# CSE 105:
# Data Structures and Algorithms-I
# (Part 2)

Instructor

Dr Md Monirul Islam

# Text Books:

- Introduction to Algorithms
  - Thomas H. Cormen and others

- Data Structures and Algorithm analysis ( either one of C++ / Java Version)
  - Clifford A Shaffer

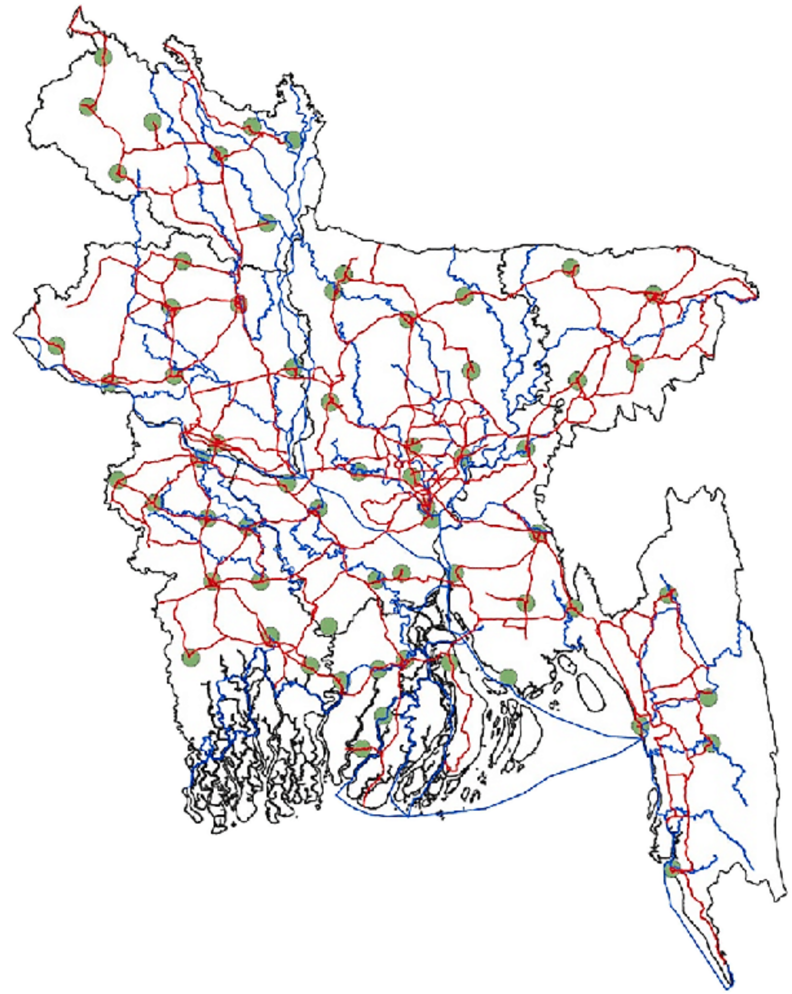
# Contents:

- trees and tree traversals; graphs and graph representations
- binary search trees;
- Heaps and priority queue
- Graph traversals: DFS, BFS, applications of DFS and BFS;

# Graphs and Trees: Representation and Search

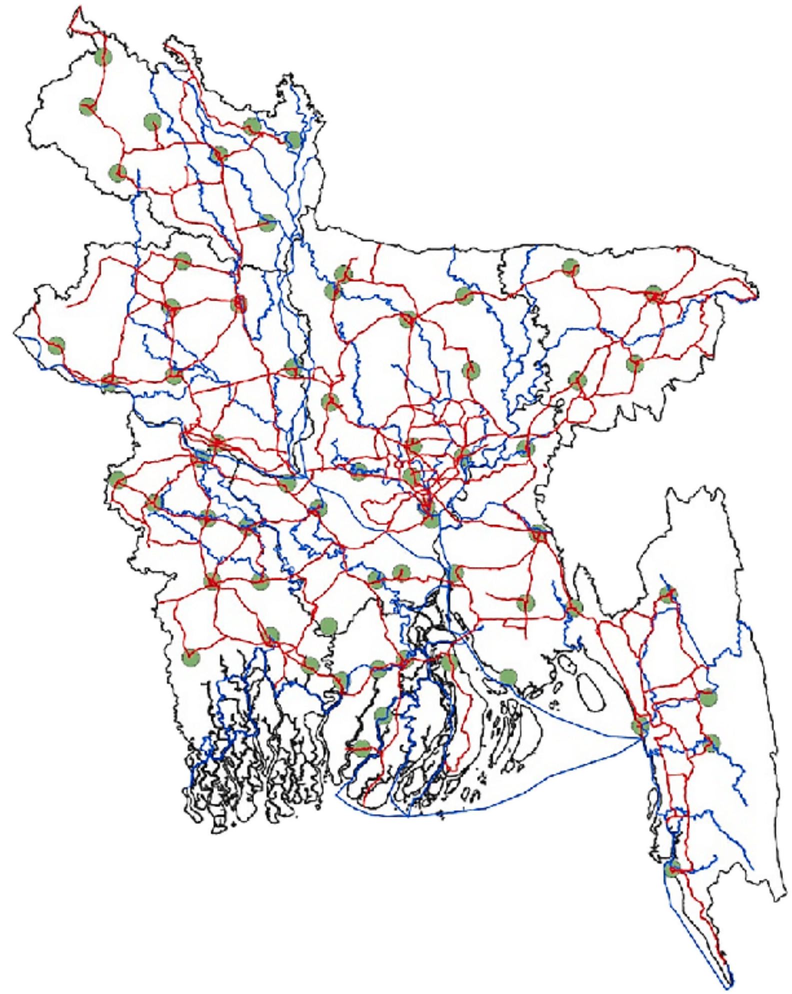
# Graph: Definition

- Bangladesh road network
  - ***Green points:*** district centroids
  - ***Red lines:*** road connecting the districts



# Graph: Definition

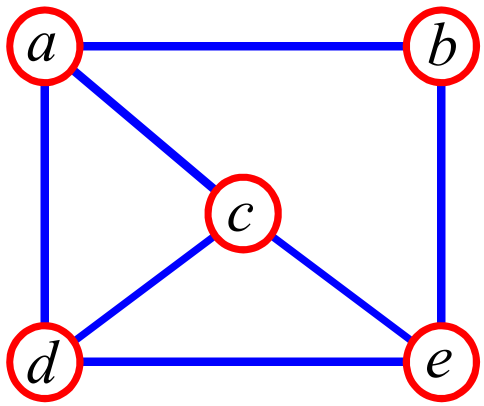
- Bangladesh road network
  - ***Green points:*** district centroids (Nodes)
  - ***Red lines:*** road connecting the districts (Edges)



A graph is a representation of a network or a relation

# Graph: Definition

- A graph is a pair $(\boldsymbol{V}, \boldsymbol{E})$, where
  - $\boldsymbol{V}$ is a set of nodes, called vertices
  - $\boldsymbol{E}$ is a collection of pairs of vertices, called edges
- *V*(*G*) and *E*(*G*) represent the sets of vertices and edges of *G*, respectively
- Example:

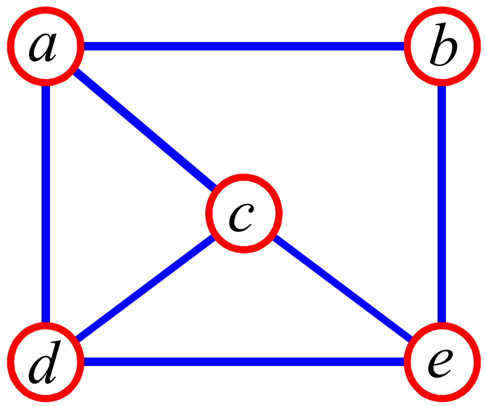



$V = \{a, b, c, d, e\}$

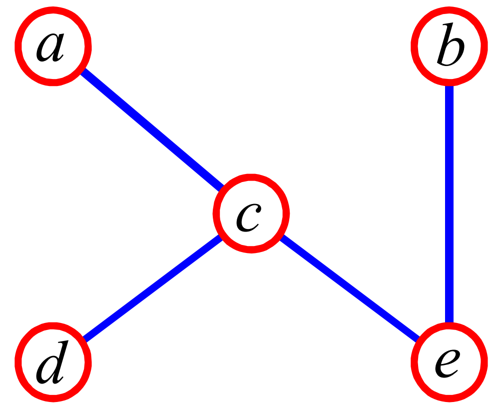
$E = \{(a, b), (a, c), (a, d), (b, e), (c, d), (c, e), (d, e)\}$

# Graph: Definition

- A tree is a special type of graph!
- A tree is a graph that is connected and acyclic.



