Calculus and Analytical Geometry

Lecture no. 10

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April 2022

Topic: Techniques of Differentiation

Outline of the lecture:

i. The product rule
ii. Quotient rule
iii. Chain rule
iv. Examples
v. Practice questions

Derivatives of trigonometric ratios:

$$\frac{d}{dx}(sinx) = cosx \qquad \frac{d}{dx}(cscx) = -cscxcotx$$

$$\frac{d}{dx}(cosx) = -sinx \qquad \frac{d}{dx}(secx) = secxtanx$$

$$\frac{d}{dx}(tanx) = sec^2x \qquad \frac{d}{dx}(cotx) = -csc^2x$$

Rules of Differentiation:

Some basic rules for differentiation are given below.

1. **The Product Rule:**

If f and g are differentiable at x, then so is the product $f.g$, and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Example 1.1: Find the derivative of $f(x) = (4x^2 - 1)(7x^3 + x)$

Solution:

Method 1.1: (simplify and then take derivative)

$$f(x) = (4x^2 - 1)(7x^3 + x)$$

$$f'(x) = \frac{d}{dx}[(4x^2)(7x^3) + 4x^2(x) - 1(7x^3) - 1x]$$

$$= \frac{d}{dx}[(28x^5) + 4x^3 - (7x^3) - x]$$

$$= \frac{d}{dx}[28x^5 - 3x^3 - x]$$

$$= 140x^{5-1} - 9x^{3-1} - 1$$

$$= 140x^4 - 9x^2 - 1$$

Method 2: (The product rule)

$$f(x) = (4x^2 - 1)(7x^3 + x)$$

$$f'(x) = \frac{d}{dx}[(4x^2 - 1)(7x^3 + x)]$$

$$= (4x^2 - 1)\frac{d}{dx}(7x^3 + x) + (7x^3 + x)\frac{d}{dx}(4x^2 - 1)$$

$$= (4x^2 - 1)(21x^2 + 1) + (7x^3 + x)(8x)$$

$$= [(4x^2)(21x^2) + 4x^2(1) - 1(21x^2) - 1(1)] + [(7x^3)(8x) + x(8x)]$$

$$= 84x^4 + 4x^2 - 21x^2 - 1 + 56x^4 + 8x^2$$

$$= 140x^4 - 9x^2 - 1$$

Example 1.2: Differentiate $f(x) = (x^2 + 1)(x^2 - 1)$

Solution:

$$f'(x) = (x^2 + 1)\frac{d}{dx}(x^2 - 1) + (x^2 - 1)\frac{d}{dx}(x^2 + 1)$$

$$= (x^2 + 1)(2x) + (x^2 - 1)(2x)$$

$$= 2x^3 + 2x + 2x^3 - 2x$$

$$= 4x^3$$

Example 1.3: Differentiate $f(x) = (x^3 + 7x^2 - 8)(2x^{-3} + x^{-4})$

Solution:

$$f'(x) = (x^3 + 7x^2 - 8)\frac{d}{dx}(2x^{-3} + x^{-4}) + (2x^{-3} + x^{-4})\frac{d}{dx}(x^3 + 7x^2 - 8)$$

$$= (x^3 + 7x^2 - 8)(-6x^{-4} - 4x^{-5}) + (2x^{-3} + x^{-4})(3x^2 + 14x)$$

$$= [x^3(-6x^{-4} - 4x^{-5}) + 7x^2(-6x^{-4} - 4x^{-5}) - 8(-6x^{-4} - 4x^{-5})] +$$
$$[2x^{-3}(3x^2 + 14x) + x^{-4}(3x^2 + 14x)]$$

$$= [-6x^{-1} - 4x^{-2} - 42x^{-2} - 28x^{-3} + 48x^{-4} + 32x^{-5}] + [6x^{-1} + 28x^{-2} + 3x^{-2}$$
$$+ 14x^{-3}]$$

