Directional Derivative

The Gradient Vector:

If f is a function of two variables x and y, then the gradient of f is a vector function denoted by ∇f and is defined as

$$\boldsymbol{\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} = f_x\hat{\imath} + f_y\hat{\jmath}}$$

Example: If $f(x,y) = \sin x + e^{xy}$, then find $\nabla f(x,y)$.

Solution:

$$f_x = \cos x + ye^{xy}$$

$$f_y = xe^{xy}$$

$$\nabla f(x,y) = f_x\hat{\imath} + f_y\hat{\jmath}$$

$$\nabla f(x,y) = (\cos x + ye^{xy})\hat{\imath} + (xe^{xy})\hat{\jmath}$$

Directional Derivative

i. Let $f(x,y)$ be a differentiable function at (a,b) and let $\vec{u} = \langle u_1, u_2\rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a,b) in the direction of $\vec{u}$ is

$$D_{\vec{u}}f(\mathrm{a},\mathrm{b}) = \nabla f(a,b)\cdot\vec{u}$$

ii. Let f be a function of three variables and is differentiable function at $(\mathrm{a},\mathrm{b},\mathrm{c})$ and let $\vec{u} = \langle u_1, u_2, u_3\rangle$ be a unit vector in the xy-plane. The directional derivative of f at (a,b,c) in the direction of $\vec{u}$ is

$$D_{\vec{u}}f(\mathrm{a},\mathrm{b},\mathrm{c}) = \nabla f(a,b,c)\cdot\vec{u}$$

where

$$\nabla f(a,b,c) = f_x(a,b,c)\hat{\imath} + f_y(a,b,c)\hat{\jmath} + f_z(a,b,c)\hat{k}$$

Example1: Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $\vec{v} = 2\hat{\imath} + 5\hat{\jmath}$.

Solution:We have to find

$$D_{\vec{u}}f(2,-1) = \nabla f(2,-1) \cdot \vec{u}$$

Step-1<u>Gradient of f</u>

$$f_x = 2xy^3$$

$$f_y = 3x^2y^2 - 4$$

$$\nabla f(x,y) = f_x(x,y)\hat{\imath} + f_y(x,y)\hat{\jmath}$$

$$\nabla f(x,y) = 2xy^3\hat{\imath} + (3x^2y^2 - 4)\hat{\jmath}$$

At $(2,-1)$

$$\nabla f(2,-1) = (2(2)(-1)^3)\hat{\imath} + (3(2)^2(-1)^2 - 4)\hat{\jmath}$$

$$\nabla f(2,-1) = -4\hat{\imath} + 8\hat{\jmath}$$

Step-2<u>Unit Vector</u>

The unit vector $\vec{u}$ parallel to $\vec{v}$is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$|\vec{v}| = \sqrt{(2)^2 + (5)^2} = \sqrt{29}$$

$$\vec{u} = \frac{2\hat{\imath}+5\hat{\jmath}}{\sqrt{29}} = \frac{2}{\sqrt{29}}\hat{\imath}+\frac{5}{\sqrt{29}}\hat{\jmath}$$

Step-3 <u>Directional Derivative</u>

$$D_{\vec{u}}f(2,-1) = \nabla f(2,-1) \cdot \vec{u}$$

$$D_{\vec{u}}f(2,-1) = (-4\hat{\imath} + 8\hat{\jmath}) \cdot (\frac{2}{\sqrt{29}}\hat{\imath}+\frac{5}{\sqrt{29}}\hat{\jmath})$$

$$D_{\vec{u}}f(2,-1) = (-4)(\frac{2}{\sqrt{29}})+(8)(\frac{5}{\sqrt{29}})$$

$$D_{\vec{u}}f(2,-1) = \frac{-8+40}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

Example2: If $(x, y) = \frac{y^2}{x^2}$, P(1,2) and $\vec{u} = \frac{2}{3}\hat{\imath} + \frac{\sqrt{5}}{3}\hat{\jmath}$, find

a) the gradient of f,
b) the gradient of f at P and
c) the rate of change of f in the direction of vector $\vec{u}$ at P.

Solution:

a) Gradient of f

$$\nabla f(x, y) = f_x(x, y)\hat{\imath} + f_y(x, y)\hat{\jmath}$$

$$f_x = \frac{-2y^2}{x^3}, \quad f_y = \frac{2y}{x^2}$$

$$\nabla f(x, y) = \frac{-2y^2}{x^3}\hat{\imath} + \frac{2y}{x^2}\hat{\jmath}$$

b) Gradient of f at P(1,2)

$$\nabla f(1,2) = \frac{-2(2)^2}{(1)^3}\hat{\imath} + \frac{2(2)}{(1)^2}$$

$$\nabla f(1,2) = -8\hat{\imath} + 4\hat{\jmath}$$

c) Rate of Change of f in the Direction of Vector $\vec{u}$ at P

$$D_{\vec{u}}f(1,2) = \nabla f(1,2) \cdot \vec{u}$$

$$D_{\vec{u}}f(1,2) = (-8\hat{\imath} + 4\hat{\jmath}) \cdot (\frac{2}{3}\hat{\imath} + \frac{\sqrt{5}}{3}\hat{\jmath})$$

$$D_{\vec{u}}f(1,2) = (-8)(\frac{2}{3}) + (4)(\frac{\sqrt{5}}{3})$$

$$D_{\vec{u}}f(1,2) = \frac{-16 + 4\sqrt{5}}{3}$$