pi}{2}\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)$	
16.	$\sin(\Omega_0 n)\,u[n]$	$\frac{e^{j\Omega}\sin(\Omega_0)}{e^{j2\Omega}-2\cos(\Omega_0)e^{j\Omega}+1}+\frac{\pi}{2j}\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_0-2\pi k)-\delta(\Omega+\Omega_0-2\pi k)$	

Selected DTFT Pairs Using the Fundamental Band

	$x[n]$	$X(\Omega)$ for $-\pi \leq \Omega < \pi$	
1.	$\delta[n-k]$	$e^{-jk\Omega}$	integer k
2.	$\gamma^n u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\|\gamma\| < 1$
3.	$-\gamma^n u[-n-1]$	$\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$	$\|\gamma\| > 1$
4.	$\gamma^{\|n\|}$	$\frac{1-\gamma^2}{1-2\gamma\cos(\Omega)+\gamma^2}$	$\|\gamma\| < 1$
5.	$n\gamma^n u[n]$	$\frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}$	$\|\gamma\| < 1$
6.	$\|\gamma\|^n \cos(\Omega_0 n+\theta)u[n]$	$\frac{e^{j\Omega}[e^{j\Omega}\cos(\theta)-\|\gamma\|\cos(\Omega_0-\theta)]}{e^{j2\Omega}-2\|\gamma\|\cos(\Omega_0)e^{j\Omega}+\|\gamma\|^2}$	$\|\gamma\| < 1$
7.	$u[n]-u[n-L_x]$	$\frac{\sin(L_x\Omega/2)}{\sin(\Omega/2)}\, e^{-j\Omega(L_x-1)/2}$	
8.	$\frac{B}{\pi}\,\text{sinc}\left(\frac{Bn}{\pi}\right)$	$\Pi\left(\frac{\Omega}{2B}\right)$	$0 < B \leq \pi$
9.	$\frac{B}{2\pi}\,\text{sinc}^2\left(\frac{Bn}{2\pi}\right)$	$\Lambda\left(\frac{\Omega}{2B}\right)$	$0 < B \leq \pi$
10.	1	$2\pi\delta(\Omega)$	
11.	$u[n]$	$\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\delta(\Omega)$	
12.	$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0)$	$\|\Omega_0\| < \pi$
13.	$\cos(\Omega_0 n)$	$\pi\left[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)\right]$	$\|\Omega_0\| < \pi$
14.	$\sin(\Omega_0 n)$	$\frac{\pi}{j}\left[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)\right]$	$\|\Omega_0\| < \pi$
15.	$\cos(\Omega_0 n)\,u[n]$	$\frac{e^{j2\Omega}-e^{j\Omega}\cos(\Omega_0)}{e^{j2\Omega}-2\cos(\Omega_0)e^{j\Omega}+1} + \frac{\pi}{2}\left[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)\right]$	$\|\Omega_0\| < \pi$
16.	$\sin(\Omega_0 n)\,u[n]$	$\frac{e^{j\Omega}\sin(\Omega_0)}{e^{j2\Omega}-2\cos(\Omega_0)e^{j\Omega}+1} + \frac{\pi}{2j}\left[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)\right]$	$\|\Omega_0\| < \pi$

Selected Laplace Transform Pairs

	$x(t)$	$X(s)$	ROC
1.	$\delta(t)$	1	All s
2.	$u(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
3.	$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\text{Re}\{s\} > 0$
4.	$e^{\lambda t} u(t)$	$\frac{1}{s-\lambda}$	$\text{Re}\{s\} > \text{Re}\{\lambda\}$
5.	$t^k e^{\lambda t} u(t)$	$\frac{k!}{(s-\lambda)^{k+1}}$	$\text{Re}\{s\} > \text{Re}\{\lambda\}$
6.	$\cos(\omega_0 t) u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
7.	$\sin(\omega_0 t) u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
8.	$e^{-at}\cos(\omega_0 t) u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} > -a$
9.	$e^{-at}\sin(\omega_0 t) u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} > -a$
10.	$e^{-at}\cos(\omega_0 t + \theta) u(t)$	$\frac{\cos(\theta)s + a\cos(\theta) - \omega_0 \sin(\theta)}{s^2+2as+(a^2+\omega_0^2)}$	$\text{Re}\{s\} > -a$
		$= \frac{0.5e^{j\theta}}{s+a-j\omega_0} + \frac{0.5e^{-j\theta}}{s+a+j\omega_0}$	
11.	$u(-t)$	$-\frac{1}{s}$	$\text{Re}\{s\} < 0$
12.	$t^k u(-t)$	$-\frac{k!}{s^{k+1}}$	$\text{Re}\{s\} < 0$
13.	$e^{\lambda t} u(-t)$	$-\frac{1}{s-\lambda}$	$\text{Re}\{s\} < \text{Re}\{\lambda\}$
14.	$t^k e^{\lambda t} u(-t)$	$-\frac{k!}{(s-\lambda)^{k+1}}$	$\text{Re}\{s\} < \text{Re}\{\lambda\}$
15.	$\cos(\omega_0 t) u(-t)$	$-\frac{s}{s^2+\omega_0^2}$	$\text{Re}\{s\} < 0$
16.	$\sin(\omega_0 t) u(-t)$	$-\frac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}\{s\} < 0$
17.	$e^{-at}\cos(\omega_0 t) u(-t)$	$-\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} < -a$
18.	$e^{-at}\sin(\omega_0 t) u(-t)$	$-\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} < -a$
19.	$e^{-at}\cos(\omega_0 t + \theta) u(-t)$	$-\frac{\cos(\theta)s + a\cos(\theta) - \omega_0 \sin(\theta)}{s^2+2as+(a^2+\omega_0^2)}$	$\text{Re}\{s\} < -a$
		$= -\frac{0.5e^{j\theta}}{s+a-j\omega_0} + \frac{0.5e^{-j\theta}}{s+a+j\omega_0}$	

