Linear Algebra

Lecture No. 1

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Presentation Overview

Assessment Instruments with Weights

- Class Participation 10%
- Quizzes 15%
- Assignments 10%
- Midterm 20%
- Final 45%

Class Rules

- Class attendance is mandatory.
- You may miss up to 6 class sessions. On the seventh absence, you will be withdrawn from the course.
- As a courtesy to the instructor and other students, be prepared to arrive at class and be in your seat on time.
- Cell phones should be powered off/silent.
- Eatables are not allowed in the class
- Intolerance of any disruptive behavior in the class
- The Dress Code has to be observed, no warnings will be given, and violators will be asked politely to leave the class and consequently will be marked absent
- Bring notebook and pen
- Bring lecture notes

Introduction

- Linear Algebra is a branch of Mathematics, but the truth of it is that linear algebra is the Mathematics of Data.
- When you take a digital photo with your phone or transform the image in Photo-shop, when you play a video game or watch a movie with digital effects, when you do a web search or make a phone call, you are using technologies that build upon linear algebra.
- Matrices and Vectors are the languages of data.

History of Linear Algebra

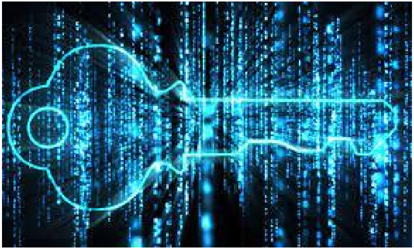
- The beginnings of matrices and determinants goes back to the second century BC although traces can be seen back to the fourth century BC. But, the ideas did not make it to mainstream math until the late 16^{th} century.
- The Babylonians around 300 BC studied problems which lead to simultaneous linear equations.
- The Chinese, between100 BC and 200 BC, came much closer to matrices then the Babylonian. Indeed, the text **Nine Chapters on the Mathematical Art** written during the **Han Dynasty** gives the first known example of matrix methods.
- In Europe, two-by-two determinants were considered by Cardano at the end of the 16^{th} century and larger ones by Leibniz and, in Japan, by Seki about 100 years later.

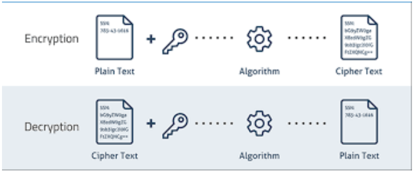
Why is Linear Algebra important?

- Linear Algebra is vital in multiple areas of computer science because linear equations are so easy to solve.
- It converts large number of problems to matrix and thus we solve the matrix.

Applications of Linear Algebra

- Machine Learning.
- Network Flow.
- Cryptography.
- Audio,video and image processing.
- Signal Processing.
- Video Games.
- Electrical Circuits.

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Plain Text
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Decryption
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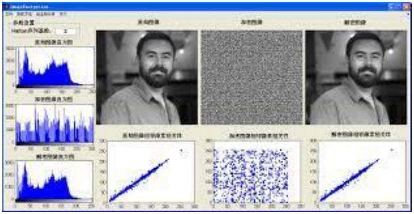
Algorithm
Plain Text



Linear Algebra

- System of Linear Equations
- Matrices
- Gaussian Elimination and Gaussian Jordan Method
- Cryptography
- Matrix Transformation
- Vector Spaces
- Eigen Values and Eigen Vectors

Linear Equations

Definitions (Linear Equations)

A linear equation is one where all the variables such as x,y,z have index(power) of 1 or 0 only, for example,

Example

$x+2y+z=5$

is a linear equation. The following are also linear equations:

Example

- $x=3$. (One Variable)
- $x+2y=5$ (Two Variables)
- $3x+y+z+w=-8$ (Four Variables)
- $x_1+x_2+...+x_n=b$ (n Variables)

Continue...

The following are not linear equations:

Example

- $x^2 - 1 = 0$
- x+y^4+$\sqrt{z}$=9
- sin(x)-y+z=3

Why?
In equation**1** the index(power) of the variable x is 2, so this is a quadratic equation.
In equation **2** the index of y is **4** and z is $\frac{1}{2}$.
In equation **3** the variable x is an argument of the trignometric function sine.

General Linear Equations

Definition

A Linear Equation in the variables x_1,x_2,...,x_n is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b \qquad (1)$$

where b and the coefficients $a_1, a_2, ..., a_n$ are real or complex numbers.

