CHAPTER 2

Introduction to Number Theory

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Appendix 2A The Meaning of Mod

LEARNING OBJECTIVES

After studying this chapter, you should be able to:

- Understand the concept of divisibility and the division algorithm.
- Understand how to use the Euclidean algorithm to find the greatest common divisor.
- Present an overview of the concepts of modular arithmetic.
- Explain the operation of the extended Euclidean algorithm.
- Discuss key concepts relating to prime numbers.
- Understand Fermat's theorem.
- Understand Euler's theorem.
- Define Euler's totient function.
- Make a presentation on the topic of testing for primality.
- Explain the Chinese remainder theorem.
- Define discrete logarithms.

Number theory is pervasive in cryptographic algorithms. This chapter provides sufficient breadth and depth of coverage of relevant number theory topics for understanding the wide range of applications in cryptography. The reader familiar with these topics can safely skip this chapter.

The first three sections introduce basic concepts from number theory that are needed for understanding finite fields; these include divisibility, the Euclidian algorithm, and modular arithmetic. The reader may study these sections now or wait until ready to tackle Chapter 5 on finite fields.

Sections 2.4 through 2.8 discuss aspects of number theory related to prime numbers and discrete logarithms. These topics are fundamental to the design of asymmetric (public-key) cryptographic algorithms. The reader may study these sections now or wait until ready to read Part Three.

The concepts and techniques of number theory are quite abstract, and it is often difficult to grasp them intuitively without examples. Accordingly, this chapter includes a number of examples, each of which is highlighted in a shaded box.

2.1 DIVISIBILITY AND THE DIVISION ALGORITHM

Divisibility

We say that a nonzero b **divides** a if $a = mb$ for some m, where a, b, and m are integers. That is, b divides a if there is no remainder on division. The notation $b|a$ is commonly used to mean b divides a. Also, if $b|a$, we say that b is a **divisor** of a.

The positive divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24.
$13 | 182$; $-5 | 30$; $17 | 289$; $-3 | 33$; $17 | 0$

Subsequently, we will need some simple properties of divisibility for integers, which are as follows:

- If $a | 1$, then $a = \pm 1$.
- If $a | b$ and $b | a$, then $a = \pm b$.
- Any $b \neq 0$ divides 0.
- If $a | b$ and $b | c$, then $a | c$:

$11 | 66$ and $66 | 198 \Rightarrow 11 | 198$

- If $b | g$ and $b | h$, then $b | (mg + nh)$ for arbitrary integers m and n.

To see this last point, note that

- If $b | g$, then g is of the form $g = b \times g_1$ for some integer g_1.
- If $b | h$, then h is of the form $h = b \times h_1$ for some integer h_1.

So

$$mg + nh = mbg_1 + nbh_1 = b \times (mg_1 + nh_1)$$

and therefore b divides $mg + nh$.

$b = 7; g = 14; h = 63; m = 3; n = 2$
$7 | 14$ and $7 | 63$.
To show $7 | (3 \times 14 + 2 \times 63)$,
we have $(3 \times 14 + 2 \times 63) = 7(3 \times 2 + 2 \times 9)$,
and it is obvious that $7 | (7(3 \times 2 + 2 \times 9))$.

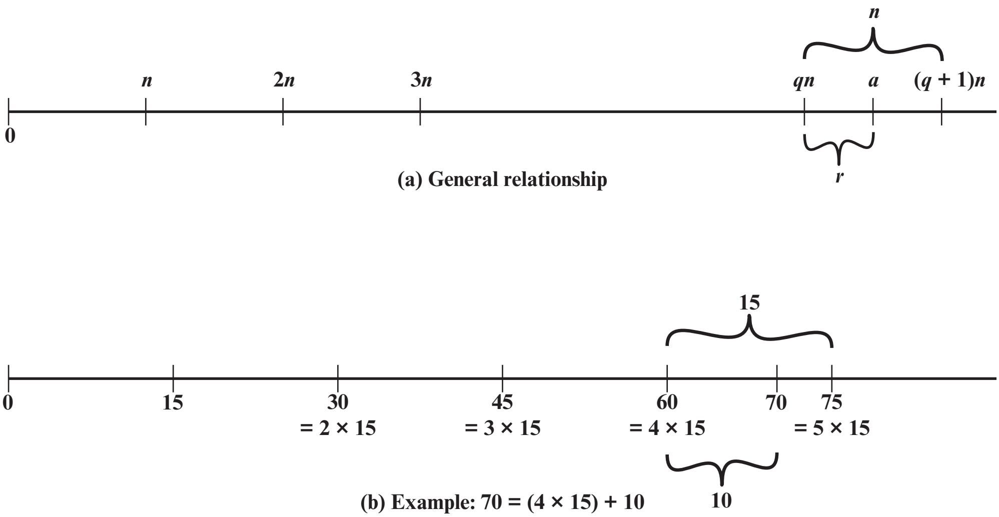
The Division Algorithm

Given any positive integer n and any nonnegative integer a, if we divide a by n, we get an integer quotient q and an integer remainder r that obey the following relationship:

$$a = qn + r \qquad 0 \leq r < n; q = \lfloor a/n \rfloor \tag{2.1}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Equation (2.1) is referred to as the division algorithm.[1]

[1] Equation (2.1) expresses a theorem rather than an algorithm, but by tradition, this is referred to as the division algorithm.



(a) General relationship

(b) Example: 70 = (4 × 15) + 10

Figure 2.1 The Relationship $a = qn + r; 0 \leq r < n$

Figure 2.1a demonstrates that, given a and positive n, it is always possible to find q and r that satisfy the preceding relationship. Represent the integers on the number line; a will fall somewhere on that line (positive a is shown, a similar demonstration can be made for negative a). Starting at 0, proceed to n, $2n$, up to qn, such that $qn \leq a$ and $(q + 1)n > a$. The distance from qn to a is r, and we have found the unique values of q and r. The remainder r is often referred to as a **residue**.

$a = 11;$	$n = 7;$	$11 = 1 \times 7 + 4;$	$r = 4$	$q = 1$
$a = -11;$	$n = 7;$	$-11 = (-2) \times 7 + 3;$	$r = 3$	$q = -2$

Figure 2.1b provides another example.

2.2 THE EUCLIDEAN ALGORITHM

One of the basic techniques of number theory is the Euclidean algorithm, which is a simple procedure for determining the greatest common divisor of two positive integers. First, we need a simple definition: Two integers are **relatively prime** if and only if their only common positive integer factor is 1.

Greatest Common Divisor

Recall that nonzero b is defined to be a divisor of a if $a = mb$ for some m, where a, b, and m are integers. We will use the notation gcd(a, b) to mean the **greatest common divisor** of a and b. The greatest common divisor of a and b is the largest integer that divides both a and b. We also define gcd(0, 0) = 0.

More formally, the positive integer c is said to be the greatest common divisor of a and b if

1. c is a divisor of a and of b.
2. any divisor of a and b is a divisor of c.

An equivalent definition is the following:

$$\gcd(a, b) = \max[k, \text{ such that } k|a \text{ and } k|b]$$

Because we require that the greatest common divisor be positive, $\gcd(a, b) = \gcd(a, -b) = \gcd(-a, b) = \gcd(-a, -b)$. In general, $\gcd(a, b) = \gcd(|a|, |b|)$.

$$\gcd(60, 24) = \gcd(60, -24) = 12$$

Also, because all nonzero integers divide 0, we have $\gcd(a, 0) = |a|$.

We stated that two integers a and b are relatively prime if and only if their only common positive integer factor is 1. This is equivalent to saying that a and b are relatively prime if $\gcd(a, b) = 1$.

8 and 15 are relatively prime because the positive divisors of 8 are 1, 2, 4, and 8, and the positive divisors of 15 are 1, 3, 5, and 15. So 1 is the only integer on both lists.

Finding the Greatest Common Divisor

We now describe an algorithm credited to Euclid for easily finding the greatest common divisor of two integers (Figure 2.2). This algorithm has broad significance in cryptography. The explanation of the algorithm can be broken down into the following points:

1. Suppose we wish to determine the greatest common divisor d of the integers a and b; that is determine $d = \gcd(a, b)$. Because $\gcd(|a|, |b|) = \gcd(a, b)$, there is no harm in assuming $a \geq b > 0$.
2. Dividing a by b and applying the division algorithm, we can state:

$$a = q_1 b + r_1 \qquad 0 \leq r_1 < b \qquad \textbf{(2.2)}$$

3. First consider the case in which $r_1 = 0$. Therefore b divides a and clearly no larger number divides both b and a, because that number would be larger than b. So we have $d = \gcd(a, b) = b$.
4. The other possibility from Equation (2.2) is $r_1 \neq 0$. For this case, we can state that $d|r_1$. This is due to the basic properties of divisibility: the relations $d|a$ and $d|b$ together imply that $d|(a - q_1 b)$, which is the same as $d|r_1$.
5. Before proceeding with the Euclidian algorithm, we need to answer the question: What is the $\gcd(b, r_1)$? We know that $d|b$ and $d|r_1$. Now take any arbitrary integer c that divides both b and r_1. Therefore, $c|(q_1 b + r_1) = a$. Because c divides both a and b, we must have $c \leq d$, which is the greatest common divisor of a and b. Therefore $d = \gcd(b, r_1)$.

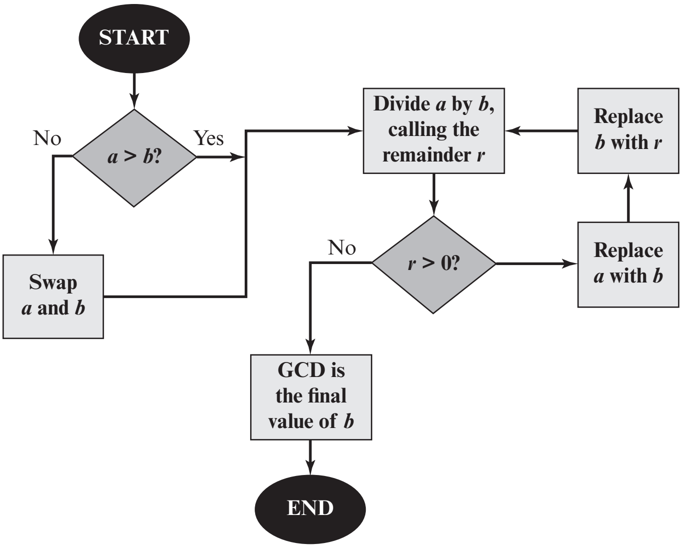


Figure 2.2 Euclidean Algorithm

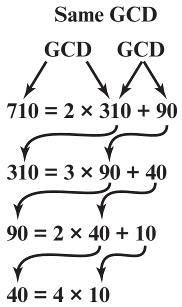


Figure 2.3 Euclidean Algorithm Example: gcd(710, 310)

Let us now return to Equation (2.2) and assume that $r_1 \neq 0$. Because $b > r_1$, we can divide b by r_1 and apply the division algorithm to obtain:

$$b = q_2 r_1 + r_2 \qquad 0 \leq r_2 < r_1$$

As before, if $r_2 = 0$, then $d = r_1$ and if $r_2 \neq 0$, then $d = \gcd(r_1, r_2)$. Note that the remainders form a descending series of nonnegative values and so must terminate when the remainder is zero. This happens, say, at the $(n + 1)$th stage where r_{n-1} is divided by r_n. The result is the following system of equations:

$$\left.\begin{array}{ll} a = q_1 b + r_1 & 0 < r_1 < b \\ b = q_2 r_1 + r_2 & 0 < r_2 < r_1 \\ r_1 = q_3 r_2 + r_3 & 0 < r_3 < r_2 \\ \quad\bullet & \quad\bullet \\ \quad\bullet & \quad\bullet \\ \quad\bullet & \quad\bullet \\ r_{n-2} = q_n r_{n-1} + r_n & 0 < r_n < r_{n-1} \\ r_{n-1} = q_{n+1} r_n + 0 & \\ d = \gcd(a, b) = r_n & \end{array}\right\} \tag{2.3}$$

At each iteration, we have $d = \gcd(r_i, r_{i+1})$ until finally $d = \gcd(r_n, 0) = r_n$. Thus, we can find the greatest common divisor of two integers by repetitive application of the division algorithm. This scheme is known as the Euclidean algorithm. Figure 2.3 illustrates a simple example.

