Calculus and Analytical Geometry

Lecture no. 25

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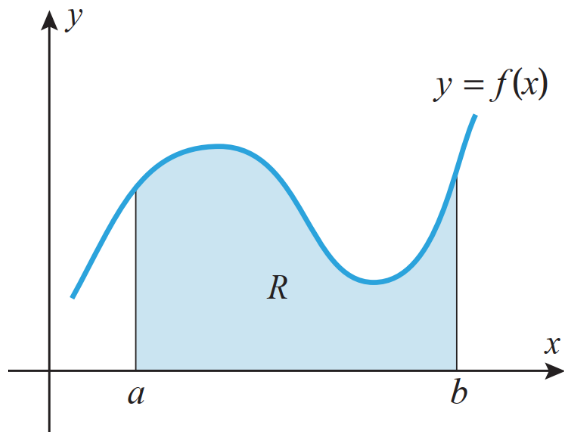
June 2022

Topic: Application of definite integral: Area under the curve

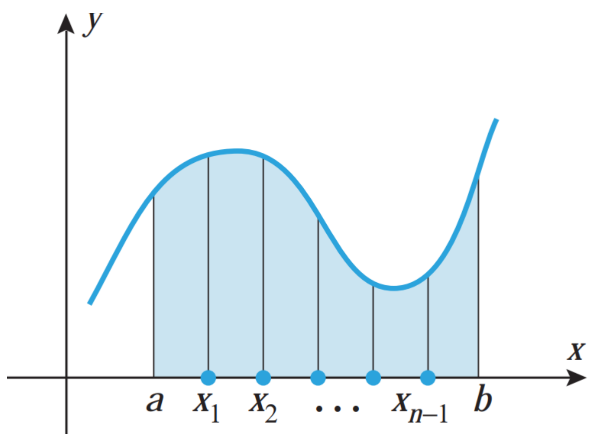
- Definition of Area
- Net signed area
- Theorem
- End point approximation for x_k^*
- Formulas
- Examples
- Practice questions

1. Definition of Area

Suppose that the function f is continuous and non-negative on the interval $[a\ , b]$, and let R denote the region bounded below by the x-axis, bounded on the sides by the vertical lines $x = a$ and $x = b$, and bounded above by the curve $y = f(x)$ as shown in the figure.



Divide the interval $[a;\ b]$ into n equal subintervals by inserting $n-1$ equally spaced points between $a \text{ and } b$, and denote those points by $x_1, x_2, x_3, \ldots, x_{n-1}$as shown in figure below



Each of these subintervals has width $\frac{b-a}{n}$, which is denoted by

$$\Delta x = \frac{b-a}{n}$$

Over each subinterval construct a rectangle whose height is the value of f at an arbitrarily selected point in the subinterval. Thus, if

$$x_1^*, x_2^*, \cdots, x_{n-1}^*$$

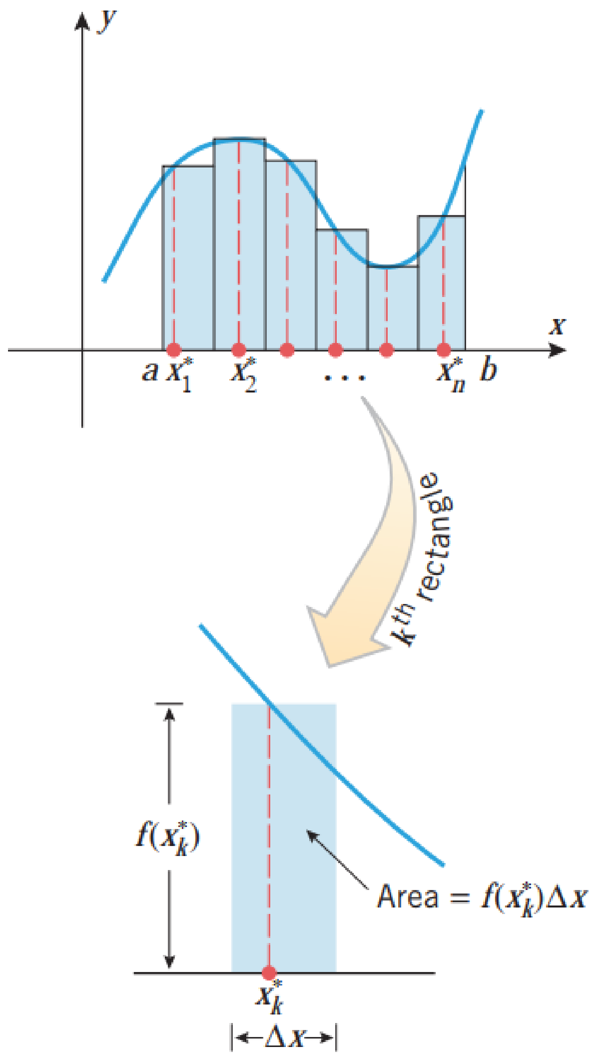
denote the points selected in the subintervals, then the rectangles will have heights:

$$f(x_1^*), f(x_2^*) \cdots f(x_n^*)$$

and areas

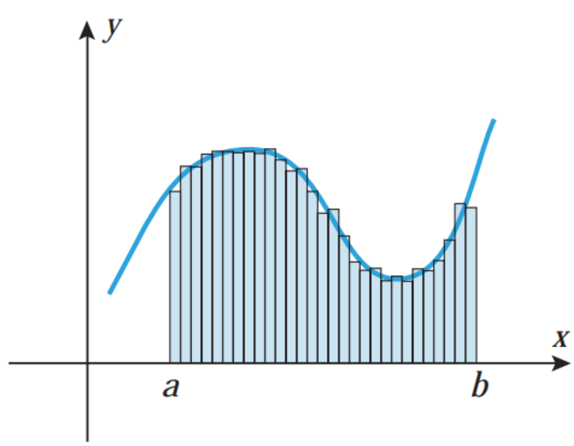
$$f(x_1^*)\Delta x, f(x_2^*)\Delta x, \cdots, f(x_n^*)\Delta x$$

as shown in figure.



The union of the rectangles forms a region Rn whose area can be regarded as an approximation to the area *A* of the region *R*; that is,

$$A = Area(R) \approx Area(R_n) = f(x_1^*)\Delta x, f(x_2^*)\Delta x, \cdots, f(x_n^*)\Delta x$$



This can be expressed in sigma notation as

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Repeat the process using more and more subdivisions, and define the area of R to be the **limit** of the areas of the approximating regions R_n as n increases without bound. That is, we define the area A as

$$A \approx \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

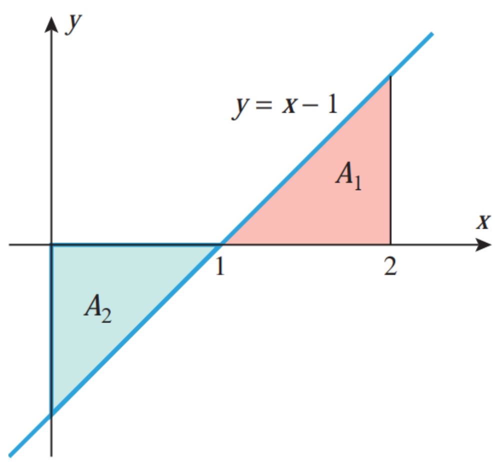
2. Net Signed Area

If the function $f(x)$ is continuous on $[a, b]$, then the net signed area A between $y = f(x)$ and the interval $[a, b]$ is defined by

$$A \approx \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Example 2.1: Let the graph of $f(x) = x - 1$ over the interval $[0, 2]$. It is geometrically evident from the figure shown below that the areas A_1 and A_2 in that figure are equal, so we expect the net signed area between the graph of f and the interval $[0, 2]$ to be zero.

