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Project In Mathematics (40cr)

Topological Groups

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Chapter 1

Introduction

Groups and Topologies are both fundamental and powerful algebraic structures in their own rights, helping us codify symmetry and continuity respectively. While the concepts behind the study of topological groups can be traced back to the Erlangen program by Felix Klein in 1872 and Sophus Lie's work on continuous groups, the first formal definition of abstract topological groups was published in 1926 by Józef Schreier. (Higgins 1974)

Within this project we will develop the properties and uses of topological groups, to begin with we will construct both groups and topologies from the ground up. Using this we will then construct topological groups as their own structure and develop their properties through definitions and examples, consequently we will also discuss a particular subset of topological groups, the matrix groups, and how they can be used to describe certain continuous transforms, and their relation to a particularly powerful type of topological group, the Lie group. Finally we will develop a theory of Locally Compact Abelian groups, leading up to proving a theorem, the Pontryagin Duality Theorem.

Chapter 2

Definitions and Nomenclature

2.1 Notations

Before we begin to formalise definitions I will give a brief run down of some of the notation I will be using. In general I will be using standard notation of Zermelo-Fraenkel set theory:

Notation	Meaning
$\{x : P(x)\}$	Set containing x with property $P(x)$
$\in$	Inclusion
$\emptyset$	The empty set
$\cup$	Union,$A \cup B = \{x : x \in A \text{ or } x \in B\}$
$\cap$	Intersection, $A \cap B = \{x : x \in A \text{ and } x \in B\}$
$\subseteq$	Subset
$\subset$	Proper subset

We will use

$$\bigcup_{\lambda \in \Lambda} u_\lambda = \{x : x \in u_\lambda \text{ for some } \lambda \in \Lambda\}$$

and

$$\bigcap_{\lambda \in \Lambda} u_\lambda = \{x : x \in u_\lambda \text{ for all } \lambda \in \Lambda\}$$

to denote the union and intersection of arbitrary families of sets indexed by an ordinal Λ,.

We will also be using standard logical symbols:

Notation	Meaning
$\forall$	Universal quantifier (or "For all")
$\exists$ $\exists!$	Existential quantifier (or "There exists") Unique existential quantifier
$\neg$	Negation
$\vee$ $\veebar$	Or (inclusive) Or (exclusive)
$\wedge$	And
$\implies$ $\impliedby$	Implies Implied by
$\iff$	Logical Equivalence (or "If and Only If"/"Iff")

We also will take the Axioms of ZF(C):

Axiom	Description
Extensionality	Given sets, A, B, $A = B$ iff A, B have the same elements
Empty Set	$\emptyset = \{\}$ is a set
Pairing	If x, y are sets,$\{x, y\}$ is a set
Union	If A is a set $\bigcup A$, the union of elements of the set, is a set
Powerset	If A is a set $\mathcal{P}(A)$, the set of all subsets of A, is a set
Separation	If A is a set $\{x : x \in A$ and $P(x)$, of members of A with the property P, is a set
Foundation	Every non-empty set A has a member $x \in A$ such that $x \cap A = \emptyset$
Replacement	If P is a property in two variables and A a set, if for all $x \in A$ there is at most one y such that $P(x, y)$ holds, then the collection $\{y : P(x, y)$ holds for some $x \in A\}$ is a set
Choice	For every set A of non-empty sets, there is a choice function f; namely, a function f such that $f(x) \in x$ for all $x \in A$

2.2 Definitions and Preliminary Theorems

Topological groups are the logical connection of Groups and Topologies. Topologies have more moving parts that will take a considerable amount of work to establish, so for this first section we will establish some preliminary notions of groups.

2.2.1 Groups

Definition 2.1. Truss (2019) A ***Group*** $G = (G, \circ)$ is a pair, containing a set which we usually denote G, along with a binary operation $\circ : G \times G \to G$, which meets the following axioms:

1. ***Closure:*** For any two elements $a, b \in G$, $a \circ b \in G$
2. ***Associativity:*** For all $a, b, c \in G$, we have $(a \circ b) \circ c = a \circ (b \circ c)$
3. ***Identity element:*** There exists a (unique) element in the group, usually denoted e, such that for all $a \in G, a \circ e = e \circ a = a$.
4. ***Inverse elements:*** For each element $a \in G$ there exists an element, usually denoted a^{-1}, such that $a \circ a^{-1} = a^{-1} \circ a = e$, the identity of the group.

A ***Subgroup*** of G is a subset H of G which becomes a group under the same operation $\circ$.

Remark. We have assumed that the identity element is unique. This is true for all groups. Let e, i be identity elements, then $ie = e$ as i is an identity, and $ie = i$ as e is an identity, so $e = i$.

NB: We will often use one of 2 types of notation, additive or product notation. Most often we will use product notation, in which we write $a \circ b$ simply as ab, we also will use a^n for $\underbrace{a \circ a \ldots \circ a}_{n \text{ times}}$, the identity element is written as 1 (or 1_G), and the inverse element written as either a^{-1}. In additive notation the identity element is usually written as 0 and the inverse element written as $-a$, when dealing with these inverses we also use the shorthand of $a - b = a + (-b)$.

Definition 2.2. Truss (2019) The ***Order*** of a group G, denoted $|G|$ is the number of elements in the group, i.e. the cardinality of the underlying set. The ***order*** of an element a of G, denoted $|a|$ is the the smallest non-zero positive integer n such that $a^n = 1_G$

Example 2.3. One example of a group would be $(\mathbb{Z}_6, +)$, that being the integers modulo 6 along with addition modulo 6 defined as normal. To check it's a group:

1. ***Closure:*** Closure is trivial by the definition of modulo arithmetic.
2. ***Associativity:*** As addition over the integers is associative, we know that for any two integers a, b we have $(a + b) + c = a + (b + c)$ for some integer c. Now since $(a + b) + c = q \mod 6$, that is $(a + b) + c = 6n + q$ for some $n \in \mathbb{N}$,so $(a + b) + c = 6n + q = a + (b + c)$ thus $(a + b) + c = a + (b + c) = q \mod 6$ and we have associativity.
3. ***Identity element:*** 0 is such an identity element by definition, $a + 0 = 0 + a = a \ \forall a \in \mathbb{Z}_6$.

4. ***Inverse elements:*** We can see the inverses of each element by the table below:

a	0	1	2	3	4	5
$-a$	0	5	4	3	2	1

Another interesting, infinite, group example would be the so called ***General Linear Group***, a group which we label $GL_n(\mathbb{R})$, the set is the set of the $n \times n$ real valued matrices with non-zero determinant, these matrices are also called ***invertible*** as they have a well defined inverse (**N.B.** this inverse is obtained using echelon form reduction on an augmented matrix, proof that every non-zero determinant matrix is invertible and that such an inverse is unique will not be covered here but will be assumed). The group operation would be matrix multiplication, if we once again check our axioms

1. ***Closure:*** We must check that given two invertible $n \times n$ matrices, A, B, their products are also invertible. so let $A, B \in GL_n(\mathbb{R})$, then we have that

$$AA^{-1} = I$$
$$BB^{-1} = I$$

and so given (AB) I suppose that $(AB)^{-1} = (B^{-1}A^{-1})$, to prove so we see that:

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= A(I)A^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

as well as:

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}(I)B \\ &= B^{-1}B \\ &= I\end{aligned}$$

and so given any two invertible matrices, their product is also invertible.

2. ***Associativity:*** Matrix multiplication is associative among matrices in general, hence it is

associative within the general linear group.

3. ***Identity element:*** The identity matrix (i.e. the diagonal matrix with all entries on the diagonal 1) is such an identity element.

4. ***Inverse elements:*** Every matrix in the general linear group has an inverse matrix, by definition of the set.

Definition 2.4. Truss (2019) We call a group G ***abelian*** if the group operation commutes, i.e. $\forall a, b \in G, a \circ b = b \circ a,$.

Remark. Of the above groups $(\mathbb{Z}_6, +)$ is abelian, but $GL_n(\mathbb{R})$ is notably only abelian for $n = 1$.

Definition 2.5. Truss (2019) A ***homomorphism*** $f : G \to H$, is a function between the underlying sets of 2 groups $G = (G, \circ), H = (H, *)$ such that for all $a, b \in G$:

$$f(a \circ b) = f(a) * f(b)$$

If a homomorphism is a bijection it is called an ***isomorphism***.

Remark 2.6. In general discussions of groups we will often equate groups "up to isomorphism" as if there is an isomorphism between two groups they are functionally just relabellings of one another. One example is the *Dihedral Group of order 6,* D_3, which represents the symmetries of a triangle (the identity, 2 rotations, and 3 reflections), which is the same **up to isomorphism** as the *Symmetric Group on 3 Elements*, S_3, which represents all permutations on the set $\{1, 2, 3\}$. These are not isomorphic, however, to $\mathbb{Z}_6$. We can check this by contradiction, let f be an isomorphism from D_3 to $\mathbb{Z}_6$, and let $a, b \in D_3$ be such that $a \circ b \neq b \circ a$, now since f is an isomorphism we have that

$$f(a \circ b) = f(a) \circ f(b)$$
$$f(b \circ a) = f(b) \circ f(a)$$

and since f is an isomorphism, and hence injective, we have that $f(a \circ b) \neq f(b \circ a)$, however as we remarked earlier, $\mathbb{Z}_6$ is abelian, so $f(a \circ b) = f(a) \circ f(b) = f(b) \circ f(a) = f(b \circ a)$, a contradiction.

Definition 2.7. Truss (2019) Given a group G with subgroup H and an element $a \in G$ we call the ***right coset of H in G***

$$Ha = \{a \circ g : g \in G\}$$

and the ***left coset of H in G***

$$aH = \{g \circ a : g \in G\}$$

Definition 2.8. Pontriagin (1939) Given a group G we call a subgroup H a ***normal subgroup of*** G if for all $g \in G$ and $h \in H$

$$ghg^{-1} \in H$$

If H is a normal subgroup of G we will write $H \trianglelefteq G$

Lemma 2.9. *Let G be a group and H be a subgroup of G, then the following are equivalent*

1. *H is a normal subgroup of G*

2. *For all $g, g' \in G$ we have if $x \in Hg$ and $y \in Hg'$ then $xy \in H(gg')$*

Proof. Suppose (1) holds, then let $x = hg$ and $y = h'g'$, then $xy = h(gh'g^{-1})gg' \in H(gg')$ as $h(gh'g^{-1}) \in H$. Now if (2) holds and $h \in H$ then $g^{-1} \in Hg^{-1}$ and $hg \in Hg$ so $g^{-1}hg \in H(g^{-1}g) = H(e) = H$ □

Definition 2.10. Truss (2019) If H is a normal subgroup of G we call the group with the set of cosets of H in G the ***quotient group of*** G ***by*** H and we equip it with the following group operation:

$$(Hg)(Hg') = H(gg')$$

we denote this group by G/H.

2.2.2 Metric Spaces

Before moving onto topological spaces, we will first establish another type of space, one in which we can measure distances between points.

Definition 2.11. (Farashahi 2020)A ***metric space*** (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}$ that satisfies the following properties:

1. **Positive Definiteness** $d(x, y) \geq 0; d(x, y) = 0 \iff x = y$

2. **Symmetry** $d(x, y) = d(y, x)$

3. **Triangle Inequality** $d(x, z) \leq d(x, y) + d(y, z)$

The function d is called the ***metric***.

Example 2.12. $\mathbb{R}^n$ is a metric space with the metric of euclidean distance:

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
$$d(\bar{x}, \bar{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $\bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ which we can check,

1. d is positive definite, as $\sqrt{(x_i - y_i)^2} \geq 0$ as $(x_i - y_i)^2$ is real and positive, and hence has a real, positive root. We also have that $\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0 \implies \sum_{i=1}^{n} (x - y)^2 = 0 \implies x_i = y_i \forall x_i \in \bar{x}$.

2. d is symmetric as $d(\bar{x}, \bar{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} (-(x_i - y_i))^2} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} = d(\bar{y}, \bar{x})$

3. Finally if we check the triangle inequality let $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^n$,

$$\begin{aligned}
(d(\bar{x}, \bar{z}))^2 &= \sum_{i=1}^{n} (x_i - z_i)^2 \\
&= \sum_{i=1}^{n} (x_i - y_i + y_i - z_i)^2 \\
&= \sum_{i=1}^{n} (x_i^2 + y_i^2 - 2x_i y_i) + (y_i^2 + z_i^2 - 2y_i z_i) + 2x_i y_i + 2y_i z_i - 2y_i^2 - 2x_i z_i \\
&= \sum_{i=1}^{n} (x_i^2 + y_i^2 - 2x_i y_i) + \sum_{i=1}^{n} (y_i^2 + z_i^2 - 2y_i z_i) + 2\sum_{i=1}^{n} y_i z_i - y_i^2 - x_i z_i \\
&= \sum_{i=1}^{n} (x_i - y_i)^2 + \sum_{i=1}^{n} (y_i - z_i)^2 + 2\sum_{i=1}^{n} (x_i - y_i)(y_i - z_i) \\
&\leq \sum_{i=1}^{n} (x_i - y_i)^2 + \sum_{i=1}^{n} (y_i - z_i)^2 + 2\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} \\
&= \left(\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2}\right)^2 \\
&= (d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}))^2 \\
\implies d(\bar{x}, \bar{z}) &\leq d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z})
\end{aligned}$$

Example 2.13. (Farashahi 2020)We can equip any set with a metric, in particular given a set S we can equip it with the ***discrete metric***

$$d : S \times S \to \mathbb{R}$$
$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

we will prove this is a metric (though it is mostly trivial):

1. Positive definiteness is trivial from definition
2. Symmetry is also trivial, if $x = y$ then $d(x, y) = 0 = d(y, x)$ and if $x \neq y$ $d(x, y) = 1 = d(y, x)$
3. The triangle inequality holds, if $x = y = z$ then $d(x, z) = d(x, y) + d(y, z) = 0$. If, WLOG, $x = y \neq z$ then $d(x, z) = 1, d(x, y) = 0, d(y, z) = 1 \implies d(x, z) = d(x, y) + d(y, z)$, and finally if $x \neq y \neq z$ then $d(x, z) = 1 < d(x, y) + d(y, z) = 2$ and so $d(x, z) \leq d(x, y) + d(y, z)$

Definition 2.14. (Farashahi 2020)A ***normed space*** $(V, \|\cdot\|)$ is a real vector space V equipped with a map, called the ***norm***, $\|\cdot\| : V \to \mathbb{R}$ satisfying

1. $\|\bar{v}\| \geq 0, \text{ and } \|\bar{v}\| = 0 \iff \bar{v} = \bar{0}$
2. $\|\lambda \bar{v}\| = |\lambda| \, \|\bar{v}\| \, \forall \lambda \in \mathbb{R}$
3. $\|\bar{v} + \bar{w}\| \leq \|\bar{v}\| + \|\bar{w}\|$

Problem 2.15. (Farashahi 2020)Let V be a normed space, prove that V is a metric space with metric

$$d(\bar{v}, \bar{w}) := \|\bar{v} - \bar{w}\|$$

Solution. We check the axioms of a metric:

1. Positive definiteness is guaranteed by the definition of the norm.
2. For symmetry, let $\bar{v}, \bar{w} \in V$, we have that

$$\begin{aligned} d(\bar{v}, \bar{w}) &= \|\bar{v} - \bar{w}\| \\ &= \|(-1)(\bar{w} - \bar{v})\| \\ &= |-1| \, \|\bar{w} - \bar{v}\| \\ &= \|\bar{w} - \bar{v}\| \\ &= d(\bar{w}, \bar{v}) \end{aligned}$$

3. For the triangle inequality, let $\bar{u}, \bar{v}, \bar{w} \in V$, we have that

$$\begin{aligned} d(\bar{u}, \bar{w}) &= \|\bar{u} - \bar{w}\| \\ &= \|\bar{u} - \bar{v} + \bar{v} - \bar{w}\| \\ &\leq \|\bar{u} - \bar{v}\| + \|\bar{v} - \bar{w}\| \\ &= d(\bar{u}, \bar{v}) + d(\bar{v}, \bar{w}) \end{aligned}$$

Hence $d(\bar{u}, \bar{w})$ is a metric, owing to its origin we call this the ***metric induced by a norm***.

Definition 2.16. (Farashahi 2020)An ***inner product space*** $(V, \langle\cdot,\cdot\rangle)$ is a real vector space V with a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ satisfying

1. $\langle \lambda\bar{v}, \bar{w}\rangle = \lambda\langle \bar{v}, \bar{w}\rangle$

2. $\langle \bar{v}_1 + \bar{v}_2, \bar{w}\rangle = \langle \bar{v}_1, \bar{w}\rangle + \langle \bar{v}_2, \bar{w}\rangle$

3. $\langle \bar{v}, \bar{w}\rangle = \langle \bar{w}, \bar{v}\rangle$

4. $\langle \bar{v}, \bar{v}\rangle \geq 0$, and $\langle \bar{v}, \bar{v}\rangle = 0 \iff \bar{v} = \bar{0}$

Problem 2.17. (Farashahi 2020) Let $(V, \langle\cdot,\cdot\rangle)$, prove that this is a normed space with norm

$$\|\bar{v}\| := \langle \bar{v}, \bar{v}\rangle^{\frac{1}{2}}$$

Solution. We check the axioms of a norm:

1. This is guaranteed by the definition of the inner product, as $\langle \bar{v}, \bar{v}\rangle$ is real and positive, hence has a real positive root, and is only 0 at $\bar{v} = \bar{0}$

2. We have that $\|\lambda\bar{v}\| = \langle \lambda\bar{v}, \lambda\bar{v}\rangle^{\frac{1}{2}} = (\lambda^2\langle \bar{v}, \bar{v}\rangle)^{\frac{1}{2}} = |\lambda|\langle \bar{v}, \bar{v}\rangle^{\frac{1}{2}} = |\lambda|\,\|\bar{v}\|$as required.

