بسم الله الرحمن الرحيم

Welcome to the Course

Linear Algebra [D17]

Course Instructor:

Ali Raza

University of Central Punjab, FOST, Lahore

Email id: alirazao1@ucp.edu.pk

Office: Room F304 A Building (3RD Floor)

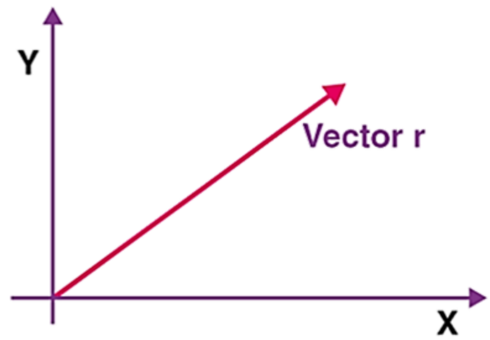
Reference Book: Linear Algebra with supplemental Applications (10th Edition) by Howard Anton

What is Linear Algebra?

Linear Algebra

Linear algebra is a branch of mathematics that deals with the study of vectors, matrices, vector spaces, linear transformations and system of linear equations .

Vectors:



Matrix of order n by n:

$$\begin{array}{cc} & \text{n columns} \\ A = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & b_{i3} & \cdots & b_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix} & \text{m rows} \end{array}$$

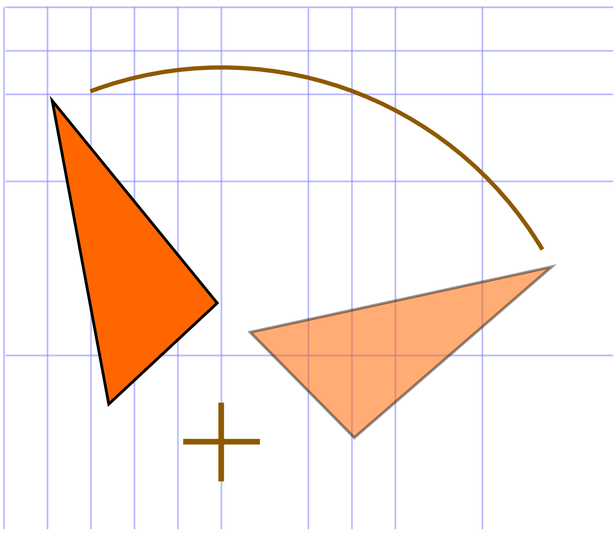
System of Linear Equations:

Coefficient of y

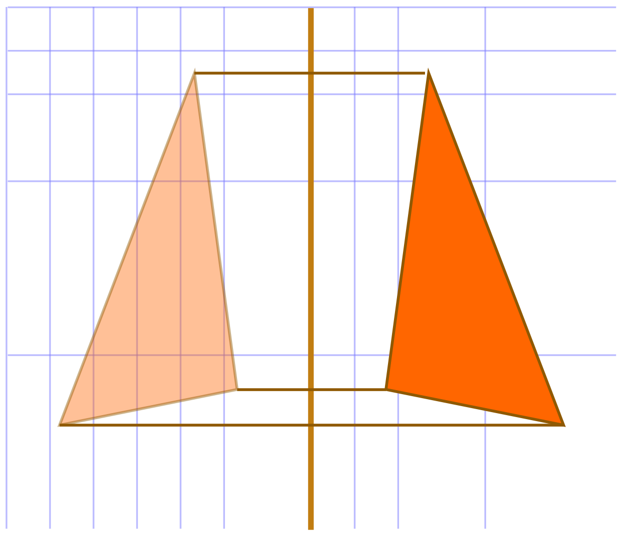
$$\text{Coefficient of x} \left[\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array} \right] \text{Constant}$$

What is Linear Algebra?

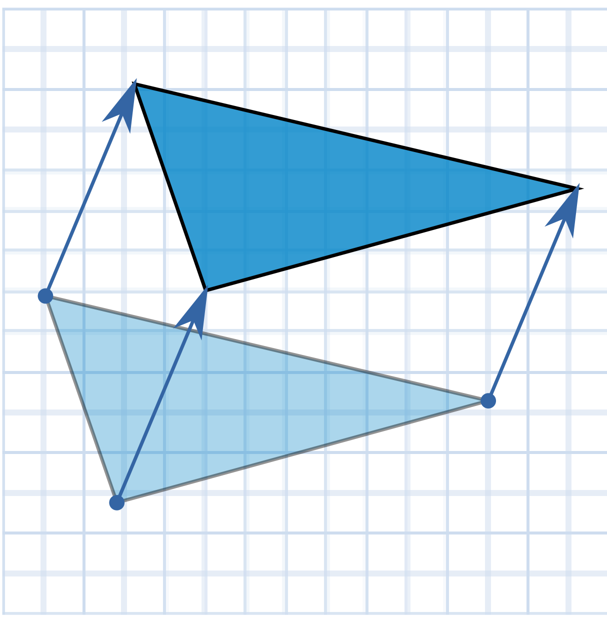
Linear Transformations:
(Rotation, reflection, translation, scaling etc.)



