Calculus and Analytical Geometry

Lecture no. 12

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April 2022

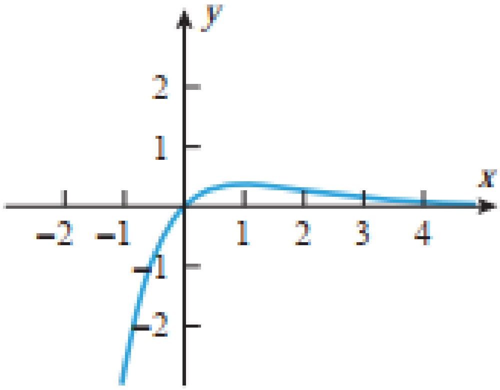
Topic: Relative extrema, first and second derivative tests, Extreme values

Outline of the lecture:

i. Examples of inflection points.
ii. Relative Maxima and Relative Minima
iii. Critical and stationary points
iv. First Derivative Test for Relative Maxima and Minima
v. Second Derivative Test for Relative Maxima and Minima
vi. Practice Questions

1. EXAMPLES OF INFLECTION POINT:

Example 1.1: *Consider the graph of the function* $f(x) = xe^{-x}$



Use the first and second derivative test to determine the intervals on which the function is increasing decreasing, concave up, concave down. Locate all the inflection points.

Solution:

Step 1: [Find the first derivative]

$$\begin{aligned} f'(x) &= x\frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(x) \\ &= x\,.\,e^{-x}(-1) + e^{-x}(1) \\ &= -x\,.\,e^{-x} + e^{-x} \\ &= e^{-x}(-x+1) = e^{-x}(1-x) \end{aligned}$$

Step 2: [Find the increasing and decreasing intervals]

Interval	$f'(x)$	Conclusion
$x < 1$	$+$	Increasing on $(-\infty, 1]$
$x > 1$	$-$	Decreasing on $[1, +\infty)$

Step 3: [Second derivative test]

$$\begin{aligned} f''(x) &= (1-x)\frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(1-x) \\ &= (1-x).\,e^{-x}(-1) + e^{-x}(-1) \\ &= (1-x).\,e^{-x}(-1) + e^{-x}(-1) \\ &= e^{-x}(-1)[(1-x)+1] \\ &= e^{-x}(-1)[1-x+1] \\ &= e^{-x}(-1)[2-x] \\ &= e^{-x}(x-2) \end{aligned}$$

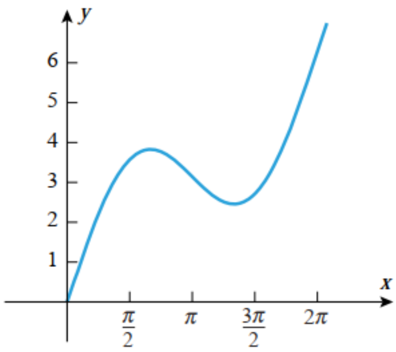
Step 4: [Find the intervals on which the function is concave up and concave down]

Interval	$f''(x)$	Conclusion
$x < 2$	$-$	Concave down on $(-\infty, 2)$
$x > 2$	$+$	Concave up on $(2, +\infty)$

Step 5: [Find inflection points]

The second table shows there is an inflection point at $x = 2$. *Since the function changes from concave down to concave up so, the inflection point is* $\left(2, f(2)\right) = (2, 2e^{-2}) = (2, 0.2707)$.

Example 1.2: *Consider the graph of the function* $f(x) = x + 2sinx$



Use the first and second derivative test to determine the intervals on which the function is increasing decreasing, concave up, concave down. Locate all the inflection points.