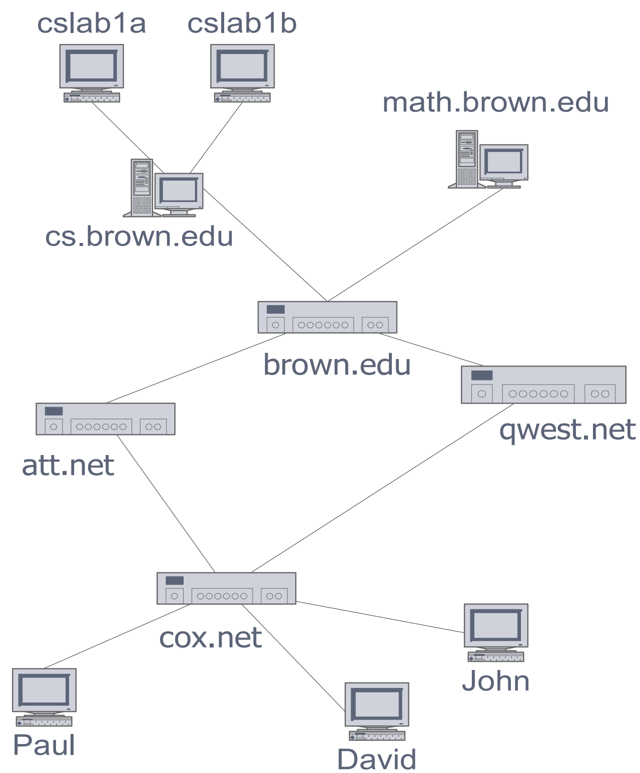

Graph




Tree

# Applications

- **Electronic circuits**
  - Printed circuit board
  - Integrated circuit
- **Transportation networks**
  - Highway network
  - Flight network
- **Computer networks**
  - Local area network
  - Internet
  - Web
- **Databases**
  - Entity-relationship diagram



# What can we do with graphs?

- Find a *path* from one place to another
- Find the *shortest path* from one place to another
- Determine connectivity
- Find the “weakest link” (min cut)
    - check amount of redundancy in case of failures
- Find the amount of flow that will go through them

# Edge and Graph Types

- Directed edge
  - ordered pair of vertices (***a***, ***b***)
  - first vertex ***a*** is the origin
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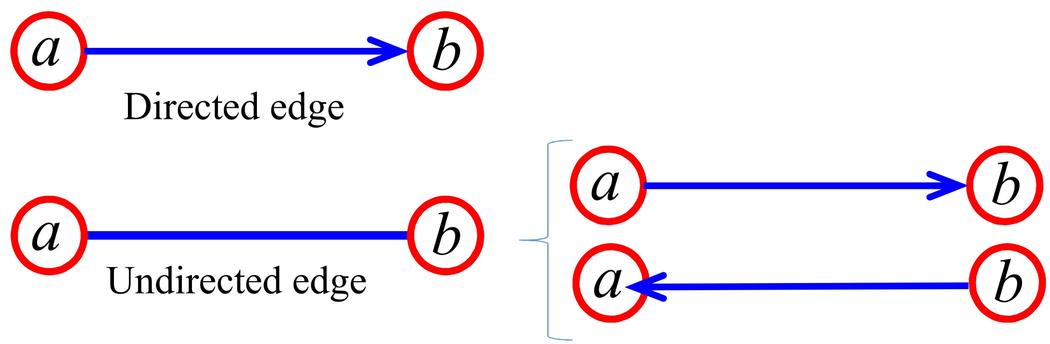

# Edge and Graph Types

- **Directed edge**
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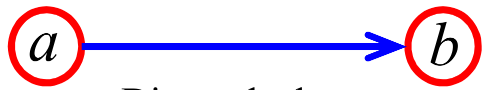
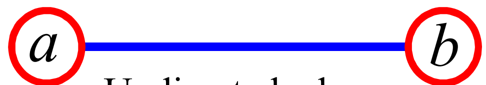
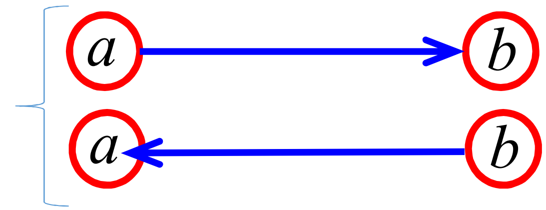
  - first vertex ***a*** is the origin
  - second vertex ***b*** is the destination
- **Undirected edge**
  - unordered pair of vertices (***a***, ***b***)
- **Directed graph (Digraph)**
  - all the edges are directed
- **Undirected graph**
  - all the edges are undirected
- **Mixed graph**
  - some edges are undirected and some edges are directed

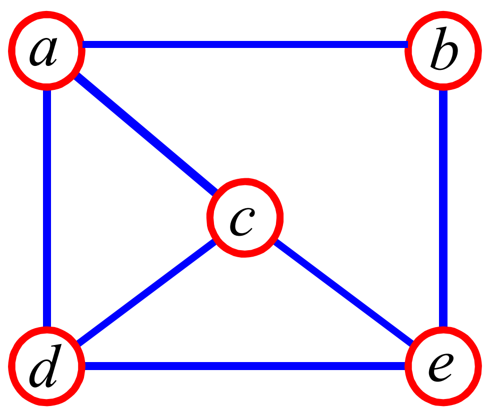



Directed edge

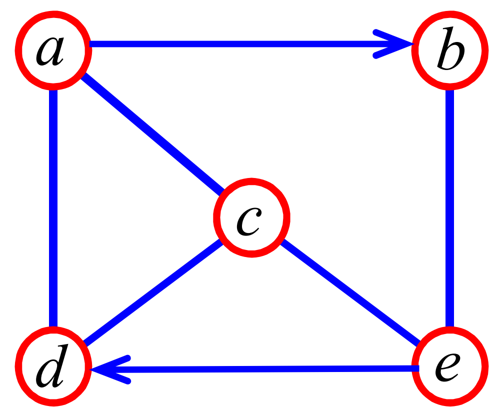



Undirected edge



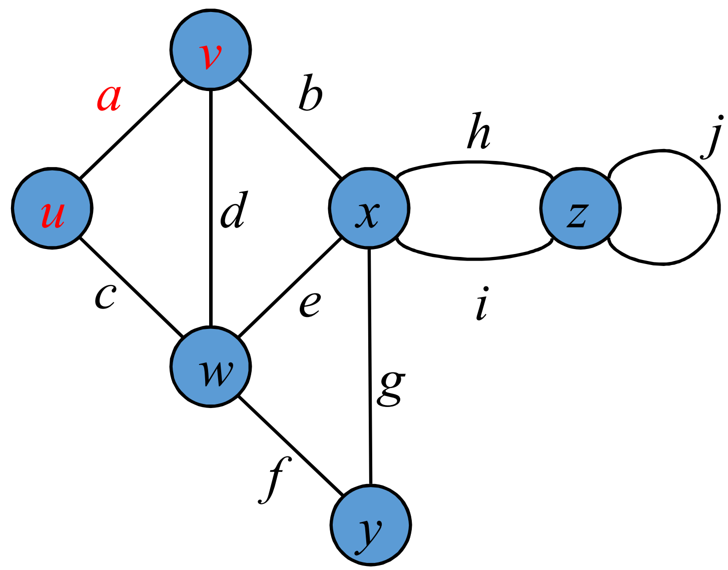

Undirected graph




Mixed graph

# Terminology

- End vertices (or endpoints) of an edge
  - $u$ and $v$ are the *endpoints* of edge $a$



# Terminology

- End vertices (or endpoints) of an edge
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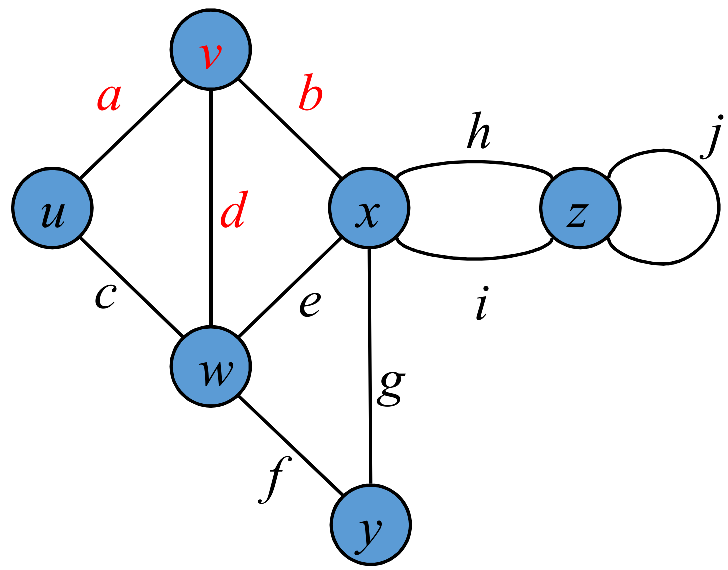
- Edges incident to a vertex
  - *a, d*, and *b* are *incident* to vertex *v*