We have essentially argued from the top down that the final result is the gcd(a, b). We can also argue from the bottom up. The first step is to show that r_n divides a and b. It follows from the last division in Equation (2.3) that r_n divides r_{n-1}. The next to last division shows that r_n divides r_{n-2} because it divides both terms on the right. Successively, one sees that r_n divides all r_i's and finally a and b. It remains to show that r_n is the largest divisor that divides a and b. If we take any arbitrary integer that divides a and b, it must also divide r_1, as explained previously. We can follow the sequence of equations in Equation (2.3) down and show that c must divide all r_i's. Therefore c must divide r_n, so that $r_n = \gcd(a, b)$.

Let us now look at an example with relatively large numbers to see the power of this algorithm:

To find $d = \gcd(a, b) = \gcd(1160718174, 316258250)$		
$a = q_1b + r_1$	1160718174 = 3 × 316258250 + 211943424	$d = \gcd(316258250, 211943424)$
$b = q_2r_1 + r_2$	316258250 = 1 × 211943424 + 104314826	$d = \gcd(211943424, 104314826)$
$r_1 = q_3r_2 + r_3$	211943424 = 2 × 104314826 + 3313772	$d = \gcd(104314826, 3313772)$
$r_2 = q_4r_3 + r_4$	104314826 = 31 × 3313772 + 1587894	$d = \gcd(3313772, 1587894)$
$r_3 = q_5r_4 + r_5$	3313772 = 2 × 1587894 + 137984	$d = \gcd(1587894, 137984)$
$r_4 = q_6r_5 + r_6$	1587894 = 11 × 137984 + 70070	$d = \gcd(137984, 70070)$
$r_5 = q_7r_6 + r_7$	137984 = 1 × 70070 + 67914	$d = \gcd(70070, 67914)$
$r_6 = q_8r_7 + r_8$	70070 = 1 × 67914 + 2156	$d = \gcd(67914, 2156)$
$r_7 = q_9r_8 + r_9$	67914 = 31 × 2156 + 1078	$d = \gcd(2156, 1078)$
$r_8 = q_{10}r_9 + r_{10}$	2156 = 2 × 1078 + 0	$d = \gcd(1078, 0) = 1078$
Therefore, $d = \gcd(1160718174, 316258250) = 1078$		

In this example, we begin by dividing 1160718174 by 316258250, which gives 3 with a remainder of 211943424. Next we take 316258250 and divide it by 211943424. The process continues until we get a remainder of 0, yielding a result of 1078.

It will be helpful in what follows to recast the above computation in tabular form. For every step of the iteration, we have $r_{i-2} = q_ir_{i-1} + r_i$, where r_{i-2} is the dividend, r_{i-1} is the divisor, q_i is the quotient, and r_i is the remainder. Table 2.1 summarizes the results.

Table 2.1 Euclidean Algorithm Example

Dividend	Divisor	Quotient	Remainder
a = 1160718174	b = 316258250	q_1 = 3	r_1 = 211943424
b = 316258250	r_1 = 211943434	q_2 = 1	r_2 = 104314826
r_1 = 211943424	r_2 = 104314826	q_3 = 2	r_3 = 3313772
r_2 = 104314826	r_3 = 3313772	q_4 = 31	r_4 = 1587894
r_3 = 3313772	r_4 = 1587894	q_5 = 2	r_5 = 137984
r_4 = 1587894	r_5 = 137984	q_6 = 11	r_6 = 70070
r_5 = 137984	r_6 = 70070	q_7 = 1	r_7 = 67914
r_6 = 70070	r_7 = 67914	q_8 = 1	r_8 = 2156
r_7 = 67914	r_8 = 2156	q_9 = 31	r_9 = 1078
r_8 = 2156	r_9 = 1078	q_{10} = 2	r_{10} = 0

2.3 MODULAR ARITHMETIC

The Modulus

If a is an integer and n is a positive integer, we define $a \bmod n$ to be the remainder when a is divided by n. The integer n is called the **modulus**. Thus, for any integer a, we can rewrite Equation (2.1) as follows:

$$a = qn + r \qquad 0 \leq r < n; q = \lfloor a/n \rfloor$$

$$a = \lfloor a/n \rfloor \times n + (a \bmod n)$$

$$11 \bmod 7 = 4; \qquad -11 \bmod 7 = 3$$

Two integers a and b are said to be **congruent modulo *n***, if $(a \bmod n) = (b \bmod n)$. This is written as $a \equiv b \pmod{n}$.[2]

$$73 \equiv 4 \pmod{23}; \qquad 21 \equiv -9 \pmod{10}$$

Note that if $a \equiv 0 \pmod{n}$, then $n | a$.

Properties of Congruences

Congruences have the following properties:

1. $a \equiv b \pmod{n}$ if $n | (a - b)$.
2. $a \equiv b \pmod{n}$ implies $b \equiv a \pmod{n}$.
3. $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ imply $a \equiv c \pmod{n}$.

To demonstrate the first point, if $n | (a - b)$, then $(a - b) = kn$ for some k. So we can write $a = b + kn$. Therefore, $(a \bmod n)$ = (remainder when $b + kn$ is divided by n) = (remainder when b is divided by n) = $(b \bmod n)$.

$23 \equiv 8 \pmod{5}$	because	$23 - 8 = 15 = 5 \times 3$
$-11 \equiv 5 \pmod{8}$	because	$-11 - 5 = -16 = 8 \times (-2)$
$81 \equiv 0 \pmod{27}$	because	$81 - 0 = 81 = 27 \times 3$

The remaining points are as easily proved.

[2] We have just used the operator *mod* in two different ways: first as a **binary operator** that produces a remainder, as in the expression $a \bmod b$; second as a **congruence relation** that shows the equivalence of two integers, as in the expression $a \equiv b \pmod{n}$. See Appendix 2A for a discussion.

Modular Arithmetic Operations

Note that, by definition (Figure 2.1), the (mod n) operator maps all integers into the set of integers $\{0, 1, \ldots, (n - 1)\}$. This suggests the question: Can we perform arithmetic operations within the confines of this set? It turns out that we can; this technique is known as **modular arithmetic**.

Modular arithmetic exhibits the following properties:

1. $[(a \bmod n) + (b \bmod n)] \bmod n = (a + b) \bmod n$
2. $[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$
3. $[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$

We demonstrate the first property. Define $(a \bmod n) = r_a$ and $(b \bmod n) = r_b$. Then we can write $a = r_a + jn$ for some integer j and $b = r_b + kn$ for some integer k. Then

$$\begin{aligned}(a + b) \bmod n &= (r_a + jn + r_b + kn) \bmod n \\ &= (r_a + r_b + (k + j)n) \bmod n \\ &= (r_a + r_b) \bmod n \\ &= [(a \bmod n) + (b \bmod n)] \bmod n\end{aligned}$$

The remaining properties are proven as easily. Here are examples of the three properties:

$11 \bmod 8 = 3; 15 \bmod 8 = 7$

$[(11 \bmod 8) + (15 \bmod 8)] \bmod 8 = 10 \bmod 8 = 2$

$(11 + 15) \bmod 8 = 26 \bmod 8 = 2$

$[(11 \bmod 8) - (15 \bmod 8)] \bmod 8 = -4 \bmod 8 = 4$

$(11 - 15) \bmod 8 = -4 \bmod 8 = 4$

$[(11 \bmod 8) \times (15 \bmod 8)] \bmod 8 = 21 \bmod 8 = 5$

$(11 \times 15) \bmod 8 = 165 \bmod 8 = 5$

Exponentiation is performed by repeated multiplication, as in ordinary arithmetic.

To find $11^7 \bmod 13$, we can proceed as follows:

$$\begin{aligned}11^2 &= 121 \equiv 4 \pmod{13} \\ 11^4 &= (11^2)^2 \equiv 4^2 \equiv 3 \pmod{13} \\ 11^7 &= 11 \times 11^2 \times 11^4 \\ 11^7 &\equiv 11 \times 4 \times 3 \equiv 132 \equiv 2 \pmod{13}\end{aligned}$$

Thus, the rules for ordinary arithmetic involving addition, subtraction, and multiplication carry over into modular arithmetic.

Table 2.2 provides an illustration of modular addition and multiplication modulo 8. Looking at addition, the results are straightforward, and there is a regular pattern to the matrix. Both matrices are symmetric about the main diagonal in conformance to the commutative property of addition and multiplication. As in ordinary addition, there is an additive inverse, or negative, to each integer in modular arithmetic. In this case, the negative of an integer x is the integer y such that $(x + y) \bmod 8 = 0$. To find the additive inverse of an integer in the left-hand column, scan across the corresponding row of the matrix to find the value 0; the integer at the top of that column is the additive inverse; thus, $(2 + 6) \bmod 8 = 0$. Similarly, the entries in the multiplication table are straightforward. In modular arithmetic mod 8, the multiplicative inverse of x is the integer y such that $(x \times y) \bmod 8 = 1 \bmod 8$. Now, to find the multiplicative inverse of an integer from the multiplication table, scan across the matrix in the row for that integer to find the value 1; the integer at the top of that column is the multiplicative inverse; thus, $(3 \times 3) \bmod 8 = 1$. Note that not all integers mod 8 have a multiplicative inverse; more about that later.

Properties of Modular Arithmetic

Define the set Z_n as the set of nonnegative integers less than n:

$$Z_n = \{0, 1, \ldots, (n - 1)\}$$

Table 2.2 Arithmetic Modulo 8

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

(a) Addition modulo 8

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

(b) Multiplication modulo 8

w	$-w$	w^{-1}
0	0	—
1	7	1
2	6	—
3	5	3
4	4	—
5	3	5
6	2	—
7	1	7

(c) Additive and multiplicative inverse modulo 8

This is referred to as the **set of residues**, or **residue classes** (mod n). To be more precise, each integer in Z_n represents a residue class. We can label the residue classes (mod n) as $[0], [1], [2], \ldots, [n-1]$, where

$$[r] = \{a\colon a \text{ is an integer}, a \equiv r \pmod{n}\}$$

The residue classes (mod 4) are

$$
\begin{aligned}
[0] &= \{\ldots, -16, -12, -8, -4, 0, 4, 8, 12, 16, \ldots\} \\
[1] &= \{\ldots, -15, -11, -7, -3, 1, 5, 9, 13, 17, \ldots\} \\
[2] &= \{\ldots, -14, -10, -6, -2, 2, 6, 10, 14, 18, \ldots\} \\
[3] &= \{\ldots, -13, -9, -5, -1, 3, 7, 11, 15, 19, \ldots\}
\end{aligned}
$$

Of all the integers in a residue class, the smallest nonnegative integer is the one used to represent the residue class. Finding the smallest nonnegative integer to which k is congruent modulo n is called **reducing *k* modulo *n***.

