2
CHAPTER

Analytic Functions

The theory of functions of a complex variable is of great importance in solving a large number of problems in the field of engineering and science. In this chapter, we shall define and discuss functions of complex variable and will discuss limit, continuity and differentiability of the complex valued functions, but our aim is to develop the theory of analytic functions as these functions have wide range of applications in complex analysis. We shall also deal here with harmonic functions and their relationship with analytic functions.

2.1 FUNCTIONS OF A COMPLEX VARIABLE

Let s be a set of complex numbers and z be one of the number of s. Let $z = x + iy$ be a complex variable where $i = \sqrt{-1}$. Now if for each value of z in s, we can find a second complex variable w in s, then w is a function of complex variable z and s and is denoted by

$$w = \delta(z)$$

where $w = u + iv$, u and v are real and imaginary parts of $f(z)$.

Here the set s usually has the same domain. It is called the domain of definition of the function w and the totality of values of $f(z)$ which we get corresponding to each z in s′ is called range of the function w. When the domain of definition is not defined, we take the largest possible set as domain for f on which f is well-defined.

For the function $f(z) = \dfrac{1}{z^2+1}$, here $f(z)$ is not defined at $z = \pm i$, thus the domain of definition will be entire complex plane except $z = \pm i$.

The domain of definition of $f_1(z) = z^3 + 2iz - 3$ and $f_2(z) = |z|$ is the entire complex plane.

Multiple valued function: If the given functions take more than one value at some or all points of the domain of the definition, it is called multiple valued function. For example,

$$f(z) = z^{1/2} = \pm\sqrt{r}\, e^{i\theta/2}, -\pi < \theta < \pi$$

is a multiple valued function, having two values $\pm\sqrt{re}^{\,i\theta}/_2$ in the interval $-\pi < \theta < \pi$.

Also every function can be written as $f(z) = u(r, \theta) + iv(v, \theta)$. Let

$$f(z) = z + \frac{1}{z},\ z \neq 0$$

then
$$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

Whenever we speak about a function we mean single valued function unless otherwise specified. Suppose $f(z_1) = f(z_2) \Rightarrow z_1 = z_2\ \forall$ $z_1, z_2 \in A$. Where $f(z)$ is a complex valued function defined on A.

$$z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2)\ \forall\ z_1, z_2 \in A.$$

If the above conditions are satisfied, mapping $w = f(z)$ is called one-to-one mapping. In such a case the function is called univalent. The given function will be univalent of $z = z_0$ if it is univalent in the same neighbourhood of z_0. Thus, we can define inverse function as

$$z = f^{-1}(w) \text{ if } w = f(z).$$

The composition of complex valued function can also be defined as

$$(gf)(z) = g(f(z))$$

If $$f(z) = u(x, y) + iv(x, y)$$

then $$\overline{f(z)} = u(x, y) - iv(x, y)$$

Thus, $f(z)$ and $\overline{f(z)}$ are different functions.

The function $f(z) = 2z + z^2$, $|z| < 1$ is univalent in the domain as if we take z_1 and z_2 any two points in $|z| < 1$ then

$$f(z_1) = f(z_2) = 2z_1 + z_1^2 = 2z_2 + z_2^2$$

$$\Rightarrow \quad (z_1 - z_2)(z_1 + z_2 + 2) = 0$$

$$\Rightarrow \quad z_1 = z_2 \text{ or } z_1 - z_2 = 0$$

As $z_1 + z_2 + 2 \neq 0$ in $|z| < 1$ as $\text{Re}(z_1 + z_2 + 2) = \text{Re1}\, z_1 + \text{Re1}\, z_2 + 2 > -1 + 1 + 2 = 0$.

Thus, $f(z)$ is one to one in $|z| < |$ and hence it is univalent function.

2.2 LIMITS

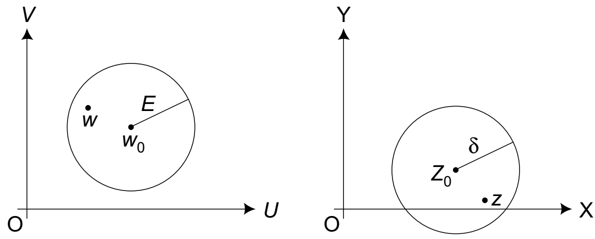
Let $f(z)$ be a function which is defined at all points in some neighbourhood of a point z_0. The function $f(z)$ is said to have limit w_0 at $z = z_0$ if for a given $\in > 0$, we can find a number $\delta > 0$ such that

$$|f(z) - w_0| < \in, \text{ whenever } 0 < |z - z_0| < \delta. \qquad ...(2.1)$$

Also we can write

$$\lim_{z \to z_0} f(z) = w_0 \qquad ...(2.2)$$

Graphically, the definition (2.1) requires that for each positive number $\in$, some positive number δ exists, such that all points except z_0, interior to the circle $|z - z_0| = \delta$ in the z-plane have image point $w = f(z)$ that are defined to the interior of circle $|w - w_0| = \in$ as shown in Fig. (2.1).



Example 1: Show that $\lim\limits_{z \to 1} \dfrac{z^2 - 1}{z - 1} = 2.$

To verify the limit, we have to find $\delta > 0$, for a given $\in > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow \left| \frac{z^2 - 1}{z - 1} - 2 \right| = |z - 1|, z \neq 1$$

Thus, $|f(z) - 2| < \in$. Whenever $0 < |z - 1| < \in$.

Thus, the condition (2.2) is satisfied for every positive number $\in$ if $\delta = \in$. This proves that $\lim\limits_{z \to 1} \dfrac{z^2 - 1}{z - 1} = 2.$

In another example we shall show that

$$\lim_{z \to z_0} az^2 + bz + c = az_0^2 + bz_0 + c,$$

where a, b, c are complex constants.