A solution to Linear Equation (1) is a sequence of n numbers s_1,s_2,...,s_n which has the property that (1) is satisfied when x_1=s_1,x_2=s_2,...,x_n=s_n are substituted in (1).

Continue...

For example

$$6x - 3y + 4z = -13 \tag{2}$$

has a solution $x = 2, y = 3, z = -4.$
since

$$6(2) - 3(3) + 4(-4) = -13$$
$$12 - 9 - 16 = -13$$
$$-13 = -13$$

System of Linear Equations in Two Variables

Definition

A system of linear equations in two variables x and y will have the form

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Here $a_i, b_i, c_i (i = 1, 2)$ are real numbers.

The two techniques to solve linear systems are:
1. Elimination Method.
2. Substitution Method.
In the Elimination Method, any of the coefficients is first equated and eliminated. After elimination, the equations are solved to obtain the other equation. Below is an example of solving linear equations using the elimination method for better understanding.

Example 1:

Find the solution of the linear system by using elimination method.

$$2x + 3y = 9 \tag{3}$$

$$x - y = 3 \tag{4}$$

Continue...

Solution:
Here, if equation (4) is multiplied by 2, the coefficient of “x” will become the same and can be subtracted.
So, multiply equation (4) $\times$ 2 and then subtract equation (3)

$$2x + 3y = 9$$

$$-(2x - 2y = 6)$$

$$-5y = -3$$

$$y = \frac{3}{5} = 0.6$$

Now, put the value of $y = 0.6$ in equation (4). So,

$$x - 0.6 = 3$$

Thus,

$$x = 3 + 0.6 = 3.6$$

So, $(0.6, 3.6)$ is the solution of the given system.

Substitution Method

To solve a linear equation using the Substitution Method, first, isolate the value of one variable from any of the equations. Then, substitute the value of the isolated variable in the second equation and solve it. Take the same equations again for example.

Example 2:

Find the solution of the linear system by using elimination method.

$$2x + 3y = 9 \tag{5}$$

$$x - y = 3 \tag{6}$$

Solution:

Now, consider equation (6) and isolate the variable “x”.

So, equation (6) becomes,

$$x = 3 + y$$

Continue...

Now, substitute the value of x in equation (5). So, equation (5) will be-

$$2x + 3y = 9$$

$$\implies 2(3 + y) + 3y = 9$$

$$\implies 6 + 2y + 3y = 9$$

$$\implies 5y = 9 - 6 \qquad Or \qquad 5y = 3$$

$$\implies y = \frac{3}{5} = 0.6$$

Now, substitute "y" value in equation (6).

$$x - y = 3$$

$$\implies x = 3 + y = 3 + 0.6 = 3.6$$

Thus, (x, y) = $(3.6, 0.6)$.

Question: Solve linear system by using elimination method.

$$\begin{aligned} x - y + z &= 5 && (7)\\ 2x + y - z &= -2 && (8)\\ 3x - y - z &= -7 && (9) \end{aligned}$$

Solution:

Solution:

We want to eliminate y and z , so add equation (7) & (8)

$$x - y + z = 5$$
$$2x + y - z = -2$$

$$3x = 3 \quad ----\rightarrow (10)$$

x=1

Take equation (7) and (9) and eliminate the variable z by adding equation (7) and (9).

$$x - y + z = 5$$
$$3x - y - z = -7$$

$$4x - 2y = -2 \quad ----\rightarrow (11)$$

Put value of x from equation (10) and (11).

$$4(1) - 2y = -2$$
$$4 - 2y = -2$$
$$4 + 2 = 2y$$
$$6 = 2y$$
$$y = 3$$

Now put value of x & y in equation (7).

$$1 - 3 + z = 5$$
$$-2 + z = 5$$
$$z = 5 + 2$$
$$z = 7$$

So $(1, 3, 7)$ is solution of given system.

Consistent and Inconsistent Linear System

Definitions (Consistent System)

If the system of linear equation has a solution then it is consistent.

Definitions (Inconsistent System)

If the system of linear equation has no solution then it is inconsistent.

So the system in example 1 and 2 is consistent.
Example 3: Find the solution of the linear system by using substitution method.

$$x - 3y = -7 \tag{10}$$

$$2x - 6y = 7 \tag{11}$$

Continue...

