Intervals and Inequalities

Lecture Notes

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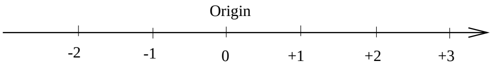
October 2018

Number Line. A line with origin, positive direction, and unit distance is called the *number line.*



Number Line

- Corresponding to every real number there is a point on the number line.
- Corresponding to every point on the number line there is a real number.
- If r is the real number that corresponds to a point P on the number line, then r is called the coordinate of P, written as $P(r)$.

Intervals. An interval is the set of all real numbers between two fixed numbers a and b, called the endpoints.

- Interval is called *closed* if it contains both of its endpoints.

$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

- Interval is called *open* if it contains neither of its endpoints.

$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

- Interval is called *half open* (or *half closed*) if it contains only one endpoint.

$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

Inequalities. Expressions of the form $x > 2, x < -1, x + 1 \geq 2x - 3$, etc. are all inequalities.

Properties of Inequalities. If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow ac > bc$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

Example 1. Solve the inequality $2x - 1 < x + 3$ and represent its solution set on the number line.

Solution.
Step 1. (Simplification)

$$\begin{aligned} 2x - x &< 3 + 1 \\ x &< 4 \end{aligned}$$

Step 2. (Solution set and its graph)

S.S.: $(-\infty, 4)$

Graph:

——————————*Solution Ends*——————————

Example 2. Solve the inequality $\frac{-x}{3} < 2x + 1$ and represent its solution set on the number line.

Solution.
Step 1. (Simplification)

$$\begin{aligned}
\frac{-x}{3} - 2x &< 1 \\
\frac{-x-6x}{3} &< 1 \\
\frac{-7x}{3} &< 1 \\
(\frac{-3}{7}) \times \frac{-7x}{3} &> (\frac{-3}{7}) \times 1 \\
x &> \frac{-3}{7}
\end{aligned}$$

Step 2. (Solution set and its graph)

S.S.: $(\frac{-3}{7}, +\infty)$

Graph:

——————————*Solution Ends*——————————

Example 3. Solve the inequality $\frac{6}{x-1} \geq 5$ and represent its solution set on the number line.

Solution.
Step 1. (Gathering terms)

$$\begin{aligned}
\frac{6}{x-1} - 5 &\geq 0 \\
\frac{6-5(x-1)}{x-1} &\geq 0 \\
\frac{6-5x+5}{x-1} &\geq 0 \\
\frac{11-5x}{x-1} &\geq 0
\end{aligned}$$

Step 2. (Boundary points)
Solving the equations $11 - 5x = 0$ and $x - 1 = 0$ we get $x = \frac{11}{5}$ and $x = 1$. These points divide the number line into three intervals $(-\infty, 1), (1, \frac{11}{5})$, and $(\frac{11}{5}, +\infty)$.

Take $x = 0, x = 2$, and $x = 3$ as test points in $(-\infty, 1), (1, \frac{11}{5})$, and $(\frac{11}{5}, +\infty)$, respectively.

- Since the test point $x = 0$ does not satisfy the inequality $\frac{11-5x}{x-1} \geq 0$, the whole interval $(-\infty, 1)$ does not satisfy the inequality.
- Since the test point $x = 2$ satisfies the inequality $\frac{11-5x}{x-1} \geq 0$, the whole interval $(1, \frac{11}{5})$ satisfies the inequality.
- Since the test point $x = 3$ does not satisfy the inequality $\frac{11-5x}{x-1} \geq 0$, the whole interval $(\frac{11}{5}, +\infty)$ does not satisfy the inequality.
- Since the boundary point $x = 1$ does not satisfy the inequality $\frac{11-5x}{x-1} \geq 0$, it is not part of the solution set.
- Since the boundary point $x = \frac{11}{5}$ does satisfy the inequality $\frac{11-5x}{x-1} \geq 0$, it is part of the solution set.

Step 4. (Solution set and its graph)

S.S.: $(1, \frac{11}{5}]$

Graph:

Solution Ends

Example 4. Find all those numbers whose squares are greater or equal to two less than thrice the numbers.

Solution.
Step 1. (Forming the inequality) Let x be the number that satisfies the condition. Then

$$x^2 \geq 3x - 2.$$

Step 2. (Gathering terms)

$$
\begin{aligned}
x^2 - 3x + 2 &\geq 0 \\
x^2 - 2x - x + 2 &\geq 0 \\
x(x-2) - 1(x-2) &\geq 0 \\
(x-2)(x-1) &\geq 0
\end{aligned}
$$

Step 3. (Boundary points)
Solving the equations $x - 2 = 0$ and $x - 1 = 0$ we get the boundary points $x = 2$ and $x = 1$. These points divide the number line into three intervals $(-\infty, 1), (1, 2)$, and $(2, +\infty)$.

Take $x = 0, x = 1.5$, and $x = 3$ as test points in $(-\infty, 1), (1, 2)$, and $(2, +\infty)$, respectively.

- Since the test point $x = 0$ satisfies the inequality $(x-2)(x-1) \geq 0$, the whole interval $(-\infty, 1)$ satisfies the inequality.

- Since the test point $x = 1.5$ does not satisfy the inequality $(x-2)(x-1) \geq 0$, the whole interval $(1, 2)$ does not satisfy the inequality.

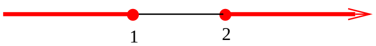
- Since the test point $x = 3$ satisfies the inequality $(x-2)(x-1) \geq 0$, the whole interval $(2, +\infty)$ satisfies the inequality.

- Note that both the boundary points $x = 1$ and $x = 2$ satisfy the inequality $(x-2)(x-1) \geq 0$, and hence are part of the solution set.

Step 4. (Solution set and its graph)

S.S.: $(-\infty, 1] \cup [2, +\infty)$

Graph:



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Practice Problems

1. $x^2 > x + 2$
2. $\frac{-2}{x-1} \leq -x$
3. $\frac{1}{x+2} < \frac{2}{x-3}$
4. Find all numbers whose squares are less or equal to six more than the numbers.

Book Link Calculus Book

(**I shall welcome your suggestions to improve these notes.**)