Example 1 Solve the linear system of equations using Gauss elimination method:

$$\begin{cases} x + z + 2w = 6 \dashrightarrow (1) \\ y - 2z = -3 \dashrightarrow (2) \\ x + 2y - z = -2 \dashrightarrow (3) \\ 2x + y + 3z - 2w = 0 \dashrightarrow (4) \end{cases}$$

Solution: Do yourself and match your answer with the solution obtained above.

Howard Anton (Exercise 1.2)

Q1. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

Q3. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution ©:

$$x_4 + 3x_5 = 9\text{-------------}x_4 = 9 - 3x_5$$

$$x_3 + x_4 + 6x_5 = 5$$

$$x_1 + 7x_2 - 2x_3 - 8x_5 = -3$$

Put value of x_4 in eq 2

$$x_3 + 9 - 3x_5 + 6x_5 = 5$$

$$x_3 = -4 - 3x_5$$

Put value of x_3 in eq 3

$$x_1 + 7x_2 - 2(-4 - 3x_5) - 8x_5 = -3$$

$$x_1 + 7x_2 + 8 + 6x_5 - 8x_5 = -3$$

$$x_1 + 7x_2 = -11 + 2x_5$$

$$x_1 = 11 + 2x_5 - 7x_2$$

Let $x_2 = t, x_5 = s$

$$x_1 = 11 + 2s - 7t$$

$$x_3 = -4 - 3s$$

$$x_4 = 9 - 3s$$

Work to do:

Q5 Solve the linear system of equation by using Gauss elimination method

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Solution of Word Problems using Gauss-Jordan Method

Example 1

Ali and Sara are shopping for chocolate bars. Ali observes, “If I add half my money to yours, it will be enough to buy two chocolate bars.” Sara naively asks, “If I add half my money to yours, how many can we buy?” Ali replies, “One chocolate bar.” How much money did Ali have?

Solution: Let a = Ali’s money
s = Sara’s money
c = Cost of chocolate

$$\begin{cases} \frac{1}{2}a + s = 2c - - - - \longrightarrow (1) \\ a + \frac{1}{2}s = c - - - - - \longrightarrow (2) \end{cases}$$

Or $\begin{cases} a + 2s = 4c \\ 2a + s = 2c \end{cases}$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 4c \\ 2 & 1 & 2c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4c \\ 0 & -3 & -6c \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 4c \\ 0 & 1 & 2c \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad -\frac{1}{3}R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2c \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_1 - 2R_2$$

Solution is $(a, s) = (0, 2c)$. It means Ali has no money.

Example 2

Three Alto, two Suzuki, and four City can be rented for $106 per day. At the same rates two Alto, four Suzuki, and three City cost $I07 per day, whereas four Alto, three Suzuki, and two City cost $102 per day. Find the rental rates for all three kinds of cars?

Solution:

$$3\text{a} + 2\text{s} + 4\text{c} = 106$$
$$2\text{a} + 4\text{s} + 3\text{c} = 107$$
$$4\text{a} + 3\text{s} + 2\text{c} = 102$$

Its Augmented matrix is

$$\begin{bmatrix} 3 & 2 & 4 & :106 \\ 2 & 4 & 3 & :107 \\ 4 & 3 & 2 & :102 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & :-1 \\ 2 & 4 & 3 & :107 \\ 4 & 3 & 2 & :102 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_1 - R_2$$

$$\begin{bmatrix} 1 & -2 & 1 & :-1 \\ 0 & 8 & 1 & :109 \\ 4 & 3 & 2 & :102 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_2 - 2R_1$$

$$\begin{bmatrix} 1 & -2 & 1 & :-1 \\ 0 & 8 & 1 & :109 \\ 0 & 11 & -2 & :106 \end{bmatrix} \quad \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_3 - 4R_1$$

$$\begin{bmatrix} 1 & -2 & 1 & :-1 \\ 0 & -3 & 3 & :3 \\ 0 & 11 & -2 & :106 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_2 - R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & :-1 \\ 0 & 1 & -1 & :-1 \\ 0 & 11 & -2 & :106 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad -\frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & :-1 \\ 0 & 1 & -1 & :-1 \\ 0 & 0 & 9 & :117 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_3 - 11R_2$$

$$\begin{bmatrix} 1 & -2 & 1 & :-1 \\ 0 & 1 & -1 & :-1 \\ 0 & 0 & 1 & :13 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad \frac{1}{9}R_3$$

$$\begin{bmatrix} 1 & -2 & 1 & :-1 \\ 0 & 1 & 0 & :12 \\ 0 & 0 & 1 & :13 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_2 + R_3$$

$$\begin{bmatrix} 1 & -2 & 0 & :-14 \\ 0 & 1 & 0 & :12 \\ 0 & 0 & 1 & :13 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_1 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & :10 \\ 0 & 1 & 0 & :12 \\ 0 & 0 & 1 & :13 \end{bmatrix} \sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim \quad R_1 + 2R_3$$

Hence, the rental rates for Alto, Suzuki, and City cars are $\$\mathbf{10}, \$\mathbf{12}$ and $\$\mathbf{13}$ per day, respectively.

Example 3

A restaurant owner plans to use x tables seating 4, y tables seating 6 and z tables seating 8, for a total 20 tables. When fully occupied, the tables seat 108 customers. If only half of the x tables, half of the y tables and one-fourth of the z tables are used, each fully occupied, then 46 customers will be seated. Find x, y, and z.

Solution:

$$x + y + z = 20$$

$$4x + 6y + 8z = 108$$

$$4\left(\frac{x}{2}\right) + 6\left(\frac{y}{2}\right) + 8\left(\frac{z}{4}\right) = 46$$

Simplifying the system, we have

$$x + y + z = 20$$

$$2x + 3y + 4z = 54$$

$$2x + 3y + 2z = 46$$

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The answer is: $\boldsymbol{x = 10}$, $\boldsymbol{y = 6}$ and $\boldsymbol{z = 4}$