Calculus Lecture 21

Amina Komal
amina.komal@ucp.edu.pk

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Integration by partial fraction

Outline of the lecture

In this lecture we are going to study about:

- Partial fraction
- Linear Factors
- Examples
- Improper rational fractions
- Practice questions

1 Definition

A way of "breaking apart" fractions with polynomial in them.

1.1 Linear Factors

If all the factors of $Q(x)$ are linear then the partial fraction decomposition $P(x)/Q(x)$ can be determined by using the following rule:

For each factor of the form $(ax+b)^m$, the partial fraction decomposition contain the following sum of m partial fraction:

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+...+\frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, ..., A_m$ are constants to be determined. In this case $m=1$, only the first term appears.

Example 1

$$Evaluate \int \frac{dx}{x^2+x-2}$$

Solution The integrant is a proper rational fraction which can be written as

$$\frac{1}{x^2+x-2}=\frac{1}{(x-1)(x+2)}$$

because

$$x^2+x-2=x^2+2x-x-2=x(x+2)-1(x+2)$$

$$=> (x-1)(x+2)$$

Thus the expression has the form

$$\frac{1}{x^2+x-2}=\frac{A}{(x-1)}+\frac{B}{x+2} \tag{1}$$

Where A and B are constants to be determined. Multiply this expression by $(x-1)(x+2)$ yields

$$1=A(x+2)+B(x-1) \tag{2}$$

Now, to find A substitute $x=1$ in (2)

$$1=A(1+2)+B(1-1) => 1=A(3)$$

$$A = \frac{1}{3}$$

Similarly, for B substitute $x = -2$

$$1 = A(-2+2) + B(x-1) => 1 = B(-3)$$

$$B = \frac{1}{-3}$$

Now, equation (2) become,

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{\frac{-1}{3}}{x+2}$$

$$\int \frac{dx}{(x-1)(x+2)} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2}$$

After integrating,

$$= \frac{1}{3}ln|x-1| - \frac{1}{3}ln|x+2| + C$$

$$= \frac{1}{3}ln|\frac{x-1}{x+2}| + C$$

Example 2

$$Evaluate \int \frac{2x+4}{x^3-2x^2}dx$$

Solution

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

Here,

$$\frac{A}{x} + \frac{B}{x^2}$$

and

$$C = \frac{c}{x-2}$$

so, the partial decomposition is

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

Multiplying $x^2(x-2)$ we get

$$2x+4 = A(x)(x-2) + B(x-2) + C(x^2) \tag{3}$$

$$2x+4 = Ax^2 - 2Ax + Bx - 2B + x^2C \tag{4}$$

$$2x+4=(A+C)x^2+(-2A+B)x-2B \tag{5}$$

substituting, $x=0$ in (4) gives $C=-2$

$$2(0)+4=A(0)^2-2A(0)+B(0)-2B+(0)^2C$$

$$4=-2B => B=-2$$

By substituting the values back in the main equation we get,

$$\frac{2x+4}{x^2(x-2)}=\frac{-2}{x}+\frac{-2}{x^2}+\frac{2}{x-2}$$

$$\int\frac{2x+4}{x^2(x-2)}=-2\int\frac{dx}{x}-2\int\frac{dx}{x^2}+2\int\frac{dx}{x-2}$$

$$=-2lnx-2(\frac{-1}{x})+2ln|x-2|+C$$

$$=-2lnx+\frac{2}{x}+2ln|x-2|+C$$

Example 3

$$Evaluate\int\frac{x^2}{(x+2)^3}dx$$

Solution By using the partial fraction,

$$\frac{x^2}{(x+2)^3}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3} \tag{6}$$

Multiply $(x+2)^3$ on both sides of equation

$$x^2=A(x+2)^2+B(x+2)+C(x+2) \tag{7}$$

Simplify the equation

$$x^2=Ax^2+4Ax+4A+Bx+2B+C \tag{8}$$

Group the terms

$$x^2 = Ax^2 + (4A + B)x + (4A + 2B + C) \tag{9}$$

Equate the coefficients

$$0 = 4A + 2B + C$$

$$0 = 4A + B$$

$$1 = A$$

From here, we can get

$$(A, B, C) = (1, -4, 4)$$

By substituting the values of A, B, C equation (6) became,

$$\frac{x^2}{(x+2)^3} = \frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{4}{(x+2)^3} \tag{10}$$

