4.3 Homogeneous Linear Equations with Constant Coefficients

As a means of motivating the discussion in this section, let us return to first differential equations—more specifically, to *homogeneous* linear equations $ay' + by = 0$ order, where the coefficients $a \neq 0$ and b are constants. This type of equation can be solved by another solution method, one that uses only algebra. Before illustrating this alternative method, we make one observation:
If we substitute $y = e^{mx}$ and $y' = me^{mx}$ into $ay' + by = 0$, we get

$$ame^{mx} + be^{mx} = 0 \qquad or \quad (am + b)e^{mx} = 0$$

Since e^{mx} is never zero for real values of x, the last equation is satisfied only when m is a solution or root of the first-degree polynomial equation $am + b = 0$. For this single value of m, $y = e^{mx}$ is a solution of the DE.

To illustrate, consider the constant-coefficient equation $2y' + 5y = 0$. It is not necessary to go through the differentiation and substitution of $y = e^{mx}$ into the DE; we merely have to form the equation $2m + 5 = 0$ and solve it for m. From $m = -\frac{5}{2}$ we conclude that $y = e^{-\frac{5}{2}x}$ is a solution and its general solution on the interval $(-\infty, \infty)$is $y = c_1e^{-\frac{5}{2}x}$.

In this section we will see that the foregoing procedure can produce exponential solutions for homogeneous linear higher-order DEs,

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0y = 0 \qquad (1)$$

Subject to: $y(x_0) = y_0,\ y'(x_0) = y_1, \dots,\ y^{(n-1)}(x_0) = y_{n-1}$.

where the coefficients a_i, $i = 0, 1, \dots, n$ are real constants and $a_n \neq 0$.

Auxiliary Equations

We begin by considering the special case of the second order equation

$$ay'' + by' + cy = 0 \qquad (2)$$

where a, b, and c are constants. If we try to find a solution of the form $y = e^{mx}$, then after substitution of $y' = me^{mx}$ and $y'' = m^2e^{mx}$, equation (2) becomes

$$a\,m^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$or \quad e^{mx}(a\,m^2 + bm + c) = 0.$$

As in the introduction we argue that because $e^{mx} \neq 0$ for all x, it is apparent that the only way $y = e^{mx}$can satisfy the differential equation (2) is when m is chosen as a root of the quadratic equation

$$a\,m^2 + bm + c = 0 \qquad (3)$$

This last equation is called the **auxiliary equation** of the differential equation (2). Since the two roots of (3) are $m_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $m_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$, there will be three forms of the general solution of (2) corresponding to the three cases:

- m_1 and m_2 real and distinct ($b^2 - 4ac > 0$),
- m_1 and m_2 real and equal ($b^2 - 4ac = 0$), and
- m_1 and m_2 conjugate complex numbers ($b^2 - 4ac < 0$).

We discuss each of these cases in turn.

Case I: Distinct Real Roots

Under the assumption that the auxiliary equation (3) has two unequal real roots m_1 and m_2, we find two solutions $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$. We see that these functions are linearly independent on $(-\infty, \infty)$ and hence form a fundamental set. It follows that the general solution of (2) on this interval is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{4}$$

Case II: Repeated Real Roots

When $m_1 = m_2$, we have one exponential solution $y_1 = e^{m_1 x}$ (which can't be repeat in general solution), so the second solution will be $y_2 = x e^{m_1 x}$. The general solution is then

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}. \tag{5}$$

Case III: Conjugate Complex Roots

If m_1 and m_2 are complex, then we can write $m_1 = \propto + i\beta$ and $m_2 = \propto - i\beta$, where $\propto$ and $\beta > 0$ are real and $i^2 = -1$. Formally, there is no difference between this case and Case I, and hence

$$y = c_1 e^{(\propto + i\beta)x} + c_2 e^{(\propto - i\beta)x}.$$

However, we prefer to work with real functions instead of complex exponentials. To this end we use **Euler's formula:**

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ is any real number. It follows from this formula that

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \qquad \text{and} \qquad e^{-i\beta x} = \cos\beta x - i\sin\beta x, \tag{6}$$

Consequently, the general solution is

$$y = c_1 e^{\propto x}\cos\beta x + c_2 e^{\propto x}\sin\beta x = e^{\propto x}(c_1\cos\beta x + c_2\sin\beta x). \tag{7}$$

EXAMPLE 1 Second-Order DEs

Solve the following differential equations.

(a) $2y'' - 5y' - 3y = 0$ **(b)** $y'' - 10y' + 25y = 0$ **(c)** $y'' + 4y' + 7y = 0$

SOLUTION We give the auxiliary equations, the roots, and the corresponding general solutions.

(a) $2m^2 - 5m - 3 = (2m + 1)(m - 3) = 0, \quad m_1 = -\frac{1}{2}, m_2 = 3$

From (4), $y = c_1e^{-x/2} + c_2e^{3x}$.

(b) $m^2 - 10m + 25 = (m - 5)^2 = 0, \quad m_1 = m_2 = 5$

From (6), $y = c_1e^{5x} + c_2xe^{5x}$.

(c) $m^2 + 4m + 7 = 0, \; m_1 = -2 + \sqrt{3}i, \quad m_2 = -2 - \sqrt{3}i$

From (8) with $\alpha = -2, \beta = \sqrt{3}$, $y = e^{-2x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$. ■

Exercise 4.3

In Problems 1–14 find the general solution of the given second-order differential equation.

1. $4y'' + y' = 0$

2. $y'' - 36y = 0$

3. $y'' - y' - 6y = 0$

4. $y'' - 3y' + 2y = 0$

5. $y'' + 8y' + 16y = 0$

6. $y'' - 10y' + 25y = 0$

7. $12y'' - 5y' - 2y = 0$

8. $y'' + 4y' - y = 0$

9. $y'' + 9y = 0$

10. $3y'' + y = 0$

11. $y'' - 4y' + 5y = 0$

12. $2y'' + 2y' + y = 0$

13. $3y'' + 2y' + y = 0$

14. $2y'' - 3y' + 4y = 0$