3. now, to satisfy the final axiom we will need the ***Cauchy-Schwartz inequality***:

$$|\langle \bar{v}, \bar{w}\rangle|^2 \leq \langle \bar{v}, \bar{v}\rangle\langle \bar{w}, \bar{w}\rangle$$

and so, to begin let us start by considering:

$$\begin{aligned}
\langle \bar{v} + \lambda \bar{w}, \bar{v} + \lambda \bar{w} \rangle &\geq 0 \\
\langle \bar{v}, \bar{v} + \lambda \bar{w} \rangle + \langle \lambda \bar{w}, \bar{v} + \lambda \bar{w} \rangle &\geq 0 \\
\langle \bar{v}, \bar{v} \rangle + \langle \bar{v}, \lambda \bar{w} \rangle + \langle \lambda \bar{w}, \bar{v} \rangle + \langle \lambda \bar{w}, \lambda \bar{w} \rangle &\geq 0 \\
\langle \bar{v}, \bar{v} \rangle + 2\lambda \langle \bar{v}, \bar{w} \rangle + \lambda^2 \langle \bar{w}, \bar{w} \rangle &\geq 0
\end{aligned}$$

now, if $\bar{w} \neq \bar{0}$ this is a polynomial of degree 2 in λ, and since it is always non-negative we have that the discriminant is 0 or negative, that is

$$\begin{aligned}
4\langle \bar{v}, \bar{w} \rangle^2 - 4(\langle \bar{v}, \bar{v} \rangle \langle \bar{w}, \bar{w} \rangle) \leq 0 \\
\langle \bar{v}, \bar{w} \rangle^2 \leq \langle \bar{v}, \bar{v} \rangle \langle \bar{w}, \bar{w} \rangle
\end{aligned}$$

and so the Cauchy-Schwartz inequality holds for this case, now if $\bar{w} = \bar{0}$ then we our statement of the inequality will be:

$$|\langle \bar{v}, \bar{0} \rangle|^2 \leq 0$$

now as $|\langle \bar{v}, \bar{0} \rangle|^2 \geq 0$ we would need to have $|\langle \bar{v}, \bar{0} \rangle|^2 = 0 \implies \langle \bar{v}, \bar{0} \rangle = 0$ which follows as

$$\begin{aligned}
\langle \bar{v}, 0\bar{0} \rangle &= 0\langle \bar{v}, \bar{0} \rangle \\
&= 0
\end{aligned}$$

so the inequality holds, and we can proceed with the final axiom of the norm:

$$\begin{aligned}
\|\bar{v} + \bar{w}\| &= \langle \bar{v} + \bar{w}, \bar{v} + \bar{w} \rangle^{\frac{1}{2}} \\
\|\bar{v} + \bar{w}\|^2 &= \langle \bar{v} + \bar{w}, \bar{v} + \bar{w} \rangle \\
&= \langle \bar{v}, \bar{v} + \bar{w} \rangle + \langle \bar{w}, \bar{v} + \bar{w} \rangle \\
&= \langle \bar{v}, \bar{v} \rangle + \langle \bar{v}, \bar{w} \rangle + \langle \bar{w}, \bar{v} \rangle + \langle \bar{w}, \bar{w} \rangle \\
&= \langle \bar{v}, \bar{v} \rangle + 2\langle \bar{v}, \bar{w} \rangle + \langle \bar{w}, \bar{w} \rangle \\
&\leq \langle \bar{v}, \bar{v} \rangle + 2(\langle \bar{v}, \bar{v} \rangle \langle \bar{w}, \bar{w} \rangle)^{\frac{1}{2}} + \langle \bar{w}, \bar{w} \rangle \\
&= (\langle \bar{v}, \bar{v} \rangle^{\frac{1}{2}} + \langle \bar{w}, \bar{w} \rangle^{\frac{1}{2}})^2 \\
&= (\|\bar{v}\| + \|\bar{w}\|)^2 \\
\implies \|\bar{v} + \bar{w}\| &\leq \|\bar{v}\| + \|\bar{w}\|
\end{aligned}$$

and we're done. This norm is called the ***norm induced by the inner product***. With this and the result from Problem 2.13 we have that any inner product space is also a metric space.

Definition 2.18. (Farashahi 2020) A sequence (x_n) in a metric space (X, d) ***converges*** to $x \in X$ if given $\epsilon > 0$ there exists a natural number N such that $d(x_n, x) < \epsilon$ whenever $n \geq N$. A sequence is ***Cauchy*** if for any $\epsilon > 0$ there is an N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 2.19. (Farashahi 2020) A metric space (X, d) is ***complete*** if every Cauchy sequence in X converges to a limit in X. A normed space that is complete as a metric space is called a ***Banach space***, an inner product space that is complete as a metric space is called a ***Hilbert Space***.

Chapter 3

Topological Spaces

3.1 Basic Definitions, Examples, and Theorems

Definition 3.1. Martin (2021) A ***Topological Space*** (X, τ) is a pair of a set X along with a subset τ of $\mathcal{P}(X)$, called a ***topology,*** a topology must meet the following axioms:

1. $\emptyset, X \in \tau$
2. Given any family of subsets in τ, $(U_\lambda)_{\lambda \in \Lambda}$, the union $\bigcup_{\lambda \in \Lambda} U_\lambda$ is also in τ.
3. Given any finite family of subsets in τ, $(U_\lambda)_{\lambda \in \Lambda}$, the intersection $\bigcap_{\lambda \in \Lambda} U_\lambda$ is also in τ.

the elements of τ are called ***open subsets*** of X and a subset of X is called ***closed*** if it's complement is an open subset of X. **N.B.** that a set can be both open and closed, take for example X (the set in it's entirety), by definition it must be open but also the complement of $X, \emptyset$, is also open. Hence it is both open and closed.

Definition 3.2. Martin (2021) Given a topological space X, and $x \in X$. A ***neighbourhood*** U of x is an open set of X with $x \in U$.

Definition 3.3. Martin (2021) Let X be a topological space and let A be a subset of X. The ***closure*** $\overline{A}$ of A is the intersection of all closed sets containing A. The ***interior*** of A is the union of all subsets of A that are open in X. A subset A of X is called ***dense in*** X if $\overline{A} = X$.

Corollary 3.4. *Martin (2021) A set A is closed iff $A = \overline{A}$.*

Proof. We prove this in two steps, first the forwards case that A closed $\implies A = \overline{A}$. Now if A is itself closed then any A is in the family of closed sets containing A, and hence $\overline{A} = A \cap (\bigcap_{\lambda \in \Lambda}(C_\lambda))$ where $A \subseteq C_\lambda$ for all $\lambda \in \Lambda$, and so $\overline{A} = A$ as required.

Now we show the backwards case A closed $\impliedby A = \overline{A}$. Now this proof will rely on De Morgan's Law for Intersections:

$$X \setminus \bigcap_{\lambda \in \Lambda} C_\lambda = \bigcap_{\lambda \in \Lambda} (X \setminus C_\lambda)$$

which we will take a brief aside to prove.

$$\begin{aligned} x \in X \setminus \bigcap_{\lambda \in \Lambda} C_\lambda &\iff x \in X \text{ and } x \notin \bigcap_{\lambda \in \Lambda} C_\lambda \\ &\iff x \in X \text{ and } \forall \lambda \in \Lambda, x \notin C_\lambda \\ &\iff \forall \lambda \in \Lambda, x \in X \setminus C_\lambda \\ &\iff x \in \bigcap_{\lambda \in \Lambda} (X \setminus C_\lambda) \end{aligned}$$

and so from this we can get that for a family of closed subsets (C_λ) their intersection will also be closed as by definition $X \setminus C_\lambda$ will be open, and since the intersection of open subsets is open we have that $\bigcap_{\lambda \in \Lambda}(X \setminus C_\lambda)$ is open hence ,by De Morgans Law, $X \setminus \bigcap_{\lambda \in \Lambda} C_\lambda$ is also open, and so the complement of this, $\bigcap_{\lambda \in \Lambda} C_\lambda$ must be closed, hence if $A = \overline{A}$ then it must be closed, and we're done.s □

Definition 3.5. Martin (2021)Let X be a topological space and let A be a subset of X. A point $x \in X$ is called a ***limit point*** of A if for every neighbourhood U of x, $U \cap A \neq \emptyset$

Lemma 3.6. *Martin (2021)Let X be a topological space, and let A be a subset of X. Then the closure of A is precisely the set of all limit points of A*

Proof. In order to do this will take two steps, first we shall show that $\overline{A} \subseteq \{x : x \text{ is a limit point of } A\}$. Let $x \in \overline{A}$ then for any closed set $A \subseteq C$ we have $x \in C$, now suppose towards a contradiction that x is not a limit point of A. Then there exists a neighbourhood U of x such that $U \cap A = \emptyset$. Then $K = X \setminus U$ is closed and $A \subset K$, however as $x \in U$ then $x \notin K$, contradicting $x \in \overline{A}$ and so any $x \in \overline{A}$ is a limit point of A.

Next we will show that $\{x : x \text{ is a limit point of } A\} \subseteq \overline{A}$, so let x be a limit point of A, and suppose towards a contradiction that $x \notin \overline{A}$, then there exists a closed set C such that $A \subset C$ and $x \notin C$. Therefor $K = X \setminus C$ is an open set containing x, hence $K \cap A = \emptyset$ contradicting x being a limit point of A, hence $\{x : x \text{ is a limit point of } A\} \subseteq \overline{A}$ and thus, together with the previous step $\{x : x \text{ is a limit point of } A\} = \overline{A}$ as required. □

Definition 3.7. Martin (2021) Let X be a topological space, let $x \in X$ and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence of points in X. The sequence x_n is said to ***converge to*** x and x is called a ***limit*** of the sequence x_n if for every neighbourhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. We write $x_n \to x$

Lemma 3.8. *Martin (2021) If C is a closed set in a topological space, then we have that for all convergent sequences $\{x_n\} \subset C$ such that $x_n \to x$ we have that $x \in C$.*

Proof. Let $\{x_n\} \subset C$ such that $x_n \to x \in X$, then x is a limit point of C since for all neighbourhoods U of x there exists some N such that $n \geq N$ implies $x_n \in U$, i.e. $U \cap C \neq \emptyset$ for all such U. □

Example 3.9. The most commonly discussed topological space would be that of "Euclidean n-Space" which is the set $\mathbb{R}^n$ equipped with the topology formed from the unions of open balls $B_r(p)$:

$$B_r(p) = \{x \in \mathbb{R}^n : d(p, x) < r\}$$

for $r > 0, p \in \mathbb{R}^n$. This topology is called either the ***usual topology,*** the ***euclidean topology,*** or the ***metric topology***. The last of these comes from the use of the metric on $\mathbb{R}^n$, in fact we can define this topology on any metric space, hence they all become a topological space. Interestingly for us, we can also use this to give us a topology to equip to $M_n(\mathbb{R})$, the set of all real-valued $n \times n$ matrices. We can view $M_n(\mathbb{R})$ as simply another way of writing an element of $\mathbb{R}^{n^2}$, for example if we have $v \in \mathbb{R}^{n^2} = (v_1, v_2, \ldots v_{n^2})$ then we can rewrite this as a matrix A with entry (i, j):

$$(A)_{ij} = v_{n(i-1)+j}$$

and so we can equip it with a similar (if relabeled) topology and make it a topological space.

All sets permit at least two topologies, so these are given the names of the ***discrete topology*** which is $\mathcal{P}(S)$ and the ***trivial topology*** which is $\{\emptyset, S\}$. These are respectively the largest and smallest possible topologies. So we can, for example, take the set $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and equip it with the trivial topology and it would become a topological space, however these spaces are not too useful or interesting to discuss, so in general we will have a more complex topology.

Corollary 3.10. *Martin (2021) Let (X, d) be any metric space with metric d becomes a topological space when equipped with the topology where a set is open if it can be expressed as the union of open balls:*

$$B_r(x) = \{y \in X : d(x, y) < r\}$$

where $x \in X$ and $r > 0$ is real.

Proof. We will check the axioms:

1. If we fix some $m \in \mathbb{R}$ then the union of the family open balls:

$$\{B_m(p) : p \in X\}$$

   is the whole set X, and if we take any two disjoint open balls their union will be empty, so $\emptyset, X \in \tau_d$

2. Let $(U_\lambda)_{\lambda\in\Lambda}$ be a family of open subsets, then we have that for each U_λ there is some family of open balls $(B_\gamma)_{\gamma\in\Gamma_\lambda}$ whose union equals U_λ,so we have that

$$\bigcup_{\lambda\in\Lambda} U_\lambda = \bigcup_{\lambda\in\lambda}\left(\bigcup_{\gamma\in\Gamma_\lambda} B_\gamma\right)_\lambda$$

which is a union of open balls, and hence $\bigcup_{\lambda\in\Lambda} U_\lambda$ is open.

3. For this it suffices to prove that the intersection of any two open subsets is open, and then by induction we will have that any intersection of an arbitrary finite family is open. So let U, V be open. Now we have two cases, if $U \cap V = \emptyset$ then this is trivially open, so we are only interested in the case that $U \cap V \neq \emptyset$. Let U, V be open with $U \cap V \neq \emptyset$, so let $x \in U \cap V$, now we can find open balls $B_{\epsilon_1}(x) \subseteq U, B_{\epsilon_2}(x) \subseteq V$ such that if we take $\epsilon = \min(\epsilon_1, \epsilon_2)$ we have that $B_\epsilon(x) \subseteq U \cap V$, so since we can find an open ball containing each point of $U \cap V$ which is contained entirely within $U \cap V$ that implies that $U \cap V$ is the union of all such open balls, hence by definition of the metric topology it is open.

□

Definition 3.11. Martin (2021) Given two topologies τ, π of a set S, we say that τ is ***coarser*** than π if $\tau \subseteq \pi$, equivalently we can say that π is ***finer*** than τ. From this we can see that the discrete topology is finer than any other topology and the trivial topology is coarser than any other topology.

Definition 3.12. Martin (2021) A topological space X is called ***separated*** or ***Hausdorff*** if for each pair of x, y of distinct points of X there exists disjoint neighbourhoods U, V such that $x \in U, y \in V$.

Problem 3.13. Prove that $\mathbb{R}^n$ with the usual topology is separated.

Solution: Let x, y be distinct points in $\mathbb{R}^n$, thus $d(x, y) > 0$. So define $\epsilon = \frac{d(x,y)}{2}$. Now we can define the open balls:

$$B_x = B_\epsilon(x) = \{a \in \mathbb{R}^n : d(x, a) < \epsilon\}$$
$$B_y = B_\epsilon(y) = \{a \in \mathbb{R}^n : d(y, a) < \epsilon\}$$

Now $x \in B_x$, $y \in B_y$ is trivially true, so we must now show they are disjoint. Towards a contradiction, assume $B_x \cap B_y \neq \emptyset$ and let $z \in B_x \cap B_y$. Using the triangle inequality we see that:

$$\begin{aligned} & d(x,y) \leq d(x,z) + d(z,y) < \epsilon + \epsilon = d(x,y) \\ \implies & d(x,y) < d(x,y) \end{aligned}$$

which is a contradiction. Hence $B_x \cap B_y$ is empty, so they are disjoint sets and so $\mathbb{R}^n$ with the usual topology is separated. N.B. This is actually true of all metric spaces equipped with their respective metric topology. $\square$

Definition 3.14. Martin (2021) Let X be a topological space. If there are two non-empty disjoint open subsets U, V such that $X = U \cap V$, we say that U and V ***partition*** X. We call X ***disconnected*** if it admits a partition and ***connected*** if it does not.

3.2 Generating Topologies

Definition 3.15. Martin (2021) Let X be any set. A ***basis for a topology*** on X is a collection β of subsets of X such that:

1. For each $x \in X$ there is at least one $B \in \beta$ such that $x \in B$
2. If $B_1, B_2 \in \beta$ and $x \in B_1 \cap B_2$ then there exists $B_3 \in \beta$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$

The sets $B_i \in \beta$ are called ***basis elements***. A collection β_p of neighbourhoods of a point p is called a ***neighbourhood base at*** p if every neighbourhood of p contains a member of β_p

Definition 3.16. Martin (2021) Let β be a basis for a topology on a set X. The ***topology generated by*** β, τ is

$$\tau = \left\{ U \subset X : \exists \left\{ B_\lambda \right\}_{\lambda \in \Lambda} \subset \beta \quad s.t. \quad U = \cup_\lambda B_\lambda \right\}$$

which, in English, can be said as "The subsets U of X such that there exists a family of basis elements whose union equals U."

This leads us to being able to define a powerful topology. Given two spaces $(X, \tau), (Y, \pi)$ we define:

$$\beta_{\tau\pi} = \{U \times V : U \in \tau, V \in \pi\}$$

which is a basis, we call the topology generated by this basis the ***product topology*** and can apply it to the space $X \times Y$

Definition 3.17. Martin (2021) Given a topological space (X, τ), we can equip a take a subset $X' \subset X$ and equip it with the following topology:

$$\tau' = \{t \cap X' : t \in \tau\}$$

which we call the ***subspace topology***. We can check that this is a valid topology for X':

1. As $\emptyset \cap X' = \emptyset$ and $X \cap X' = X'$ we have the first axiom satisfied

2. We know that any given family of subsets $(t_\lambda)_{\lambda \in \Lambda}$ their union will also be in τ, so that is $\bigcup_{\lambda \in \Lambda} t_\lambda = t \in \tau$, as such $\bigcup_{\lambda \in \Lambda}(t_\lambda \cap X') = t \cap X' \in \tau'$

3. Similarly we know that any given a finite family of subsets $(t_\lambda)_{\lambda \in \Lambda}$ their intersection will also be in τ, so that is $\bigcap_{\lambda \in \Lambda} t_\lambda = t \in \tau$, as such $\bigcap_{\lambda \in \Lambda}(t_\lambda \cap X') = t \cap X' \in \tau'$

Lemma 3.18. *Martin (2021) Let (X, τ) be a topological space and Y be any set, let $f : X \to Y$ be any function. Then*

$$\tau_f := \{U \subset Y : f^{-1}(U) \in \tau\}$$

is a topology on Y.

Proof. Let us check the axioms:

1. We have that $f^{-1}(Y) = X \in \tau$ and $f^{-1}(\emptyset) = \emptyset \in \tau$, so $\emptyset, Y \in \tau_f$

2. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a family of open subsets in τ_f, then $f^{-1}(\bigcup_{\lambda \in \Lambda} U_\lambda) = \bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda) \in \tau$, so $\bigcup_{\lambda \in \Lambda} U_\lambda \in \tau_f$.

