4.6 Variation of Parameters

Variation of parameters will solve any non-homogenous linear differential equation provided that the solutions to homogeneous equation are known.

To adapt the method of **variation of parameters** to a linear second-order differential equation suppose we have a second order non-homogenous differential equation in standard form

$$y'' + P(x)y' + Q(x)y = f(x). \qquad (1)$$

where $P(x)$ and $Q(x)$ are continuous on some interval I.

Further, let $y_c = c_1y_1(x) + c_2y_2(x)$ be the solution of homogenous second order differential equation (1).

In the same manner as in reduction of order, if we define

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x),$$

it follows that

$$y_p{}'(x) = u_1'(x)\, y_1(x) + u_1(x)\, y_1'(x) + u_2'(x)\, y_2(x) + u_2(x)\, y_2'(x)$$

$$y_p''(x) = u_1''(x)\, y_1(x) + 2\, u_1'(x)\, y_1'(x) + u_1(x)\, y_1''(x) + u_2''(x)\, y_2(x) + 2\, u_2'(x)\, y_2'(x) + u_2(x)\, y_2''(x)$$

So equation (1) becomes

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= u_1''(x)\, y_1(x) + 2\, u_1'(x)\, y_1'(x) + u_1(x)\, y_1''(x) + u_2''(x)\, y_2(x) \\ &+ 2\, u_2'(x)\, y_2'(x) + u_2(x)\, y_2''(x) \\ &+ P(x)[u_1'(x)\, y_1(x) + u_1(x)\, y_1'(x) + u_2'(x)\, y_2(x) + u_2(x)\, y_2'(x)] \\ &+ Q(x)\left(u_1(x)\, y_1(x) + u_2(x)\, y_2(x)\right) = f(x) \end{aligned}$$

After rearranging, we have

$$u_1(x)[y_1''(x) + P(x)y_1'(x) + Q(x)\; y_1(x)] + u_2(x)[y_2''(x) + P(x)y_2'(x) + Q(x)\; y_2(x)] + u_1''(x)\, y_1(x) + u_1'(x)y_1'(x) + u_2''(x)\, y_2(x) + u_2'(x)y_2'(x) + P(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' = f(x), \qquad (2)$$

But

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \quad \text{and} \quad y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

So (2) reduces to

$u_1''(x)\,y_1(x) + u_1'(x)y_1'(x) + u_2''(x)\,y_2(x) + u_2'(x)y_2'(x) + P(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' = f(x),$

$$u_1''(x)\,y_1(x) + u_1'(x)y_1'(x) + u_2''(x)\,y_2(x) + u_2'(x)y_2'(x) + P(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' = f(x), \tag{3}$$

But

$$\frac{d}{dx}\big(u_1'(x)y_1(x)\big) = u_1''(x)\,y_1(x) + u_1'(x)y_1'(x)$$

And

$$\frac{d}{dx}\big(u_2'(x)y_2(x)\big) = u_2''(x)\,y_2(x) + u_2'(x)y_2'(x)$$

So (3) becomes

$$\frac{d}{dx}\big(u_1'(x)y_1(x)\big) + \frac{d}{dx}\big(u_2'(x)y_2(x)\big) + P(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' = f(x), \tag{4}$$

Equation (4) can be written as

$$\frac{d}{dx}(u_1'y_1 + u_2'y_2) + P(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' = f(x), \tag{5}$$

Since we seek to determine two unknown functions u_1 and u_2, so we require only two equations. We can obtain these by making the further assumption that the functions u_1 and u_2 satisfy $y_1u_1' + y_2u_2' = 0$. So Eq. (5) further reduces to $y_1'u_1' + y_2'u_2' = f(x)$.

So we have

$$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases} \tag{6}$$

By Cramer's Rule, the solution of the system can be expressed in terms of determinants.

$$u_1' = \frac{W_1}{W} = -\frac{y_2(x)f(x)}{W} \quad \text{and } u_2' = \frac{W_2}{W} = \frac{y_1(x)f(x)}{W}$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},\ W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \text{ and } W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

NOTE: $W\big(y_1(x), y_2(x)\big) \neq 0$ for every x in the interval.

Hence finally we have

$$u_1 = \int \frac{W_1}{W}\,dx = -\int \frac{y_2(x) f(x)}{W}\,dx$$

and

$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{y_1(x) f(x)}{W}\,dx$$

So the particular solution is

$$y_p = u_1 y_1 + u_2 y_2$$

Hence the general solution will be

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$$

EXAMPLE 1 **General Solution Using Variation of Parameters**

Solve $y'' - 4y' + 4y = (x + 1)e^{2x}$.

SOLUTION From the auxiliary equation $m^2 - 4m + 4 = (m - 2)^2 = 0$ we have $y_c = c_1 e^{2x} + c_2 x e^{2x}$. With the identifications $y_1 = e^{2x}$ and $y_2 = xe^{2x}$, we next compute the Wronskian:

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

Since the given differential equation is already in form (6) (that is, the coefficient of y'' is 1), we identify $f(x) = (x + 1)e^{2x}$. From (10) we obtain

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x + 1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x + 1)xe^{4x}, \qquad W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x + 1)e^{2x} \end{vmatrix} = (x + 1)e^{4x},$$

and so from (9)

$$u_1' = -\frac{(x + 1)xe^{4x}}{e^{4x}} = -x^2 - x, \qquad u_2' = \frac{(x + 1)e^{4x}}{e^{4x}} = x + 1.$$

It follows that $u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$ and $u_2 = \frac{1}{2}x^2 + x$. Hence

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

and

$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}.$$

Exercise 4.6

In Problems 1–10 solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$

2. $y'' + y = \tan x$

3. $y'' + y = \sin x$

4. $y'' + y = \sec\theta \tan\theta$

5. $y'' + y = \cos^2 x$

6. $y'' + y = \sec^2 x$

7. $y'' - y = \cosh x$

8. $y'' - y = \sinh 2x$

9. $y'' - 4y = \dfrac{e^{2x}}{x}$

10. $y'' - 9y = \dfrac{9x}{e^{3x}}$