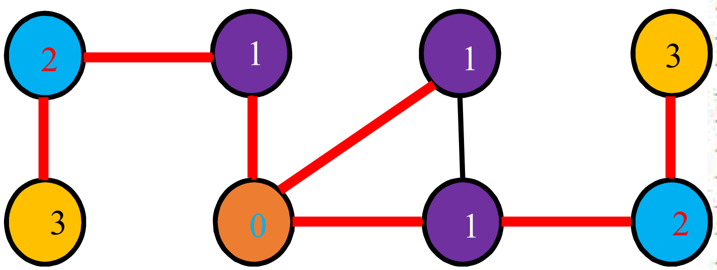
CSE 105:
Data Structures and Algorithms-I
(Part 2)

Instructor

Dr Md Monirul Islam

Graph Searching

Breadth-First Search




```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = Ø
ENQUEUE(Q, s)
while Q ≠ Ø
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Review

Whitening

Enqueue the root

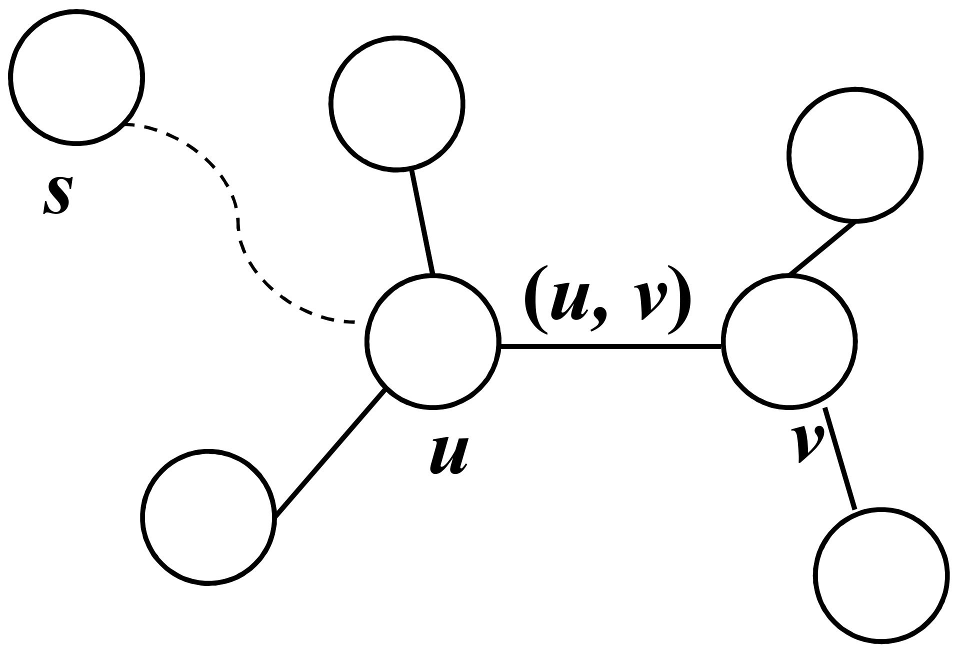
runs until queue is empty

Lemma 22.1

Let G = (V, E) be a directed or undirected graph, and let $s \in V$ be an arbitrary vertex. Then, for any edge $(u, v) \in E$,

$\delta(s, v) \leq \delta(s, u) + 1$

Review



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s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
  Next      v.π = u
  EnQs      ENQUEUE(Q, v)
    u.color = BLACK
```

Lemma 22.2

Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex s Ɛ V. Then upon termination, for each vertex v Ɛ V, the value $v.d$ computed by BFS satisfies P $v.d \geq \delta(s, v)$

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```

Next EnQs

Lemma 22.2

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$v.d \geq \delta(s, v)$

Assume, before an EnQ, P holds

Then show, after the next EnQ,
P still holds

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Next EnQs

Assume, before an EnQ, P holds

Then show, after the next EnQ, P still holds

Lemma 22.2

Then upon termination, for each vertex v Ɛ V, the value $v.d$ computed by BFS satisfies $v.d \geq \delta(s, v)$ P

Basis:

$s.d = 0 = \delta(s, s)$ ⇒ $s.d \geq \delta(s, s)$

and

$v.d = \infty \geq \delta(s, v)$ for all other vertices v

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Next EnQs

Assume, before an EnQ, P holds

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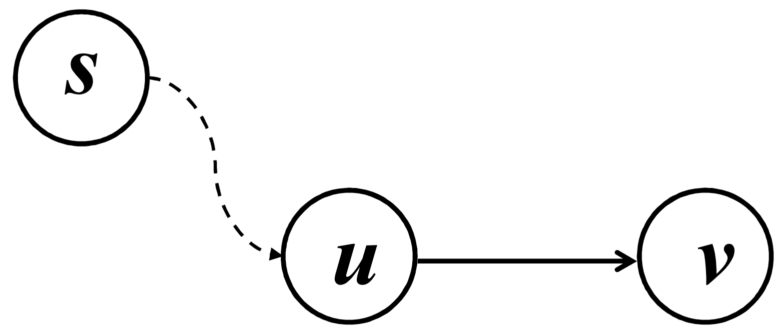
Lemma 22.2

Then upon termination, for each vertex $v \in V$, the value $v.d$ computed by BFS satisfies $v.d \geq \delta(s, v)$ P

Induction:

Let, white vertex v is discovered from u



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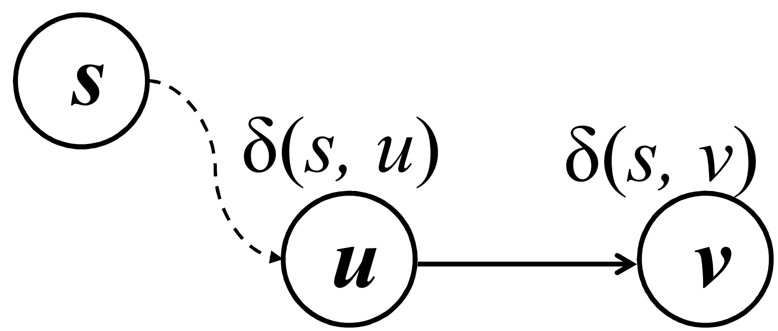

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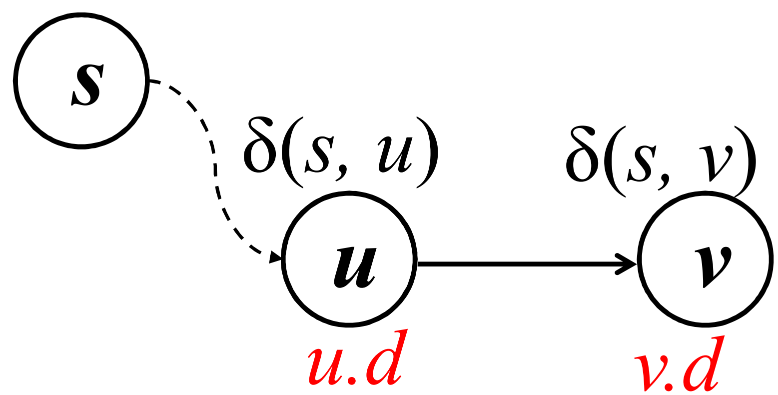
Then upon termination, for each vertex $v \in V$, the value $v.d$ computed by BFS satisfies $v.d \geq \delta(s, v)$ P

Induction:

Let, white vertex v is discovered from u

Now, by induction: $u.d \geq \delta(s, u)$



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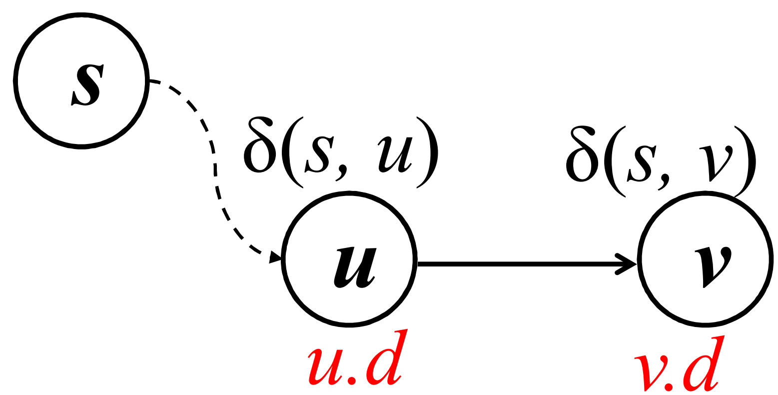
Now, by induction: $u.d \geq \delta(s, u)$

