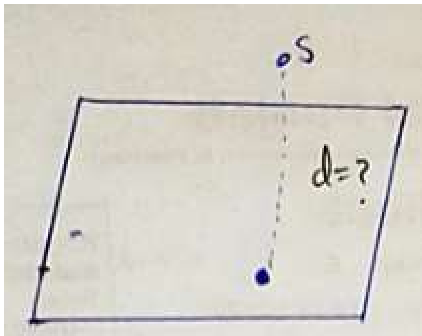
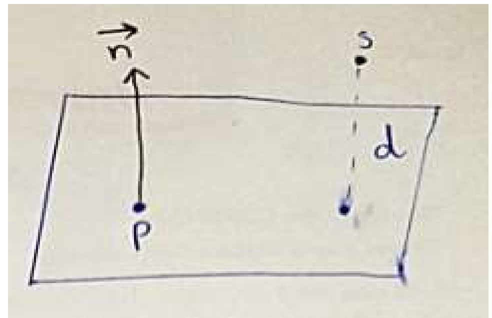
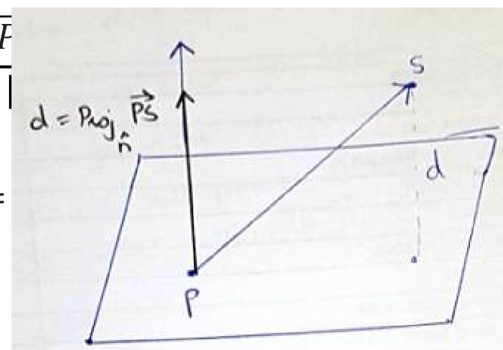
Distance from a Point to a Plane

If P is a point on the plane with normal $\vec{n}$, then the distance from any point S to the plane is the length of the vector projection $\overrightarrow{PS}$ onto $\vec{n} = A\hat{\imath} + B\hat{\jmath} + C\hat{k}$.





$$Proj_{\vec{n}}\overrightarrow{PS} = \Big[$$



$$d = \left|Proj_{\vec{n}}\overrightarrow{PS}\right| =$$

$$\mathrm{d} = \frac{|\overrightarrow{PS}.\vec{n}|}{|\vec{n}|^2}.\, |\vec{n}|$$

$$d = \frac{\left|\overrightarrow{PS}.\vec{n}\right|}{|\vec{n}|}$$

Example:

Find the distance from the point S(1,1,3) to the plane $3x + 2y + 6z = 6$.

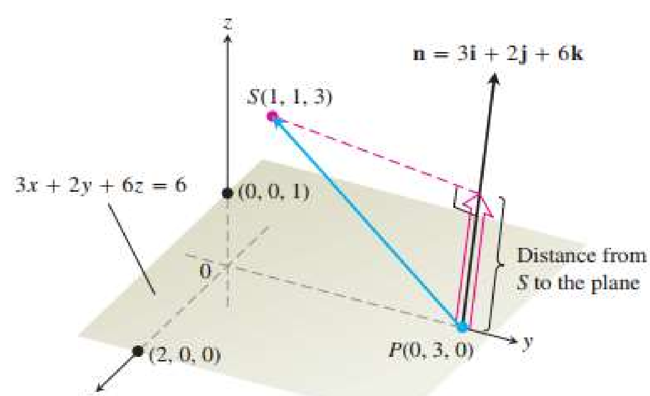
Solution:$\boldsymbol{\vec{n} = 3\vec{\imath} + 2\vec{\jmath} + 6\vec{k}}$

$$\boldsymbol{S = (1, 1, 3)}$$

$$\boldsymbol{P} = ?$$

We find a point P in the plane andcalculate the length of the vector projection

of $\overrightarrow{PS}$ onto a vector $\vec{n}$ normal to the plane. The point on plane easiest to find from the plane's equation are the intercepts.



If we take P to be the y-intercept (0,3,0) then

$\overrightarrow{PS} = (1 - 0)\vec{\imath} + (1 - 3)\vec{\jmath} + (3 - 0)\vec{k}$

$= \vec{i} - 2\vec{j} + 3\vec{k}$

$|\vec{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7$

The distance from S to the plane is

$$d = \frac{|\vec{PS}.\vec{n}|}{|\vec{n}|}$$

$= \left|(\vec{i} - 2\vec{j} + 3\vec{k}).(\frac{3}{7}\vec{i} + \frac{2}{7}\vec{j} + \frac{6}{7}\vec{k})\right| = \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}$

Ex. 12.5: 39-44, 45, 46

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4)$, $x + 2y + 2z = 13$

40. $(0, 0, 0)$, $3x + 2y + 6z = 6$

41. $(0, 1, 1)$, $4y + 3z = -12$

42. $(2, 2, 3)$, $2x + y + 2z = 4$

43. $(0, -1, 0)$, $2x + y + 2z = 4$

44. $(1, 0, -1)$, $-4x + y + z = 4$

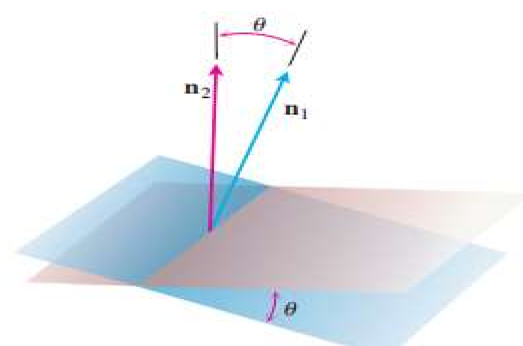
45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

46. Find the distance from the line $x = 2 + t, y = 1 + t, z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

Angle between Two Planes

The angle between two intersecting planes is defined to be the angle betweentheir normal vectors.

$\theta = Cos^{-1}\left(\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)$



Example: Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution:

$\vec{n_1} = 3\vec{i} - 6j\text{-}2\vec{k}$

$\vec{n_2}=2\vec{i}+\vec{j}-2\vec{k}$

$\vec{n_1}\,.\,\vec{n_2}=6\text{-}6\text{+}4 =4$

$|\vec{n_1}|=\sqrt{(3)^2+(-6)^2+(-2)^2}=\sqrt{9+36+4}=7$

$|\vec{n_2}| = \sqrt{(2)^2+(1)^2+(-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$

So $\theta = Cos^{-1}\left(\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)$

$\theta = Cos^{-1}\left(\frac{4}{7*3}\right)$= 79 degrees

Ex. 12.5: 47-52

Angles

Find the angles between the planes in Exercises 47 and 48.

47. $x + y = 1, \quad 2x + y - 2z = 2$

48. $5x + y - z = 10, \quad x - 2y + 3z = -1$

Use a calculator to find the acute angles between the planes in Exercises 49–52 to the nearest hundredth of a radian.

49. $2x + 2y + 2z = 3, \quad 2x - 2y - z = 5$

50. $x + y + z = 1, \quad z = 0$ (the xy-plane)

51. $2x + 2y - z = 3, \quad x + 2y + z = 2$

52. $4y + 3z = -12, \quad 3x + 2y + 6z = 6$