$$= -15x^{-2} - 14x^{-3} + 48x^{-4} + 32x^{-5}$$

Example 1.4: find the derivative of $f(x) = (x^2 + 1)secx$

Solution:

$$f'(x) = (x^2 + 1)\frac{d}{dx}(secx) + secx\frac{d}{dx}(x^2 + 1)$$

$$= (x^2 + 1)secxtanx + secx(2x)$$

Example 1.5: Find the derivative of the function $f(x) = e^{sinx}\ln(cosx)$

Solution:

$$f'(x) = \left(e^{sinx}\right)\frac{d}{dx}(\ln(cosx)) + \ln(cosx)\frac{d}{dx}\left(e^{sinx}\right)$$

$$= e^{sinx}.\frac{-sinx}{cosx} + [\ln(cosx)]e^{sinx}.cosx$$

$$= e^{sinx}.-tanx + [\ln(cosx)]e^{sinx}.cosx$$

$$= e^{sinx}.-tanx + [\ln(cosx)]e^{sinx}.cosx$$

$$= e^{sinx}[\ln(cosx) - tanx]$$

2. The Quotient Rule:

If f and g are differentiable at x, and if $g(x) \neq 0$ then $\frac{f}{g}$ is differentiable at x and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example 2.1: Differentiate $f(x) = \frac{x^3+4x^2-1}{x+5}$

Solution:

$$\frac{d}{dx}\left[\frac{x^3+4x^2-1}{x+5}\right] = \frac{x+5\frac{d}{dx}[x^3+4x^2-1] - (x^3+4x^2-1)\frac{d}{dx}[x+5]}{[x+5]^2}$$

$$= \frac{(x+5)(3x^2+8x) - (x^3+4x^2-1)(1)}{[x+5]^2}$$

$$= \frac{x(3x^2+8x) + 5(3x^2+8x) - x^3 - 2x^2 + 1}{[x+5]^2}$$

$$= \frac{3x^3+8x^2+15x^2+40x-x^3-2x^2+1}{[x+5]^2}$$

$$= \frac{2x^3+21x^2+40x+1}{[x+5]^2}$$

Example 2.2: Differentiate $y = \frac{3x+4}{x^2-1}$

Solution:

$$\frac{d}{dx}\left[\frac{3x+4}{x^2-1}\right] = \frac{(x^2-1)\frac{d}{dx}[3x+4] - (3x+4)\frac{d}{dx}[x^2-1]}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(3) - (3x+4)(2x)}{(x^2-1)^2}$$

$$= \frac{(3x^2-3) - (6x^2+8x)}{(x^2-1)^2}$$

$$= \frac{-3x^2-8x-3}{(x^2-1)^2}$$

Example 2.3: Differentiate $y = \frac{x-2}{x^4+x+1}$

Solution:

$$\frac{d}{dx}\left[\frac{x-2}{x^4+x+1}\right]=\frac{(x^4+x+1)\frac{d}{dx}[x-2]-(x-2)\frac{d}{dx}[x^4+x+1]}{(x^4+x+1)^2}$$

$$=\frac{(x^4+x+1)(1)-(x-2)(4x^3+1)}{(x^4+x+1)^2}$$

$$=\frac{x^4+x+1-4x^4-x+8x^3+2}{(x^4+x+1)^2}$$

$$=\frac{-3x^4+8x^3+3}{(x^4+x+1)^2}$$

Example 2.4: Differentiate $f(x)=\frac{secx}{1+tanx}$

Solution:

$$\frac{d}{dx}\left[\frac{secx}{1+tanx}\right]=\frac{(1+tanx)\frac{d}{dx}(secx)-(secx)\frac{d}{dx}(1+tanx)}{(1+tanx)^2}$$