By lemma 22.1: $\delta(s, v) \leq \delta(s, u) + 1$



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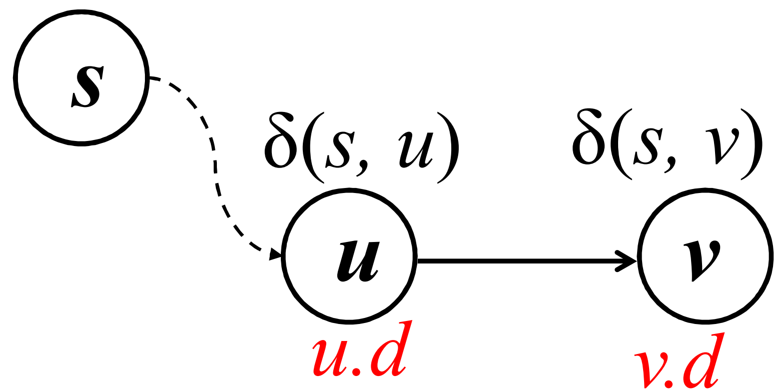
By lemma 22.1: $\delta(s, v) \leq \delta(s, u) + 1$

Now, $v.d = u.d + 1$



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Induction:

Let, white vertex v is discovered from u

Now, by induction: $u.d \geq \delta(s, u)$

By lemma 22.1: $\delta(s, v) \leq \delta(s, u) + 1$

$$
\begin{aligned}
\text{Now, } v.d &= u.d + 1 \\
&\geq \delta(s, u) + 1 \\
&\geq \delta(s, v)
\end{aligned}
$$

```
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Lemma 22.3

Suppose that during the execution of BFS on a graph $G = (V, E)$, the queue Q contains the vertices $<v_1, v_2, \cdots, v_r>$ where v_1 is the head of Q and v_r is the tail. Then (1) $v_r.d \leq v_1.d + 1$ and (2) $v_i.d \leq v_{i+1}.d$ for $i = 1, 2, \cdots, r-1$

v_1	v_2	v_3	.	..	.	.	.		v_r

Vertices in Queue

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BFS(G, s)
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        if v.color == WHITE
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            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Lemma 22.3

during the execution, the queue $Q = <v_1, v_2, \cdots, v_r>$ where v_1 = head and v_r = tail

Then (1) $v_r.d \le v_1.d + 1$ and

(2) $v_i.d \le v_{i+1}.d$ for $i = 1, 2, \cdots, r-1$

v_1	v_2	v_3	.	..	.	.	.		v_r

Vertices in Queue

We have to prove:

2. $v_1.d \le v_2.d \le v_3.d \ldots \le v_r.d$

1. $v_r.d \le v_1.d + 1$

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Lemma 22.3

$Q = <v_1, v_2, \cdots, v_r>$ where v_1 = head and v_r = tail

Prove: (1) $v_r.d \le v_1.d + 1$

(2) $v_i.d \le v_{i+1}.d$ for $i = 1, 2, \cdots, r\text{-}1$

v_1	v_2	v_3	.	..	.	.	.		v_r

Basis:

It is true., as queue contains only s.

```
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Lemma 22.3

$Q = \langle v_1, v_2, \cdots, v_r \rangle$ where v_1 = head and v_r = tail

Prove: (1) $v_r.d \le v_1.d + 1$

(2) $v_i.d \le v_{i+1}.d$ for $i = 1, 2, \cdots, r\text{-}1$

v_1	v_2	v_3	.	..	.	.	.		v_r

We will prove both for Dequeue and Enqueue operations

1. $v_1.d \le v_2.d \le v_3.d \ldots \le v_r.d$

2. $v_r.d \le v_1.d + 1$

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Lemma 22.3

Prove: (1) $v_r.d \leq v_1.d + 1$

(2) $v_1.d <= v_2.d <= v_3.d <= \cdots <= v_r.d$

v_1	v_2	v_3	.	..	.	.	.		v_r

Before DEQUEUE

Induction

Before DEQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$

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Lemma 22.3

Prove: (1) $v_r.d \leq v_1.d + 1$

(2) $v_1.d <= v_2.d <= v_3.d <= \cdots <= v_r.d$

v_1	v_2	v_3	.	..	.	.	.		v_r	Before DEQUEUE

v_2	v_3	.	..	.	.	.		v_r	After DEQUEUE

Induction

Before DEQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$

After DEQUEUE (of v1):

$v_2.d <= v_3.d \ldots <= v_r.d$ (Okay)

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Lemma 22.3

Prove: (1) $v_r.d \leq v_1.d + 1$

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v_1	v_2	v_3	.	..	.	.	.		v_r

Before DEQUEUE

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Induction

Before DEQUEUE, IH holds:

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$v_r.d <= v_1.d + 1$

After DEQUEUE (of v1):

$v_2.d <= v_3.d \ldots <= v_r.d$ (Okay)

$v_r.d <= v_1.d + 1$ (from previous relation)

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Lemma 22.3

Prove: (1) $v_r.d \leq v_1.d + 1$

(2) $v_1.d <= v_2.d <= v_3.d <= \cdots <= v_r.d$

v_1	v_2	v_3	.	..	.	.	.		v_r	Before DEQUEUE
	v_2	v_3	.	..	.	.	.		v_r	After DEQUEUE

Induction

Before DEQUEUE, IH holds:

$\boxed{v_1.d <= v_2.d} <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$

After DEQUEUE (of v1):

$v_2.d <= v_3.d \ldots <= v_r.d$ (Okay)

$v_r.d <= \boxed{v_1.d + 1 <= v_2.d + 1}$

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Lemma 22.3

Prove: (1) $v_r.d \leq v_1.d + 1$

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v_1	v_2	v_3	.	..	.	.	.		v_r

Before DEQUEUE

v_2	v_3	.	..	.	.	.		v_r

After DEQUEUE

Induction

Before DEQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$

After DEQUEUE (of v1):

$v_2.d <= v_3.d \ldots <= v_r.d$ (Okay)

$v_r.d <= v_1.d + 1 <= v_2.d + 1$ (Okay)

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Before ENQUEUE

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Before ENQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$

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v_1	v_2	v_3	.	..	.	.	.		v_r	Before ENQUEUE

v_1	v_2	v_3	.	..	.	.	.		v_r	v_{r+1}	After ENQUEUE

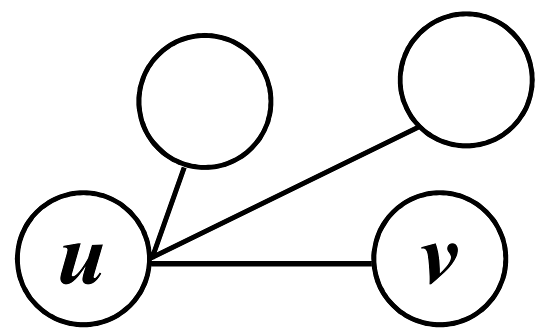
v

Induction

Before ENQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$



After ENQUEUE (of v):

Let, we enqueue v from u

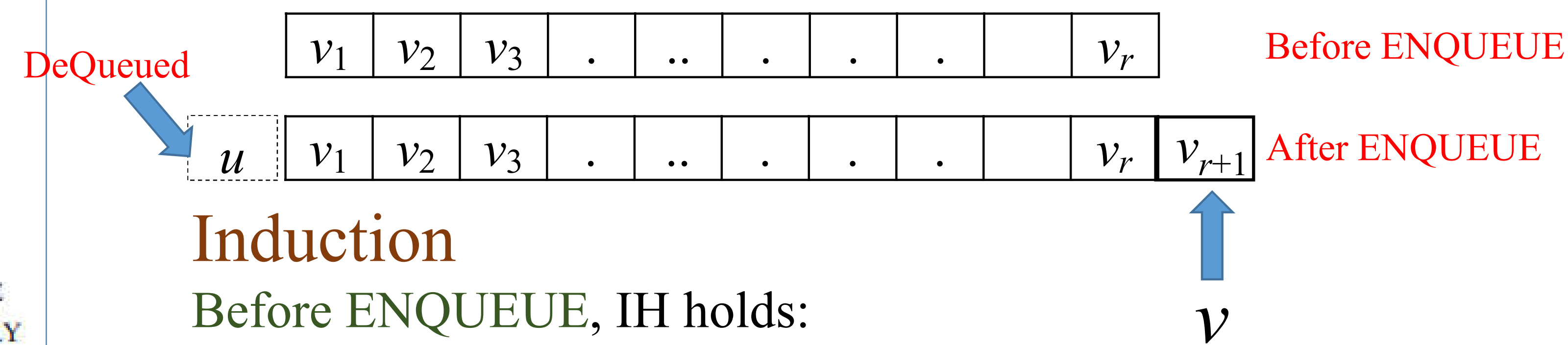
v becomes v_{r+1}.

