Calculus and Analytical Geometry

Lecture no. 15

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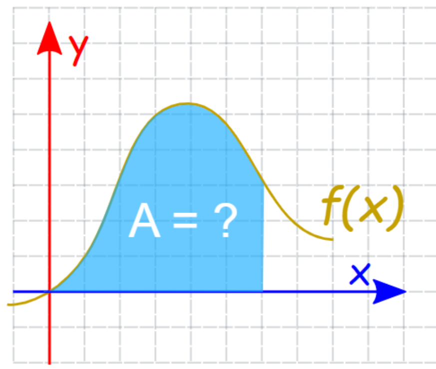
Topic: The indefinite integral of algebraic functions

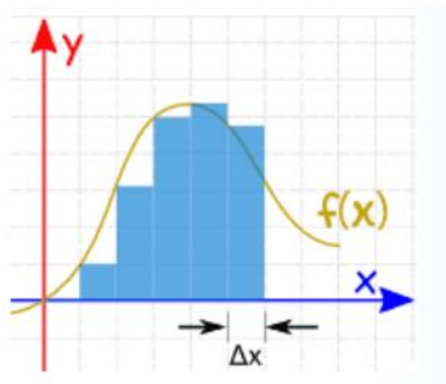
- Integration
- Properties of integration
- Integration formulas
- Examples
- Initial value problem
- Practice questions

1) **Integration**:

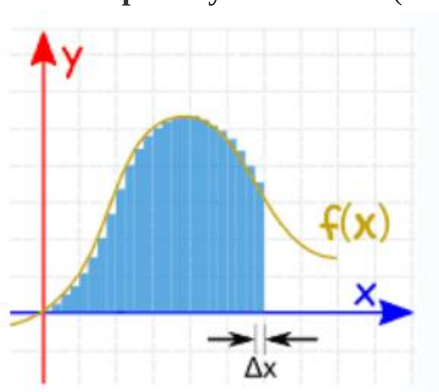
Integration is the process of finding the area of a region under a curve.



This is done by drawing as many small rectangles covering up the area and summing up their areas. The sum approaches a limit that is equal to the region under the curve of a function.
We could calculate the function at a few points and **add up slices of width Δx** like this (but the answer won't be very accurate):

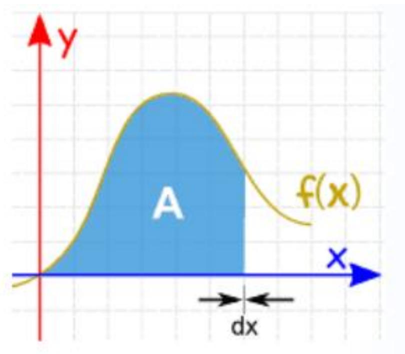


We can make **Δx** a lot smaller and **add up many small slices** (answer is getting better):



And as the slices **approach zero in width**, the answer approaches the **true answer**.

We now write **dx** to mean the **Δx** slices are approaching zero in width.



Theorem: The process of finding antiderivatives is called ***antidifferentiation*** or ***integration***. Thus, if

$$\frac{d}{dx}[F(x)] = f(x) \qquad (1)$$

then ***integrating*** (or ***antidifferentiating***) the function $f(x)$ produces an antiderivative of the form $F(x) + C$. To emphasize this process, Equation (1) is recast using ***integral notation***,

$$\int f(x)\,dx = F(x) + C$$

where C is understood to represent an arbitrary constant. The expression $\int f(x)\,dx$ is called indefinite integral.

PROPERTIES OF INDEFININTE INTEGRAL:

Suppose that F (x) and G(x) are antiderivatives of f(x) and g(x), respectively, and that c is a constant. Then:

(a) A constant factor can be moved through an integral sign; that is,

$$\int cf(x)\,dx = cF(x) + C$$

(b) An antiderivative of a sum is the sum of the antiderivatives; that is,

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(c) An antiderivative of a difference is the difference of the antiderivatives; that is,

$$\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$$

(d) $\int [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]dx = c_1 \int f_1(x)dx + c_2 \int f_2(x)dx + \ldots + c_n \int f_n(x)dx$

