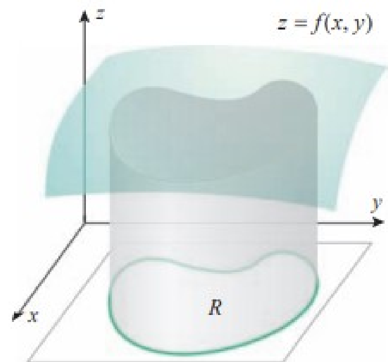
Multiple Integrals

Volume under the surface $z = f(x, y)$:

If f is a function of two variables that is continuous on a region R in the xy-plane, then the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R is defined by

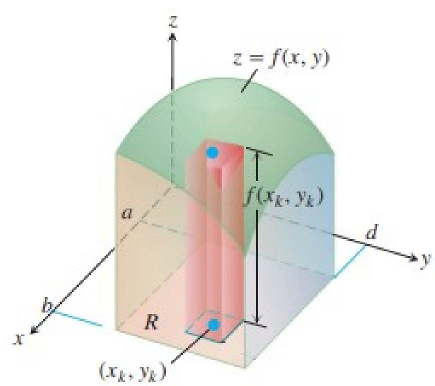
$$V = \iint_R f(x, y) dA$$



Evaluating Double Integrals over Rectangular Region:

If the case when the region $R = \{(x, y): a \le x \le b, c \le y \le d\}$ is a rectangular region, the double integral can be evaluated as:

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$



or

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

Examples:

1. Evaluate the integral $\int_0^3 \int_0^2 (\mathbf{4} - \boldsymbol{y}^2)\, \boldsymbol{dy}\, \boldsymbol{dx}$.

$$= \int_0^3 \left(\int_0^2 4dy - \int_0^2 y^2 dy \right) dx$$

$$= \int_0^3 \left[|4y|_0^2 - \left| \frac{y^3}{3} \right|_0^2 \right] dx$$

$$= \int_0^3 \left[(4(2) - 4(0)) - \left(\frac{2^3}{3} - \frac{0^3}{3} \right) \right] dx$$

$$= \int_0^3 \left[\frac{24 - 8}{3} \right] dx = \int_0^3 \frac{16}{3} dx$$

$$= \left| \frac{16}{3} x \right|_0^3 = \frac{16}{3}(3) - \frac{16}{3}(0) = 16$$

2. Find the value of $\int_0^3 \int_{-2}^0 (\boldsymbol{x}^2\boldsymbol{y} - \mathbf{2}\boldsymbol{xy})\boldsymbol{dydx}$.

$$= \int_0^3 \left[\int_{-2}^0 x^2 y dy - \int_{-2}^0 2xy dy \right] dx$$

$$= \int_0^3 \left[x^2 \left| \frac{y^2}{2} \right|_{-2}^0 - 2x \left| \frac{y^2}{2} \right|_{-2}^0 \right] dx =$$

$$= \int_0^3 \left[x^2 \left(\frac{0^2}{2} - \frac{(-2)^2}{2} \right) - 2x \left(\frac{0^2}{2} - \frac{(-2)^2}{2} \right) \right] dx$$

$$= \int_0^3 x^2 \left(-\frac{4}{2} \right) - 2x \left(-\frac{4}{2} \right) dx = \int_0^3 (-2\mathrm{x}^2 + 4x)\, dx$$

$$= \left| -\frac{2x^3}{3} \right|_0^3 + \left| \frac{4x^2}{2} \right|_0^3 = -\frac{2}{3}[3^3 - 0^3] + 2[3^3 - 0^3]$$

$$= -\frac{2}{3}(27) + 2(9) = -18 + 18 = 0$$

3.Solve $\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y)dxdy.$

$$\int_{\pi}^{2\pi} \left[\int_0^{\pi} \sin x \, dx + \int_0^{\pi} \cos y \, dx \right] dy = \int_{\pi}^{2\pi} [|-\cos x|_0^{\pi} + \cos y \,(|x|_0^{\pi})] \, dy$$

$$= \int_{\pi}^{2\pi} [-(\cos \pi - \cos 0) + \cos y \,(\pi - 0)] \, dy$$

$$= \int_{\pi}^{2\pi} [-(-1 - 1) + \pi \cos y] \, dy = \int_{\pi}^{2\pi} [2 + \pi \cos y] dy$$

$$= |2y|_{\pi}^{2\pi} + \pi|\sin y|_{\pi}^{2\pi} = 2(2\pi - \pi) + \pi(\sin 2\pi - \sin \pi) = 2\pi + 0 = 2\pi$$

