Lecture # 3

HOMOGENEOUS LINEAR SYSTEM

A system of linear equation is homogeneous if its constant term is equal to zero.

Example
$$\begin{cases} 3x + 4y = 0 \\ -2x + 5y = 0 \end{cases}$$

- Homogeneous systems always has the trivial solution

TRIVIAL SOLUTION :-
A system having the solution $x = 0, y = 0$ $(0, 0)$

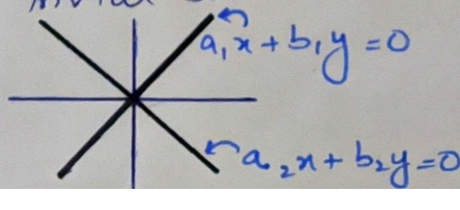
(If there are other solutions then they are Non-trivial solution)

- Homogeneous systems has infinitely many solutions in addition to the trivial solutions.
  (when system involves more unknowns than equations)

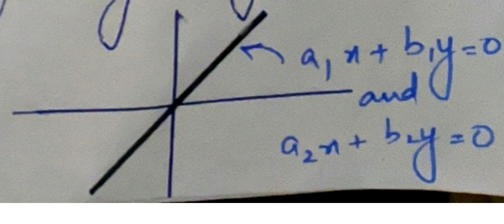
→ Homogeneous Linear system of two equations in two unknowns :

$$a_1 x + b_1 y = 0$$
$$a_2 x + b_2 y = 0$$

Only trivial solution



Infinitely many solutions.



NON-HOMOGENEOUS LINEAR SYSTEM

e.g. $$3x + 4y = 3$$
$$4x + 5y = 9$$

Note :-

- Non-homogeneous system having
  No. of equations < No. of unknowns
  ↳ need not to be CONSISTENT

  If CONSISTENT

  then it has
  infinitely many solutions.

Example.

Solve the linear system of equation

$$\begin{cases} x + z + 2w = 6 & \text{— ①} \\ y - 2z = -3 & \text{— ②} \\ x + 2y - z = -2 & \text{— ③} \\ 2x + y + 3z - 2w = 0 & \text{— ④} \end{cases}$$

Adding ① & ④

$$\begin{array}{r} x + z + 2w = 6 \\ + \quad 2x + y + 3z - 2w = 0 \\ \hline 3x + y + 4z = 6 \end{array} \quad \text{— ⑤}$$

Multiplying eq ③ by 3 and subtracting from ⑤

$$\begin{array}{r} \cancel{3x} + y + 4z = 6 \\ \cancel{3x} + 6y - 3z = -6 \\ - \quad\ - \quad\ + \quad\ + \\ \hline -5y + 7z = 12 \end{array} \quad \text{— ⑥}$$

Multiplying equation ② by 5 and adding to eq ⑥

$$\begin{array}{r} \cancel{-5y} + 7z = 12 \\ + \quad \cancel{5y} - 10z = -15 \\ \hline -3z = -3 \end{array} \Longrightarrow \boxed{z = 1}$$

Putting $z = 1$ in eq 2

$$y - 2(1) = -3$$

$$y - 2 = -3$$

$$\boxed{y = -1}$$

Putting $y = -1$ & $z = 1$ in eq 3

$$x + 2(-1) - (1) = -2$$

$$x - 2 - 1 = -2$$

$$x - 3 = -2$$

$$\boxed{x = 1}$$

Putting values of x, y, z in eq 1

$$x + z + 2w = 6$$

$$(1) + (1) + 2w = 6$$

$$2w = 4$$

$$\boxed{w = 2}$$

Thus, Solution of the linear system is

$$\boxed{(1, -1, 1, 2)}$$

AUGMENTED MATRICES and
ELEMENTARY ROW OPERATIONS

More the number of equations and unknowns in a linear system $\rightarrow$ complexity of algebra increases.

So, system of equations $\xrightarrow[\text{into}]{\text{changed}}$ MATRICES

solve by

(1) **Gauss Elimination Method**
i.e Row echelon form

(2) **Gauss Jorden Method**
i.e Reduced Row Echelon form

System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \quad \dots \\ \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

Can be written in rectangular array of numbers called **Augmented Matrix** for the system.

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & \ddots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Example:

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_1 + 4x_2 - 3x_3 = 1$$

$$3x_1 + 6x_2 - 5x_3 = 0$$

Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 2 & 4 & -3 & : & 1 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$$

Elementary row operations: (to solve the matrix)

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

Example. Use elementary row operations to solve the linear system.

$$x_1 + x_2 + 2x_3 = 9$$

$$2x_1 + 4x_2 - 3x_3 = 1$$

$$3x_1 + 6x_2 - 5x_3 = 0$$

Augmented matrix
$$\begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 2 & 4 & -3 & : & 1 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$$

$$R_2 - 2R_1 \longrightarrow \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 2 & -7 & : & -17 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$$

$$R_3 - 3R_1 \longrightarrow \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 2 & -7 & : & -17 \\ 0 & 3 & -11 & : & -27 \end{bmatrix}$$

$$\tfrac{1}{2} R_2 \longrightarrow \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & -7/2 & : & -17/2 \\ 0 & 3 & -11 & : & -27 \end{bmatrix}$$

$$R_3 - 3R_2 \longrightarrow \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & -7/2 & : & -17/2 \\ 0 & 0 & -1/2 & : & -3/2 \end{bmatrix}$$

$$-2R_3 \longrightarrow \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & -7/2 & : & -17/2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Echelon form

$$\boxed{x_3 = 3}$$

$$x_2 - \frac{7}{2}x_3 = -17/2 \longrightarrow x_2 - \frac{7}{2}(3) = -17/2$$

$$x_2 = \frac{-17}{2} + \frac{21}{2}$$

$$\boxed{x_2 = 2}$$

$$x_1 + x_2 + 2x_3 = 9$$

$$\downarrow$$

$$x_1 + (2) + 2(3) = 9$$

$$\boxed{x_1 = 1}$$

So $(1, 2, 3)$ is solution of the above system.

Note: Solving the solution of linear system by converting augmented matrix into Echelon form

↳ "Gauss Elimination Method"

Finding the same solution using "Gauss Jordan Method" i.e reduced echelon form

continuing with the recent echelon form

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_2 + \frac{7}{2} R_3 \longrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 - 2R_3 \longrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 - R_2 \longrightarrow \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Hence we get $x_1 = 1$
$x_2 = 2$
$x_3 = 3$

$(1, 2, 3)$ is solution of the above system.

$$\left[\begin{array}{ccc:c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc:c} \textcircled{1}\,1 & \textcircled{9}\,0 & \textcircled{8}\,0 & 1 \\ \textcircled{2}\,0 & \textcircled{4}\,1 & \textcircled{7}\,0 & 2 \\ \textcircled{3}\,0 & \textcircled{5}\,0 & \textcircled{6}\,1 & 3 \end{array}\right]$$

order in which entries are preferably changed.

we get $x_1 = 1$

$x_2 = 2$

$x_3 = 3$