The Dot Product

We have seen how to add vectors? Can we multiply two vectors together?

Next, we will see two different ways of doing so,

1. The Scalar Product (Dot Product) → Produces Scalar
2. The Vector Product (Cross Product)→ Produces Vector

Dot Product (Scalar Product)

Let $\boldsymbol{v} = v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k}$ and $\boldsymbol{w} = w_1\mathbf{i} + w_2\,\mathbf{j} + w_3\mathbf{k}$

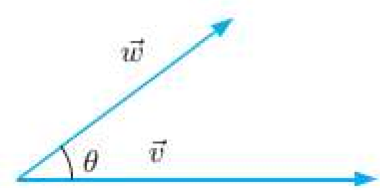
Then the dot product of $\boldsymbol{v}$and$\boldsymbol{w}$can be defined in two ways:

The dot product links geometry and algebra. We already know how to calculate the length of a vectorfrom its components; the dot product gives us a way of computing the angle between two vectors·

• **Geometric definition**

$\vec{v}.\vec{w}$ = $|\vec{v}||\vec{w}|\cos\theta$

where θ is angle between$\vec{v}$ and $\vec{w}$ and 0 ≤θ≤ π.



• **Algebraic definition**

$\vec{v}.\vec{w}$ = $v_1w_1 + v_2\,w_2\; + v_3w_3$

Note: The dot product of two vectors is a **number**, not a vector.

Example 1 Suppose $\vec{v} = \vec{i}$ and $\vec{w} = 2\vec{i} + 2\vec{j}$. Compute $\vec{v} \cdot \vec{w}$ both geometrically and algebraically.

Solution To use the geometric definition, see Figure 13.27. The angle between the vectors is $\pi/4$, or $45°$, and the lengths of the vectors are given by

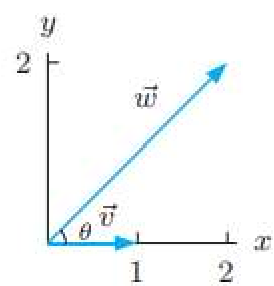
$$\|\vec{v}\| = 1 \quad \text{and} \quad \|\vec{w}\| = 2\sqrt{2}.$$

Thus,

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta = 1 \cdot 2\sqrt{2}\cos\left(\frac{\pi}{4}\right) = 2.$$

Using the algebraic definition, we get the same result:

$$\vec{v} \cdot \vec{w} = 1 \cdot 2 + 0 \cdot 2 = 2.$$



Orthogonal Vectors:

Two non-zero vectors $\vec{v}$ and $\vec{w}$ are perpendicular, or orthogonal, if and only if

$$\vec{v}.\vec{w} = 0$$

For example: $\vec{i}.\vec{j}$= 0, $\vec{j}.\vec{k}$= 0, $\vec{i}.\vec{k}$= 0.

Remarks:
1) If we take the dot product of a vector with itself, then $\theta = 0$ and $\cos\theta = 1$.

For example:

$\vec{i}.\vec{i}$= 1, $\vec{j}.\vec{j}$= 1, $\vec{k}.\vec{k}$= 1.

2) Magnitude and dot product are related as follows:

$$\vec{v}.\vec{v} = ||\vec{v}||^2$$

Example 1: Which pairs from the following list of 3-dimensional vectors are perpendicular to one another?

$$\vec{u}=\vec{\imath}+\sqrt{3}\vec{k},\ \vec{v}=\vec{\imath}+\sqrt{3}\vec{\jmath},\ \vec{w}=\sqrt{3}\vec{\imath}+\ \vec{\jmath}-\vec{k}.$$

Solution: The geometric definition tells us that two vectors are perpendicular if and only if their dot product is zero. Since the vectors are given in components, we calculate dot products using the algebraicdefinition:

$$\begin{aligned}
\vec{v}\cdot\vec{u} &= (\vec{i}+\sqrt{3}\vec{j}+0\vec{k})\cdot(\vec{i}+0\vec{j}+\sqrt{3}\vec{k}) = 1\cdot1+\sqrt{3}\cdot0+0\cdot\sqrt{3}=1,\\
\vec{v}\cdot\vec{w} &= (\vec{i}+\sqrt{3}\vec{j}+0\vec{k})\cdot(\sqrt{3}\vec{i}+\vec{j}-\vec{k}) = 1\cdot\sqrt{3}+\sqrt{3}\cdot1+0(-1)=2\sqrt{3},\\
\vec{w}\cdot\vec{u} &= (\sqrt{3}\vec{i}+\vec{j}-\vec{k})\cdot(\vec{i}+0\vec{j}+\sqrt{3}\vec{k}) = \sqrt{3}\cdot1+1\cdot0+(-1)\cdot\sqrt{3}=0.
\end{aligned}$$

So, the only two vectors that are perpendicular are $\vec{v}$and$\vec{w}$**.**

Example 2: Compute the angle between the vectors $\vec{v}$and $\vec{w}$.