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$

(2) $v_1.d <= v_2.d <= v_3.d <= \cdots <= v_r.d$

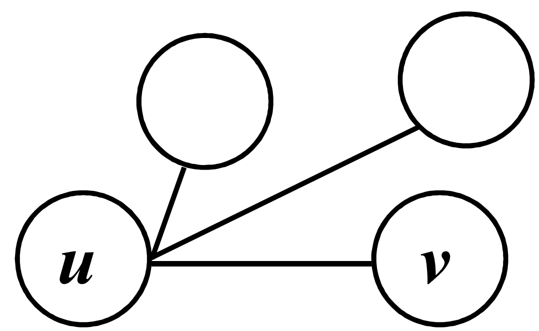


Induction

Before ENQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$



After ENQUEUE (of v):

u was is IN queue but dequeued

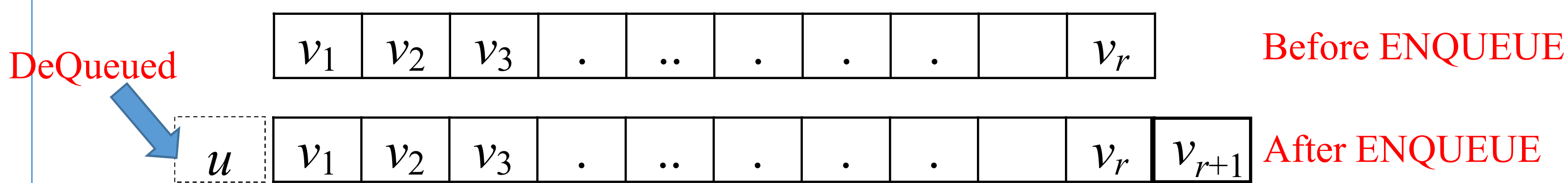
$u.d \le v_1.d$

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Lemma 22.3

Prove: (1) $v_r.d \leq v_1.d + 1$

(2) $v_1.d <= v_2.d <= v_3.d <= \cdots <= v_r.d$

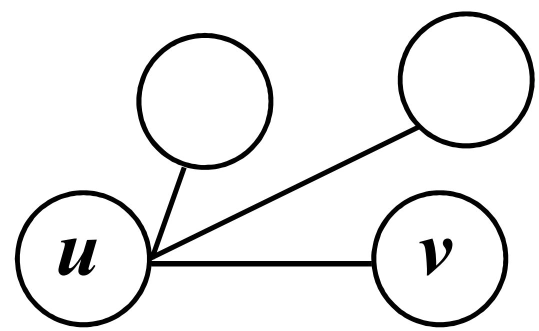


Induction

Before ENQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$



After ENQUEUE (of v):

u was is IN queue but dequeued

$u.d \leq v_1.d$

$v_{r+1}.d = v.d = u.d + 1$

```
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for each vertex u ∈ G.V − {s}
    u.color = WHITE
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    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
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Lemma 22.3

Prove: (1) $v_r.d \leq v_1.d + 1$

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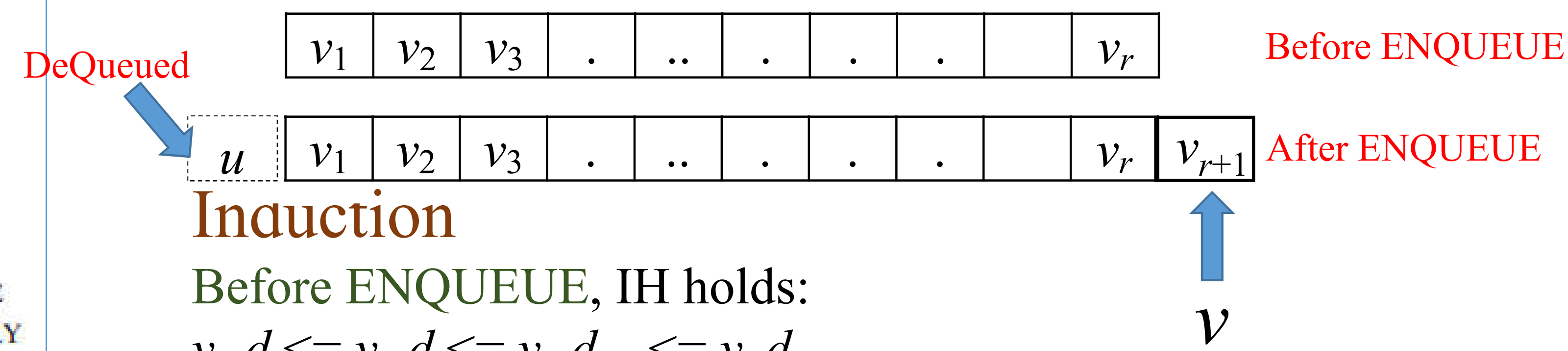
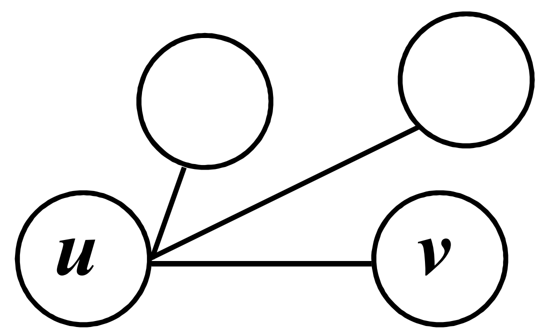


Induction

Before ENQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$



After ENQUEUE (of v):

u was is IN queue but dequeued

$u.d \leq v_1.d$

$v_{r+1}.d = v.d = u.d + 1 \leq v_1.d + 1$

```
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for each vertex u ∈ G.V − {s}
    u.color = WHITE
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            v.color = GRAY
            v.d = u.d + 1
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    u.color = BLACK
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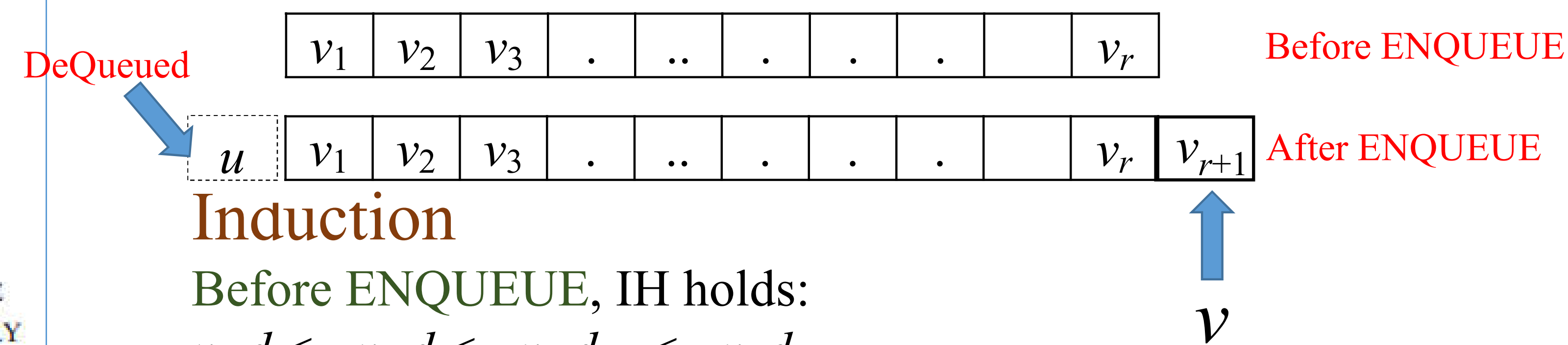
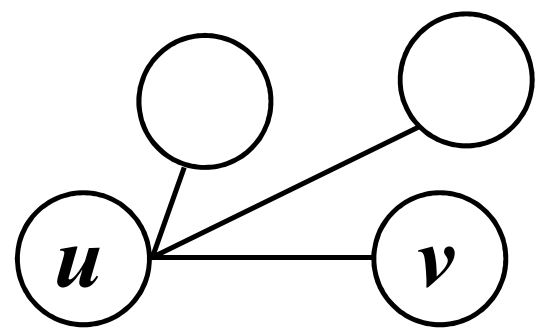