In order to verify, let $\delta > 0$ be any positive number for a given $\in > 0$, such that

$$0 < |z - z_0| < \delta \Rightarrow |(az_0^2 + bz + c) - (az_0^2 + bz_0 + c)| < \in \quad \text{...(2.3)}$$

$$= |(z - z_0)(a(z + z_0) + b)| < \in$$

$$\leq |z - z_0| \, (|a| \, |z + z_0| + |b|) < \in$$

Now restricting $|z - z_0| < 1$, we see that

$$|a||z + z_0| + |b| = |a||z - z_0 + 2z_0| + |b|$$

$$\leq |a|(|z - z_0| + 2|z_0| + |b| < |a|(1 + 2|z_0|) + |b|$$

If we choose $\delta = \min\left\{1, \dfrac{\in}{|a|(1+2|z_0|+|b|)}\right\}$ then (2.3) holds.

Properties of Limits

If $\qquad \lim_{z\to z_0} f(z) = A \quad$ and $\quad \lim_{z\to z_0} g(z) = B$

Then

(i) $\lim_{z\to z_0} [f(z) \pm g(z)] = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z) = A \pm B$

(ii) $\lim_{z\to z_0} [f(z)g(z)] = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z) = AB.$

(iii) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \lim_{z\to z_0} f(z) \Big/ \lim_{z\to z_0} g(z) = A/B$ if $B \neq 0$.

The proofs of above are simple and can be proved easily.

Limit point at infinity: Let us consider extended complex number system by adjoining an ideal point denoted by symbol ∞. This extended complex plane is denoted by C_∞.

Let $f(z)$ a function which is defined on an unbounded set in complex plane. Now if for a given $\in > 0$, there exists an $R > 0$ such that $|f(z) - w| < \in$, whenever $z \in E$ and $|z| > R$, where E is an unbounded set, then we say that $f(z) \to w$ or $z \to \infty$ and we write it as

$$\operatorname*{Lt}_{z\to\infty} f(z) = w \text{ or } \lim_{|z|\to\infty} f(z) = w.$$

Let us show that $\lim_{z\to\infty} \dfrac{1}{z^2} = 0.$

Now the function $\dfrac{1}{z^2}$ is defined everywhere except $z = 0$. Thus, for a given $\in > 0$.

$$\left|\frac{1}{z^2}\right| < \in \Leftrightarrow |z| > \frac{1}{\sqrt{\in}}$$

Thus, in this case we have

$$R = \frac{1}{\sqrt{\in}}.$$

Infinite limit: Let $f(z)$ be defined in some neighbourhood of a point z_0. If for every $N > 0$, we can find $\delta > 0$ each that $|f(z)| > N$ whenever $0 < |z - z_0| < \delta$ then we say that $f(z) \to \infty$ as $z \to z_0$ and symbolically we write it as

$$\lim_{z \to z_0} f(z) = \infty$$

Exercise 2.1

Q. 1 Determine the domain and range of a function.

(a) $\dfrac{z}{z + \bar{z}}$ (b) $\dfrac{1}{1 - |z|^2}$

Q. 2 Find which are univalent functions

(a) $z + 2z^2$, $|z| < 1$ (b) $\dfrac{z}{1 - z^2}$, $|z| < 1$

(c) $\dfrac{az + b}{cz + d}$, $z \neq d/c$

Q. 3 Prove that $\lim\limits_{z \to z_0} \dfrac{\bar{z}}{z}$ does not exist.

Q. 4 Prove that $\lim\limits_{z \to 1+i} (z^2 - 5z + 10) = 5 - 3i$.

Q. 5 Using the definition of limit show that

(a) $\lim\limits_{z \to i} z^4 = 1$ (b) $\lim\limits_{z \to 1} z^3 = 1$

Q. 6 If $p(z)$ is bounded and $\lim\limits_{z \to z_0} g(z) = 0$, prove that $\lim\limits_{z \to z_0} g(z)p(z) = 0$.

Q. 7 Evaluate $\lim\limits_{z \to i} \dfrac{iz^3 + 1}{z + i}$.

Answers

Q. 1 (a) Domain : $e \backslash Y$-axis, rang : e

(b) Domain : $e \backslash |z| = 1$, range : $e \backslash \{0\}$.

Q. 2 (a) Not univalent (b) univalent

(c) univalent.

Q. 7 0

2.3 CONTINUITY

A complex valued function $f(z)$ is said to be continuous at a point z_0 if for every $\in > 0$, we can find a number $\delta > 0$ such that

$$|f(z) - f(z)| < \in \text{ whenever } |z - z_0| < \delta$$

Thus, we can say that $f(z)$ is continuous if it is defined through some neighbourhood of the point z_0 and

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If two functions are continuous their sum and product are also continuous, and their quotient is continuous except for those values of z for which the denominator vanishes.

Exercise 2.2

Q. 1 (a) Prove that $f(z) = z^2$ is continuous at $z = z_0$.

(b) Prove that $f(z) = \begin{cases} z^2 & z \neq z_0 \\ 0, & z = z_0 \end{cases}$, where $z_0 \neq 0$ is discontinuous at $z = z_0$.

Q. 2 Is the function $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ continuous at $z = i$?

Q. 3 Prove that if $f(z)$ and $g(z)$ are continuous at $z = z_0$ then

(a) $f(z) + g(z)$ (b) $f(z)g(z)$

(c) $\dfrac{g(z)}{g(z)}$, $g(z_0) \neq 0$

are also continuous at $z = z_0$.

