Calculus and Analytical Geometry

Lecture no. 08

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Topic: The derivatives

1. Average rate of change
2. Instantaneous rate of change
3. Derivative
4. Examples
5. Practice questions

1. **Rate of change**:

   Rate of change tells you how quickly something is changing.

2. **Average rate of change**:

   The rate of change at which something was changing for the **longer period of time**, is known as average rate of change. In calculus it is known as secant line slope.

3. **Instantaneous rate of change:**

   The rate of change at which something is changing for a **precise moment of time**, is known as instantaneous rate of change. In calculus this change is known as tangent line slope or derivative.

4. **Derivative:**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 1: Consider the function $f(x) = -4x + 1$

a) Find the derivative of function with respect to x.

b) Use the result of (a) to find the instantaneous rate of change of function at x=2

Solution:

<u>Step 1</u>: Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[-4(x+h) + 1] - [-4x + 1]}{h}$$

$$= \lim_{h \to 0} \frac{[-4x - 4h + 1] - [-4x + 1]}{h}$$

$$= \lim_{h \to 0} \frac{[-4x - 4h + 1] + 4x - 1}{h} = \lim_{h \to 0} \frac{-4h}{h}$$

$$= -4$$

Step 2: instantaneous rate of change

The instantaneous rate of change of f at $x = 2$ is $f'(2) = -4$. This means x decreases instantaneously 4 units per one unit increase in x at $x = 2$.

Example 2: consider the function $f(x) = 3x^2 + 2x + 11$, find

a) The derivative of function with respect to x.

b) Use the result of a) to find the instantaneous rate of change at $x = 1$

c) Find the equation of tangent line of the function at $x = 1$.

Solution:

a) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{[3(x+h)^2 + 2(x+h) + 11] - [3x^2 + 2x + 11]}{h}$$

$$= \lim_{h \to 0} \frac{[3(x^2 + 2xh + h^2) + 2x + 2h + 11] - 3x^2 - 2x - 11]}{h}$$

$$= \lim_{h \to 0} \frac{[3x^2 + 6xh + 3h^2 + 2x + 2h + 11] - 3x^2 - 2x - 11]}{h}$$

$$= \lim_{h \to 0} \frac{[6xh + h^2 + 2h]}{h}$$

$$= \lim_{h \to 0} \frac{h(6x + h + 2)}{h}$$

$$= 6x + 2$$

b) $f'(x) = 6x + 2 = 6(1) + 2 = 8$

c) At $x_o = 1, f(x_o) = 3(1)^2 + 2(1) + 11 = 3 + 2 + 11 = 16$

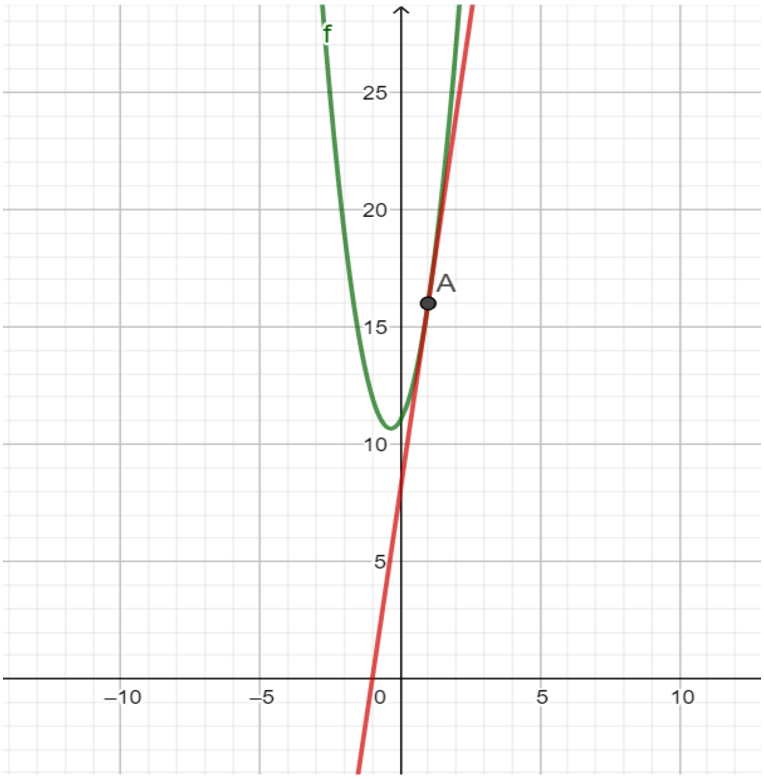
$$y - f(x_o) = m_{sec}(x - x_o)$$

$$y - 16 = 8(x - 1)$$

$$y - 16 = 8x - 8$$

$$y = 8x + 8$$

d) Graphical representation:



Example 3: Consider the function $f(x) = x^3 + 3$, find

a) The derivative of function with respect to x.

b) Use the result of a) to find the instantaneous rate of change at $x = 1$

c) Find the equation of tangent line of the function at $x = 1$.

