Optimization

Question 1:

A manufacturing company produces two products which are sold in two separate markets. The company's economists analyze the two markets and determine that the two quantities q_1 and q_2 demanded by the consumers and prices P_1 and P_2 (in $) of each item are related by the equation

$$P_1 = 600 - 0.3q_1$$

$$P_2 = 500 - 0.2q_2$$

If the price for either item increases the demand for it decreases. The company's total production cost is given by

$$C = 16 + 1.2q_1 + 1.5q_2 + 0.2q_1q_2$$

If the company wants to maximize its total profit, how much of each product should it produce? What is the maximum profit?

Solution:

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$\text{AsRevenue} = \text{Price} \times \text{quantity}$

$$= P_1q_1 + P_2q_2$$

$$= (600 - 0.3q_1)q_1 + (500 - 0.2q_2)q_2$$

$$= 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2$$

So

$$f = \text{Profit} = \text{Revenue} - \text{Cost}$$

$$f = 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2 - (16 + 1.2q_1 + 1.5q_2 + 0.2q_1q_2)$$

$$f = 600q_1 - 0.3q_1^2 + 500q_2 - 0.2q_2^2 - 16 - 1.2q_1 - 1.5q_2 - 0.2q_1q_2$$

$$f(q_1, q_2) = -0.3q_1^2 - 0.2q_2^2 + 598.8q_1 + 498.5q_2 - 0.2q_1q_2 - 16$$

$$f_{q_1} = \frac{\partial f}{\partial q_1} = -0.6q_1 + 598.8 - 0.2q_2$$

$$f_{q_2} = \frac{\partial f}{\partial q_2} = -0.4q_2 + 498.8 - 0.2q_1$$

To find critical points put

$$f_{q_1} = f_{q_2} = 0$$

$$-0.6q_1 + 598.8 - 0.2q_2 = 0 \Rightarrow \; 0.6q_1 + \; 0.2q_2 \; = \; 598.8 \; \ldots (1)$$

$$-0.4q_2 + 498.5 - 0.2q_1 = 0 \; \Rightarrow \; 0.2q_1 + 0.4q_2 = 498.5 \ldots (2)$$

Multiply equation (2) by 3 & subtract from (1)

$$\begin{array}{c} 0.6q_1 + 0.2q_2 = 598.8 \\ \pm 0.6q_1 \pm 0.2q_2 = \pm 1495.5 \\ \hline -q_2 = -896.7 \end{array}$$

$$q_2 = 896.7$$

Put in (1)

$$0.6q_1 + 0.2(896.7) = 598.8$$

$$0.6q_1 + 179.34 = 598.8$$

$$0.6q_1 = 419.46$$

$$q_1 = 699.1$$

So $(\mathbf{699}.\,\mathbf{1}, \mathbf{896}.\,\mathbf{7})$ is the critical point.

To see whether $(\mathbf{699}.\,\mathbf{1}, \mathbf{896}.\,\mathbf{7})$ is a maximum, minimum or neither,

$$f_{q_1} = -0.6q_1 + 598.8 - 0.2q_2 \quad \Rightarrow \quad f_{q_1q_1} = -0.6; \quad f_{q_1q_2} = -0.2$$

$$f_{q_1q_1}(699.1, 896.7) = -0.6\,; \quad f_{q_1q_2}(699.1, 896.7) = -0.2$$

$$f_{q_2} = -0.4q_2 + 498.8 - 0.2q_1 \quad \Rightarrow \quad f_{q_2q_2} = -0.4$$

$$f_{q_2q_2}\,(699.1, 896.7) = -0.4$$

$$D \; = \; f_{q_1q_1} * \; f_{q_2q_2} - \left(f_{q_1q_2}\right)^2$$

$$= (-0.6) * (-0.4) - (-0.2)^2 = 0.2 \; > \; 0$$

Since$D$> 0 &$f_{q_1q_1} = -0.6 < 0$, so f has local maximum at $(\mathbf{699}.\mathbf{1}, \mathbf{896}.\mathbf{7})$.
The company should produce 699 units of$q_1$and 897units of $q_2$and the maximum profit would be

$$f(699, 897) = \$\,432{,}797.$$

Question21. (Ex 9.5):

A company operates two plants which manufacture the same item and whose total functions are

$$C_1 = 8.5 + 0.03q_1^2 \text{and} C_2 = 5.2 + 0.04q_2^2$$

where $q_1$and q_2 are the quantities produced by each plant. The company is a monopoly. The total quantity demanded,$q = q_1 + q_2$,is related to the price, P, by

$$P = 60 - 0.04q.$$

How much should each plant produce in order to maximize the company's profit?

Solution:

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

As $\text{Revenue} = \text{Price} \times \text{Quantity}$

$$= P \times q = (60 - 0.04q) \times (q)$$

$$= 60q - 0.04q^2$$

$$= 60(q_1 + q_2) - 0.04(q_1 + q_2)^2$$

$$= 60q_1 + 60q_2 - 0.04(q_1^2 + q_2^2 + 2q_1q_2)$$

$$\text{Revenue} = 60q_1 + 60q_2 - 0.04q_1^2 - 0.04q_2^2 - 0.08q_1q_2$$

$$\text{Profit} = 60q_1 + 60q_2 - 0.04q_1^2 - 0.04q_2^2 - 0.08q_1q_2 - (8.5 + 0.03q_1^2 + 5.2 + 0.04q_2^2)$$

$$= 60q_1 + 60q_2 - 0.04q_1^2 - 0.04q_2^2 - 0.08q_1q_2 - 13.7 - 0.03q_1^2 - 0.04q_2^2$$

$$f(q_1, q_2) = -0.07q_1^2 - 0.08q_2^2 - 0.08q_1q_2 + 60q_1 + 60q_2 - 13.7$$

$$f_{q_1} = -0.14q_1 - 0.08q_2 + 60$$

$$f_{q_2} = -0.16q_2 - 0.08q_1 + 60$$

$f_{q_1} = f_{q_2} = 0$implies

$$-0.14q_1 - 0.08q_2 + 60 = 0$$
$$-0.16q_2 - 0.08q_1 + 60 = 0$$

$$0.14q_1 + 0.08q_2 = 60 \ldots (1)$$
$$0.08q_1 + 0.16q_2 = 60 \ldots (2)$$

Multiply equation (1) by 0.08 & equation (2) by 0.14 then subtract it from equation (1).

$$0.0112q_1 + 0.0064q_2 = 4.8$$
$$\underline{\pm 0.0112q_1 \pm 0.0224q_2 = \pm 8.4}$$
$$-0.016q_2 = -3.6$$
$$q_2 = 225$$

Put in (1)

$$0.14q_1 + 0.08(225) = 60$$
$$0.14q_1 = 60 - 18$$
$$q_1 = \frac{42}{0.14} = 300$$
$$q_1 = 300$$

So $(q_1, q_2) = (300, 225)$ is a critical point of f.

$$f_{q_1q_1} = -0.14 \qquad f_{q_1q_2} = -0.08 \qquad f_{q_2q_2} = -0.16$$

$$D = f_{q_1q_1} * f_{q_2q_2} - \left(f_{q_1q_2}\right)^2 = (-0.14) * (-0.16) - (-0.8)^2$$
$$= -0.0224 - 0.0064 = 0.016 > 0$$

As,D> 0 and $f_{q_1q_1} = -0.14 < 0$ so,$f$has a local maximum at (300,225)

The plant should produce 300 units of q_1 and 225 units of q_2.

Practice Problems:

Ex. 9.5: 18–21.