# Topological Groups

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- So a topological group can help us represent systems of continuous symmetry, which make them particularly adept at modelling certain real-world phenomona.
- Within this presentation I will explain the general topological group, and then go over the two types of topological group with interesting properties. I will be assuming many results and definitions of Group Theory and Topology, all of which can be found in Chapter 2 and 3 of my project.

# Definitions

## Definition

A Topological Group $(X, \circ, \tau)$ is a set $X$ with an operator $\circ$ such that $(X, \circ)$ is a group, $(X, \tau)$ is a topological space, and that $\circ$ is a continuous function from $X \times X \to X$, where $X \times X$ has the product topology, and that the inversion map $^{-1} : X \to X$ that maps $x \mapsto x^{-1}$ is also continuous.

## Example

$(\mathbb{R}, +, \tau)$ where $+$ is addition as usually defined, and $\tau$ is the usual topology on $\mathbb{R}$

# Matrix Groups

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$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n^2-n} \\ x_{n^2-n+1} \\ \vdots \\ x_{n^2} \end{pmatrix} \to \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x_{n+1} & \ddots & \ddots & x_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ x_{n^2-n+1} & x_{n^2+1} & \dots & x_{n^2} \end{pmatrix}$$

# Matrix Groups

### Groups

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### Example

$$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}$$
$$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$$

# Matrix Groups

## Groups

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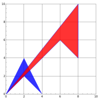
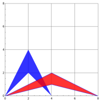
## Example

$$T_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in GL_2(\mathbb{R})$$

$$T_2 = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in SL_2(\mathbb{R})$$



$T_1$



$T_2$

# Matrix Groups

## Complex Matrices

We can broaden our view matrix groups by considering matrices with complex entries, $M_n(\mathbb{C})$, and its subgroups. In particular we will focus on complex general linear group, $GL_n(\mathbb{C})$, and it's subgroups. There are particular subgroups we will be interested in:

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## Definition

A ***Matrix Lie Group*** is a subgroup $H$ of $GL_n(\mathbb{C})$ where every sequence of matrices $(A_n)$ that converges to some matrix $A$ then either $A \in H$ or $A$ is not invertible. This is equivalent to saying a Matrix Lie Group is a closed subgroup of $GL_n(\mathbb{C})$

## Example

The condition of convergence does exclude some fairly basic groups, such as the general linear group with rational entries.

## Lie Groups

Sophus Lie's early work is seen as the beginning of study of topological groups (though he called them continuous groups), and his name is often associated with what many consider to be one of the most powerful type of topological groups. In order to discuss the general Lie group we require a whistlestop tour of a couple other areas of topology, and then drawing in one of the classic tools of analysis, smooth maps.

Figure:

# Lie Groups

## Manifolds

In brief, an $n-$***manifold*** is a topological space that locally "behaves" like a real Euclidean hyperplane of degree $n$. In particular we require that a manifold be generated by a finite basis, that it be seperable, and that each element $x$ has an open neighbourhood $U_x$ homeomorphic to an open ball of radius 1 centred on 0, where $x$ maps to 0.

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## Differentiation

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A ***smooth atlas*** is a collection of charts $\{U_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ such that $\{U_\lambda\}_{\lambda\in\Lambda}$ covers $M$, that for all $\lambda, \gamma \in \Lambda$, $\varphi_\lambda(U_\lambda \cap U_\gamma)$ is open in $\mathbb{R}^n$ and that $\varphi_\gamma\varphi_{\lambda,}^{-1} : \varphi_\lambda(U_\lambda \cap U_\gamma) \to \varphi_\gamma(U_\lambda \cap U_\gamma)$ is smooth, and that its inverse is smooth. If a manifold has a smooth atlas it is said to be ***differentiable*** or ***smooth.***

# Lie Groups

## Definition

A map between two smooth manifolds $F : M \to N$ is ***smooth*** or ***differentiable*** if for each point $x \in M$ and chart $(U_\alpha, \varphi_\alpha)$ in $M$ with $x \in U_\alpha$and chart $(V_i, \psi_i)$ in $N$ with $F(x) \in V_i$ the set $F^{-1}(V_i)$ is open and the compositie function

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## Example

The General Linear Group with complex entries is a Lie Group, as it is simply $\mathbb{R}^{2n^2}$, and trivially any real space is a manifold, and as matrix multiplication is polynomial, it is smooth.

# Lie Groups

## An interesting property

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This fact is not immediately obvious, indeed the proof of it requires knowledge of Lie groups beyond what is in the scop of this project. The general idea is to first prove that a closed subgroup of a Lie group is itself a smooth manifold, which requires us to use the ***Lie Algebra*** and ***Exponential Function***, both powerful tools in the study of Lie groups.