3. It suffices to show that for $U \cap V \in \tau_f$ for $U, V \in \tau_f$, as if that is true then we can inductively take the union of any finite number of intersections. So let $U, V \in \tau_f$ then $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \in \tau$, hence $U \cap V \in \tau_f$

□

and we're done. This topology is usually called the ***final topology of*** $f : X \to Y$.

Definition 3.19. Martin (2021) Given a topological space $X = (X, \tau)$ and an equivalence relation $\sim$ on X, the set of all equivalence classes is denoted by $X/\sim$, then the projection map

$$q : X \to X/\sim$$
$$x \mapsto [x]$$

defines a topology on $X/\sim$ called the ***quotient topology in*** $X/\sim$:

$$\tau_\sim = \{q(U) : U \in \tau\}$$

3.3 Continuity

Definition 3.20. Martin (2021) Let X and Y be two topological spaces and let $f : X \to Y$ be a function. The function f is called ***continuous*** if for every open set $A \subset Y$, $f^{-1}(A)$ is an open set of X where

$$f^{-1}(A) = \{x \in X : f(x) \in A\}$$

Definition 3.21. Martin (2021) If $f : X \to Y$ where X, Y are metric spaces with metric d, d' respectively, then a function is ***metric continuous*** at a point $x \in X$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that:

$$B_\delta(x) \subseteq f^{-1}(B'_\epsilon(f(x)))$$

where $B_\delta(x)$ is an open ball with the metric from X and $B'_\epsilon(x)$ is an open ball with metric from Y. If f is metric continuous at all $x \in X$ then we say that f is metric continuous.

Lemma 3.22. *Martin (2021) Let $f : X \to Y$ be a function between topological spaces with metric topologies, then f is metric continuous if and only if f is continuous.*

Proof. Suppose $f : X \to Y$ is a function between topological spaces with metric topologies. To begin with we demonstrate the the backwards case f metric continuous $\Longleftarrow$ f continuous. So let f be continuous. Then for all $\epsilon > 0$ we have $f^{-1}(B'_\epsilon(f(x)))$ is open as $B'_\epsilon(f(x))$ is open in Y. As $x \in f^{-1}(B'_\epsilon(f(x)))$ then there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B'_\epsilon(f(x)))$ as all open sets in metric topologies are the union of open balls of their elements, then we have that f is metric continuous.

Now the forwards case f metric continuous $\implies$ f continuous. Let f be metric continuous and let U be open in Y, choose $x \in f^{-1}(U)$, that is $f(x) \in U$. Since U is open there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset U$ and since f is metric continuous there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x))) \subset f^{-1}(U)$, and since there is a $B_\delta(x)$ for each $x \in f^{-1}(U)$ and $B_\delta(x) \subset f^{-1}(U)$ is open in X we can view $f^{-1}(U)$ as the union of all such $B_\delta(x)$, and since the union of open sets is itself open, we have that $f^{-1}(U)$ is open, hence f is continuous and we are done. $\square$

Lemma 3.23. *Martin (2021) If $f : X \to Y, g : Y \to Z$ is continuous then the composition of* *those functions*

$$\begin{aligned} g \circ f &: X \to Z \\ (g \circ f)(x) &= g(f(x)) \end{aligned}$$

is continuous.

Proof. Let A be an open subset of Z, then

$$\begin{aligned} (g \circ f)^{-1}(A) &= (g(f(A))^{-1} \\ &= f^{-1}(g^{-1}(A)) \end{aligned}$$

and we know, as f and g are continuous that $g^{-1}(A)$ is an open subset of Y, and then $f^{-1}(g^{-1}(A))$ is an open subset of X, hence for all open subsets of $Z, (g \circ f)^{-1}(A)$ is open in X and we're done. □

Definition 3.24. Martin (2021) Let X and Y be two topological spaces and let $f : X \to Y$. Then f is called a ***homeomorphism*** if it is bijective and both f and f^{-1} are continuous. In this case we say X is ***homeomorphic*** to Y, denoted by $X \cong Y$.

Theorem 3.25. *All real polynomials are continuous on* $\mathbb{R}$ *with the usual topology*

Proof. First we note that the identity map

$$\begin{aligned} I &: X \to X \\ x &\mapsto x \end{aligned}$$

is trivially continuous as $I^{-1}(U) = U$ where U is open in X and constant maps:

$$\begin{aligned} c &: X \to Y \\ x &\to c \end{aligned}$$

for some $c \in Y$ are also continuous as for $U \in \tau_Y$:

$$c^{-1}(U) = \begin{cases} \emptyset & c \notin U \\ X & c \in U \end{cases}$$

which are open by definition.

We will also need to show that given continuous functions, $f : X \to Y, g : A \to B$ that the Cartesian product

$$\begin{aligned} f \times g : X \times A \to Y \times B \\ (f \times g)(x, a) = (f(x), g(a)) \end{aligned}$$

is continuous where $X \times A$ and $Y \times B$ have the product topology, so let $U \times D$ be open in $Y \times B$, that is that U is open in Y and D is open in B, and so

$$(f \times g)^{-1}(U, D) = \left(f^{-1}(U), g^{-1}(D)\right)$$

and since f, g are continuous $f^{-1}(U)$ is open in X and $g^{-1}(D)$ is open in A and so $(f^{-1}(U), g^{-1}(D))$ is open in $X \times A$, hence $f \times g$ is continuous.

Now we need to show that

$$\begin{aligned} + : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ +(x, y) = x + y \end{aligned}$$

is continuous where $\mathbb{R} \times \mathbb{R}$ has the product topology and $\mathbb{R}$ has the usual topology. Since $+$ is a function between metric spaces, it suffices to show it is metric continuous, so let $\epsilon > 0$ and let $B_\epsilon(+(x, y)) = (x + y - \epsilon, x + y + \epsilon)$ be an open ball of $\mathbb{R}$, then if we let $\delta < \frac{\epsilon}{2}$ we would have that

$$\begin{aligned} B_\delta(x, y) &= B_\delta(x) \times B_\delta(y) \\ &= (x - \delta, x + \delta) \times (y - \delta, y + \delta) \end{aligned}$$

and since $+^{-1}(B_\epsilon(+(x, y)))$ is the ball containing all sets of points who's sum is less than ϵ away from $x+y$, and if we chose any point $(a, b) \in B_\delta(x, y)$ we would have that $x+y-\epsilon < a+b < x+y+\epsilon$, hence

$$B_\delta(x, y) \subseteq +^{-1}(B_\epsilon(+(x, y)))$$

and so $+$ is metric continuous, and so is continuous.

Next we need that multiplication

$$\begin{aligned} \circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ \circ(x, y) = xy \end{aligned}$$

is continuous, in fact the proof is much the same as the above proof, we have $B_\epsilon(\circ(x,y)) = (xy-\epsilon, xy+\epsilon)$, we can see that $\circ^{-1}(B_\epsilon(\circ(x,y)))$ is the set containing all points in $\mathbb{R}\times\mathbb{R}$ who's product is less than ϵ away from xy, so we need to construct a ball with radius $\delta > 0$ that is a subset of this, however we must be more careful with how we define δ than we were before. Essentially we need that δ satisfy certain conditions, in particular that

$$\begin{aligned}(x+\delta)(y+\delta) &< xy+\epsilon\\ (x-\delta)(y-\delta) &> xy-\epsilon\end{aligned}$$

expanding these we get that we need δ to satisfy:

$$\begin{aligned} xy+\delta(x+y)+\delta^2 &< xy+\epsilon\\ xy-\delta(x+y)+\delta^2 &> xy-\epsilon\\ \implies \delta(x+y)-xy-\delta^2 &< \epsilon-xy\\ \implies 2\delta(x+y) &< 2\epsilon\\ \implies \delta(x+y) &< \epsilon\end{aligned}$$

Now if $x+y>0$ then we simply need $\delta < \frac{\epsilon}{x+y}$ for these to be satisfied. However if $x+y\leq 0$ then $\delta(x+y)\leq 0$ regardless of our choice of δ, so we would have that

$$\begin{aligned} xy+\delta(x+y)+\delta^2 &\leq xy+\delta^2\\ xy-\delta(x+y)+\delta^2 &\geq xy+\delta^2\end{aligned}$$

so we will need only then that

$$\begin{aligned} xy+\delta^2 &< xy+\epsilon\\ xy+\delta^2 &> xy-\epsilon\end{aligned}$$

which means we need $\delta^2 < \epsilon \implies 0 < \delta < \sqrt{\epsilon}$. So with that we can obtain a $\delta > 0$ given $(x,y)\in\mathbb{R}\times\mathbb{R}$ and $\epsilon > 0$ such that

$$B_\delta(x,y) \subseteq \circ^{-1}(B_\epsilon(\circ(x,y)))$$

so $\circ(x,y)$ is metric continuous, hence continuous.

Finally, we need to simply combine all of these results with the fact that composition of continuous functions is continuous, and proceed by induction. Let P_n be the set of real polynomials in x of degree n, we wish to prove that any $f(x) \in P_n$ is continuous for all $n \geq 0$. Begin with the base case that $n = 0$, but we have already shown these are all continuous as P_0 is simply the set of all constant-valued functions. Now assume as our inductive hypothesis that for some $n \geq 0$, all $f(x) \in P_n$ is continuous. Now let $g(x) \in P_{n+1}$. We can see that, for some $f(x) \in P_n, c \in \mathbb{R}$ we must have

$$g(x) = xf(x) + c$$

Now, we can rewrite this as:

$$g(x) = +(\circ(I(x), f(x)), c(x))$$

where $c(x)$ is the constant map sending x to c and $I(x)$ is the identity map as shown above. Since this is the composition of continuous functions, by Lemma 3.17, this is continuous, and so all $g(x) \in P_{n+1}$ are continuous if all $f(x) \in P_n$ are continuous, and so we're done. $\square$

3.4 Manifolds

Definition 3.26. Martin (2021) A topological space (X, τ) is said to be ***second countable*** if τ is generated by a basis with countably many elements,

Definition 3.27. Martin (2021) A topological space M is called a ***topological manifold of dimension*** n or $n-$***dimensional topological manifold*** if:

1. M is separated
2. M is second countable
3. For all $x \in M$ there exists a neighbourhood U_x and a homeomorphism $f : U_x \to B_1^n(0) \subset \mathbb{R}^n$ with $f(x) = 0$

That is that (locally) M behaves as an n-dimensional Euclidean hyper-plane.

3.5 Compactness

Definition 3.28. Martin (2021) Let X be a topological space and A be an subset of X. An ***open cover*** for A is an indexed family $\{U_\lambda\}_{\lambda \in \Lambda}$ of open subsets of X such that

$$A \subset \bigcup_{\lambda \in \Lambda} U_\lambda$$

An open cover is called ***finite*** if Λ contains finitely-many elements. If $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover for A and $\Lambda' \subset \Lambda$ is such that

$$A \subset \bigcup_{\lambda \in \Lambda'} U_\lambda$$

then $\{U_\lambda\}_{\lambda \in \Lambda'}$ is called a ***subcover*** of $\{U_\lambda\}_{\lambda \in \Lambda}$.

Definition 3.29. Martin (2021) Let X be a topological space and let A be a subset of X, we call A ***compact*** (sometimes ***compact in*** X) if every open cover for A has a finite subcover. We call X a ***compact topological space*** if the set X is a compact subset of X.

Definition 3.30. Let X be a topological space, we call the set of all bounded complex-valued functions of X by $C(X)$. The ***support*** of a complex function f on X is precisely the closure of the set

$$\{x \in X : f(x) \neq 0\}$$

the set of all $f \in C(X)$ whose support is compact is denoted by $C_c(X)$.

If, for each $\varepsilon > 0$ the inequality $|f(p)| < \varepsilon$ holds for all p in the complement of some compact set, then f is said to ***vanish at infinity***, the set of all $f \in C(X)$ which vanish at infity is denoted by $C_0(X)$. If X is compact then $C(X) = C_0(X) = C_c(X)$

Theorem 3.31. *Martin (2021) The continuous image of a compact set is compact.*

Proof. Let X, Y be topological spaces with $A \subseteq X$ a compact set of X, and let $f : X \to Y$ be a continuous function. We wish to show that $f(A)$ is compact in Y. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover for $f(A)$, then $\{f^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is an open cover for A as

$$\begin{aligned} A &\subseteq f^{-1}(f(A)) \\ &\subseteq f^{-1}\left(\bigcup_{\lambda \in \Lambda} U_\lambda\right) \\ &= \bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda) \end{aligned}$$

Now since A is compact there is a finite subcover $\{f^{-1}(U_\lambda)\}_{\lambda \in \Lambda'}$. Then $\{U_\lambda\}_{\lambda \in \Lambda'}$ is a finite subcover for $f(A)$ since:

$$\begin{aligned} f(A) &\subseteq f\left(\bigcup_{\lambda\in\Lambda'} f^{-1}(U_\lambda)\right) \\ &= f\left(f^{-1}\left(\bigcup_{\lambda\in\Lambda'} U_\lambda\right)\right) \\ &\subseteq \bigcup_{\lambda\in\Lambda'} U_\lambda \end{aligned}$$

and we're done. □

Theorem 3.32. *A closed subset C of a compact space X is itself compact.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover for C. We wish to find some finite subcover. We know that $\{U_\lambda : \lambda \in \Lambda\} \cup \{X \setminus C\}$ is an open cover for X , therefor there is a finite subcover of the form $\{U_\lambda : \lambda \in \Lambda'\} \cup \{X \setminus C\}$ where Λ' is a finite subset of Λ. Hence $\{U_\lambda : \lambda \in \Lambda'\}$is a finite open cover for C because $C \cap (X \setminus C) = \emptyset$ □

Tychonoff's Theorem. *Given an arbitrary set of topological spaces $\{X_\lambda\}_{\lambda\in\Lambda}$ their product*

$$\prod_{\lambda\in\Lambda} X_\lambda = X_{\lambda_1} \times X_{\lambda_2} \times \dots$$

equipped with the product topology is compact if and only if X_λ is compact for all $\lambda \in \Lambda$

In order to prove this, we will first start with a lemma

Lemma 3.33. *Martin (2021) Let X, Y be topological spaces and $X \times Y$ be a space with the product topology, then the projection maps:*

$$\begin{aligned} \pi_X &: X \times Y \to X \\ \pi_Y(x, y) &= x \\ \pi_Y &: X \times Y \to Y \\ \pi_Y(x, y) &= y \end{aligned}$$

are continuous.

Proof. Let U be open in X, then $\pi_X^{-1}(U) = U \times Y$, which is trivially open in $X \times Y$ so π_X is continuous. The proof for π_Y is trivially the same proof. □

Now we can proceed with the proof for a weaker version of Tychonoff's theorem, before proceeding with the whole thing

Theorem. *Martin (2021) Given an arbitrary **finite** set of topological spaces* $\{X_\lambda\}_{\lambda\in\Lambda}$ *their product*

$$\prod_{\lambda\in\Lambda} X_\lambda = X_{\lambda_1} \times X_{\lambda_2} \times \dots$$

equipped with the product topology is compact if and only if X_λ *is compact for all* $\lambda \in \Lambda$*:*

Proof. If $(X_\lambda)_{\lambda\in\Lambda}$ is a finite family then it suffices to that that given two topological spaces, X, Y, that their product space is also compact if and only if $X \times Y$ is compact with the product topology, and then it will follow by induction that $\prod_{\lambda\in\Lambda} X_\lambda$ is compact iff X_λ is compact for all $\lambda \in \Lambda$.

Begin with the case $X \times Y$ compact $\implies X, Y$ compact. So let $X \times Y$ be compact for some topological spaces X, Y. Then from the prior lemma, X, Y are both the image of the continuous functions π_X, π_Y respectively.

Now we have the case $X \times Y$ compact $\impliedby X, Y$ compact. So let X, Y be compact topological spaces and $X \times Y$ their product space and let $\{W_\lambda\}_{\lambda\in\Lambda}$ be an open cover for $X \times Y$. For each $(x, y) \in X \times Y$ there exists $\lambda_{(x,y)} \in \Lambda$ such that $(x, y) \in W_{\lambda_{(x,y)}}$. Then, by definition, there exists open sets $U_{(x,y)} \subset X$ and $V_{(x,y)} \subset Y$ such that $(x, y) \in U_{(x,y)} \times V_{(x,y)} \in W_{\lambda_{(x,y)}}$. Now we consider the open cover

$$\left\{U_{(x,y)} \times V_{(x,y)}\right\}_{(x,y)\in X\times Y}$$

We will show that this has a finite subcover $\left\{U_{(x,y)} \times V_{(x,y)}\right\}_{(x,y)\in\Phi}$ where Φ is a finite subset of $X \times Y$, and it will then follow that $\{W_\lambda\}_{\lambda\in\Lambda}$ will have finite subcover $\left\{W_{\lambda_{(x,y)}}\right\}_{(x,y)\in\Phi}$.

So, for each $b \in Y$ the family $\left\{U_{(x,b)}\right\}_{x\in X}$ is an open cover for X, therefor there is a finite subcover $\left\{U_{(x,b)}\right\}_{x\in\iota_b}$ where ι_b is a finite subset of X, so define

$$V_b = \bigcap_{x\in\iota_b} V_{(x,b)}$$

. Now $\{V_y\}_{y\in Y}$ is an open cover for Y, therefor there is a finite subcover $\{V_y\}_{y\in\Upsilon}$ where Υ is a finite subset of Y. We then claim that

$$\left\{U_{(x,y)} \times V_{(x,y)} : y \in \Upsilon, x \in \iota_y\right\}$$

is a finite open cover of $X \times Y$. Note that

$$X \times V_y \subset \bigcup_{x \in \iota_y} U_{(x,y)} \times V_y \subset \bigcup_{x \in \iota_y} U_{(x,y)} \times V_{(x,y)}$$

from which it follows that

$$X \times Y \subset \bigcup_{y \in \Upsilon} X \times V_y \subset \bigcup_{y \in \Upsilon} \bigcup_{x \in \iota_y} U_{(x,y)} \times V_{(x,y)}$$

and so given any two topological sets X, Y, their product $X \times Y$ is compact if and only if X, Y are compact. So it follows that for any finite family $\{X_\lambda\}_{\lambda \in \Lambda}$ of topological spaces that $\prod_{\lambda \in \Lambda} X_\lambda$ is compact if and only if each $X_\lambda$is also compact. $\square$

We will now need to contend with the (much harder) infinite case of Tychonoff's theorem. This proof relies on some alternate definitions of compactness, and some related lemmas. One lemma we must accept is the so called Teichmüller–Tukey Lemma which is equivalent to taking the Axiom of Choice, since we assumed back in Chapter 1 that we took the axiom of choice, we shall accept lemma without proof. Since Tychonoff's Theorem, as Stephen Willard put it, "can lay good claim to being the most important theorem in general (nongeometric) topology", it is almost certainly worth taking choice as an axiom in order to have it as a tool on our belt.