Induction

Before ENQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$



After ENQUEUE (of v):

u was is IN queue but dequeued

$u.d \le v_1.d$

$v_{r+1}.d \le v_1.d+1$

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
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ENQUEUE(Q, s)
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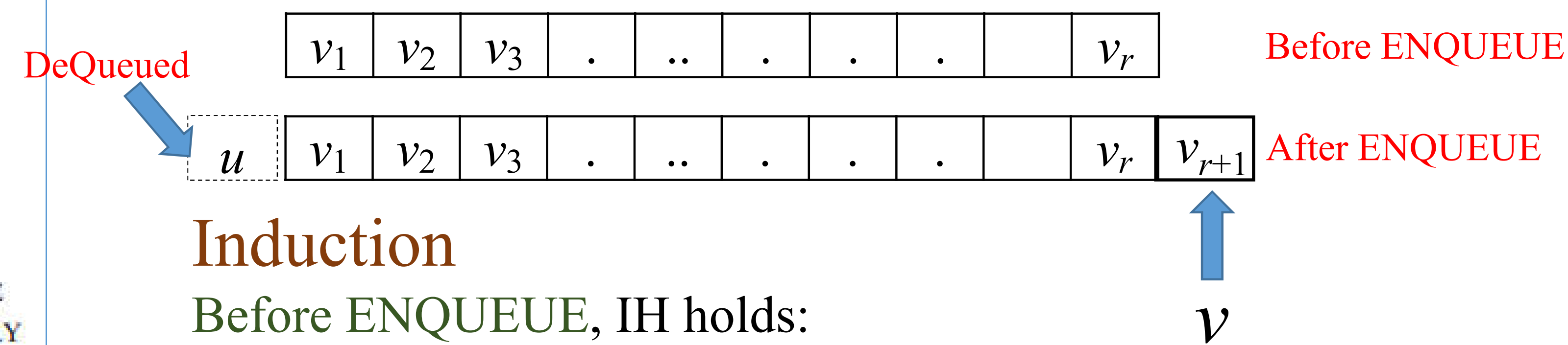
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$

(2) $v_1.d <= v_2.d <= v_3.d <= \cdots <= v_r.d$

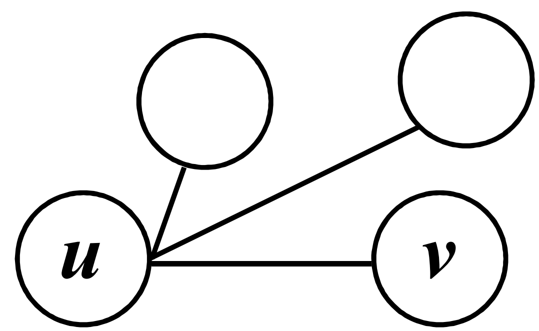


Induction

Before ENQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$



After ENQUEUE (of v):

By induction

$v_r.d \le u.d+1$

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
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Q = ∅
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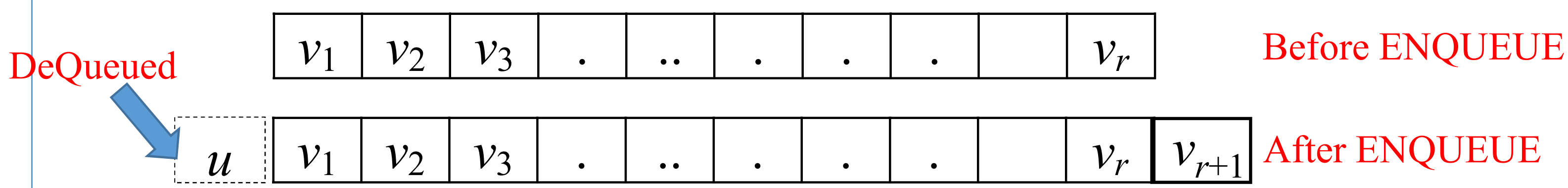
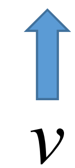
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$

(2) $v_1.d <= v_2.d <= v_3.d <= \cdots <= v_r.d$



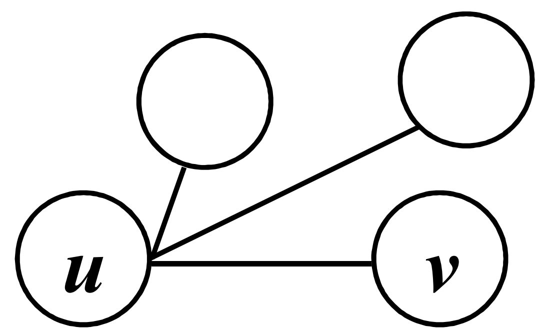


Induction

Before ENQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$



After ENQUEUE (of v):

By induction

$v_r.d \le u.d+1 = v.d = v_{r+1}.d$

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
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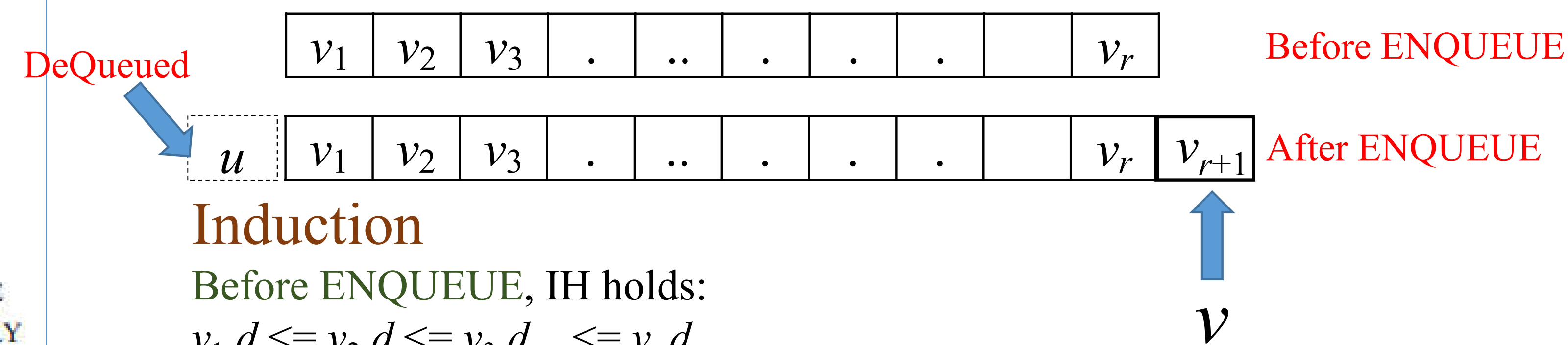
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$

(2) $v_1.d <= v_2.d <= v_3.d <= \cdots <= v_r.d$

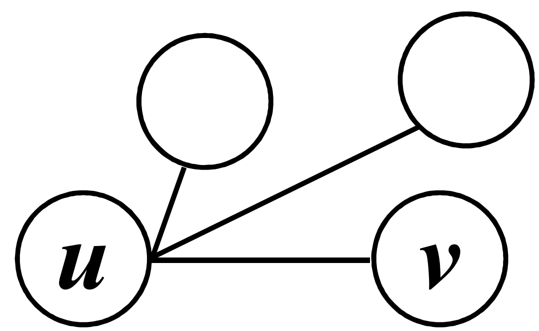


Induction

Before ENQUEUE, IH holds:

$v_1.d <= v_2.d <= v_3.d \ldots <= v_r.d$

$v_r.d <= v_1.d + 1$



After ENQUEUE (of v):

By induction

$v_r.d \le u.d{+}1 = v.d = v_{r+1}.d$

That means, $v_r.d \le v_{r+1}.d$

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
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    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$

(2) $v_1.d <= v_2.d <= v_3.d <= \cdots <= v_r.d$

Corollary 22.4

Suppose that vertices v_i and v_j are enqueued during the execution of BFS, and that v_i is enqueued before v_j. Then $v_i.d \le v_j.d$ at the time that v_j is enqueued.

v_1	v_2	v_3	.	v_i	..	v_j	.		v_r

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
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while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

Theorem 22.5 (Correctness of breadth-first search)

Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then, during its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$. Moreover, for any vertex $v \neq s$ that is reachable from s, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
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ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
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Let some vertex v receives other distance

$$v.d \neq \delta(s,v)$$

v is not s.