$\vec{v}=\vec{\imath}+\sqrt{3}\vec{\jmath}$,$\vec{w}=\sqrt{3}\vec{\imath}+\ \vec{\jmath}-\vec{k}$

Solution:$\vec{v}.\vec{w}=||\vec{v}|||\vec{w}||Cos\theta$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}.$$

$$\vec{v}\cdot\vec{w} = (\vec{i}+\sqrt{3}\vec{j}+0\vec{k})\cdot(\sqrt{3}\vec{i}+\vec{j}-\vec{k}) = 1\cdot\sqrt{3}+\sqrt{3}\cdot1+0(-1)=2\sqrt{3},$$

$$\cos\theta = \frac{2\sqrt{3}}{\|\vec{v}\|\|\vec{w}\|} = \frac{2\sqrt{3}}{\sqrt{1^2+(\sqrt{3})^2+0^2}\sqrt{(\sqrt{3})^2+1^2+(-1)^2}} = \frac{\sqrt{3}}{\sqrt{5}}$$

$$\text{so}\quad \theta = \arccos\left(\frac{\sqrt{3}}{\sqrt{5}}\right) = 39.2315^\circ.$$

Question: Compute the angle between the vectors $\vec{v}$and $\vec{w}$.

$\vec{v}=\vec{\imath}-2\vec{\jmath}-2\vec{k}$,$\vec{w}=\overrightarrow{6\imath}+3\vec{\jmath}+2\vec{k}$

Finding the Angle of a Triangle

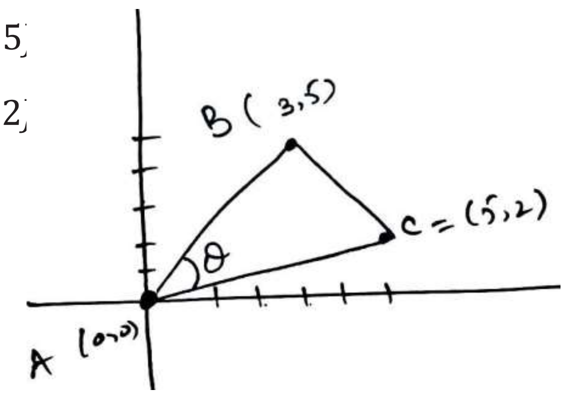
Example: Find the angle in the triangle **ABC** determined by the vertices A=(0,0), B= (3,5), C=(5,2).

Solution:

The angleθ is the angle between the vectors$\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = 3\vec{\imath} + 5\vec{\jmath}$$

$$\overrightarrow{AC} = 5\vec{\imath} + 2\vec{\jmath}$$



$$Cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{15+10}{(\sqrt{3^2+5^2})\sqrt{5^2+2^2})}$$

$$= \frac{25}{(\sqrt{34})\sqrt{29})} = 35.9°$$

Vector Projection

$$Proj_{\vec{b}}\vec{a} = \vec{u} = |\vec{u}|\hat{u}$$

First, we find the magnitude of projection vector $\vec{u}$.

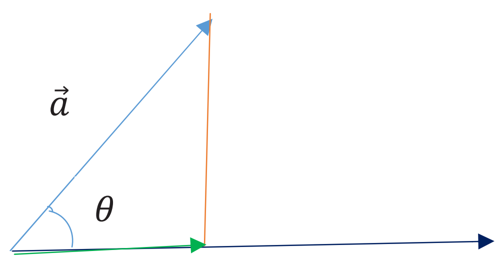
Magnitude of $\vec{u}$:

$$\text{Cos}\theta = \frac{base}{Hyp} = \frac{|\overrightarrow{u}|}{|\overrightarrow{a}|}$$

As $Cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

$$\Rightarrow \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{|\overrightarrow{u}|}{|\overrightarrow{a}|}$$

$\vec{b}$



$Proj_{\vec{b}}\vec{a}$

As $\hat{u} = \hat{b} = \frac{\vec{b}}{|\vec{b}|}$

So $Proj_{\vec{b}}\vec{a} = \vec{u} = |\vec{u}|\hat{u} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}.\frac{\vec{b}}{|\vec{b}|}$

$$Proj_{\vec{b}}\vec{a} = (\frac{\vec{a}.\vec{b}}{\left|\vec{b}\right|^2})\vec{b}$$

Similarly,

$$Proj_{\vec{a}}\vec{b} = (\frac{\vec{a}.\vec{b}}{|\vec{a}|^2})\vec{a}$$

Example: Find the vector projection of $\vec{a}= 6\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$ onto $\vec{b}=\vec{\imath} - 2\vec{\jmath} - 2\vec{k}$.

Solution:

$$Proj_{\vec{b}}\vec{a} = (\frac{\vec{a}.\vec{b}}{\left|\vec{b}\right|^2})\vec{b}$$

$\vec{a}.\vec{b} = \left(6\vec{\imath} + 3\vec{\jmath} + 2\vec{k}\right).\left(\vec{\imath} - 2\vec{\jmath} - 2\vec{k}\right)$

=6-6-4=-4

$\overrightarrow{|b|}=\sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9}$ =3

$\overrightarrow{|b|}^2 = 3^2 = 9$

$Proj_{\vec{b}}\vec{a} = \frac{-4}{9}\left(\vec{\imath} - 2\vec{\jmath} - 2\vec{k}\right)$

Practice Problems:

Question 1: Find $\vec{a}.\vec{b}$, $|\vec{a}|$, $\left|\vec{b}\right|$ and $Proj_{\vec{b}}\vec{a}$,

$\vec{a}=\vec{\imath} + \vec{\jmath} + \vec{k}$, $\vec{b}= 2\vec{\imath} + 10\vec{\jmath} - 11\vec{k}$

Question 2: Find the angle between the vectors

$\vec{a}= 2\vec{\imath} + \vec{\jmath}$, $\vec{b}=\vec{\imath} + 2\vec{\jmath} - \vec{k}$.

Question 3: Find the measures of the angles of the triangle having vertices $A = (-1,0), B = (2,1)$ and $C = (1,-20)$.

Ex. 12.3: 1-13