Definition 3.34. Engelking (1989)A property P of subsets of a set A is said to be ***a property of finite character*** if the empty set has the property and a set $A \subset X$ has property P if and only if all finite subsets of A have the property.

Teichmüller–Tukey Lemma. *Engelking (1989)If P is a property of finite character pertaining to subset of a set X, then every set $A \subset X$ which has the property P is contained in a set $B \subset X$ which has property P and is such that any set $C \subset X$ with $B \subsetneq C$ we have that $B = C$ (B is **maximal**)*

Definition 3.35. Engelking (1989)A non-empty family $\{F_\lambda\}_{\lambda \in \Lambda}$ has ***the finite intersection property*** if $\bigcap_{\lambda \in \Lambda'} F_\lambda \neq \emptyset$ for every finite set $\Lambda' \subset \Lambda$

Theorem 3.36. *Engelking (1989)A topological space X is compact if and only if every family of closed subsets of X which has the finite intersection property has non-empty intersection.*

Proof. Let X be a topological space, we start with the forwards case X compact $\implies$ every family of closed subsets of X which has the finite intersection property has non-empty intersection.

Let X be compact, and let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a family of closed subset of X such that $\bigcap_{\lambda\in\Lambda} F_\lambda = \emptyset$. Consider the family of subsets $\{U_\lambda\}_{\lambda\in\Lambda}$ where $U_\lambda = X \setminus F_\lambda$, we have that

$$\bigcup_{\lambda\in\Lambda} U_\lambda = \bigcup_{\lambda\in\Lambda}(X \setminus F_\lambda) = X \setminus \bigcap_{\lambda\in\Lambda} F_\lambda = X$$

so $\{U_\lambda\}_{\lambda\in\Lambda}$ is an open cover of X, since X is compact there is a finite subcover $\{U_\lambda\}_{\lambda\in\Lambda'}$. Hence

$$X = \bigcup_{\lambda\in\Lambda'} U_\lambda = \bigcup_{\lambda\in\Lambda'}(X \setminus F_\lambda) = X \setminus \bigcap_{\lambda\in\Lambda'} F_\lambda$$

which implies that $\bigcap_{\lambda\in\Lambda'} F_\lambda = \emptyset$ for some $\Lambda' \subset \Lambda$, hence if $\{F_\lambda\}_{\lambda\in\Lambda}$ has the finite intersection property, $\bigcap_{\lambda\in\Lambda} F_\lambda \neq \emptyset$.

Now the backwards case, X compact $\Longleftarrow$ every family of closed subsets of X which has the finite intersection property has non-empty intersection. So let X be a topological space such that every collection of closed subsets of X with the finite intersection property has non-empty intersection and let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open cover for X. Now assume, towards a contradiction, that X is not compact, then for all $\Lambda' \subset \Lambda$ we have that

$$\bigcup_{\lambda\in\Lambda'} U_\lambda \neq X$$

which would give

$$X \setminus \left(\bigcup_{\lambda\in\Lambda'} U_\lambda\right) \neq X \setminus X$$
$$\bigcap_{\lambda\in\Lambda'} X \setminus U_\lambda \neq \emptyset$$

hence $\{X \setminus U_\lambda\}_{\lambda\in\Lambda}$ is a family of closed subsets of X which has the finite intersection property, so

$$\bigcap_{\lambda\in\Lambda} X \setminus U_\lambda \neq \emptyset$$
$$X \setminus \bigcap_{\lambda\in\Lambda} X \setminus U_\lambda \neq X \setminus \emptyset$$
$$\bigcup_{\lambda\in\Lambda} U_\lambda \neq X$$

which contradicts $\{U_\lambda\}_{\lambda\in\Lambda}$ being an open cover of X. Hence if every collection of closed subsets of X with the finite intersection property has non-empty intersection then X is compact. $\square$

We are now ready to tackle the full proof of Tychonoff's Theorem:

Proof. Engelking (1989) Let $(X_\lambda)_{\lambda\in\Lambda}$ be an arbitrary family of topological spaces. We will start with the forwards case, $\prod_{\lambda\in\Lambda} X_\lambda$ compact $\implies X_\lambda$ compact for each $\lambda \in \Lambda$.

So let $X = \prod_{\lambda\in\Lambda} X_\lambda$ be compact for some arbitrary family of topological spaces $\{X_\lambda\}_{\lambda\in\Lambda}$. The proof is essentially the same as the finite case, define a projection mapping $\pi_\lambda : X \to X_\lambda$ for each $\lambda \in \Lambda$ which sends $x \in X$ to it's λ^{th} co-ordinate, this is continuous by Lemma 3.33, and hence each X_λ is the continuous image of compact sets, hence each are compact.

Now we consider the backwards case $\prod_{\lambda\in\Lambda} X_\lambda$ compact $\impliedby X_\lambda$ compact for each $\lambda \in \Lambda$.

So let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary family of compact topological spaces, and let $X = \prod_{\lambda\in\Lambda} X_\lambda$. Consider then a family $\mathcal{F}_0$ of closed subsets of X which has the finite intersection property. Since the finite intersection property is of finite character it follows from Teichmüller-Tukey that the family $\mathcal{F}_0$ is contained in some maximal family $\mathcal{F}$ of subsets of X which has the finite intersection property.

We then need to show that $\cap\mathcal{F}_0 \neq \emptyset$, it will suffice then to show there exists a point $x \in X$ such that $x \in \overline{A}$ for every $A \in \mathcal{F}$. Now since $\mathcal{F}$ is maximal then if $A_1, A_2, \ldots, A_k \in \mathcal{F} \implies \bigcap_{i=1}^k A_i \in \mathcal{F}$ and, if we have $A_0 \subset X$ with $A_0 \cap A \neq \emptyset$ for every $A \in \mathcal{F}$ then $A_0 \in \mathcal{F}$. As $\mathcal{F}$ has the finite intersection property the family $\mathcal{F}_\lambda = \left\{\overline{\pi_\lambda(A)}\right\}_{A\in\mathcal{F}}$ of closed subsets of X_λ also has this property for every $\lambda \in \Lambda$. Hence for every $\lambda \in \Lambda$ there exists a point

$$x_\lambda \in \bigcap_{A\in\mathcal{F}} \overline{\pi_\lambda(A)} \subset X_\lambda$$

Now let W_λ be a neighbourhood of x_λ in X_λ. By our definition of x_λ, $W_\lambda \cap \pi_\lambda(A) \neq \emptyset$ for every $A \in \mathcal{F}$, i.e.

$$\pi_\lambda^{-1}(W_\lambda) \cap A \neq \emptyset, \forall A \in \mathcal{F}$$

So we can see then that $\pi_\lambda^{-1}(W_\lambda) \in \mathcal{F}$, and, from $A_1, A_2, \ldots, A_k \in \mathcal{F} \implies \bigcap_{i=1}^k A_i \in \mathcal{F}$, it follows that all members of the canonical base for X that contain the point $x = \{x_\lambda\}$ belongs to the family $\mathcal{F}$. As $\mathcal{F}$ has the finite intersection property every $A \in \mathcal{F}$ intersects all members of the canonical base for X that contain the point x, hence there is an $x \in X$ such that $x \in \overline{A}$ for every $A \in \mathcal{F}$. $\square$

This is a very powerful result, one useful consequence of this is the Heine-Borel theorem

Heine-Borel. *(Martin 2021) A nonempty subset of $\mathbb{R}^n$ with usual topology is compact iff it is closed and bounded*

Proof. We begin with the forwards case, First the closed condition, indeed this is true in any Hausdorff space. Let X be any Hausdorff space and A a compact subset of X, let $x \in X \setminus A$. For any $a \in A$ there exists disjoint open sets U_a, V_a such that $a \in U_a$ and $x \in V_a$. The family $\{U_a\}_{a \in A}$ forms an open cover for A, and since A is compact it admits a finite subcover, call it $\{U_{a_i}\}_{1 \leq i \leq n}$ where $a_1, \ldots, a_n$ are finitely many points in A. We define

$$V_x = \bigcap_{i=1}^{n} V_{a_i}$$

V_x is the intersection of finitely many open sets, hence it is itself open. As well, $x \in V_x$ and $V_x \cap A = \emptyset$. So we

$$X \setminus A = \bigcup_{x \in X \setminus A} V_x$$

we can see that $X \setminus A \subset \bigcup_{x \in X \setminus A} V_x$ because every $x \in X \setminus A$ is in at least 1 V_x, and $X \setminus A \supset \bigcup_{x \in X \setminus A} V_x$ because $V_x \cap A = \emptyset$ for all x. So $X \setminus A$ is closed. Next we check the bounded condition, indeed this is also true in any metric space with the metric topology. So let X be a metric space with metric topology, and A be a compact subset of X. Let x be a point in A, we can cover A with the family $(B_n(x))_{n \in \mathbb{Z}^+}$ because any $y \in A$ will have an n such that $y \in B_n(x)$ as long as $n > d(x, y)$. Now since A is compact there is a finite subset $\Gamma \subset \mathbb{Z}^+$ such that $(B_n(x))_{n \in \lneqq}$ is a finite subcover. We can then take the maximal $M \in \Gamma$ and conclude that $A \subset B_M(x)$, hence A must be bounded from above. So a compact subset of $\mathbb{R}^n$ must be both closed and bounded.

Now for the backwards case. Let A be a bounded and closed subset of $\mathbb{R}^n$. Since A is bounded there exists an $a \in A$ and $M > 0$ such that $A \subset B_M(a)$ but

$$B_M(a) \subset [a_1 - M, a_1 + M] \times [a_2 - M, a_2 + M] \times \ldots \times [a_n - M, a_n + M]$$

now each $[a_i - M, a_i + M]$ is compact. So by Tychonoff's theorem $B_M(a)$ is compact, therefor A is a closed subset of a compact set, hence compact and we're done. $\square$

Chapter 4

Basic Properties of Topological Groups

Now that we have a basic understanding of Groups and Topological Spaces, let us combine them with the following construct:

Definition 4.1. Given a topological space $G = (G, \tau)$ we say that it is a ***Topological Group*** if it is also a group and the group operation

$$\begin{aligned} \circ : G \times G &\to G \\ (x, y) &\to x \circ y \end{aligned}$$

and the inversion map

$$\begin{aligned} ^{-1} : G &\to G \\ x &\mapsto x^{-1} \end{aligned}$$

are both continuous. We view $G \times G$ here as a topological space with the product topology, we write $G = (G, \tau, \circ)$.

Example 4.2. One topological group we will become very familiar with soon is that of $GL_n(\mathbb{R})$, we know from Example

that this is in fact a group, now to consider what topology we will use, consider what we showed in Example 3.9, that the set of $n \times n$ matrices is a topological space, equipped with the euclidean metric. So as $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ we can equip it with the subspace topology inherited from $M_n(\mathbb{R})$ and so the general linear group is a topological group if and only if matrix multiplication and the inversion map are continuous.

Lemma. *Real Matrix multiplication and inversion are continuous*

Proof. Matrix multiplication and inversion reduce to polynomials in their entries, and since real polynomials are continuous by Theorem 3.25 these are continuous as well. □

Example 4.3. We can equip any generic group with the discrete topology to create a topological group, we can see this as under the discrete topology any function is continuous (as all possible subsets are open). So being a topological group is not necessarily special, like with topological spaces the interesting part is how the topology applies to the space.

Problem 4.4. Prove that *left translation by g*:

$$\begin{aligned} l_g : G &\to G \\ x &\mapsto gx \end{aligned}$$

is continuous.

Solution. The identity and constant maps are continuous (Theorem 3.25) and so the mapping:

$$\begin{aligned} f_g : G &\to G \times G \\ x &\mapsto (g, x) \end{aligned}$$

is also continuous. Then by composition of functions

$$\circ(f_g(x)) = gx = l_g(x)$$

is also continuous.

As with groups and topological spaces we can discuss a subsets with similar structure.

Definition 4.5. Higgins (1974) Let G be a topological group, a subset H is then said to be a ***subgroup of*** G if the underlying algebraic subgroup of H a subgroup of the underlying algebraic group of G. It has the topology inherited from G as a subspace. If H is open in the topology of G we call it an ***open subgroup***

Example 4.6. From $GL_n(\mathbb{R})$ we can take the subgroup called the ***Special Linear Group*** $SL_n(\mathbb{R})$, which is all $n \times n$ real valued matrices with determinant 1. It is relatively easy to show that this is a subgroup and the continuity of the group operation and inversion map is inherited from $GL_n(\mathbb{R})$

Lemma 4.7. *All open subgroups are also closed*

In order to prove the above lemma we must discuss the topological group equivalent of the quotient group, coset spaces.

Definition 4.8. Higgins (1974) Let G be a topological group and H a subgroup. Then define G/H as

$$G/H \coloneqq \{xH : x \in G\}$$

the set of all left cosets. We can then define the projection map:

$$\begin{aligned} q : G &\to G/H \\ x &\mapsto xH \end{aligned}$$

to imbue this with a (quotient) topology. We call this by ***the left coset space of*** G ***with respect to*** H***.*** If $H \lhd G$ then this space is a topological group (with group operation as normally defined for quotient groups)

Proof. Now to prove the above lemma, Let H be an open subgroup of a topological group G. I claim that any left coset of H is open. In order to do so I note that left translation:

$$\begin{aligned} l_g : G &\to G \\ x &\mapsto gx \end{aligned}$$

is continuous for any $g \in G$, so if H is open then $l_g(H) = gH$ is open. So all elements of the left coset space are open of H are open. Now ultimately we wish to discuss $G \setminus H$. So we consider $y \in G \setminus H$ and take $l_y(H) = yH$. This is disjoint from H as if $h \in H$ then $yh \notin H$ as that would imply that $yhh^{-1} = y \in H$ which contradicts $y \in G \setminus H$. As $1_g \in H$ then $y \in yH$ so we have:

$$\bigcup_{y \in G \setminus H} yH = G \setminus H$$

and as since $yH \in G/H$ they are all open. Since the union of all open subsets is open then $G \setminus H$ is open, hence H is closed in G. □

Corollary 4.9. *If G admits an open subgroup H, then G is not connected as it must admit a partition (H and $G \setminus H$). Consequently this means any connected topological group has no open subgroups.*

Remark. While any open subgroup is closed, it is not necessarily true that any closed subgroup is open. Take for example $\mathbb{R}_+$ (The reals under addition with the usual topology) which is connected, so any subgroup is closed and not open, take for an example $\mathbb{Z}_+$ (The integers under addition).

Definition 4.10. Higgins (1974) Given two topological groups G, H we call a function $f : G \to H$ a ***morphism*** if it is both a group homomorphism and continuous, we call it an ***isomorphism*** if it is also both bijective and its inverse $f^{-1} : H \to G$ is also continuous. An isomorphism that acts within the group (i.e. $f : G \to G$) is called an ***automorphism***.

Definition 4.11. Deane Montgomery (2018) We call a topological group G ***homogenous*** if for any any two elements $a, b \in G$ there exists an automorphism $f : G \to G$ such that $a \mapsto b$.

Definition 4.12. Deane Montgomery (2018) Let M be a separated topological space, and G be a topological group such that each $g \in G$ acts (on the left) as an automorphism on M, with

$$f(g; x) = g(x) = x' \quad x, x' \in M, g \in G$$

then we call (G, M) or sometimes just G the ***topological transformation group*** if for every pair $g_1, g_2 \in G$ and every $x \in M$:

$$g_1(g_2(x)) = (g_1 g_2)(x)$$

and $f(g; x)$ is continuous in both $x \in M$ and $g \in G$

Chapter 5

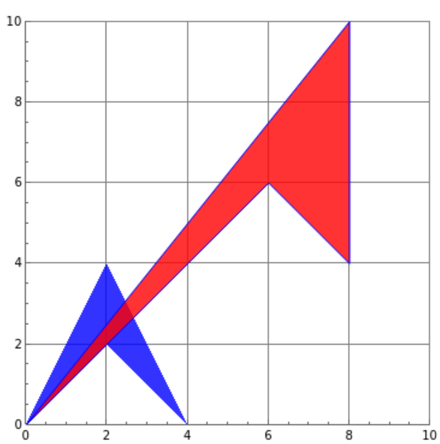
Matrix Groups and Introductory Lie Groups

In discussion of topological groups, one cannot avoid discussing a particular set of them all involving (invertible) matrices. In this section we will first construct, and then discuss the properties of, certain groups of matrices. This will then lead us into a particularly powerful subtype of topological group, known as a Lie group.

Example 5.1. Let us begin with the topological matrix groups we have constructed previously, those being the general and special linear groups, these groups are all the real $n \times n$ matrices with non-zero determinant or determinant 1 respectively. These, along with the (usual) space of $\mathbb{R}^n$ form a topological transformation group. In particular the general linear group is the group of all possible linear transforms of an object in $\mathbb{R}^n$ (in fact, this is where it obtains the name!), where as the special linear group is the set of all transformations that preserve both the volume and orientation of the object. Let us use view some examples in a simple space, $\mathbb{R}^2$:

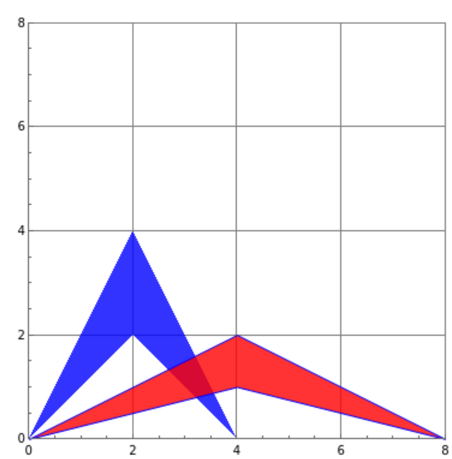
$$T_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in GL_2(\mathbb{R})$$

$$T_2 = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in SL_2(\mathbb{R})$$



[a]

T_1



T_2

[a]Images Generated using WolframCloud Notebooks

Example 5.2. It is generally common for groups to have a relationship with symmetries. Typically one would define symmetry as preserving the "look" of the object. To get closer to this in a matrix group we must look at another subgroup of the general linear group: the ***orthogonal group*** $O(n)$, an orthogonal matrix is a matrix who's inverse is it's transpose. This group is all of the distance (and angle) preserving linear transforms (Higgins 1974), these matrices all have determinant 1 or -1 as $\det(A^\top) = \det(A)$ and so since $1 = \det(I) = \det(AA^\top) = \det(A)\det(A^\top) = (\det(A))^2$, $\det(A) = \pm 1$. We can also consider the ***special orthogonal group*** $SO(n)$ which is specifically all the orthogonal matrices with determinant 1. This group is also called the ***rotation group*** [1], in $\mathbb{R}^2$ this group has elements of the form:

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

which makes it obvious it is a group of rotations in $\mathbb{R}^2$ and hence the symmetry group of the 1-sphere or circle. We can note that the orthogonal group is not connected, it is partitioned by the special orthogonal group and the orthogonal matrices with determinant -1.

