Linear Algebra

Lecture No. 2

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Presentation Overview

Homogeneous Linear System

A homogeneous system of linear equations is a linear system of equations in which there are no constant terms. i.e, a homogeneous linear system of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

This is a system in 'n' unknowns $(x_1, x_2, ..., x_n)$, and in each equation, the constant term is O.

Example

$$2x - 3y = 0$$
$$-4x + 6y = 0$$

- Note that homogeneous systems are always consistent. This is because all of the variables can be set equal to zero. i.e., $x = 0$ and $y = 0$ to satisfy all of the equations. This special solution, $(0, 0, ..., 0)$, is called the **trivial solution**. Any other solution of a homogeneous system is called a **non-trivial solution**.

Continue...

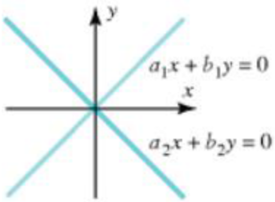
- A homogeneous system always has the trivial solution. There are only two possibilities for its solutions:
  - The system has only the trivial solution.
  - The system has infinite many solutions in addition to the trivial solutions(This happened when the system involves more unknowns than equations.).
- A homogeneous linear system of two equations in two unknowns are of the form:

$$a_1x + b_1y = 0$$

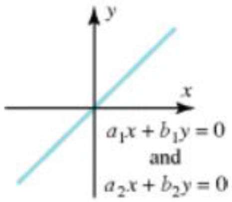
$$a_2x + b_2y = 0$$

Continue...

- The graph of the equations are the lines passing through the origin.



Only the trivial solution



Infinitely many solutions

Non-homogeneous Linear System

A non-homogeneous system has the form:

$$Ax = b$$

Example:

$$3x - 2y = 6$$
$$-x + 7y = 3$$

- A non-homogeneous system with more unknowns than equations need not to be consistent.
- If a non-homogeneous system with more unknowns then equations is consistent then it has infinitely many solutions.

Example

Example 1:

Solve the linear system of equation:

$$\begin{cases} y + 3z = 4 \quad ----\rightarrow (1) \\ x - y + z = 1 ----\rightarrow (2) \\ 3x - y + 9z = 11 ----\rightarrow (3) \end{cases}$$

Solution:

Multiply equation (2) by 3 and subtract equation (3)

$$\begin{array}{rcrcrcr} 3x & - & 3y & + & 3z & = & 3 \\ 3x & - & y & + & 9z & = & 11 \\ - & & + & & - & & - \end{array}$$

$$-2y - 6z = -8 ----\rightarrow (4)$$

Multiply equation (1) by 2 and add in equation (4)

$$2y + 6z = 8$$
$$-2y - 6z = -8$$

$$0 = 0$$

This means this system has infinitely many solutions.
Now from (1)

$$y = 4 - 3z - - - - \longrightarrow (5)$$

$$x = 1$$

Put value of y in equation (1).

$$x - (4 - 3z) + z = 1$$

$$x - 4 + 3z + z = 1$$

$$x + 4z = 1 + 4$$

$$x + 4z = 5$$

$$x = 5 - 4z$$

Let z=t an arbitrary parameter, then the solution can be written as,

$$x = 5 - 4t, y = 4 - 3t, z = t$$

S.S=$\{(5-4t, 4 - 3t, t)\}, t \in R$

If t=0 then S.S={5,4,0}
If t=1 then S.S={1,1,1}

Definitions (Augmented Matrix)

An augmented matrix for a system of equations is a matrix of numbers in which each row represents the constants from one equation(both the coefficients and the constant on the other side of the equal sign) and each column represents all the coefficients for a single variable.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Continue...

The above system can be written in rectangular array of numbers:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_m \end{bmatrix}$$

This is called the augmented matrix for the system.

Example 1:

Solve the linear system of equation:

$$\begin{cases} x - 2y + 3z = 7 \;-----\rightarrow (1) \\ 2x + y + z = 4 \;-----\rightarrow (2) \\ -3x + 2y - 2z = -10 \;-----\rightarrow (3) \end{cases}$$

Here is the augmented matrix for this system

$$\begin{bmatrix} 1 & -2 & 3 & : & 7 \\ 2 & 1 & 1 & : & 4 \\ -3 & 2 & -2 & : & -10 \end{bmatrix}$$

Now, we need to discuss elementary row operations. There are three of them and we will give both the notation used for each one as well as an example using the augmented matrix given above.

1. **Interchange Two Rows:**

With this operation we will change all the entries in row i and row j. The notation we will use here is $R_i \longleftrightarrow R_j$. Here is an example

$$\begin{bmatrix} 1 & -2 & 3 & : & 7 \\ 2 & 1 & 1 & : & 4 \\ -3 & 2 & -2 & : & -10 \end{bmatrix} \quad R_1 \leftrightarrow R_3 \quad \begin{bmatrix} -3 & 2 & -2 & : & -10 \\ 2 & 1 & 1 & : & 4 \\ 1 & -2 & 3 & : & 7 \end{bmatrix}$$

Continue...

