Gaussian Elimination and the RREF

Homogeneous Linear System

A system of linear equation is said to be homogeneous if it's constant term is equal to zero.

Example:

$$3x + 4y = 0$$

$$-2x + 5y = 0$$

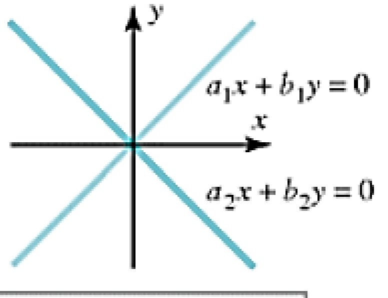
- **Every system homogenous of linear equations is consistent because all such systems have $x = 0, y = 0$ as a solution. This solution is called trivial solution. If there are other solutions, they are called non trivial solutions.**

- **A homogenous system always has the trivial solution. There are only two possibilities for its solutions:**
  - **The system has only the trivial solution.**
  - **The system has infinite many solutions in addition to the trivial solutions (This happened when the system involves more unknown than equations).**

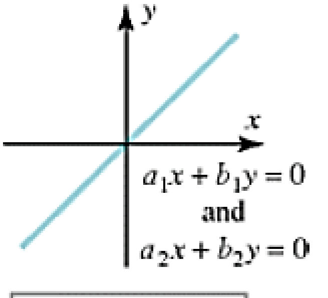
- **A homogeneous linear system of two equations in two unknowns are of the form**

$$a_1 x + b_1 y = 0$$
$$a_2 x + b_2 y = 0$$

- **The graph of the equations are the lines through the origin.**



Only the trivial solution



Infinitely many solutions

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Non-homogenous linear system

$$3x + 4y = 3$$

$$4x + 5y = 9$$

- A non-homogenous system with more unknowns than equations need not to be consistent.
- If a non-homogenous system with more unknowns than equations is consistent then it has infinitely many solutions.

Example 1: Solve the linear system of equation:

$$\begin{cases} x + z + 2w = 6 \dashrightarrow (1) \\ y - 2z = -3 \dashrightarrow (2) \\ x + 2y - z = -2 \dashrightarrow (3) \\ 2x + y + 3z - 2w = 0 \dashrightarrow (4) \end{cases}$$

Solution: Subtracting (1) and (4), we get

$$x + z + 2w = 6$$

$$2x + y + 3z - 2w = 0$$

$$\Rightarrow\ 3x + y + 4z = 6 \dashrightarrow (5)$$

Subtracting equation (3) and (5), we get

$$3x + 6y - 3z = -6$$

$$3x + y + 4z = 6$$

$$\Rightarrow\ 5y - 7z = -12 \dashrightarrow (6)$$

Subtracting equation (2) and (6), we get

$$5y - 10z = -15$$

$$5y - 7z = -12$$

$$\Rightarrow -3z = -3 \Rightarrow z = 1$$

Solution (Continue):

Put $z = 1$ in equation, we get $\Rightarrow y = -1$.

Put value of y and z in equation (3).