Q. 4 Prove that $f(z) = z^2$ is continuous in the region $|z| \leq 1$.

Q. 5 Prove that continuous function of a continuous function is continuous.

Q. 6 If a function is continuous in a domain D, prove that $|f(z)|$ is also continuous in the domain.

Q. 7 If $f(z)$ is continuous at $z = z_0$, prove that $\{f(z)\}^2$, $\{f(z)\}^3$ are also continuous at $z = z_0$.

Answer

Q. 2 No.

2.4 DIFFERENTIABILITY

Let $f(z)$ be complex valued function which is defined in some neighbourhood of z_0, then we say that $f(z)$ is differentiable at $z = z_0$ if $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exist and finite.

In other way we say that if $f(z)$ is single valued in some region R of the z-plane, the derivative of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided the limit exists and independent of the manner in which $\Delta z \to 0$. In such case we say that $f(z)$ is differentiable at z.

Differentiability implies continuity but converse is not true.

Exercise 2.3

Q. 1 A $f(z) = z^3 + 2z^2 + i$, find $f'(z)$.

Q. 2 Prove that every differentiable function is continuous.

Q. 3 Prove that $f(z) = |z|$ is nowhere differentiable.

Q. 4 Find $\dfrac{d}{dz}\left(\dfrac{1+z}{1-z}\right)$.

Q. 5 Show that $\dfrac{d\bar{z}}{dz}$ does not exist everywhere.

Answers

Q. 1 $3z^2 + 4z$ **Q. 4** $\dfrac{2}{(1-z)^2}$

2.5 THE CAUCHY-RIEMANN EQUATION (C – R EQUATION)

Before discussing Cauchy-Riemann equation ($C - R$ equation), we may discuss about uniform continuity of the complex valued function.

Definition: A complex valued function $f(z)$ is said to be uniformly continuous as a given set C if for a given $\in > 0$, we can find a number $\delta > 0$ such that $|f(z_1) - f(z_2)| < \in$ whenever $|z_1 - z_2| < \delta$ for each z_1, $z_2 \in e$. The choice of δ is independent of z_1 and z_2 in C.

Example: Show that $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$, but $f(z) = \dfrac{1}{z}$ is not uniformly continuous in that region.

Let $f(z) = z^2$ is uniformly continuous in the given region. Then for a given $\in > 0$ we can find a $\delta > 0$ such that $|z^2 - z_0^2| < \in$ whenever $|z - z_0| < \delta$, where δ depends only on $\in$ and a particular choice of the point z_0. If z and z_0 are any points in

$$|z| < 1 \text{ then } |z - z_0{}^2| = |z + z_0|\,|z - z_0| \leq (|z| + |z_0|)|z - z_0| < 2|z - z_0|.$$

Thus, if we choose $\delta = \in/2$, then $|z^2 - z_0^2| < \in$. Thus, $f(z) = z^2$ is uniformly continuous in $|z| < 1$.

Now let $f(z) = \dfrac{1}{z}$ is uniformly continuous in $|z| < 1$. Thus, for a given $\in > 0$, there exists a $\delta > 0$ between 0 and 1, such that $|f(z) - f(z_0)| < \in$ whenever $|z - z_0| < \delta$ for all z and z_0 in $|z| < 1$. Let $z = \delta$ and $z = \delta/(1 + \in)$. Then points z and z_0 are in $|z| < 1$ and we see that

$$|z - z_0| = \left|\delta - \frac{\delta}{1+\in}\right| = \frac{\in}{1+\in}\delta < \delta$$

but $$\left|\frac{1}{z}-\frac{1}{z_0}\right| = \left|\frac{1}{\delta}-\frac{1+\in}{\delta}\right| = \frac{\in}{\delta} > \in \text{ as } 0<\delta<1$$

This gives us contradiction and hence the function $f(z) = \frac{1}{z}$ cannot be uniformly continuous in the given region.

If $f(z) = u(x, y) + iv(x, y)$ be a complex valued function where $u(x, y)$ and $v(x, y)$ are real valued functions of two variables, then

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \text{ and } \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

or $$u_x = v_y \text{ and } u_y = -v_x$$

are called Cauchy-Riemann equations.

Here, we shall try to establish $C - R$ equation. Let $f(z) = u(x, y) + iv(x, y)$ and let $f'(z_0)$ exists at $z_0 = x_0 + iy_0$. Now

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad ...(1)$$

The above limit is finite and whatever path we shall adapt to let $\Delta z \to 0$, it must be unique.

Now $\Delta z = \Delta x + i\Delta y$ and from (1), when $\Delta x \Rightarrow 0$ and $\Delta y = 0$, we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{u(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$f'(z_0) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \qquad ...(2)$$

In a similar way let $\Delta x = 0$ and $\Delta y \to 0$ then

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y}$$

$$= -i \lim_{\Delta y \to 0} \frac{u(x_0\ y_0 + \Delta y,) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta y}$$

$$= -i\left[\lim \cdot \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} + i\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y}\right]$$

$$f'(z_0) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad ...(3)$$

As $f'(z_0)$ exists so equations (2) and (3) are same. Thus, in view of (2) and (3) equating real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

or $\qquad u_x = v_y$ and $u_y = -v_x$.

Here it is noted that for the existence of the derivative, $C - R$ equations are necessary but they are not sufficient for the existence of the derivative. Then we can say that if $f(z)$ is differentiable at $z = z_0$, then the $C - R$ equations are satisfied at that point but the converse is not true.

Now let u and v be single valued functions of x and y, which together with their partial derivatives of the first order are continuous at each point (x_0, y_0). If these partial derivatives satisfy the $C - R$ equation at that point, then the derivative $f'(z_0)$ of $f(z)$ exists. Let us prove this.