Example 5.3. Matrices, however, need not be restricted to the reals, let us consider the general linear group as discussed before, but instead with complex entries, that is all $n \times n$ matrices with non-zero determinant and entries in $\mathbb{C}$, denoted $GL_n(\mathbb{C})$. This is a group by similar reasoning to that given for $GL_n(\mathbb{R})$, and it inherits a similar topology as $\mathbb{C}$ can be viewed as (a relabeling of) $\mathbb{R}^2$ (i.e. $GL_n(\mathbb{C})$ is viewed as a $\mathbb{R}^{2n^2}$ space). Complex matrices come with their own properties and definitions, so we define an invertible complex matrix U to be ***unitary*** if it's inverse U^{-1} is it's conjugate transpose, i.e.

$$U^{-1} = \left(\overline{U}\right)^\top$$

where

$$(\overline{U})_{ij} = \overline{(U_{ij})}$$

And we can form a subgroup of $GL_n(\mathbb{C})$ called the ***unitary group*** $U(n)$ (Chevalley 2016). The unitary group can be seen as the complex equivalent of the orthogonal group, however the unitary group is connected unlike the orthogonal group. In the unitary group the norm of the determinant is always 1, we can construct the subgroup similar to the special orthogonal group, the ***special unitary group***, of unitary matrices with determinant equal to 1.

This gives us a basis to discuss a particularly interesting sub-type of topological groups, all the above are an example of a ***Lie group***, so named for Sophus Lie who's work on continuous

[1] *Rotation Group. From MathWorld—A Wolfram Web Resource*

groups they derive from. We will eventually define a general Lie group but first we must build some foundation.

Definition 5.4. Hall (2000) Let A_n be a sequence of complex matrices, we say that A_n ***converges*** to a matrix A if each entry of A_n converges to the corresponding entry of A in the usual sense of convergence.

Definition 5.5. Hall (2000) A ***matrix Lie group*** is any subgroup H of $GL_n(\mathbb{C})$ such that for any sequence A_n in H that converges, then either $A \in H$ or A is not invertible. Equivalently then, we can say that a matrix Lie group is a closed subgroup of $GL_n(\mathbb{C})$.

Example 5.6. While it may appear that most of the "usual" subgroups of $GL_n(\mathbb{C})$ are indeed matrix Lie groups, one can easily find a counter example. A relatively simple one is that of $GL_2(\mathbb{Q})$, the group of invertible 2×2 matrices with rational entries. The proof this is a group follows the same as $GL_n(\mathbb{R})$ from Example 2.3. Now we have that for natural n

$$\begin{pmatrix} \left(1+\frac{1}{n}\right)^n & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q})$$

and the sequence $\left(\begin{pmatrix} \left(1+\frac{1}{n}\right)^n & 0 \\ 0 & 1 \end{pmatrix}\right)_n$ is a sequence in $GL_2(\mathbb{Q})$. Now 3 of the entries of this sequence are constant, so sequence to converge to an element in $GL_2(\mathbb{Q})$ we would need that $\left(1+\frac{1}{n}\right)^n$ converges to a rational number, however famously $(1+\frac{1}{n})^n$ converges to $e \notin \mathbb{Q}$ so this is a sequence in $GL_2(\mathbb{Q})$ that does not converge in $GL_2(\mathbb{Q})$, so this is not a matrix Lie group

Now in order to fully define the general Lie group we will have to move away from matrix groups for the time being, head into the world of manifolds.

Definition 5.7. (Hitchin 2014)A ***co-ordinate chart*** (U, φ) on a manifold M is an open subset $U \subseteq M$ together with a homeomorphism φ such that $\varphi(U)$ is an open set in $\mathbb{R}^n$.

Definition 5.8. A subset N of an m-dimensional manifold is called an ***n-dimensional*** (or sometimes $(m-n)$-codimensional) ***submanifold*** of M if every point $p \in N$ there is a chart (U, φ) of M such that $p \in U$ and $\phi(N \cap U) = \mathbb{R}^n$.

Definition 5.9. (Hitchin 2014)A ***smooth atlas*** on X is a collection of co-ordinate charts $\{U_\lambda, \varphi_\lambda\}_{\lambda \in \Lambda}$ such that

1. X is covered by $\{U_\lambda\}_{\lambda \in \Lambda}$

2. for each $\lambda, \gamma \in \Lambda, \varphi_\lambda (U_\lambda \cap U_\gamma)$ is open in $\mathbb{R}^n$

3. the map

$$\varphi_\gamma \varphi_{\lambda,}^{-1} : \varphi_\lambda(U_\lambda \cap U_\gamma) \to \varphi_\gamma(U_\lambda \cap U_\gamma)$$

   is ***smooth*** (that is to say that the partial derivative of any order exists for each co-ordinate), and that its inverse is also smooth.

A topological manifold M is said to be ***smooth, differentiable*** or ***possesses differential structure*** if there exists a smooth atlas on M.

Definition 5.10. (Hitchin 2014) If M is a differentiable manifold, we say that a real valued function f is ***differentiable at*** x if for $x \in U_\lambda$ we have that $f \circ \varphi_\lambda^{-1}(x)$ is differentiable at x in the usual meaning.

Definition 5.11. A map between two smooth manifolds $F : M \to N$ is ***smooth*** or ***differentiable*** if for each point $x \in M$ and chart $(U_\alpha, \varphi_\alpha)$ in M with $x \in U_\alpha$ and chart (V_i, ψ_i) in N with $F(x) \in V_i$ the set $F^{-1}(V_i)$ is open and the compositie function

$$\psi_i \circ F \circ \varphi_\alpha^{-1}$$

is smooth. An isomorphism f between smooth manifolds is called a ***diffeomorphism*** if both f and f^{-1} are differentiable.

Definition 5.12. A map $f : N \to M$ between smooth manifolds is called an ***embedding*** of N into M if $f(N)$ is a submanifold of M and $f : N \to f(N)$ is a diffeomorphism.

Definition 5.13. Hall (2000) A ***Lie group*** is a differentiable manifold G which is also a group such that the group product

$$\begin{aligned} \circ : G \times G &\to G \\ (x, y) &\to x \circ y \end{aligned}$$

and the inversion map

$$\begin{aligned} {}^{-1} : G &\to G \\ x &\mapsto x^{-1} \end{aligned}$$

are differentiable.

Clearly, as a Lie group is a manifold and the product and inversion are differentiable (hence continuous), Lie groups are a type of topological group. Often a Lie group is defined by what is called its underlying ***Lie algebra***, which is the tangent space of the identity of a Lie group, however further exploration of these are beyond the scope of this project.

Theorem. *Hall (2000) All matrix Lie groups are Lie groups*

Proof. To begin, if we show that $GL_n(\mathbb{C})$ is a Lie group, the problem will then reduce to proving a closed subgroup of a Lie group is a Lie group itself.

First we can view $GL_n(\mathbb{C})$ as $\mathbb{R}^{2n^2}$ with the usual topology, now we know that this space is trivially a $2n^2$-dimensional manifold, and as the group function and inverse are both polynomials of the elements they are also differentiable. So $GL_n(\mathbb{C})$ is a Lie group. Now we need to show that any closed subgroup of a Lie group is also a Lie group. This is somewhat difficult and requires some more ground work. $\square$

Theorem 5.14. *A subgroup H of the underlying algebraic group of a Lie group G is a submanifold of G iff H is closed in G*

Proof. First, the forwards case, so let H be a submanifold of G then H is locally closed in G. Thus we can find a neighbourhood U of the identity of G such that $H \cap U$ is closed in U, given $y \in \overline{H}$, let $x \in yU^{-1} \cap H$, so $x \in H$ and $y \in xU$. Thus $y \in \overline{H} \cap xU$ which implies that $x^{-1}y \in \overline{H} \cap U = H \cap U$ hence $y \in H$ and thus $H = \overline{H}$, that is H is closed.

Now, the backwards case, the process of this will be to find some neighbourhood of the identity, U, such that $H \cap U$ is a submanifold of U and then translating such a neighbourhood to produce G. This requires us to use the Lie algebra, as stated before these are beyond the scope of this project. In brief then, the proof involves finding a specific subspace of the Lie algebra of G which "looks like" the potential Lie algebra of H. When placed under a specific function called the ***exponential*** function, which is a continuous smooth mapping from a Lie algebra to its Lie group, we will be able to exponentiate and map the Lie algebra of G to U and this potential Lie algebra of H to $U \cap H$, which will then show H to be a submanifold due to properties of the Lie algebra. The full proof is available in Brocker (1985), and is itself not too difficult once the properties of Lie algebras and the exponential function are developed. $\square$

Corollary 5.15. *A closed subgroup H of a Lie group G is itself a Lie group*

Proof. This is a consequence of the above theorem, since H will inherit the differentiable manifold from G, and the group function and inverse are merely restricted to $H \times H$ and so will retain their properties on the submanifold that is likewise restricted. Hence a closed subgroup of a Lie group is itself a Lie group, and Theorem 5.13 is also proven, all matrix Lie groups are indeed Lie groups. □

Let us take pause for a moment and discuss the implications and usage of what we have discussed thus far. Lie groups are a particularly connected and powerful piece of modern mathematics, without going into detail of each (as they are far beyond the scope of this project) Lie groups and their analogues are used in:

- Physics as the representations of symmetries of a physical system
- Number theory as analogues for automorphic forms
- Analysis as symmetries of differential equations

and much more, indeed Lie groups are connected to almost every branch of modern mathematics!

Chapter 6

Commutative Groups

The prior chapter established the notion of a Lie group, a type of topological group intrinsically linked with the notion of differentiability. Now we will discuss the concept of Locally Compact Abelian group, which is closely linked to integrals, and in particular Fourier analysis.

6.1 Locally Compact Abelian Groups

Definition 6.1. A topological group $G = (G, \circ, \tau)$ is said to be ***abelian*** if the underlying group $(G, \circ)$ is abelian in the usual sense and ***compact*** if the underlying topology (G, τ) is a compact space in the usual sense.

Definition 6.2. (Pontriagin 1939)Let $G = (G, \circ, \tau)$ be a topological group. G is said to be ***locally compact*** if for each $x \in G$ there exists a compact neighbourhood of x in τ.

For the following section, we will also assume that any locally compact topological group is Hausdorff, while not true in general it is true for most locally compact spaces, and is true for all the groups we are interested in discussing.

Corollary 6.3. *All compact topological groups are locally compact.*

Proof. Given a compact topological group $G = (G, \circ, \tau)$ and any $x \in G$ we have that G is an open neighbourhood of x which is also compact. □

Proposition 6.4. *(Pontriagin 1939)Given a topological group G, to show it is locally compact it suffices to find a compact neighbourhood of the identity.*

Proof. Let G be a topological group and let U_e be a compact neighbourhood of the identity, then we can find an open neighbourhood of any $g \in G$ by left translation, $gU_e = U_g$, any cover of U_g can be expressed as a left translate of a cover of U_e, and since U_e is compact then U_g is also compact. □

Lemma 6.5. *Gleason (2010) Let G be a topological group, let K be a compact subset of G and let U be an open subset of G such that $K \subseteq U$, then there is an open set V containing the identity such that $KV \subseteq U$*

Proof. For each $x \in K$, let $W_x = x^{-1}U$, because $x \in U, W_x$ is an open neighbourhood of the identity. Then we pick V_x to be an open neighbourhood of the identity such that $V_x V_x \subseteq W_x$. So $\{xV_x : x \in K\}$ is an open cover of K, so there is a finite collection of points $x_1, \ldots, x_n$ such that $K \subseteq \bigcup_{k=1}^{n} x_k V_{x_k}$. Define $V = \bigcap_{k=1}^{n} V_{x_k}$. Let $x \in K$ then there is some $x_k$such that $x \in x_k V_{x_k}$,so:

$$xV \subseteq x_k V_{x_k} V_{x_k} \subseteq x_k W_{x_k} = U$$

□

Problem 6.6. Find a compact neighbourhood of identity in $\mathbb{R}^n$ with addition and the usual topology:

Solution. By the Heine-Borel theorem a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded, as such since any open ball containing 0 contains a (smaller) closed ball we have that any open ball containing 0 is a compact neighbourhood of 0

Lemma 6.7. *Gleason (2010) Let X be a Hausdorff space, let K be a compact subset of X and let U_1, U_2 be open subsets of X such that $K \subseteq U_1 \cup U_2$. Then there are compact sets K_1, K_2 of X such that $K_1 \subseteq U_1, K_2 \subseteq U_2$ and $K = K_1 \cup K_2$.*

Proof. Define $L_1 = K \setminus U_1$ and $L_2 = K \setminus U_2$. K is closed because X is Hausdorff, so L_1, L_2 are both also closed, and hence compact. We have that $L_1 \cap L_2 = \emptyset$ as $K \subseteq U_1 \cup U_2$, now as L_1, L_2 are disjoint compact subsets of X, we can find disjoint open subsets V_1, V_2 containing L_1, L_2 respectively. Now we define $K_1 = K \setminus V_1$ and $K_2 \setminus V_2$. Similarly as before both K_1 and K_2 are compact. We have, by this definition

$$K_1 = K \setminus V_1 \subseteq K \setminus L_1 = K \setminus (K \setminus U_1) = K \cap (K \cap U_1^C)^C = K \cap (K^C \cup U_1) \subseteq U_1$$

and the same argument leads us to $K_2 \subseteq U_2$, as well $K_1 \cup K_2 = K \setminus (V_1 \cap V_2) = K$ as required. □

Definition 6.8. Rudin (1990)Let G be a locally compact abelian (LCA) topological group, a ***character*** γ of G is a homomorphism:

$$\gamma : G \to T$$

where $T = \{z \in \mathbb{C} : |z| = 1\}$. The set of all continuous characters forms a group Γ, which we call the ***Dual Group of*** G, with the group operation being the point-wise product of functions, the inverse of $\gamma \in \Gamma$ is $\bar{\gamma}$ (the complex conjugate of γ). We typically will denote the value of a character γ at a point $x \in G$ as (x, γ)

Now in order to discuss the topology on this group we must establish the notion of a *measure*

6.2 Measures

Definition 6.9. (Cohn 2013) Let X be a set, a ***σ-algebra on*** X is a collection of subsets A such that, $X \in A$, A is closed under countable union, closed under countable intersection, and is closed under complementation.

Definition 6.10. (Rudin 1990) Let X be a locally compact Hausdorff (LCH) space, let B be the smallest family of subsets of X which contains all open subsets of X and is a σ-algebra on X, We call the members of B the ***Borel sets*** of X.

Definition 6.11. (Cohn 2013) A ***measure*** on a set X is a set function μ defined on some σ-algebra Σ of X with $\mu(\emptyset) = 0$ and which is countably additive, that is if $E \in \Sigma$ is the union of the countable family of pairwise disjoint sets $\{E_i\}$ where each $E_i \in \Sigma$ then

$$\mu(E) = \sum \mu(E_i)$$

We call μ a ***Borel measure*** if the σ-algebra it is defined on is that of the Borel sets.

Definition 6.12. (Rudin 1990) If

$$|\mu|(E) = \sup |\mu|(K) = \inf |\mu|(V)$$

for every set $E \in \Sigma$, where K ranges over all compact subsets of E and V ranges over all open supersets of E then we call μ ***regular***. We define $M(X)$ as all complex-valued regular μ such that $||\mu||$ is finite. If $|\mu|(E) = \sup |\mu|(K)$ over compact subsets of E then we call it ***inner regular***, if $|\mu|(E) = \inf |\mu|(V)$ over open supersets of E then we call it ***outer regular***

Definition 6.13. (Cohn 2013) Let X be a set, an ***outer measure*** on X is a function

$$\mu^* : \mathcal{P}(X) \to [0, \infty)$$

such that

1. $\mu^*(\emptyset) = 0$

2. If $A \subseteq B \subseteq X$ then $\mu^*(A) \leq \mu^*(B)$

3. If $\{A_n\}$ is an infinite sequence of subsets of X then

$$\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n)$$

this property is called ***countable subadditivity***.

Definition 6.14. (Cohn 2013) We say that $B \subseteq X$ is μ^****-measurable*** if, for every $A \subseteq X$

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^C)$$

a μ^*-measurable set is sometimes called a ***Carathedory measurable set***.