$$x + 2y - z = -2$$

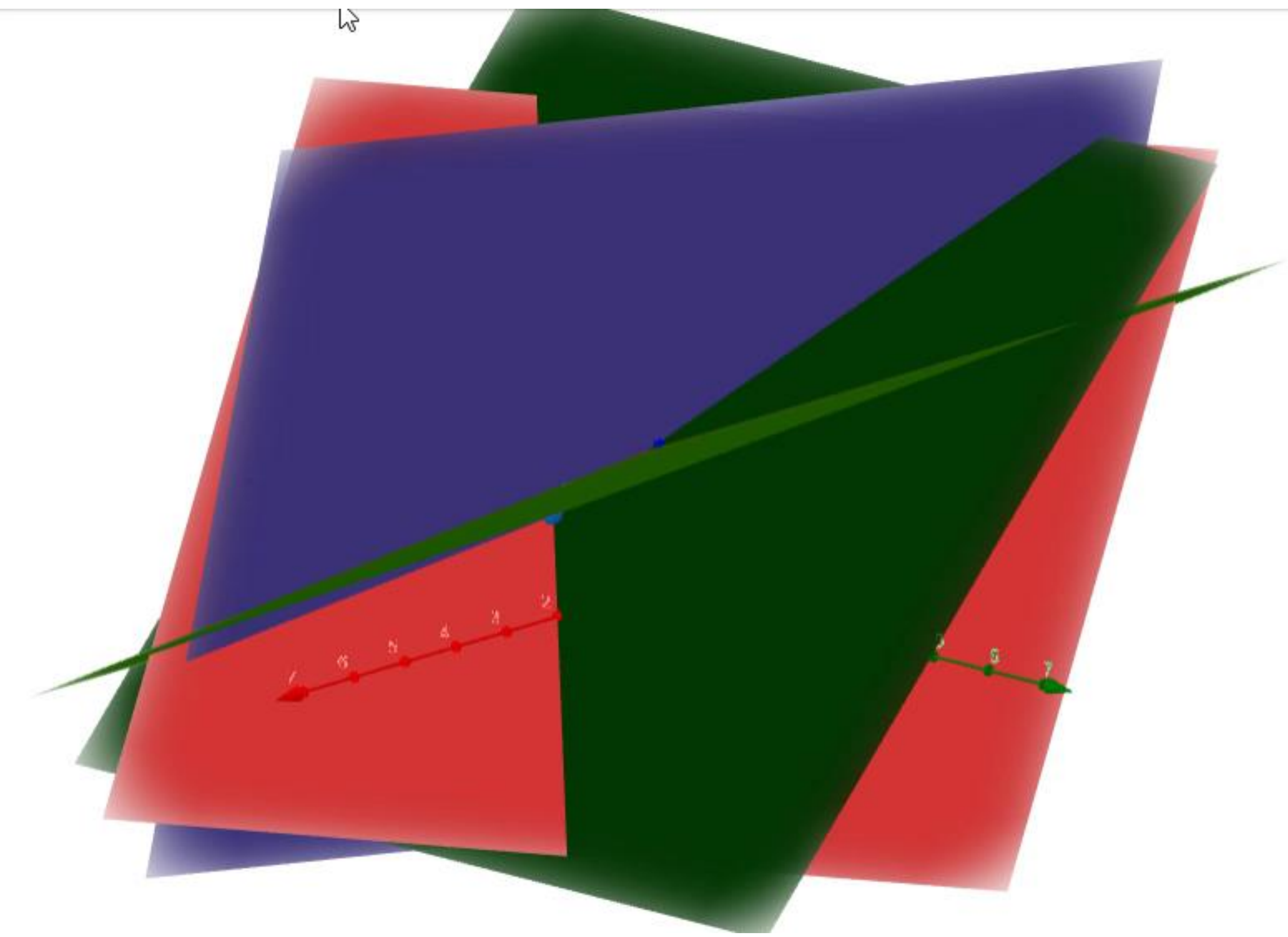
$$x + 2(-1) - (1) = -2 \Rightarrow x = 1$$

Put value of x, y and z in equation (1), we get

$$x + z + 2w = 6$$

$$2w = 6 - x - 2 \Rightarrow w = 2$$

So (1,-1,1,2) is solution of above system of four equations with four unknowns. Geometrically it clearly can be seen that three planes intersect at unique point (1,-1,1,2).



Gaussian Elimination and the RREF

Augmented Matrices and Elementary Row Operations

As the number of equations and unknowns in a linear system increases so does the complexity of the algebra involved in finding solutions. So we write system of equations in the form of matrices and solve it by

- **row echelon form (Gauss Elimination Method) or**
- **reduced row echelon form (Gauss Jorden Method).**

These methods are very useful to find the solutions of system linear equations.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\quad\quad\quad\quad \cdots\cdot \\ \quad\quad\quad\quad\quad \cdots\cdot \\ \quad\quad\quad\quad\quad \cdots\cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

