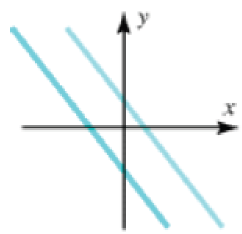
Linear Systems with Two Unknowns Geometrically

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

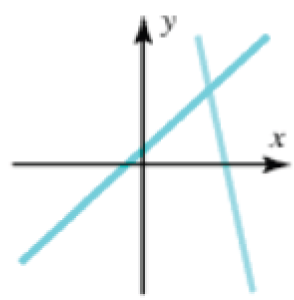
in which the graphs of the equations are lines in the xy-plane. Each solution (x, y) of this system corresponds to a point of intersection of the lines, so there are three possibilities:

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.



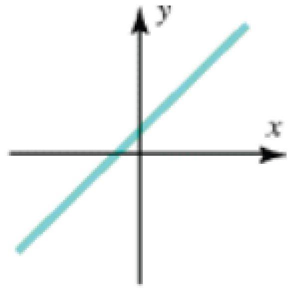
No solution

2. The lines may intersect at only one point, in which case the system has exactly one solution.



One solution

3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.



Infinitely many solutions (coincident lines)

EXAMPLE 1 **A Linear System with One Solution:**

Solve the linear system

$$\begin{cases} x - y = 1 \\ 2x + y = 6 \end{cases}$$

Solution:

We want to eliminate x from the second equation by adding -2 times the first equation to the second.

$$2x - 2y = 2$$
$$2x + y = 6$$
$$- \quad - \qquad -$$

$$-3y = -4$$
$$y = \frac{4}{3}$$

Put $y = \frac{4}{3}$ in the first equation we obtain

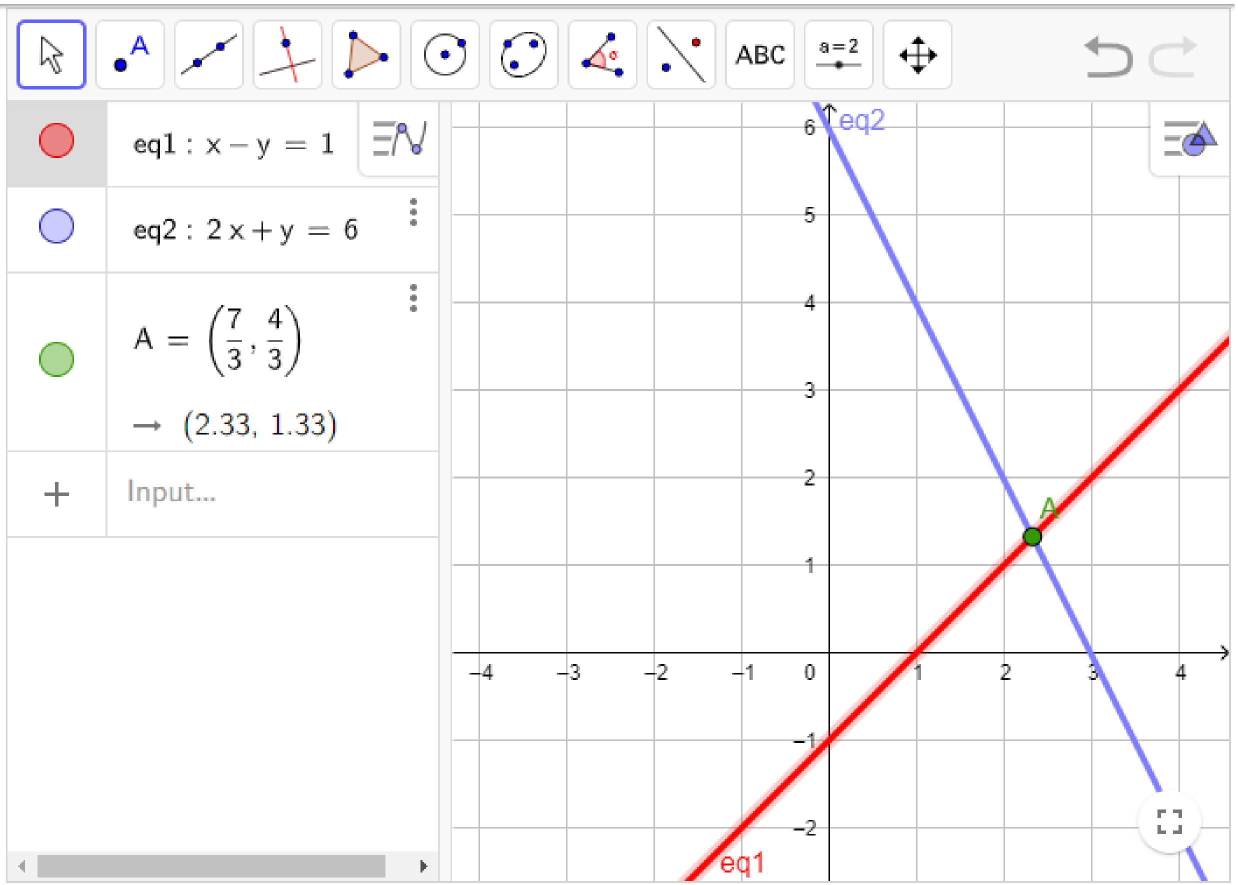
$$x = 1 + y = 1 + \frac{4}{3} = \frac{7}{3}$$