Since u and its partial derivative of the first order are continuous at (x_0, y_0), those functions are defined throughout some neighbourhood of that point. When $(x_0 + \Delta x, y_0 + \Delta y)$ is a point in the neighbourhood, we can write

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$

Here $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are the values of partial derivatives at that point (x_0, y_0) and $\in_1, \in_2 \to 0$ as both Δx and $\Delta y \to 0$.

In a similar way we can write

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)$$

$$= \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \in_3 \Delta x + \in_4 \Delta y.$$

Therefore, we have

$$\Delta f = f(z_0 + \Delta z) - f(z_0) = \Delta u + i\Delta v$$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \in_1 \Delta x + \in_2 \Delta y$$

$$+ i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \in_3 \Delta x + \in_4 \Delta y\right) \qquad ...(4)$$

Let Cauchy-Riemann conditions are satisfied at (x_0, y_0), we replace $\partial u/\partial y$ by $-\partial v/\partial x$ and $\partial v/\partial y$ by $\partial u/\partial x$, equation (4) can be written as

$$\Delta f = \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y) + \in_1 \Delta x + i\in_3 \Delta x + \in_2 \Delta y + i\in_4 \Delta y$$

$$= \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y) + \delta_1 \Delta x + \delta_2 \Delta y$$

where $\delta_1 = \in_1 + i\in_3$ and $\delta_2 = \in_2 + i\in_4$ and $\delta_1, \delta_2 \to 0$ as $\Delta z \to 0$. Thus, we have

$$\frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \delta_1\frac{\Delta x}{\Delta z} + \delta_2\frac{\Delta y}{\Delta z} \qquad ...(5)$$

Since $\quad |\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$

we have

$$\left|\frac{\Delta x}{\Delta z}\right| \le 1 \text{ and } \left|\frac{\Delta y}{\Delta z}\right| \le 1$$

Thus, the last two terms in equation (5) tends to zero or $\Delta z \to 0$. Therefore at $z = z_0$ we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{...(6)}$$

This shows that $f'(z)$ exists. Thus, we say that if $f(z) = u + iv$ be defined in the same neighbourhood of z and that first order partial derivative of u and v are continuous at $z = z_0$ and satisfy C – R equation then $f'(z)$ exists at $z = z_0$. This gives the sufficient condition for the existence of the derivative.

Polar form of C – R Equation

The C – R equation in Cartesian form is given by $u_x = v_y$, $u_y = -v_x$. Now we transform this into polar form from (x, y) coordinate system to (r, θ) system, may be obtained as follows: Let $x = r\cos\theta$, $y = r\sin\theta$.

Now we have

$$u_r = u_x\frac{\partial x}{\partial r} + u_y\frac{\partial y}{\partial r},\ u_\theta = u_x\frac{\partial x}{\partial \theta} + u_y\frac{\partial y}{\partial \theta}$$

Using $x = r\cos\theta$ and $y = r\sin\theta$ we obtain

$$u_r = \cos\theta\ u_x + \sin\theta\ u_y,\ u_\theta = -r\sin\theta\ u_x + r\cos\theta\ u_y$$

Also

$$u_r = \cos\theta\ u_x + \sin\theta\ u_y,\ v_\theta = -r\sin\theta\ u_x + r\cos\theta\ v_y \qquad \text{...(7)}$$

Using $u_x = u_y$ and $u_y = -v_x$ in the above equation, we get

$$u_r = -\cos\theta\ u_y + \sin\theta\ u_x,\ v_\theta = r\sin\theta\ u_y + v\cos\theta\ u_x. \qquad \text{...(8)}$$

Now in view of (7) and (8), we obtain

$$u_r = \frac{1}{r}v_\theta \text{ and } \frac{1}{r}u_\theta = -v_r \qquad \text{...(9)}$$

which is the polar form of C – R equation.

2.6 ANALYTIC FUNCTION

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighbourhood A z_0.

The terms regular and holomorphic are also used for analytic function.

In section 2.5 we have introduced the idea of Cauchy-Riemann equations. Thus, if $f(z)$ is analytic in a domain D, those partial derivatives u_x, u_y, v_x, v_y exist and satisfy.

$$u_x = v_y,\ u_y = -v_x$$

everywhere in D. Thus, if $f(z)$ is analytic in D, C – R equations are satisfied. Also we have shown in section (2.5). that if C – R equations are satisfied and their partial derivatives are continuous. Then $f'(z_0)$ exists at $z = z_0$. Thus, Cauchy-Riemann equations are fundamental because they are not only necessary but also sufficient for a given function to be analytic.

Example: (1) Let $f(z) = z^3$

Now $\quad z = x + iy \Rightarrow f(z) = u + iv = z^3 = (x + iy)^3$

Then $\quad u = x^3 - 3xy^2,\ v = 3x^2y - y^3$

$$u_x = 3x^2 - 3y^2 = v_y \text{ and } u_y = -6xy = -\,v_x$$

The C – R equations are satisfied for every z. Hence, $f(z)$ is analytic for all values of z.

(2) Let $\quad f(z) = \text{rel}\,z = x \qquad \text{...(10)}$

have $\quad f(z) = u + iv = x \Rightarrow u = x,\ v = c$

Thus, $$\frac{\partial u}{\partial x} = 1,\ \frac{\partial v}{\partial y} = 0,\ \frac{\partial u}{\partial x} = 0,\ \frac{\partial v}{\partial y} = 0$$

Hence, $$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}.$$

This shows that $f(z) = \text{Re}(z)$ is not analytic.

C – R Equation in Polar Form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r},\ r > 0$$

The above C – R equation can be obtained by using the polar form $z = r(\cos\theta + i\sin\theta)$ and set $f(z) = u(r, \theta) + iv(r, \theta)$.

Entire function: A complex valued function $f(z)$ is said to be entire if it is analytic in the whole complex plane. Obviously, the sum, difference and product of the two or more entire functions are entire functions. Moreover, the superposition of an entire function is an entire function.