Solution:

Step 1: [find the first derivative]

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x + 2\sin(x))$$

$$f'(x) = \frac{d}{dx}(x) + 2\frac{d}{dx}(\sin(x))$$

$$= 1 + 2\text{cosx}$$

Substitute $f'(x) = 0$

$$1 + 2\cos(x) = 0$$

$$2\cos(x) = -1$$

$$\cos(x) = -\frac{1}{2}$$

Since cos(x) is negative and cos(x) is negative in II and III Quadrant. So, we can write

$$\cos(x) = \frac{1}{2}$$

$$x = \cos^{-1}\left[\frac{1}{2}\right]$$

$$x = \frac{\pi}{3}$$

Angles:

For Quadrant II	*For Quadrant III*
$x_1 = \pi - x$ $x_1 = \pi - \frac{\pi}{3}$ $x_1 = \frac{3\pi - \pi}{3}$ $\boxed{x_1 = \frac{2\pi}{3}}$	$x_2 = \pi + x$ $x_2 = \pi + \frac{\pi}{3}$ $x_2 = \frac{3\pi + \pi}{3}$ $\boxed{x_2 = \frac{4\pi}{3}}$

Step 2: [find the increasing and decreasing intervals]

Interval	$f'(x)$	Conclusion
$0 < x < \frac{2\pi}{3}$	$+$	Increasing on $[0, \frac{2\pi}{3}]$
$\frac{2\pi}{3} < x < \frac{4\pi}{3}$	$-$	Decreasing on $[\frac{2\pi}{3}, \frac{4\pi}{3}]$
$\frac{4\pi}{3} < x < 2\pi$	$+$	Increasing on $[\frac{4\pi}{3}, 2\pi]$

Step 3: [Second derivative test]

$$f''(x) = \frac{d}{dx}(1 + 2\cos(x))$$

$$= 0 + 2(-\sin(x))$$

$$= -2\sin(x)$$

Since $f''(x)$ is a continuous function. So, its sign changes in the interval will occur only at values of x at which $f''(x) = 0$.

$$f''(x) = 0$$

$$-2\sin(x) = 0$$

$$\sin(x) = 0$$

$$x = sin^{-1}(0)$$

$$x = \pi$$

Step 4: [find the intervals on which the function is concave up and concave down]

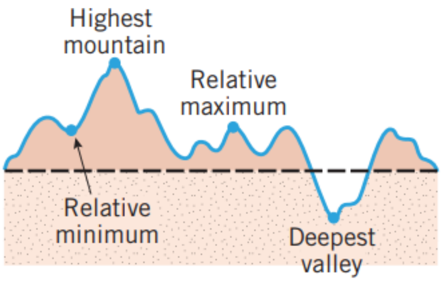
Interval	$f''(x)$	Conclusion
$\boldsymbol{0 < x < \pi}$	$-$	Concave down on $(0, \pi)$
$\boldsymbol{\pi < x > 2\pi}$	$+$	Concave up on $(\pi, 0)$

Step 5: [Find inflection points]

The second table shows there is an inflection point at $x = \pi$.

2. RELATIVE MAXIMA AND RELATIVE MINIMA:

If we imagine the graph of a function *f* to be a two-dimensional mountain range with **hills** and **valleys**, then the **tops** of the **hills** are called **relative maxima**, and the **bottoms** of the **valleys** are called **relative minima** as shown in figure.

