Eigen values and Eigen vectors

If A is $n \times n$ matrix, then a non-zero vector $\vec{x}$ in R^n is called an Eigen vector of A if $A\vec{x}$ is a scalar multiple of$\vec{x}$that is

$$A\vec{x} = \lambda\vec{x} => A\vec{x} - \lambda\vec{x} = 0$$

$$=> (A - \lambda I)\vec{x} = 0$$

For some scalar λ, the scalar λ is called an Eigen value of A. $\vec{x}$ is said to be an Eigenvector corresponding to λ.

Note: If A is $n \times n$ matrix, then λ is an eigenvaluye of A if and only if it satisfies the equation

$$det(A - \lambda I) = 0$$

This is called the **charactristic equation** of A.

Example1: Find eigenvalue/s of the matrix A

$$\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Solution: The eigenvalue/s of A are the solution of the equation

$$det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 - \lambda & 0 \\ 8 & -1 - \lambda \end{bmatrix}$$

$$det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 0 \\ 8 & -1 - \lambda \end{vmatrix}$$

$$= (3-\lambda)(-1-\lambda) - 8(0)$$

$$= -(3-\lambda)(1+\lambda) - 0$$

$$= -(3-\lambda)(1+\lambda)$$

Put $\quad det(A-\lambda I) = 0,$

$$-(3-\lambda)(1+\lambda) = 0$$

$$(3-\lambda) = 0 \qquad , \qquad (1+\lambda) = 0$$

$$\lambda = 3 \qquad , \qquad \lambda = -1$$

So, eigenvalues are $\boxed{\lambda = 3}$ & $\boxed{\lambda = -1}$

Example 2: Find eigenvalue/s of the matrix A

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution:

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -17 & 8-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 4 & 8-\lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & -\lambda \\ 4 & -17 \end{vmatrix}$$

$$= -\lambda[-\lambda(8-\lambda)+17] - 1[0(8-\lambda)-4(1)] + 0[0(-17)-(-\lambda)4)]$$

$$= -\lambda[8\lambda + \lambda^2 + 17] - 1[-4]$$

$$= 8\lambda^2 - \lambda^3 - 17\lambda + 4$$

$$= -\lambda^3 + 8\lambda^2 - 17\lambda + 4$$

To find eigenvalue put $det(A - \lambda I) = 0$,

$$-\lambda^3 + 8\lambda^2 - 17\lambda + 4 = 0 \quad \Rightarrow \quad \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

$\boldsymbol{\lambda} = 4$ is the one solution so by synthetic division,

$$\begin{array}{c|cccc} 4 & 1 & -8 & 17 & -4 \\ & & 4 & -16 & 4 \\ \hline & 1 & -4 & 1 & 0 \end{array}$$

$$\lambda^2 - 4\lambda + 1 = 0$$

from quadratic equation,

$$a = 1, \quad b = -4, \quad c = 1$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$\lambda = \frac{4 \pm \sqrt{16-4}}{2} \quad => \lambda = \frac{4 \pm \sqrt{12}}{2}$$

$$\lambda = \frac{4 \pm 2\sqrt{3}}{2}$$

$$\lambda = 2 \pm \sqrt{3}$$

$\lambda = 2 + \sqrt{3}$, $\lambda = 2 - \sqrt{3}$ & $\lambda = 4$

So the eigenvalues of A are

$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \lambda = 2 - \sqrt{3}$

Exercise:

Find the eigenvalues of the following matrices.

a) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$
b) $\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$
c) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$
d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
e) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
f) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
g) $\begin{bmatrix} 3 & 0 & -5 \\ 1/5 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$
h) $\begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$
i) $\begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$

Example 3: Find the eigenvalues and the corresponding eigenvectors of the following matrix.

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Solution: The eigenvalue/s of A are the solution of the equation

$$det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(-1-\lambda) - 8(0)$$

$$= -(3-\lambda)(1+\lambda)$$

Put $\quad det(A - \lambda I) = 0,$

$$-(3-\lambda)(1+\lambda) = 0$$

$$(3-\lambda) = 0 \qquad , \qquad (1+\lambda) = 0$$

$$\lambda = 3 \qquad , \qquad \lambda = -1$$

For $\lambda = 3,$ eigenvector is

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = 3,$

$$\begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0x + 0y = 0$$

$$8x - 4y = 0$$

$$8x = 4y \qquad \Rightarrow \; x = \frac{4}{8} y$$

$$x = \frac{1}{2} y$$

Let $y = t$,

$$x = \frac{1}{2} t$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

So, $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 3$.

Now for $\lambda = -1$, eigenvector is

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda = -1,$

$$\begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x + 0y = 0 \qquad \dots (1)$$

$$8x - 0y = 0 \qquad \dots (2)$$

$$\Rightarrow \quad x = 0 \ \textit{from both equations} \text{ (1) and (2)}$$

As y is free variable so put $y = t$,

$$y = t$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda = -1$.

Example 4: Find the eigenvalues and the corresponding eigenvectors of the following matrix.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution: The eigenvalue/s of A are the solution of the equation

$$det(A - \lambda I) = 0$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(2-\lambda)-2] - 2[2-\lambda-1] + 1(2-3+\lambda) = 0$$

$$(2-\lambda)(6-3\lambda-2\lambda+\lambda^2-\lambda) - 2[1-\lambda] + (-1+\lambda) = 0$$

$$(2-\lambda)(\lambda^2-5\lambda+4) - 2(1-\lambda) + (\lambda-1) = 0$$

$$2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda - 2 + 2\lambda + \lambda - 1 = 0$$

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$\lambda = 1$ is one solution of above equation because

$$1^3 - 7(1)^2 + 11(1) - 5 = 0$$

$$1 - 7 + 11 - 5 = 0$$

$$0 = 0$$

So by using synthetic division we find other,

$$\begin{array}{c|cccc} 1 & 1 & -7 & 11 & -5 \\ & & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & 0 \end{array}$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\lambda^2 - 5\lambda - \lambda + 5 = 0$$

$$\lambda(\lambda - 1) - 5(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda = 1\ ,\ \lambda = 5$$

So eigenvalues of A are

$$\lambda = 1\ , \lambda = 1\ ,\ \lambda = 5$$

For $\lambda = 1$, eigenvector is

$$(A - \lambda I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put $\lambda = 1$,

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y + z = 0$$

$$x + 2y + z = 0$$

$$x + 2y + z = 0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \sim R_2 - R_1, R_3 - R_1$$

$$x + 2y + z = 0$$

$$x = -2y - z$$

Since y & z are free variables, So put $y = s, z = t$

$$\Rightarrow \qquad x = -2s - t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors for $\lambda = 1$.

Find the eigenvectors for $\lambda = 5$.