Thus, the system has the unique solution

$$x = \frac{7}{3}, \qquad y = \frac{4}{3}$$

Geometrically, this means that the lines represented by the equations in the system intersect at the single point $(\frac{7}{3}, \frac{4}{3})$.

y



EXAMPLE 2 **A Linear System with No Solutions:**

Solve the linear system

$$\begin{cases} x + y = 4 \\ 3x + 3y = 6 \end{cases}$$

Solution: We can eliminate x from the second equation by adding −3 times the first equation to the second equation.

$$3x + 3y = 12$$
$$3x + 3y = 6$$

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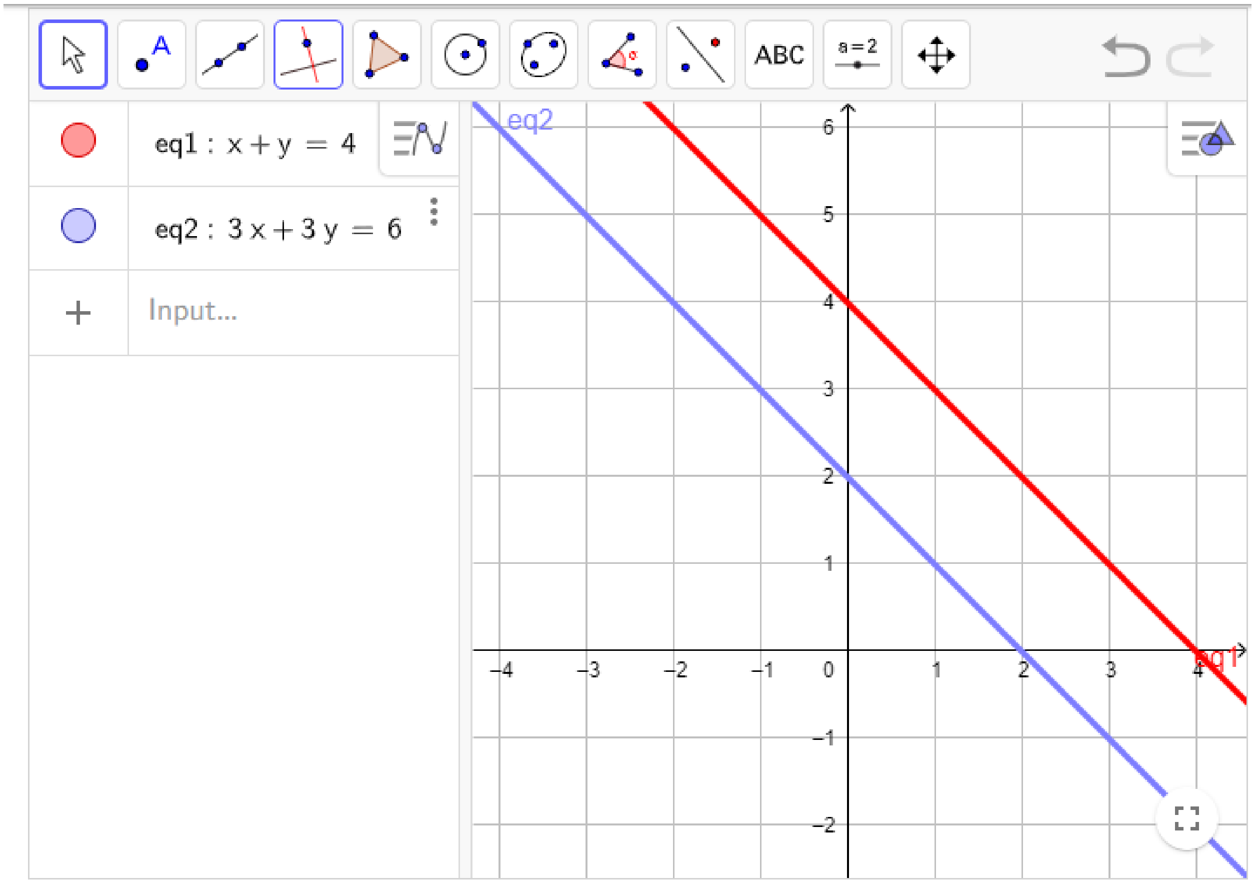
$$0 = 6$$

This is not possible, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct.



Graphically, it shows that the lines have the same slope but different *y*-intercepts.

EXAMPLE 3 A Linear System with Infinitely many Solutions

Solve the linear system

$$\begin{cases} 4x - 2y = 1 \\ 16x - 8y = 4 \end{cases}$$

Solution: We can eliminate *x* from the second equation by adding −4 times the first equation to the second.

$$16x - 8y = 4$$
$$16x - 8y = 4$$
$$- \quad + \quad -$$

$$0 = 0$$

Thus, the solutions of the system are those values of *x* and *y* that satisfy the single equation:

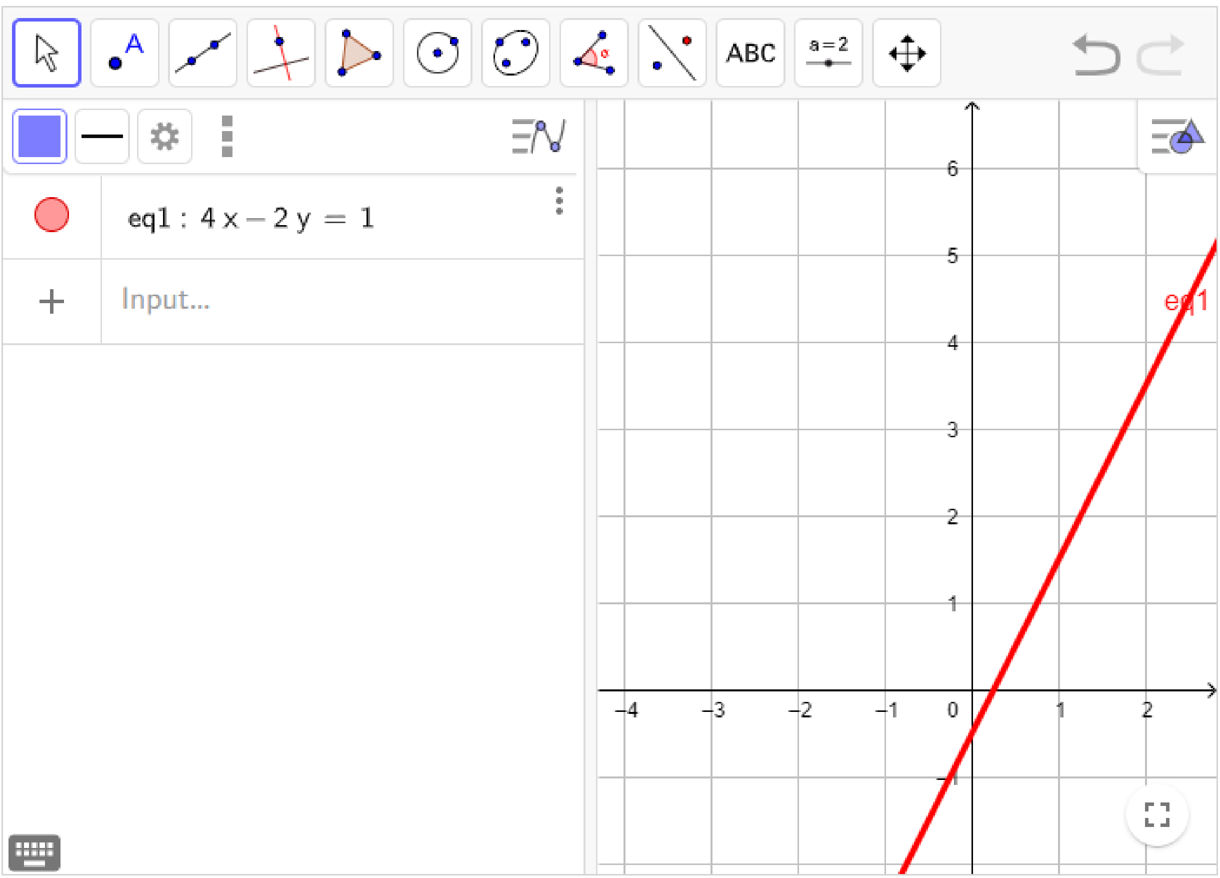
$$4x - 2y = 1 \Rightarrow x = \frac{1}{4} + \frac{1}{2}y$$