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

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Let some vertex v receives other distance

$$v.d \neq \delta(s,v)$$

v is not s. *Why?*

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
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s.color = GRAY
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ENQUEUE(Q, s)
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    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

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Let some vertex v receives other distance

$$v.d \neq \delta(s,v)$$

v is not s.

By Lemma 22.2, $v.d \geq \delta(s,v)$

Lemma 22.2

Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then upon termination, for each vertex $v \in V$, the value $v.d$ computed by BFS satisfies $v.d \geq \delta(s, v)$.

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
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    u.color = BLACK
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Theorem 22.5 (Correctness of breadth-first search)

Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then, during its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$. Moreover, for any vertex $v \neq s$ that is reachable from s, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

Let some vertex v receives other distance

$$v.d \neq \delta(s,v)$$

v is not s.

By Lemma 22.2, $v.d \geq \delta(s,v)$

Therefore, $v.d > \delta(s,v)$

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = Ø
ENQUEUE(Q, s)
while Q ≠ Ø
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
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    u.color = BLACK
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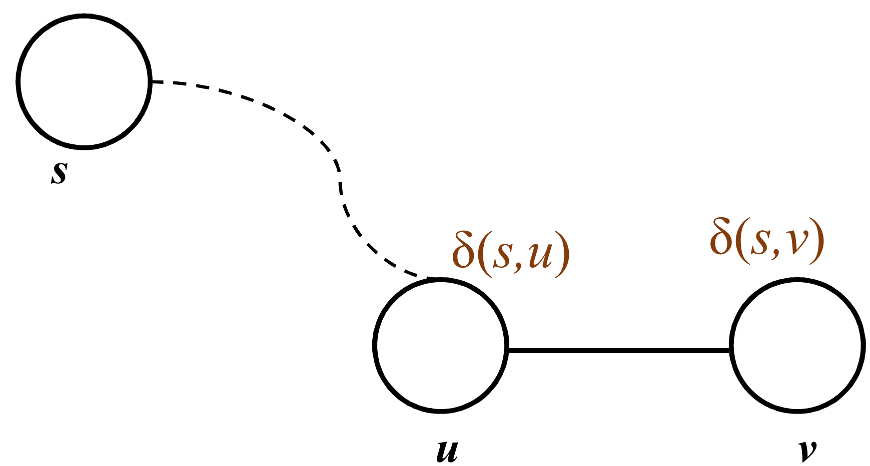
Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then, during its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$. Moreover, for any vertex $v \neq s$ that is reachable from s, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

Let some vertex v receives other distance

$$v.d \neq \delta(s,v)$$

We got, $v.d > \delta(s,v)$

Let u be just the previous vertex on the shortest path to v



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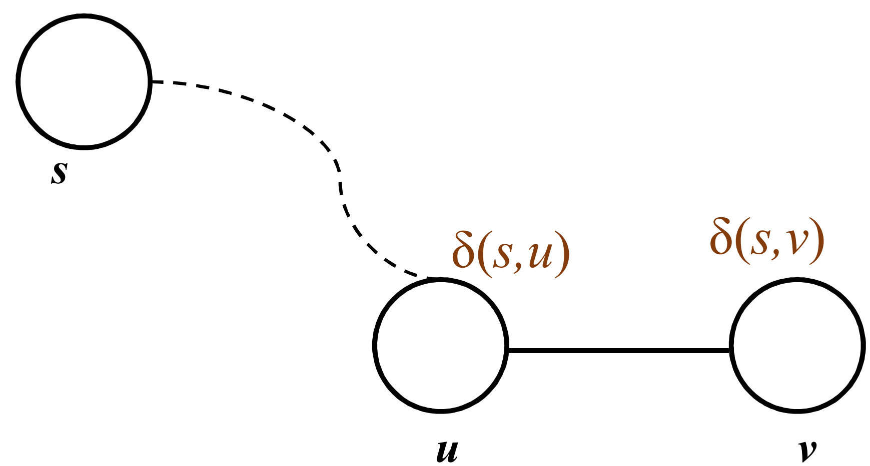
Let some vertex v receives other distance

$$v.d \neq \delta(s,v)$$

We got, $v.d > \delta(s,v)$

If u is just the previous vertex on the shortest path to v,

$$\delta(s,v) = \delta(s,u) + 1 \text{ and so, } \delta(s,u) < \delta(s,v)$$



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Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then, during its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$. Moreover, for any vertex $v \neq s$ that is reachable from s, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