Solution: From equation 7 isolate the value of x.

$$x = -7 + 3y$$

Now equation 8 becomes.

$$2(-7 + 3y) - 6y = 7$$

$$\implies 2(-7 + 3y) - 6y = 7$$

$$\implies -14 + 6y - 6y = 7$$

$$\implies 0y = 7 + 14$$

$$\implies 0 = 21$$

which is not possible. This means that the given system has no solution. Therefore, it is inconsistent system.

Continue...

Note!

- A consistent linear system of two or more equations in two or more unknowns has either one solution or infinitely many solutions, there are no other possibilities.

Linear Systems with Two Unknowns Geometrically

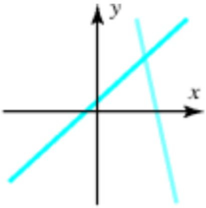
- The set of all possible solutions is called the Solution Set of the linear system.
- Two linear systems are called equivalent if they have the same solution set.
- That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.
- Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines.

For example, consider the linear system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Each solution (x,y) of this system corresponds to any point of intersection of the lines, so there are three possible solutions:

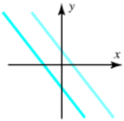
- The points may intersect at only one point, so in this case there is only one solution.



One solution

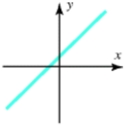
Continue...

- The lines may be parallel and distinct, in this case there is no intersection and consequently no solution.



No solution

- The lines coincide, in this case there are infinitely many points of intersection and hence there are infinitely many solutions.



Infinitely many solutions (coincident lines)

A linear system with no solution

Example 1:

Solve the linear system

$$\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 9 \end{cases}$$

Solution:

Multiply (1) by 2 and then subtract equation (2):

$$\begin{aligned} 4x + 6y &= 12 \\ -4x - 6y &= -9 \end{aligned}$$

$$- - - - - - - - - - - -$$

$$0 = 3$$

which is not possible. Therefore these equations have no solution.

Graph of linear system with no solution

Examples

A linear system with only one solution

Example 2:

Solve the linear system

$$\begin{cases} x - 2y = -1 \\ -x + 3y = 3 \end{cases}$$

Solution:

Add both equations (1) & (2)

$$\begin{aligned} x - 2y &= -1 \\ -x + 3y &= 3 \\ \hline y &= 2 \end{aligned}$$

Now put value of y in equation (1) and we get

$$x - 2(2) = -1$$

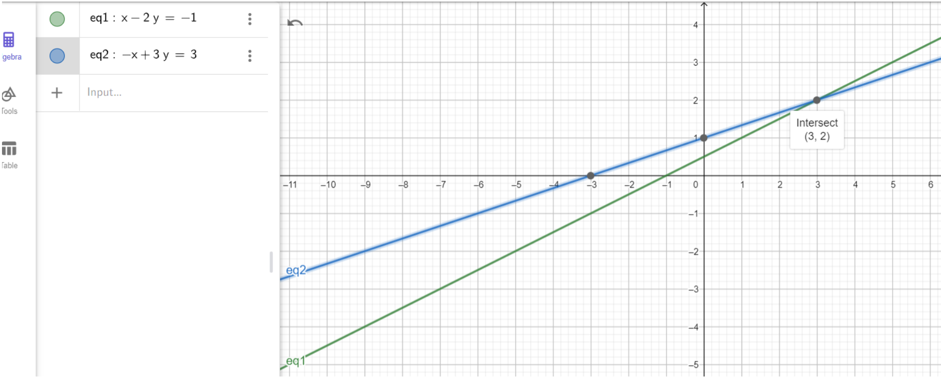
$$x - 4 = -1$$

$$x = -1 + 4$$

$$x = 3$$

The solution of given linear system is unique i.e., it has only one solution (x,y)=(3,2)

Graph of a linear system with one solution only



A linear system with infinite solution

Example 3:

Solve the linear system

$$\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 12 \end{cases}$$

Solution:

Multiply equation (1) by 2 and subtract both equations

$$\begin{array}{r} 4x + 6y = 12 \\ -4x - 6y = -12 \\ \hline 0 = 0 \end{array}$$

Thus, the solutions of the system are those values of x and y that satisfy the single equation:

$$2x + 3y = 6 \Longrightarrow 2x = 6 - 3y \Longrightarrow x = 3 - \frac{3}{2}y$$

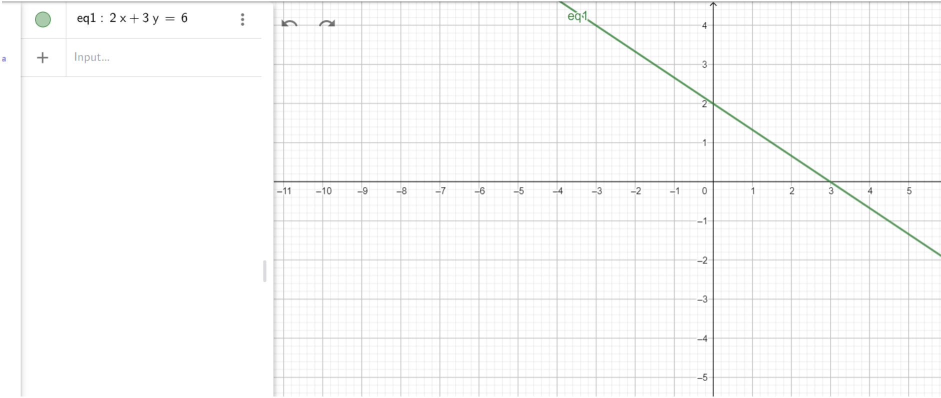
Let y=t, So

$$x = 3 - \frac{3}{2}t$$

$\left(3 - \frac{3}{2}t, t\right), t \in R$ is the solution of given system.

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter t. For example, t=0 gives (3,0), t=1 gives $\left(\frac{3}{2}, 1\right)$.

Graph of system with infinite solution



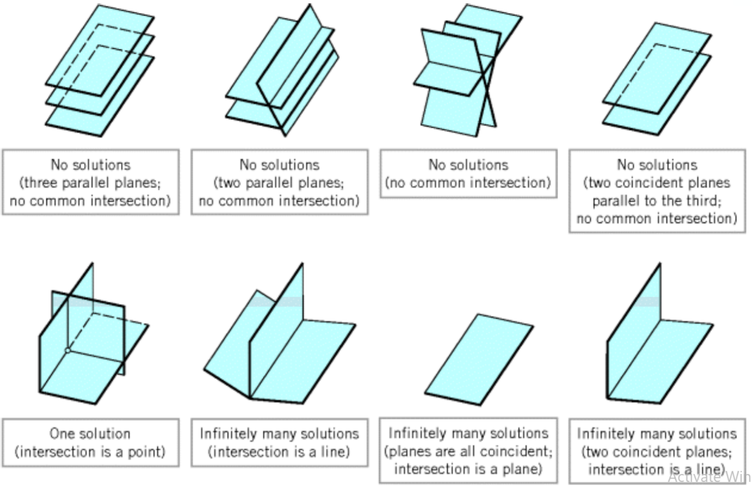

eq1 : 2 x + 3 y = 6
eq2 : 4 x + 6 y = 12
Input...
eq2

Linear Systems with Three Unknowns Geometrically

In case of three equations in three unknowns a linear system can be written as:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities- no solutions, one solution, or infinitely many solutions:


No solutions (three parallel planes; no common intersection)
No solutions (two parallel planes; no common intersection)
No solutions (no common intersection)
No solutions (two coincident planes parallel to the third; no common intersection)
One solution (intersection is a point)
Infinitely many solutions (intersection is a line)
Infinitely many solutions (planes are all coincident; intersection is a plane)
Infinitely many solutions (two coincident planes; intersection is a line)

Examples

Question:
Solve the linear system

$$\begin{cases} x + 2y + 4z = 7 & ------\rightarrow (1) \\ 3x + 7y + 2z = -11 & -------- (2) \\ 2x + 3y + 3z = 1 & ----------- (3) \end{cases}$$

Solution:
We want to eliminate x, so multiply equation (1) by 2 and the subtract equation (3).

$$\begin{array}{r} 2x + 4y + 8z = 14 \\ 2x + 3y + 3z = 1 \\ - \quad - \quad - \quad - \\ \hline 0 + \; y + 5z = 13 \end{array} \quad ----\rightarrow (4)$$

After eliminating x the equation is $y + 5z = 13$
Now multiply equation (1) by 3 and then subtract equation(2).