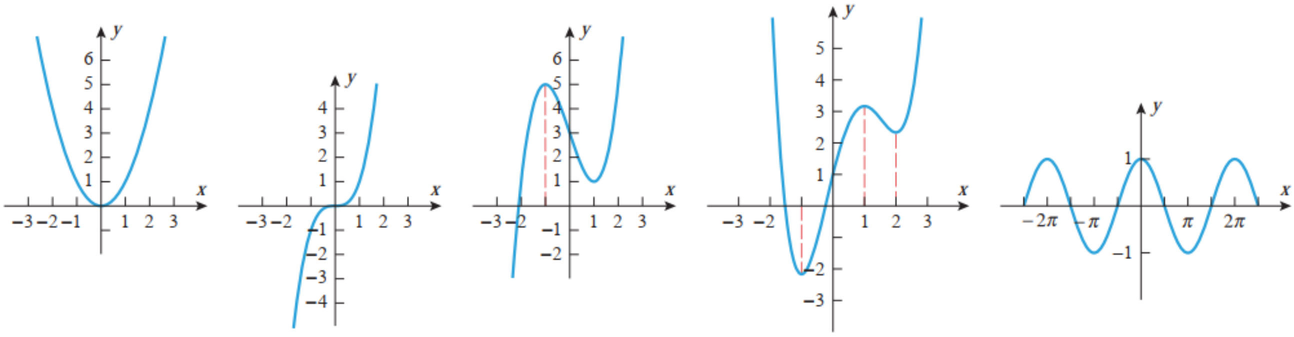


The **relative maxima** are the **high points** in their **immediate vicinity**, and the **relative minima** are the **low points**.
A **relative maximum** need not be the highest point in the entire mountain range, and a **relative minimum** need not be the lowest point, they are just high and low points relative to the nearby other points.

THEOREM: Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

- A function **f** is said to have a **relative maxima** at x_o, if **f** has the **largest value** at x_o
- A function **f** is said to have a **relative minima** at x_o, if **f** has the **smallest value** at x_o
- If the function have neither relative maxima nor minima at x_o, then f is said to have relative extremum at x_o
- The points where the function have maximum or minimum values are called the points of extrema.

Exam



$y = x^2$ $y = x^3$ $y = x^3 - 3x + 3$ $y = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ $y = \cos x$

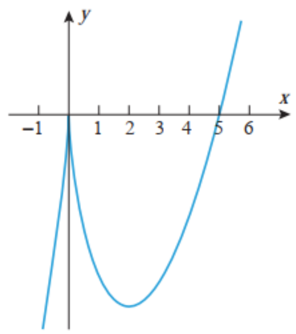
Solution:

- *The function* $y = x^2$ *has no relative maxima or minima.*
- *The function* $y = x^3$*has no relative maxima or minima.*
- *The function* $y = x^3 - 3x + 3$ *has relative maxima at* $x = -1$ *and relative minima at* $x = 1$.
- *The function* $y = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ *has relative maxima at* $x = 1$ *and relative minima at* $x = -1$ *and* $x = 2$.
- *The function* $y = cos\,x$ *has relative maxima at* $, \ldots - 2\pi, 0, 2\pi, \ldots$ *and relative minima at,* $\ldots - \pi, \pi, \ldots$

3. CRITICAL POINT AND STATIONARY POINT:

- Critical point:
  A point x_o at which the function is either not differentiable or $f'(x) = 0$.
- Stationary point:
  It is a special case of critical point. It is a point at which $f'(x) = 0$.

Example 3.1: *find all the critical points of the graph of function* $f(x) = 3x^{\frac{3}{5}} - 15x^{\frac{2}{3}}$



Solution:

Step 1: [Find the first derivative]

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}\right]$$
$$f'(x) = \frac{d}{dx}\left[3x^{\frac{5}{3}}\right] - \frac{d}{dx}\left[15x^{\frac{2}{3}}\right]$$
$$f'(x) = 3\frac{d}{dx}\left[x^{\frac{5}{3}}\right] - 15\frac{d}{dx}\left[x^{\frac{2}{3}}\right]$$
$$f'(x) = 3.\left(\frac{5}{3}\right)x^{\frac{5}{3}-1} - 15.\left(\frac{2}{3}\right)x^{\frac{2}{3}-1}$$
$$f'(x) = 5x^{\frac{2}{3}} - 5.(2)x^{-\frac{1}{3}}$$
$$f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}}$$
$$f'(x) = 5\left[x^{\frac{2}{3}} - \frac{2}{x^{\frac{1}{3}}}\right]$$
$$f'(x) = 5\left[\frac{x^{\frac{2}{3}}\ x^{\frac{1}{3}} - 2}{x^{\frac{1}{3}}}\right]$$
$$\boxed{f'(x) = 5\left[\frac{(x-2)}{x^{\frac{1}{3}}}\right]}$$