If we perform modular arithmetic within Z_n, the properties shown in Table 2.3 hold for integers in Z_n. We show in the next section that this implies that Z_n is a commutative ring with a multiplicative identity element.

There is one peculiarity of modular arithmetic that sets it apart from ordinary arithmetic. First, observe that (as in ordinary arithmetic) we can write the following:

$$\textbf{if } (a + b) \equiv (a + c) \pmod{n} \textbf{ then } b \equiv c \pmod{n} \qquad \textbf{(2.4)}$$

$$\boxed{(5 + 23) \equiv (5 + 7)(\text{mod } 8);\ 23 \equiv 7(\text{mod } 8)}$$

Equation (2.4) is consistent with the existence of an additive inverse. Adding the additive inverse of a to both sides of Equation (2.4), we have

$$
\begin{gathered}
((-a) + a + b) \equiv ((-a) + a + c)(\text{mod } n) \\
b \equiv c \pmod{n}
\end{gathered}
$$

Table 2.3 Properties of Modular Arithmetic for Integers in Z_n

Property	Expression
Commutative Laws	$(w + x) \bmod n = (x + w) \bmod n$ $(w \times x) \bmod n = (x \times w) \bmod n$
Associative Laws	$[(w + x) + y] \bmod n = [w + (x + y)] \bmod n$ $[(w \times x) \times y] \bmod n = [w \times (x \times y)] \bmod n$
Distributive Law	$[w \times (x + y)] \bmod n = [(w \times x) + (w \times y)] \bmod n$
Identities	$(0 + w) \bmod n = w \bmod n$ $(1 \times w) \bmod n = w \bmod n$
Additive Inverse ($-w$)	For each $w \in Z_n$, there exists a z such that $w + z \equiv 0 \bmod n$

However, the following statement is true only with the attached condition:

$$\textbf{if } (a \times b) \equiv (a \times c)(\text{mod } n) \textbf{ then } b \equiv c(\text{mod } n) \textbf{ if } a \text{ is relatively prime to } n \qquad \textbf{(2.5)}$$

Recall that two integers are **relatively prime** if their only common positive integer factor is 1. Similar to the case of Equation (2.4), we can say that Equation (2.5) is consistent with the existence of a multiplicative inverse. Applying the multiplicative inverse of a to both sides of Equation (2.5), we have

$$((a^{-1})ab) \equiv ((a^{-1})ac)(\text{mod } n)$$
$$b \equiv c(\text{mod } n)$$

To see this, consider an example in which the condition of Equation (2.5) does not hold. The integers 6 and 8 are not relatively prime, since they have the common factor 2. We have the following:

$$6 \times 3 = 18 \equiv 2(\text{mod } 8)$$
$$6 \times 7 = 42 \equiv 2(\text{mod } 8)$$

Yet $3 \not\equiv 7 \ (\text{mod } 8)$.

The reason for this strange result is that for any general modulus n, a multiplier a that is applied in turn to the integers 0 through $(n - 1)$ will fail to produce a complete set of residues if a and n have any factors in common.

With $a = 6$ and $n = 8$,

Z_8	0	1	2	3	4	5	6	7
Multiply by 6	0	6	12	18	24	30	36	42
Residues	0	6	4	2	0	6	4	2

Because we do not have a complete set of residues when multiplying by 6, more than one integer in Z_8 maps into the same residue. Specifically, $6 \times 0 \text{ mod } 8 = 6 \times 4 \text{ mod } 8$; $6 \times 1 \text{ mod } 8 = 6 \times 5 \text{ mod } 8$; and so on. Because this is a many-to-one mapping, there is not a unique inverse to the multiply operation.

However, if we take $a = 5$ and $n = 8$, whose only common factor is 1,

Z_8	0	1	2	3	4	5	6	7
Multiply by 5	0	5	10	15	20	25	30	35
Residues	0	5	2	7	4	1	6	3

The line of residues contains all the integers in Z_8, in a different order.

In general, an integer has a multiplicative inverse in Z_n if and only if that integer is relatively prime to n. Table 2.2c shows that the integers 1, 3, 5, and 7 have a multiplicative inverse in Z_8; but 2, 4, and 6 do not.

Euclidean Algorithm Revisited

The Euclidean algorithm can be based on the following theorem: For any integers $a, b,$ with $a \geq b \geq 0$,

$$\gcd(a, b) = \gcd(b, a \bmod b) \tag{2.6}$$

$$\gcd(55, 22) = \gcd(22, 55 \bmod 22) = \gcd(22, 11) = 11$$

To see that Equation (2.6) works, let $d = \gcd(a, b)$. Then, by the definition of gcd, $d|a$ and $d|b$. For any positive integer b, we can express a as

$$a = kb + r \equiv r \pmod{b}$$
$$a \bmod b = r$$

with k, r integers. Therefore, $(a \bmod b) = a - kb$ for some integer k. But because $d|b$, it also divides kb. We also have $d|a$. Therefore, $d|(a \bmod b)$. This shows that d is a common divisor of b and $(a \bmod b)$. Conversely, if d is a common divisor of b and $(a \bmod b)$, then $d|kb$ and thus $d|[kb + (a \bmod b)]$, which is equivalent to $d|a$. Thus, the set of common divisors of a and b is equal to the set of common divisors of b and $(a \bmod b)$. Therefore, the gcd of one pair is the same as the gcd of the other pair, proving the theorem.

Equation (2.6) can be used repetitively to determine the greatest common divisor.

$$\gcd(18, 12) = \gcd(12, 6) = \gcd(6, 0) = 6$$
$$\gcd(11, 10) = \gcd(10, 1) = \gcd(1, 0) = 1$$

This is the same scheme shown in Equation (2.3), which can be rewritten in the following way.

Euclidean Algorithm	
Calculate	**Which satisfies**
$r_1 = a \bmod b$	$a = q_1 b + r_1$
$r_2 = b \bmod r_1$	$b = q_2 r_1 + r_2$
$r_3 = r_1 \bmod r_2$	$r_1 = q_3 r_2 + r_3$
• • •	• • •
$r_n = r_{n-2} \bmod r_{n-1}$	$r_{n-2} = q_n r_{n-1} + r_n$
$r_{n+1} = r_{n-1} \bmod r_n = 0$	$r_{n-1} = q_{n+1} r_n + 0$ $d = \gcd(a, b) = r_n$

We can define the Euclidean algorithm concisely as the following recursive function.

```
Euclid(a,b)
    if (b=0) then return a;
    else return Euclid(b, a mod b);
```

The Extended Euclidean Algorithm

We now proceed to look at an extension to the Euclidean algorithm that will be important for later computations in the area of finite fields and in encryption algorithms, such as RSA. For given integers a and b, the extended Euclidean algorithm not only calculates the greatest common divisor d but also two additional integers x and y that satisfy the following equation.

$$ax + by = d = \gcd(a, b) \tag{2.7}$$

It should be clear that x and y will have opposite signs. Before examining the algorithm, let us look at some of the values of x and y when $a = 42$ and $b = 30$. Note that $\gcd(42, 30) = 6$. Here is a partial table of values[3] for $42x + 30y$.

x y	−3	−2	−1	0	1	2	3
−3	−216	−174	−132	−90	−48	−6	36
−2	−186	−144	−102	−60	−18	24	66
−1	−156	−114	−72	−30	12	54	96
0	−126	−84	−42	0	42	84	126
1	−96	−54	−12	30	72	114	156
2	−66	−24	18	60	102	144	186
3	−36	6	48	90	132	174	216

Observe that all of the entries are divisible by 6. This is not surprising, because both 42 and 30 are divisible by 6, so every number of the form $42x + 30y = 6(7x + 5y)$ is a multiple of 6. Note also that $\gcd(42, 30) = 6$ appears in the table. In general, it can be shown that for given integers a and b, the smallest positive value of $ax + by$ is equal to $\gcd(a, b)$.

Now let us show how to extend the Euclidean algorithm to determine (x, y, d) given a and b. We again go through the sequence of divisions indicated in Equation (2.3), and we assume that at each step i we can find integers x_i and y_i that satisfy $r_i = ax_i + by_i$. We end up with the following sequence.

$$\begin{array}{rlrl}
a &= q_1 b + r_1 & r_1 &= ax_1 + by_1 \\
b &= q_2 r_1 + r_2 & r_2 &= ax_2 + by_2 \\
r_1 &= q_3 r_2 + r_3 & r_3 &= ax_3 + by_3 \\
&\vdots & &\vdots \\
r_{n-2} &= q_n r_{n-1} + r_n & r_n &= ax_n + by_n \\
r_{n-1} &= q_{n+1} r_n + 0 & &
\end{array}$$

[3] This example is taken from [SILV06].

Now, observe that we can rearrange terms to write

$$r_i = r_{i-2} - r_{i-1}q_i \tag{2.8}$$

Also, in rows $i - 1$ and $i - 2$, we find the values

$$r_{i-2} = ax_{i-2} + by_{i-2} \quad \text{and} \quad r_{i-1} = ax_{i-1} + by_{i-1}$$

Substituting into Equation (2.8), we have

$$\begin{aligned} r_i &= (ax_{i-2} + by_{i-2}) - (ax_{i-1} + by_{i-1})q_i \\ &= a(x_{i-2} - q_ix_{i-1}) + b(y_{i-2} - q_iy_{i-1}) \end{aligned}$$

But we have already assumed that $r_i = ax_i + by_i$. Therefore,

$$x_i = x_{i-2} - q_ix_{i-1} \quad \text{and} \quad y_i = y_{i-2} - q_iy_{i-1}$$

We now summarize the calculations:

Extended Euclidean Algorithm			
Calculate	**Which satisfies**	**Calculate**	**Which satisfies**
$r_{-1} = a$		$x_{-1} = 1; y_{-1} = 0$	$a = ax_{-1} + by_{-1}$
$r_0 = b$		$x_0 = 0; y_0 = 1$	$b = ax_0 + by_0$
$r_1 = a \bmod b$ $q_1 = \lfloor a/b \rfloor$	$a = q_1b + r_1$	$x_1 = x_{-1} - q_1x_0 = 1$ $y_1 = y_{-1} - q_1y_0 = -q_1$	$r_1 = ax_1 + by_1$
$r_2 = b \bmod r_1$ $q_2 = \lfloor b/r_1 \rfloor$	$b = q_2r_1 + r_2$	$x_2 = x_0 - q_2x_1$ $y_2 = y_0 - q_2y_1$	$r_2 = ax_2 + by_2$
$r_3 = r_1 \bmod r_2$ $q_3 = \lfloor r_1/r_2 \rfloor$	$r_1 = q_3r_2 + r_3$	$x_3 = x_1 - q_3x_2$ $y_3 = y_1 - q_3y_2$	$r_3 = ax_3 + by_3$
• • •	• • •	• • •	• • •
$r_n = r_{n-2} \bmod r_{n-1}$ $q_n = \lfloor r_{n-2}/r_{n-1} \rfloor$	$r_{n-2} = q_nr_{n-1} + r_n$	$x_n = x_{n-2} - q_nx_{n-1}$ $y_n = y_{n-2} - q_ny_{n-1}$	$r_n = ax_n + by_n$
$r_{n+1} = r_{n-1} \bmod r_n = 0$ $q_{n+1} = \lfloor r_{n-1}/r_n \rfloor$	$r_{n-1} = q_{n+1}r_n + 0$		$d = \gcd(a, b) = r_n$ $x = x_n; y = y_n$

We need to make several additional comments here. In each row, we calculate a new remainder r_i based on the remainders of the previous two rows, namely r_{i-1} and r_{i-2}. To start the algorithm, we need values for r_0 and r_{-1}, which are just a and b. It is then straightforward to determine the required values for x_{-1}, y_{-1}, x_0, and y_0.