Selected z-Transform Pairs

	$x[n]$	$X(z)$	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
3.	$\gamma^n u[n]$	$\frac{z}{z-\gamma}$	$\lvert z\rvert > \lvert\gamma\rvert$
4.	$\gamma^{n-1} u[n-1]$	$\frac{1}{z-\gamma}$	$\lvert z\rvert > \lvert\gamma\rvert$
5.	$n\gamma^n u[n]$	$\frac{\gamma z}{(z-\gamma)^2}$	$\lvert z\rvert > \lvert\gamma\rvert$
6.	$n^2\gamma^n u[n]$	$\frac{\gamma z(z+\gamma)}{(z-\gamma)^3}$	$\lvert z\rvert > \lvert\gamma\rvert$
7.	$\frac{n!}{(n-m)!m!}\gamma^{n-m}u[n]$	$\frac{z}{(z-\gamma)^{m+1}}$	$\lvert z\rvert > \lvert\gamma\rvert$
8.	$\lvert\gamma\rvert^n \cos(\beta n)u[n]$	$\frac{z[z-\lvert\gamma\rvert\cos(\beta)]}{z^2-2\lvert\gamma\rvert\cos(\beta)z+\lvert\gamma\rvert^2}$	$\lvert z\rvert > \lvert\gamma\rvert$
9.	$\lvert\gamma\rvert^n \sin(\beta n)u[n]$	$\frac{z\lvert\gamma\rvert\sin(\beta)}{z^2-2\lvert\gamma\rvert\cos(\beta)z+\lvert\gamma\rvert^2}$	$\lvert z\rvert > \lvert\gamma\rvert$
10.	$\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\frac{z[z\cos(\theta)-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-2\lvert\gamma\rvert\cos(\beta)z+\lvert\gamma\rvert^2}$ $= \frac{(0.5e^{j\theta})z}{z-\lvert\gamma\rvert e^{j\beta}} + \frac{(0.5e^{-j\theta})z}{z-\lvert\gamma\rvert e^{-j\beta}}$	$\lvert z\rvert > \lvert\gamma\rvert$
11.	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$ $r=\sqrt{\frac{a^2\lvert\gamma\rvert^2+b^2-2abc}{\lvert\gamma\rvert^2-c^2}}$ $\beta=\cos^{-1}\left(\frac{-c}{\lvert\gamma\rvert}\right)$ $\theta=\tan^{-1}\left(\frac{ac-b}{a\sqrt{\lvert\gamma\rvert^2-c^2}}\right)$	$\frac{z(az+b)}{z^2+2cz+\lvert\gamma\rvert^2}$	$\lvert z\rvert > \lvert\gamma\rvert$
12.	$\delta[n-k]$	z^{-k}	$\lvert z\rvert > 0 \quad k>0$ $\lvert z\rvert < \infty \quad k<0$
13.	$-u[-n-1]$	$\frac{z}{z-1}$	$\lvert z\rvert < 1$
14.	$-\gamma^n u[-n-1]$	$\frac{z}{z-\gamma}$	$\lvert z\rvert < \lvert\gamma\rvert$
15.	$-n\gamma^n u[-n-1]$	$\frac{z\gamma}{(z-\gamma)^2}$	$\lvert z\rvert < \lvert\gamma\rvert$