Step 2: [find critical and stationary point]

$$f'(x) = 0$$

So we have

$$x - 2 = 0 \Rightarrow \boxed{x = 2} \qquad and \qquad x^{\frac{1}{3}} = 0 \Rightarrow \boxed{x = 0}$$

Function	Critical points	Stationary points
$f(x) = 3x^{\frac{3}{5}} - 15x^{\frac{2}{3}}$	$x_o = 0,2$	$x_o = 0,2$

4. FIRST DERIVATIVE TEST FOR RELATIVE MAXIMA AND MINIMA:

Suppose $f(x)$ is **continuous** at a critical point x_0, then

• If $f'(x_0) > 0$ on an open interval (a, b) for all points before x_0, and $f'(x_0) < 0$ on and open interval (a, b) for all points after x_0, then $f(x)$ has a **relative maxima** at $x0$.

• If $f'(x_0) < 0$ on an open interval (a, b) for all points before $x0$, and $f'(x_0) > 0$ on an open interval (a, b) for all points after than x_0, then $f(x)$ has a **relative minima** at x_0.

• If $f'(x_0)$ has **same sign** on an open interval (a, b) for all points before and after x_0, then $f(x)$ does not have **relative extremum** at x_0,.

Example 4.1: *Find the intervals on which the function $f(x) = x^2 - 4x + 3$ is increasing and decreasing using the first derivative test.*

Solution:

Step 1: [find the derivative]

$$f'(x) = \frac{d}{dx}(x^2 - 4x + 3)$$
$$= 2x - 4$$
$$= 2(x - 2)$$

Step 2:[find the critical points and stationary points]

$$f'(x) = 2(x - 2) = 0$$
$$x = 2$$

Interval	$f'(x)$	Conclusion
$x < 2$	$-$	Increasing on $(-\infty, 2]$
$x > 2$	$+$	Decreasing on $[2, +\infty)$

- *From the first derivative test the sign of $f'(x)$ changes from negative to positive at $x = 2$ so, there is a relative minima at that point.*

Example 4.2: *Find the intervals on which the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing and decreasing using the first derivative test.*

Solution:

Step 1: [find the derivative]

$$f'(x) = \frac{d}{dx}(3x^4 + 4x^3 - 12x^2 + 2)$$
$$= 12x^3 + 12x^2 - 24x$$
$$= 12x(x^2 + x - 2)$$

Step 2:[find the critical points and stationary points]

$$f'(x) = 12x(x^2 + x - 2) = 0$$
$$= 12x(x+2)(x-1)$$
$$x = 0,1,-2$$

Step 3:[Find the increasing and decreasing intervals]

Interval	$f'(x)$	Conclusion
$\boldsymbol{x < -2}$	$-$	Increasing on $(-\infty, -2]$
$\boldsymbol{-2 < x < 0}$	$+$	Increasing on $[-2,0]$
$\boldsymbol{0 < x < 1}$	$-$	Decreasing on $[0,1]$
$\boldsymbol{1 < x}$	$+$	Increasing on $[1,+\infty)$

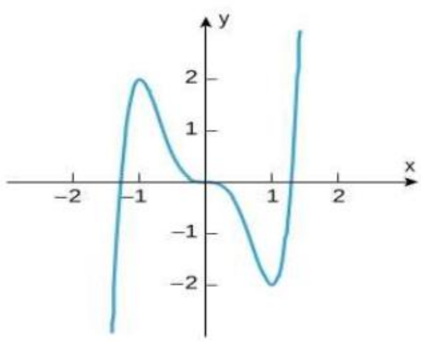
- *Sign of $f'(x)$ changes from negative to positive at $x = -2$. So, there is a relative minima at the point.*
- *Sign of $f'(x)$ changes from positive to negative at $x = 0$. So, there is a relative maxima at the point.*
- *Sign of $f'(x)$ changes from negative to positive at $x = 1$. So, there is a relative minima at the point.*

5. SECOND DERIVATIVE TEST FOR RELATIVE MAXIMA AND RELATIVE MINIMA:

Suppose $f(x)$ is **twice differentiable** at a critical point x_0, then
- If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has relative minima at x_0.
- If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has relative maxima at x_0.
- If $f'(x_0) = 0$ *and* $f''(x_0) = 0$, then the test is inconclusive.