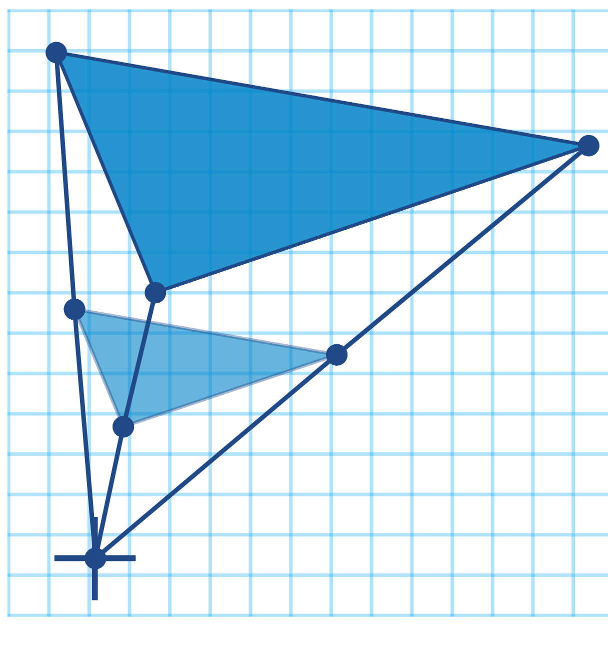
Rotation



Reflection



Translation



Scaling

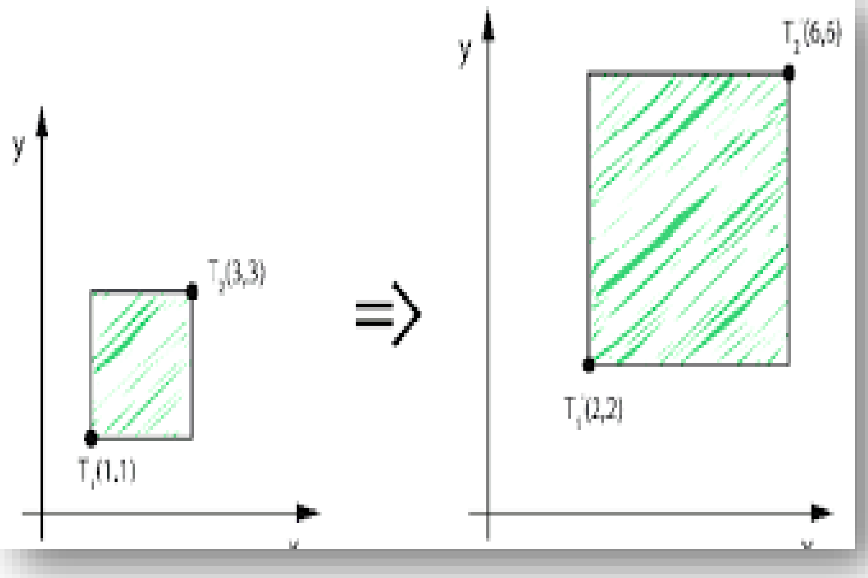
Why We Study Linear Algebra in Computer Science?

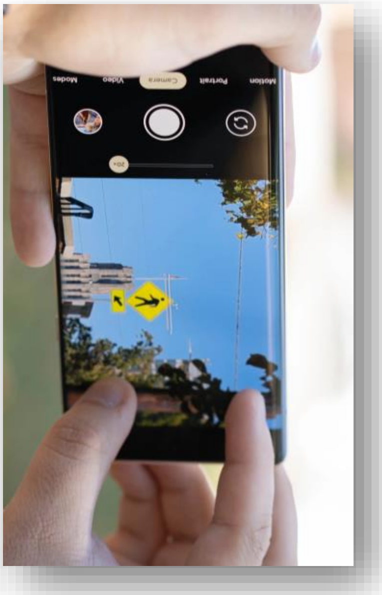
Linear algebra provides concepts that are crucial to many areas of computer science, including

- **graphics,**
- **image processing,**
- **cryptography,**
- **machine learning,**
- **computer vision,**
- **optimization,**
- **graph algorithms,**
- **quantum computation,**
- **computational biology,**
- **information retrieval and**
- **web search.**

Linear algebra in turn is built on two basic elements, the matrix and the vector.

Linear algebra is used when resizing images, such as during the process of zooming in or out on a picture.





Solving Linear Systems

Chapter 1 (From Howard Anton) Linear Equation

An equation whose exponent is one is called linear equation.

Examples (One, Two, Three, Four and n-variables):

- $2x + 1 = 0$ → **One Variable**
- $x + 3y = 7$ → **Two Variables**
- $\frac{1}{2}x - y + 3z = -1$ → **Three Variables**
- $x_1 - 2x_2 - 3x_3 + x_4 = 2$ → **Four Variables**
- $x_1 + x_2 + x_3 + \cdots + x_n = 1$ → **n Variables**

Linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear, for example, as arguments of trigonometric, logarithmic, or exponential functions.

The following equations are not linear equations (which term is making it nonlinear?):

- $x + 3y^2 = 4$ (y^2 **make it nonlinear**)
- $3x + 2y - xy = 5$ (**"xy" make it nonlinear**)
- $\sin x + y = 0$ (**sin(x) make it nonlinear**)
- $\sqrt{x_1} + x_2 + x_3 = 1$ ($\sqrt{x}$ **make it nonlinear**)

Solving Linear Systems

1. **In each part, determine whether the equation is linear in x_1, x_2 and x_3.**

a) $x_1 + 5x_3 - \sqrt{2x_3} = 1$
b) $x_1 + 3x_2 + x_1x_3 = 2$
c) $x_1^{-2} + x_2 + 8x_3 = 5$
d) $x_1 = -7x_2 + 3x_3$
e) $x_1^{\frac{3}{5}} - 2x_2 + x_3 = 4$

General Form of Linear Equation

The equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \qquad \rightarrow (1)$$

which expresses the real quantity b in terms of the unknowns $x_1, x_2, x_3, \dots, x_n$ and the real constants $a_1, a_2, a_3, \dots, a_n$ is called a linear equation.