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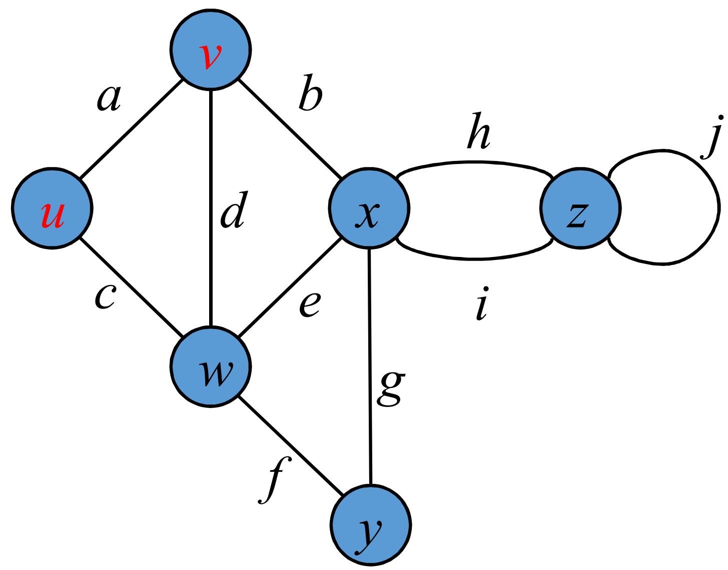
- Adjacent vertices
  - *u* and *v* are *adjacent*



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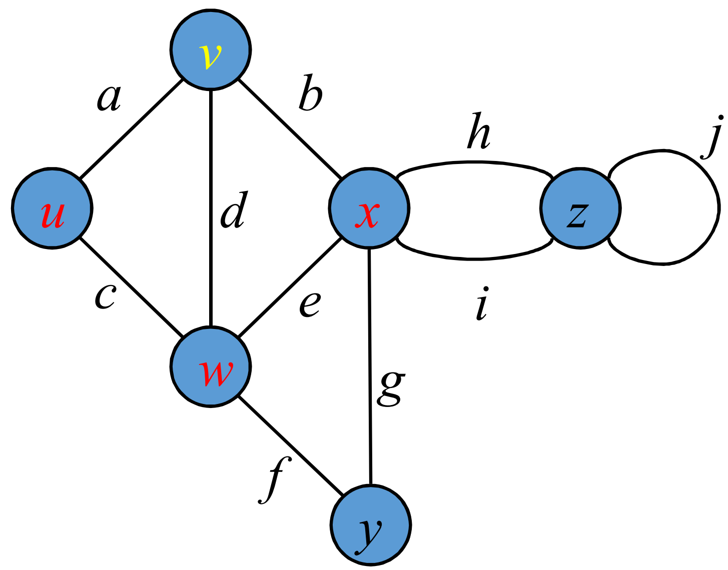
- Adjacent vertices
  - *u*, *x* and *w* are *adjacent vertices of v*



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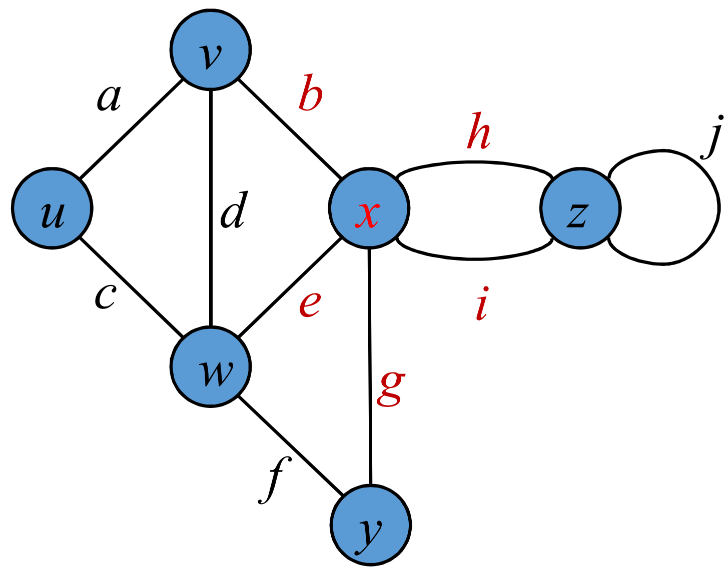
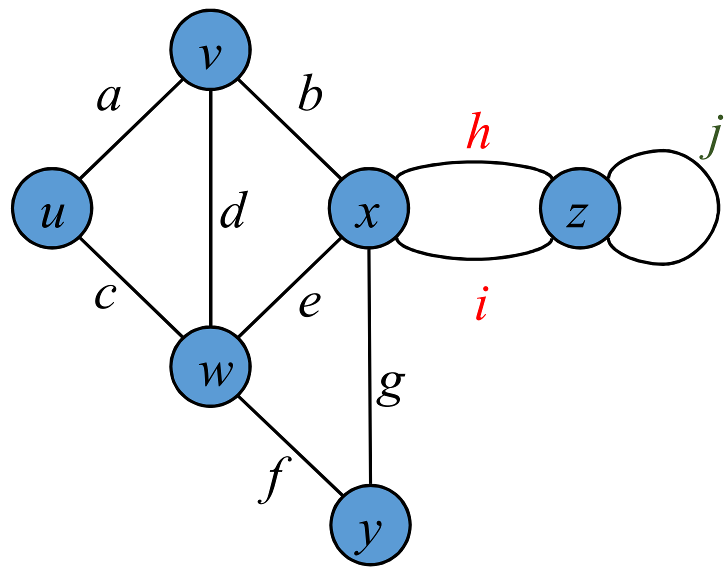
- Degree of a vertex
  - *x* has *degree* 5



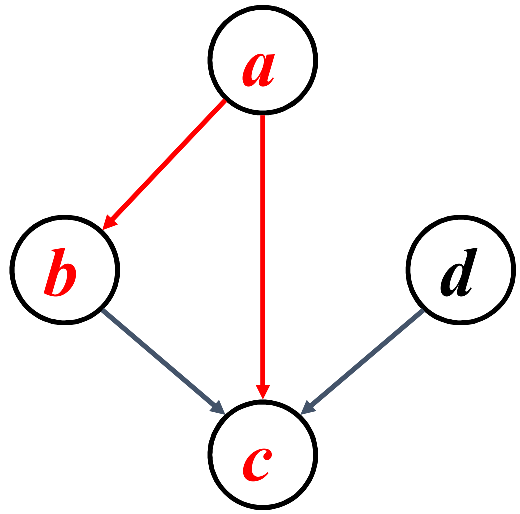
# Terminology

- Adjacent vertices
  - *u* and *v* are *adjacent*
- Degree of a vertex
  - *x* has *degree* 5
- Parallel edges
  - *h* and *i* are *parallel edges*
- Self-loop
  - *j* is a *self-loop*



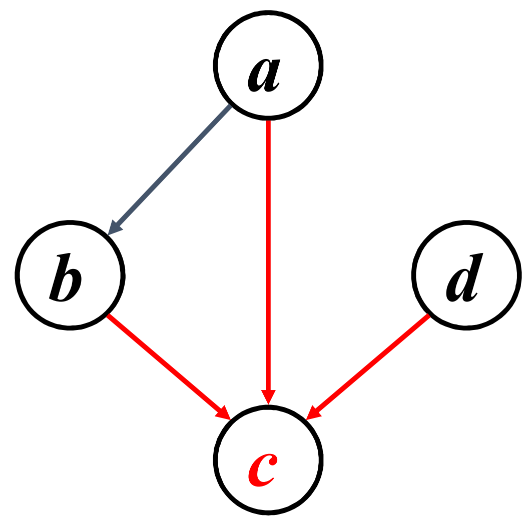
# Terminology (cont.)

- Outgoing edges of a vertex
  - ($a$, $b$) and ($a$, $c$) are outgoing edges of vertex $a$



# Terminology (cont.)

- Outgoing edges of a vertex
  - ($a$, $b$) and ($a$, $c$) are outgoing edges of vertex $a$
- Incoming edges of a vertex
  - ($b$, $c$), ($d$, $c$) and ($a$, $c$) are incoming edges of vertex $c$



# Terminology (cont.)

- Outgoing edges of a vertex
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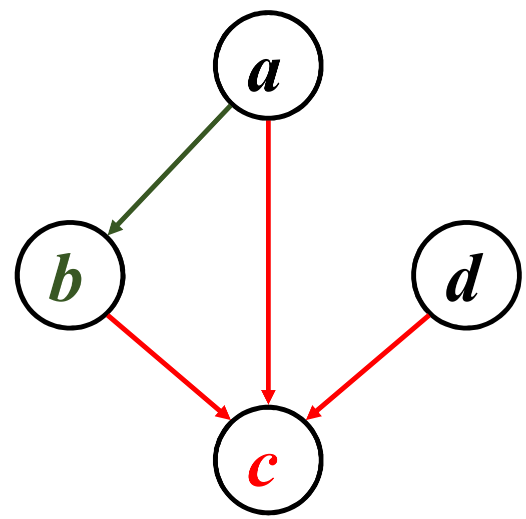
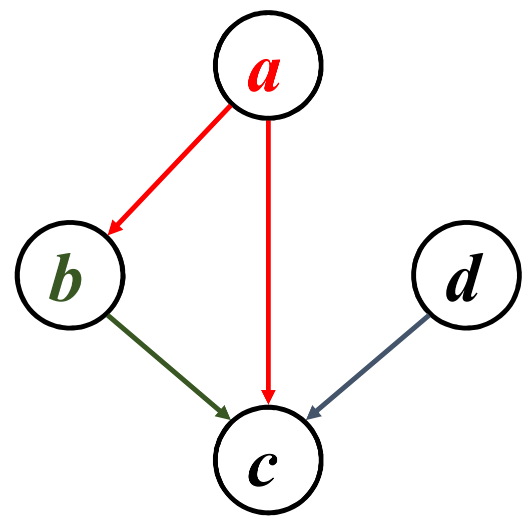
- Incoming edges of a vertex
  - ($b$, $c$), ($d$, $c$) and ($a$, $c$) are incoming edges of vertex $c$
- In-degree of a vertex
  - $c$ has *in-degree* 3
  - $b$ has *in-degree* 1