Example: *Find the intervals on which the function $f(x) = 3x^5 - 5x^3$ is increasing and decreasing using the second derivative test.*

Solution:



Step 1: [find the derivative]

$$f'^{(x)} = \frac{d}{dx}(3x^5 - 5x^3\ )$$
$$= 15x^4 - 15x^2$$
$$= 15x^2(x^2 - 1) = 15x^2(x-1)(x+1)$$

Step 2:[find the 2nd derivative]

$$f''^{(x)} = \frac{d}{dx}[15x^2(x^2 - 1)]$$
$$= 15x^2\frac{d}{dx}(x^2 - 1) + (x^2 - 1)\frac{d}{dx}(15x^2)$$
$$= 15x^2(2x) + (x^2 - 1)(30x)$$
$$= 30x^3 + 30x^3 - 30x$$
$$= 60x^3 - 30x$$
$$= 30x(2x^2 - 1)$$

Step 3:[find the critical and stationary points]

$$15x^2(x-1)(x+1) = 0$$

$$15x^2 = 0, \quad x - 1 = 0, \quad x + 1 = 0$$

$$x = 0, x = \pm 1$$

So, stationary points are 0,-1,+1

Step 4: [Find the relative maxima or minima]

Stationary points	$30x(2x^2-1)$	$f''(x)$	*2nd derivative test*
$x = -1$	*-30*	-	*Relative maxima*
$x = 0$	*0*	*0*	*Inconclusive*
$x = 1$	*30*	+	*Relative minima*

So, the test is inconclusive for x=0, therefore we are going to use 1st derivative test at this point.

$$15x^2(x-1)(x+1) = 0$$

From here the stationary points are -1,0,1

Intervals	*Test points*	$f'(x)$
$-1 < x < 0$	*-0.5*	*Negative*
$0 < x < 1$	*+0.5*	*Negative*

According to the first derivative test there is no change in sign so, there is no relative maxima or minima at the point.

Practice Questions:

Question no. 1

Find the intervals on which f is increasing and the intervals on which it is decreasing using **first derivative test**.

- $f(x) = x^3 - 3x^2 + 1$.
- $f(x) = 3x^4 - 4x^3$.
- $f(x) = x^3 - x^2 - 2x$.

Question no. 2

Find the relative extremum of the following functions using **second derivative test**.

- $f(x) = x^3 - 3x^2 + 1$.
- $f(x) = 3x^5 - 5x^3$.
- $f(x) = x^3 - x^2 - 2x$.

Question no. 3

- Use both the **first and second derivative tests** to show that $f(x) = 3x^2 - 6x + 1$ has a **relative minimum** at $x = 1$.
- Use both the **first and second derivative tests** to show that $f(x) = x^3 - 3x + 3$ has a **relative minimum** at $x = 1$ and a **relative maximum** at $x = -1$.

Question no. 4

- Use both the first and second derivative tests to show that $f(x) = sin^2(x)$ has a relative minimum at $x = 0$.
- Use both the first and second derivative tests to show that $g(x) = tan^2 x$ has a relative minimum at $x = 0$.

Question no. 5 Locate the **critical points** and identify which **critical points are stationary points** for $f(x) = 4x^4 - 16x^2 + 17$.

Question no. 6 Locate the **critical points** and identify which **critical points are stationary points** for $g(x) = 3x^4 + 12x$.