$$\begin{array}{r} 3x + 6y + 12z = 21 \\ 3x + 7y + 2z = -11 \\ - \quad - \quad - \quad + \\ \hline 0 - y + 10z = 32 \end{array} \quad ----\rightarrow (5)$$

Again after elimination of x the new equation is $-y + 10z = 32$
Now solve the two simultaneous equations (4) & (5) that we have obtained

$$y + 5z = 13$$
$$-y + 10z = 32$$

$$0 + 15z \quad = 45$$
$$z = \frac{45}{15} = 3$$

Put $z = 3$ in equation (4)

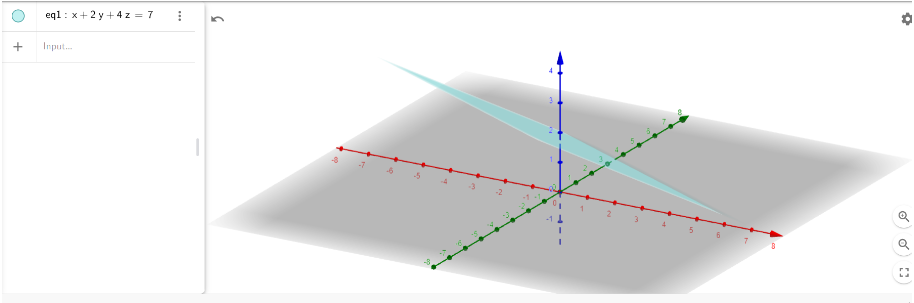
$$y + 5z = 13$$
$$y + 5(3) = 13$$
$$y + 15 = 13$$
$$y = 13 - 15 = -2$$

Put $z = 3, y = -2$ in equation (1)

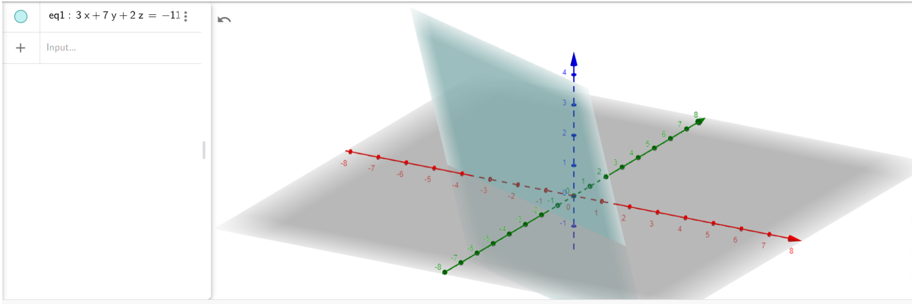
$$x + 2(-2) + 4(3) = 7$$

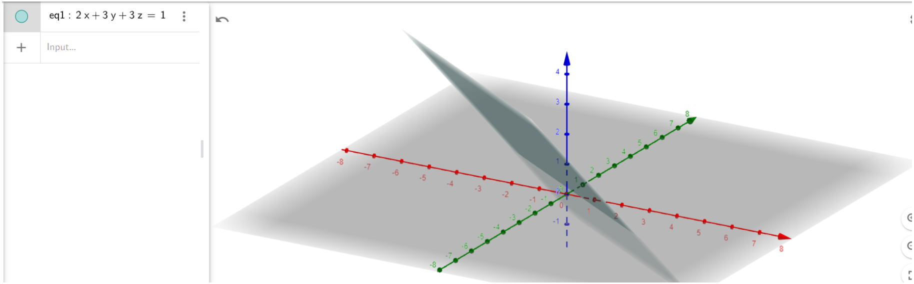
$$x - 4 + 12 = 7$$
$$x + 8 = 7$$
$$x = 7 - 8 = -1$$

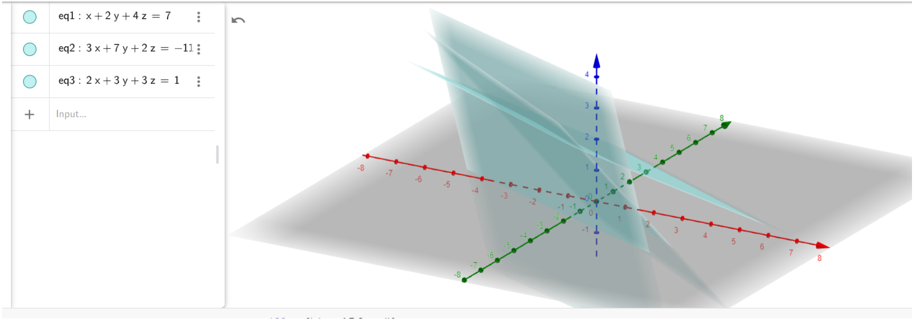
So $(-1, -2, 3)$ is solution of given system, which is unique. Geometrically it represents that three planes intersect at a unique point.



