Calculus and Analytical Geometry

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Conic Sections: Parabola

Outline of the lecture:

The following topics will be discussed in this lecture

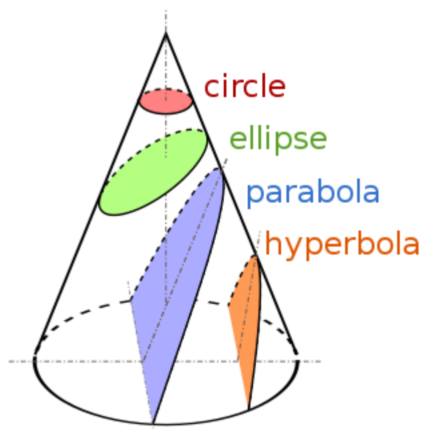
- Conic Sections
- Parabola
- Standard equations of parabola
- Examples
- Practice Questions

1. Conic Section

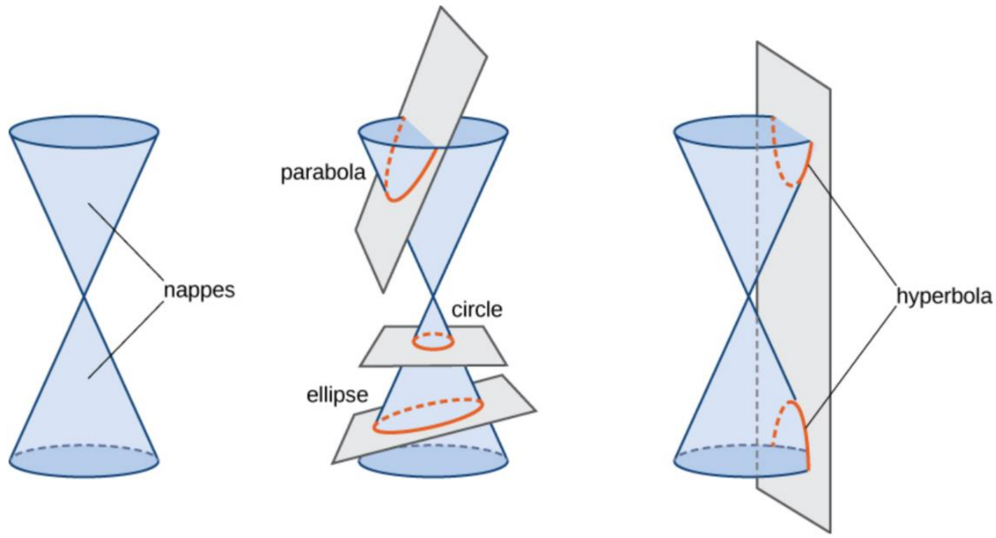
The intersection of a cone and a plane is called a conic section.

• There are four types of curves that result from these intersections that are of particular interest:

1. Parabola
2. Circle
3. Ellipse
4. Hyperbola



Conic sections can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called nappes. One nappe is what most people mean by **cone**, and has the shape of a party hat.



A cone and conic sections: The nappes and the four conic sections. Each conic is determined by the angle the plane makes with the axis of the cone.

• Conic sections are generated by the intersection of a plane with a cone.

• If the plane is parallel to the axis of revolution (the *y-axis*), then the conic section is a hyperbola.

• If the plane is parallel to the generating line, the conic section is a parabola.

• If the plane is perpendicular to the axis of revolution, the conic section is a circle.

• If the plane intersects one nappe at an angle to the axis (other than 90^0), then the conic section is an ellipse

2. Parabola

A parabola is the set of all points in the plane that are equidistant from a fixed line, called the **directrix**, and a fixed point, called the **focus**.

• The line through the focus and perpendicular to the directrix is called the **axis of symmetry** of the parabola.

• The point of intersection of the axis of symmetry and parabola is called the **vertex** of the parabola.

