EXACT EQUATIONS

Although the simple first-order equation

$$y\,dx + x\,dy = 0$$

is separable, we can solve the equation in an alternative manner by recognizing that the expression on the left-hand side of the equality is the differential of the function

$$f\,(x, y) = xy$$

that is,

$$d(xy) = y\,dx + x\,dy.$$

In this topic, we examine first-order equations in differential form

$$M(x,y)dx + N(x,y)dy = 0.$$

By applying a simple test to M and N, we can determine whether $M(x,y)dx + N(x,y)dy$ is a differential of a function $f\,(x, y)$. If the answer is yes, we can construct f by partial integration and its solution will be of the form $f\,(x, y) = c$.

What is an exact equation?

A differential expression $M(x,y)dx + N(x,y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f\,(x, y)$ defined in R. A first-order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be an **exact equation** if the expression on the left-hand side is an exact differential.

Criterion for an Exact Equation

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

In simple words, a 1st order ODE be in the form:

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \; .$$

Check whether the following equations are exact or not:

1. $2xydx = (x^2 - 1)dy$
2. $(x^3 + y^3)dx - 3xy^2dy = 0$
3. $(e^{2y} - ycosxy)dx + (2xe^{2y} - xcosxy + 2y)dy = 0$
4. $(4t^3y - 15t^2 - y)dt + (t^4 + 3y^2 - t)dy = 0$
5. $(5y - 2x)y' - 2y = 0$

Solving an Exact DE

Method 1

Example 1:

Solve $2xy\,dx + (x^2 - 1)\,dy = 0$.

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and so the criterion suggests that there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2y + g(y).$$

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y$. Hence $f(x, y) = x^2y - y$, so the solution of the equation in implicit form is $x^2y - y = c$. The explicit form of the solution is easily seen to be $y = c/(1 - x^2)$ and is defined on any interval not containing either $x = 1$ or $x = -1$. ■

Method 2

Formula for solving EXACT equations

$$\int \boldsymbol{M\,dx} + \int (\boldsymbol{Terms\ of\ N\ without\ x})\,\boldsymbol{dy} = \boldsymbol{c}$$

Example 1:

Solve $2xy\,dx + (x^2 - 1)\,dy = 0$.

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus, the equation is exact.

Putting values in the formula:

$$\int \boldsymbol{M\,dx} + \int (\boldsymbol{Terms\ of\ N\ without\ x})\,\boldsymbol{dy} = \boldsymbol{c}$$

$$\int 2xy\,dx + \int (-1)\,dy = c$$

$$x^2y - y = c.$$

Example 2:

Solve $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$.

SOLUTION The equation is exact because

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Putting values in the formula:

$$\int \boldsymbol{M\,dx} + \int (\boldsymbol{Terms\ of\ N\ without\ x})\,\boldsymbol{dy} = \boldsymbol{c}$$

$$\int (e^{2y} - ycosxy)\,dx + \int (2y)\,dy = c$$

$$xe^{2y} - y\frac{sinxy}{y} + y^2 = c$$

$$xe^{2y} - sinxy + y^2 = c.$$

Practice Questions

[Exercise 2.4 of Book: Differential Equations by D.G. Zill]

In Problems 1–20 determine whether the given differential equation is exact. If it is exact, solve it.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$

2. $(2x + y)\,dx - (x + 6y)\,dy = 0$

3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$

4. $(\sin y - y\sin x)\,dx + (\cos x + x\cos y - y)\,dy = 0$

5. $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$

6. $\left(2y - \frac{1}{x} + \cos 3x\right)\frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$

7. $(x^2 - y^2)\,dx + (x^2 - 2xy)\,dy = 0$

8. $\left(1 + \ln x + \frac{y}{x}\right)dx = (1 - \ln x)\,dy$

9. $(x - y^3 + y^2\sin x)\,dx = (3xy^2 + 2y\cos x)\,dy$

10. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$

11. $(y\ln y - e^{-xy})\,dx + \left(\frac{1}{y} + x\ln y\right)dy = 0$

12. $(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0$

13. $x\frac{dy}{dx} = 2xe^x - y + 6x^2$

14. $\left(1 - \frac{3}{y} + x\right)\frac{dy}{dx} + y = \frac{3}{x} - 1$

15. $\left(x^2y^3 - \frac{1}{1 + 9x^2}\right)\frac{dx}{dy} + x^3y^2 = 0$

16. $(5y - 2x)y' - 2y = 0$

17. $(\tan x - \sin x\sin y)\,dx + \cos x\cos y\,dy = 0$

18. $(2y\sin x\cos x - y + 2y^2e^{xy^2})\,dx = (x - \sin^2 x - 4xye^{xy^2})\,dy$

19. $(4t^3y - 15t^2 - y)\,dt + (t^4 + 3y^2 - t)\,dy = 0$

20. $\left(\frac{1}{t} + \frac{1}{t^2} - \frac{y}{t^2 + y^2}\right)dt + \left(ye^y + \frac{t}{t^2 + y^2}\right)dy = 0$