The above system can be written in rectangular array of numbers:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdot & a_{1n} & : b_1 \\ a_{21} & a_{22} & \cdots\cdot & a_{2n} & : b_2 \\ \vdots & & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots\cdot & a_{mn} & : b_m \end{bmatrix}.$$

This is called the augmented matrix for the system. For example, the augmented matrix for the system of equations

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_1 + 4x_2 - 3x_3 = 1$$

$$3x_1 + 6x_2 - 5x_3 = 0$$

Is

$$\begin{bmatrix} 1 & 1 & 2 & : 9 \\ 2 & 4 & -3 & : 1 \\ 3 & 6 & -5 & : 0 \end{bmatrix}.$$

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The basic method for solving a linear system is to perform appropriate algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are as follows:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called **elementary row operations** on a matrix.

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Example 2:

Use elementary row operations to solve the linear system:

$$x_1 + x_2 + 2x_3 = 9$$
$$2x_1 + 4x_2 - 3x_3 = 1$$
$$3x_1 + 6x_2 - 5x_3 = 0$$

Solution:

$$\begin{bmatrix} 1 & 1 & 2 & : 9 \\ 2 & 4 & -3 & : 1 \\ 3 & 6 & -5 & : 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 2 & -7 & : & -17 \\ 3 & 6 & -5 & : & 0 \end{bmatrix} \quad \longrightarrow \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 2 & -7 & : & -17 \\ 0 & 3 & -11 & : & -27 \end{bmatrix} \quad \longrightarrow \quad R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & \frac{-7}{2} & : & -\frac{17}{2} \\ 0 & 3 & -11 & : & -27 \end{bmatrix} \quad \longrightarrow \quad \frac{1}{2}R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & \frac{-7}{2} & : & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & : & -\frac{3}{2} \end{bmatrix} \quad \longrightarrow \quad R_3 - 3R_2$$

Thus we obtained the echelon form:

$$\begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & \frac{-7}{2} & : & -\frac{17}{2} \\ 0 & 0 & 1 & : & 3 \end{bmatrix} \quad \longrightarrow \quad -2R_3$$

Its corresponding system can be represented as

$$\begin{cases} x_3 = 3 \\ x_2 - \dfrac{7}{2}x_3 = -\dfrac{17}{2} \\ x_1 + x_2 + 2x_3 = 9 \end{cases}$$

Solution (Continue):

$$\begin{bmatrix} 1 & 1 & 2 & : 9 \\ 0 & 1 & \dfrac{-7}{2} : & -\dfrac{17}{2} \\ 0 & 0 & 1 & : 3 \end{bmatrix}$$

Its corresponding system can be represented as

$$\begin{cases} x_3 = 3 \\ x_2 - \dfrac{7}{2}x_3 = -\dfrac{17}{2} \\ x_1 + x_2 + 2x_3 = 9 \end{cases}$$

Put $x_3 = 3$ in equation (2)

$$x_2 - \frac{7}{2}(3) = -\frac{17}{2} \Rightarrow x_2 = -\frac{17+21}{2} \quad \Rightarrow x_2 = 2$$

Put $x_3 = 3, x_2 = 2$ in equation (3), we get

$$x_1 + 2 + 2(3) = 9 \Rightarrow x_1 = 1$$

So (1, 2, 3) is solution of above system.

Finding the solution of linear system by converting the augmented matrix into echelon form is called Gauss Elimination Method.

(This system already has been solved by elimination method)

Gaussian Elimination and the RREF

Gauss Jordan Method

Finding the solution of linear system by converting the augmented matrix into reduced echelon form is called Gauss Jorden Method. For this I take the matrix of echelon form:

$$\begin{bmatrix} 1 & 1 & 2 & : 9 \\ 0 & 1 & \frac{-7}{2} & : -\frac{17}{2} \\ 0 & 0 & 1 & : 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & : 9 \\ 0 & 1 & 0 & : 2 \\ 0 & 0 & 1 & : 3 \end{bmatrix} \longrightarrow R_2 + \frac{7}{2}R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & : 3 \\ 0 & 1 & 0 & : 2 \\ 0 & 0 & 1 & : 3 \end{bmatrix} \longrightarrow R_1 - 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & : 1 \\ 0 & 1 & 0 & : 2 \\ 0 & 0 & 1 & : 3 \end{bmatrix} \longrightarrow R_1 - R_2$$

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 3$$

So (1, 2, 3) is solution of above system.