Corollary 6.15. *(Cohn 2013) A set B is μ^*-measurable if*

$$\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \cap B^C)$$

Proof. As μ^* is countably subadditive,

$$\mu^*(A) \leq \mu^*(A \cap B) + \mu^*(A \cap B^C)$$

for all $A, B \subseteq X$ so if

$$\mu^*(A) \geq (A \cap B) + \mu^*(A \cap B^C)$$

then $\mu^*(A) \geq (A \cap B) + \mu^*(A \cap B^C)$ $\square$

Theorem 6.16. *(Cohn 2013) The set $\mathcal{C}$ of Carathedory measurable sets of an outer measure μ^* on a set X is a σ-algebra*

Proof. For all $A \in X$ we have that $A \cap X = A, A \cap X^C = A \cap \emptyset$, and so we have that

$$\begin{aligned} \mu^*(A \cap X) + \mu^*(A \cap X^C) &= \mu^*(A) + 0 \\ &= \mu^*(A) \end{aligned}$$

hence X is in $\mathcal{C}$. We can also note that for any $B \in \mathcal{C}$

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap B) + \mu^*(A \cap B^C) \\ &= \mu^*(A \cap B^C) + \mu^*(A \cap B) \\ \implies B^C &\in \mathcal{C} \end{aligned}$$

so $\mathcal{C}$ is closed under complementation. Now let $B_1, B_2 \in \mathcal{C}$ and let A be some subset of X. Since $B_1 \in \mathcal{C}$ we have that

$$\begin{aligned}\mu^*(A \cap (B_1 \cup B_2)) &= \mu^*(A \cap (B_1 \cup B_2) \cap B_1) + \mu^*(A \cap (B_1 \cup B_2) \cap B_1^C) \\ &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^C \cap B_2)\end{aligned}$$

Now, we wish to show that $B_1 \cup B_2 \in \mathcal{C}$, so

$$\begin{aligned}\mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^C) &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^C \cap B_2) + \mu^*(B_1 \cap B_1^C \cap B_2^C) \\ &= \mu^*(A \cap B_1) + \mu^*(A \cap B_1^C) \quad \text{as } B_2 \in \mathcal{C} \\ &= \mu^*(A)\end{aligned}$$

and so $\mathcal{C}$ is closed under union of two elements. Now together with closure under complementation this also proves that $\mathcal{C}$ is closed under intersection of two elements as

$$\begin{aligned}B_1^C, B_2^C \in \mathcal{C} &\implies B_1^C \cup B_2^C \in \mathcal{C} \\ &\implies (B_1^C \cup B_2^C)^C \in \mathcal{C} \\ &\implies B_1 \cap B_2 \in \mathcal{C}\end{aligned}$$

It then follows by induction then that $\mathcal{C}$ is closed under finite union and intersection.

Now, let $\{B_i\}$ be a countable sequence of disjoint members of $\mathcal{C}$, we will show by induction that

$$\mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*(A \cap (\bigcap_{i=1}^{n} B_i^C)) \quad (I)$$

holds for each subset A of X and each positive integer n. The base case of $n = 1$ is simply a restatement of the definition of $B_1 \in \mathcal{C}$. So we assume for induction that this holds for some $n \geq 1$. Now as $B_{n+1} \in \mathcal{C}$ and the sequence being disjoint implies

$$\begin{aligned}\mu^*(A \cap (\bigcap_{i=1}^{n} B_i^C)) &= \mu^*(A \cap (\bigcap_{i=1}^{n} B^C) \cap B_{n+1}) + \mu^*(A \cap (\bigcap_{i=1}^{n} B_i^C) \cap B_{n+1}^C) \\ &= \mu^*(A \cap B_{n+1}) + \mu^*(A \cap (\bigcap_{i=1}^{n+1} B_i^C)) \\ \implies \mu^*(A) &= \sum_{i=1}^{n+1} \mu^*(A \cap B_i) + \mu^*(A \cap (\bigcap_{i=1}^{n+1} B_i^C))\end{aligned}$$

and so by induction it is proven. We can not that we do not increase the RHS of (I) if we replace $\mu^*(A \cap (\bigcap_{i=1}^n B_i^C))$ with $\mu^*(A \cap (\bigcap_{i=1}^\infty B_i^C) = \mu^*(A \cap (\bigcup_{i=1}^\infty B_i)^C)$. and so by letting n approach infinity we find that

$$\mu^*(A) \geq \sum_{i=1}^\infty \mu^*(A \cap B_i) + \mu^*(A \cap (\bigcup_{i=1}^\infty B_i)^C)$$

and so, with the countable subadditivity of μ^* we have that

$$\begin{aligned}
\mu^*(A) &\geq \sum_{i=1}^\infty \mu^*(A \cap B_i) + \mu^*(A \cap (\bigcup_{i=1}^\infty B_i)^C) \\
&\geq \mu^*(A \cap (\bigcup_{i=1}^\infty B_i)) + \mu^*(A \cap (\bigcup_{i=1}^\infty B_i)^C) \\
&\geq \mu^*(A)
\end{aligned}$$

and so each prior inequality must be in fact an inequality, hence

$$\mu^*(A) = \sum_{i=1}^\infty \mu^*(A \cap B_i) + \mu^*(A \cap (\bigcup_{i=1}^\infty B_i)^C)$$

and so $(\bigcup_{i=1}^\infty B_i) \in \mathcal{C}$, and so $\mathcal{C}$ is closed under the union of disjoint sequences of set. However since the union of an arbitrary countable sequence $\{B_i\} \in \mathcal{C}$ is the union of a disjoint sequence of sets, namely

$$B_1, B_1^C \cap B_2, \ldots, B_1^C \cap B_2^C \cap \ldots \cap B_n, \ldots$$

we have that $\mathcal{C}$ is closed under the formation of countable unions, and by similar reasoning to before this means it is closed under the countable intersection, hence $\mathcal{C}$ is a σ-algebra. □

Theorem 6.17. *(Cohn 2013) The restriction of μ^* to $\mathcal{C}$ is a measure.*

Proof. To show that μ^* restricted to $\mathcal{C}$ is a measure, we need only show that it is countably additive for any pairwise disjoint sequence $\{B_i\} \in \mathcal{C}$. So let $\{B_i\}$ be such a sequence. So let $A = \bigcup B_i$, then from the above proof we have that

$$\begin{aligned}
\mu^*(\bigcup B_i) &\geq \sum_i \mu^*(A \cap B_i) + \mu^*(A \cap (\bigcup_{i=1} B_i)^C) \\
&= \sum_i \mu^*(B_i)
\end{aligned}$$

and by countable subadditivity we have

$$\mu^*(\bigcup B_i) \leq \sum_i \mu^*(B_i)$$

and thus $\mu^*(\bigcup B_i) = \sum_i \mu^*(B_i)$, Hence μ^* is a measure when restricted to $\mathcal{C}$ $\square$

Definition 6.18. (Rudin 1990) A ***Borel function*** f is a function

$$f : X \to \mathbb{C}$$

such that if $f^{-1}(V)$ is a Borel set for every open set V in $\mathbb{C}$ (with the usual topology). If $\mu \in M(X)$ all bounded Borel functions on X are integrable with respect to μ, and the inequality

$$\left| \int_X f d\mu \right| \leq \|\mu\| \cdot \sup_{x\in X} |f(x)|$$

holds.

If m is a non-negative measure on X and if $1 \leq p$, $L^p(m)$ denotes the set of all Borel functions f on X for which the norm

$$\|f\|_p = \left\{ \int_X |f|^p dm \right\}^{1/p}$$

is finite. I claim, without proof of completeness, that each $L^p(m)$ is a Banach space normed by $\|\cdot\|_p$, and that for $p = 2$ it is a Hilbert space with inner product $\langle f, g\rangle = \int f\overline{g}dm$. If G is a LCA group, then we will write $L^p(G)$ in place of $L^p(m)$. We will use $L^\infty(G)$ to denote the space of all bounded Borel functions on X normed by

$$\|f\|_\infty = \operatorname{ess\,sup}_{x\in X} |f(x)|$$

where this ***essential supremum*** is the smallest λ such that $m(\{x : f(x) > \lambda\}) = 0$.

Remark. L^p-norms are translation invariant, i.e. that

$$\|f_x\|_p = \|f\|_p$$

where f_x is the translate of f defined by

$$f_x(y) = f(y - x) \quad y \in G$$

Theorem 6.19. *Suppose $1 \leq p < \infty$ and $f \in L^p(G)$ then the map*

$$x \to f_x$$

is uniformly continuous map of G into $L^p(G)$.

Proof. Let $\varepsilon > 0$ be given. Since $C_c(G)$ is dense in $L^1(G)$ then there exists $g \in C_c(G)$, with compact support K, such that $\|g - f\|_p < \frac{\varepsilon}{3}$, and the uniform continuity of g implies that there is a neighbourhood V of 0 in G such that

$$\|g - g_x\|_\infty < \frac{\varepsilon}{3} [m(K)]^{-1/p}$$

for all $x \in V$. Hence $\|g - g_x\|_p < \frac{\varepsilon}{3}$ and so

$$\|f - f_x\|_p \leq \|f - g\|_p + \|g - g_x\|_p + \|g_x - f_x\|_p < \varepsilon$$

if $x \in V$. Finally $f_x - f_y = (f - f_{y-x})_x$ so that $\|f_x - f_y\|_p < \epsilon$ if $y - x \in V$ and the proof is complete. Note that this same proof applies for $C_0(G)$ in place of $L^p(G)$ but that it is false for $L^\infty(G)$ unless G is discrete. $\square$

Theorem 6.20. *(Gleason 2010) On every Locally Compact topological group G there exists a non-negative regular measure $m \in M(G)$, which we call the* **Left-Haar measure** *of G, which is not identically 0 and which is left translation-invariant, i.e.*

$$m(xE) = m(E)$$

for every $x \in G$ and every Borel set E.

Proof. In order to accomplish this, we will first construct a ***covering number*** of a set. Let G be a locally compact topological group and let $S, T \subseteq G$ we define the covering number of T by S

$$[T : S] \coloneqq \inf\{\#I : T \subseteq \bigcup_{x \in I} xS, I \subseteq G\}$$

that is, it is the least number of left translates of S which cover T.

Now we begin to construct the measure, to begin let $\mathcal{K}$ be the collection of compact subsets of G and let $\mathcal{U}$ be the collection of open subsets of G containing the identity. Since G is locally compact there is a compact subset of G with non-empty interior: let us call this set K_0. For each

$U \in \mathcal{U}$ define a function $\mu_U : \mathcal{K} \to \mathbb{R}$ such that:

$$\mu_U(K) = \frac{(K : U)}{(K_0 : U)}$$

Since K_0 is nonempty, $(K_0 : U) \neq 0$ and this is well-defined.

In order to construct the Haar measure from μ_U we must show that $0 \leq \mu_U(K) \leq (K : K_0)$. Non-negativity is guaranteed by the fact that $(K : U) > 0$ for all K, U, so we now must show that

$$(K : U) \leq (K : K_0)(K_0 : U)$$

for $K \in \mathcal{K}$ and $U \in \mathcal{U}$. Let us call $(K : K_0) = m, (K : U) = n$ that is we have sequences $g_1, \ldots, g_m \in G$ and $h_1, \ldots, h_n \in G$ such that $K \subseteq \bigcup_{k=1}^m g_k K_0$ and $K_0 \subseteq \bigcup_{k=1}^n h_k U$, hence

$$K \subseteq \bigcup_{i=1}^m \left[\bigcup_{j=1}^n g_i h_j U \right]$$

that is that K be covered by mn cosets of U, so $(K : U) \leq mn = (K : K_0)(K_0 : U)$, hence $0 \leq \mu_U(K) \leq (K : K_0)$.

Define $X = \prod_{K \in \mathcal{K}} [0, (K : K_0)]$, that is the product space of these intervals with the product topology, as $0 \leq \mu_U(K) \leq (K : K_0)$ each μ_U may be thought of as a point in X, with $\mu_U(K)$ the K^{th} co-ordinate, so for each $V \in \mathcal{U}$ define $C(V) = \overline{\{\mu_U : U \in \mathcal{U}, U \subseteq V\}}$. We want to show that the collection $\{C(V) : V \in \mathcal{U}\}$ possess the finite intersection property, so let $V_1, \ldots, V_n \in \mathcal{U}$ then $\mu_{\bigcap_{k=1}^n V_k} \in \bigcap_{k=1}^n C(V_K)$, so that $\bigcap_{k=1}^n C(V_K)$ is non-empty, and because X is compact by Tychonoffs Theorem, it follows that $\bigcap_{k=1}^n C(V_K)$ is non-empty, so for our Haar Measure we may pick some $\mu \in \bigcap_{V \in \mathcal{U}} C(V)$.

Now let us establish some properties of μ to guarantee it is in fact a Haar measure, to begin with let us show $\mu(K_1) \leq \mu(K_2)$ if $K_1 \subseteq K_2$. Now for each $U \in \mathcal{U}$, trivially we have $\mu_U(K_1) \leq \mu_U(K_2)$ as the covering of K_2 by cosets of U is also a covering of K_1, so $(K_1 : U) \leq (K_2 : U) \implies \mu_U(K_1) \leq \mu_U(K_2) \quad \forall U \in \mathcal{U}$.

Now if we think of elements f of X as functions from $\mathcal{K}$ to $\mathbb{R}$, consider the map that sends $f \in X$ to $f(K_2) - f(K_1)$. This is a composition of continuous functions, and hence is continuous. This is also non-negative on each $C(V)$ because $\mu_U(K_1) \leq \mu_U(K_2) \quad \forall U \in \mathcal{U}$. It follows that this map is also non-negative at μ, so $\mu(K_2) - \mu(K_1) \geq 0$ hence $\mu(K_2) \geq \mu(K_1)$.

Now let us show that $\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$, we will once again start by showing that $\mu_U(K_1 \cup K_2) \leq \mu_U(K_1) + \mu_U(K_2)$ for each $U \in \mathcal{U}$. Now if we have a covering of K_1 with $(K_1 : U)$ cosets of U together with a covering of K_2 with $(K_2 : U)$ cosets of U, then that is a covering of

$K_1 \cup K_2$ with $(K_1 : U) + (K_2 : U)$ cosets of U, so $\mu_U(K_1 \cup K_2) \leq \mu_U(K_1) + \mu_U(K_2)$.

By similar argument to above $f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is continuous and non-negative on each $C(V)$ and hence is non-negative for $\mu \in X$. Thus $\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2)$.

Next, we will show that $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$ if $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$ ($KU^{-1} = \{ku^{-1} : k \in K, u \in U\}$). Let $g_1, \dots, g_n$ be such that $n = (K_1 \cup K_2 : U)$ and $K_1 \cup K_2 \subseteq \bigcup_{k=1}^{n} g_k U$. If some $g_k U$ intersects both $K_1$and K_2, then $g_k \in K_1 U^{-1} \cap K_2 U^{-1}$, contradicting it being empty. Thus there is some m with $0 \leq m \leq n$ and re-index (g_k) such that $K_1 \subseteq \bigcup_{k=1}^{m} g_k U$ and $K_1 \subseteq \bigcup_{k=m+1}^{n} g_k U$, so $\mu_U(K_1 \cup K_2) \geq \mu_U(K_1) + \mu_U(K_2)$, together with the previous result, we have that $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$ if $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$

Now let us consider disjoint $K_1, K_2 \in \mathcal{K}$, so we can find disjoint open sets U_1 and U_2 such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$, then by Lemma 6.5 there are open neighbourhoods of the identity V_1, V_2 such that $K_1 V_1 \subseteq U_1$ and $K_2 V_2 \subseteq U_2$. Now define $V = V_1 \cap V_2$, then $K_1 V$ and $K_2 V$ are disjoint because U_1 and U_2 are disjoint, thus for any $U \in \mathcal{U}$ with $U \subseteq V^{-1}$ we have that $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$, so by our previous result $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$. Hence the continuous map that sends $f \in X$ to $f(K_1) + f(K_2) - f(K_1 \cup K_2)$ is 0 for each $f \in C(V^{-1})$. In particular, $\mu(K_1) + \mu(K_2) = \mu(K_1 \cup K_2)$.

Next we will extend μ to all subsets of G. Let's start by extending to all opens sets, so let $U \subseteq G$ open, define

$$\mu(U) = \sup\{\mu(K) : K \subseteq U, K \in \mathcal{K}\}$$

now we must show that this definition matches with our previous one when U is also compact. So let K' be an open compact subset of G, since $\mu(K') \in \{\mu(K) : K \subseteq K', K \in \mathcal{K}\}$ we have that $\mu(K')$ we have that $\mu(K') \leq \sup\{\mu(K) : K \subseteq K', K \in \mathcal{K}\}$. We also have, by a previous step, that $\{\mu(K) : K \subseteq K', K \in \mathcal{K}\}$ is bounded above by $\mu(K')$, and so $\mu(K') \geq \sup\{\mu(K) : K \subseteq K', K \in \mathcal{K}\}$, hence

$$\mu(K') = \sup\{\mu(K) : K \subseteq K', K \in \mathcal{K}\}$$

so this definition agrees, as well it follows that $\mu(U_1) \leq \mu(U_2)$ if $U_1 \subseteq U_2$. Now let us extend this to all subsets $A \subseteq G$ by defining:

$$\mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ open}\}$$

by similar arguments as above, if A is open then this definition will agree as $\mu(A) \in \{\mu(U) : A \subseteq U, U \text{ open}\}$ so $\mu(A) \geq \inf\{\mu(U) : A \subseteq U, U \text{ open}\}$, and $\{\mu(U) : A \subseteq U, U \text{ open}\}$ is bounded from below by $\mu(A)$ so $\mu(A) \leq \inf\{\mu(U) : A \subseteq U, U \text{ open}\}$ hence $\mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ open}\}$ if A open, and so this definition agrees with our above definition, and so we have extended μ to all subsets of G, and we have also maintained the property that $\mu(A_1) \leq \mu(A_2)$ if $A_1 \subseteq A_2$.

I claim then, that μ extended to all sets is an outer measure of G. We have that $\mu(\emptyset) = 0$ as $(\emptyset : U) = 0$ for every $U \in \mathcal{U}$. Since we already have then that $\mu(A_1) \leq \mu(A_2)$ if $A_1 \subseteq A_2$, all that is required is to show countable subadditivity. We will start by showing that μ is countably subadditive for each collection of open sets $\{U_n\}$, that is

$$\mu\left(\bigcup_n U_n\right) \leq \sum_n \mu(U_n)$$

So let $\{U_n\}$ be some countable collection of G. Let K be a compact subset of $\bigcup_n U_n$. Then $K \subseteq \bigcup_{i=1}^k U_i$ for some natural k. Then by applying Lemma 6.7 inductively we can find compact sets $K_1, \dots, K_k$ such that $K = \bigcup_{i=1}^k K_i$ and $K_i \subseteq U_i$ for $1 \leq i \leq k$. Then by applying a prior step inductively, we have that

$$\mu(K) \leq \sum_{i=1}^k \mu(K_i) \leq \sum_{i=1}^k \mu(U_i) \leq \sum_n \mu(U_n)$$

It follows then that

$$\mu\left(\bigcup_n U_n\right) \leq \sup\left\{\mu(K) : K \subseteq \bigcup_n U_n, K \in \mathcal{K}\right\} \leq \sum_n \mu(U_n)$$

Now let $\{A_n\}$ be an arbitrary collection of subsets of G. If $\sum_n \mu(A_n) = \infty$ then trivially

$$\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$$

so suppose that $\sum_n \mu(A_n) < \infty$. Let $\epsilon > 0$, and for each $n \in \mathbb{N}$ pick an open set U_n such that $A_n \subseteq U_n$ and $\mu(U_n) \leq \mu(A_n) + \frac{\epsilon}{2^n}$. Then we have

$$\mu(\bigcup_n A_n) \leq \mu(\bigcup_n U_n) \leq \sum_n \mu(U_n) \leq \sum_n \mu(A_n) + \epsilon \sum_n \frac{1}{2^n} = \sum_n \mu(A_n) + \epsilon$$

but as $\epsilon > 0$ is arbitrary then we have that

$$\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$$

so μ is an outer measure on G.