The polynomial $a_0 + a_1 z + ... + a_x z^x$, $a_n \neq 0$ and $f(z) = e^x(\cos y + i \sin y)$ are entire functions. The proof is left as an exercise.

Singularity of function: If a function $f(z)$ fails to be analytic at a point $z = z_0$ but in every neighbourhood of z_0 there exists at least one point where the function is analytic, then $z = z_0$ is called singular point or singularity of $f(z)$.

$f(z) = \dfrac{1}{z}$ is analytic except at $z = 0$ and in each deleted neighbourhood $f(z)$ is analytic. So by definition $z = 0$ is a singularity.

$$f(z) = \frac{z^4 + 8}{(z^2 - 2z + z)}(z - 4)$$

the point $z = 4$, $z = 1 + i$, $1 - i$ are singular points of $f(z)$ at the function $f(z)$ is analytic except at $z = 1 + i$, $1 - i$ and $z = 4$.

Laplace Equation, Harmonic Functions

We shall study harmonic functions and laplace equation, which has great practical importance of complex analysis in engineering mathematics results from the fact that both real and imaginary part of analytic function $f(z) = u + iv$ satisfy the most important differential equation of physics, known as Laplace equation, which occurs in gravitation, electrostatics, fluid flow, heat conduction, etc.

Let $f(z) = u(x, y) + iv(x, y)$, which is analytic in some domain of the z-plane; then we have at every point

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(11)$$

and therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \qquad ...(12)$$

provided these second order derivatives exist. Thus, from (12) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(13)}$$

Similarly, we can have

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

The equations (12) and (13) are known as Laplace's equation. Solution of Laplace's equation having continuous second order partial derivatives are called harmonic functions, and the theory based on harmonic function is called potential theory. The real and imaginary parts of $f(z) = u(x, y) + iv(x, y)$, which is analytic, are known as harmonic functions. Thus, $u(x, y)$ and $v(x, y)$ are harmonic functions. $v(x, y)$ is said to be a conjugate harmonic function of $u(x, y)$ in D and $u(x, y)$ is conjugate harmonic to $v(\mathrm{x}, y)$.

The curves of u = const. are called equipotential lines or level curves of u. They form a family of curves. The same is true in case of curve v = const. The two families together form an orthogonal net.

Example: 1. Prove that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane. Find its conjugate harmonic.

Solution: $4_x = 2x, 4_{xx} = 2, 4_y = -2y - 1, 4_{yy} = -2$

Thus, $4_{xx} + 4_{yy} = 0$

This show that u is harmonic function.

Now $4_x = 2x = v_y$ by C – R equation

and $u_y = -2y - 1 = -v_x$

Hence, $v(x, y) = 2xy + k(x)$

$$\Rightarrow \qquad \frac{\partial v}{\partial x} = 2y + \frac{dk}{dx} = 2y + 1$$

$$\Rightarrow \qquad \frac{dk}{dx} = 1 \Rightarrow k(x) = n + k.$$

Thus, conjugate harmonic $v(x, y) = 2xy + x + k$ and most general analytic function $f(z)$ is

$$f(z) = x^2 - y^2 - y + i(2xy + x + k)$$

Milne-Thomson Method

This method is used to construct an analytic function when its real or imaginary components are known, and for finding harmonic conjugates.

If a given function $f(z)$ is analytic in a given domain the $f(z)$ can be integrated in the domain using anti-derivative, that is by finding $F(z)$ such that $F'(z) = f(z)$. Now we have $z = x + iy$, so that $x = \dfrac{z+\bar{z}}{2}$ and $y = \dfrac{z+\bar{z}}{2i}$

Now $w = f(z) = u + iv = u(x, y) + iv(x, y)$ or $f(z) = 4\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right)$ $+ iv\left(\dfrac{z+\bar{z}}{2}, \dfrac{z-\bar{z}}{2i}\right)$

Now by setting $x = z$, $y = 0$ so that $\bar{z} = z$ we have,

$$f(z) = u(z, 0) + iv(z, 0)$$

$$\therefore \qquad f'(z) = \frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial y} \text{ by } C-R \text{ equation}$$

Let $$\frac{\partial u}{\partial x} = \varphi_1(x, y) = \phi_1(z, 0)$$

and $$\frac{\partial u}{\partial y} = \varphi_2(x, y) = \varphi_2(\text{z}, 0)$$

we obtain $$f'(z) = \varphi_1(z, 0) - i\varphi_2(z, 0)$$

$$\Rightarrow \qquad f(z) = \int \{\varphi_1(z, 0) - i\varphi_2(z, 0)\}\, dz + c$$

Where c is a constant. Now we can calculate $f(z)$ directly if $u(x, y)$ is given. In a similar way, we can find

$$f(z) = \int \{\psi_1(z, 0) + i\psi_2(z, 0)\}\, dz + c_1$$

where $$\psi_1 = \frac{\partial v}{\partial v},\ \psi_2 = \frac{\partial v}{\partial x}$$

Example: Given $u = e^x(x \cos y - y \sin y)$, find $f(z)$ by Milne Thomson method.