Solution:



Theorem 2.1: If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the area A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A \approx \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Construction:

Step 1: Divide $[a, b]$ into n Subintervals

Divide the interval $[a, b]$ into n equal subintervals $[x_o, x_1], [x_1, x_2] \cdots [x_{n-1}, x_n]$ by inserting $n - 1$ equally spaced points $x_1, x_2, x_3, \ldots, x_{n-1}$ between a and b. Each of these subintervals has width $\Delta x = \frac{b-a}{n}$

Step 2: Construct Rectangles

Over each subinterval construct a rectangle whose height is the value of f at an arbitrarily selected point in the subinterval. Thus, if $x_1^*, x_2^*, \cdots, x_{n-1}^*$ denote the points selected in the subintervals, then the rectangles will have heights $f(x_1^*), f(x_2^*) \cdots f(x_n^*)$ and areas

$$f(x_1^*)\Delta x, f(x_2^*)\Delta x, \cdots, f(x_n^*)\Delta x$$

Step 3: Sum of Areas of Rectangles

The sum of areas of n rectangles can be regarded as an approximation to the area A under the curve $y = f(x)$, that is,

$$A \approx f(x_1^*)\Delta x, f(x_2^*)\Delta x, \cdots, f(x_n^*)\Delta x$$

This can be expressed more compactly in sigma notation as

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x$$

This is called **Riemann sum**.

Step 4: Area as a Limit

Repeat the process using more and more subdivisions, and define the area A to be the limit of the sum of areas of rectangles as n increases without bound. That is, we define the area A as

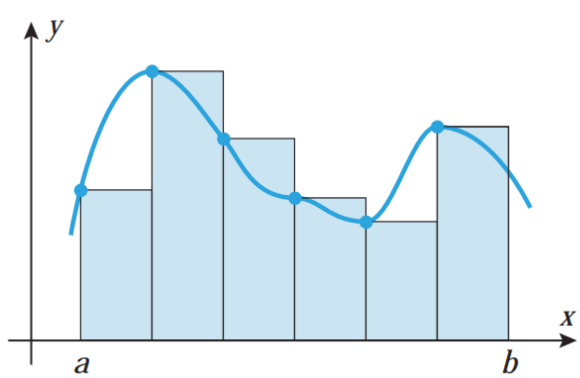
$$A \approx \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

3 Endpoints approximations for x_k^*

1. The **left endpoint approximation** for $x*k$ is given by the formula

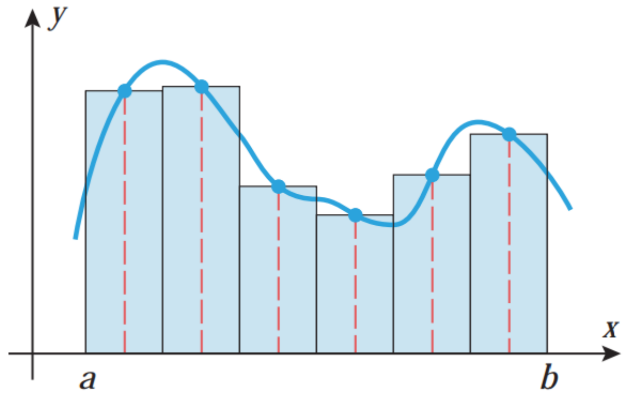
$$x_k^* = x_{k-1} = a + (k - 1)\Delta x$$

Graphically, it can be represented as



2. The **mid-point approximation** for x_k^* is given by the formula

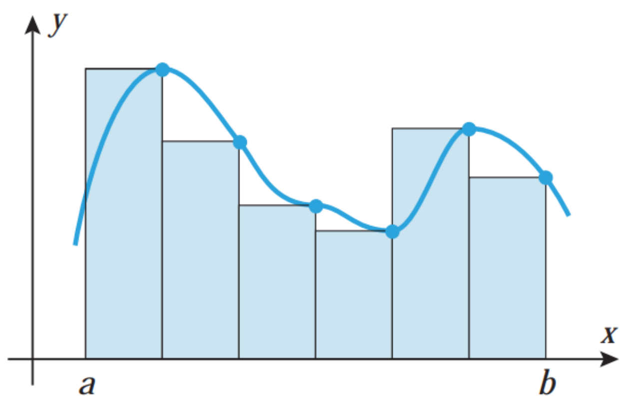
$$x_k^* = \frac{1}{2}(x_{k-1} + x_k) = a + \left(k - \frac{1}{2}\right)\Delta x$$



3. The **right endpoint approximation** for x_k^* is given by the formula

$$x_k^* = x_k = a + k\Delta x$$

Graphically, it can be represented as



4. Summation Formulas

1. $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ (c does not depend on k)

2. $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

3. $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

4. $\sum_{k=1}^{n}(1) = \underbrace{1+1+1+\ldots+1}_{n-times} = n$

5. $\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$

6. $\sum\limits_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

7. $\sum\limits_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

8. $\lim\limits_{n\to+\infty} \frac{1}{n^k} = 0, \quad k \in \mathbb{N}$

<u>Example 4.1:</u> Find the area under the graph of $f(x) = 3x$ over the interval $[0\,,2]$, taking n subintervals and taking x_k^* as the right end point of each subinterval.