Example 1 Solve the linear system of equations using Gauss elimination method:

$$\begin{cases} x + z + 2w = 6 ----\rightarrow (1) \\ y - 2z = -3 \ \ ------\rightarrow (2) \\ x + 2y - z = -2 -----\rightarrow (3) \\ 2x + y + 3z - 2w = 0 ----\rightarrow (4) \end{cases}$$

Solution: Do yourself and match your answer with the solution obtained above.

Gaussian Elimination and the RREF

Howard Anton (Exercise 1.2)

Q1. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both or neither.

(a) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

Q2. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Work to do:

Q3 Solve the linear system of equation by using Gauss elimination method

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Gaussian Elimination and the RREF

Solution of Word Problems using Gauss-Jordan Method

Example 1

Ali and Sara are shopping for chocolate bars. Ali observes, "If I add half my money to yours, it will be enough to buy two chocolate bars." Sara naively asks, "If I add half my money to yours, how many can we buy?" Ali replies, "One chocolate bar." How much money did Ali have?

Solution: **Let** **a = Ali's money**

s = Sara's money

c = Cost of chocolate

$$\begin{cases} \frac{1}{2}a + s = 2c & (1) \\ a + \frac{1}{2}s = c & (2) \end{cases},$$

Or $$\begin{cases} a + 2s = 4c \\ 2a + s = 2c \end{cases}$$

The augmented matrix is:

$$\begin{bmatrix} 1 & 2 & : & 4c \\ 2 & 1 & : & 2c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & : & 4c \\ 0 & -3 & : & -6c \end{bmatrix} \quad \longrightarrow \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & : & 4c \\ 0 & 1 & : & 2c \end{bmatrix} \quad \longrightarrow \quad -\frac{1}{3}R_2$$

$$\begin{bmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & 2c \end{bmatrix} \quad \longrightarrow \quad R_1 - 2R_2$$

Solution is $(a, s) = (0, 2c)$. **It means Ali has no money.**

Gaussian Elimination and the RREF

Example 2

Three Alto, two Suzuki, and four City can be rented for \$106 per day. At the same rates two Alto, four Suzuki, and three City cost \$l07 per day, whereas four Alto, three Suzuki, and two City cost \$102 per day. Find the rental rates for all three kinds of cars?

Solution:

$$3a + 2s + 4c = 106$$

$$2a + 4s + 3c = 107$$

$$4a + 3s + 2c = 102$$

Its Augmented matrix is

$$\begin{bmatrix} 3 & 2 & 4 & : & 106 \\ 2 & 4 & 3 & : & 107 \\ 4 & 3 & 2 & : & 102 \end{bmatrix} \xrightarrow{R_1 - R_2}$$

$$\begin{bmatrix} 1 & -2 & 1 & : & -1 \\ 2 & 4 & 3 & : & 107 \\ 4 & 3 & 2 & : & 102 \end{bmatrix} \xrightarrow{R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & -2 & 1 & : & -1 \\ 0 & 8 & 1 & : & 109 \\ 4 & 3 & 2 & : & 102 \end{bmatrix} \xrightarrow{R_3 - 4R_1}$$

$$\begin{bmatrix} 1 & -2 & 1 & : & -1 \\ 0 & 8 & 1 & : & 109 \\ 0 & 11 & -2 & : & 106 \end{bmatrix} \xrightarrow{R_2 - R_3}$$

$$\begin{bmatrix} 1 & -2 & 1 & : & -1 \\ 0 & -3 & 3 & : & 3 \\ 0 & 11 & -2 & : & 106 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_3}$$