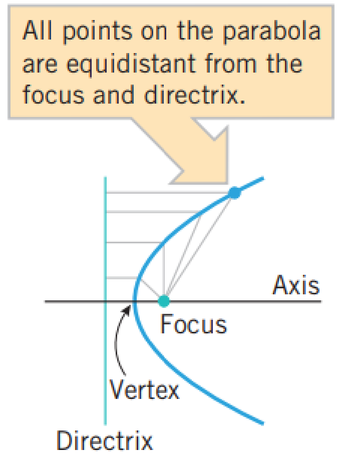
• The distance from the vertex to focus is called the **focal length**.

• A line joining two distinct points on a parabola is called a **chord** of the parabola.

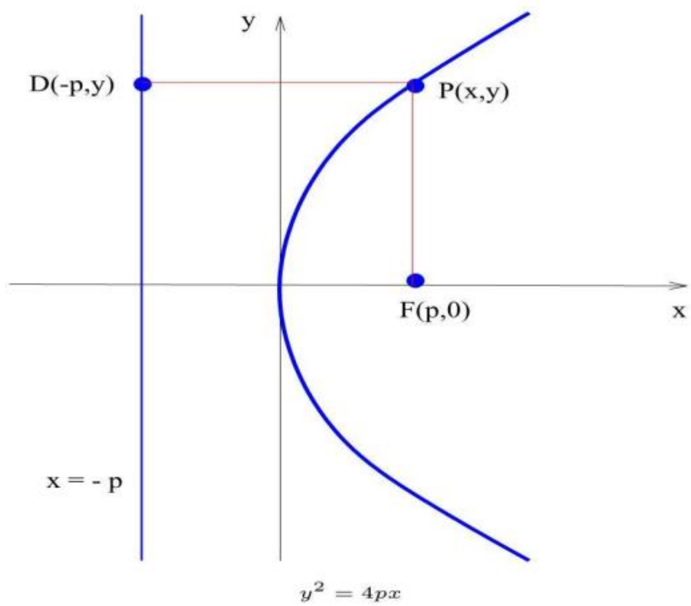
• A chord passing through the focus of a parabola is called a **focal chord** of the parabola.

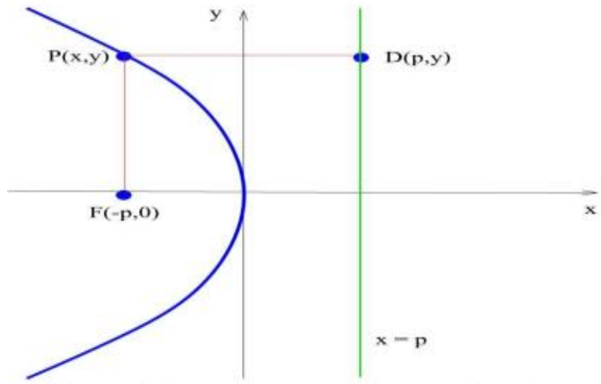
• The focal chord perpendicular to the axis of the parabola is called **latusrectum** of the parabola.

• A line used to construct and define a conic section is called **directrix**. A parabola has one directrix, ellipses and hyperbolas have two directrices.

