Nonhomogeneous linear DE- Superposition Approach

To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(x) \qquad (1)$$

we must do two things;

- find the complementary function $\boldsymbol{y_c}$ of the associated homogeneous equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

- find *any* particular solution $\boldsymbol{y_p}$ of the nonhomogeneous equation

Then, **general solution** for non-homogeneous DE will be:

$$\boldsymbol{y = y_c + y_p}$$

Method of undetermined coefficients

The method for obtaining a particular solution y_p for a nonhomogeneous linear DE is called the **method of undetermined coefficients.** The general method is limited to linear DEs such as (1) where:

1. *the coefficients $a_i, i = 0, 1, \ldots, n$ are constants and*
2. *$g(x)$ is a constant k, a polynomial function, an exponential function e^{ax}, a sine or cosine function sin bx or cos bx, or finite sums and products of these functions.*

The following functions are some examples of the types of inputs $g(x)$ that are appropriate for this discussion:

$$g(x) = 10, \quad g(x) = x^2 - 5x, \quad g(x) = 15x - 6 + 8e^{-x},$$
$$g(x) = \sin 3x - 5x \cos 2x, \quad g(x) = xe^x \sin x + (3x^2 - 1)e^{-4x}.$$

That is, $g(x)$ is a linear combination of functions of the type:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$$P(x)\, e^{\alpha x}, \quad P(x)\, e^{\alpha x} \sin \beta x, \; P(x)\, e^{\alpha x} \cos \beta x,$$

where n is a nonnegative integer and α, β are real numbers.

The method of undetermined coefficients is not applicable to equations of form (1) when

$$g(x) = \ln x, \quad g(x) = \frac{1}{x}, \quad g(x) = \tan x, \quad g(x) = \sin^{-1} x,$$ and so on.

Idea behind the Method:

The set of functions that consists of constants, polynomials, exponentials, sines, and cosines has the remarkable property that derivatives of their sums and products are again sums and products of constants, polynomials, exponentials, sines, and cosines. Because the linear combination of derivatives

$$a_n y_p^{(n)} + \cdots + a_1 y_p' + a_0 y_p = g(x)$$

must be identical to $g(x)$, it seems reasonable to assume that ***y_p has the same form as $g(x)$.***

Example 1: General Solution using Undetermined Coefficients

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$ (2)

SOLUTION **Step 1.** We first solve the associated homogeneous equation $y'' + 4y' - 2y = 0$. From the quadratic formula we find that the roots of the auxiliary equation $m^2 + 4m - 2 = 0$ are $m_1 = -2 - \sqrt{6}$ and $m_2 = -2 + \sqrt{6}$. Hence the complementary function is

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

Step 2. Now, because the function $g(x)$ is a quadratic polynomial, let us assume a particular solution that is also in the form of a quadratic polynomial:

$$y_p = Ax^2 + Bx + C.$$

We seek to determine *specific* coefficients A, B, and C for which y_p is a solution of (2). Substituting y_p and the derivatives

$$y_p' = 2Ax + B \quad \text{and} \quad y_p'' = 2A$$

into the given differential equation (2), we get

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6.$$

$$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

Because the last equation is supposed to be an identity, the coefficients of like powers of x must be equal:

equal

$$\boxed{-2A}\,x^2 + \boxed{8A - 2B}\,x + \boxed{2A + 4B - 2C} = 2x^2 - 3x + 6$$

That is, $\quad -2A = 2, \qquad 8A - 2B = -3, \qquad 2A + 4B - 2C = 6.$

Solving this system of equations leads to the values $A = -1$, $B = -\frac{5}{2}$, and $C = -9$. Thus a particular solution is

$$y_p = -x^2 - \frac{5}{2}x - 9.$$

Step 3. The general solution of the given equation is

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_1 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$$

Example 2: Particular Solution using Undetermined Coefficients

Find a particular solution of $y'' - y' + y = 2\sin 3x$.

SOLUTION A natural first guess for a particular solution would be $A\sin 3x$. But because successive differentiations of $\sin 3x$ produce $\sin 3x$ *and* $\cos 3x$, we are prompted instead to assume a particular solution that includes both of these terms:

$$y_p = A\cos 3x + B\sin 3x.$$

Differentiating y_p and substituting the results into the differential equation gives, after regrouping,

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$$

$$-9A\cos 3x - 9B\sin 3x + (3A\sin 3x - 3B\cos 3x) + (A\cos 3x + B\sin 3x) = 2\sin 3x$$

equal

$$\boxed{-8A - 3B}\cos 3x + \boxed{3A - 8B}\sin 3x = 0\cos 3x + 2\sin 3x.$$

From the resulting system of equations,

$$-8A - 3B = 0, \qquad 3A - 8B = 2,$$

we get $A = \frac{6}{73}$ and $B = -\frac{16}{73}$. A particular solution of the equation is

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x.$$

Example 3: **Forming y_p using Superposition**

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$. (3)

SOLUTION **Step 1.** First, the solution of the associated homogeneous equation $y'' - 2y' - 3y = 0$ is found to be $y_c = c_1e^{-x} + c_2e^{3x}$.

Step 2. Next, the presence of $4x - 5$ in $g(x)$ suggests that the particular solution includes a linear polynomial. Furthermore, because the derivative of the product xe^{2x} produces $2xe^{2x}$ and e^{2x}, we also assume that the particular solution includes both xe^{2x} and e^{2x}. In other words, g is the sum of two basic kinds of functions:

$$g(x) = g_1(x) + g_2(x) = \textit{polynomial} + \textit{exponentials}.$$

$$y_p = y_{p_1} + y_{p_2},$$

where $y_{p_1} = Ax + B$ and $y_{p_2} = Cxe^{2x} + Ee^{2x}$. Substituting

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

into the given equation (3) and grouping like terms gives

$$\begin{aligned} y_p'' - 2y_p' - 3y_p &= 4Ce^{2x} + 4Cxe^{2x} + 4Ee^{2x} - 2(A + Ce^{2x} + 2Cxe^{2x} + 2Ee^{2x}) \\ &\quad - 3(Ax + B + Cxe^{2x} + Ee^{2x}) = 4x - 5 + 6xe^{2x} \\ &= -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}. \quad (4) \end{aligned}$$

From this identity we obtain the four equations

$$-3A = 4, \qquad -2A - 3B = -5, \qquad -3C = 6, \qquad 2C - 3E = 0.$$

The last equation in this system results from the interpretation that the coefficient of e^{2x} in the right member of (4) is zero. Solving, we find $A = -\frac{4}{3}$, $B = \frac{23}{9}$, $C = -2$, and $E = -\frac{4}{3}$. Consequently,

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}.$$

Step 3. The general solution of the equation is

$$y = c_1e^{-x} + c_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}.$$

Following is the table of various examples for the formation of particular solution:

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

Exercise 4.4 D.G. Zill

In Problems 1–26 solve the given differential equation by undetermined coefficients.

1. $y'' + 3y' + 2y = 6$

2. $4y'' + 9y = 15$

3. $y'' - 10y' + 25y = 30x + 3$

4. $y'' + y' - 6y = 2x$

5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$

6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$

7. $y'' + 3y = -48x^2e^{3x}$

8. $4y'' - 4y' - 3y = \cos 2x$

9. $y'' - y' = -3$

10. $y'' + 2y' = 2x + 5 - e^{-2x}$

11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$

12. $y'' - 16y = 2e^{4x}$

13. $y'' + 4y = 3\sin 2x$

14. $y'' - 4y = (x^2 - 3)\sin 2x$

15. $y'' + y = 2x\sin x$