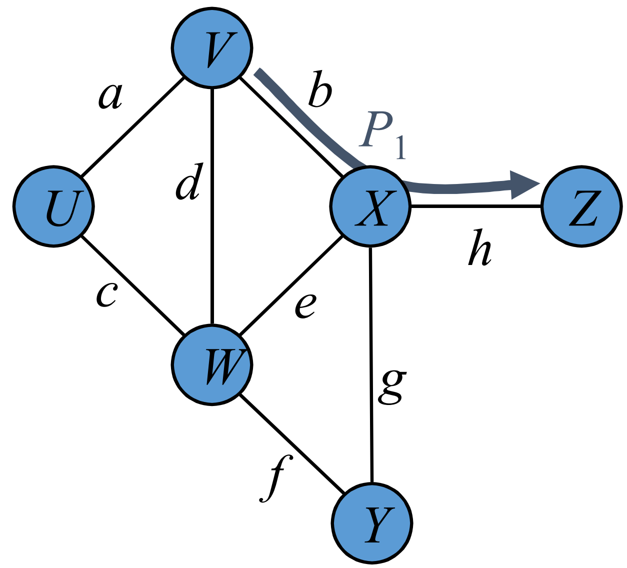
# Terminology (cont.)

- Outgoing edges of a vertex
  - ($a$, $b$) and ($a$, $c$) are outgoing edges of vertex $a$
- Incoming edges of a vertex
  - ($b$, $c$), ($d$, $c$) and ($a$, $c$) are incoming edges of vertex $c$
- In-degree of a vertex
  - $c$ has *in-degree* 3
  - $b$ has *in-degree* 1
- Out-degree of a vertex
  - $a$ has *out-degree* 2
  - $b$ has *out-degree* 1



# Terminology (cont.)

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints



# Terminology (cont.)

- Path
  - sequence of alternating vertices and edges
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  - ends with a vertex
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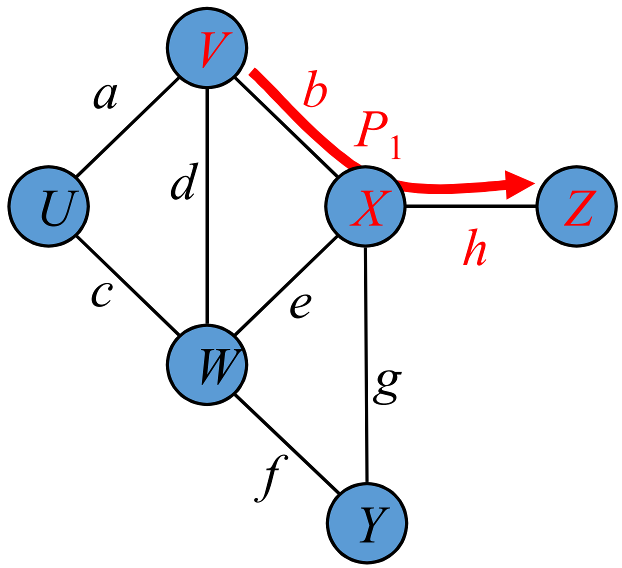
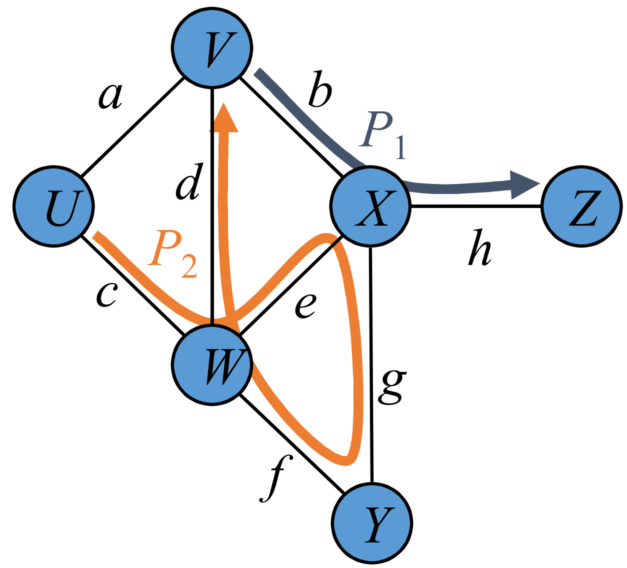
- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - $P_1=(V, b, X, h, Z)$ is a simple path



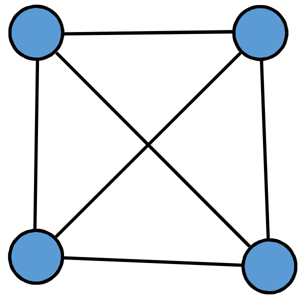
# Terminology (cont.)

- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - $P_1=(V, b, X, h, Z)$ is a simple path
  - $P_2=(U, c, W, e, X, g, Y, f, W, d, V)$ is a path that is NOT simple



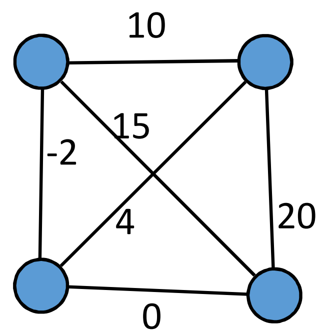
# Terminology (cont.)

- *Dense* graph: $|E| \approx |V|^2$; *Sparse* graph: $|E| \approx |V|$ or $|E| << |V|$



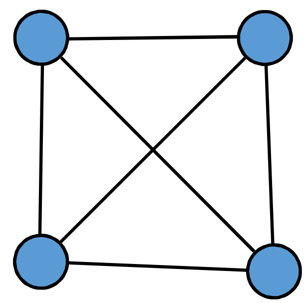
# Terminology (cont.)

- *Dense* graph: $|E| \approx |V|^2$; *Sparse* graph: $|E| \approx |V|$ or $|E| << |V|$
- A *weighted graph* associates weights with either the edges or the vertices



# Terminology (cont.)

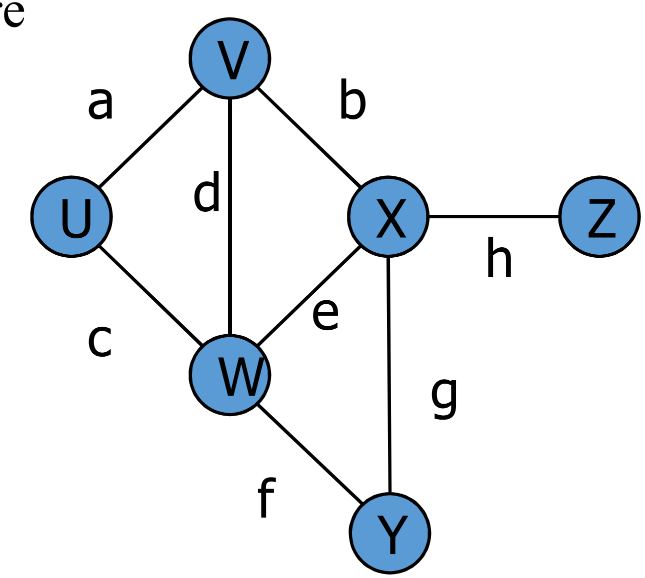
- *Dense* graph: $|E| \approx |V|^2$; *Sparse* graph: $|E| \approx |V|$ or $|E| << |V|$
- A *weighted graph* associates weights with either the edges or the vertices
- A complete graph is a graph that has the maximum number of edges
  - for undirected graph with $n$ vertices, the maximum number of edges is $n(n-1)/2$
  - for directed graph with $n$ vertices, the maximum number of edges is $n(n-1)$



A complete undirected graph

# Terminology (cont.)

- Cycle
  - A cycle is a path whose start and end vertices are the same
  - each edge is preceded and followed by its endpoints