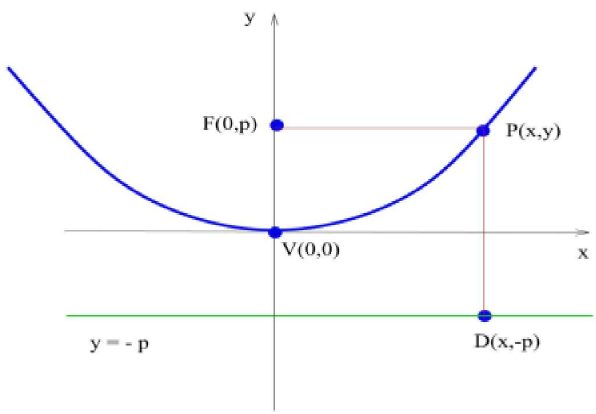


3. Standard Equations of Parabolas

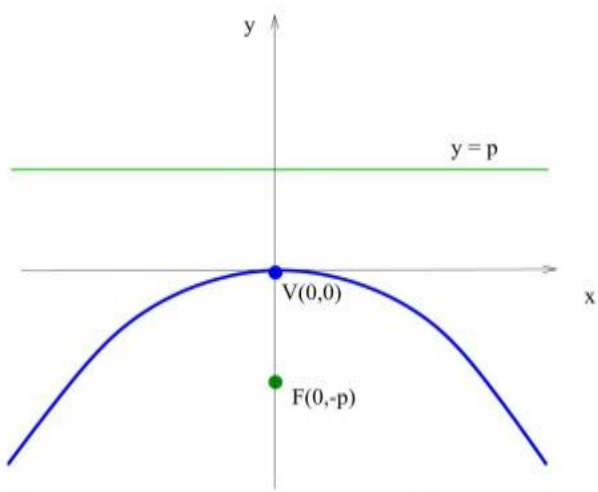




Horizontal Parabola Opening on the Left, $y^2 = -4px$



Vertical Parabola Opening Upwards, $x^2 = 4py$



Vertical Parabola Opening Downwards, $x^2 = -4py$

3.1 Symmetries of Parabolas

• A parabola is **symmetric** about the x-axis if it involves y^2.

• A parabola is **symmetric** about the y-axis if it involves x^2.

3.2 Summary of standard parabolas

Equation	$y^2 = 4px$	$y^2 = -4px$	$x^2 = 4py$	$x^2 = -4py$
Focus	$(p, 0)$	$(-p, 0)$	$(0, p)$	$(0, -p)$
Vertex	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Directrix	$x = -p$	$x = p$	$y = -p$	$y = p$
Lectus Rectum	$x = p$	$x = -p$	$y = p$	$y = -p$
Length of L.R	$4a$	$4a$	$4a$	$4a$

Example 3.1. Find the **focus** and **directrix** of the parabola $y^2 = 10x$. Also, sketch the parabola along with the focus and directrix.

Solution:

→ **Step 1: Symmetry and opening**

• **Symmetry:** Since the parabola involves y^2, therefore, the parabola is **symmetric** about the x-axis.

• **Opening:** Since the coefficient of x (which is actually the linear term) is positive, the parabola opens on the right.

→ **Step 2: Vertex**

By comparing the standard equation

$$(y - k)^2 = 4p(x - h)$$

with the given equation of parabola

$$y^2 = 10x$$

from here, we can write as

$h = 0, k = 0$

Hence $V\ (h,k)\ =\ V\ (0,0)$

→ Step 3: Focus and directrix: Find the value of p

Since $V\ (h,k)\ =\ V\ (0,0)$This implies that the general form of parabola becomes

$$y^2 = 4px$$

By comparing this equation with the given equation of parabola

$$y^2 = 10x$$

We can write as

$$4p\ =\ 10$$

$$p\ = \frac{5}{2}$$

$$p\ =\ 2.5$$

• **Focus:** The focus is $F(2.5,0)$

• **Directrix:** The directrix is $x\ =\ -2.5$

→ Step 4: Sketch the parabola

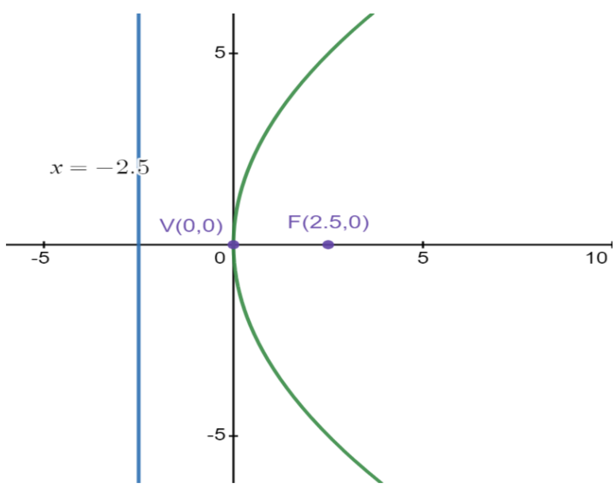
Follow the following steps:

Step 1: Locate Vertex

Step 2: Sketch according to opening

Step 3: Moves inside the parabola p-times from vertex to locate focus.