Now to make μ a measure on the Borel sets of G, we need only show that the open sets of G are μ-measurable, as those sets form a σ-algebra, now as μ. So let $U \subseteq G$ be and let $A \subseteq G$, if $\mu(A) = \infty$ then trivially $\mu(A) \geq \mu(A \cap U) + \mu(A \cap U^C)$, so assume $\mu(A) < \infty$ and let $\epsilon > 0$. Choose open $V \subseteq G$ such that $A \subseteq V$ and $\mu(V) \leq \mu(A) + \epsilon$. Let K be a compact subset of $V \cap U$ such that $\mu(V \cap U) - \epsilon \leq \mu(K)$ and let L be a compact subset of $V \cap K^C$ such that $\mu(V \cap K^C) - \epsilon \leq \mu(L)$. Since $K \subseteq U, V \cap U^C \subseteq V \cap K^C$ so

$$\mu(V \cap U^C) - \epsilon \leq \mu(V \cap K^C) - \epsilon \leq \mu(L)$$

thus we have by a prior step

$$\begin{aligned}
\mu(A \cap U) + \mu(A \cap U^C) - 2\epsilon &\leq \mu(V \cap U) + \mu(V \cap U^C) - 2\epsilon \leq \mu(K) + \mu(L) \\
&= \mu(K \cup L) \\
&\leq \mu((V \cap U) \cup (V \cap K^C)) \\
&\leq \mu(V) \\
&\leq \mu(A) + \epsilon
\end{aligned}$$

hence

$$\mu(A \cap U) + \mu(A \cap U^C) \leq \mu(A) + 3\epsilon$$

and since $\epsilon > 0$ is arbitrary we have that $\mu(A \cap U) + \mu(A \cap U^C) \leq \mu(A)$, so U is measurable. It follows then from Theorem 6.17 that μ restricts to a measure on the Borel subsets of G, so μ is a Borel measure. From here μ will refer to the constructed Borel measure.

By our construction of the extension of μ to open and general subsets, we have that

$$\mu(A) = \sup \mu(K) = \inf \mu(V)$$

for compact subsets K and open supersets V of A, so for regularity we simply need that $\mu(A) < \infty$, and as G is compact this is trivially true so $\mu \in M(X)$.

Finally we shall show translation invariance, so fix $g \in G$. Now let $x_1, \dots, x_n$ be such that $K \subseteq \bigcup_{i=1}^n x_i U$ for some open U, now clearly we have that $gK \subseteq g \bigcup_{i=1}^n x_i U = \bigcup_{i=1}^n (gx_i)U$, and so $(K : U) = (gK : U)$ for each $U \in \mathcal{U}$,hence $\mu_U(K) = \mu_U(gK)$ for each $U \in \mathcal{U}$. Naturally then it follows that the map that take $f \in X$ to $f(K) - f(gK)$ is 0 on each $C(U)$ and hence $\mu(K) = \mu(gK)$, so μ is translation invariant and, thus, a Haar measure. $\square$

Theorem 6.21. *(Rudin 1990) If μ, ν are two Haar measures on a space X then $\nu = \lambda\mu$ for some*

$\lambda \in (0, \infty)$

Proof. Fix $g \in C_c(G)$ so that $\int_G g d\mu = 1$. Define λ by

$$\int_G g(-x) d\nu(x) = \lambda$$

for any $f \in C_c(G)$ we then have

$$\begin{aligned}
\int_G f d\nu &= \int_G g(y) d\mu(y) \int_G f(x) d\nu(x) \\
&= \int_G g(y) d\mu(y) \int_G f(x+y) d\nu(x) \\
&= \int_G d\nu(x) \int_G g(y) f(x+y) d\mu(y) \\
&= \int_G d\nu(x) \int_G g(y-x) f(y) d\mu(y) \\
&= \int_G f(y) d\mu(y) \int_G g(y-x) d\nu(x) \\
&= \lambda \int_G f d\mu
\end{aligned}$$

Hence $\nu = \lambda\mu$ $\square$

Remark 6.22. (Rudin 1990) Since we have, up to scaling, a unique Haar measure on any topological space, we will change our notation from

$$\int_G f dm$$

to

$$\int_G f(x) dx$$

and assume we are integrating with respect to the Haar measure.

Additionally, given we are integrating w.r.t translation invariant measure, we will have that

$$\int_G f(-x) dx = \int_G f(x) dx$$

Since we are concerned with Abelian groups it is trivial that this left-Haar measure is also a right-Haar measure. Our next goal to establish the topology of the dual group is to identify Γ with another space.

6.3 Banach Algebras

Definition 6.23. (Rudin 1990) A vector space A over the complex field is a ***commutative algebra*** if it possesses a "multiplication" $* : A \times A \to A$ which is commutative, associative, and distributive. If A has a norm that makes it a Banach space and that $\|xy\| \leq \|x\| \cdot \|y\|$then we say it is a ***commutative Banach algebra.***

Definition 6.24. (Rudin 1990) An ***ideal*** I is a subalgebra of a commutative algebra A, such that for all $x \in A, y \in I$, $xy \in I$. We say that if $I \neq A$ then I is a ***proper ideal*** of A, a ***maximal ideal*** is a proper ideal that is not contained in any larger proper ideals.

Definition 6.25. (Rudin 1990) A complex homomorphism h of a commutative Banach algebra A is a function $h : A \to \mathbb{C}$ such that $h(xy) = h(x)h(y)$. Let Δ be the set of all complex homomorphisms of A which are not identically 0. For each $x \in A$ define a function on Δ

$$\hat{x}(h) = h(x)$$

called the ***Gelfand transform of*** x***.*** The ***Gelfand topology*** on Δ is the finest topology on Δ that makes all $\hat{x}$ continuous. Δ with the Gelfand topology is called the ***Maximal Ideal Space*** of X.

Theorem 6.26. *Rudin (1990) The maximal ideal space of a Banach space X is a compact and Hausdorff.*

Proof. First we will establish a notion of the *dual space* of X (not to be confused with the prior established dual group), this is the space of bounded linear transformations from X to the complex field normed by the absolute value. This space is denoted X^*.

Now each $x \in X$ can be viewed as a function on X^* whose value at a point $T \in X^*$ is Tx, the finest topology on X^*which makes all X continuous is called the ***weak*-topology*** of X^*. We now show that the unit ball, S^*,of this space is a compact Hausdorff space. To do so we can view it as a subset of the topological product space generated by the set of open disks $D_x = \{z \in \mathbb{C} : |z| \leq \|x\|\}$ indexed by $x \in X$, that is $D = \prod_{x \in X} D_x$, since each of these disks is a compact subset of a Hausdorff space, it is itself compact and Hausdorff by Tychonoff's theorem, and S^* is a closed subset of D so it is itself Hausdorff and compact.

Finally, then, we can see that the Gelfand topology coincides with the subspace topology Δ inherits from X^*, and that Δ is a subset of S^*. In particular $\Delta \cup \{0\}$ is a closed subset of S^*, hence it follows that Δ is compact (hence locally compact) and Hausdorff. $\square$

Definition 6.27. (Rudin 1990) The space $L^1(G)$ is a commutative Banach algebra, with multiplication defined by convolution:

$$(f * g)(x) = \int_G f(x-y)g(y)dy$$

We can quickly check that convolution meets the criteria:

1. **Commutative:** *Let $f, g \in L^1(G)$, then by replacing y with $y + x$ we obtain*

$$\begin{aligned}(f * g)(x) &= \int_G f(x-y)g(y)dy \\ &= \int_G f(-y)g(y+x)dy \\ &= \int_G f(y)g(-y+x)dy \\ &= (g * f)(x)\end{aligned}$$

2. **Associativite:** *Let $f, g, h \in L^1(G)$,*

$$\begin{aligned}(h * (f * g))(x) &= \int_G (f * g)(y)h(x-y)dy \\ &= \int_G \left[\int_G (f(y-z)g(z)dz\right] h(x-y)dy \\ &= \int_G \int_G f(y-z)g(z)h(x-y)dzdy\end{aligned}$$

by Fubini's Theorem, we can change the order of integration:

$$
\begin{aligned}
(h * (f * g))(x) &= \int\limits_G \int\limits_G f(y-z)g(z)h(x-y)dydz \\
&= \int\limits_G g(z) \left[\int\limits_G f(y-z)h(x-y)dy \right] dz
\end{aligned}
$$

we can then observe, as we are integrating w.r.t a translation-invariant measure

$$
\begin{aligned}
\int\limits_G f(y-z)h(x-y)dy &= \int\limits_G f(y-z+z)h(x-y+z)dy \\
&= \int\limits_G f(y-z+z)h(x-(y+z))dy \\
&= \int\limits_G f(y)h((x-z)-y)dy \\
&= h * f(x-z)
\end{aligned}
$$

hence:

$$
\begin{aligned}
(h * (f * g))(x) &= \int\limits_G (h * f)(x-z)g(z)dz \\
&= ((h * f) * g)(x)
\end{aligned}
$$

3. **Distibutive:** *Let* $f, g, h \in L^1(G)$,

$$
\begin{aligned}
(f * (g(x) + h(x))) &= \int\limits_G f(x-y)(g(y)+h(y))dy \\
&= \int\limits_G f(x-y)g(y) + f(x-y)h(y)dy \\
&= \int\limits_G f(x-y)g(y)dy + \int\limits_G f(x-y)h(y)dy \\
&= (f * g)(x) + (f * h)(x)
\end{aligned}
$$

4. Let $f, g \in L^1(G)$. We wish to show that

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

First we show that that the equation in Definition 6.27 is a Borel function on $G \times G$. Fix an open set V in the plane, put $E = f^{-1}(V), E' = E \times G$ and let $E'' = \{(x, y) : x - y \in E\}$. Then $E'$is a Borel set in $G \times G$, and since the homeomorphism of $G \times G$ onto itself which carries (x, y) to $(x + y, y)$ maps E' onto E'', it is also a Borel set. Since $f(x - y) \in V$ iff $(x, y) \in E''$, we see that $f(x - y)$ is a Borel function on $G \times G$, and so is the product $f(x - y)g(y)$. By Fubini's theorem,

$$\|(f * g)(x)\|_1 = \int\limits_G \int\limits_G |f(x - y)g(y)| dx dy = \|f\|_1 \|g\|_1$$

Setting $\phi(x) = \int_G |f(x - y)g(y)| dy$, it follows that $\phi \in L^1(G)$. In particular, $\phi(x) < \infty$ for almost all x, and so $(f * g)(x)$ exists for almost all x. Finally, $|(f * g)| \leq \phi(x)$, and the proof is complete.

Remark 6.28. Rudin (1990) Let G be a LCA Group, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The bounded linear functionals T on $L^p(G)$ are in one-to-one correspondence with the members g of $(L^q(G))^*$: each $T \in (L^p(G))^*$ is of the form

$$Tf = \int\limits_G f(x)g(x)dx$$

and $\|T\| = \|g\|_q$. Thus $L^q = (L^p)^*$. We will also have $L^\infty = (L^1)^*$

Theorem 6.29. *(Rudin 1990) Let G be a LCA topological group and Γ its dual. If $\gamma \in \Gamma$ and if*

$$\hat{f}(\gamma) = \int\limits_G f(x)(-x, \gamma)dx \quad f \in L^1(G)$$

then the map $f \to \hat{f}(\gamma)$ is a complex homomorphism of $L^1(G)$ and is not identically 0. *Conversely every non-zero complex homomorphism of $L^1(G)$ is obtained in this way, and distinct characters induce distinct homomorphisms.*

Proof. Let $f, g \in L^1(G)$, and $k = f * g$ Then

$$\begin{aligned}
\hat{k}(\gamma) &= \int_G (f*g)(x)(-x,\gamma)dx \\
&= \int_G (-x,\gamma) \int_G f(x-y)(g(y)dy \\
&= \int_G g(y)(-y,\gamma)dy \int_G f(x-y)(y-x,\gamma)dx \\
&= \hat{g}(\gamma)\hat{f}(\gamma)
\end{aligned}$$

thus the map $f \to \hat{f}(\gamma)$ is multiplicative on $L^1(G)$, and since it is clearly linear it is a homomorphisim.

For the converse, suppose h is a non-zero complex homomorphism of $L^1(G)$. Then h is a bounded linear functional of norm 1 so that

$$h(f) = \int_G f(x)\phi(x)dx \quad f \in L^1(G)$$

for some $\phi \in L^\infty(G)$ with $\|\phi\|_\infty = 1$. If $f$and g are in $L^1(G)$ then we have

$$\begin{aligned}
\int_G h(f)g(y)\phi(y)dy &= h(f)h(g) \\
&= h(f*g) \\
&= \int_G (f*g)(x)\phi(x)dx
\end{aligned}$$

$$= \int_G g(y)dy \int_G f(x-y)\phi(x)dx$$
$$\int_G g(y)h(f_y)dy$$

where

$$h(f)\phi(y) = h(f_y)$$

for almost all $y \in G$. By Theorem 6.19 and the continuity of h, $h(f_y)$ is a continuous function G, for each $f \in L^1(G)$. Choosing f so that $h(f) \neq 0$ it also shows that $\phi(y)$ coincides with a continuous function almost everywhere, and hence we may assume that ϕ is continuous. So $h(f)\phi(y) = h(f_y)$ holds for all $y \in G$

If we replace y by $x+y$ and then f by f_x in $h(f)\phi(y) = h(f_y)$ we obtain

$$\begin{aligned} h(f_x)\phi(x+y) &= h(f_{x+y}) \\ &= h((f_x)_y) \\ &= h(f_x)\phi(y) \\ &= h(f)\phi(x)\phi(y) \end{aligned}$$

so that

$$\phi(x+y) = \phi(x)\phi(y) \quad (x, y \in G)$$

since $|\phi(x)| \leq 1$ for all x and since the above imples that $\phi(-x) = \phi(x)^{-1}$, it follows that $|\phi(x)| = 1$, hence $\phi \in \Gamma$

Finally, if $\hat{f}(\gamma_1) = \hat{f}(\gamma_2)$ for all $f \in L^1(G)$ then □

Definition 6.30. (Rudin 1990) For all $f \in L^1(G)$ the function $\hat{f}$ defined on Γ by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \quad (\gamma \in \Gamma)$$

is called the ***Fourier transform*** of f. It is precisely the Gelfand transform of f. The set of all functions $\hat{f}$ obtained this way will be denoted $A(\Gamma)$

Corollary 6.31. *(Rudin 1990) Γ is the maximal ideal space of $L^1(G)$ when endowed with the Gelfand topology, i.e. the finest topology making all Fourier transforms continuous.*

Lemma 6.32. *(Rudin 1990) The dual group Γ of a locally compact abelian topological group G is also a locally compact abelian topological group.*

Proof. We rely on a alternative description of the topology of Γ, so we will prove: □

1. (x, γ) is a continuous function on $G \times \Gamma$

2. Let K and C be compact subsets of G and Γ, respectively let U_r be the set of all complex numbers z with $|1 - z| < r$ and put

$$N(K, r) = \{\gamma : (x, \gamma) \in U_r \forall x \in K)$$
$$N(C, r) = \{x : (x, \gamma) \in U_r \forall \gamma \in C\}$$

then $N(K,r)$ and $N(C,r)$ are open subsets of Γ and G respectively.

3. The family of all sets $N(K,r)$ and their translates is a base for the topology of Γ

4. Γ is a LCA group.

The equation in Theorem 6.29 rewritten as

$$\hat{f}(\gamma)(x,\gamma) = \hat{f}_x(\gamma) \quad (x \in G, \gamma \in \Gamma)$$

implies (1) if $\hat{f}_x(\gamma)$ is a continuous function on $G \times \Gamma$ for every $f \in L^1(G)$

Fix x_0, γ_0 and $\epsilon > 0$. There are neighbourhoods V of x_0 and W of γ_0

$$\|f_x - f_{x_0}\|_1 < \epsilon$$
$$\left\|\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)\right\|_1 < \epsilon$$

for all $x \in V$, $\gamma \in W$ such that by Theorem 6.19 and the continuity of $\hat{f}_{x_0}$. Since $|\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)| \leq \|f_x - f_{x_0}\|_1$ it follows that $|\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)| < 2\epsilon$ if $x \in V$ and $\gamma \in W$, and (1) is proved.

Choose a compact set K in G, choose $r > 0$ and fix $\gamma_0 \in N(K,r)$. To every $x_0 \in K$ there correspond neighbourhoods V of x_0 and γ_0 such that $(x,\gamma) \in U_r$, if $x \in V$ and $\gamma \in W$; this follows from (1). Since K is compact, finitely many of these sets V cover K and if W^* is the intersection of the corresponding sets W, then $W^* \subset N(K,r)$. Since $W^*$is a neighbourhood of γ_0, $N(K,r)$ is open. The same proof then applies to $N(C,r)$ and (2) is proven.