# Terminology (cont.)

- Cycle
  - A cycle is a path whose start and end vertices are the same
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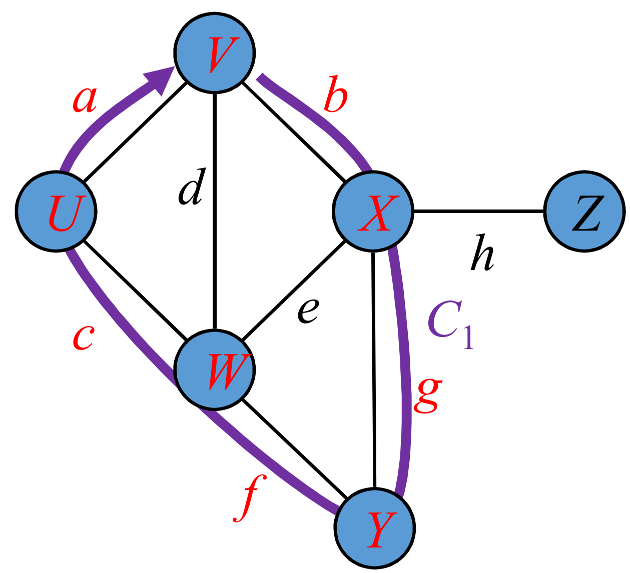
- Simple cycle
  - A cycle is simple if each edge is distinct and each vertex is distinct, except for the first and the last one
- Examples
  - $C_1=(V, b, X, g, Y, f, W, c, U, a, V)$ is a simple cycle



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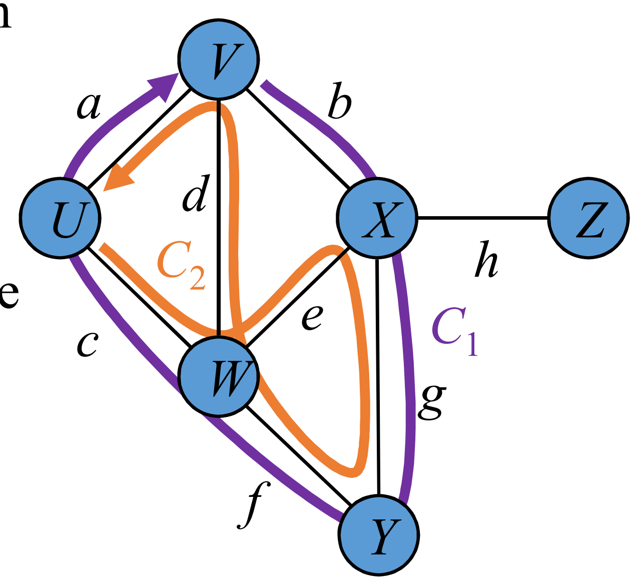
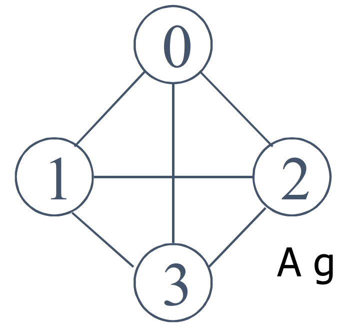
- Examples
  - $C_1=(V, b, X, g, Y, f, W, c, U, a, V)$ is a simple cycle
  - $C_2=(U, c, W, e, X, g, Y, f, W, d, V, a, U)$ is a cycle that is not simple

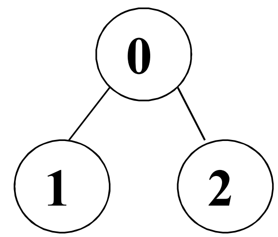


# Terminology (cont.)

- A subgraph of $G$ is a graph $G'$ such that
  - $V(G')$ is a subset of $V(G)$ [$V(G') \subseteq V(G)$] and
  - $E(G')$ is a subset of $E(G)$ [$E(G') \subseteq E(G)$]




A graph, G




A subgraph, G′

# Terminology (cont.)

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  - V(G’) is a subset of V(G) [V(G’) $\subseteq$ V(G)] and
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- A spanning subgraph G’ of G is a subgraph of G that contains all the vertices of G, that is
  - V(G’) is equal to V(G) [V(G’) = V(G)] and
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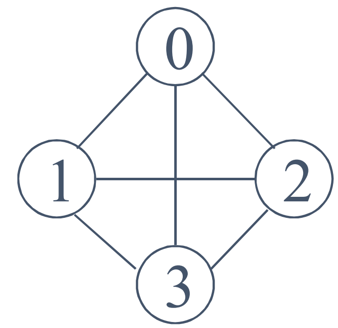
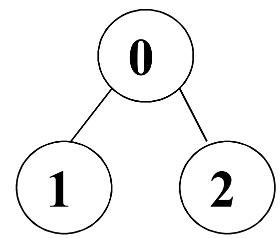
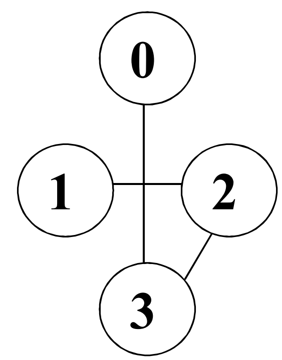



A subgraph




A spanning subgraph (tree)

# Terminology (cont.)

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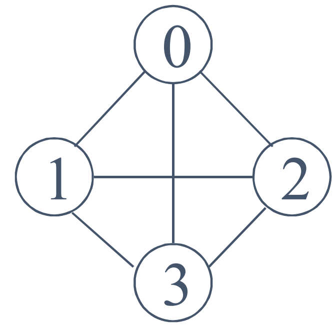
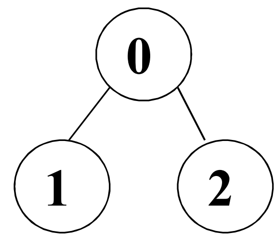
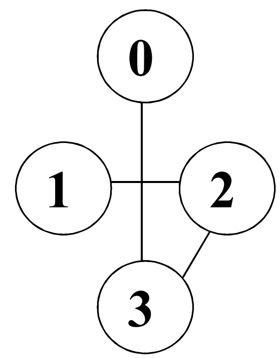
  - V(G’) is equal to V(G) [V(G’) = V(G)] and
  - E(G’) is a subset of E(G) [E(G’) $\subseteq$ E(G)]
- A forest is a graph without cycles.
- A (free) tree is a connected forest, that is, a connected graph without cycles.
- A spanning tree of a graph G is a spanning subgraph that is a (free) tree.




A subgraph




A spanning subgraph (tree)

# Properties

## Property 1

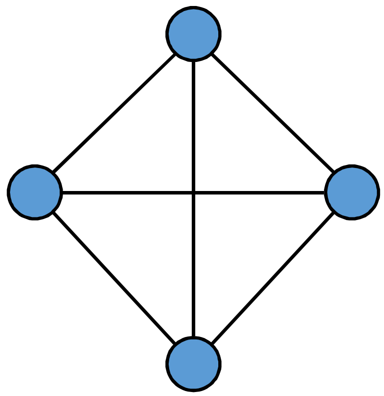
For an undirected graph

$$\Sigma_v \deg(v) = 2m$$

Proof: each edge is counted twice

### Notation

| | |
|---|---|
| $n$ | number of vertices |
| $m$ | number of edges |
| $\deg(v)$ | degree of vertex $v$ |



# Properties

Property 1

For an undirected graph

$$\Sigma_v \deg(v) = 2\boldsymbol{m}$$

Proof: each edge is counted twice

Property 2

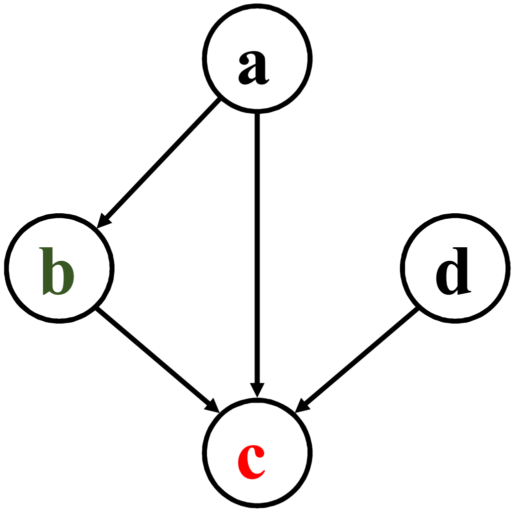
For a directed graph

$$\Sigma_v \mathrm{indeg}(v) = \Sigma_v \mathrm{outdeg}(v) = \boldsymbol{m}$$

Proof: each edge is counted once for in-degree and once for out-degree

Notation

| | |
|---|---|
| $\boldsymbol{n}$ | number of vertices |
| $\boldsymbol{m}$ | number of edges |
| $\deg(\boldsymbol{v})$ | degree of vertex $\boldsymbol{v}$ |



# Properties

## Property 2

For a directed graph

$$\Sigma_v \operatorname{indeg}(\boldsymbol{v}) = \Sigma_v \operatorname{outdeg}(\boldsymbol{v}) = \boldsymbol{m}$$

Proof: each edge is counted once for in-degree and once for out-degree

## Property 3

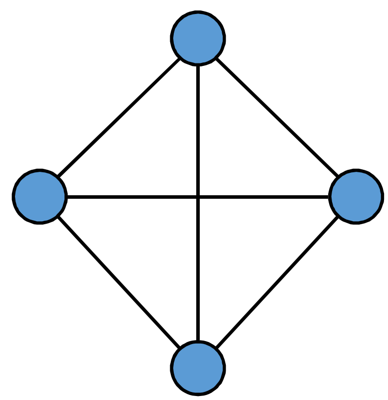
If G is a simple undirected graph, then $\boldsymbol{m} \leq \boldsymbol{n}\,(\boldsymbol{n}-1)/2$,and

if G is a simple directed graph, then $\boldsymbol{m} \leq \boldsymbol{n}\,(\boldsymbol{n}-1)$.