Step 3: Moves outside the parabola p-times to draw the directrix line.



→ Step 5: Gather all information

Symmetry: About x-axis

Opening: Right side

Vertex: $V\ (h, k)\ =\ V\ (0, 0)$

Value of p: $p\ =\ 2.5$

Focus: $F(2.5\ , 0)$

Directrix: At $x\ =\ -2.5$

Axis of symmetry: At $y\ =\ 0$

Example 3.2. Find the **focus** and **directrix** of the parabola $x^2 = 12y$. Also, sketch the parabola along with the focus and directrix.

Solution:

→ **Step 1: Symmetry and opening**

• **Symmetry:** Since the parabola involves x^2, therefore, the parabola is **symmetric** about the y**-axis**.

• **Opening:** Since the coefficient of y (which is actually the linear term) is positive, the parabola opens on the **upward** (on the positive y-axis).

→ **Step 2: Vertex**

By comparing the standard equation

$$(x - h)^2 = 4p(y - k)$$

with the given equation of parabola

$$x^2 = 12y$$

from here, we can write as

$h\ =\ 0\ , k\ =\ 0$

$Hence\ \ V\ (h, k) =\ V\ (0, 0)$

→ **Step 3: Focus and directrix: Find the value of** p

Since $V\ (h, k) =\ V\ (0, 0)$This implies that the general form of parabola becomes

$$x^2 = 4py$$

By comparing this equation with the given equation of parabola

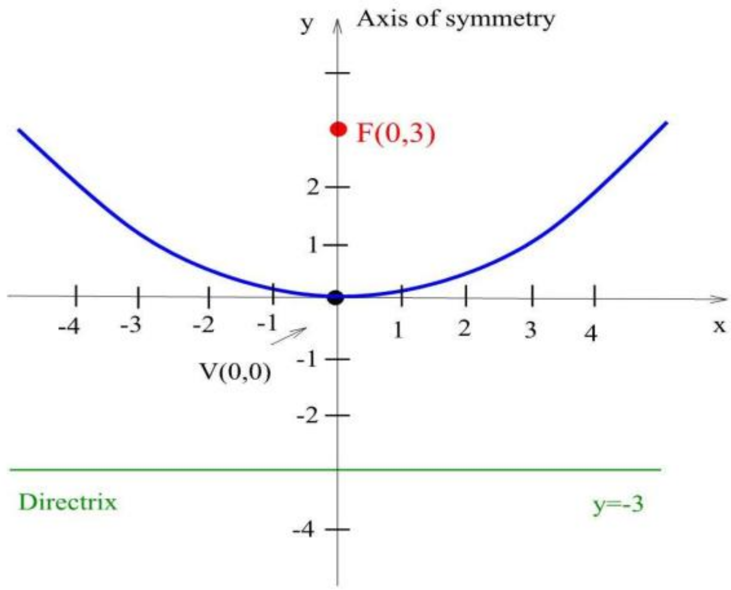
$$y^2 = 10x$$

We can write as

$$4p\ =\ 12$$

$$p\ = 3$$

- **Focus:** The focus is $F(0,3)$
- **Directrix:** The directrix is $x = -3$



Example 3.3. Describe the parabola $y^2 - 8x - 6y - 23 = 0$.