eq1 : x + 2 y + 4 z = 7
Input...


eq1 : 3 x + 7 y + 2 z = −11
Input...


eq1 : 2 x + 3 y + 3 z = 1
Input...


eq1 : x + 2 y + 4 z = 7
eq2 : 3 x + 7 y + 2 z = −11
eq3 : 2 x + 3 y + 3 z = 1
Input...

Question:

Solve the linear system

$$\begin{cases} y + 4z = -5 & \dashrightarrow (1) \\ x + 3y + 5z = -2 & ------- (2) \\ 3x + 7y + 7z = 6 & --------- (3) \end{cases}$$

Solution:

From equation (1) isolate y

$$y = -5 - 4z \dashrightarrow (4)$$

Now put value of y in equation (2)

$$x + 3(-5 - 4z) + 5z = -2$$

$$x - 15 - 12z + 5z = -2$$

$$x - 7z = -2 + 15$$

$$x - 7z = 13$$

$$x = 13 + 7z \dashrightarrow (5)$$

Now put value of x and y from equations (4) and (5) in equation (3)

$$3(13+7z)+7(-5-4z)+7z=6$$

$$39+21z-35-28z+7z=6$$

$$-7z+7z+4=6$$

$$0=6-4$$

$$0=2$$

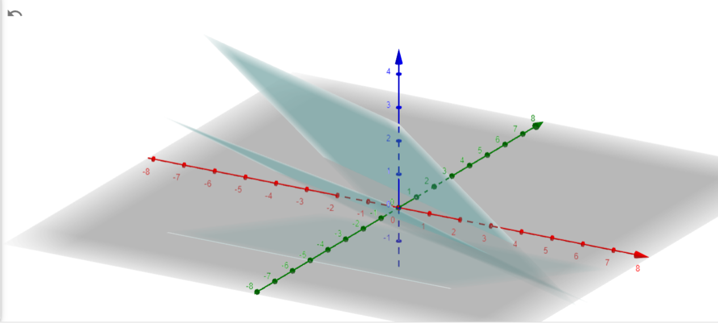
which is not possible, therefore the system of equations has no solution. So the system is inconsistent.

eq1 : $y + 4z = -5$

eq2 : $x + 3y + 5z = -2$

eq3 : $3x + 7y + 7z = 6$

Input...



Question:

Solve the linear system

$$\begin{cases} 2x + y - 3z = 0 & \text{------}\rightarrow (1) \\ 4x + 2y - 6z = 0 & \text{--------} (2) \\ x - y + z = 0 & \text{-----------} (3) \end{cases}$$

Solution:

First, we multiply equation (1) by -2 and add it to equation (2).

$$-4x - 2y + 6z = 0$$
$$4x + 2y - 6z = 0$$

0=0

We do not need to proceed any further. The result we get is an identity, 0=0 , which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2 and adding it to equation (1). We then perform the same steps as above and find the same result, $0 = 0$. When a system is dependent, we can find general expressions for the solutions. Adding equations (1) & (2), we have

$$2x + y - 3z = 0$$
$$x - y + z = 0$$

$$3x - 2z = 0$$

We then solve the resulting equation for z

$$3x - 2z = 0 \quad \Longrightarrow 3x = 2z \Longrightarrow z = \frac{3}{2}x \quad ------\rightarrow (4)$$

Now back substitute the expression for z into one of the equations and solve for y.