Now, integrate equation (10)

$$\int \frac{x^2}{(x+2)^3} = \int \frac{1}{x+2} - \int \frac{4}{(x+2)^2} + \int \frac{4}{(x+2)^3}$$

$$= ln|x+2| - 4(\frac{(x+2)^{-2+1}}{-2+1} + 4\frac{(x+2)^{-3+1}}{-3+1} + C$$

After simplification,

$$\int \frac{x^2}{(x+2)^3} = ln|x+2| + \frac{1}{x+2} - \frac{2}{(x+2)^2} + C$$

2 Improper rational fractions

Definition: If the numerator $P(x)$ has degree greater than or equal to the degree of the denominator $Q(x)$, then the rational function $P(x)/Q(x)$ is called improper.

2.1 Integrating Improper Rational Fractions

An improper rational function can be integrated by performing a long division and expressing the function as the quotient plus the remainder over the divisor. The remainder over the divisor will be a proper rational function.

Example

$$Evaluate \int \frac{x^5 + x^2 + 2}{x^3 - x} dx$$

Solution: As the function is improper function, we are going to use long division here:

$$\begin{array}{r|l} & x^2 + 1 \\ x^3 - x & x^5 \quad + x^2 + 2 \\ & \underline{x^5 - x^3} \\ & \quad x^3 + x^2 \quad + 2 \\ & \quad \underline{x^3 \quad - x} \\ & \qquad x^2 + x + 2 \end{array}$$

Now,

$$\frac{x^5 + x^2 + 2}{x^3 - x} = x^2 + 1 + \frac{x^2 + x + 2}{x^3 - x} \tag{11}$$

from equation (11) the part $x^2 + x + 2/x^3 - x$ will be solved by partial fraction

$$\frac{x^2 + x + 2}{x^3 - x} = \frac{x^2 + x + 2}{x(x^2 - 1)} = \frac{x^2 + x + 2}{x(x + 1)(x - 1)} \tag{12}$$

from here,

$$\frac{x^2 + x + 2}{x(x + 1)(x - 1)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x - 1} \tag{13}$$

by multiplying eq 13 with $x(x + 1)(x - 1)$

$$x^2 + x + 2 = A(x + 1)(x - 1) + B(x)(x - 1) + C(x)(x + 1)$$

$$x^2 + x + 2 = Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx$$

$$x^2 + x + 2 = Ax^2 + Bx^2 + Cx^2 - Bx + Cx - A$$
$$x^2 + x + 2 = (A + B + C)x^2 + (-B + C)x - A$$

now, equate the coefficients

$$-2 = A$$
$$1 = -B + C$$
$$1 = A + B + C$$

from here,

$$(A, B, C) = (-2, 1, 2)$$

Equation (13) after substituting the points:

$$\frac{x^2 + x + 2}{x(x+1)(x-1)} = \frac{-2}{x} + \frac{1}{x+1} + \frac{2}{x-1} \tag{14}$$

Substitute equation (13) in equation (12)

$$\frac{x^5 + x^2 + 2}{x^3 - x} = x^2 + 1 + \frac{-2}{x} + \frac{1}{x+1} + \frac{2}{x-1} \tag{15}$$

By integrating equation (15)

$$\int \frac{x^5 + x^2 + 2}{x^3 - x} dx = \int x^2 dx + \int 1 dx + \int \frac{-2}{x} dx + \int \frac{1}{x+1} dx + \int \frac{2}{x-1} dx$$

$$\int \frac{x^5 + x^2 + 2}{x^3 - x} dx = \frac{1}{3}x^3 + x - 2lnx + ln|x+1| + 2|x-1| + C$$

Practice Questions:
Integrate the following:

- $\frac{2x-3}{x^3-x^2}$
- $\frac{3x}{(x-1)(x^2+6)}$
- $\frac{4x^3-x}{(x^2+5)^2}$
- $\frac{dx}{x^2-3x-4}$
- $\frac{x^5-4x^3+1}{x^3-4x}$