Let some vertex v receives other distance

$$v.d \neq \delta(s,v)$$

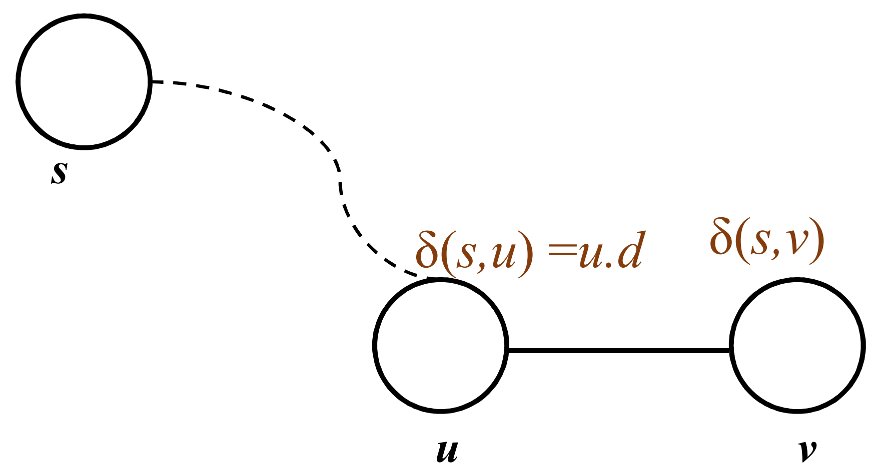
We got, $v.d > \delta(s,v)$

If u is just the previous vertex on the shortest path to v,

$$\delta(s,v) = \delta(s,u) + 1 \text{ and so, } \delta(s,u) < \delta(s,v)$$

Assume v is the ONLY unlucky vertex: $v.d \neq \delta(s,v)$

For others $u.d = \delta(s,u)$



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while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
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    u.color = BLACK
```

Theorem 22.5 (Correctness of breadth-first search)

Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then, during its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$. Moreover, for any vertex $v \neq s$ that is reachable from s, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

Let some vertex v receives other distance

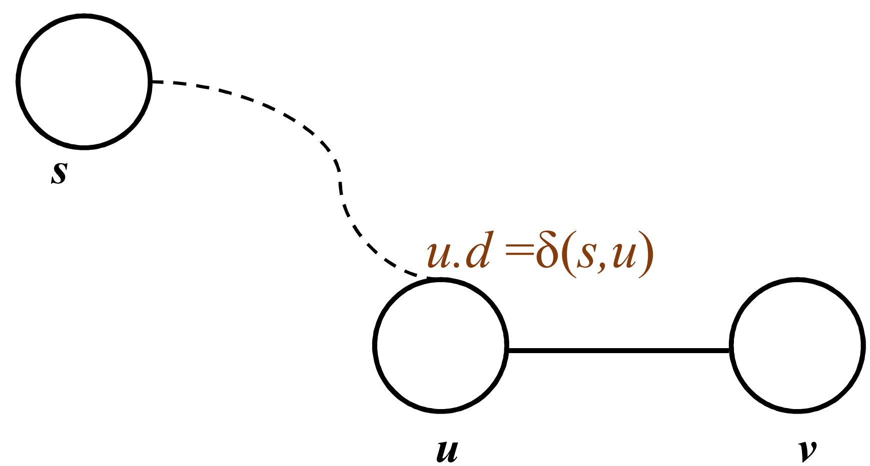
$$v.d \neq \delta(s,v)$$

We got, $v.d > \delta(s,v)$

If u is just the previous vertex on the shortest path to v,

$$\delta(s,v) = \delta(s,u) + 1 \text{ and so, } \delta(s,u) < \delta(s,v)$$

Now, $v.d > \delta(s,v) = \delta(s,u) + 1 = u.d + 1$



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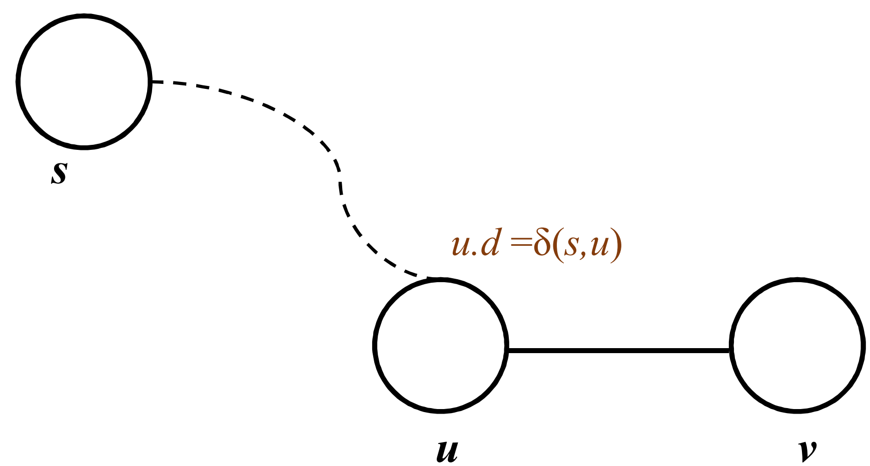

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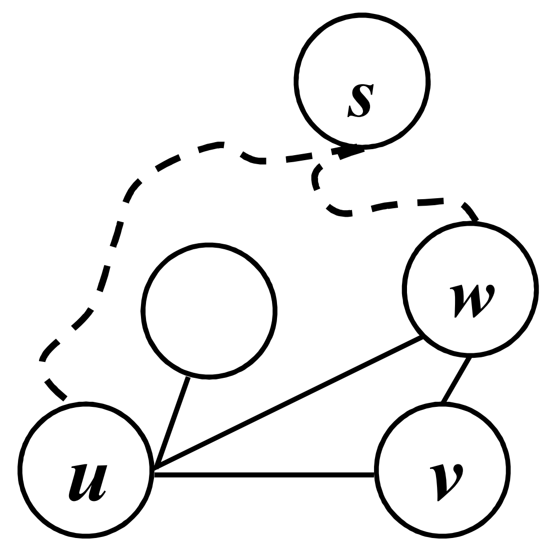
When u is dequeued from Q:

Case 1: v is white

- $v.d = u.d + 1$ [Contradicts 22.1]