Solution: Now $u = e^x(x \cos y - y \sin y)$

$$\therefore \quad \frac{\partial u}{\partial x} = e^x u + e^x \cos y$$

$$\frac{\partial u}{\partial y} = e^x[-x \sin y - \sin y - y \cos y]$$

$$\Rightarrow \quad u_x \text{ at } y = 0 = e^x x + e^x = e^x (x + 1)$$

$$u_y \text{ at } y = 0 = 0$$

$$\therefore \quad \varphi_1(x, 0) = u_x \text{ at } y = 0 = e^x (x + 1)$$

$$\varphi_2(x, 0) = u_y \text{ at } y = 0 = 0.$$

$$\Rightarrow \quad f(z) = \int \{\varphi_1 (z, 0) - i\varphi_2 (z, 0)\}\, dz + c$$

$$f(z) = \int \{e^2 (z + 1) - \text{i}.0\}\, dz + c$$

$$\therefore \quad f(z) = ze^2 + c$$

Exercise 2.5

Q. 1 Find most general analytic function corresponding to

$$u(x, y) = y^3 - 3x^2 y$$

Q. 2 Prove that each of these functions is entire

(a) $f(z) = 3x + y + i(3y - x)$

(b) $f(z) = e^{-y} (\cos x + i \sin x)$

Q. 3 Show why each of these functions is nowhere analytic.

(a) $f(z) = xy + iy$

(b) $f(z) = e^y (\cos x + i \sin x)$

Q. 4 If in some domain $f = u + iv$ and its complex conjugate $\bar{f} = u - iv$ are both analytic, then prove that f is constant.

Q. 5 In the domain $r > 0$, $0 < \theta < 2\pi$, show that $u = \log r$ is harmonic and find its harmonic conjugate.

Q. 6 Determine a and b such that the given functions are harmonic and find a conjugate harmonic.

(a) $u = ax^3 + by^3$

(b) $u = e^{ax} \cos 5y$

Q. 7 Find analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that

$$v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$$

Q. 8 Show that $f(z) = \sqrt{|xy|}$ satisfies Cauchy-Riemann equation at the origin but is not analytic at that point.

Q. 9 Prove that $\dfrac{x - iy}{x^2 + y^2}$ is analytic or not.

Q. 10 For what values of z do the function defined by the following equation ceases to be analytic. $f(z) = \dfrac{1}{z^2 - 1}$.

Q. 11 Show that $u = \dfrac{1}{2} \log (x^2 + y^2)$ is harmonic. Find its harmonic conjugate v.

Answers

Q. 1 $f(z) = i(z^3 + C)$ **Q. 5** $v = \theta + C$

Q. 6 (a) $a = b = 0$, $v = C$

(b) $a = \pm 5$, $v = \pm e^{\pm 5x} \sin 5y + C$

Q. 7 $u = -r^2$, $\sin 2\theta + r \sin \theta + C$

$f(z) = u + iv = i(r^2 e^{2i\theta} - re^{i\theta}) + 2i + C.$

Q. 9 Not analytic **Q. 10** $z = \pm 1$

Q. 11 $v(x, y) = \tan^{-1} y/x + C.$

Miscellaneous Solved Exercise 2

Q. 1 Prove that $\bar{z}$ is not differentiable

Sol. Let $f(z) = \bar{z} = x - iy$

$\Delta z = \Delta x + i\Delta y$, so we have

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{(\overline{z + \Delta z}) - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta 2} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim \cdot \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$ and $\Delta \to 0$, we get from above we have Limit *al* +1 and when $\Delta x = 0$ and $\Delta y \to 0$, we get Limit *al* –1. Hence the Limit comes out different along different paths. Thus Limit does not exist and so $\bar{z}$ is not differentiable.

Q. 2 Determine whether the following functions are continuous inside a unit circle

(a) $\dfrac{1}{1+z^2}$ (b) $\dfrac{1}{z-1}$.

Sol: (a) $f(z) = \dfrac{1}{1+z^2}$ is continuous except at where $1 + z^2$ is zero, that is, a point $z = \pm\, i$, for unit circle $|z| < 1$, $z = \pm\, i$ are excluded. Thus the given function is continuous inside $|z| < 1$.

(b) Similar method is applied for $f(z) = \dfrac{1}{z-1}$. This one is also continuous except at $z = 1$ but $|z| < 1$ as the domain. Thus $\dfrac{1}{z-1}$ is continuous in $|z| < 1$.

Q. 3 Is $f(z) = z/|z|$ continuous at origin, where $f(z)$ is defined for $z \neq 0$ and $f(0) = 0$.

Sol: Now, $\lim\limits_{z \to 0} f(z) = \lim\limits_{z \to 0} \dfrac{z}{|z|}$

$$= \lim_{z \to 0} \frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x \to iy} \frac{x+iy}{\sqrt{x^2+y^2}} +$$

Let first $y = 0$ and $x \to 0$, then $\lim\limits_{x \to iy} \dfrac{x}{x} = 1$

Again when $x = 0$ and $y \to 0$ $\lim \dfrac{z}{|z|} = i$

As the Limit along different paths are 1 and i, that is, limit does not exist. $f(z)$ is dis continuous at $z_0 = 0$.

Q. 4 Show that $f(z) = \text{Re}z = x$ is continuous but not differentiable

Sol: As $\lim\limits_{z \to z0} f(z) = \lim\limits_{z \to z0} = \lim\limits_{\substack{x \to x_0 \\ y \to y_0}} x = x_0 = f(z_0)$

the function is continuous at $z = z_0$.

Now $$f'(z) = \lim_{\Delta_z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta_x \to 0 \\ \Delta_y \to 0}} \frac{\Delta x}{\Delta x + i\Delta y} = 0$$

while $$f'(z) = \lim_{\substack{\Delta y \to 0 \\ \Delta_x \to 0}} \frac{\Delta x}{\Delta x + i\Delta y} = 1$$

Hence Limit does not exist, that $f'(z) = 0$ and 1 along different paths. Thus $f(z)$ is not differentiable.

