Power Series Solutions Continued…

The power series method of solving a linear DE with variable coefficients is often described as "the method of undetermined *series* coefficients." In brief, here is the idea: We substitute $y = \sum_{n=0}^{\infty} c_n\, x^n$ into the DE combine series as we did in Example 1(of last lecture), and then equate all coefficients to the right-hand side of the equation to determine the coefficients c_n. But because the right-hand side is zero, the last step requires, by the identity property in the preceding bulleted list, that all coefficients of x must be equated to zero. No, this does *not* mean that all coefficients *are* zero; this would not make sense. Theorem 6.1.1 guarantees that we can find two solutions. Example 3 illustrates how the single assumption that leads to two sets of coefficients, so we have two distinct power series $y_1(x)$ and $y_2(x)$, both expanded about the ordinary point $x = 0$. The general solution of the DE is $y = C_1y_1(x) + C_2y_2(x)$; indeed, it can be shown that $C_1= c_0$ and $C_2 = c_1$.

EXAMPLE 3 Power Series Solutions

Solve $y'' + xy = 0$.

Solution. Since there are no finite singular points, Theorem 6.1.1 guarantees two power series solutions centered at 0, convergent for $|x| < \infty$. Substituting $y = \sum_{n=0}^{\infty} c_n\, x^n$ and the second derivative $y'' = \sum_{n=2}^{\infty} n(n-1)c_n\, x^{n-2}$ into the DE gives

$$y'' + xy = \sum_{n=2}^{\infty} n(n-1)c_n\, x^{n-2} + x\sum_{n=0}^{\infty} c_n\, x^n = \sum_{n=2}^{\infty} n(n-1)c_n\, x^{n-2} + \sum_{n=0}^{\infty} c_n\, x^{n+1} \quad (7)$$

In Example 1 we already added the last two series on the right-hand side of the equality in (7) by shifting the summation index. From the result given in (4),

$$y'' + xy = 2c_2 + \sum_{k=1}^{\infty} [(k+1)(k+2)c_{k+2} + c_{k-1}]\, x^k = 0 \qquad (8)$$

At this point we invoke the identity property. Since (8) is identically zero, it is necessary that the coefficient of each power of x be set equal to zero—that is, $2c_2 = 0$ (it is the coefficient of x_0), and

$$(k+1)(k+2)c_{k+2} + c_{k-1} = 0, \qquad k = 1, 2, 3, \ \ldots \qquad (9)$$

Now $2c_2 = 0$ obviously dictates that $c_2 = 0$. But the expression in (9), called a **recurrence relation,** determines the c_k in such a manner that we can choose a certain subset of the set of coefficients to be *nonzero*. Since $(k + 1)(k + 2) = 0$ for all values of k, we can solve (9) for c_{k+2} in terms of c_{k-1}:

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}, \qquad k = 1, 2, 3, \ \ldots \qquad (10)$$

This relation generates consecutive coefficients of the assumed solution one at a time as we let k take on the successive integers indicated in (10):

$$k = 1, \qquad c_3 = -\frac{c_0}{(2)(3)}$$

$$k = 2, \qquad c_4 = -\frac{c_1}{(3)(4)}$$

$$k = 3, \qquad c_5 = -\frac{c_2}{(4)(5)} = 0$$

$$k = 4, \qquad c_6 = -\frac{c_3}{(5)(6)} = \frac{1}{2.3.5.6}\ c_0$$

$$k = 5, \qquad c_7 = -\frac{c_4}{(6)(7)} = \frac{1}{3.4.6.7}\ c_1$$

$$k = 6, \qquad c_8 = -\frac{c_5}{(7)(8)} = 0$$

$$k = 7, \qquad c_9 = -\frac{c_6}{(8)(9)} = \frac{1}{2.3.5.6.8.9}\ c_0$$

$$k = 8, \qquad c_{10} = -\frac{c_7}{(9)(10)} = \frac{1}{3.4.6.7.9.10}\ c_1$$

$$k = 9, \qquad c_{11} = -\frac{c_8}{(10)(11)} = 0$$

and so on. Now substituting the coefficients just obtained into the original assumption

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + \cdots$$

We get

$$y = c_0 + c_1 x + 0 - \frac{c_0}{(2)(3)} x^3 - \frac{c_1}{(3)(4)} x^4 + 0 + \frac{c_0}{2.3.5.6}\ x^6 + \frac{c_1}{3.4.6.7}\ x^7 + 0$$
$$+ \frac{c_0}{2.3.5.6.8.9}\ x^9 + \frac{c_1}{3.4.6.7.9.10}\ x^{10} + 0 + \cdots$$

After grouping the terms containing c_0 and the terms containing c_1, we obtain $y = c_0 y_1(x) + c_1 y_2(x)$ as

$$y = c_0\left(1 - \frac{1}{(2)(3)} x^3 + \frac{1}{2.3.5.6}\ x^6 - \frac{1}{2.3.5.6.8.9}\ x^9 + \cdots\right)$$
$$+ c_1\left(x - \frac{1}{(3)(4)} x^4 + \frac{1}{3.4.6.7}\ x^7 - \frac{1}{3.4.6.7.9.10}\ x^{10} + \cdots\right)$$

where

$$y_1(x) = 1 - \frac{1}{(2)(3)} x^3 + \frac{1}{2.3.5.6}\ x^6 - \frac{1}{2.3.5.6.8.9}\ x^9 + \cdots$$

$$y_2(x) = x - \frac{1}{(3)(4)} x^4 + \frac{1}{3.4.6.7}\ x^7 - \frac{1}{3.4.6.7.9.10}\ x^{10} + \cdots$$

Because the recursive use of (10) leaves c_0 and c_1 completely undetermined, they can be chosen arbitrarily. As already mentioned, the linear combination $y = c_0 y_1(x) + c_1 y_2(x)$, actually represents the general solution of the differential equation.

