Computing Partial Derivatives Algebraically

First order partial derivatives

Q1: Find the first order partial derivatives of

$$\boldsymbol{f(x,y) = x^2 + 5y^2}$$

Solution: $f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 5y^2)$

$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(5y^2)$

As we find partial derivative with respect to x, y is considered as constant.

$= 2x + 0 = 2x$

$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 5y^2)$

$= \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(5y^2)$

$= 0 + 5\frac{\partial}{\partial y}(y^2)$

$= 5(2y) = 10y$

Q2: Find $\boldsymbol{f_x(3,2)}$ and $\boldsymbol{f_y(3,2)}$ for $\boldsymbol{f(x,y) = x^2 + 5y^2}$.

Solution: $f_x(x,y) = 2x$

$f_x(3,2) = 2(3) = 6$

$f_y(x,y) = 10y$

$f_y(3,2) = 10(2) = 20$

Q3: Find both partial derivatives of each of the following functions:

i. $\boldsymbol{f(x,y) = 3x + e^{-5y}}$
ii. $\boldsymbol{f(x,y) = x^2 y}$
iii. $\boldsymbol{f(u,v) = u^2 e^{2v}}$

Solution:

i. $\boldsymbol{f(x,y) = 3x + e^{-5y}}$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x + e^{-5y})$$

$$= \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial x}(e^{-5y})$$

$$= \boldsymbol{3 + 0 = 3}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x + e^{-5y})$$

$$= \frac{\partial}{\partial y}(3x) + \frac{\partial}{\partial y}(e^{-5y})$$

$$= \boldsymbol{0 + e^{-5y}(-5) = e^{-5y}}$$

ii. $\boldsymbol{f(x,y) = x^2 y}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 y) = y\frac{\partial}{\partial x}(x^2)$$

$$\boldsymbol{\frac{\partial f}{\partial x} = y(2x) = 2xy}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 y) = x^2\frac{\partial}{\partial y}(y)$$

$$\boldsymbol{\frac{\partial f}{\partial y} = x^2(1) = x^2}$$

iii. $\boldsymbol{f(u,v) = u^2e^{2v}}$

$$\frac{\partial f}{\partial u} = \frac{\partial}{\partial u}(u^2e^{2v})$$

$$\frac{\partial f}{\partial u} = e^{2v}\frac{\partial}{\partial u}(u^2)$$

$$\frac{\partial f}{\partial u} = e^{2v}(2u)$$

$$\boldsymbol{\frac{\partial f}{\partial u} = 2ue^{2v}}$$

$$\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}(u^2e^{2v})$$

$$\frac{\partial f}{\partial v} = u^2\frac{\partial}{\partial v}(e^{2v})$$

$$\frac{\partial f}{\partial v} = u^2e^{2v}(2)$$

$$\boldsymbol{\frac{\partial f}{\partial v} = 2u^2e^{2v}}$$

Question: Let's consider a small printing business where N is the number of workers; v is the value of equipments (in units of \$25000) and P is the production, measured in thousands of pages per day.

$$\boldsymbol{P = f(N, v) = 2N^{0.6}v^{0.4}}$$

a) **If this company has a labor force of $\mathbf{100}$ workers and $\mathbf{200}$ units worth of equipments. What is the production of company?**
b) **Find$\boldsymbol{f_N(100, 200)\,and\ f_v(100, 200)}$. Interpret your answers in terms of production.**

Solution: **a)** $N = 100, v = 200$

$$P = f(100{,}200) = 2(100)^{0.6}(200)^{0.4}$$

$$= 2639\ thousand\ pages\ per\ day$$

b) To find$\boldsymbol{f_N}$, we treat $\boldsymbol{v}$ as a constant,

$$\frac{\partial f}{\partial N} = \frac{\partial}{\partial N}(2N^{0.6}v^{0.4})$$

$$\frac{\partial f}{\partial N} = 2v^{0.4}\frac{\partial}{\partial N}(N^{0.6}) = 2v^{0.4}(0.6)N^{0.6-1}(1)$$

$$\frac{\partial f}{\partial N} = 1.2v^{0.4}N^{-0.4}$$

$$\frac{\partial f}{\partial N}(100{,}200) = 1.2(200)^{0.6}100^{-0.4}$$

$$\boxed{\boldsymbol{\frac{\partial f}{\partial N}(100, 200) = 1.583\ thousands\ per\ worker}}$$

This tells us if we have 200 units of equipments and increase the no of worker by 1 from 100 to 101 the productions output will go up by 1.58 units or 1580 units by per pages.