2. **Multiply a Row by a Constant:**
In this operation we will multiply row i by a constant c and the notation will use here is $c \times R_i$. Note that we can also divide a row by a constant using the notation $\frac{1}{c} \times R_i$,Here is an example

$$\begin{bmatrix} 1 & -2 & 3 & : & 7 \\ 2 & 1 & 1 & : & 4 \\ -3 & 2 & -2 & : & -10 \end{bmatrix} \quad -4R_3 \rightarrow \quad \begin{bmatrix} 1 & -2 & 3 & : & 7 \\ 2 & 1 & 1 & : & 4 \\ 12 & -8 & 8 & : & 40 \end{bmatrix}$$

|

3. **Add a Multiple of a Row to Another Row:**
In this operation we will replace row i with the sum of row i and a constant c times row j. The notation will use this operation is $R_i + c \times R_j \rightarrow R_i$. To perform this operation we will take an entry from row i and add to it c, times the corresponding entry from row j and put the result back into row i.Here is an example of this operation.

$$\begin{bmatrix} 1 & -2 & 3 & : & 7 \\ 2 & 1 & 1 & : & 4 \\ -3 & 2 & -2 & : & -10 \end{bmatrix} \quad R_3 - 4R_1 \rightarrow R_3 \quad \begin{bmatrix} 1 & -2 & 3 & : & 7 \\ 2 & 1 & 1 & : & 4 \\ -7 & 10 & -14 & : & -38 \end{bmatrix}$$

Lets check the individual computation.
$-3 - 4(1) = -7$
$2 - 4(-2) = 10$
$-2 - 4(3) = -14$
$-10 - 4(7) = -38$

Continue...

Example 2:

Solve the linear system using elementary row operations.

$$\begin{cases} -3x_1 + 2x_2 + 4x_3 = 12 \\ x_1 - 2x_3 = -4 \\ 2x_1 - 3x_2 + 4x_3 = -3 \end{cases}$$

Solution:

The Augmented Matrix is:

$$\begin{bmatrix} -3 & 2 & 4 & : & 12 \\ 1 & 0 & -2 & : & -4 \\ 2 & -3 & 4 & : & -3 \end{bmatrix}$$

Interchange Row 1 (R_1) and Row 2(R_2) : $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & -2 & : & -4 \\ -3 & 2 & 4 & : & 12 \\ 2 & -3 & 4 & : & -3 \end{bmatrix}$$

Continue...

Add $3R_1$ to R_2 : $3R_1+R_2$

$$\begin{bmatrix} 1 & 0 & -2 & : & -4 \\ 0 & 2 & -2 & : & 0 \\ 2 & -3 & 4 & : & -3 \end{bmatrix}$$

Add $-2R_1$ to R_3 : $-2R_1+R_3$

$$\begin{bmatrix} 1 & 0 & -2 & : & -4 \\ 0 & 2 & -2 & : & 0 \\ 0 & -3 & 8 & : & 5 \end{bmatrix}$$

Continue...

Multiply R_2 by $\frac{1}{2}$: $\frac{1}{2}R_2$

$$\begin{bmatrix} 1 & 0 & -2 & : & -4 \\ 0 & 1 & -1 & : & 0 \\ 0 & -3 & 8 & : & 5 \end{bmatrix}$$

Add $3R_2$ to R_3 : $3R_2+R_3$

$$\begin{bmatrix} 1 & 0 & -2 & : & -4 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 5 & : & 5 \end{bmatrix}$$

Continue...

Multiply R_3 by $\frac{1}{5}$: $\frac{1}{5}R_3$

$$\begin{bmatrix} 1 & 0 & -2 & : & -4 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \longrightarrow \text{Echelon Form}$$

By Back Substitution

$$x_3 = 1$$

$$x_2 - x_3 = 0 \Longrightarrow x_2 = x_3 = 1$$

$$x_1 - 2x_3 = -4 \Rightarrow x_1 - 2(1) = -4 \Rightarrow x_1 = -4 + 2 = -2$$

Continue...

So $(-2,1,1)$ is the solution of the above system.

Finding the solution of linear system by converting the augmented matrix into echelon form is called **Gauss Elimination Method**.

Gauss Jordan Method

Finding the solution of linear system by converting the augmented matrix into reduced echelon form is called Gauss Jordan Method. For this consider the above example.

$$\begin{bmatrix} 1 & 0 & -2 & : & -4 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Add Row 3 multiplied by 2 to row 1: R_1+2R_3

$$\begin{bmatrix} 1 & 0 & 0 & : & -2 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Add Row 3 to row 2: R_2+R_3

$$\begin{bmatrix} 1 & 0 & 0 & : & -2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

So, $x=-2, y=1, z=1$

So $(-2,1,1)$ is the solution of the above system.