Proof: each vertex has degree at most $(\boldsymbol{n}-1)$. Then use Property 1 and Property 2.

### Notation

| | |
|---|---|
| $\boldsymbol{n}$ | number of vertices |
| $\boldsymbol{m}$ | number of edges |
| $\deg(\boldsymbol{v})$ | degree of vertex $\boldsymbol{v}$ |

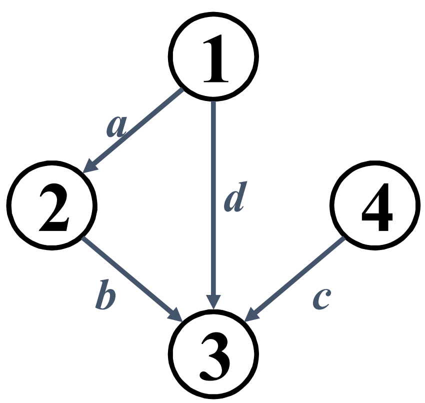


# Graph Representation

- For graphs to be computationally useful, they have to be conveniently represented in programs
- There are two computer representations of graphs:
  - **Adjacency matrix representation**
  - **Adjacency lists representation**

# Adjacency Matrix Representation

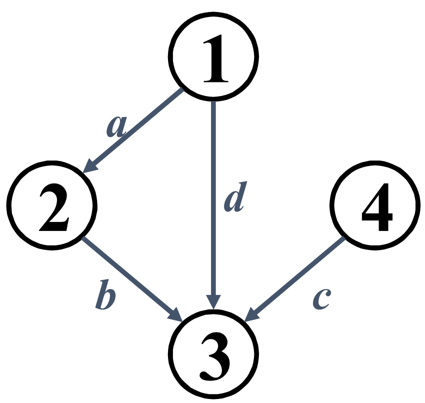
- Assume V = {1, 2, …, $n$}
- An *adjacency matrix* represents the graph as a $n$ x $n$ matrix A:
  - A[$i$, $j$] = 1 if edge $(i, j) \in E$ (or weight of edge)
    = 0 if edge $(i, j) \notin E$




| A | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

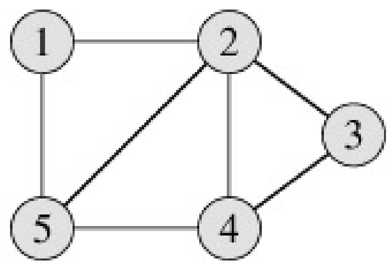
# Adjacency Matrix Representation

- Assume V = {1, 2, …, *n*}
- An *adjacency matrix* represents the graph as a *n* x *n* matrix A:
  - A[*i*, *j*] = 1 if edge (*i*, *j*) $\in$ E (or weight of edge)
    = 0 if edge (*i*, *j*) $\notin$ E




| A | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 1 | 0 |

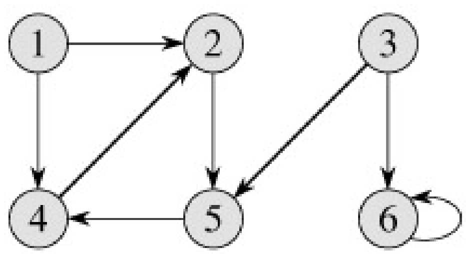
# Adjacency Matrix Representation




Undirected Graph

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 \\
2 & 1 & 0 & 1 & 1 & 1 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 1 & 0 & 1 \\
5 & 1 & 1 & 0 & 1 & 0 \\
\hline
\end{array}
$$

Symmetric matrix




Directed Graph

$$
\begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0 & 1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 1 \\
4 & 0 & 1 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

may NOT be symmetric

# Adjacency Matrix Representation

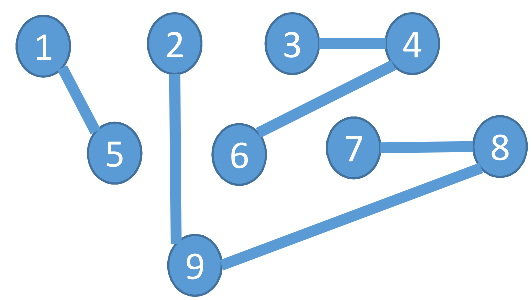
- Pros:
  - Simple to implement
  - Easy and fast to tell if a pair (i, j) is an edge: simply check if A[i, j] is 1 or 0
  - Can be very efficient for small graphs
  - Good for dense graphs (why?)
- Cons:
  - No matter how few edges the graph has, the matrix takes O($n^2$), i.e., $O(|V|^2)$ in memory

$|V| = n$

$|V| = n$

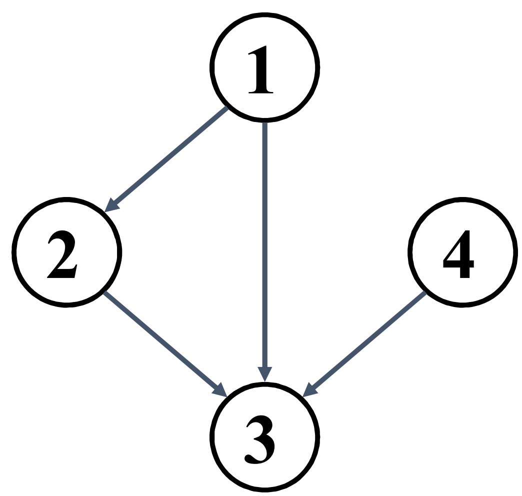
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| **2** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| **3** | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| **4** | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| **5** | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **6** | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| **7** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| **8** | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| **9** | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

$n \times n$ matrix

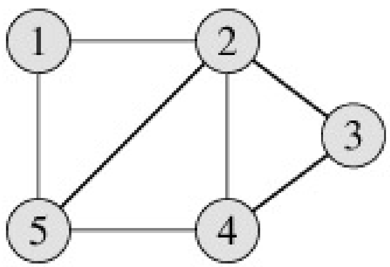


# Adjacency Lists Representation

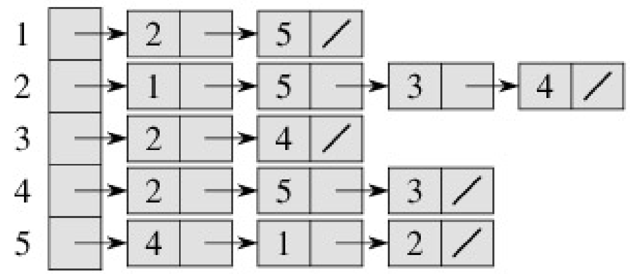
- A graph is represented by a one-dimensional array L of linked lists, where
  - L[i] is the linked list containing all the nodes adjacent to node i.
  - The nodes in the list L[i] are in NO particular order

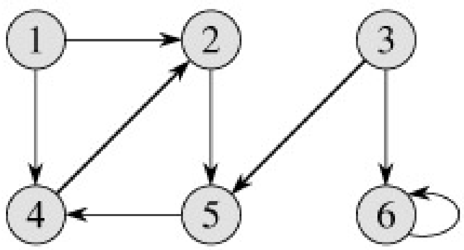


# Adjacency Lists Representation

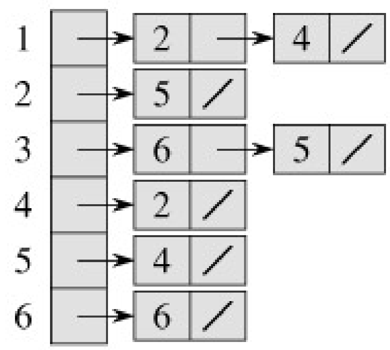



Undirected Graph



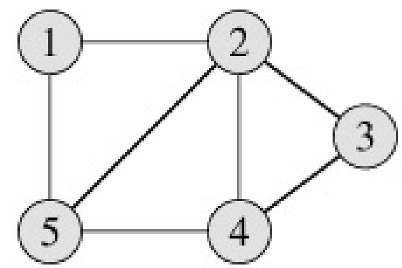

Directed Graph



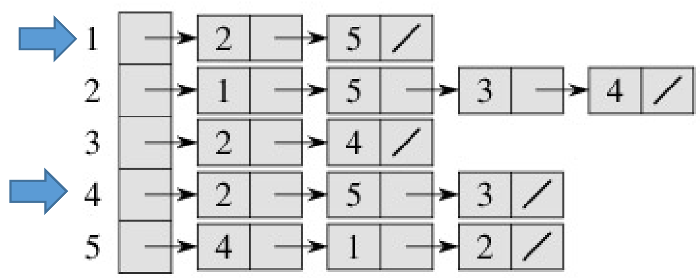
# Adjacency Lists Representation

- Pros:
  - Saves on space (memory): the representation takes O(|V|+|E|) memory.
  - Good for large, sparse graphs (e.g., planar maps)

How to find whether there is an edge (4,1)?