A solution to Linear Equation (1) is a sequence of n numbers $s_1, s_2, \dots, s_n$ which has the property that (1) is satisfied when $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ are substituted in (1).

Example:

The equation $6x_1 - 3x_2 + 4x_3 = -13$ has the solution $x_1 = 2$, $x_2 = 3,\ x_3 = -4$.

Solving Linear Systems

System of Linear Equation in two Variables

A system of linear equations in two variables x and y will have the form

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

Here a_i, b_i, c_i $(i = 1,2)$ are real numbers. To find a solution to a linear system, we already know two techniques called the

1. **Method of Elimination**
2. **Method of Substitution**

In elimination method, we eliminate some variables by adding a multiple of one equation to another equation. Elimination merely amounts to the development of a new linear system that is equivalent to the original system but is much simpler to solve.

Similarly, in substitution method we can use value of one variable from one equation in other equation to get a simplified equation.

Example 1 **Find the solution of the linear system by using method of elimination.**

$$\begin{cases} 5x + y = 3 \\ 2x - y = 4 \end{cases}$$

Solution: We want to eliminate y, so by adding equation 1 and 2 we get:

$$7x = 7 \Rightarrow x = 1$$

Put value of $x = 1$ in equation 1:

$$5(1) + y = 3 \qquad \Rightarrow y = -2$$

So (1, -2) is solution of the given system.

Example 2 **Find the solution of the linear system by using method of elimination.**

$$\begin{cases} x - 3y = -7 \\ 2x - 6y = 7 \end{cases}$$

Solution: We want to eliminate x, so by multiplying equation 1 by ''2'' and subtracting from 2, we get:

$$\begin{array}{r} 2x - 6y = -14 \\ -2x + 6y = -7 \\ \hline 0 = -21 \end{array}$$

which makes no sense. This means that the given system has no solution.

Consistent and Inconsistent Linear System

(in the framework of statements)

If the linear system has no solution, it is said to be inconsistent, if it has a solution. it is called consistent.

So the system in example 1 is consistent and in example 2 is inconsistent.

Note!

A consistent linear system of two equations in two unknowns has either

- one solution or
- infinitely many solutions

And there are no other possibilities.

Solving Linear Systems

Exercise In each part, determine whether the given point is a solution of the linear system

$$\begin{cases} 2x - 4y - z = 1 \\ x - 3y + z = 1 \\ 3x - 5y - 3z = 1 \end{cases}$$

a) (3, 1, 1)

b) (3, -1, 1)

c) (13, 5, 2)

d) $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$

e) $(17, 7, 5)$

Exercise In each part, determine whether the given point is a solution of the linear system

$$\begin{cases} x + 2y - 2z = 3 \\ 3x - y + z = 1 \\ -x + 5y - 5z = 5 \end{cases}$$

a) $\left(\frac{5}{7}, \frac{8}{7}, 0\right)$

b) $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$

c) $(5, 8, 1)$

d) $\left(\frac{5}{7}, \frac{10}{7}, \frac{2}{7}\right)$

e) $\left(\frac{5}{7}, \frac{22}{7}, 2\right)$

Work to do:

[Elementary Linear Algebra by Howard Anton]

Exercise 1.1: Q 1-9

Solving Linear Systems

Question:

Solve linear system by using elimination method

$$\begin{cases} 2x-3y+4z=\ -12 \quad ------\rightarrow (1) \\ x-2y+z=-5 \quad -------(2) \\ 3x+y+2z=1 \ ---------(3) \end{cases}$$

Solution:

We want to eliminate x, so multiply equation (2) by 2 and subtract from (1).

$$2x-3y+4z=-12$$

$$2x-4y+2z=-10$$

- + - +

$$y+2z=-2 \quad ---\rightarrow (4)$$

multiply equation (2) by 3 and subtract from (3).

$$3x+y+2z=1$$

$$3x-6y+3z=-15$$

- + - +

$$7y-z=16 \quad ---\rightarrow (5)$$

multiply equation (5) by 2 and add in (4).

$$y+2z=-2$$

$$14y-2z=32$$

$$15y=30$$

$$y=2$$

Put $y = 2$ in equation (4)

$$2 + 2z = -2$$

$$2z = -4$$
$$z = -2$$

Put $z = -2, y = 2$ in equation (1)

$$2x - 3(2) + 4(-2) = -12$$
$$2x - 6 - 8 = -12$$
$$2x = -12 + 14$$
$$2x = 2$$
$$x = 1$$

So $(1, 2, -2)$ is solution of given system.