To find$f_v(100{,}200)$, we treat N as constant and differentiate f with respect to v.

$$\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}(2N^{0.6}v^{0.4})$$

$$= 2N^{0.6}(0.4)v^{0.4-1}(1)$$

$$= 0.8N^{0.6}v^{-0.6}$$

$$\frac{\partial f}{\partial V}(100{,}200) = \; 0.8\,(100)^{0.6}\,(200)^{-0.6}$$

$$\boxed{\boldsymbol{\frac{\partial f}{\partial V}(100, 200) = 0.53\ thousand\,\frac{pages}{unit}\,of\ equipment}}$$

This tells us that if we have 100 workers and increase the value of equipment by 1 unit ($25000) from 200 to 201 units, the production goes up by about 0.53 units or 530 pages per day.

Question: Find the partial derivatives of

$$\boldsymbol{f(x, y) = xy^2 + 3x^2e^y}$$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy^2 + 3x^2e^y)$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial x}(3x^2e^y) = y^2\frac{\partial}{\partial x}(x) + 3e^y\frac{\partial}{\partial x}(x^2)$$

$$\frac{\partial f}{\partial x} = y^2(1) + 3e^y(2x) = y^2 + 6xe^y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy^2 + 3x^2e^y)$$

$$= x\frac{\partial}{\partial y}(y^2) + 3x^2\frac{\partial}{\partial y}(e^y) = x(2y) + 3x^2(e^y)$$

$$\boxed{\boldsymbol{\frac{\partial f}{\partial y} = 2xy + 3x^2(e^y)}}$$

Question: Find partial derivatives of

$$\boldsymbol{f(x,y) = 10x^2e^{3y}}$$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(10x^2e^{3y})$$

$$= 10e^{3y}\frac{\partial}{\partial x}(x^2) = 10e^{3y}(2x)$$

$$\boxed{\boldsymbol{f_x = 20xe^{3y}}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(10x^2e^{3y})$$

$$= 10x^2\frac{\partial}{\partial y}(e^{3y}) = 10x^2e^{3y}(3)$$

$$\boxed{\boldsymbol{f_y = 30x^2e^{3y}}}$$

Question: Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the following functions.

i. $f(x,y) = x^2 - xy + y^2$
ii. $f(x,y) = (x^2 - 1)(y + 2)$
iii. $f(x,y) = (xy - 1)^2$
iv. $f(x,y) = \sqrt{x^2 + y^2}$
v. $f(x,y) = \boldsymbol{\frac{1}{x+y}}$
vi. $f(x,y) = \boldsymbol{e^{x+y+1}}$
vii. $f(x,y) = \boldsymbol{e^{-x}}\sin(x + y)$
viii. $f(x,y) = \ln(x + y)$
ix. $f(x,y) = \boldsymbol{e^{xy}}\ln(y)$
x. $f(x,y) = \boldsymbol{\frac{x}{x^2+y^2}}$
xi. $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
xii. $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$
xiii. $h(x,y) = xe^y + y + 1$
xiv. $g(x,y) = x^2y + cosy + ysinx$

Calculate the 1st order partial derivative of the following:

i. $f(x, y, z) = 1 + xy^2 - 2z^2$
ii. $f(x, y, z) = xy + yz + zx$
iii. $f(x, y, z) = x - \sqrt{y^2 + z^2}$
iv. $f(x, y, z) = (x^2 + y^2 + z^2)^{\frac{-1}{2}}$
v. $f(x, y, z) = \ln(x + 2y + 3z)$
vi. $f(x, y, z) = e^{-(x^2+y^2+z^2)}$
vii. $f(x, y, z) = e^{-xyz}$
viii. $f(x, y, z) = yz \ln xy$