Solution:

a)
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{[(x+h)^3 + 3] - (x^3 + 3)}{h}$$

$$= \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 + 3] - (x^3 + 3)}{h}$$

$$= \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 + 3] - x^3 - 3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$

b) $f'(x) = 3x^2 = 3(1)^2 = 3$

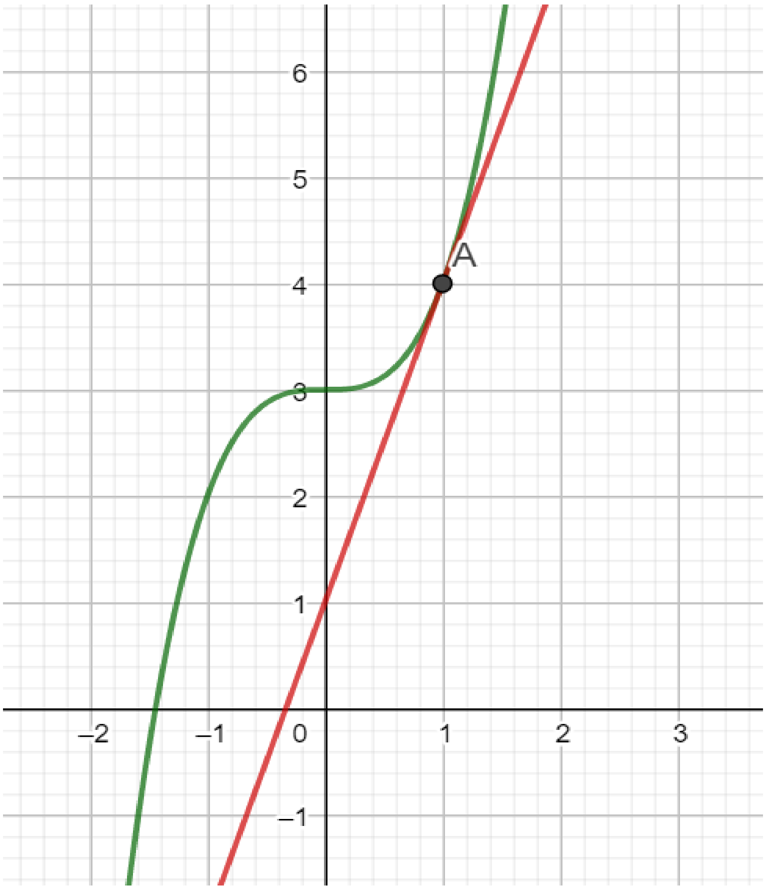
c) At $x_o = 1, f(x_o) = (1)^3 + 3 = 4$

$$y - f(x_o) = m_{sec}(x - x_o)$$

$$y - 4 = 3(x - 1)$$

$$y - 4 = 3x - 3$$

$$y = 3x + 1$$



Example 4: Consider the function $f(x) = \frac{1}{x^2}$, find

a) The derivative of function with respect to x.

b) Use the result of a) to find the instantaneous rate of change at $x = 1$

c) Find the equation of tangent line of the function at $x = 1$.

Solution:

a)
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} [f(x+h) - f(x)]$$

$$= \lim_{h \to 0} \frac{1}{h} [f(2+h) - f(2)]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{(x+h)^2} - \frac{1}{x^2} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2 - (x^2 + 2xh + h^2)}{x^2(x+h)^2} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-2xh - h^2}{x^2(x+h)^2} \right]$$

$$= \lim_{h \to 0} \left[\frac{-2x - h}{x^2(x+h)^2} \right]$$

$$= \frac{-2x}{x^4} = \frac{-2}{x^3} = -2x^{-3}$$

b) $f'(x) = -2x^{-3} = -2(1)^{-3} = -2$

c) At $x_o = 1, f(x_o) = \frac{1}{1^2} = 1$

$$y - f(x_o) = m_{sec}(x - x_o)$$

$$y - 1 = -2(x - 1)$$

$$y - 1 = -2x + 2$$

$$y = -2x + 3$$

Practice questions:

1. Consider the function $f(x) = 2x^3 + 4$, find
   - The derivative of function with respect to x.
   - Use the result of a) to find the instantaneous rate of change at $x = 0$
   - Find the equation of tangent line of the function at $x = 0$.
2. Consider the function $f(x) = \frac{1}{x-2}$, find
   - The derivative of function with respect to x.
   - Use the result of a) to find the instantaneous rate of change at $x = 1$
   - Find the equation of tangent line of the function at $x = 1$ and show on the graph.
3. Consider the function $f(x) = \frac{1}{x}$, find
   - The derivative of function with respect to x.
   - Use the result of a) to find the instantaneous rate of change at $x = 3$
   - Find the equation of tangent line of the function at $x = 3$.
4. Consider the function $f(x) = \frac{3x+1}{2x-5}$, find
   - The derivative of function with respect to x.
   - Use the result of a) to find the instantaneous rate of change at $x = 3$
   - Find the equation of tangent line of the function at $x = 3$.
5. Consider the function $f(x) = x^2 + 2$, find
   - The derivative of function with respect to x.
   - Use the result of a) to find the instantaneous rate of change at $x = 2$
   - Find the equation of tangent line of the function at $x = 2$ and show it on graph