Example 2 (Continue)

$$\begin{bmatrix} 1 & -2 & 1 & : & -1 \\ 0 & 1 & -1 & : & -1 \\ 0 & 11 & -2 & : & 106 \end{bmatrix} \xrightarrow{R_3 - 11R_2}$$

$$\begin{bmatrix} 1 & -2 & 1 & : & -1 \\ 0 & 1 & -1 & : & -1 \\ 0 & 0 & 9 & : & 117 \end{bmatrix} \xrightarrow{\frac{1}{9}R_3}$$

$$\begin{bmatrix} 1 & -2 & 1 & : & -1 \\ 0 & 1 & -1 & : & -1 \\ 0 & 0 & 1 & : & 13 \end{bmatrix} \xrightarrow{R_2 + R_3}$$

$$\begin{bmatrix} 1 & -2 & 1 & : & -1 \\ 0 & 1 & 0 & : & 12 \\ 0 & 0 & 1 & : & 13 \end{bmatrix} \xrightarrow{R_1 - R_3}$$

$$\begin{bmatrix} 1 & -2 & 0 & : & -14 \\ 0 & 1 & 0 & : & 12 \\ 0 & 0 & 1 & : & 13 \end{bmatrix} \xrightarrow{R_1 + 2R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & : & 10 \\ 0 & 1 & 0 & : & 12 \\ 0 & 0 & 1 & : & 13 \end{bmatrix}$$

Hence, the rental rates for Alto, Suzuki, and City cars are $\$10$, $\$12$ and $\$13$ per day, respectively.

Example 3 **A restaurant owner plans to use x tables seating 4, y tables seating 6 and z tables seating 8, for a total 20 tables. When fully occupied, the tables seat 108 customers. If only half of the x tables, half of the y tables and one-fourth of the z tables are used, each fully occupied, then 46 customers will be seated. Find x, y, and z.**

Solution:

$$x + y + z = 20$$

$$4x + 6y + 8z = 108$$

$$4\left(\frac{x}{2}\right) + 6\left(\frac{y}{2}\right) + 8\left(\frac{z}{4}\right) = 46$$

Simplifying the system, we have

$$x + y + z = 20$$

$$2x + 3y + 4z = 54$$

$$2x + 3y + 2z = 46$$

.

.

.

The answer is: $x = 10, y = 6 \text{ and } z = 4$

Gaussian Elimination and the RREF

Our true End-goal: the row reduced echelon form (rref)

Example: the augmented matrix on the left corresponds to the system on the right

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 6 \\ 0 & 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftrightarrow \begin{matrix} x_1 + 3x_3 + x_5 & = 6 \\ x_2 + 2x_3 & = 5 \\ x_4 + 2x_5 & = 2 \\ 0 & = 1 \\ 0 & = 0 \end{matrix} \quad \text{or} \quad \begin{matrix} x_1 = 6 - 3x_3 - x_5 \\ x_2 = 5 - 2x_3 \\ x_4 = 2 - 2x_5 \\ 0 = 1 \\ 0 = 0 \end{matrix}$$

General conditions for the rref:

- The first non-zero term in each row (the "leading 1") is a 1 ; there are zeros above and below it.
- The leading 1 in any row is to the right of the leading 1 in the row above it.
- Any rows containing all zeros (to the left of the vertical bar) appear at the bottom.

For each of the following problems…

- Determine whether or not the matrix given is in rref form. If it is not, use elementary row operations to put it in rref form.
- Determine whether the system represented by the matrix has solutions (is consistent) or has no solutions (is inconsistent).
- If the system is consistent, write down the solutions to the system.

Gaussian Elimination and the RREF

Work to do:

Q1. Students are buying books for the new semester. Asma buys the linear algebra book and the differential equation book for $178. Aiman, who is buying books for herself and her friend, spends $319 on two linear algebra books, one differential equation book, and one educational psychology book. Sara buys the educational psychology book and the differential equation book for $147 in total. How much does each book cost?