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## Summing up

Lie groups naturally come up a lot in mathematics, particularly those studying physics as they are very useful in representing the symmetries of physical systems. Lie groups arise naturally in discussions of Hamiltonian systems through the ***Lie Bracket***, which provides a natural smooth map. However in terms of pure algebra the Lie group has been used by mathematicians like Borel and Chevalley to connect algebraic geometry and group theory, giving rise to the study of ***Algebraic Groups***. Analgoues of Lie groups also are linked into number thoery through the study of automorphic forms, which was put forward in Langland's program

# Locally Compact Groups

## What is Local Compactness?

### Definition

A topological group $G$ is said to be ***locally compact*** if for each $x \in G$ there exists a compact neighbourhood of $x$ in $\tau$.
N.B. Due to the fact that (left-)translation is continuous it suffices to show that the identity has a compact neighbourhood to show a topological group is locally compact.'
Throughout this discussion topological groups will only refer to locally compact groups, and in particular ***Hausdorff and Abelian*** locally compact groups.

## Why should we care?

Through a series of processes we can perform ***Fourier Analysis*** on any locally compact group, broadening the scope of this powerful tool

## Dual Groups

### Definition

If $G$ is a LCA topological group, a ***character*** $\gamma$ of $G$ is a homomorphism:

$$\gamma : G \to T$$

where $T = \{z \in \mathbb{C} : |z| = 1\}$. The set of all continuous characters forms a group $\Gamma$, which we call the ***Dual Group of*** $G$, with the group operation being the point-wise product of functions, the inverse of $\gamma \in \Gamma$ is $\bar{\gamma}$ (the complex conjugate of $\gamma$). We typically will denote the value of a character $\gamma$ at a point $x \in G$ as $(x, \gamma)$ This dual of a group is closely linked to the group itself. Particularly through another space from which it inherits a topology. This other space requires a touch more discussion and going further afield.

## Measures

### Definition

A ***measure*** on a set $X$ is a set function $\mu$ defined on some $\sigma$-algebra $\Sigma$ of $X$ with $\mu(\emptyset) = 0$ and which is countably additive, that is if $E \in \Sigma$ is the union of the countable family of pairwise disjoint sets $\{E_i\}$ where each $E_i \in \Sigma$ then

$$\mu(E) = \sum \mu(E_i)$$

We call $\mu$ a ***Borel measure*** if the $\sigma$-algebra it is defined on is that of the Borel sets. Measure theory, developed largely by Henri Lebesgue, allows us to integrate function that are ***measurable.***

# Measures

## $L^p$ space

### Definition

A ***Borel function*** $f$ is a function

$$f : X \to \mathbb{C}$$

such that if $f^{-1}(V)$ is a Borel set for every open set $V$ in $\mathbb{C}$ (with the usual topology). If $\mu \in M(X)$ all bounded Borel functions on $X$ are integrable with respect to $\mu$, and the inequality

$$\left| \int\limits_X f d\mu \right| \leq \|\mu\| \cdot \sup_{x \in X} |f(x)|$$

holds.

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holds.

### Definition

If $m$ is a non-negative measure on $X$ and if $1 \leq p$, $L^p(m)$ denotes the set of all Borel functions $f$ on $X$ for which the norm

$$\|f\|_p = \left\{ \int_X |f|^p dm \right\}^{1/p}$$

is finite.

In fact, this normed space is complete, making it a Banach space. $L^1(m)$ will be of particular note to us. We must only now guarantee that the space we're discussing actually have measures.

## Haar Measure

### Theorem

*On every Locally Compact topological group $G$ there exists a non-negative regular measure $m \in M(G)$, which we call the **Left-Haar measure** of $G$, which is not identically $0$ and which is left translation-invariant, i.e.*

$$m(xE) = m(E)$$

*for every $x \in G$ and every Borel set $E$. Usually this measure is denoted $\mu$*
*The proof of this theorem is long, but not inelegant. The gist is that we may measure each open set by it's "covering number", the least number of translates of an open subset that covers it. The proof is then just that this is measure indeed has all the properties we require of it.*
*What's interesting is also that the Haar measure is unique up to scaling, that is if we have two Haar measures $\mu, \nu$ then $\mu = \lambda\nu$ where $\lambda$ is some positive real constant.*
*The uniqueness of the Haar measure allows us to change the notation we used previously, making $L^1(G)$ the set of Borel functions on a LC group and implicitly using this Haar measure.*

# The Duality Theorem

## $\Gamma$ and $L^1(G)$

Through quite a bit of work (Definition 6.23 - Corollary 6.31) we can associate $\Gamma$ and $L^1(G)$ in such a way that $\Gamma$ becomes a locally compact topological space, then through by reworking the definition slightly one comes to the conclusion that the dual group $\Gamma$ of an LCH group is itself an LCH group.

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The fact that the dual of an LCH group is itself an LCH group raises an interesting question. What is the dual of a dual? The answer is that the dual group of the dual group of $G$ is (or at least is isomorphic too) $G$. This theorem, first proven in an somewhat primitive stage by Lev Pontryagin in 1934 and later expanded by Egbert van Kampen in 1935 and Weil in 1940, is the crowning jewel of Fourier analysis, this allows us to perform Fourier analysis on generic groups and use their specific Fourier transforms.