Q. 5 If $\lim\limits_{z \to z0} f(z)$ exist, prove that it must be unique

Sol: Let $\lim\limits_{z \to z0} f(z) = l_1$ and $\lim\limits_{z \to z0} f(z) = l_2$

then for uniqueness we must show that $l_1 = l_2$. Now for given $\in > 0$, we can find a number $\delta > 0$ such that

$$|f(z) - l_1| < \in/2 \text{ when } 0 < |z - z_0| < \delta$$

and $$|f(z) - l_2| < \in/2 \text{ when } 0 < |z - z_0| < \delta$$

Then $$|l_1 - l_2| = |l_1 - f(z) + f(z) - l_2|$$

$$\leq |l_1 - f(z)| + |-l_2 + f(z)|$$

$$\leq \in/2 + \in/2$$

$$= \in$$

$\Rightarrow |l_1 - l_2| < \in$ (very-very small number) and so must be zero. Thus we have $l_1 = l_2$ Hence we get the derived result.

Q. 6 If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, Prove that $\lim_{n\to\infty} (a_n + b_n) = A + B$.

Sol: In view of the definition and for given $\in$ we can find N. such that

$$|a_n - A) < \in/2, \ |b_n - B| < \in/2 \text{ for } n > N$$

Thus for $n > N$

$$|a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B) \le |a_n - A| + |b_n - B|$$
$$\le \in/2 + \in/2 = \in$$

which gives the desired result.

Q. 7 Is $f(z) = z^3$ analytic?

Sol: We have $\quad f(z) = z^3 = (x + iy)^3$

$$= x^2 - 3xy^2 + i(3x^2y - y^3)$$

Thus $\quad u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

Also $\quad u_x = 3x^2 - 3y^2 = v_y$ and $u_y = -6xy = -v_x$

Thus Cauchy-Riemann equation is satisfied for every z. Hence $f(z) = z^3$ analytic for all z.

Q. 8 Prove that an analytic function of constant absolute value is constant.

Sol: Given $f(z)$ is analytic and $|f(z)| = k$ (constant). Now we have to show that $f(z)$ is also constant. since $|f(z)| = k \Rightarrow u^2 + v^2 = k^2$

Thus we have from $u^2 + v^2 = k^2$, $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$.

As $u_x = v_y$ and $u_y = -v_x$. We have from these equations and above equations

$$(u^2 + v^2)u_x = 0 \text{ and } (u^2 + v^2)u_y = 0$$

If $k^2 = u^2 + v^2 = 0$ then $u = v = 0 \Rightarrow (f(z) = 0$. and if $k \ne 0$, then $u_x = u_y = 0$. Hence by $C - R$ equations also $v_x = v_y = 0$. This gives us that u = const. and v = cons. Hence $f(z)$ is constant.

Q. 9 Prove that e^z is and entire function.

Sol: Now $\quad f(z) = e^z = e^{x+iy}$

$$= e^x [\cos y + i \sin y]$$

Thus we have $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$

$\Rightarrow \qquad u_x = e^x \cos y = u_y$

and $\qquad u_y = -\, e^x \sin y = -u_x.$

That is, $C - R$ equations are satisfied for all x and y. Therefore e^z is analytic everywhere. Thus e^z is an entire function.

Q. 10 If $u(r, \theta) = \left(r - \frac{1}{r}\right) \sin \theta,\ r \neq 0$, find an analytic function $f(z) = u + iv$.

sol: In view of $C - R$ equation in polar form, $u_r = u_\theta$ and $u_\theta = -r\, u_r$, we have from the given $u(r, \theta)$,

$$u_\theta = -r u_r = -r\left(1 + \frac{1}{r^2}\right)\sin\theta = -\left(1 + \frac{1}{r^2}\right)\sin\theta$$

Integrating above w.r.t. θ, we get

$$u(r, \theta) = \left(r + \frac{1}{r}\right)\cos\theta + c(r)$$

Now differentiating above w.r.t. r and using $u_r = \frac{1}{r} u_\theta$, we get

$$\frac{dc}{dr} = 0 \Rightarrow c(r) = \text{constant} = c_1$$

Hence $\qquad u(r, \theta) = \left(r + \frac{1}{r}\right)\cos\theta + c_1$

Thus $\qquad f(z) = u\,(r, \theta) + iv(r, \theta) + c$

Q. 11 For what values of z the function w defined by the equation ceases to be analytic?

$$z = \log r + i\,\theta,\ w = re^{i\theta}$$

Sol: Given $\qquad z = \log r + i\,\theta \qquad ...(1)$

$\qquad w = re^{i\theta} = r\,(\cos\theta + i \sin\theta) \qquad ...(2)$

from the above equations (1) we have

$$z = \log re^{i\theta} \Rightarrow re^{i\theta} = e^z$$

$\Rightarrow \qquad e^z = w$ from (2)

$$\Rightarrow \qquad \frac{dw}{dz} = e^z = re^{i\theta}$$

This explains that w will be analytic of z if r is finite. Hence w is analytic function in a finite domain. Thus the given function ceases to be analytic if $r = \infty$.

Q. 12 For what values of z the function $z = \sin h \cos v + i \cos hu \sin v$, $w = u + iv$ ceases to be analytic.

Sol: Now as $\cos ix = \cos hx$ and $\sin ix = i \sin hx$

$$\sin h\,(u + iv) = \frac{1}{i} \sin i\,(u + iv) = i \sin (iu - v)$$

$$= -i\,(\sin \text{hu} \cos v - \cos u \sin v)$$

$$\therefore \qquad z = \sin h\,(u + iv) \sin hw$$

$$\Rightarrow \qquad w = \sin h^{-1} z$$

$$\Rightarrow \qquad \frac{dw}{dz} = \frac{1}{\sqrt{1+z^2}}$$

$$\Rightarrow \qquad \frac{dw}{dz} = \infty \text{ at } z = \pm i$$

$\Rightarrow$ w is not analytic at $z = \pm i$

Q. 13 Prove that an analytic function with constant real part is constant.

Sol: Given $f(z) = u + iv$ is analytic

also $u = \text{constant} = c$ (given)

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}$$

but $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and

Thus from the above relations, we have

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0$$

$\Rightarrow v = \text{constant}$

Hence the result.