Q2. A soap manufacturer wants to spend 60 Lac rupees on radio, magazine, and TV advertising. If he spends as much on TV advertisement as on magazines and radio together, and the amount spend on magazines and TV combined equals 5 times that spent on radio, what is the amount to be spent on each type of advertising?

Q3. Three merchants find a purse lying in the road. One merchant says "If I keep the purse, I shall have twice as much money as the two of you together". "Give me the purse and I shall have three times as much as the two of you together", said the second merchant. The third merchant said, "I shall be much better off than either of you if I keep the purse, I shall have five times as much as the two of you together." If there are 60 coins (of equal value) in the purse, how much money does each merchant have?

Gaussian Elimination and the RREF

Example 1: Solve the linear system of equation:

$$\begin{cases} x + 4y - z = 12 \\ 3x + 8y - 2z = 4 \end{cases}$$

Solution: Using (1) and (2) and eliminating x implies

$$4y = 32 + z$$

$$y = 8 + \frac{1}{4}z$$

Put value of y in equation (1), we get

$$x = -20$$

Let $z = t$,

$$y = 8 + \frac{1}{4}t$$

So, Solution of this system is $\left(-20, 8 + \frac{1}{4}t, t\right)$.

Note: Solve the above linear system of equations using Gauss-Jordan method.

Example 2: Solve the following systems of linear equations using Gauss Jordan elimination method:

$$\begin{cases} x + 2y + z = 12 \\ 5x + 2y - 3z = 1 \\ 2x + y - z = 2 \end{cases}$$

Solution: $\begin{bmatrix} 1 & 2 & 1 & : 12 \\ 5 & 2 & -3 & : 1 \\ 2 & 1 & -1 & : 2 \end{bmatrix} \xrightarrow{R_2 - 5R_1} \begin{bmatrix} 1 & 2 & 1 & : 12 \\ 0 & -8 & -8 & : -59 \\ 2 & 1 & -1 & : 2 \end{bmatrix}$

$$\xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & : 12 \\ 0 & -8 & -8 & : -59 \\ 0 & -3 & -3 & : -22 \end{bmatrix} \xrightarrow{-\frac{1}{8}R_2} \begin{bmatrix} 1 & 2 & 1 & : 12 \\ 0 & 1 & 1 & : \frac{59}{8} \\ 0 & -3 & -3 & : -22 \end{bmatrix}$$

$$\xrightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & 2 & 1 & : 12 \\ 0 & 1 & 1 & : \frac{59}{8} \\ 0 & 0 & 0 & : \frac{1}{8} \end{bmatrix}$$

So the above system has no solution.

Gaussian Elimination and the RREF

Example 3: Solve the linear system of equation:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & :6 \\ 2 & -3 & 2 & :14 \\ 3 & 1 & -1 & :-2 \end{bmatrix} \xrightarrow{R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & -7 & -4 & :2 \\ 3 & 1 & -1 & :-2 \end{bmatrix} \xrightarrow{R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & -7 & -4 & :2 \\ 0 & -5 & -10 & :-20 \end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix} 1 & 0 & 0 & :1 \\ 0 & 1 & 0 & :-2 \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

So the above system has solution (1, -2, 3).

Example 4: Solve the linear system of equation:

$$\begin{cases} x + 2y - 3z = -4 \\ 2x + y - 3z = 4 \end{cases}$$

Solution: **The above system has solution (4+t, t-4, t).**

Gaussian Elimination and the RREF

Work to do: Howard Anton (Exercise 1.2)

In Exercises 21–23, solve the given homogeneous linear system by any method.

21.

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

Answer:

$$w = t, x = -t, y = t, z = 0$$

22.

$$\begin{aligned} x_1 + 3x_2 + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

23.

$$\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$$

Answer:

$$I_1 = -1, I_2 = 0, I_3 = 1, I_4 = 2$$

Determine the values of a for which the system has no solutions, exactly one solution, or infinite many solutions.

25.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

Answer:

If $a = 4$, there are infinitely many solutions; if $a = -4$, there are no solutions; if $a \neq \pm 4$, there is exactly one solution.