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\, dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\, dx = \sin x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\, dx = -\cos x + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\, dx = -\cot x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + C$
8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\, dx = -\csc x + C$
9. $\frac{d}{dx}[e^x] = e^x$	$\int e^x\, dx = e^x + C$
10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \quad (0 < b, b \neq 1)$	$\int b^x\, dx = \frac{b^x}{\ln b} + C \quad (0 < b, b \neq 1)$
11. $\frac{d}{dx}[\ln \lvert x\rvert] = \frac{1}{x}$	$\int \frac{1}{x}\, dx = \ln \lvert x\rvert + C$
12. $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$
13. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$
14. $\frac{d}{dx}[\sec^{-1} \lvert x\rvert] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} \lvert x\rvert + C$

Examples:

1. Evaluate the integral $\int 2x^3\, dx$

$$\int 2x^3\, dx = 2\int x^3\, dx$$

$$= 2\left(\frac{x^{3+1}}{3+1}\right) = 2\left(\frac{x^4}{4}\right) + C$$

$$= \frac{x^4}{2} + C$$

2. Evaluate the integral $\int x^3\sqrt{x}\, dx$

$$\int x^3\sqrt{x}\, dx = \int x^{3+\frac{1}{2}}\, dx$$

$$= \int x^{\frac{7}{2}}\ dx$$

$$= \left(\frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1}\right) = \frac{2}{9}x^{\frac{9}{2}} + C$$

3. Evaluate the integral $\int (3x^6 - 2x^2 + 7x + 1)\ dx$

$$\int (3x^6 - 2x^2 + 7x + 1)dx = 3\int x^6 dx - 2\int x^2 dx + 7\int x\ dx + \int 1\ dx$$

$$= 3\left(\frac{x^{6+1}}{6+1}\right) - 2\left(\frac{x^{3+1}}{3+1}\right) + 7\left(\frac{x^{1+1}}{1+1}\right) + x + C$$

$$= 3\left(\frac{x^7}{7}\right) - 2\left(\frac{x^4}{4}\right) + 7\left(\frac{x^2}{2}\right) + x + C$$

$$= \frac{3x^7}{7} - \frac{2x^4}{4} + \frac{7x^2}{2} + x + C$$

4. Evaluate the integral $\int \frac{3-x^2}{(x^2+3)^2}$

Since,

$$\frac{d}{dx}\left[\frac{x}{x^2+3}\right] = \frac{3-x^2}{(x^2+3)^2}$$

So,

$$\int \frac{3-x^2}{(x^2+3)^2} = \frac{x}{x^2+3} + C$$

5. Evaluate the integral$\int \frac{x^2}{x^2+1}$

$$\int \frac{x^2}{x^2+1}\ dx = \int \frac{x^2+1-1}{x^2+1}\ dx$$

$$= \int \left(\frac{x^2}{x^2+1} - \frac{1}{x^2+1}\right) dx$$

$$= \int \left(1 - \frac{1}{x^2+1}\right)\ dx$$

$$= \int 1\, dx - \int \frac{1}{x^2+1}\ dx$$

$$= x - tan^{-1} x + C$$

6. Evaluate the integral $\int \left[\frac{10}{y^{\frac{3}{4}}} - \sqrt[3]{y} + \frac{4}{\sqrt{y}}\right] dy$

$$\int \left[\frac{10}{y^{\frac{3}{4}}} - \sqrt[3]{y} + \frac{4}{\sqrt{y}}\right] dy = 10 \int \frac{1}{y^{\frac{3}{4}}}\, dy - \int \sqrt[3]{y}\, dy + 4 \int \frac{1}{\sqrt{y}}\, dy$$

$$= 10 \int y^{-\frac{3}{4}}\, dy - \int y^{\frac{1}{3}}\, dy + 4 \int y^{-\frac{1}{2}}\, dy$$

$$= 10 \int y^{-\frac{3}{4}}\, dy - \int y^{\frac{1}{3}}\, dy + 4 \int y^{-\frac{1}{2}}\, dy$$

$$= 10 \left(\frac{y^{-\frac{3}{4}+1}}{-\frac{3}{4}+1}\right) - \left(\frac{y^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right) + 4 \left(\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + C$$

$$= 10 \left(\frac{y^{\frac{1}{4}}}{\frac{1}{4}}\right) - \left(\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right) + 4 \left(\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= 10(4)y^{\frac{1}{4}} - \frac{3}{4} y^{\frac{4}{3}} + 4(2)y^{\frac{1}{2}} + C$$