We know from the original Euclidean algorithm that the process ends with a remainder of zero and that the greatest common divisor of a and b is $d = \gcd(a, b) = r_n$. But we also have determined that $d = r_n = ax_n + by_n$. Therefore, in Equation (2.7), $x = x_n$ and $y = y_n$.

As an example, let us use $a = 1759$ and $b = 550$ and solve for $1759x + 550y = \gcd(1759, 550)$. The results are shown in Table 2.4. Thus, we have $1759 \times (-111) + 550 \times 355 = -195249 + 195250 = 1$.

Table 2.4 Extended Euclidean Algorithm Example

i	r_i	q_i	x_i	y_i
−1	1759		1	0
0	550		0	1
1	109	3	1	−3
2	5	5	−5	16
3	4	21	106	−339
4	1	1	−111	355
5	0	4		

Result: $d = 1; x = -111; y = 355$

2.4 PRIME NUMBERS[4]

A central concern of number theory is the study of prime numbers. Indeed, whole books have been written on the subject (e.g., [CRAN01], [RIBE96]). In this section, we provide an overview relevant to the concerns of this book.

An integer $p > 1$ is a prime number if and only if its only divisors[5] are ± 1 and $\pm p$. **Prime numbers** play a critical role in number theory and in the techniques discussed in this chapter. Table 2.5 shows the primes less than 2000. Note the way the primes are distributed. In particular, note the number of primes in each range of 100 numbers.

Any integer $a > 1$ can be factored in a unique way as

$$a = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_t^{a_t} \tag{2.9}$$

where $p_1 < p_2 < \ldots < p_t$ are prime numbers and where each a_i is a positive integer. This is known as the fundamental theorem of arithmetic; a proof can be found in any text on number theory.

$$\begin{aligned} 91 &= 7 \times 13 \\ 3600 &= 2^4 \times 3^2 \times 5^2 \\ 11011 &= 7 \times 11^2 \times 13 \end{aligned}$$

It is useful for what follows to express this another way. If P is the set of all prime numbers, then any positive integer a can be written uniquely in the following form:

$$a = \prod_{p \in \mathrm{P}} p^{a_p} \quad \text{where each } a_p \geq 0$$

[4] In this section, unless otherwise noted, we deal only with the nonnegative integers. The use of negative integers would introduce no essential differences.

[5] Recall from Section 2.1 that integer a is said to be a divisor of integer b if there is no remainder on division. Equivalently, we say that a divides b.

Table 2.5 Primes Under 2000

2	101	211	307	401	503	601	701	809	907	1009	1103	1201	1301	1409	1511	1601	1709	1801	1901
3	103	223	311	409	509	607	709	811	911	1013	1109	1213	1303	1423	1523	1607	1721	1811	1907
5	107	227	313	419	521	613	719	821	919	1019	1117	1217	1307	1427	1531	1609	1723	1823	1913
7	109	229	317	421	523	617	727	823	929	1021	1123	1223	1319	1429	1543	1613	1733	1831	1931
11	113	233	331	431	541	619	733	827	937	1031	1129	1229	1321	1433	1549	1619	1741	1847	1933
13	127	239	337	433	547	631	739	829	941	1033	1151	1231	1327	1439	1553	1621	1747	1861	1949
17	131	241	347	439	557	641	743	839	947	1039	1153	1237	1361	1447	1559	1627	1753	1867	1951
19	137	251	349	443	563	643	751	853	953	1049	1163	1249	1367	1451	1567	1637	1759	1871	1973
23	139	257	353	449	569	647	757	857	967	1051	1171	1259	1373	1453	1571	1657	1777	1873	1979
29	149	263	359	457	571	653	761	859	971	1061	1181	1277	1381	1459	1579	1663	1783	1877	1987
31	151	269	367	461	577	659	769	863	977	1063	1187	1279	1399	1471	1583	1667	1787	1879	1993
37	157	271	373	463	587	661	773	877	983	1069	1193	1283		1481	1597	1669	1789	1889	1997
41	163	277	379	467	593	673	787	881	991	1087		1289		1483		1693			1999
43	167	281	383	479	599	677	797	883	997	1091		1291		1487		1697			
47	173	283	389	487		683		887		1093		1297		1489		1699			
53	179	293	397	491		691				1097				1493					
59	181			499										1499					
61	191																		
67	193																		
71	197																		
73	199																		
79																			
83																			
89																			
97																			

The right-hand side is the product over all possible prime numbers p; for any particular value of a, most of the exponents a_p will be 0.

The value of any given positive integer can be specified by simply listing all the nonzero exponents in the foregoing formulation.

The integer 12 is represented by $\{a_2 = 2, a_3 = 1\}$.
The integer 18 is represented by $\{a_2 = 1, a_3 = 2\}$.
The integer 91 is represented by $\{a_7 = 1, a_{13} = 1\}$.

Multiplication of two numbers is equivalent to adding the corresponding exponents. Given $a = \prod_{p \in \mathrm{P}} p^{a_p}$, $b = \prod_{p \in \mathrm{P}} p^{b_p}$. Define $k = ab$. We know that the integer k can be expressed as the product of powers of primes: $k = \prod_{p \in \mathrm{P}} p^{k_p}$. It follows that $k_p = a_p + b_p$ for all $p \in \mathrm{P}$.

$k = 12 \times 18 = (2^2 \times 3) \times (2 \times 3^2) = 216$
$k_2 = 2 + 1 = 3; k_3 = 1 + 2 = 3$
$216 = 2^3 \times 3^3 = 8 \times 27$

What does it mean, in terms of the prime factors of a and b, to say that a divides b? Any integer of the form p^n can be divided only by an integer that is of a lesser or equal power of the same prime number, p^j with $j \leq n$. Thus, we can say the following.

Given

$$a = \prod_{p \in \mathrm{P}} p^{a_p}, b = \prod_{p \in \mathrm{P}} p^{b_p}$$

If $a|b$, then $a_p \leq b_p$ for all p.

$a = 12; b = 36; 12|36$
$12 = 2^2 \times 3; 36 = 2^2 \times 3^2$
$a_2 = 2 = b_2$
$a_3 = 1 \leq 2 = b_3$
Thus, the inequality $a_p \leq b_p$ is satisfied for all prime numbers.

It is easy to determine the greatest common divisor of two positive integers if we express each integer as the product of primes.

$$\boxed{\begin{aligned} 300 &= 2^2 \times 3^1 \times 5^2 \\ 18 &= 2^1 \times 3^2 \\ \gcd(18{,}300) &= 2^1 \times 3^1 \times 5^0 = 6 \end{aligned}}$$

The following relationship always holds:

$$\text{If } k = \gcd(a, b), \text{ then } k_p = \min(a_p, b_p) \text{ for all } p.$$

Determining the prime factors of a large number is no easy task, so the preceding relationship does not directly lead to a practical method of calculating the greatest common divisor.

2.5 FERMAT'S AND EULER'S THEOREMS

Two theorems that play important roles in public-key cryptography are Fermat's theorem and Euler's theorem.

Fermat's Theorem[6]

Fermat's theorem states the following: If p is prime and a is a positive integer not divisible by p, then

$$a^{p-1} \equiv 1 \pmod{p} \tag{2.10}$$

Proof: Consider the set of positive integers less than p: $\{1, 2, \ldots, p - 1\}$ and multiply each element by a, modulo p, to get the set $X = \{a \bmod p, 2a \bmod p, \ldots, (p - 1)a \bmod p\}$. None of the elements of X is equal to zero because p does not divide a. Furthermore, no two of the integers in X are equal. To see this, assume that $ja \equiv ka(\text{mod } p))$, where $1 \leq j < k \leq p - 1$. Because a is relatively prime[7] to p, we can eliminate a from both sides of the equation [see Equation (2.3)] resulting in $j \equiv k(\text{mod } p)$. This last equality is impossible, because j and k are both positive integers less than p. Therefore, we know that the $(p - 1)$ elements of X are all positive integers with no two elements equal. We can conclude the X consists of the set of integers $\{1, 2, \ldots, p - 1\}$ in some order. Multiplying the numbers in both sets (p and X) and taking the result mod p yields

$$a \times 2a \times \cdots \times (p - 1)a \equiv [(1 \times 2 \times \cdots \times (p - 1)](\text{mod } p)$$
$$a^{p-1}(p - 1)! \equiv (p - 1)! \pmod{p}$$

We can cancel the $(p - 1)!$ term because it is relatively prime to p [see Equation (2.5)]. This yields Equation (2.10), which completes the proof.

[6] This is sometimes referred to as Fermat's little theorem.

[7] Recall from Section 2.2 that two numbers are relatively prime if they have no prime factors in common; that is, their only common divisor is 1. This is equivalent to saying that two numbers are relatively prime if their greatest common divisor is 1.

$$
\begin{aligned}
&a = 7, p = 19\\
&7^2 = 49 \equiv 11 \text{ (mod 19)}\\
&7^4 \equiv 121 \equiv 7 \text{ (mod 19)}\\
&7^8 \equiv 49 \equiv 11 \text{ (mod 19)}\\
&7^{16} \equiv 121 \equiv 7 \text{ (mod 19)}\\
&a^{p-1} = 7^{18} = 7^{16} \times 7^2 \equiv 7 \times 11 \equiv 1 \text{ (mod 19)}
\end{aligned}
$$

An alternative form of Fermat's theorem is also useful: If p is prime and a is a positive integer, then

$$a^p \equiv a(\text{mod } p) \tag{2.11}$$

Note that the first form of the theorem [Equation (2.10)] requires that a be relatively prime to p, but this form does not.

$$
\begin{array}{ll}
p = 5, a = 3 & a^p = 3^5 = 243 \equiv 3(\text{mod } 5) = a(\text{mod } p)\\
p = 5, a = 10 & a^p = 10^5 = 100000 \equiv 10(\text{mod } 5) \equiv 0(\text{mod } 5) = a(\text{mod } p)
\end{array}
$$

Euler's Totient Function

Before presenting Euler's theorem, we need to introduce an important quantity in number theory, referred to as **Euler's totient function**. This function, written $\phi(n)$, is defined as the number of positive integers less than n and relatively prime to n. By convention, $\phi(1) = 1$.

Determine $\phi(37)$ and $\phi(35)$.

Because 37 is prime, all of the positive integers from 1 through 36 are relatively prime to 37. Thus $\phi(37) = 36$.

To determine $\phi(35)$, we list all of the positive integers less than 35 that are relatively prime to it:

$$
\begin{gathered}
1, 2, 3, 4, 6, 8, 9, 11, 12, 13, 16, 17, 18\\
19, 22, 23, 24, 26, 27, 29, 31, 32, 33, 34
\end{gathered}
$$

There are 24 numbers on the list, so $\phi(35) = 24$.

Table 2.6 lists the first 30 values of $\phi(n)$. The value $\phi(1)$ is without meaning but is defined to have the value 1.