EXAMPLE 4 Solve $(x^2+1)y''+xy'-y=0$.

Solution. Substituting $y=\sum_{n=0}^{\infty} c_n x^n$ and its first two derivatives lead to

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+x\sum_{n=1}^{\infty} nc_n x^{n-1}-\sum_{n=0}^{\infty} c_n x^n$$

$$=\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{n=k}+\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{n=k+2}+\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{n=k}-\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{n=k}$$

$$=\sum_{k=2}^{\infty} k(k-1)c_k x^k+\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2} x^k+\sum_{k=1}^{\infty} kc_k x^k-\sum_{k=0}^{\infty} c_k x^k$$

$$=\sum_{k=2}^{\infty} k(k-1)c_k x^k+2c_2+6c_3x+\sum_{k=2}^{\infty}(k+2)(k+1)c_{k+2} x^k+c_1x+\sum_{k=2}^{\infty} kc_k x^k$$

$$-c_0-c_1x-\sum_{k=2}^{\infty} c_k x^k$$

$$=2c_2-c_0+6c_3x+\sum_{k=2}^{\infty}[k(k-1)c_k+(k+1)(k+2)c_{k+2}+kc_k-c_k]\,x^k 2c_2$$

$$2c_2-c_0+6c_3x+\sum_{k=2}^{\infty}[(k+1)(k-1)c_k+(k+1)(k+2)c_{k+2}]\,x^k=0$$

From this identity we conclude that $2c_2-c_0=0,\;\; 6c_3=0$, and

$$(k+1)(k-1)c_k+(k+1)(k+2)c_{k+2}=0.$$

Now $c_2=\frac{1}{2}c_0, c_3=0$ and the **recurrence relation**

$$c_{k+2}=-\frac{(k-1)}{(k+2)}c_k, \qquad k=2,3,4,\;\ldots$$

Substituting $k=2,3,4,\;\ldots$ into the last formula gives

$$c_4=-\frac{1}{4}c_2=-\frac{1}{2\,.\,4}c_0, \qquad c_5=-\frac{2}{5}c_3=0,$$

$$c_6=-\frac{3}{6}c_4=\frac{3}{2\,.\,4\,.\,6}c_0, \qquad c_7=-\frac{4}{7}c_5=0,$$

$$c_8=-\frac{5}{8}c_6=-\frac{3\,.\,5}{2\,.\,4\,.\,6\,.\,8}c_0, \qquad c_9=-\frac{6}{9}c_7=0,$$

$$c_{10}=-\frac{7}{10}c_8=-\frac{3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8\,.\,10}c_0, \quad c_{11}=-\frac{8}{11}c_9=0, \quad \ldots$$

and so on. Now substituting the coefficients into the original assumption

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + \cdots$$

We get

$$y = c_0\left[1 + \frac{1}{2}x^2 - \frac{1}{2.2.2}x^4 + \frac{1}{2.2.2.2}\,x^6 - \frac{5}{2.2.2.2.2}\,x^8 + \cdots\right] + c_1 x$$

After grouping the terms containing c_0 and the terms containing c_1, we obtain $y = c_0 y_1(x) + c_1 y_2(x)$, where

$$y_1(x) = 1 + \tfrac{1}{2}x^2 - \tfrac{1}{2.2.2}x^4 + \tfrac{1}{2.2.2.2}\,x^6 - \tfrac{5}{2.2.2.2.2}\,x^8 + \cdots \quad \text{and} \quad y_2(x) = x.$$

Exercise 6.1

In Problems 17–28 find two power series solutions of the given differential equation about the ordinary point $x = 0$.

17. $y'' - xy = 0$

18. $y'' + x^2 y = 0$

19. $y'' - 2xy' + y = 0$

20. $y'' - xy' + 2y = 0$

21. $y'' + x^2 y' + xy = 0$

22. $y'' + 2xy' + 2y = 0$

23. $(x - 1)y'' + y' = 0$

24. $(x + 2)y'' + xy' - y = 0$

25. $y'' - (x + 1)y' - y = 0$

26. $(x^2 + 1)y'' - 6y = 0$

27. $(x^2 + 2)y'' + 3xy' - y = 0$

28. $(x^2 - 1)y'' + xy' - y = 0$

Answers of Q 17, 19

17. $y_1(x) = c_0\left[1 + \dfrac{1}{3 \cdot 2}x^3 + \dfrac{1}{6 \cdot 5 \cdot 3 \cdot 2}x^6 + \dfrac{1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}x^9 + \cdots\right]$

$y_2(x) = c_1\left[x + \dfrac{1}{4 \cdot 3}x^4 + \dfrac{1}{7 \cdot 6 \cdot 4 \cdot 3}x^7 + \dfrac{1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}x^{10} + \cdots\right]$

19. $y_1(x) = c_0\left[1 - \dfrac{1}{2!}x^2 - \dfrac{3}{4!}x^4 - \dfrac{21}{6!}x^6 - \cdots\right]$

$y_2(x) = c_1\left[x + \dfrac{1}{3!}x^3 + \dfrac{5}{5!}x^5 + \dfrac{45}{7!}x^7 + \cdots\right]$