Separable Differential Equation

A first order equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have separable variables.

(i.e if we can separate the equation into two terms, one containing only x and other containing y only)

For example, the equations

$$\frac{dy}{dx} = y^2xe^{3x+4y} \qquad \text{and} \qquad \frac{dy}{dx} = y + \sin x$$

are separable and nonseparable, respectively. In the first equation we can factor $f(x, y) = y^2xe^{3x+4y}$ as

$$f(x, y) = y^2xe^{3x+4y} = \underset{\substack{\uparrow \\ g(x)}}{(xe^{3x})}\underset{\substack{\uparrow \\ h(y)}}{(y^2e^{4y})},$$

Method of Solution

- Step 1 Move all the y terms (including dy) to one side of the equation and all the x terms (including dx) to the other side.
- Step 2 Integrate one side with respect to y and the other side with respect to x. Don't forget "+ C" (the constant of integration).
- Step 3 Simplify

$$\frac{dy}{dx} = 5xy$$

$$\frac{dy}{y\,dx} = 5x$$

$$\frac{dy}{y} = 5x\,dx$$

EXAMPLE 1 Solving a Separable DE

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} \quad \leftarrow \text{laws of exponents}$$

$$= |1 + x|\,e^{c_1}$$

$$= \pm e^{c_1}(1 + x). \quad \leftarrow \begin{cases} |1 + x| = 1 + x, & x \geq -1 \\ |1 + x| = -(1 + x), & x < -1 \end{cases}$$

Relabeling $\pm e^{c_1}$ as c then gives $y = c(1 + x)$.

EXAMPLE 4 **An Initial-Value Problem**

Solve $(e^{2y} - y) \cos x \dfrac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0.$

SOLUTION Dividing the equation by $e^y \cos x$ gives

$$\frac{e^{2y} - y}{e^y}\, dy = \frac{\sin 2x}{\cos x}\, dx.$$

Before integrating, we use termwise division on the left-hand side and the trigonometric identity $\sin 2x = 2 \sin x \cos x$ on the right-hand side. Then

integration by parts →

$$\int (e^y - ye^{-y})\, dy = 2 \int \sin x\, dx$$

yields

$$e^y + ye^{-y} + e^{-y} = -2 \cos x + c. \tag{7}$$

The initial condition $y = 0$ when $x = 0$ implies $c = 4$. Thus a solution of the initial-value problem is

$$e^y + ye^{-y} + e^{-y} = 4 - 2 \cos x. \tag{8}$$

EXERCISES 2.2

In Problems 1–22 solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx} = \sin 5x$

2. $\dfrac{dy}{dx} = (x + 1)^2$

3. $dx + e^{3x}dy = 0$

4. $dy - (y - 1)^2 dx = 0$

5. $x\dfrac{dy}{dx} = 4y$

6. $\dfrac{dy}{dx} + 2xy^2 = 0$

7. $\dfrac{dy}{dx} = e^{3x+2y}$

8. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

9. $y \ln x \dfrac{dx}{dy} = \left(\dfrac{y + 1}{x}\right)^2$

10. $\dfrac{dy}{dx} = \left(\dfrac{2y + 3}{4x + 5}\right)^2$

11. $\csc y\, dx + \sec^2 x\, dy = 0$

12. $\sin 3x\, dx + 2y \cos^3 3x\, dy = 0$

13. $(e^y + 1)^2 e^{-y}\, dx + (e^x + 1)^3 e^{-x}\, dy = 0$

14. $x(1 + y^2)^{1/2}\, dx = y(1 + x^2)^{1/2}\, dy$

15. $\dfrac{dS}{dr} = kS$

16. $\dfrac{dQ}{dt} = k(Q - 70)$

17. $\dfrac{dP}{dt} = P - P^2$

18. $\dfrac{dN}{dt} + N = Nte^{t+2}$

19. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$ **20.** $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$

21. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$ **22.** $(e^x + e^{-x})\dfrac{dy}{dx} = y^2$

In Problems 23–28 find an explicit solution of the given initial-value problem.

23. $\dfrac{dx}{dt} = 4(x^2 + 1), \quad x(\pi/4) = 1$

24. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}, \quad y(2) = 2$

25. $x^2\dfrac{dy}{dx} = y - xy, \quad y(-1) = -1$

26. $\dfrac{dy}{dt} + 2y = 1, \quad y(0) = \frac{5}{2}$

27. $\sqrt{1 - y^2}\,dx - \sqrt{1 - x^2}\,dy = 0, \quad y(0) = \dfrac{\sqrt{3}}{2}$

28. $(1 + x^4)\,dy + x(1 + 4y^2)\,dx = 0, \quad y(1) = 0$