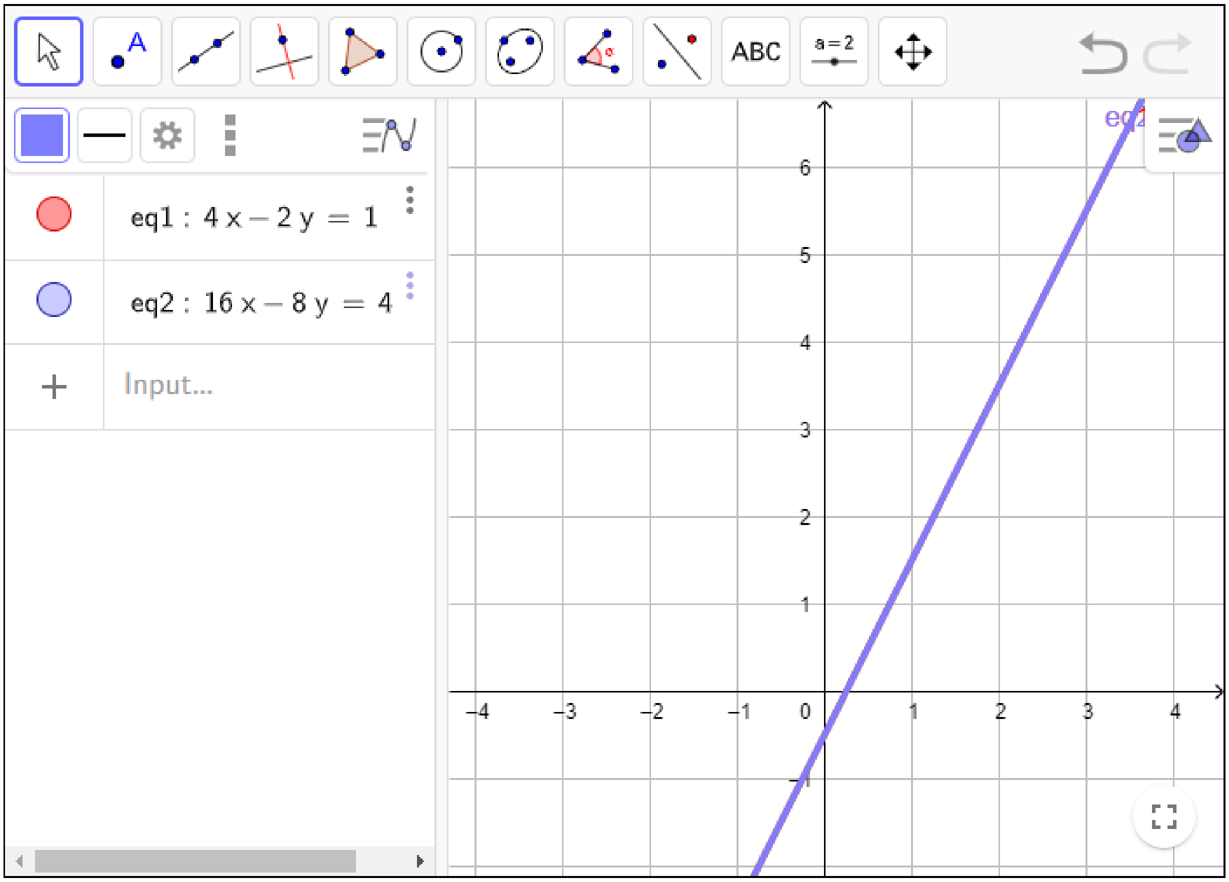
Let $y = t$, So

$$x = \frac{1}{4} + \frac{1}{2}t$$

$\left(\frac{1}{4} + \frac{1}{2}\ t,\ \ t\right), t \in \mathbb{R}$ is solution of given system.

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter t. For example, $t = 0$, yields the solution $\left(\frac{1}{4}, 0\right)$, $t = 1$, yields the solution$\left(\frac{3}{4}, 1\right)$.





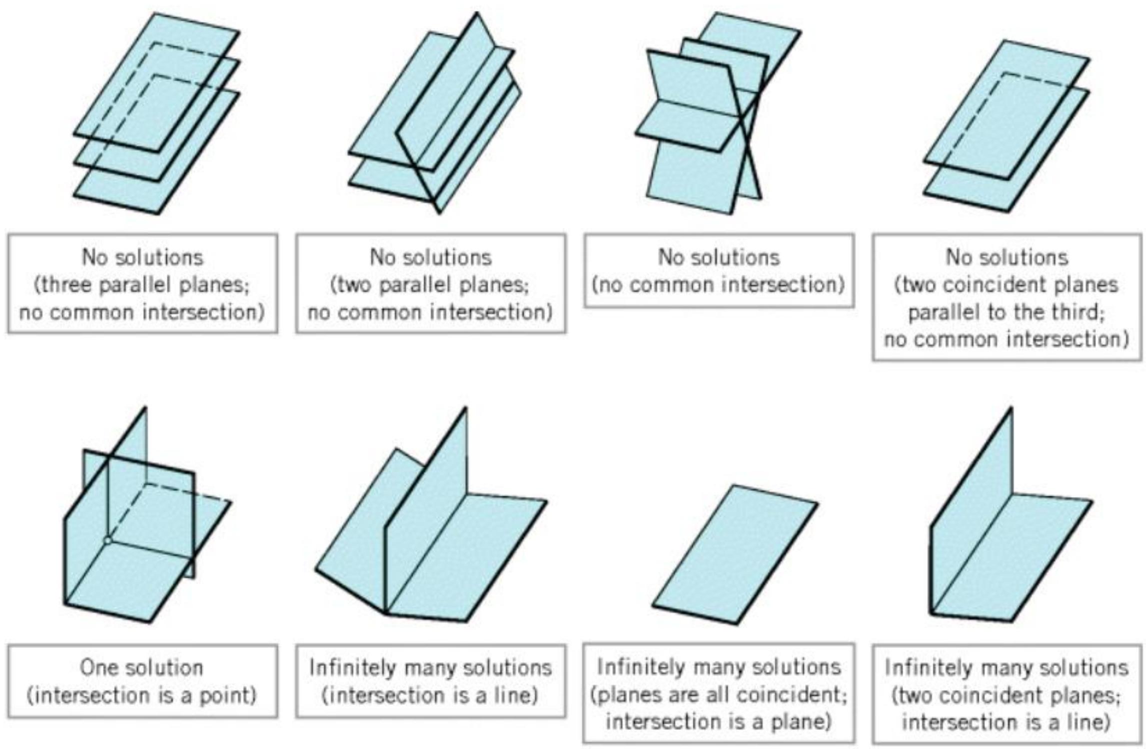
ABC
a = 2
eq1 : 4 x − 2 y = 1
eq2 : 16 x − 8 y = 4
Input...
eq2