It should be clear that, for a prime number p,

$$\phi(p) = p - 1$$

Now suppose that we have two prime numbers p and q with $p \neq q$. Then we can show that, for $n = pq$,

Table 2.6 Some Values of Euler's Totient Function $\phi(n)$

n	$\phi(n)$	n	$\phi(n)$	n	$\phi(n)$
1	1	11	10	21	12
2	1	12	4	22	10
3	2	13	12	23	22
4	2	14	6	24	8
5	4	15	8	25	20
6	2	16	8	26	12
7	6	17	16	27	18
8	4	18	6	28	12
9	6	19	18	29	28
10	4	20	8	30	8

$$\phi(n) = \phi(pq) = \phi(p) \times \phi(q) = (p - 1) \times (q - 1)$$

To see that $\phi(n) = \phi(p) \times \phi(q)$, consider that the set of positive integers less than n is the set $\{1, \ldots, (pq - 1)\}$. The integers in this set that are not relatively prime to n are the set $\{p, 2p, \ldots, (q - 1)p\}$ and the set $\{q, 2q, \ldots, (p - 1)q\}$. To see this, consider that any integer that divides n must divide either of the prime numbers p or q. Therefore, any integer that does not contain either p or q as a factor is relatively prime to n. Further note that the two sets just listed are non-overlapping: Because p and q are prime, we can state that none of the integers in the first set can be written as a multiple of q, and none of the integers in the second set can be written as a multiple of p. Thus the total number of unique integers in the two sets is $(q - 1) + (p - 1)$. Accordingly,

$$\begin{aligned}\phi(n) &= (pq - 1) - [(q - 1) + (p - 1)] \\ &= pq - (p + q) + 1 \\ &= (p - 1) \times (q - 1) \\ &= \phi(p) \times \phi(q)\end{aligned}$$

$\phi(21) = \phi(3) \times \phi(7) = (3 - 1) \times (7 - 1) = 2 \times 6 = 12$
where the 12 integers are $\{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$.

Euler's Theorem

Euler's theorem states that for every a and n that are relatively prime:

$$a^{\phi(n)} \equiv 1(\text{mod } n) \tag{2.12}$$

Proof: Equation (2.12) is true if n is prime, because in that case, $\phi(n) = (n - 1)$ and Fermat's theorem holds. However, it also holds for any integer n. Recall that

$\phi(n)$ is the number of positive integers less than n that are relatively prime to n. Consider the set of such integers, labeled as

$$R = \{x_1, x_2, \ldots, x_{\phi(n)}\}$$

That is, each element x_i of R is a unique positive integer less than n with $\gcd(x_i, n) = 1$. Now multiply each element by a, modulo n:

$$S = \{(ax_1 \bmod n), (ax_2 \bmod n), \ldots, (ax_{\phi(n)} \bmod n)\}$$

The set S is a permutation[8] of R , by the following line of reasoning:

1. Because a is relatively prime to n and x_i is relatively prime to n, ax_i must also be relatively prime to n. Thus, all the members of S are integers that are less than n and that are relatively prime to n.
2. There are no duplicates in S. Refer to Equation (2.5). If $ax_i \bmod n = ax_j \bmod n$, then $x_i = x_j$.

Therefore,

$$\begin{aligned}
\prod_{i=1}^{\phi(n)} (ax_i \bmod n) &= \prod_{i=1}^{\phi(n)} x_i \\
\prod_{i=1}^{\phi(n)} ax_i &\equiv \prod_{i=1}^{\phi(n)} x_i \pmod{n} \\
a^{\phi(n)} \times \left[\prod_{i=1}^{\phi(n)} x_i\right] &\equiv \prod_{i=1}^{\phi(n)} x_i \pmod{n} \\
a^{\phi(n)} &\equiv 1 \pmod{n}
\end{aligned}$$

which completes the proof. This is the same line of reasoning applied to the proof of Fermat's theorem.

$$\begin{array}{ll}
a = 3; n = 10; \phi(10) = 4; & a^{\phi(n)} = 3^4 = 81 = 1(\text{mod } 10) = 1(\text{mod } n) \\
a = 2; n = 11; \phi(11) = 10; & a^{\phi(n)} = 2^{10} = 1024 = 1(\text{mod } 11) = 1(\text{mod } n)
\end{array}$$

As is the case for Fermat's theorem, an alternative form of the theorem is also useful:

$$a^{\phi(n)+1} \equiv a(\text{mod } n) \qquad \textbf{(2.13)}$$

Again, similar to the case with Fermat's theorem, the first form of Euler's theorem [Equation (2.12)] requires that a be relatively prime to n, but this form does not.

[8] A permutation of a finite set of elements S is an ordered sequence of all the elements of S, with each element appearing exactly once.

2.6 TESTING FOR PRIMALITY

For many cryptographic algorithms, it is necessary to select one or more very large prime numbers at random. Thus, we are faced with the task of determining whether a given large number is prime. There is no simple yet efficient means of accomplishing this task.

In this section, we present one attractive and popular algorithm. You may be surprised to learn that this algorithm yields a number that is not necessarily a prime. However, the algorithm can yield a number that is almost certainly a prime. This will be explained presently. We also make reference to a deterministic algorithm for finding primes. The section closes with a discussion concerning the distribution of primes.

Miller–Rabin Algorithm[9]

The algorithm due to Miller and Rabin [MILL75, RABI80] is typically used to test a large number for primality. Before explaining the algorithm, we need some background. First, any positive odd integer $n \geq 3$ can be expressed as

$$n - 1 = 2^k q \qquad \text{with } k > 0, q \text{ odd}$$

To see this, note that $n - 1$ is an even integer. Then, divide $(n - 1)$ by 2 until the result is an odd number q, for a total of k divisions. If n is expressed as a binary number, then the result is achieved by shifting the number to the right until the rightmost digit is a 1, for a total of k shifts. We now develop two properties of prime numbers that we will need.

TWO PROPERTIES OF PRIME NUMBERS The **first property** is stated as follows: If p is prime and a is a positive integer less than p, then $a^2 \bmod p = 1$ if and only if either $a \bmod p = 1$ or $a \bmod p = -1 \bmod p = p - 1$. By the rules of modular arithmetic $(a \bmod p)\ (a \bmod p) = a^2 \bmod p$. Thus, if either $a \bmod p = 1$ or $a \bmod p = -1$, then $a^2 \bmod p = 1$. Conversely, if $a^2 \bmod p = 1$, then $(a \bmod p)^2 = 1$, which is true only for $a \bmod p = 1$ or $a \bmod p = -1$.

The **second property** is stated as follows: Let p be a prime number greater than 2. We can then write $p - 1 = 2^k q$ with $k > 0$, q odd. Let a be any integer in the range $1 < a < p - 1$. Then one of the two following conditions is true.

1. a^q is congruent to 1 modulo p. That is, $a^q \bmod p = 1$, or equivalently, $a^q \equiv 1 (\bmod p)$.
2. One of the numbers $a^q, a^{2q}, a^{4q}, \ldots, a^{2^{k-1}q}$ is congruent to -1 modulo p. That is, there is some number j in the range $(1 \leq j \leq k)$ such that $a^{2^{j-1}q} \bmod p = -1 \bmod p = p - 1$ or equivalently, $a^{2^{j-1}q} \equiv -1 (\bmod p)$.

Proof: Fermat's theorem [Equation (2.10)] states that $a^{n-1} \equiv 1 (\bmod n)$ if n is prime. We have $p - 1 = 2^k q$. Thus, we know that $a^{p-1} \bmod p = a^{2^k q} \bmod p = 1$. Thus, if we look at the sequence of numbers

$$a^q \bmod p, a^{2q} \bmod p, a^{4q} \bmod p, \ldots, a^{2^{k-1}q} \bmod p, a^{2^k q} \bmod p \qquad \textbf{(2.14)}$$

[9] Also referred to in the literature as the Rabin-Miller algorithm, or the Rabin-Miller test, or the Miller–Rabin test.

we know that the last number in the list has value 1. Further, each number in the list is the square of the previous number. Therefore, one of the following possibilities must be true.

1. The first number on the list, and therefore all subsequent numbers on the list, equals 1.
2. Some number on the list does not equal 1, but its square mod p does equal 1. By virtue of the first property of prime numbers defined above, we know that the only number that satisfies this condition is $p - 1$. So, in this case, the list contains an element equal to $p - 1$.

This completes the proof.

Details of the Algorithm These considerations lead to the conclusion that, if n is prime, then either the first element in the list of residues, or remainders, $(a^q, a^{2q}, \ldots, a^{2^{k-1}q}, a^{2^k q})$ modulo n equals 1; or some element in the list equals $(n - 1)$; otherwise n is composite (i.e., not a prime). On the other hand, if the condition is met, that does not necessarily mean that n is prime. For example, if $n = 2047 = 23 \times 89$, then $n - 1 = 2 \times 1023$. We compute $2^{1023} \bmod 2047 = 1$, so that 2047 meets the condition but is not prime.

We can use the preceding property to devise a test for primality. The procedure TEST takes a candidate integer n as input and returns the result `composite` if n is definitely not a prime, and the result `inconclusive` if n may or may not be a prime.

```
TEST (n)
1. Find integers k, q, with k > 0, q odd, so that
   (n – 1 = 2k q);
2. Select a random integer a, 1 < a < n - 1;
3. if aq mod n = 1 then return("inconclusive");
4. for j = 0 to k - 1 do
5.   if $a^{2^j q}$ mod n = n - 1 then return("inconclusive");
6. return("composite");
```

Let us apply the test to the prime number $n = 29$. We have $(n - 1) = 28 = 2^2(7) = 2^k q$. First, let us try $a = 10$. We compute $10^7 \bmod 29 = 17$, which is neither 1 nor 28, so we continue the test. The next calculation finds that $(10^7)^2 \bmod 29 = 28$, and the test returns `inconclusive` (i.e., 29 may be prime). Let's try again with $a = 2$. We have the following calculations: $2^7 \bmod 29 = 12$; $2^{14} \bmod 29 = 28$; and the test again returns `inconclusive`. If we perform the test for all integers a in the range 1 through 28, we get the same `inconclusive` result, which is compatible with n being a prime number.

Now let us apply the test to the composite number $n = 13 \times 17 = 221$. Then $(n - 1) = 220 = 2^2(55) = 2^k q$. Let us try $a = 5$. Then we have $5^{55} \bmod 221 = 112$, which is neither 1 nor $220(5^{55})^2 \bmod 221 = 168$. Because we have used all values of j (i.e., $j = 0$ and $j = 1$) in line 4 of the TEST algorithm, the test returns `composite`, indicating that 221 is definitely a composite number. But suppose we had selected $a = 21$. Then we have $21^{55} \bmod 221 = 200$; $(21^{55})^2 \bmod 221 = 220$; and the test returns `inconclusive`, indicating that 221 may be prime. In fact, of the 218 integers from 2 through 219, four of these will return an inconclusive result, namely 21, 47, 174, and 200.

Repeated Use of the Miller–Rabin Algorithm How can we use the Miller–Rabin algorithm to determine with a high degree of confidence whether or not an integer is prime? It can be shown [KNUT98] that given an odd number n that is not prime and a randomly chosen integer, a with $1 < a < n - 1$, the probability that TEST will return `inconclusive` (i.e., fail to detect that n is not prime) is less than 1/4. Thus, if t different values of a are chosen, the probability that all of them will pass TEST (return inconclusive) for n is less than $(1/4)^t$. For example, for $t = 10$, the probability that a nonprime number will pass all ten tests is less than 10^{-6}. Thus, for a sufficiently large value of t , we can be confident that n is prime if Miller's test always returns `inconclusive`.