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Suppose that vertices v_i and v_j are enqueued during the execution of BFS, and that v_i is enqueued before v_j. Then $v_i.d \le v_j.d$ at the time that v_j is enqueued.

Let some vertex *v* receives other distance

$$v.d \neq \delta(s,v)$$

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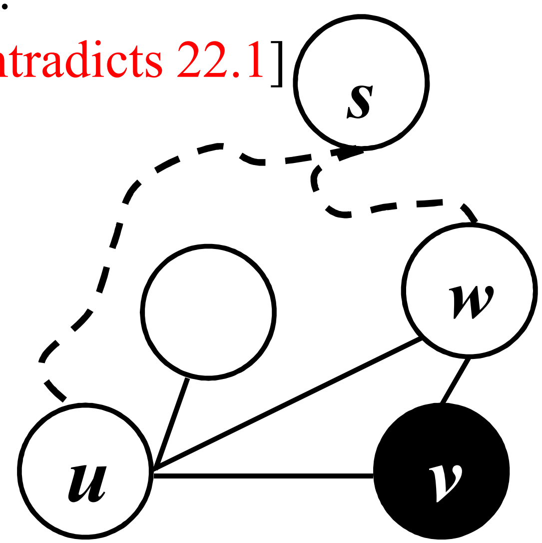
When *u* is dequeued from Q:

Case 1: *v* is white

- $v.d = u.d + 1$ [Contradicts 22.1]

Case 2: *v* is black

- *v* has been handled before *u*.
- Cor. 22.4=> $v.d <= u.d$ [Contradicts 22.1]



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Let some vertex v receives other distance

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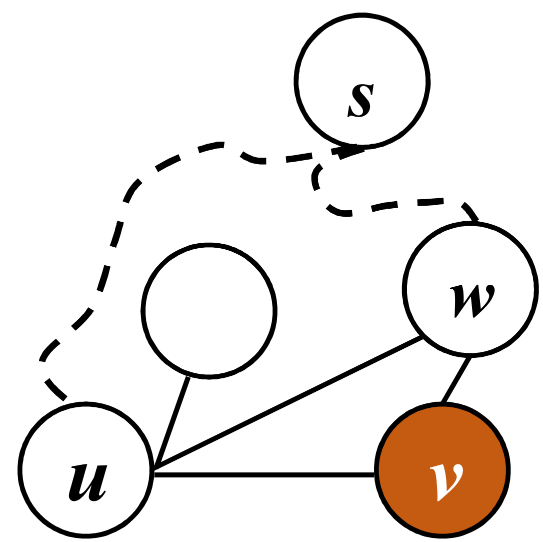
When u is dequeued from Q:

Case 3: v is GRAY



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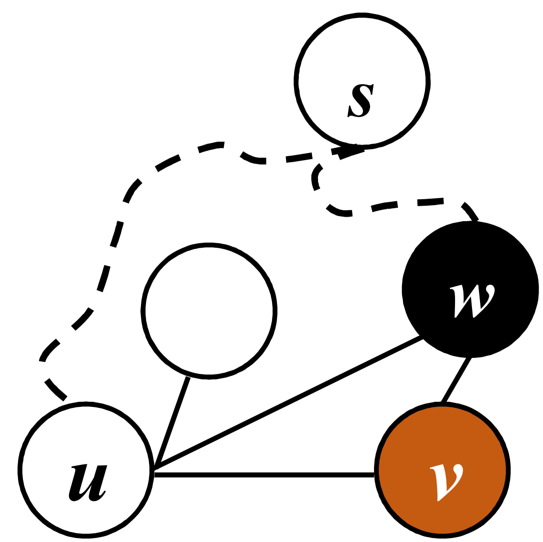

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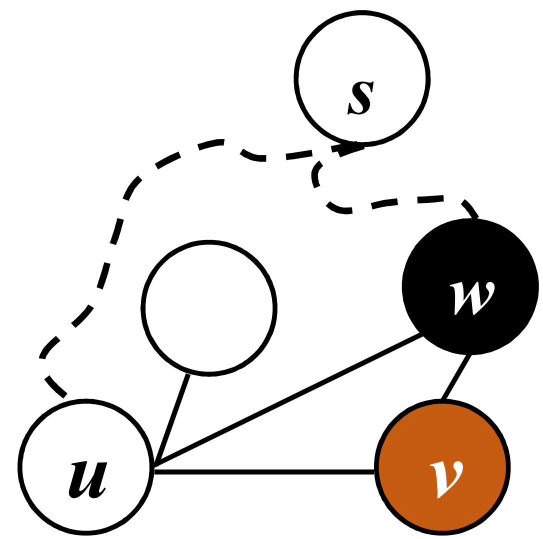

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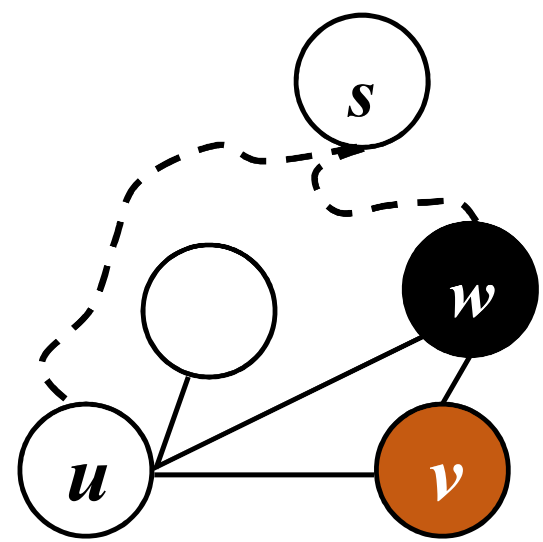
- w has been handled before u.
- Cor. 22.4=> $w.d <= u.d$ (B)



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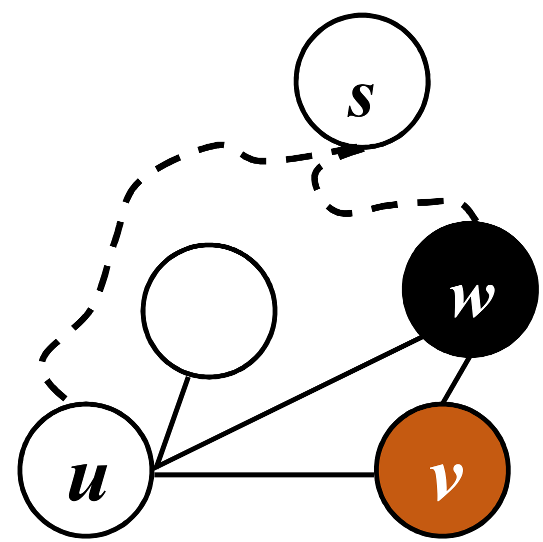
[Contradicts 22.1]



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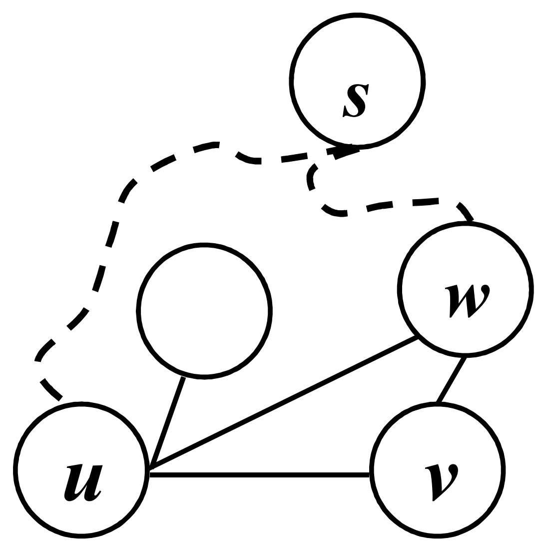

That means, $v.d = \delta(s,v)$

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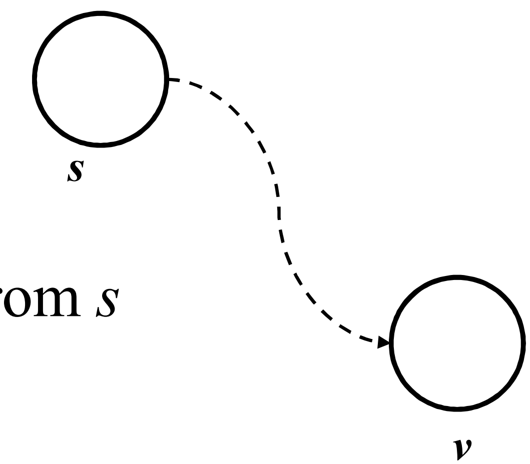

v is reachable from *s*

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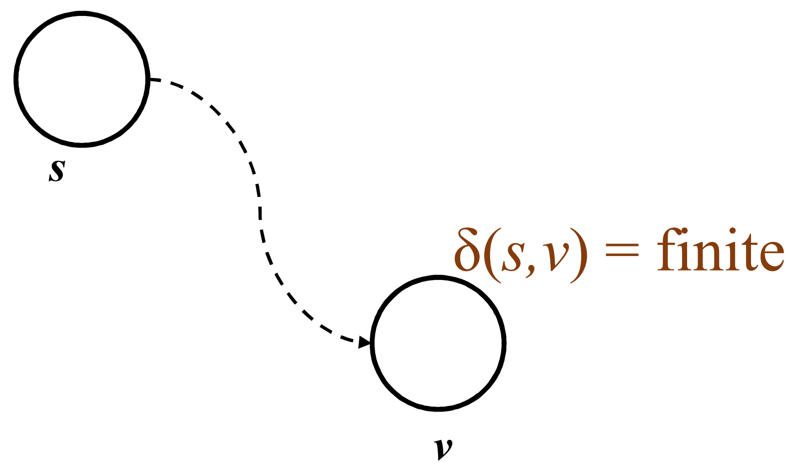