Linear Systems with Three Unknowns Geometrically

The same is true for a linear system of three equations in three unknowns

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities–no solutions, one solution, or infinitely many solutions:



Important Note!
Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

EXAMPLE 4

Solve the linear system

$$\begin{cases} x + y + 2z = 9 & \dashrightarrow (1) \\ 2x + 4y - 3z = \ 1 & \dashrightarrow (2) \\ 3x + 6y - 5z = 0 & \dashrightarrow (3) \end{cases}$$

Solution:

Using (1) and (2)

$$\begin{array}{c} 2x + 2y + 4z = 18 \\ +2x + 4y - 3z = 1 \\ - \qquad - \qquad + \qquad - \\ \hline -2y + 7z = 17 \quad \dashrightarrow (4) \end{array}$$

Using (1) and (3)

$$\begin{array}{c} 3x + 3y + 6z = 27 \\ +3x + 6y - 5z = 0 \\ - \qquad - \qquad + \qquad - \\ \hline -3y + 11z = 27 \quad \dashrightarrow (5) \end{array}$$

Using (4) and (5)

$$\begin{array}{c} -6y + 21z = 51 \\ -6y + 22z = 54 \\ + \qquad - \qquad - \\ \hline -z = -3 \\ z = 3 \end{array}$$

Put $z = 3$ in equation (4)

$$-2y + 7(3) = 17$$
$$-2y + 2 = 17$$
$$y = 2$$

Put $y = 2$ and $z = 3$ in equation (1).

$$x = 9 - 2 - 2(3)$$
$$x = 1$$

So the solution of given system is $(1, 2, 3)$, which is unique.
Geometrically it represents that three planes intersect at unique point.

EXAMPLE 5

Solve the linear system

$$\begin{cases} x - y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases}$$

Solution:

This system can be solved by inspection, since the second and third equations are multiples of the first. Geometrically, this means that the three planes coincide and that those values of *x*, *y*, and *z* that satisfy the equation