This gives us a basis for determining whether an odd integer n is prime with a reasonable degree of confidence. The procedure is as follows: Repeatedly invoke TEST (n) using randomly chosen values for a. If, at any point, TEST returns `composite`, then n is determined to be nonprime. If TEST continues to return `inconclusive` for t tests, then for a sufficiently large value of t, assume that n is prime.

A Deterministic Primality Algorithm

Prior to 2002, there was no known method of efficiently proving the primality of very large numbers. All of the algorithms in use, including the most popular (Miller–Rabin), produced a probabilistic result. In 2002 (announced in 2002, published in 2004), Agrawal, Kayal, and Saxena [AGRA04] developed a relatively simple deterministic algorithm that efficiently determines whether a given large number is a prime. The algorithm, known as the AKS algorithm, does not appear to be as efficient as the Miller–Rabin algorithm. Thus far, it has not supplanted this older, probabilistic technique.

Distribution of Primes

It is worth noting how many numbers are likely to be rejected before a prime number is found using the Miller–Rabin test, or any other test for primality. A result from number theory, known as the prime number theorem, states that the primes near n are spaced on the average one every ln (n) integers. Thus, on average, one would have to test on the order of $\ln(n)$ integers before a prime is found. Because all even integers can be immediately rejected, the correct figure is $0.5 \ln(n)$. For example, if a prime on the order of magnitude of 2^{200} were sought, then about $0.5 \ln(2^{200}) = 69$ trials would be needed to find a prime. However, this figure is just an average. In some places along the number line, primes are closely packed, and in other places there are large gaps.

The two consecutive odd integers 1,000,000,000,061 and 1,000,000,000,063 are both prime. On the other hand, $1001! + 2, 1001! + 3, \ldots, 1001! + 1000, 1001! + 1001$ is a sequence of 1000 consecutive composite integers.

2.7 THE CHINESE REMAINDER THEOREM

One of the most useful results of number theory is the **Chinese remainder theorem** (CRT).[10] In essence, the CRT says it is possible to reconstruct integers in a certain range from their residues modulo a set of pairwise relatively prime moduli.

> The 10 integers in Z_{10}, that is the integers 0 through 9, can be reconstructed from their two residues modulo 2 and 5 (the relatively prime factors of 10). Say the known residues of a decimal digit x are $r_2 = 0$ and $r_5 = 3$; that is, $x \bmod 2 = 0$ and $x \bmod 5 = 3$. Therefore, x is an even integer in Z_{10} whose remainder, on division by 5, is 3. The unique solution is $x = 8$.

The CRT can be stated in several ways. We present here a formulation that is most useful from the point of view of this text. An alternative formulation is explored in Problem 2.33. Let

$$M = \prod_{i=1}^{k} m_i$$

where the m_i are pairwise relatively prime; that is, $\gcd(m_i, m_j) = 1$ for $1 \le i, j \le k$, and $i \neq j$. We can represent any integer A in Z_M by a k-tuple whose elements are in Z_{m_i} using the following correspondence:

$$A \leftrightarrow (a_1, a_2, \ldots, a_k) \tag{2.15}$$

where $A \in Z_M$, $a_i \in Z_{m_i}$, and $a_i = A \bmod m_i$ for $1 \le i \le k$. The CRT makes two assertions.

1. The mapping of Equation (2.15) is a one-to-one correspondence (called a **bijection**) between Z_M and the Cartesian product $Z_{m_1} \times Z_{m_2} \times \ldots \times Z_{m_k}$. That is, for every integer A such that $0 \le A < M$, there is a unique k-tuple $(a_1, a_2, \ldots, a_k)$ with $0 \le a_i < m_i$ that represents it, and for every such k-tuple $(a_1, a_2, \ldots, a_k)$, there is a unique integer A in Z_M.
2. Operations performed on the elements of Z_M can be equivalently performed on the corresponding k-tuples by performing the operation independently in each coordinate position in the appropriate system.

Let us demonstrate the **first assertion**. The transformation from A to $(a_1, a_2, \ldots, a_k)$, is obviously unique; that is, each a_i is uniquely calculated as $a_i = A \bmod m_i$. Computing A from $(a_1, a_2, \ldots, a_k)$ can be done as follows. Let

[10] The CRT is so called because it is believed to have been discovered by the Chinese mathematician Sun-Tsu in around 100 A.D.

$M_i = M/m_i$ for $1 \leq i \leq k$. Note that $M_i = m_1 \times m_2 \times \ldots \times m_{i-1} \times m_{i+1} \times \ldots \times m_k$, so that $M_i \equiv 0 \pmod{m_j}$ for all $j \neq i$. Then let

$$c_i = M_i \times (M_i^{-1} \bmod m_i) \qquad \text{for } 1 \leq i \leq k \tag{2.16}$$

By the definition of M_i, it is relatively prime to m_i and therefore has a unique multiplicative inverse mod m_i. So Equation (2.16) is well defined and produces a unique value c_i. We can now compute

$$A \equiv \left(\sum_{i=1}^{k} a_i c_i \right) \pmod{M} \tag{2.17}$$

To show that the value of A produced by Equation (2.17) is correct, we must show that $a_i = A \bmod m_i$ for $1 \leq i \leq k$. Note that $c_j \equiv M_j \equiv 0 \pmod{m_i}$ if $j \neq i$, and that $c_i \equiv 1 \pmod{m_i}$. It follows that $a_i = A \bmod m_i$.

The **second assertion** of the CRT, concerning arithmetic operations, follows from the rules for modular arithmetic. That is, the second assertion can be stated as follows: If

$$A \leftrightarrow (a_1, a_2, \ldots, a_k)$$
$$B \leftrightarrow (b_1, b_2, \ldots, b_k)$$

then

$$(A + B) \bmod M \leftrightarrow ((a_1 + b_1) \bmod m_1, \ldots, (a_k + b_k) \bmod m_k)$$
$$(A - B) \bmod M \leftrightarrow ((a_1 - b_1) \bmod m_1, \ldots, (a_k - b_k) \bmod m_k)$$
$$(A \times B) \bmod M \leftrightarrow ((a_1 \times b_1) \bmod m_1, \ldots, (a_k \times b_k) \bmod m_k)$$

One of the useful features of the Chinese remainder theorem is that it provides a way to manipulate (potentially very large) numbers mod M in terms of tuples of smaller numbers. This can be useful when M is 150 digits or more. However, note that it is necessary to know beforehand the factorization of M.

To represent 973 mod 1813 as a pair of numbers mod 37 and 49, define

$$m_1 = 37$$
$$m_2 = 49$$
$$M = 1813$$
$$A = 973$$

We also have $M_1 = 49$ and $M_2 = 37$. Using the extended Euclidean algorithm, we compute $M_1^{-1} = 34 \bmod m_1$ and $M_2^{-1} = 4 \bmod m_2$. (Note that we only need to compute each M_i and each M_i^{-1} once.) Taking residues modulo 37 and 49, our representation of 973 is (11, 42), because 973 mod 37 = 11 and 973 mod 49 = 42.

Now suppose we want to add 678 to 973. What do we do to (11, 42)? First we compute $(678) \leftrightarrow (678 \bmod 37, 678 \bmod 49) = (12, 41)$. Then we add the tuples element-wise and reduce $(11 + 12 \bmod 37, 42 + 41 \bmod 49) = (23, 34)$. To verify that this has the correct effect, we compute

$$\begin{aligned}(23, 34) \leftrightarrow\ & a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} \bmod M \\ &= [(23)(49)(34) + (34)(37)(4)] \bmod 1813 \\ &= 43350 \bmod 1813 \\ &= 1651\end{aligned}$$

and check that it is equal to $(973 + 678) \bmod 1813 = 1651$. Remember that in the above derivation, M_i^{-1} is the multiplicative inverse of M_1 modulo m_1 and M_2^{-1} is the multiplicative inverse of M_2 modulo m_2.

Suppose we want to multiply 1651 (mod 1813) by 73. We multiply (23, 34) by 73 and reduce to get $(23 \times 73 \bmod 37, 34 \times 73 \bmod 49) = (14, 32)$. It is easily verified that

$$\begin{aligned}(14, 32) \leftrightarrow\ & [(14)(49)(34) + (32)(37)(4)] \bmod 1813 \\ &= 865 \\ &= 1651 \times 73 \bmod 1813\end{aligned}$$

2.8 DISCRETE LOGARITHMS

Discrete logarithms are fundamental to a number of public-key algorithms, including Diffie–Hellman key exchange and the digital signature algorithm (DSA). This section provides a brief overview of discrete logarithms. For the interested reader, more detailed developments of this topic can be found in [ORE67] and [LEVE90].

The Powers of an Integer, Modulo *n*

Recall from Euler's theorem [Equation (2.12)] that, for every a and n that are relatively prime,

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

where $\phi(n)$, Euler's totient function, is the number of positive integers less than n and relatively prime to n. Now consider the more general expression:

$$a^m \equiv 1 \pmod{n} \tag{2.18}$$

If a and n are relatively prime, then there is at least one integer m that satisfies Equation (2.18), namely, $m = \phi(n)$. The least positive exponent m for which Equation (2.18) holds is referred to in several ways:

- The order of a (mod n)
- The exponent to which a belongs (mod n)
- The length of the period generated by a

To see this last point, consider the powers of 7, modulo 19:

$$\begin{array}{lcl}
7^1 \equiv & & 7 \pmod{19} \\
7^2 = 49 = 2 \times 19 + 11 & \equiv & 11 \pmod{19} \\
7^3 = 343 = 18 \times 19 + 1 & \equiv & 1 \pmod{19} \\
7^4 = 2401 = 126 \times 19 + 7 & \equiv & 7 \pmod{19} \\
7^5 = 16807 = 884 \times 19 + 11 & \equiv & 11 \pmod{19}
\end{array}$$

There is no point in continuing because the sequence is repeating. This can be proven by noting that $7^3 \equiv 1(\text{mod } 19)$, and therefore, $7^{3+j} \equiv 7^3 7^j \equiv 7^j(\text{mod } 19)$, and hence, any two powers of 7 whose exponents differ by 3 (or a multiple of 3) are congruent to each other (mod 19). In other words, the sequence is periodic, and the length of the period is the smallest positive exponent m such that $7^m \equiv 1(\text{mod } 19)$.

Table 2.7 shows all the powers of a, modulo 19 for all positive $a < 19$. The length of the sequence for each base value is indicated by shading. Note the following:

1. All sequences end in 1. This is consistent with the reasoning of the preceding few paragraphs.
2. The length of a sequence divides $\phi(19) = 18$. That is, an integral number of sequences occur in each row of the table.
3. Some of the sequences are of length 18. In this case, it is said that the base integer a generates (via powers) the set of nonzero integers modulo 19. Each such integer is called a primitive root of the modulus 19.

More generally, we can say that the highest possible exponent to which a number can belong (mod n) is $\phi(n)$. If a number is of this order, it is referred to as a **primitive root** of n. The importance of this notion is that if a is a primitive root of n, then its powers

$$a, a^2, \ \ldots \ , a^{\phi(n)}$$

are distinct (mod n) and are all relatively prime to n. In particular, for a prime number p, if a is a primitive root of p, then

$$a, a^2, \ \ldots \ , a^{p-1}$$

are distinct (mod p). For the prime number 19, its primitive roots are 2, 3, 10, 13, 14, and 15.