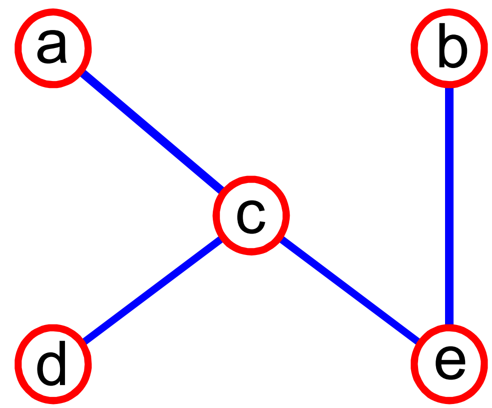



- Cons:
  - It can take up to O(n) time to determine if a pair of nodes (i, j) is an edge: one would have to search the linked list L[i], which takes time proportional to the length of L[i].



# Tree: Definition

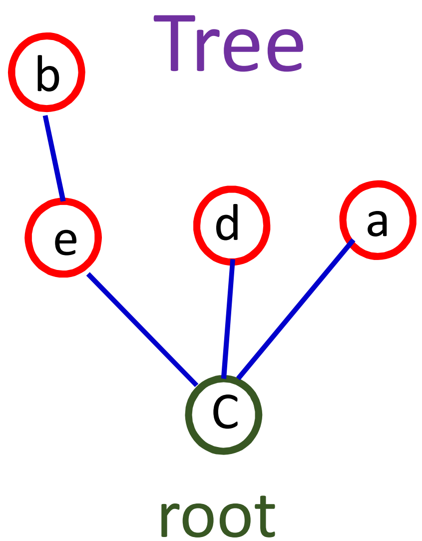
- A tree is a special type of graph!
- A tree is a graph that is connected and acyclic.
- A tree consists of one or more nodes
- a free tree has NO special node.

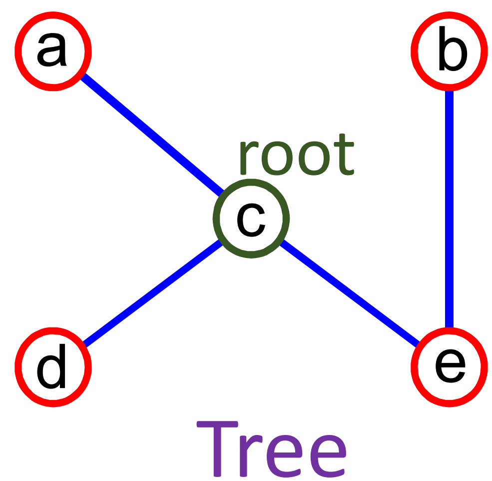



Tree

# Rooted Tree: Definition

- a rooted tree has a special node, e.g., first node  or root.
- every node has a parent except the root
- every node has zero or more children



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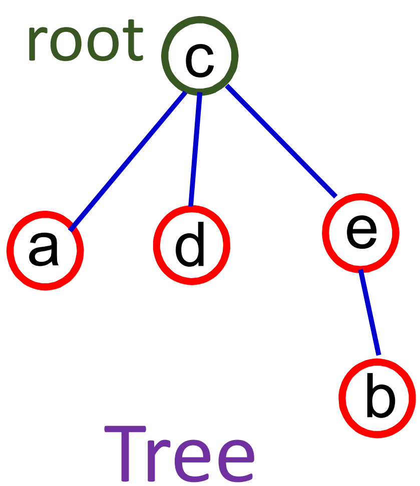
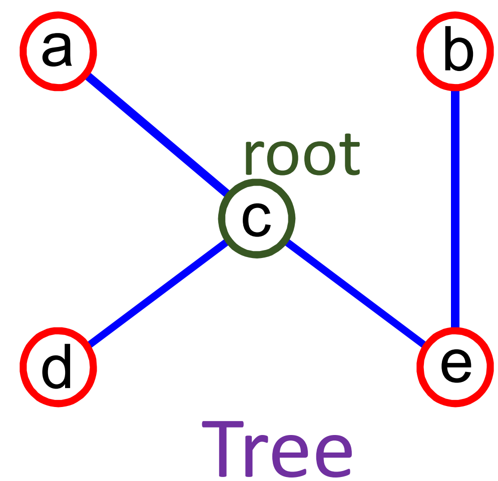


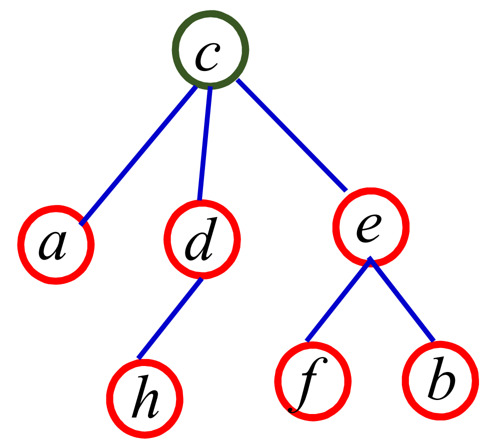
# Rooted Tree: Definition

**degree of a node:**

No. of children of a node

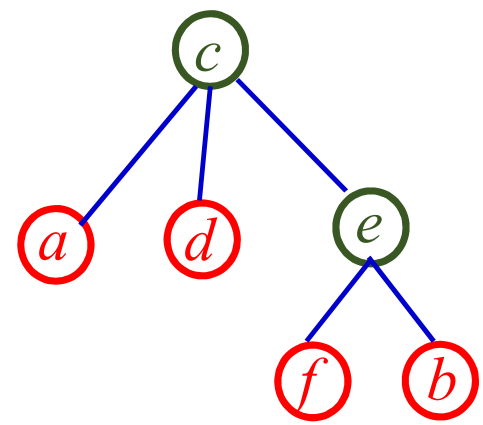
degree ($c$) =3, degree ($e$) =2 , degree ($d$) =1

others have degree 0



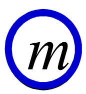
# Rooted Tree: Definition

- **Leaf** nodes: nodes with degree 0: *a, d, f, b*
- **Internal** nodes: other nodes : *c, e*

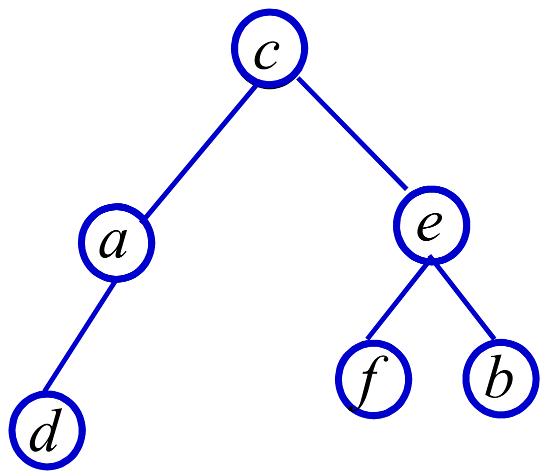


# Binary Tree: Definition

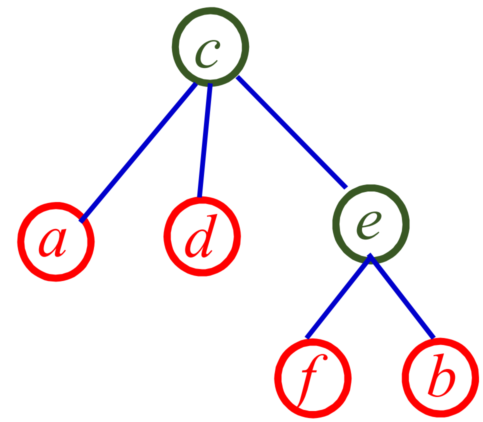
all nodes have at most 2 children




Binary tree



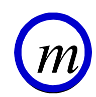

Binary tree



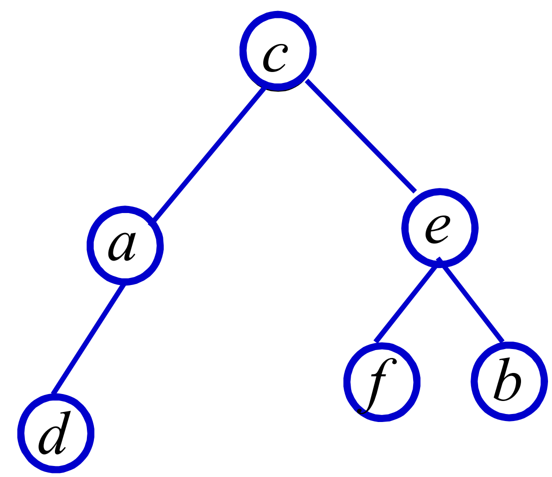

Non-binary tree

# Binary Tree: Definition

- A binary tree is made up of a finite set of nodes
- This set either is empty or consists of a node called the root together with two binary trees, called the left and right subtrees
- Subtrees are disjoint from each other and from the root