Q. 14 Show that the function $f(z) = \sqrt{(|xy|)}$ is not regular at the origin, although $C - R$ equations are satisfied at that point.

Sol: Let $$f(z) = u + iv = \sqrt{(|xy|)}$$

$$\Rightarrow \quad u = \sqrt{(|xy|)} \text{ and } v = 0$$

A the origin $$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0$$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - v(0,0)}{x} = 0$$

and $$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - v(0,0)}{y} = 0$$

$\Rightarrow C - R$ equations are satisfied at (0, 0).

Again $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\Rightarrow \quad f(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - z0} = \lim_{z \to 0} \frac{\sqrt{(|xy|)}}{x + iy}$$

$$\Rightarrow \quad f'(0) = \lim_{x \to 0} \frac{\sqrt{(|mm^2|)}}{x + imm} \text{ along } y = mn$$

$$= \frac{\sqrt{|m|}}{1 + im}$$

$\Rightarrow f'$ depends on m, hence it is not unique

Thus $f(z)$ is not analytic at the origin although $C - R$ equations are satisfied at the origin.

Q. 15 Prove that $e^{\bar{z}}$ is nowhere analytic

Sol: $f(z) = u + iv = e^{\bar{z}} = e^{x-iy} = e^x[\cos y - i \sin y]$

$\Rightarrow$ $u = e^x \cos y$ and $v = -e^x \sin y$

Now $u_x = e^x \cos y,\ u_y = -e^x \sin y$

$u_x = -e^x \sin y$ and $u_y = -e^x \sin y$

As $u_x \neq u_y$ and $u_y \neq -u_x$

This gives us that $e^{\bar{z}}$ is no where analytic.

Miscellaneous Exercise 2 (Unsolved)

Q. 1 Find out whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$, the function $f(z)$ is equal to

(a) $(Imz)/|z|$ (b) $(Rez^2)/|z|$

(c) $(Rez)/(1 + |z|)$ (d) $(Rez - Imz)/|z|^2$

Q. 2 Find an analytic function $w = u + iv$ where $u = x^3 - 3x^2y + 3x^2 - 3y^2 + 1$

Q. 3 Show that $f(z) = \sin x \cos hy + i \cos x \sin hy$ is analytic everywhere.

Q. 4 Show that $u = \cos x \cos y$ is harmonic and find its harmonic conjugate.

Q. 5 Prove that $u\ e^x(x \cos y - y \sin y)$ satisfies Laplace equation. Find $f(z) = u + iv$.

Q. 6 Find an analytic function $w = u + iv$, if

(a) $u = x^3 - 3xy^2$ (b) $u = e^x \cos y$

Q.7 If $f(z)$ is analytic show that $\left\{\frac{\partial}{\partial x}|f|\right\}^2 + \left\{\frac{\partial}{\partial y}|f|\right\}^2 = |f'|^2$

Q. 8 Determine whether $C - R$ conditions are satisfied for the given function:

(a) $f(z) = \frac{1}{2} \ln(x^2 + y^2) + i \tan^{-1} y/x$

(b) $f(z) = x + ay + i\,(bx + cy)$

(c) $f(z) = xy + iy$

(d) $f(z) = z\bar{z}$

Q. 9 Show that $u(r, \theta) = e^{-\theta} \cos(\ln r)$ is harmonic. Find its conjugate harmonic.

Q. 10 Find the conjugate harmonic function of $u(r, \theta) = -r^3 \sin 3\theta$. Also show that u is harmonic.

Q. 11 If $f'(z) = 0$, then show that $f(z)$ is constant.

Q. 12 If both $f(z)$ and $\overline{f(z)}$ are analytic, show that $f(z)$ is constant.

Q. 13 Show that if u is harmonic and v is conjugate harmonic of u, then u in conjugate harmonic of $-v$.

Q. 14 Verify if $f(z) = \dfrac{xy^2 \cdot z}{x^2 + y^4}$, $z \neq 0$

$$f(0) = 0,$$

is analytic or not?

Q. 15 Given $u(x, y) = 2xy + 2x$, construct harmonic function. Also find $f(z)$ using Milne-Thomson method.

Q. 16 Construct an analytic function where the imaginary part is $v(x, y) = 2x(y + 1) - 4$ under the condition $f(1 + i) = 2$. Also write function in terms of z.

Q. 17 Let $f(z) = u(x, y) + iv(x, y)$ be analytic.

Prove that

(a) $f(z) = 2u\left(\dfrac{z}{2}, -\dfrac{iz}{2}\right) + c$ (b) $f(z) = 2iv\left(\dfrac{z}{2}, \dfrac{-iz}{2}\right) + c_1$

Q. 18 Given $f(z)$ is an analytic function. Prove that Ln $f(z)|$ is also analytic.

Miscellaneous Answers 2

Q. 1 (a) No (b) Yes (c) No (d) No

Q. 2 $f(z) = z^3 + 3z^2 + c$

Q. 4 $v = -\sin x \sin hy + c$

Q. 5 $f(z) = ze^z + c$

Q. 6 (a) $w = z^3 + c$ (b) $w = z^3 + c$

Q. 8 (a) for all z (b) $a = -b$, $c = 1$ (c) No where (d) only at the origin.

Q. 9 $v(r, \theta) = e^{-\theta} \sin(\ln r) + c$

Q. 10 $v = r^3 \cos 3\theta + c$

Q. 14 No

Q. 15 $-iz^2 + 2z + c$

Q. 16 $z^2 + 2iz + 4 - 4i$