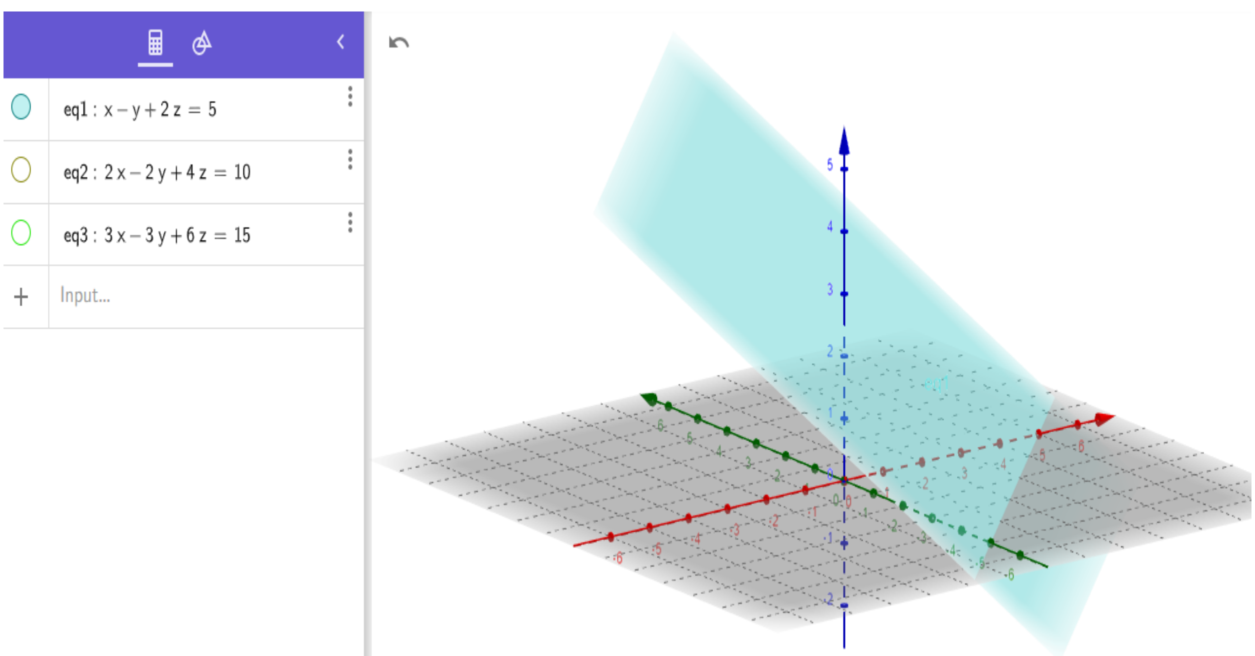
$$x - y + 2z = 5 \qquad \rightarrow \qquad (1)$$

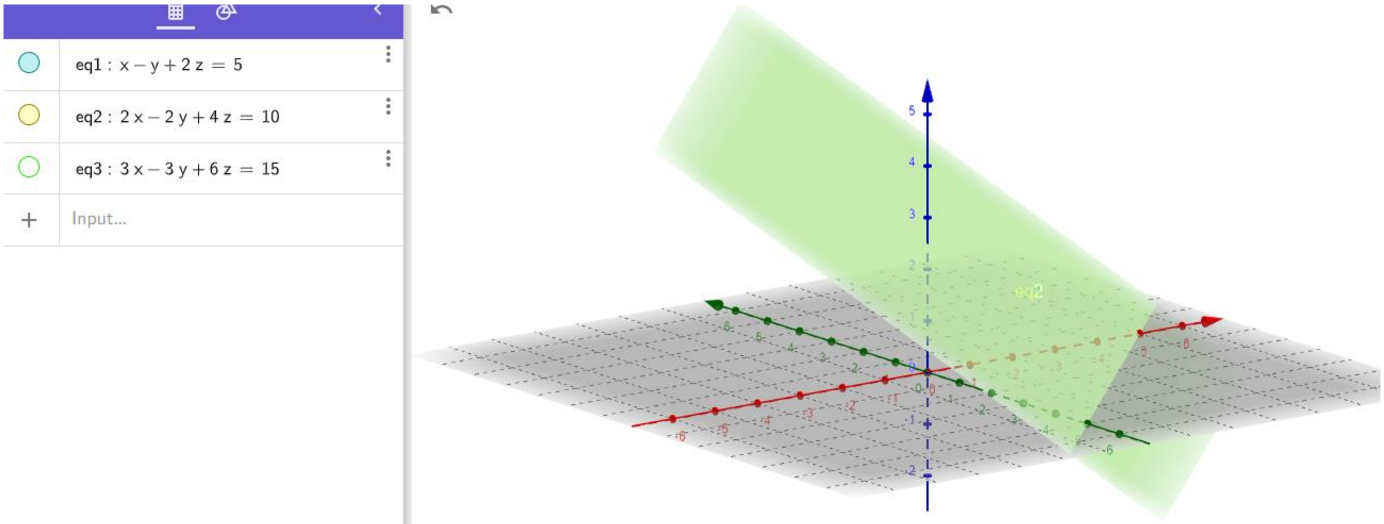
Automatically satisfy all three equations. Thus, it suffices to find the solutions of (1). We can do this by first solving (1) for *x* in terms of *y* and *z*, then assigning arbitrary values s and t (parameters) to these two variables and then expressing the solution by the three parametric equations