Question: Find the value of $\int_{-1}^{0} \int_{-1}^{1} (x + y + 1)dydx.$ (do it yourself)

4. Find the volume of the solid lying under the surface $f(x, y) = 1 - 6x^2y$ and over the region $R: 0 \leq x \leq 2, -1 \leq y \leq 1.$

Solution**:**

$$\boldsymbol{Volume} = \iint_R f(x, y)dA$$

$$= \int_0^2 \int_{-1}^1 (1 - 6x^2 y) dy dx$$

$$= \int_0^2 \left(\int_{-1}^1 1 dy - 6x^2 \int_{-1}^1 y dy \right) dx$$

$$= \int_0^2 \left(|y|_{-1}^1 - 6x^2 \left| \frac{y^2}{2} \right|_{-1}^1 \right) dx = \int_0^2 \left(\left(1 - (-1)\right) - 6x^2 \left[\frac{1^2}{2} - \frac{(-1)^2}{2} \right] \right) dx$$

$$= \int_0^2 (2 - 3x^2[1 - 1]) dx = \int_0^2 \left(2 - 3x^2(0)\right) dx$$

$$= \int_0^2 2dx = |2x|_0^2 = 2(2) - 0$$

$$= 4$$

5. Find the volume of the solid lying under the surface $f(x, y) = x + y + 1$ and over the region $R: -1 \leq x \leq 1, -1 \leq y \leq 0$.

Solution: $\boldsymbol{Volume} = \iint_R f(x, y) dA$

$$= \int_{-1}^1 \int_{-1}^0 (x + y + 1) dy dx$$

$$= \int_{-1}^1 \left(\int_{-1}^0 x dy + \int_{-1}^0 y dy + \int_{-1}^0 dy \right) dx$$

$$= \int_{-1}^1 \left(x|y|_{-1}^0 + \left| \frac{y^2}{2} \right|_{-1}^0 + |y|_{-1}^0 \right) dx$$

$$= \int_{-1}^{1} \left(x\big(0 - (-1)\big) + \left[\frac{0^2}{2} - \frac{(-1)^2}{2}\right] + \big(0 - (-1)\big) \right) dx$$

$$= \int_{-1}^{1} \left(x - \frac{1}{2} + 1 \right) dx = \int_{-1}^{1} \left(x + \frac{1}{2} \right) dx$$

$$= \int_{-1}^{1} x dx + \int_{-1}^{1} \frac{1}{2} dx = |x^2|_{-1}^{1} + \frac{1}{2} |x|_{-1}^{1}$$

$$= (1^2 - (-1)^2) + \frac{1}{2}\big(1 - (-1)\big) = 1 - 1 + \frac{1}{2}(2)$$

$$= 1$$

Practice Problems:

Evaluate the given double integrals.

1. $\int_0^1 \int_0^2 (x + 3)\, dy\, dx$
2. $\int_1^3 \int_{-1}^1 (2x - 4y)\, dy\, dx$
3. $\int_2^4 \int_0^1 x^2 y\, dx\, dy$
4. $\int_{-2}^0 \int_{-1}^2 (x^2 + y^2)\, dx\, dy$
5. $\int_0^{\ln 3} \int_0^{\ln 2} e^{x+y}\, dy\, dx$
6. $\int_0^2 \int_0^1 y \sin x\, dy\, dx$
7. $\int_{-1}^0 \int_2^5 dx\, dy$
8. $\int_4^6 \int_{-3}^7 dy\, dx$

Use double integral to find the volume.

29. The volume under the plane $z = 2x + y$ and over the rectangle $R = \{(x, y) : 3 \leq x \leq 5, 1 \leq y \leq 2\}$.

30. The volume under the surface $z = 3x^3 + 3x^2 y$ and over the rectangle $R = \{(x, y) : 1 \leq x \leq 3, 0 \leq y \leq 2\}$.

31. The volume of the solid enclosed by the surface $z = x^2$ and the planes $x = 0$, $x = 2$, $y = 3$, $y = 0$, and $z = 0$.