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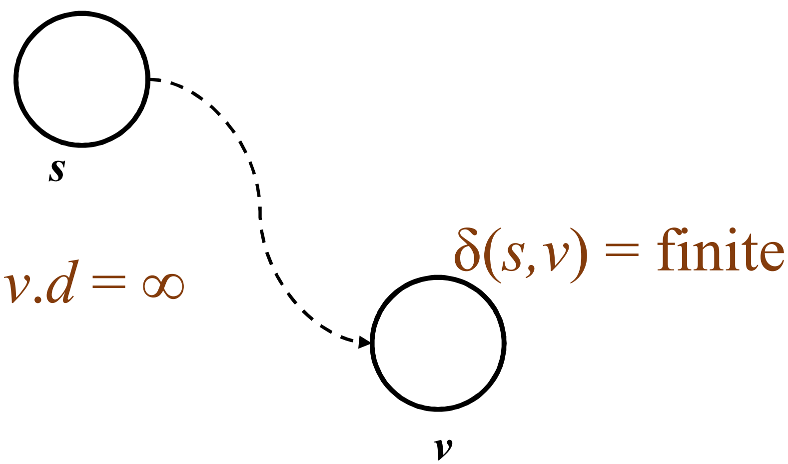
If not discovered, $v.d = \infty$



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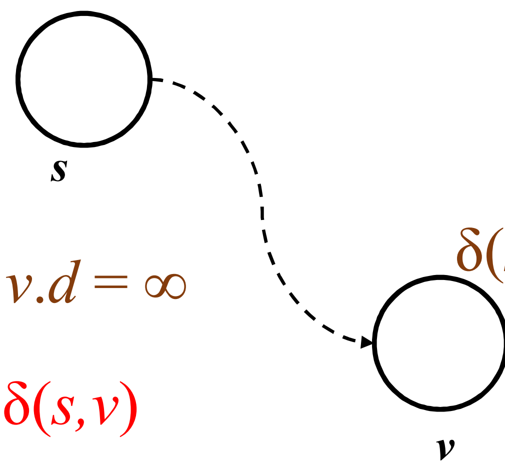

$\delta(s,v)$ = finite

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We will get $v.d > \delta(s,v)$

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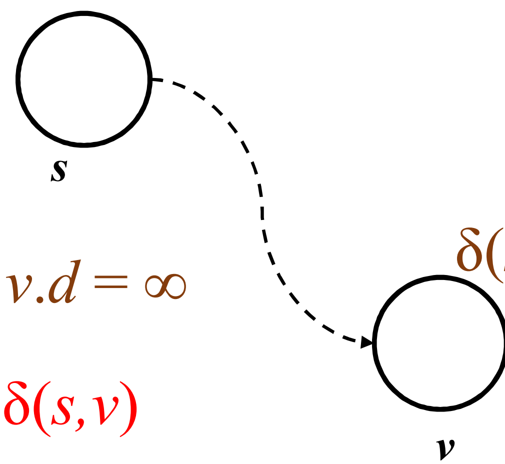

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From the 2nd statement, $v.d = \delta(s,v)$

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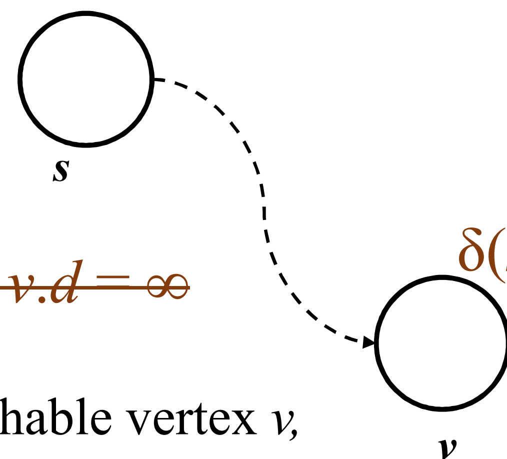
Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then, during its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$. Moreover, for any vertex $v \neq s$ that is reachable from s, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.



$\delta(s, v)$ = finite

~~If not discovered, $v.d = \infty$~~

This means, for a reachable vertex v,
$v.d$ is finite and discovered

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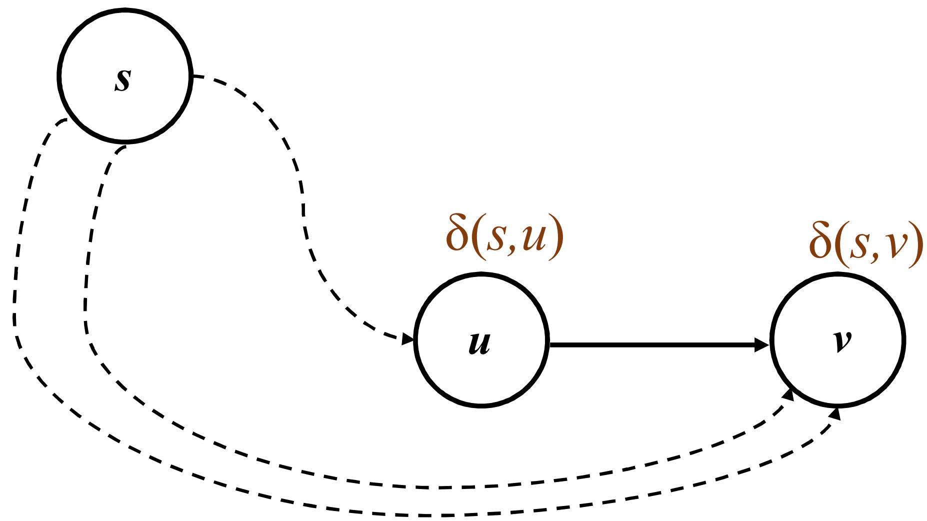

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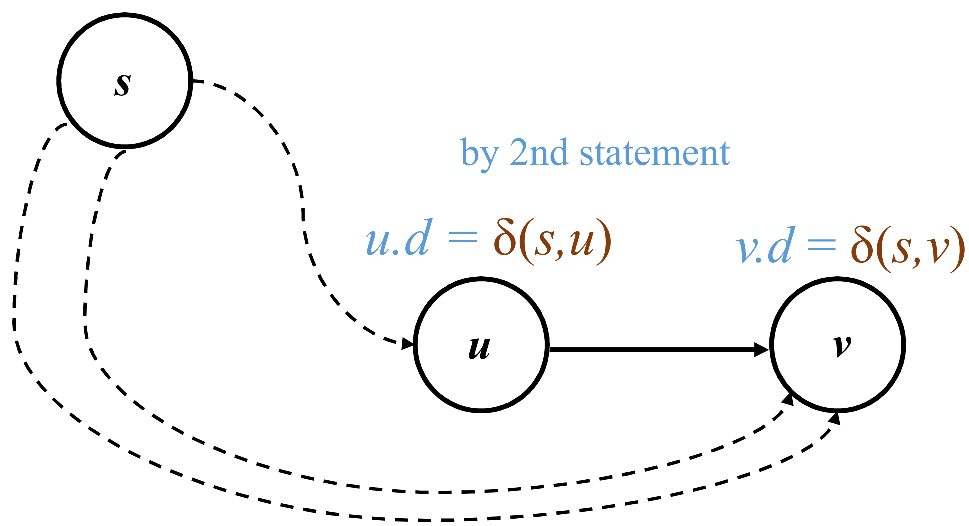

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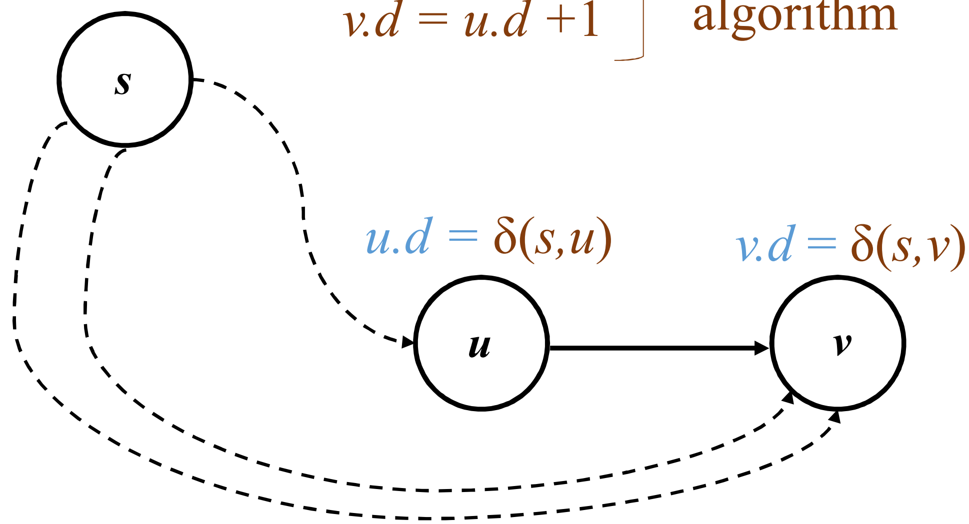

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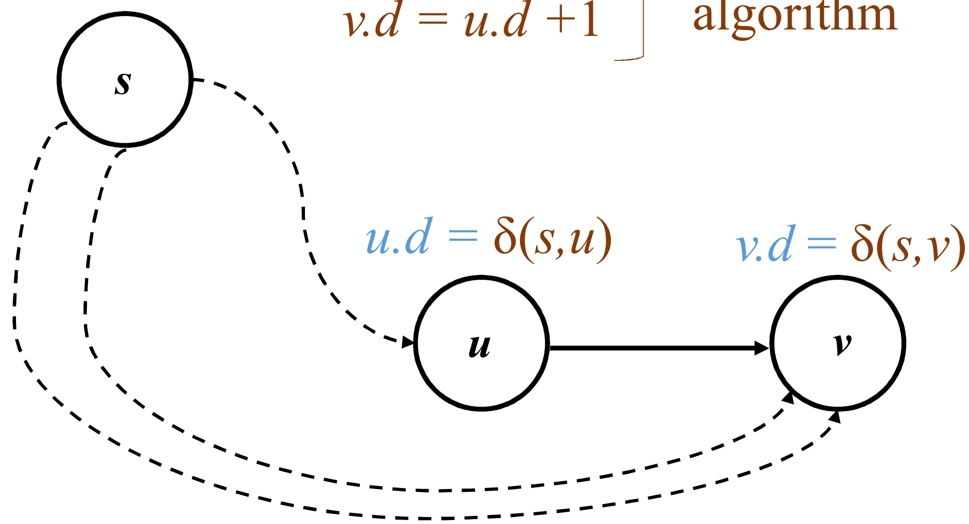

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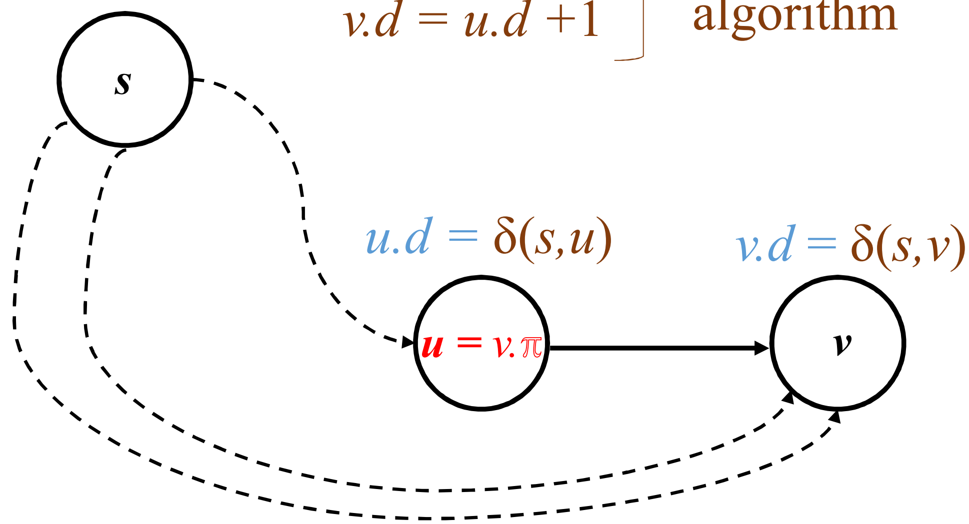

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