Not all integers have primitive roots. In fact, the only integers with primitive roots are those of the form 2, 4, p^α, and $2p^\alpha$, where p is any odd prime and α is a positive integer. The proof is not simple but can be found in many number theory books, including [ORE76].

Table 2.7 Powers of Integers, Modulo 19

a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	a^{14}	a^{15}	a^{16}	a^{17}	a^{18}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	4	8	16	13	7	14	9	18	17	15	11	3	6	12	5	10	1
3	9	8	5	15	7	2	6	18	16	10	11	14	4	12	17	13	1
4	16	7	9	17	11	6	5	1	4	16	7	9	17	11	6	5	1
5	6	11	17	9	7	16	4	1	5	6	11	17	9	7	16	4	1
6	17	7	4	5	11	9	16	1	6	17	7	4	5	11	9	16	1
7	11	1	7	11	1	7	11	1	7	11	1	7	11	1	7	11	1
8	7	18	11	12	1	8	7	18	11	12	1	8	7	18	11	12	1
9	5	7	6	16	11	4	17	1	9	5	7	6	16	11	4	17	1
10	5	12	6	3	11	15	17	18	9	14	7	13	16	8	4	2	1
11	7	1	11	7	1	11	7	1	11	7	1	11	7	1	11	7	1
12	11	18	7	8	1	12	11	18	7	8	1	12	11	18	7	8	1
13	17	12	4	14	11	10	16	18	6	2	7	15	5	8	9	3	1
14	6	8	17	10	7	3	4	18	5	13	11	2	9	12	16	15	1
15	16	12	9	2	11	13	5	18	4	3	7	10	17	8	6	14	1
16	9	11	5	4	7	17	6	1	16	9	11	5	4	7	17	6	1
17	4	11	16	6	7	5	9	1	17	4	11	16	6	7	5	9	1
18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1	18	1

Logarithms for Modular Arithmetic

With ordinary positive real numbers, the logarithm function is the inverse of exponentiation. An analogous function exists for modular arithmetic.

Let us briefly review the properties of ordinary logarithms. The logarithm of a number is defined to be the power to which some positive base (except 1) must be raised in order to equal the number. That is, for base x and for a value y,

$$y = x^{\log_x(y)}$$

The properties of logarithms include

$$\log_x(1) = 0$$
$$\log_x(x) = 1$$

$$\log_x(yz) = log_x(y) + \log_x(z) \quad \textbf{(2.19)}$$

$$\log_x(y^r) = r \times \log_x(y) \quad \textbf{(2.20)}$$

Consider a primitive root a for some prime number p (the argument can be developed for nonprimes as well). Then we know that the powers of a from

1 through $(p - 1)$ produce each integer from 1 through $(p - 1)$ exactly once. We also know that any integer b satisfies

$$b \equiv r \pmod{p} \text{ for some } r, \text{ where } 0 \leq r \leq (p - 1)$$

by the definition of modular arithmetic. It follows that for any integer b and a primitive root a of prime number p, we can find a unique exponent i such that

$$b \equiv a^i \pmod{p} \quad \text{where } 0 \leq i \leq (p - 1)$$

This exponent i is referred to as the **discrete logarithm** of the number b for the base $a \pmod{p}$. We denote this value as $\text{dlog}_{a,p}(b)$.[11]

Note the following:

$$\text{dlog}_{a,p}(1) = 0 \text{ because } a^0 \bmod p = 1 \bmod p = 1 \tag{2.21}$$

$$\text{dlog}_{a,p}(a) = 1 \text{ because } a^1 \bmod p = a \tag{2.22}$$

Here is an example using a nonprime modulus, $n = 9$. Here $\phi(n) = 6$ and $a = 2$ is a primitive root. We compute the various powers of a and find

$$\begin{array}{ll} 2^0 = 1 & 2^4 \equiv 7 \pmod{9} \\ 2^1 = 2 & 2^5 \equiv 5 \pmod{9} \\ 2^2 = 4 & 2^6 \equiv 1 \pmod{9} \\ 2^3 = 8 & \end{array}$$

This gives us the following table of the numbers with given discrete logarithms (mod 9) for the root $a = 2$:

Logarithm	0	1	2	3	4	5
Number	1	2	4	8	7	5

To make it easy to obtain the discrete logarithms of a given number, we rearrange the table:

Number	1	2	4	5	7	8
Logarithm	0	1	2	5	4	3

Now consider

$$x = a^{\text{dlog}_{a,p}(x)} \bmod p \qquad y = a^{\text{dlog}_{a,p}(y)} \bmod p$$
$$xy = a^{\text{dlog}_{a,p}(xy)} \bmod p$$

[11] Many texts refer to the discrete logarithm as the **index**. There is no generally agreed notation for this concept, much less an agreed name.

Using the rules of modular multiplication,

$$\begin{aligned} xy \bmod p &= [(x \bmod p)(y \bmod p)] \bmod p \\ a^{\mathrm{dlog}_{a,p}(xy)} \bmod p &= [(a^{\mathrm{dlog}_{a,p}(x)} \bmod p)(a^{\mathrm{dlog}_{a,p}(y)} \bmod p)] \bmod p \\ &= (a^{\mathrm{dlog}_{a,p}(x) + \mathrm{dlog}_{a,p}(y)}) \bmod p \end{aligned}$$

But now consider Euler's theorem, which states that, for every a and n that are relatively prime,

$$a^{\phi(n)} \equiv 1 (\bmod n)$$

Any positive integer z can be expressed in the form $z = q + k\phi(n)$, with $0 \le q < \phi(n)$. Therefore, by Euler's theorem,

$$a^z \equiv a^q (\bmod n) \qquad \text{if } z \equiv q \bmod \phi(n)$$

Applying this to the foregoing equality, we have

$$\mathrm{dlog}_{a,p}(xy) \equiv [\mathrm{dlog}_{a,p}(x) + \mathrm{dlog}_{a,p}(y)](\bmod \phi(p))$$

and generalizing,

$$\mathrm{dlog}_{a,p}(y^r) \equiv [r \times \mathrm{dlog}_{a,p}(y)](\bmod \phi(p))$$

This demonstrates the analogy between true logarithms and discrete logarithms.

Keep in mind that unique discrete logarithms mod m to some base a exist only if a is a primitive root of m.

Table 2.8, which is directly derived from Table 2.7, shows the sets of discrete logarithms that can be defined for modulus 19.

Calculation of Discrete Logarithms

Consider the equation

$$y = g^x \bmod p$$

Given g, x, and p, it is a straightforward matter to calculate y. At the worst, we must perform x repeated multiplications, and algorithms exist for achieving greater efficiency (see Chapter 9).

However, given y, g, and p, it is, in general, very difficult to calculate x (take the discrete logarithm). The difficulty seems to be on the same order of magnitude as that of factoring primes required for RSA. At the time of this writing, the asymptotically fastest known algorithm for taking discrete logarithms modulo a prime number is on the order of [BETH91]:

$$e^{((\ln p)^{1/3}(\ln(\ln p))^{2/3})}$$

which is not feasible for large primes.

Table 2.8 Tables of Discrete Logarithms, Modulo 19

(a) Discrete logarithms to the base 2, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{2,19}(a)$	18	1	13	2	16	14	6	3	8	17	12	15	5	7	11	4	10	9

(b) Discrete logarithms to the base 3, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{3,19}(a)$	18	7	1	14	4	8	6	3	2	11	12	15	17	13	5	10	16	9

(c) Discrete logarithms to the base 10, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{10,19}(a)$	18	17	5	16	2	4	12	15	10	1	6	3	13	11	7	14	8	9

(d) Discrete logarithms to the base 13, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{13,19}(a)$	18	11	17	4	14	10	12	15	16	7	6	3	1	5	13	8	2	9

(e) Discrete logarithms to the base 14, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{14,19}(a)$	18	13	7	8	10	2	6	3	14	5	12	15	11	1	17	16	4	9

(f) Discrete logarithms to the base 15, modulo 19

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$log_{15,19}(a)$	18	5	11	10	8	16	12	15	4	13	6	3	7	17	1	2	14	9

2.9 KEY TERMS, REVIEW QUESTIONS, AND PROBLEMS

Key Terms

bijection
composite number
commutative
Chinese remainder theorem
discrete logarithm
divisor
Euclidean algorithm
Euler’s theorem
Euler’s totient function
Fermat’s theorem
greatest common divisor
identity element
index
modular arithmetic
modulus
order
prime number
primitive root
relatively prime
residue

Review Questions

2.1 What does it mean to say that b is a divisor of a?

2.2 What is the meaning of the expression *a divides b*?

2.3 What is the difference between modular arithmetic and ordinary arithmetic?

2.4 What is a prime number?

2.5 What is Euler's totient function?

2.6 The Miller–Rabin test can determine if a number is not prime but cannot determine if a number is prime. How can such an algorithm be used to test for primality?

2.7 What is a primitive root of a number?

2.8 What is the difference between an index and a discrete logarithm?

Problems

2.1 Reformulate Equation (2.1), removing the restriction that a is a nonnegative integer. That is, let a be any integer.

2.2 Draw a figure similar to Figure 2.1 for $a < 0$.

2.3 For each of the following equations, find an integer x that satisfies the equation.

a. $4x \equiv 2 \pmod 3$

b. $7x \equiv 4 \pmod 9$

c. $5x \equiv 3 \pmod{11}$

2.4 In this text, we assume that the modulus is a positive integer. But the definition of the expression $a \bmod n$ also makes perfect sense if n is negative. Determine the following:

a. $7 \bmod 4$

b. $7 \bmod -4$

c. $-7 \bmod 4$

d. $-7 \bmod -4$

2.5 A modulus of 0 does not fit the definition but is defined by convention as follows: $a \bmod 0 = a$. With this definition in mind, what does the following expression mean: $a \equiv b \pmod 0$?

2.6 In Section 2.3, we define the congruence relationship as follows: Two integers a and b are said to be congruent modulo n if $(a \bmod n) = (b \bmod n)$. We then proved that $a \equiv b \pmod n$ if $n|(a - b)$. Some texts on number theory use this latter relationship as the definition of congruence: Two integers a and b are said to be congruent modulo n if $n|(a - b)$. Using this latter definition as the starting point, prove that, if $(a \bmod n) = (b \bmod n)$, then n divides $(a - b)$.

2.7 What is the smallest positive integer that has exactly k divisors? Provide answers for values for $1 \leq k \leq 8$.

2.8 Prove the following:

a. $a \equiv b \pmod n$ implies $b \equiv a \pmod n$

b. $a \equiv b \pmod n$ and $b \equiv c \pmod n$ imply $a \equiv c \pmod n$

2.9 Prove the following:

a. $[(a \bmod n) - (b \bmod n)] \bmod n = (a - b) \bmod n$

b. $[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$

2.10 Find the multiplicative inverse of each nonzero element in Z_5.

2.11 Show that an integer N is congruent modulo 9 to the sum of its decimal digits. For example, $723 \equiv 7 + 2 + 3 \equiv 12 \equiv 1 + 2 \equiv 3 \pmod 9$. This is the basis for the familiar procedure of "casting out 9's" when checking computations in arithmetic.