Solution:

→ **Step 1:** Writing in Standard Equation

Since the only quadratic term in the equation is y^2, we first take all the y-terms to one side:

$$y^2 - 8x - 6y - 23 = 0$$

$$y^2 - 6y = 8x + 23$$

$$(y)^2 - 2(3)y = 8x + 23$$

Next, we complete the square on the y-terms by adding 9 to both sides:

$$(y)^2 - 2(3)y + 9 = 8x + 23 + 9$$

$$(y)^2 - 2(3)y + (3)^2 = 8x + 32$$

$$(y-3)^2 = 8(x + 4)$$

$$(y-3)^2 = 4(2)(x + 4)$$

→ Step 2: Opening, Vertex, Focus and directrix

In order to find vertex, focus, and directrix we write the above equation in a new XY -coordinate system as

$$Y^2 = 4(2)X$$

Where,

$$X = x + 4, \quad \text{and} \quad Y = y - 3.$$

It follows from here that $p = 2$.

- **Opening:** Since the coefficient of X is positive, the parabola opens on the right.
- **Vertex:** Note that the vertex of the parabola

$$Y^2 = 4(2)X$$

lies at the origin. That is,

$$X = 0 \quad \text{and} \quad Y = 0$$

For $X = 0$ implies that $x + 4 = 0$. Hence $x = -4$.

For $Y = 0$ implies that $y - 3 = 0$. Hence $y = 3$.

Thus, the vertex of the parabola $y^2 - 8x - 6y - 23 = 0$ is:

$$(h,k) = (-4,3)$$

- **Focus:** Note that the focus of the parabola

$$Y^2 = 4(2)X$$

is $(p, 0)$. That is,

$$X = p \quad \text{and} \quad Y = 0$$

For $X = p = 2$ implies that $x + 4 = 2$. Hence $x = 2 - 4 = -2$.

For $Y = 0$ implies that $y - 3 = 0$. Hence $y = 3$.

Thus, the focus of the parabola $y^2 - 8x - 6y - 23 = 0$ is $F(-2,3)$.

- **Directrix:** Note that the directrix of

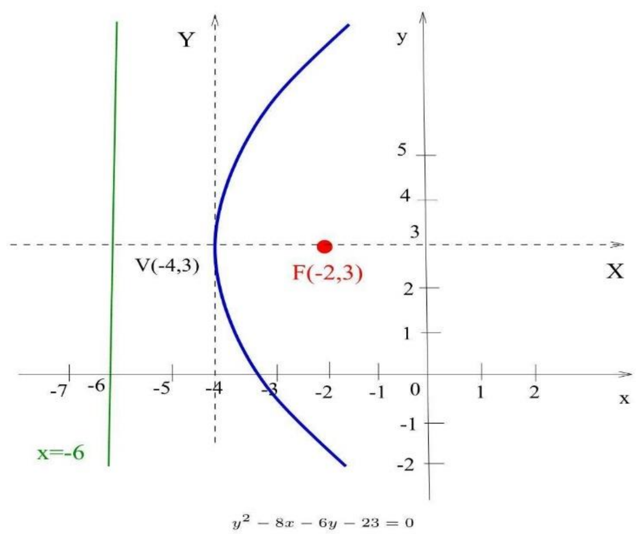
$$Y^2 = 4(2)X$$

is $X = -p$. It follows:

For $X = -p = -2$ implies that $x + 4 = -2$, then $x = -2 - 4 = -6$. Hence $x = -6$.

Thus, the directrix of the parabola $y^2 - 8x - 6y - 23 = 0$ is $x = -6$.

→ Step 3: Sketch



Example 3.4. Write the parabola $x^2 + 2x + 4y - 3 = 0$ in its standard form, and find its vertex, focus, and directrix.