Solving Linear Systems

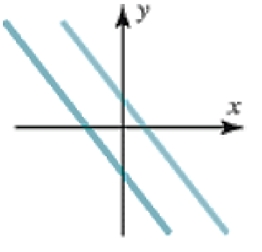
Linear Systems with Two Unknowns

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

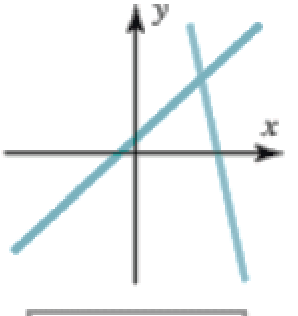
in which the graphs of the equations are lines in the xy-plane. Each solution (x, y) of this system corresponds to a point of intersection of the lines, so there are three possibilities:

1. **The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.**



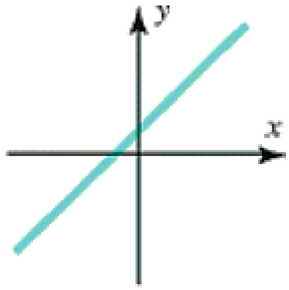
No solution

2. **The lines may intersect at only one point, in which case the system has exactly one solution**



One solution

3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.



Infinitely many solutions (coincident lines)

EXAMPLE 1 A Linear System with One Solution:

Solve the linear system

$$\begin{cases} x - y = 1 \\ 2x + y = 6 \end{cases}$$

Solution: We want to eliminate x from the second equation by adding -2 times the first equation to the second.

$$2x - 2y = 2$$

$$2x + y = 6$$

$$-3y = -4 \quad \Rightarrow \quad y = \frac{4}{3}$$

Put $y = \frac{4}{3}$ in the first equation we obtain

$$x = 1 + y = 1 + \frac{4}{3} = \frac{7}{3}$$

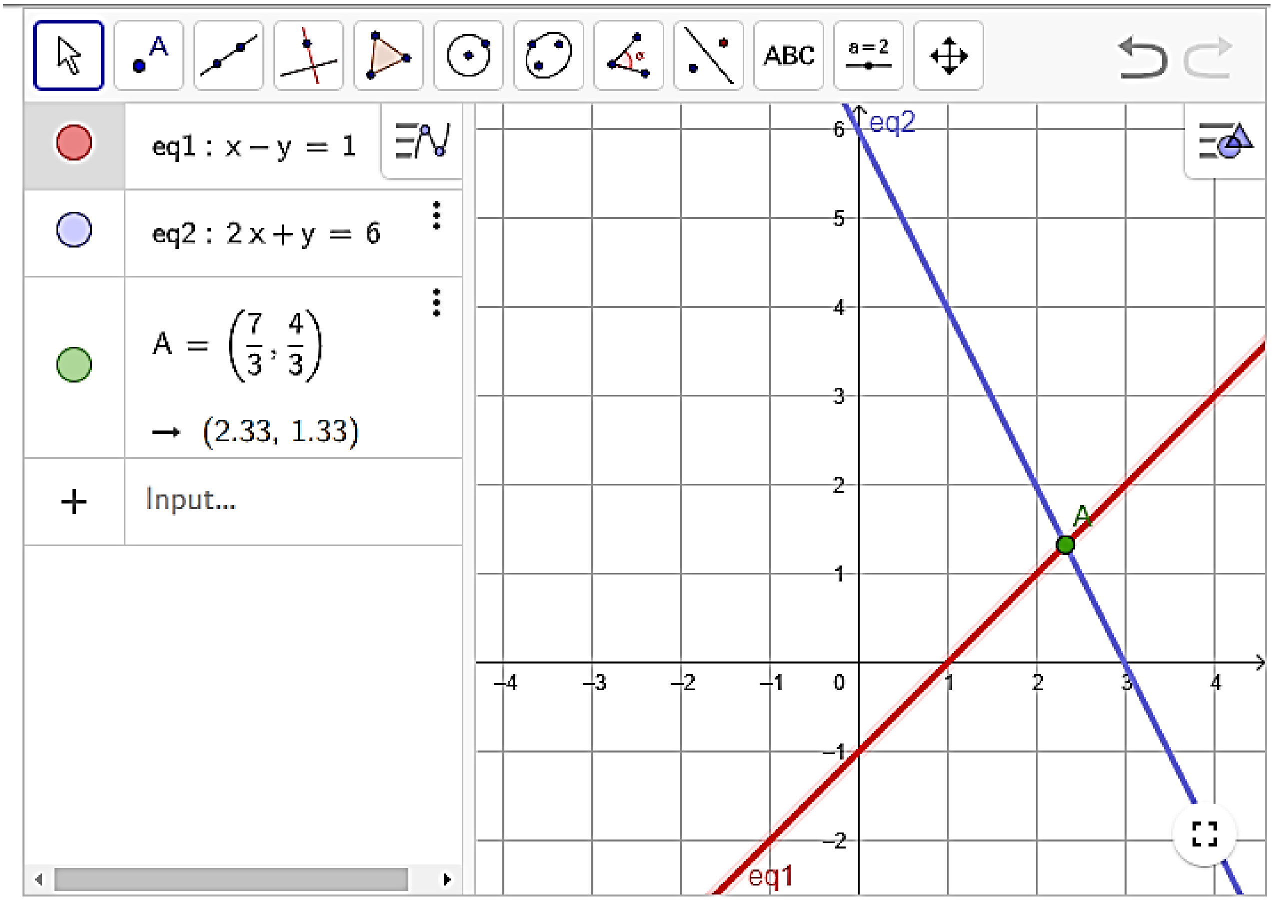
Thus, the system has the unique solution

$$x = \frac{7}{3}, \qquad y = \frac{4}{3}.$$

Solving Linear Systems

Geometrically, this means that the lines represented by the equations in the system intersect at the single point ($\frac{7}{3}, \frac{4}{3}$).