2.12 a. Determine gcd(72345, 43215)
b. Determine gcd(3486, 10292)

2.13 The purpose of this problem is to set an upper bound on the number of iterations of the Euclidean algorithm.
a. Suppose that $m = qn + r$ with $q > 0$ and $0 \leq r < n$. Show that $m/2 > r$.
b. Let A_i be the value of A in the Euclidean algorithm after the ith iteration. Show that

$$A_{i+2} < \frac{A_i}{2}$$

c. Show that if m, n, and N are integers with $(1 \leq m, n, \ \leq 2^N)$, then the Euclidean algorithm takes at most $2N$ steps to find gcd(m, n).

2.14 The Euclidean algorithm has been known for over 2000 years and has always been a favorite among number theorists. After these many years, there is now a potential competitor, invented by J. Stein in 1961. Stein's algorithms is as follows: Determine gcd(A, B) with $A, B \geq 1$.

STEP 1 Set $A_1 = A, B_1 = B, C_1 = 1$

STEP 2 For $n > 1$, (1) If $A_n = B_n$, stop. gcd$(A, B) = A_n C_n$
(2) If A_n and B_n are both even, set $A_{n+1} = A_n/2, B_{n+1} = B_n/2, C_{n+1} = 2C_n$
(3) If A_n is even and B_n is odd, set $A_{n+1} = A_n/2, B_{n+1} = B_n, C_{n+1} = C_n$
(4) If A_n is odd and B_n is even, set $A_{n+1} = A_n, B_{n+1} = B_n/2, C_{n+1} = C_n$
(5) If A_n and B_n are both odd, set $A_{n+1} = |A_n - B_n|, B_{n+1} = \min(B_n, A_n), C_{n+1} = C_n$

Continue to step $n + 1$.
a. To get a feel for the two algorithms, compute gcd(6150, 704) using both the Euclidean and Stein's algorithm.
b. What is the apparent advantage of Stein's algorithm over the Euclidean algorithm?

2.15 a. Show that if Stein's algorithm does not stop before the nth step, then

$$C_{n+1} \times \gcd(A_{n+1}, B_{n+1}) = C_n \times \gcd(A_n, B_n)$$

b. Show that if the algorithm does not stop before step $(n - 1)$, then

$$A_{n+2}B_{n+2} \leq \frac{A_n B_n}{2}$$

c. Show that if $1 \leq A, B \leq 2^N$, then Stein's algorithm takes at most $4N$ steps to find gcd(m, n). Thus, Stein's algorithm works in roughly the same number of steps as the Euclidean algorithm.
d. Demonstrate that Stein's algorithm does indeed return gcd(A, B).

2.16 Using the extended Euclidean algorithm, find the multiplicative inverse of
a. 135 mod 61
b. 7465 mod 2464
c. 42828 mod 6407

2.17 The purpose of this problem is to determine how many prime numbers there are. Suppose there are a total of n prime numbers, and we list these in order: $p_1 = 2 < p_2 = 3 < p_3 = 5 < \ \ldots \ < p_n$.
a. Define $X = 1 + p_1 p_2 \ldots p_n$. That is, X is equal to one plus the product of all the primes. Can we find a prime number P_m that divides X?
b. What can you say about m?
c. Deduce that the total number of primes cannot be finite.
d. Show that $P_{n+1} \leq 1 + p_1 p_2 \ldots p_n$.

2.18 The purpose of this problem is to demonstrate that the probability that two random numbers are relatively prime is about 0.6.

a. Let $P = \Pr[\gcd(a, b) = 1]$. Show that $P = \Pr[\gcd(a, b) = d] = P/d^2$. *Hint:* Consider the quantity $\gcd\left(\frac{a}{d}, \frac{b}{d}\right)$.

b. The sum of the result of part (a) over all possible values of d is 1. That is $\Sigma^{d \geq 1}\Pr[\gcd(a, b) = d] = 1$. Use this equality to determine the value of P. *Hint:* Use the identity $\sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{6}$.

2.19 Why is $\gcd(n, n + 1) = 1$ for two consecutive integers n and $n + 1$?

2.20 Using Fermat's theorem, find 4^{225} mod 13.

2.21 Use Fermat's theorem to find a number a between 0 and 92 with a congruent to 7^{1013} modulo 93.

2.22 Use Fermat's theorem to find a number x between 0 and 37 with x^{73} congruent to 4 modulo 37. (You should not need to use any brute-force searching.)

2.23 Use Euler's theorem to find a number a between 0 and 9 such that a is congruent to 9^{101} modulo 10. (*Note*: This is the same as the last digit of the decimal expansion of 9^{100}.)

2.24 Use Euler's theorem to find a number x between 0 and 14 with x^{61} congruent to 7 modulo 15. (You should not need to use any brute-force searching.)

2.25 Notice in Table 2.6 that $\phi(n)$ is even for $n > 2$. This is true for all $n > 2$. Give a concise argument why this is so.

2.26 Prove the following: If p is prime, then $\phi(p^i) = p^i - p^{i-1}$. *Hint:* What numbers have a factor in common with p^i?

2.27 It can be shown (see any book on number theory) that if $\gcd(m, n) = 1$ then $\phi(mn) = \phi(m)\phi(n)$. Using this property, the property developed in the preceding problem, and the property that $\phi(p) = p - 1$ for p prime, it is straightforward to determine the value of $\phi(n)$ for any n. Determine the following:

a. $\phi(29)$ **b.** $\phi(51)$ **c.** $\phi(455)$ **d.** $\phi(616)$

2.28 It can also be shown that for arbitrary positive integer a, $\phi(a)$ is given by

$$\phi(a) = \prod_{i=1}^{t}[p_i^{a_i - 1}(p_i - 1)]$$

where a is given by Equation (2.9), namely: $a = P_1^{a_1}P_2^{a_2} \ldots P_t^{a_t}$. Demonstrate this result.

2.29 Consider the function: $f(n)$ = number of elements in the set $\{a: 0 \leq a < n$ and $\gcd(a, n) = 1\}$. What is this function?

2.30 Although ancient Chinese mathematicians did good work coming up with their remainder theorem, they did not always get it right. They had a test for primality. The test said that n is prime if and only if n divides $(2^n - 2)$.

a. Give an example that satisfies the condition using an odd prime.

b. The condition is obviously true for $n = 2$. Prove that the condition is true if n is an odd prime (proving the **if** condition).

c. Give an example of an odd n that is not prime and that does not satisfy the condition. You can do this with nonprime numbers up to a very large value. This misled the Chinese mathematicians into thinking that if the condition is true then n is prime.

d. Unfortunately, the ancient Chinese never tried $n = 341$, which is nonprime $(341 = 11 \times 31)$, yet 341 divides $2^{341} - 2$ without remainder. Demonstrate that $2341 \equiv 2 \pmod{341}$ (disproving the **only if** condition). *Hint:* It is not necessary to calculate 2^{341}; play around with the congruences instead.

2.31 Show that, if n is an odd composite integer, then the Miller–Rabin test will return `inconclusive` for $a = 1$ and $a = (n - 1)$.

2.32 If n is composite and passes the Miller–Rabin test for the base a, then n is called a *strong pseudoprime to the base a*. Show that 2047 is a strong pseudoprime to the base 2.

2.33 A common formulation of the Chinese remainder theorem (CRT) is as follows: Let $m_1, \ldots, m_k$ be integers that are pairwise relatively prime for $1 \leq i, j \leq k$, and $i \neq j$. Define M to be the product of all the m_i's. Let $a_1, \ldots, a_k$ be integers. Then the set of congruences:

$$\begin{gathered} x \equiv a_1(\text{mod } m_1) \\ x \equiv a_2(\text{mod } m_2) \\ \cdot \\ \cdot \\ \cdot \\ x \equiv a_k(\text{mod } m_k) \end{gathered}$$

has a unique solution modulo M. Show that the theorem stated in this form is true.

2.34 The example used by Sun-Tsu to illustrate the CRT was

$$x \equiv 2 \ (\text{mod } 3); x \equiv 3 \ (\text{mod } 5); x \equiv 2 \ (\text{mod } 7)$$

Solve for x.

2.35 Six professors begin courses on Monday, Tuesday, Wednesday, Thursday, Friday, and Saturday, respectively, and announce their intentions of lecturing at intervals of 3, 2, 5, 6, 1, and 4 days, respectively. The regulations of the university forbid Sunday lectures (so that a Sunday lecture must be omitted). When first will all six professors find themselves compelled to omit a lecture? *Hint:* Use the CRT.

2.36 Find all primitive roots of 37.

2.37 Given 5 as a primitive root of 23, construct a table of discrete logarithms, and use it to solve the following congruences.

a. $3x^5 \equiv 2 \ (\text{mod } 23)$
b. $7x^{10} + 1 \equiv 0 \ (\text{mod } 23)$
c. $5^x \equiv 6 \ (\text{mod } 23)$

Programming Problems

2.1 Write a computer program that implements fast exponentiation (successive squaring) modulo n.

2.2 Write a computer program that implements the Miller–Rabin algorithm for a user-specified n. The program should allow the user two choices: (1) specify a possible witness a to test using the Witness procedure or (2) specify a number s of random witnesses for the Miller–Rabin test to check.

APPENDIX 2A THE MEANING OF MOD

The operator mod is used in this book and in the literature in two different ways: as a binary operator and as a congruence relation. This appendix explains the distinction and precisely defines the notation used in this book regarding parentheses. This notation is common but, unfortunately, not universal.

The Binary Operator mod

If a is an integer and n is a positive integer, we define a mod n to be the remainder when a is divided by n. The integer n is called the **modulus**, and the remainder is called the **residue**. Thus, for any integer a, we can always write

$$a = \lfloor a/n \rfloor \times n + (a \bmod n)$$

Formally, we define the operator mod as

$$a \bmod n = a - \lfloor a/n \rfloor \times n \quad \text{for } n \neq 0$$

As a binary operation, mod takes two integer arguments and returns the remainder. For example, 7 mod 3 = 1. The arguments may be integers, integer variables, or integer variable expressions. For example, all of the following are valid, with the obvious meanings:

7 mod 3

7 mod m

x mod 3

x mod m

$(x^2 + y + 1) \bmod (2m + n)$

where all of the variables are integers. In each case, the left-hand term is divided by the right-hand term, and the resulting value is the remainder. Note that if either the left- or right-hand argument is an expression, the expression is parenthesized. The operator mod is not inside parentheses.

In fact, the mod operation also works if the two arguments are arbitrary real numbers, not just integers. In this book, we are concerned only with the integer operation.

The Congruence Relation mod

As a congruence relation, mod expresses that two arguments have the same remainder with respect to a given modulus. For example, $7 \equiv 4 \pmod 3$ expresses the fact that both 7 and 4 have a remainder of 1 when divided by 3. The following two expressions are equivalent:

$$a \equiv b \pmod m \qquad \Leftrightarrow \qquad a \bmod m = b \bmod m$$

Another way of expressing it is to say that the expression $a \equiv b \pmod m$ is the same as saying that $a - b$ is an integral multiple of m. Again, all the arguments may be integers, integer variables, or integer variable expressions. For example, all of the following are valid, with the obvious meanings:

$7 \equiv 4 \pmod 3$

$x \equiv y \pmod m$

$(x^2 + y + 1) \equiv (a + 1)(\text{mod } [m + n])$

where all of the variables are integers. Two conventions are used. The congruence sign is $\equiv$. The modulus for the relation is defined by placing the mod operator followed by the modulus in parentheses.