<u>Solution:</u>

Step 1: Interval Length

Since number of subintervals is n, that is a = 0, and b = 2. The length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

Note that the right end point of the kth subinterval is

$$x_k^* = x_{k-1} = a + k\Delta x = 0 + \frac{2k}{n} = \frac{2k}{n}$$

Step 2: Riemann Sum

The Riemann sum (sum of areas of all n rectangles) is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} (3x_k^*)\Delta x$$

$$= \sum_{k=1}^{n} 3\left(\frac{2k}{n}\right)\frac{2}{n}$$

$$= \sum_{k=1}^{n} \left(\frac{12k}{n^2}\right) = \frac{12}{n^2}\sum_{k=1}^{n} k$$

$$= \frac{12}{n^2}\left[\frac{n(n+1)}{2}\right] = 6.\left[\frac{(n+1)}{n}\right]$$

$$= 6\left[1 + \frac{1}{n}\right]$$

Step 3: Applying Limits

Finally, the required area, as the limit of Riemann sum, is

$$Area = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n\to+\infty} 6\left[1 + \frac{1}{n}\right] = 6\left[1 + \frac{1}{\infty}\right]$$

$$= 6(1 + 0) = 0$$

Example 4.2: Find the area under the graph of $f(x) = 3x$ over the interval $[0, 2]$, taking n subintervals and taking x_k^* as the left end point of each subinterval.

Solution:

Step 1: Interval Length

Since number of subintervals is n, that is a = 0, and b = 2. The length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

Note that the left end point of the kth subinterval is

$$x_k^* = x_{k-1} = a + (k-1)\Delta x = 0 + (k-1)\frac{2}{n} = \frac{2(k-1)}{n} = \frac{2k-2}{n}$$

Step 2: Riemann Sum

The Riemann sum (sum of areas of all n rectangles) is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} (3x_k^*)\Delta x$$

$$= \sum_{k=1}^{n} 3\left(\frac{2k-2}{n}\right)\frac{2}{n}$$

$$= \sum_{k=1}^{n} \left(\frac{12(k-1)}{n^2}\right)$$

$$= \frac{12}{n^2}\sum_{k=1}^{n} k-1$$

$$= \frac{12}{n^2}\sum_{k=1}^{n} k - \frac{12}{n^2}\sum_{k=1}^{n} 1$$

$$= \frac{12}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{12}{n^2}(n)$$

$$= 6\left[1+\frac{1}{n}\right] - \frac{12}{n}$$

$$= 6 + \frac{6}{n} - \frac{12}{n}$$

Step 3: Applying Limits

Finally, the required area, as the limit of Riemann sum, is

$$Area = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n\to+\infty} 6 + \frac{6}{n} - \frac{12}{n} = 6 + \frac{6}{\infty} - \frac{12}{\infty}$$

$$= 6(1 + 0 - 0) = 6$$

5. Practice questions

1. Find the area under the graph of $f(x) = x$ over the interval $[\mathbf{0}, \mathbf{3}]$, taking *n* subintervals and taking x_k^* as the left end point of each subinterval.
2. Find the area under the graph of $f(x) = x$ over the interval $[\mathbf{0}, \mathbf{3}]$, taking *n* subintervals and taking x_k^* as the right end point of each subinterval.
3. Find the area under the graph of $f(x) = 6 - x$ over the interval $[\mathbf{0}, \mathbf{5}]$, taking *n* subintervals and taking x_k^*as the right end point of each subinterval.
4. Find the area under the graph of $f(x) = 2x$ over the interval $[-\mathbf{1}, \mathbf{3}]$, taking *n* subintervals and taking x_k^* as the left end point of each subinterval.