Assume V is a neighbourhood of γ_0. We have to show that $\gamma_0 + N(K,r) \subset V$ for some choice of K and r. Take WLOG $\gamma_0 = 0$. The definition of the Gelfand topology on Γ shows that there exists functions $f_1, \ldots, f_n \in L^1(G)$ and $\epsilon > 0$ so that

$$\bigcap_{i=1}^{n} \{\gamma : |\hat{f}_i(\gamma) - \hat{f}_i(0)| < \epsilon\} \subset V$$

Since $C_0(G)$ is dense in $L^1(G)$ we may assume that $f_1, \ldots, f_n$ vanish outside a compact set K in G, If

$$r < \epsilon / \max_i \|f_i\|_1$$

and if $\gamma \in N(K,r)$ then

$$|\hat{f}_i(\gamma) - \hat{f}_i(0)| \leq \int\limits_k |(-x,\gamma) - 1||f_i(x)|dx$$
$$\leq r \|f_i\|_1 < \epsilon$$

hence $N(K,r) \subset V$ and (3) follows.

Finally, given $\gamma', \gamma'' \in \Gamma$ and $N(K,r)$, the relation

$$[\gamma' + N(K,r/2)] - [\gamma'' + N(K,r/2)] \subset \gamma' - \gamma'' + N(K,r)$$

shows, by (2) and (3) that the map

$$(\gamma'.\gamma'') \to \gamma' - \gamma''$$

of $\Gamma \times \Gamma$ onto Γ is continuous, hence (4) is proven and we're done.

Pontryagin duality theorem. *(Rudin 1990) Let G be any locally compact abelian topological group, and Γ be its dual group. As Γ is itself LCA it possesses a dual, $\hat{\Gamma}$. Writing $\hat{\gamma} \in \hat{\Gamma}$ at the point $\gamma \in \Gamma$ as $(\gamma, \hat{\gamma})$. Since we can view $x \in G$ as a continuous character on Γ there is a natural map α of G into $\hat{\Gamma}$ such that $(x,\gamma) = (\gamma, \alpha(x))$ for $\gamma \in \Gamma, x \in G$. This is an isomorphic and homeomorphic map . As such we can identify G with $\hat{\Gamma}$.*

Proof. For $x, y \in G$ and $\gamma \in \Gamma$ we have

$$\begin{aligned}(\gamma, \alpha(x+y)) &= (x+y, \gamma) \\ &= (x,\gamma)(y,\gamma) \\ &= (\gamma, \alpha(x))(\gamma, \alpha(y)) \\ &= (\gamma, \alpha(x) + \alpha(y))\end{aligned}$$

and so $\alpha(x+y) = \alpha(x) + \alpha(y)$ so α is a homomorphism. Since Γ separates points on G, α is bijective, hence an isomorphism of G into $\hat{\Gamma}$. We now proceed in three stages.

To begin with let us prove that α is a homeomorphism of G in $\hat{\Gamma}$, so choose a compact set C in Γ, choose $r > 0$ and put

$$V = \{x \in G : |1 - (x,\gamma)| < r \forall \gamma \in C\}$$
$$W = \{\hat{\gamma} \in G : |1 - (\hat{\gamma},\gamma)| < r \forall \gamma \in C\}$$

By Lemma 6.32.3 and Appendix Corollary B.5 these sets V form a neighbourhood base at 0 in G, and the sets W form a neighbourhood base at 0 in $\hat{\Gamma}$. The definition of α gives us that

$$\alpha(V) = W \cap \alpha(G)$$

It follows then that both α and its inverse are continuous at the identity, and since α is an isomorphism the same result holds at any other point of G or $\alpha(G)$ and so α is a homeomorphism.

Next we prove that $\alpha(G)$ is closed in $\hat{\Gamma}$, since $\alpha(G)$ is a homeomorphism, $\alpha(G)$ is locally compact in the subspace topology $\alpha(G)$ has as a subset of $\hat{\Gamma}$. Suppose $\hat{\gamma_0}$ is in the closure of $\alpha(G)$ and let U be a neighbourhood of $\hat{\gamma_0}$whose closure $\overline{U}$ is compact. Since $\alpha(G)$ is locally compact $\alpha(G) \cap \overline{U}$is compact and hence closed in $\hat{\Gamma}$ but $\hat{\gamma_0}$ is in the closure of $\alpha(G) \cap \overline{U}$and it follows that $\hat{\gamma_0} \in \alpha(G)$. Thus $\alpha(G)$ is closed.

Finally we prove that $\alpha(G)$ is dense in $\hat{\Gamma}$. Suppose towards contradiction that $\alpha(G)$ is not dense in $\hat{\Gamma}$, then there exists a function $F \in A(\hat{\Gamma})$ which is 0 at every point of $\alpha(G)$ but is not identically 0. For some $\phi \in L^1(\Gamma)$ we have

$$F(\hat{\gamma}) = \int_{\Gamma} \phi(\gamma)(-\gamma, \hat{\gamma}) d\gamma \quad \hat{\gamma} \in \hat{\Gamma}$$

since $F(\alpha(x)) = 0$ for all $x \in G$ it follows that

$$\int_{\Gamma} \phi(\gamma)(-x, \gamma) d\gamma = \int_{\Gamma} \phi(\gamma)(-\gamma, \alpha(x)) d\gamma = 0$$

and so $\phi = 0$ by Appendix Theorem A.5. Hence $F = 0$, a contradiction. Hence we have that α is an isomorphic and homeomorphic map, and hence we can say $G = \hat{\Gamma}$. $\square$

With lets check some common examples:

Example 6.33. The Dual group of $\mathbb{R}$ is $\mathbb{R}$. The Dual group of $\mathbb{Z}$ is the circle group T, and the dual group of T is $\mathbb{Z}$ so these follow the duality theorem.

This duality is key to performing Fourier analysis on groups that are not the reals, in particular it leads itself into the Fourier series on the integers. These are very powerful analytic tools which allow us to analyse functions in a more concrete manner.

Appendix A

Fourier-Stieltjes Transforms

Definition A.1. Rudin (1990) Suppose G is an LCA Group, and $\mu, \lambda \in M(G)$ then the ***product measure*** $\mu \times \lambda$ on $G \times G$ is defined by

$$(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B) \quad A, B \subset G$$

as well, we can associate with each Borel set of G, E, the set

$$E_{(2)} = \{(x, y) \in G \times G : x + y \in E\}$$

which is a Borel set in $G \times G$, so we define $\mu * \lambda$ by

$$(\mu * \lambda)(E) = (\mu \times \lambda)(E_{(2)})$$

which we call the ***convolution of*** μ ***and*** λ.

Theorem A.2. *Rudin (1990) (a) If $\mu \in M(G)$ and $\lambda \in M(G)$ then $\mu*\lambda \in M(G)$. (b) Convolution is commutative and associative. (c) $\|\mu * \lambda\| \leq \|\mu\| \cdot \|\lambda\|$*

Proof. First, for (a) it is trivial that $\mu * \lambda$ is finite, next we can show it is inner regular. First we can see

$$(\mu * \lambda)(E) = \sum_i (\mu * \lambda)(E_i)$$

where $E_i$are disjoint Borel sets and where $i = 1, 2, 3, \ldots$. If E is a Borel set in G and if $\varepsilon > 0$, the regularity of $\mu \times \lambda$ shows that there is a compact set $K \subset E_{(2)}$such that

$$(\mu \times \lambda)(K) > (\mu \times \lambda)(E_{(2)}) - \varepsilon$$

If C is the image of K under the map $(x, y) \to x + y$, then C is a compact subset of $E, K \subset C_{(2)}$, and hence

$$(\mu * \lambda)(C) = (\mu \times \lambda)(C_{(2)}) \geq (\mu \times \lambda)(K) > (\mu * \lambda)(E) - \varepsilon$$

that is, $\mu * \lambda$ is inner regular. Now for outer regular, Let $E \subset G$ be a Borel set, since $\mu * \lambda$ is inner regular there is an $F \subset E^c$ satisfying $\mu(F) > \mu(E^c) - \varepsilon$. Hence we can define

$$U := E \subset F^c$$

and

$$\mu(U) = \mu(G) - \mu(F) < \mu(G)\,[\mu(E^c) - \varepsilon] = \mu(E) + \varepsilon$$

that is $\mu * \lambda$ is outer regular. Hence $\mu * \lambda$ is a regular measure.

Now for (b) commutativity follows trivially from the commutivity of addition, for associativity we first extend the definition of convolution to the case of n measures with $\mu_1, \ldots, \mu_n \in M(G)$. With each Borel set E associate the set

$$E_n = \{(x_1, \ldots, x_n) \in G^n : \left(\sum_{i=1}^{n} x_i\right) \in E$$

and put

$$(\mu_1 * \mu_2 * \ldots * \mu_n)(E) = (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(E_n)$$

Associativity then follows from Fubini's theorem.

Finally, for (c) Let

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \in E^c \end{cases}$$

for a Borel set E in G, χ_E is called the ***characteristic function*** of the Borel set E in G. The definition of $(\mu * \lambda)(E)$ is equivalent to the equation

$$\int_G \chi_E d(\mu * \lambda) = \int_G \int_G \chi_E(x + y) d\mu(x) d\lambda(y)$$

Hence if f is a ***simple function*** (a finite linear combination of characteristic functions of Borel sets), we have

$$\int_G f d(\mu * \lambda) = \int_G \int_G f(x + y) d\mu(x) d\lambda(y)$$

and since every bounded Borel function is the uniform limit of a sequence of simple function, the above equation holds for every bounded Borel function f. N.B. One could use the above as the definition of $\mu * \lambda$. If $|f(x)| \leq 1$ for all $x \in G$ then $|\int_G f(x+y) d\mu(x)| \leq \|\mu\|$ for all $y \in G$, hence $|\int_G f(x+y) d\mu(x)| \leq \|\mu\| \cdot \|\lambda\|$, and (c) follows. $\square$

Corollary A.3. *Rudin (1990) M(G) is a commutative Banach algebra, with multiplication defined by convolution.*

Definition A.4. Rudin (1990) If $\mu \in M(G)$ the function $\hat{\mu}$ defined on Γ by

$$\hat{\mu}(\gamma) = \int\limits_G (-x, \gamma) d\mu(x) \quad y \in \Gamma$$

is called the ***Fourier-Stieltjes transform*** of μ.

Theorem A.5. *Rudin (1990) If $\mu \in M(\Gamma)$ and if*

$$\int\limits_\Gamma (x, \gamma) d\mu(\gamma) = 0$$

for every $x \in G$ then $\mu = 0$

Proof. For every $f \in L^1(G)$,

$$\begin{aligned} \int\limits_\Gamma \hat{f}(\gamma) d\mu(\gamma) &= \int\limits_\Gamma \int\limits_G f(x)(-x, \gamma) dx d\mu(\gamma) \\ &= \int\limits_G f(x) dx \int\limits_\Gamma (-x, \gamma) d\mu(\gamma) = 0 \end{aligned}$$

Since $A(\Gamma)$ is dence in $C_0(\Gamma)$, it follows that $\int_\Gamma \phi d\mu = 0$ for every $\phi \in C_0(\Gamma)$ and hence $\mu = 0$ $\square$

Appendix B

Inversion Theorem

Definition B.1. Rudin (1990) A function ϕ defined on G is said to be ***positive-definite*** if the inequality

$$\sum_{n,m=1}^{N} z_n \overline{z_m} \phi(x_n - x_m) \geq 0$$

holds for every choice of $x_1, \ldots, x_N$ in G and for every choice of complex numbers $z_1, \ldots, z_N$. The following three relations hold on positive definite functions:

1. $\phi(-x) = \overline{\phi(x)}$
2. $|\phi(x)| \leq \phi(0)$
3. $|\phi(x) - \phi(y)|^2 \leq 2\phi(0)\Re[\phi(0) - \phi(x-y)]$

Bochner's Theorem. *Rudin (1990) A continuous function ϕ on an LCA group G is positive definite iff there is a non-negative measure $\mu \in M(\Gamma)$ such that*

$$\phi(x) = \int_G (x, \gamma) d\mu(\gamma)$$

Definition B.2. Rudin (1990) We call by $B(G)$ the set of all functions f on G which are representable in the form

$$f(x) = \int_\Gamma (x, \gamma) d\mu(\gamma)$$

We note here that Bochner's theorem implies that $B(G)$ is exactly the set of all finite linear combinations of continuous positive-definite functions on G.

Theorem B.3. *Rudin (1990) There are 2 parts to this theorem:*

(a) If $f \in L^1(G) \cap B(G)$, then $\hat{f} \in L^1(\Gamma)$

(b) If the Haar measure of G is fixed, the Haar measure of Γ can be normalised such that the inversion fomula

$$f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma) d\gamma \quad x \in G$$

is valid for every $f \in L^1(G)$

Proof. Let us write B^1 in place of $L^1(G) \cap B(G)$, and if μ is associated with f as in B.2 let us write $\mu = \mu_f$. If $f \in B^1$ and $h \in L^1(G)$ we have

$$(h * f)(0) = \int\limits_G h(-x)f(x)dx = \int\limits_\Gamma \hat{h}(\gamma)d\mu_f(\gamma)$$

and if g is also in B^1,that implies that

$$\int\limits_\Gamma \hat{h}\hat{g}d\mu_f = ((h * g) * f)(0) = ((h * f) * g)(0) = \int\limits_\Gamma \hat{h}\hat{f}d\mu_g$$

and since $A(\Gamma)$ is dense in $C_0(\Gamma)$ it follows that

$$\hat{g}d\mu_f = \hat{f}d\mu_g \quad (f, g \in B^1)$$

Now define a positive linear functional T on $C_c(\Gamma)$. Suppose K is the support of some $\psi \in C_c(\Gamma)$. To every $\gamma_0 \in K$ there corresponds a function $u \in C_c(G)$ with $\hat{u}(\gamma_0) \neq 0$, since $C_c(G)$ is dense in $L^1(G)$. The Fourier transform of $u * \tilde{u}$, where $\tilde{u}(x) = \overline{u(-x)}$, is positive at γ_0 and is nowhere negative. Since K is compact, there is a finite number of such functions, $u_1, \ldots, u_n$such that the function $g = \sum_{i=1}^{n}(u_i * \tilde{u}_i)$ has $\hat{g} > 0$ on K. Since $g \in C_c(G)$ we have that $g \in B^1$. Put

$$T\psi = \int\limits_\Gamma (\frac{\psi}{\hat{g}})d\mu_g$$

Note that $T\psi$ is well defined: if g were replaced by another function $f \in B^1$ whose Fourier transform has no zero on K, the value of $T\psi$ would not be changed since a prior result gives us

$$\int\limits_\Gamma (\frac{\psi}{\hat{f}\hat{g}})\hat{f}d\mu_g = \int\limits_\Gamma (\frac{\psi}{\hat{f}\hat{g}})\hat{g}d\mu_f$$

It is clear that L is linear. The function g in our definition of $T\psi$ positive-definite, hence $\mu_g \geq 0$ and it follows that $T\psi \geq 0$ if $\psi \geq 0$. There exists ψ and μ_f such that $\int \psi d\mu_f \neq 0$, and with g being as we have defined it, we have

$$T(\psi\hat{f}) = \int\limits_\Gamma \frac{\psi\hat{f}}{\hat{g}}d\mu_g = \int\limits_\Gamma \psi d\mu_f \neq 0$$

Thus $T \neq 0$

Fix $\psi \in C_c(\Gamma)$ and $\gamma_0 \in \Gamma$. Construct g as above, so that $\hat{g} > 0$ on K and also on $K+\gamma_0$. Setting $f(x) = (-x, \gamma_0)g(x)$, we have $\hat{f}(\gamma) = \hat{g}(\gamma + \gamma_0)$ and $\mu_f(E) = \mu_g(E - \gamma_0)$. If $\psi_0(\gamma) = \psi(\gamma - \gamma_0)$, then

$$T\psi_0 = \int_\Gamma \left[\frac{\psi(\gamma - \gamma_0)}{\hat{g}(\gamma)}\right] d\mu_g(\gamma) = \int_\Gamma \left[\frac{\psi(\gamma - \gamma_0)}{\hat{f}(\gamma)}\right] d\mu_f(\gamma) = T\psi$$

Thus T is translation-invariant, and it follws that

$$T\psi = \int_\Gamma \psi(\gamma) d\gamma \quad \gamma \in C_c(\Gamma)$$

where $d\gamma$ denotes respect to the Haar measure on Γ.

If now $f \in B^1$ and $\psi \in C_c(\Gamma)$, we can deduce from prior statements:

$$\int_\Gamma \psi d\mu_f = T(\psi\hat{f}) = \int_\Gamma \psi\hat{f} d\gamma$$

and since this holds for every $\psi \in C_c(\Gamma)$ we conclude that

$$\hat{f} d\gamma = d\mu_f \quad f \in B^1$$

Since μ_f is a finite measure it follows that $\hat{f} \in L^1(\Gamma)$, and substition of the above statement into the $\int_G (x, \gamma) d\mu(\gamma)$ gives the inversion theorem and so the proof is completed. □

Remark B.4. Rudin (1990) Let V be a neighbourhood of 0 in G, choose a compact neighbourhood W of 0 such that $W - W = \{w_1 - w_2 : w_1, w_2 \in W\} \subset V$, let f be the charactereistic function of W divided by $m(W)^{\frac{1}{2}}$, and put $g = f * \tilde{f}$. Then g is continous, postive-definite, and 0 outside $W - W$. Hence $\hat{g} = |\hat{f}|^2 \geq 0$,

$$\int_\Gamma \hat{g}(\gamma) d\gamma = g(0) = 1$$

and so it follows that there is a compact set C in Γ such that

$$\int_C \hat{g}(\gamma) d\gamma > \frac{2}{3}$$

If $x \in N(C, \frac{1}{3})$ to use notation from Lemma 6.32.2 we write

$$g(x) = \left(\int\limits_{C} + \int\limits_{C'} \right) \hat{g}(\gamma)(x, \gamma) d\gamma;$$

for $\gamma \in C$, $|1 - (x, \gamma)| < \frac{1}{3}$ hence $\Re(x, \gamma) > \frac{2}{3}$, and the integral over C is at least $\frac{2}{3} \int_C \hat{g} > \frac{4}{9}$. Since $| \int_{C'} | < \frac{1}{3}$, we see that $g(x) > \frac{1}{9}$ if $x \in N(C, \frac{1}{3})$, and our conclusion is that $N(C, \frac{1}{3}) \subset V$. Since the sets $N(C, r)$ are open in G by Lemma 6.32.2 we have the following analogue of Lemma 6.32.3:

Corollary B.5. *Rudin (1990) The family of all sets $N(C, r)$ and their translates is a base for the topology of G.*

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