$$= 40y^{\frac{1}{4}} - \frac{3}{4} y^{\frac{4}{3}} + 8\sqrt{y} + C$$

7. Evaluate the integral $\int \frac{2x^{\frac{1}{3}} - 17x^{-\frac{1}{3}}}{\sqrt{x}}\, dx$

$$\int \frac{2x^{\frac{1}{3}} - 17x^{-\frac{1}{3}}}{\sqrt{x}}\, dx = \int \frac{2x^{\frac{1}{3}}}{x^{\frac{1}{2}}}\, dx - \frac{17x^{-\frac{1}{3}}}{x^{\frac{1}{2}}}\, dx$$

$$= \int 2x^{\frac{1}{3}-\frac{1}{2}}\, dx - 17x^{-\frac{1}{3}-\frac{1}{2}}\, dx$$

$$= \int 2x^{-\frac{1}{6}}\, dx - \int 17x^{-\frac{5}{6}} dx$$

$$= 2 \left(\frac{x^{-\frac{1}{6}+1}}{-\frac{1}{6}+1}\right) - 17 \left(\frac{x^{-\frac{5}{6}+1}}{-\frac{5}{6}+1}\right) + C$$

$$= 2\left(\frac{x^{\frac{5}{6}}}{\frac{5}{6}}\right) - 17\left(\frac{x^{\frac{1}{6}}}{\frac{1}{6}}\right) + C$$

$$= 2\left(\frac{6}{5}\right)x^{\frac{5}{6}} - 17(6)x^{\frac{1}{6}} + C$$

$$= \frac{12}{5}x^{\frac{5}{6}} - 102x^{\frac{1}{6}} + C$$

2) **Differential Equation”**

Finding an antiderivative for a function f(x) means finding a function y that satisfies the equation

$$\frac{dy}{dx} = f(x)$$

This is called differential equation.

3) **Initial-Value Problem:**

The problem of finding a function $y(x)$ whose derivative is $f(x)$ and its graph passes through (x_o, y_0) is expressed as

$$\frac{dy}{dx} = f(x), \quad y(x_o) = y_o$$

This is called an initial-value problem.

Examples:

i. Solve the initial value problem

$$\frac{dy}{dx} = \cos x, \;\; y(0) = 1$$

Solution:

Step 1: (Separate variables)

$$dy = \cos x \, dx$$

Step 2: (Integrate)

$$\int dy = \int \cos x \;\, dx$$

$$y = \sin x + C$$

Step 3: (Initial Condition)

Apply initial condition $y(0) = 1$, this means $y = 1$ when $x = 0$

$$1 = \sin 0 + C$$

$$1 = 0 + C$$

This gives $C = 1$.

Step 4: (Solution)

$$y = \sin x + 1$$

ii. Find the curve that has slope $2x + 1$ and that passes through the point $(-3,1)$

Solution:

Step 1: (Initial-Value Problem)

$$\frac{dy}{dx} = 2x + 1, \quad y(-3) = 0.$$

Step 2: (Separate variables)

$$dy = (2x + 1)dx$$

Step 3: (Integrate)

$$\int dy = \int (2x + 1)dx$$

$$y = 2\int x\,dx + \int 1\ dx$$

$$y = x^2 + x + C$$

Step 4: (Solution)

Apply initial condition $y(-3) = 1$, this means $y = 1$ when $x = -3$

$$0 = (-3)^2 + (-3) + C$$

$$0 = 6 + C$$

$$C = -6$$

Step 5: (Solution)

$$y = x^2 + x - 6$$

Practice Questions:

Evaluate the following integral

- $\int 2x^{\frac{5}{7}}\,dx$
- $\int \sqrt[3]{x^2}\,dx$
- $\int \left[x^{-\frac{1}{2}} - 3x^{\frac{5}{7}} + \frac{1}{9}\right]\,dx$
- $\int \frac{x^5+2x^2-1}{x^4}\,dx$
- $\int \frac{1-2t^3}{t^3}\,dt$
- $\int \left[5x + \frac{2}{3x^5}\right]\,dx$
- Solve the initial value problem $\frac{dy}{dx} = 3x^{-\frac{2}{3}}, y(-1) = -5$.
- Solve the initial value problem $\frac{dy}{dx} = 1 + \cos t\,, s(0) = 4$.
- Find the curve that has slope $(x+1)^2$ and that passes through point $(-2,8)$.