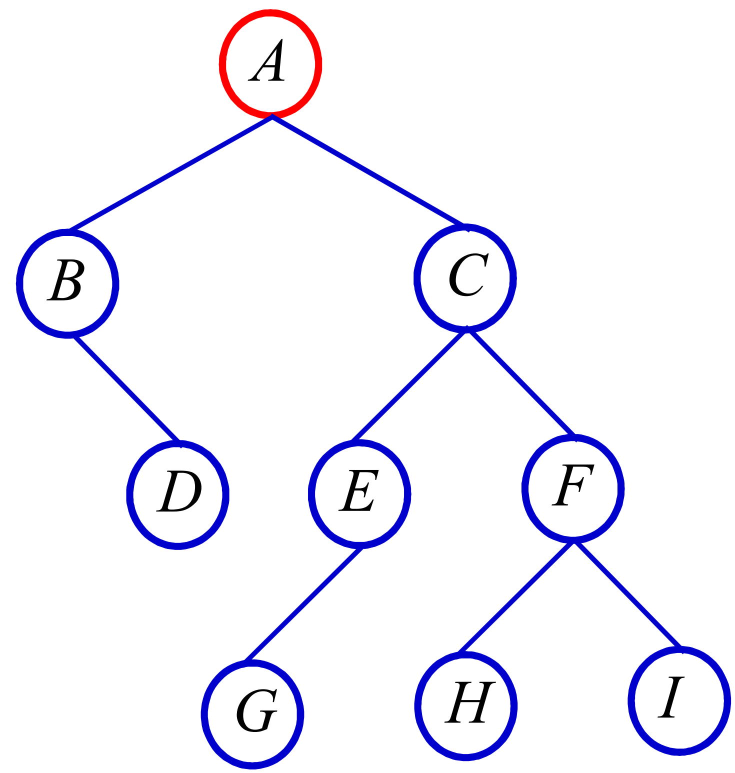

Binary tree




Binary tree

# Binary Tree: Elements

One or more Nodes, with a special node called root



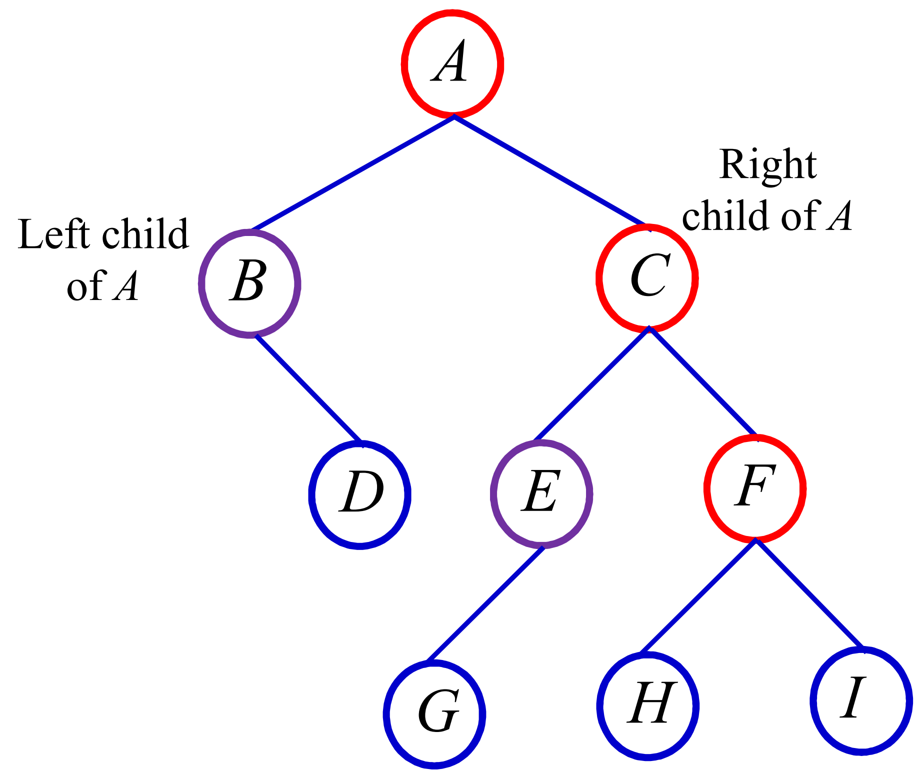
# Binary Tree: Elements

**Children**:

Every node has 0, 1 or 2 children

Leaf nodes: have no child

Internal nodes: have 1 or 2 children



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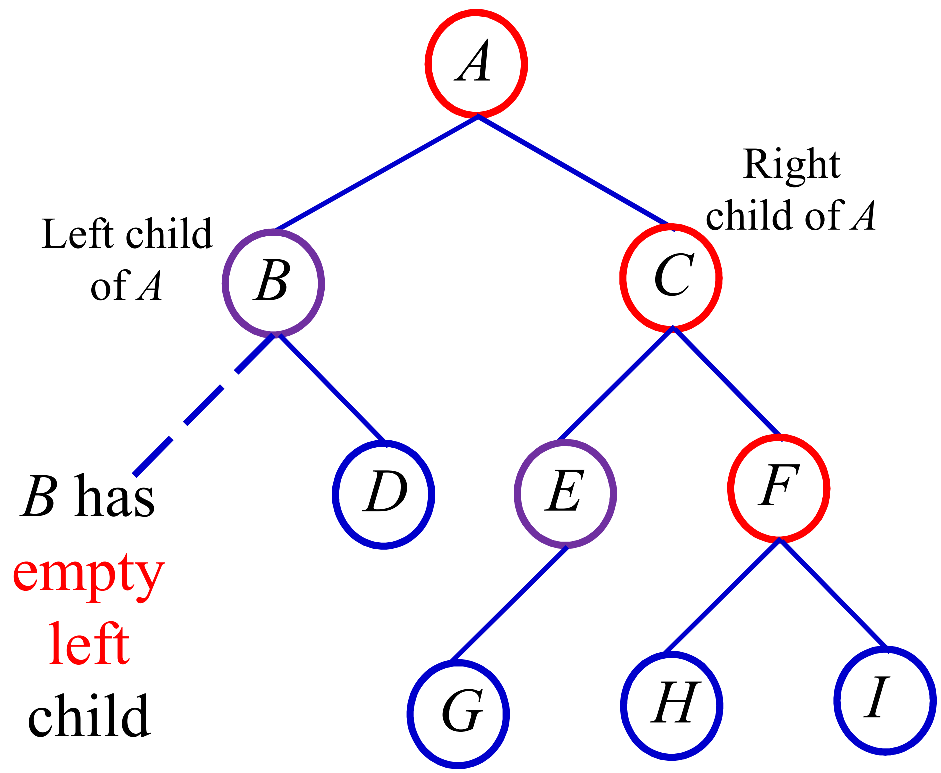
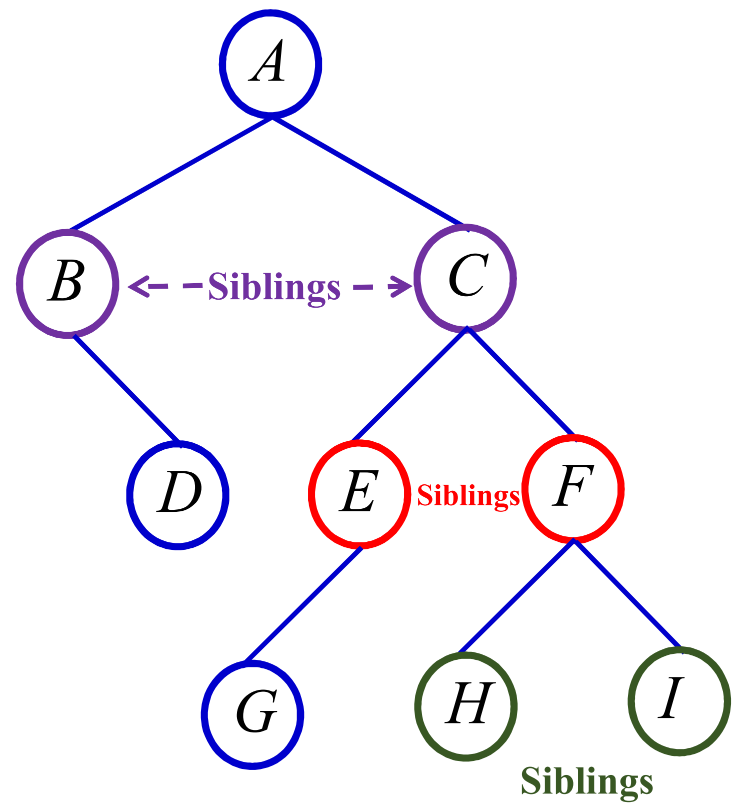


# Binary Tree: Elements

**Siblings**:

Immediate children of the same parent



# Binary Tree: Elements

**Edge:**

Each node is connected to each of its children by an edge