$$=\frac{(1+tanx)\frac{d}{dx}(secx)-(secx)\frac{d}{dx}(1+tanx)}{(1+tanx)^2}$$

$$=\frac{(1+tanx)secxtanx-(secx)sec^2x}{(1+tanx)^2}$$

$$=\frac{secxtanx+secxtan^2x-sec^3x}{(1+tanx)^2}$$

$$=\frac{secx(tanx+tan^2x-sec^2x)}{(1+tanx)^2}$$

$$\therefore 1+tan^2x=sec^2x$$

$$=\frac{secx(tanx-1)}{(1+tanx)^2}$$

Example 2.5: Differentiate $f(x)=\frac{(x^2+1)cotx}{3-cosxcscx}$

Solution:

$$\frac{d}{dx}\left[\frac{(x^2+1)cotx}{3-cosxcscx}\right]=\frac{(3-cosxcscx)\frac{d}{dx}((x^2+1)cotx)-((x^2+1)cotx)\frac{d}{dx}(3-cosxcscx)}{(3-cosxcscx)^2}$$

$$\therefore cosxcscx = cosx\left(\frac{1}{sinx}\right) = cotx$$

$$= \frac{(3-cotx)[(x^2+1)\frac{d}{dx}cotx + cotx\frac{d}{dx}(x^2+1)] - ((x^2+1)cotx)[\frac{d}{dx}3 - \frac{d}{dx}(cotx)]}{(3-cosxcscx)^2}$$

$$= \frac{(3-cotx)[(x^2+1)(-csc^2x) + cotx(2x)] - ((x^2+1)cotx)[-(-csc^2x)]}{(3-cotx)^2}$$

$$= \frac{(3-cotx)[2xcotx - (x^2+1)csc^2x] - ((x^2+1)cotx)(csc^2x)}{(3-cotx)^2}$$

$$= \frac{[6xcotx - 3(x^2+1)csc^2x - 2xcot^2x + ((x^2+1)cotx)(csc^2x)] - ((x^2+1)cotx)(csc^2x)}{(3-cotx)^2}$$

$$= \frac{6xcotx - 3(x^2+1)csc^2x - 2xcot^2x}{(3-cotx)^2}$$

3. Chain Rule:

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function fog is differentiable at x. Moreover, if

$$y = f\big(g(x)\big) \;\; and \;\; u = g(x) \text{ then } y = f(u)$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Example 3.1: Differentiate $f(x) = \tan(4x^3 + x)$

Solution:

$$f(x) = tanu, u = 4x^3 + x$$

$$= \frac{d}{du}tanu.\frac{d}{dx}4x^3 + x$$

$$= sec^2u.(12x^2 + 1)$$

$$= sec^2(4x^3 + x).(12x^2 + 1)$$

$$= (12x^2 + 1)sec^2(4x^3 + x)$$

Alternate version of chain rule:

$$\frac{d}{dx}[f(g(x)] = (fog)'(x) = f'(g(x))g'(x)$$

Example: Differentiate $f(x) = \tan(4x^3 + x)$

Solution:

$$\frac{dy}{dx} = sec^2(4x^3 + x).\frac{d}{dx}(4x^3 + x)$$

$$= sec^2(4x^3 + x).(12x^2 + 1)$$

$$= (12x^2 + 1)\, sec^2(4x^3 + x)$$

Example 3.2: Differentiate $f(x) = \sqrt{x^3 + cscx}$

Solution:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^3 + cscx}}\frac{d}{dx}(x^3 + cscx)$$

$$= \frac{1}{2\sqrt{x^3 + cscx}}.(3x^2 - cscxcotx)$$

$$= \frac{(3x^2 - cscxcotx)}{2\sqrt{x^3 + cscx}}$$

Practice Questions:

Differentiate the following.

1. $\left(\frac{1}{x}+\frac{1}{x^2}\right)(3x^3+27)$
2. $\frac{2x^2+5}{3x-4}$
3. $\frac{sinx}{x^2+sinx}$
4. $\frac{sinxsecx}{1+xtanx}$
5. $x^2cosx+4sinx$
6. $x^2sec\left(\frac{1}{x}\right)$
7. $(1+t)\sqrt{t}$
8. $x^2\ln x$
9. $e^x(1+\ln x)$
10. $\frac{e^x}{1-sinx}$