Using GeoGebra, we clearly can see that two lines are intersecting at point ($\frac{7}{3}, \frac{4}{3}$).



Solving Linear Systems

EXAMPLE 2 A Linear System with No Solutions:

Solve the linear system

$$\begin{cases} x + y = 4 \\ 3x + 3y = 6 \end{cases}$$

Solution: We can eliminate x from the second equation by adding -3 times the first equation to the second equation.

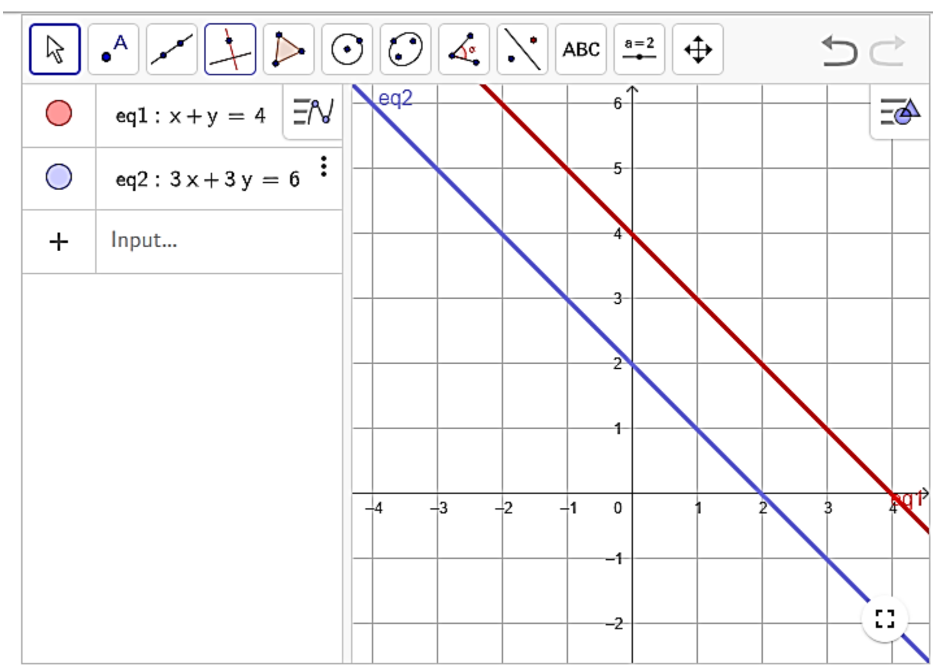
$$3x + 3y = 12$$

$$3x + 3y = 6$$

\- - - -

$$0 = 6 \text{ (Not True)}$$

This is not possible, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct.



Graphically, it shows that the lines have the same slope but different y-intercepts.

Solving Linear Systems

EXAMPLE 3 A Linear System with Infinitely many Solutions

Solve the linear system

$$\begin{cases} 4x - 2y = 1 \\ 16x - 8y = 4 \end{cases}$$

Solution: We can eliminate x from the second equation by adding -4 times the first equation to the second.

$$16x - 8y = 4$$

$$16x - 8y = 4$$

- - + -

$$0 = 0$$

Thus, the solutions of the system are those values of x and y that satisfy the single equation:

$$4x - 2y = 1 \Rightarrow x = \frac{1}{4} + \frac{1}{2}y.$$

Let $y = t$, So

$$x = \frac{1}{4} + \frac{1}{2}t.$$

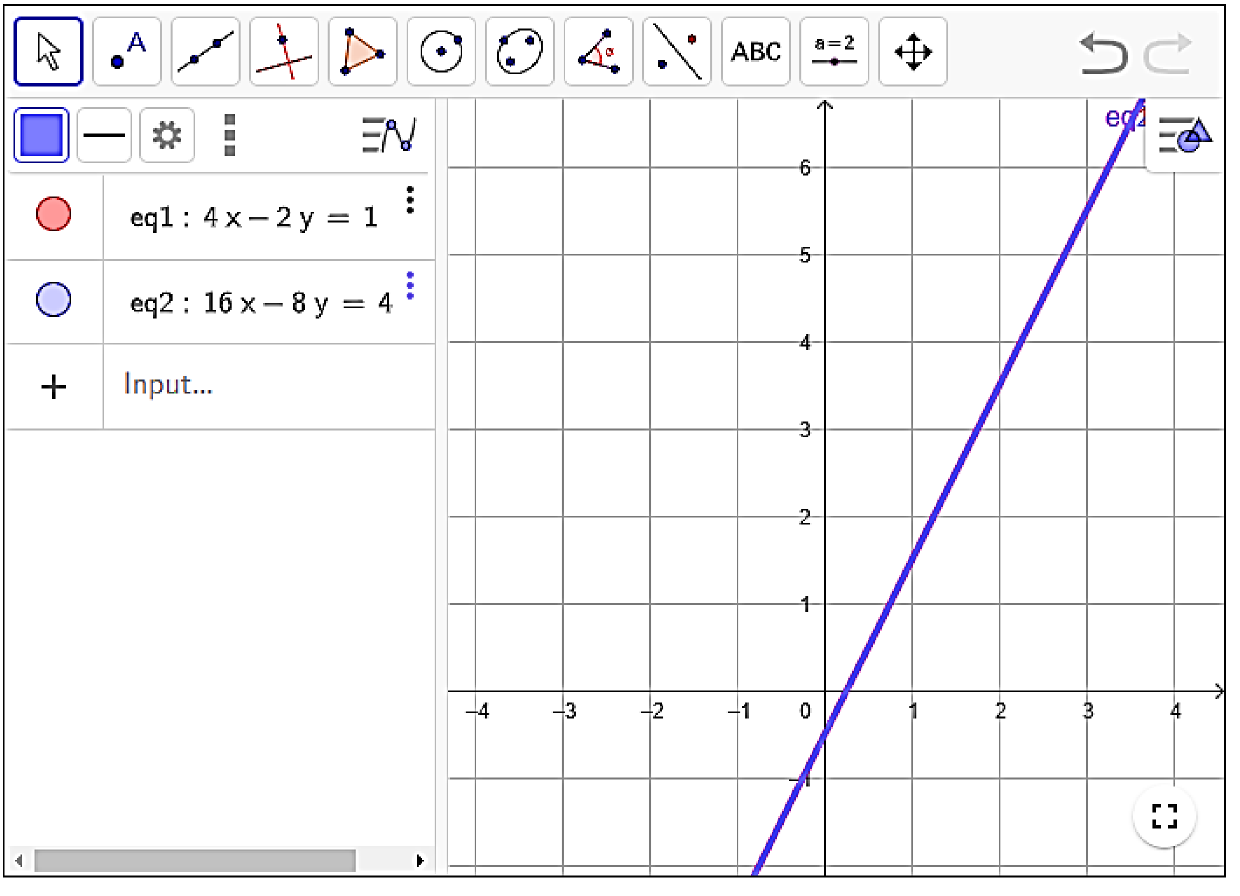
$\left(\frac{1}{4} + \frac{1}{2}\,t,\ t\right), t \in \mathbb{R}$ is solution of given system.

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter t. For example, $t = 0$, yields the solution $\left(\frac{1}{4}, 0\right)$, $t = 1$, yields the solution $\left(\frac{3}{4}, 1\right)$

Solving Linear Systems

$$\begin{cases} 4x - 2y = 1 \\ 16x - 8y = 4 \end{cases}$$

Graphically, it shows that the lines are overlapping. It means there are infinite points which satisfies the above system of linear equations.



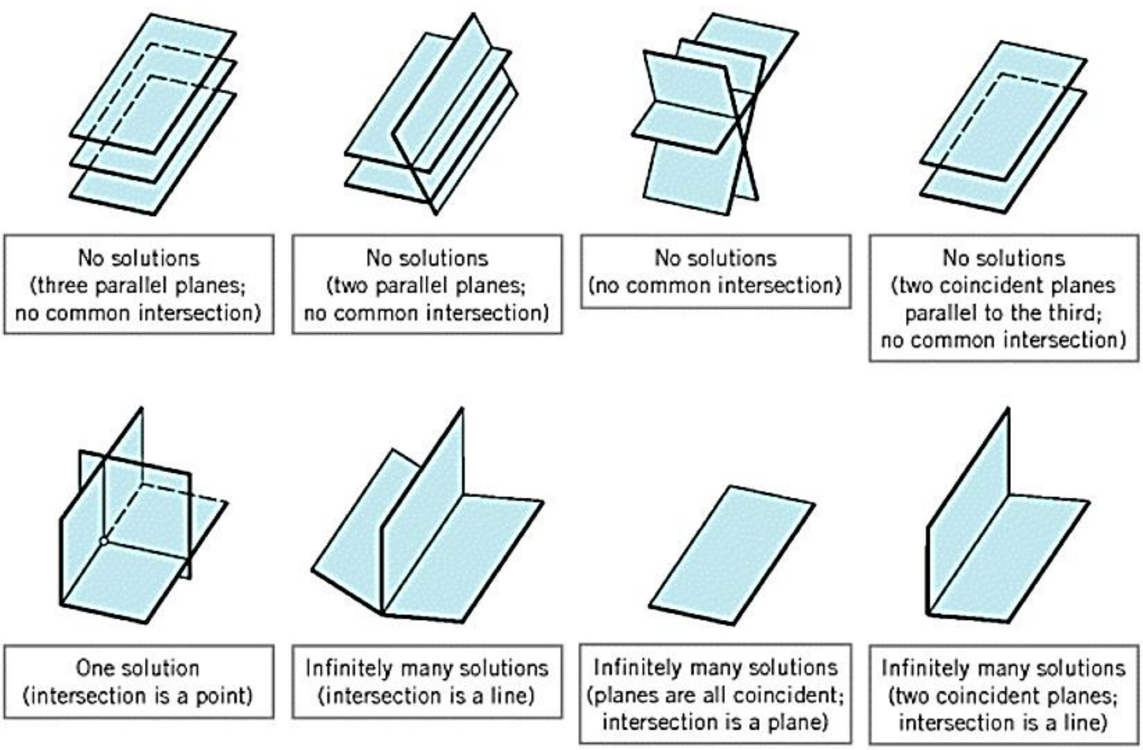
Solving Linear Systems

Linear Systems with Three Unknowns Geometrically

The same is true for a linear system of three equations in three unknowns:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities — no solutions, one



Important Note!

Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Solving Linear Systems

EXAMPLE 4 **Solve the linear system**

$$\begin{cases} x + y + 2z = 9 & \dashrightarrow (1) \\ 2x + 4y - 3z = 1 & \dashrightarrow (2) \\ 3x + 6y - 5z = 0 & \dashrightarrow (3) \end{cases}$$

Solution: **Subtracting (1) and (2), we get**

$$2x + 2y + 4z = 18$$
$$+2x + 4y - 3z = 1$$

\- $\Rightarrow -2y + 7z = 17 \quad \dashrightarrow (4)$

Subtracting (1) and (3), we get

$$3x + 3y + 6z = 27$$
$$+3x + 6y - 5z = 0$$

\- $\Rightarrow -3y + 11z = 27 \quad \dashrightarrow (5)$

Subtracting (4) and (5), we get

$$-6y + 21z = 51$$
$$-6y + 22z = 54$$

$$\Rightarrow -z = -3$$
$$z = 3$$

Put $z = 3$ **in equation (4), we get**

$$-2y + 7(3) = 17$$

$$-2y + 2 = 17$$
$$y = 2$$

Put $y = 2$ **and** $z = 3$ **in equation (1).**

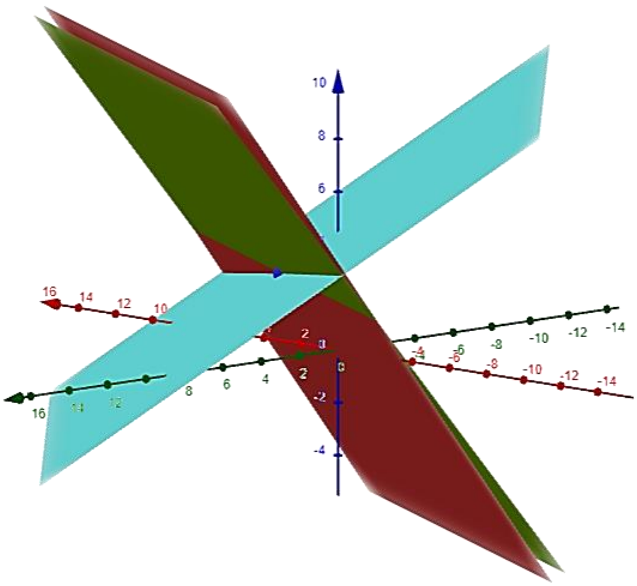
$$x = 9 - 2 - 2(3)$$
$$x = 1$$

So the solution of given system is $(1, 2, 3)$, **which is unique.**

Solving Linear Systems

Geometrically it represents that three planes intersect at unique point. **(1,2,3).**

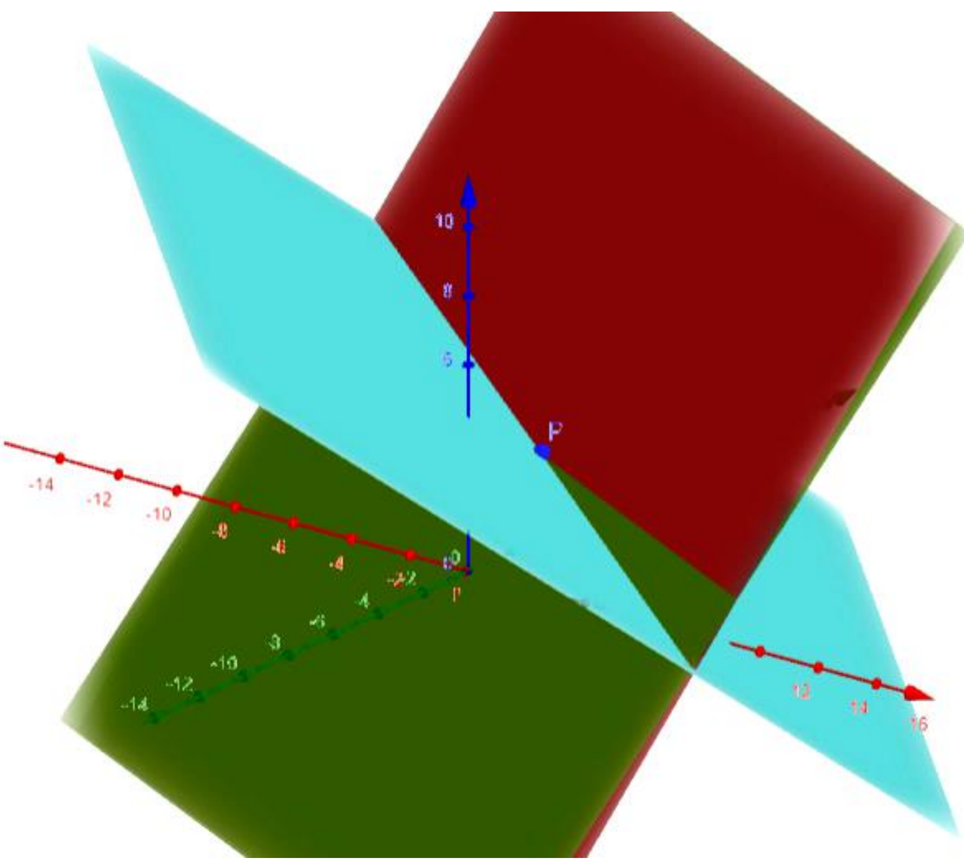
Position-I



GeoGebra 3D Calculator

eq1	$x + y + 2z = 9$
eq2	$2x + 4y - 3z = 1$
eq3	$3x + 6y - 5z = 0$
P	$P = (1, 2, 3)$

GeoGebra Command



Solving Linear Systems

EXAMPLE 5 Solve the linear system

$$\begin{cases} x - y + 2z = 5 \\ 2x - 2y + 4z = 10 \\ 3x - 3y + 6z = 15 \end{cases}$$

Solution: This system can be solved **by inspection**, since the second and third equations are multiples of the first.