$$2x + y - 3\left(\frac{3}{2}\right)x = 0$$

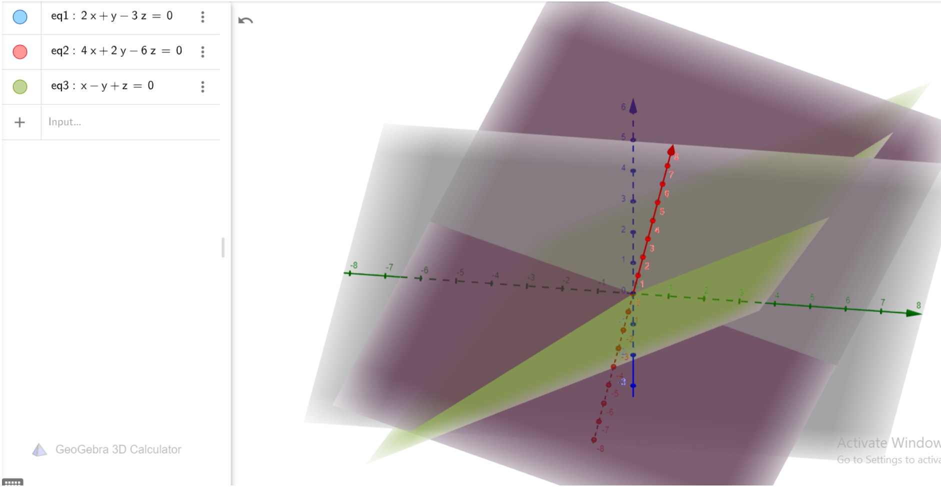
$$2x + y - \frac{9}{2}x = 0$$

$$y = \frac{9}{2}x - 2x = \frac{5}{2}x ------\rightarrow (5)$$

Now let $x = t$ put value of x in equation (4) & (5) and we get

$y = \frac{5}{2}t, z = \frac{3}{2}t$

$\left(t, \frac{5}{2}t, \frac{3}{2}t\right)$, $t \in R$ is the solution of the given system.


eq1 : 2 x + y − 3 z = 0
eq2 : 4 x + 2 y − 6 z = 0
eq3 : x − y + z = 0
Input...
GeoGebra 3D Calculator
Activate Window
Go to Settings to activa

1.In each part, determine whether the equation is linear in x_1, x_2 and x_3.

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$

(b) $x_1 + 3x_2 + x_1x_3 = 2$

(c) $x_1 = -7x_2 + 3x_3$

(d) $x_1^{-2} + x_2 + 8x_3 = 5$

(e) $x_1^{\frac{3}{5}} - 2x_2 + x_3 = 4$

(f) $\pi x_1 - \sqrt{2}x_2 = 7^{\frac{1}{3}}$

2. In each part, determine whether the given point is a solution of the linear system.

$$2x_1 - 4x_2 - x_3 = 1$$

$$x_1 - 3x_2 + x_3 = 1$$

$$3x_1 - 5x_2 - 3x_3 = 1$$

Exercise(Howard Anton Ex 1.1)

(a) $(3, 1, 1)$ (b) $(3, -1, 1)$ (c) $(13, 5, 2)$ (d)$(\frac{13}{2}, \frac{5}{2}, 2)$ (e)$(17, 7, 5)$

3. In each part, determine whether the given point is a solution of the linear system.

$$x + 2y - 2z = 3$$

$$3x - y + z = 1$$

$$-x + 5y - 5z = 5$$

(a) $(\frac{5}{7}, \frac{8}{7}, 1)$ (b) $(\frac{5}{7}, \frac{8}{7}, 0)$ (c) $(5, 8, 1)$ (d)$(\frac{5}{7}, \frac{10}{7}, \frac{2}{7})$ (e)$(\frac{5}{7}, \frac{22}{7}, 2)$

1. Solve the following linear system of equations:

a) $$\begin{cases} 2x + 4y + 6z = -12 \\ 2x - 3y - 4z = 15 \\ 3x + 4y + 5z = -8 \end{cases}$$

b) $$\begin{cases} x + y = 5 \\ 3x + 3y = 10 \end{cases}$$

c) $$\begin{cases} 2x + 3y = 13 \\ x - 2y = 3 \\ 5x + 2y = 27 \end{cases}$$

2. In each part, find the solution set of the linear equation by using parameters as necessary.
   a) $7x - 5y = 3$
   b) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

Exercise 1.1 Q 1 to 10

Reference Books

1. Linear Algebra with supplemented Applications by Howard Anton/ Chris Rorres, 10th Edition.

2. Introductory Linear Algebra with Applications by Bernard Kolman, David R. Hill.

3. Linear Algebra with applications by Otto Bretscher, 4th edition.

4. Linear Algebra with Applications by Steven J. Leon.


Thank you