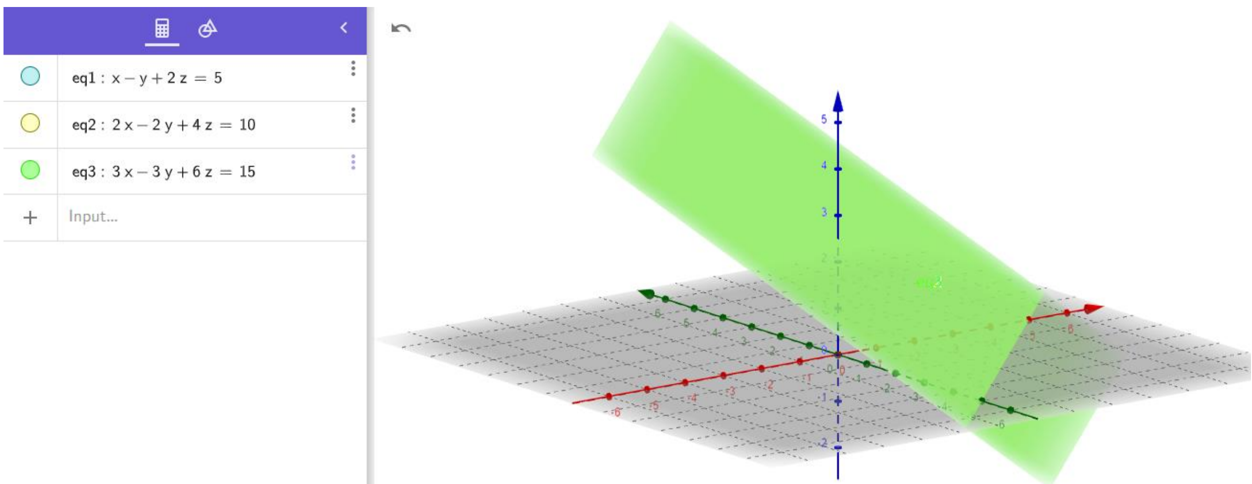
$$y = t, z = s \Rightarrow x = 5 + t - 2s$$

So $(5 + t - 2s, t, s)$ is the solution of the above system.

Specific solutions can be obtained by choosing numerical values for the parameters t and s. For example, taking $t = 1$ and $s = 0$ yields the solution $(6, 1{,}0)$.







Exercise:

1. Solve the following linear system of equations:

a)
$$\begin{cases} 2x + 4y + 6z = -12 \\ 2x - 3y - 4z = 15 \\ 3x + 4y + 5z = -8 \end{cases}$$

b)
$$\begin{cases} x + y = 5 \\ 3x + 3y = 10 \end{cases}$$

c)
$$\begin{cases} 2x + 3y = 13 \\ x - 2y = 3 \\ 5x + 2y = 27 \end{cases}$$

2. In each part, find the solution set of the linear equation by using parameters as necessary.
   a) $7x - 5y = 3$
   b) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

Work to do

Exercise 1.1 **Q6-14**