Nonhomogeneous linear DE Continued…

The form that we assume for the particular solution y_p is an educated guess; it is not a blind guess. This educated guess must take into consideration not only the types of functions that make up $g(x)$ but also, as we shall see in Example 4, the functions that make up the complementary function y_c.

Example 4: A Glitch in the Method

Find a particular solution of $y'' - 5y' + 4y = 8e^x$.

SOLUTION Differentiation of e^x produces no new functions. Therefore proceeding as we did in the earlier examples, we can reasonably assume a particular solution of the form $y_p = Ae^x$. But substitution of this expression into the differential equation yields the contradictory statement $0 = 8e^x$, so we have clearly made the wrong guess for y_p.

The difficulty here is apparent on examining the complementary function $y_c = c_1e^x + c_2e^{4x}$. Observe that our assumption Ae^x is already present in y_c. This means that e^x is a solution of the associated homogeneous differential equation, and a constant multiple Ae^x when substituted into the differential equation necessarily produces zero.

Let's see whether we can find a particular solution of the form:

$$y_p = Axe^x.$$

Substituting $y_p' = Axe^x + Ae^x$ and $y_p'' = Axe^x + 2Ae^x$ into the differential equation and simplifying gives

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x.$$

From the last equality we see that the value of A is now determined as $A = -\frac{8}{3}$. Therefore a particular solution of the given equation is $y_p = -\frac{8}{3}xe^x$. ■

Example 5: Initial Value Problem

Solve $y'' + y = 4x + 10 \sin x$, $y(\pi) = 0$, $y'(\pi) = 2$.

Solution:

Let $$y_p = Ax + B + C\cos x + E\sin x.$$

But there is an obvious duplication of the terms $\cos x$ and $\sin x$ in this assumed form and two terms in the complementary function. This duplication can be eliminated by

$$y_p = Ax + B + Cx\cos x + Ex\sin x.$$

Differentiating this expression and substituting the results into the differential equation gives

$$y_p'' + y_p = Ax + B - 2C\sin x + 2E\cos x = 4x + 10\sin x,$$

$$y_p = 4x - 5x\cos x.$$

$$y = y_c + y_p = c_1\cos x + c_2\sin x + 4x - 5x\cos x.$$

We now apply the prescribed initial conditions to the general solution of the equation. First, $y(\pi) = c_1\cos\pi + c_2\sin\pi + 4\pi - 5\pi\cos\pi = 0$ yields $c_1 = 9\pi$, since $\cos\pi = -1$ and $\sin\pi = 0$. Next, from the derivative

$$y' = -9\pi\sin x + c_2\cos x + 4 + 5x\sin x - 5\cos x$$

and $$y'(\pi) = -9\pi\sin\pi + c_2\cos\pi + 4 + 5\pi\sin\pi - 5\cos\pi = 2$$

we find $c_2 = 7$. The solution of the initial-value is then

$$y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x.$$ ■

Exercise 4.4 D.G. Zill

16. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$

17. $y'' - 2y' + 5y = e^x\cos 2x$

18. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

19. $y'' + 2y' + y = \sin x + 3\cos 2x$

20. $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$

21. $y''' - 6y'' = 3 - \cos x$

22. $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$

23. $y''' - 3y'' + 3y' - y = x - 4e^x$

24. $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$

25. $y^{(4)} + 2y'' + y = (x - 1)^2$

26. $y^{(4)} - y'' = 4x + 2xe^{-x}$

In Problems 27–36 solve the given initial-value problem.

27. $y'' + 4y = -2, \quad y\left(\frac{\pi}{8}\right) = \frac{1}{2}, y'\left(\frac{\pi}{8}\right) = 2$

28. $2y'' + 3y' - 2y = 14x^2 - 4x - 11, \quad y(0) = 0, y'(0) = 0$

29. $5y'' + y' = -6x, \quad y(0) = 0, y'(0) = -10$

30. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \quad y(0) = 2, y'(0) = 5$

31. $y'' + 4y' + 5y = 35e^{-4x}, \quad y(0) = -3, y'(0) = 1$