Solution:

→ Step 1: Writing in Standard Equation

Since the only quadratic term in the equation is x^2, we first take all the x-terms to one side:

$$x^2 + 2x + 4y - 3 = 0$$

$$x^2 + 2x = -4y + 3$$

$$(x)^2 + 2(1)(x) = -4y + 3$$

Next, we complete the square on the y-terms by adding 1 to both sides:

$$x^2 + 2(1)(x) + 1 = -4y + 3 + 1$$

$$(x)^2 + 2(1)(x) + (1)^2 = -4y + 4$$

$$(x + 1)^2 = -4(y - 1)$$

$$(x + 1)^2 = -4(1)(y - 1)$$

→ Step 2: Opening, Vertex, Focus and directrix

In order to find vertex, focus, and directrix we write the above equation in a new XY -coordinate system as

$$X^2 = -4(1)Y$$

where

$$X = x + 1, \quad \text{and} \quad Y = y - 1.$$

It follows from here that $p = 1$.

- **Opening:** Since the coefficient of Y is negative, the parabola opens downward. • Vertex: Note that the vertex of the parabola

$$X^2 = -4(1)Y$$

lies at the origin. That is,

$$X = 0 \quad \text{and} \quad Y = 0$$

For $X = 0$ implies that $x + 1 = 0$. Hence $x = -1$.

For $Y = 0$ implies that $y - 1 = 0$. Hence $y = 1$.

Thus, the vertex of the parabola $y^2 - 8x - 6y - 23 = 0$ is

$$(h, k) = (-1,1)$$

- **Focus:** Note that the focus of the parabola

$$X^2 = -4(1)Y$$

is $(0, -p)$. That is,

$$X = 0 \quad \text{and} \quad Y = -p$$

For $X = 0$ implies that $x + 1 = 0$. Hence $x = -1$.

For $Y = -p = -1$ implies that $y - 1 = -1$, then $y = -1 + 1 = 0$. Hence $y = 0$.

Thus, the focus of the parabola $x^2 + 2x + 4y - 3 = 0$ is $F(-1,0)$.

- **Directrix:** Note that the directrix of

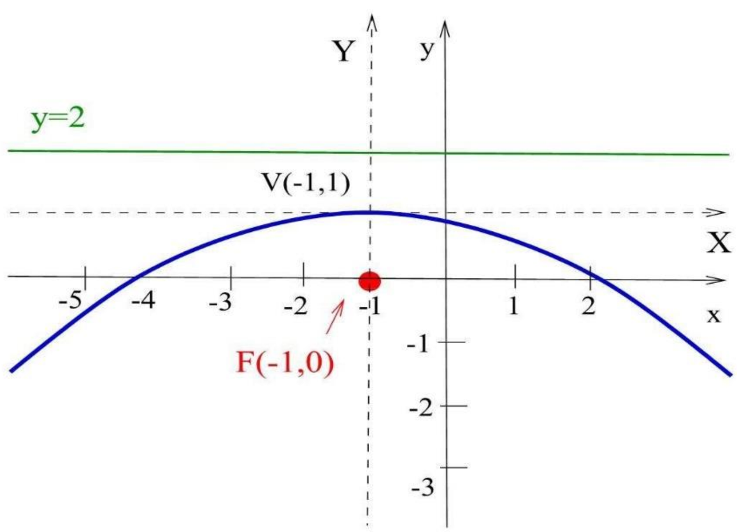
$$X^2 = -4(1)Y$$

is $Y = p$. It follows:

For $Y = p = 1$ implies that $y - 1 = 1$. Hence $y = 2$.

Thus, the directrix of the parabola $x^2 + 2x + 4y - 3 = 0$ is $y = 2$.

→ **Step 3: Sketch**



$x^2 + 2x + 4y - 3 = 0$

4. Practice questions

1. Sketch the parabola $y^2 = 4x$, and label its focus, vertex, and directrix.
2. Sketch the parabola $x^2 = -8y$, and label its focus, vertex, and directrix.
3. Sketch the parabola $(y-1)^2 = -12(x + 4)$, and label its focus, vertex, and directrix.
4. Sketch the parabola $y = 4x^2 + 8x + 5$, and label its focus, vertex, and directrix.
5. Find an equation of the parabola that is symmetric about the y-axis, has its vertex at the origin, and passes through the point (5,2).