Geometrically, this means that the three planes coincide and that those values of **x, y, and z** that satisfy the equation

$$x - y + 2z = 5 \qquad \rightarrow \qquad (1)$$

Automatically satisfy all three equations.

Thus, it suffices to find the solutions of (1). We can do this by first solving (1) for **x** in terms of **y** and **z**, then assigning arbitrary values s and t (parameters) to these two variables and then expressing the solution by the three parametric equations

$$y = t, z = s \Rightarrow x = 5 + t - 2s.$$

So $(5 + t - 2s, t, s)$ is the solution of the above system.

Specific solutions can be obtained by choosing numerical values for the **parameters t and s**. For example, taking $t = 1$ and $s = 0$ yields the solution $(6, 1, 0)$.

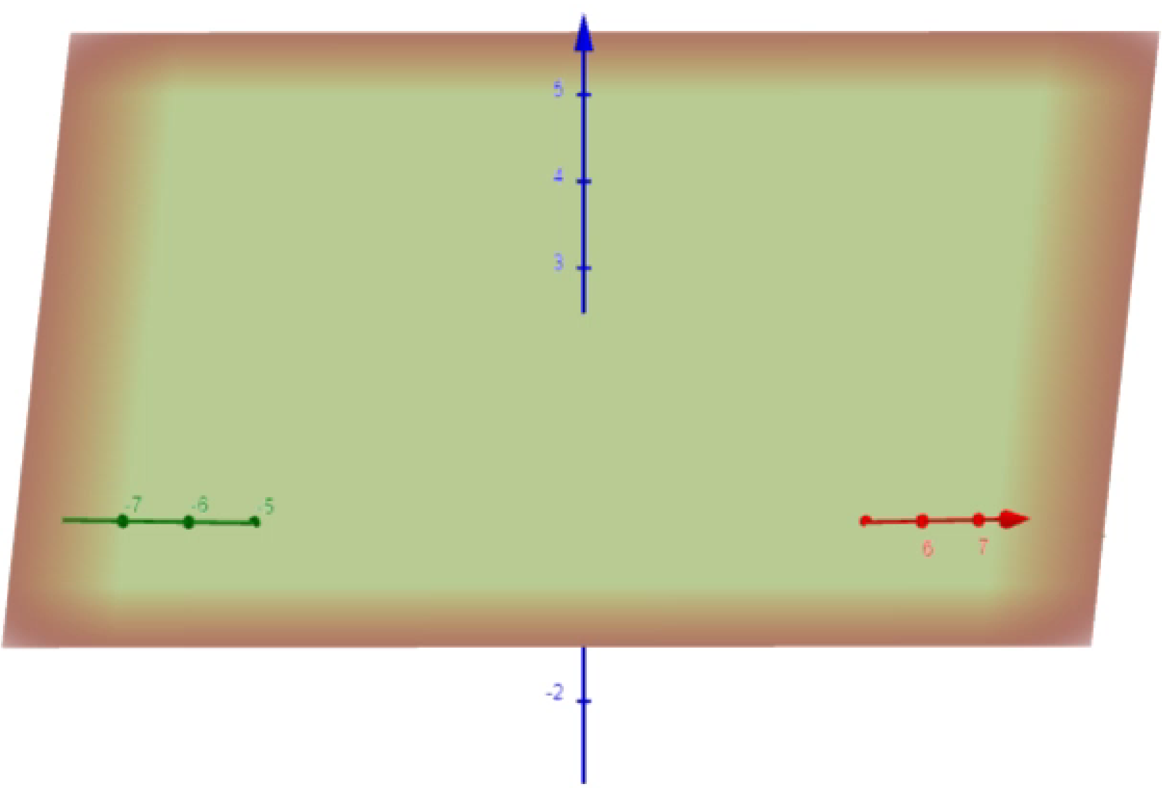
Geometrically it represents that three planes intersect at unique point. (1,2,3).

GeoGebra Command

$eq1: x - y + 2z = 5$

$eq2: 2x - 2y + 4z = 10$

$eq3: 3x - 3y + 6z = 15$



Solving Linear Systems

Exercise:

1. **Solve the following linear system of equations:**

a)
$$\begin{cases} 2x + 4y + 6z = -12 \\ 2x - 3y - 4z = 15 \\ 3x + 4y + 5z = -8 \end{cases}$$

b)
$$\begin{cases} x + y = 5 \\ 3x + 3y = 10 \end{cases}$$

c)
$$\begin{cases} 2x + 3y = 13 \\ x - 2y = 3 \\ 5x + 2y = 27 \end{cases}$$

2. **In each part, find the solution set of the linear equation by using parameters as necessary.**

a) $7x - 5y = 3$

b) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

Work to do

Exercise 1.1 **Q**6-14