Example 3:

Solve the linear system of equation.

$$\begin{cases} x + 4y - z = 12 \\ 3x + 8y - 2z = 4 \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 4 & -1 & : & 12 \\ 3 & 8 & -2 & : & 4 \end{bmatrix}$$

Subtract row 1 multiplied by 3 from row 2: $R_2 - 3R_1$

$$\begin{bmatrix} 1 & 4 & -1 & : & 12 \\ 0 & -4 & 1 & : & -32 \end{bmatrix}$$

Divide row 2 by -4: $\frac{1}{-4}(R_2)$

Continue...

$$\begin{bmatrix} 1 & 4 & -1 & : & 12 \\ 0 & 1 & \frac{-1}{4} & : & 8 \end{bmatrix}$$

Subtract row 2 multiplied by 4 from row 1: $R_1 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 & : & -20 \\ 0 & 1 & \frac{-1}{4} & : & 8 \end{bmatrix}$$

By back substitution

$$y - \frac{1}{4}z = 8 \Longrightarrow y = \frac{1}{4}z + 8$$

$$x = -20$$

Let $z = t$

$$y = \frac{1}{4}t + 8$$

So the solution of this system is $(-20, 8 + \frac{1}{4}t, t)$

Examples

Example 4:

$$\begin{cases} 3x - 4y - z = 1 \\ 2x - 3y + z = 3 \\ x - 2y + 3z = 2 \end{cases}$$

Solution:

$$\begin{bmatrix} 3 & -4 & -1 & : & 1 \\ 2 & -3 & 1 & : & 3 \\ 1 & -2 & 3 & : & 2 \end{bmatrix}$$

Divide Row 1 by 3:$\frac{1}{3}R_1$

$$\begin{bmatrix} 1 & \frac{-4}{3} & \frac{-1}{3} & : & \frac{1}{3} \\ 2 & -3 & 1 & : & 3 \\ 1 & -2 & 3 & : & 2 \end{bmatrix}$$

|

Subtract Row 1 multiplied by 2 from Row 2:$R_2 - 2R_1$

$$\begin{bmatrix} 1 & \frac{-4}{3} & \frac{-1}{3} & : & \frac{1}{3} \\ 0 & \frac{-1}{3} & \frac{5}{3} & : & \frac{1}{3} \\ 1 & -2 & 3 & : & 2 \end{bmatrix}$$

Subtract Row 1 from Row 3:$R_3 - R_1$

$$\begin{bmatrix} 1 & \frac{-4}{3} & \frac{-1}{3} & : & \frac{1}{3} \\ 0 & \frac{-1}{3} & \frac{5}{3} & : & \frac{1}{3} \\ 0 & \frac{-2}{3} & \frac{10}{3} & : & \frac{5}{3} \end{bmatrix}$$

Multiply Row 2 by -3:-3 R_2

$$\begin{bmatrix} 1 & \frac{-4}{3} & \frac{-1}{3} & : & \frac{1}{3} \\ 0 & 1 & -5 & : & -7 \\ 0 & \frac{-2}{3} & \frac{10}{3} & : & \frac{5}{3} \end{bmatrix}$$

Examples

Add Row 2 multiplied by $\frac{4}{3}$ to Row 1:$R_1 + \frac{4}{3}R_2$

$$\begin{bmatrix} 1 & 0 & -7 & : & -9 \\ 0 & 1 & -5 & : & -7 \\ 0 & \frac{-2}{3} & \frac{10}{3} & : & \frac{5}{3} \end{bmatrix}$$

Add Row 2 multiplied by $\frac{2}{3}$ to Row 3:$R_3 + \frac{2}{3}R_2$

$$\begin{bmatrix} 1 & 0 & -7 & : & -9 \\ 0 & 1 & -5 & : & -7 \\ 0 & 0 & 0 & : & -3 \end{bmatrix}$$

Now by back substitution $0 = -3$ which is not possible. So the system has no solution.

Examples

Example 5:

$$\begin{cases} 2x + 7y = -5 \\ x + 4y = 2 \end{cases}$$

Solution:

$$\begin{bmatrix} 2 & 7 & : & -5 \\ 1 & 4 & : & 2 \end{bmatrix}$$

Divide Row 1 by 2: $\frac{1}{2}R_1$

$$\begin{bmatrix} 1 & \frac{7}{2} & : & -\frac{5}{2} \\ 1 & 4 & : & 2 \end{bmatrix}$$

Subtract Row 1 from row 2: $R_2 - R_1$

$$\begin{bmatrix} 1 & \frac{7}{2} & : & -\frac{5}{2} \\ 0 & \frac{1}{2} & : & \frac{9}{2} \end{bmatrix}$$

Examples

Multiply Row 2 by 2: $2R_2$

$$\begin{bmatrix} 1 & \frac{7}{2} & : & -\frac{5}{2} \\ 0 & 1 & : & 9 \end{bmatrix}$$

Subtract Row 2 multiplied by $\frac{7}{2}$ from Row 1: $R_1 - \frac{7}{2}R_2$

$$\begin{bmatrix} 1 & 0 & : & -34 \\ 0 & 1 & : & 9 \end{bmatrix}$$

By Back Substitution

$$x = -34, y = 9$$

So $(-34{,}9)$ is the solution of the above system.

Exercise

Question Solve the linear system of equations using Gauss elimination method:

$$\begin{cases} x + z + 2w = 6 \dashrightarrow (1) \\ y - 2z = -3 \dashrightarrow (2) \\ x + 2y - z = -2 \dashrightarrow (3) \\ 2x + y + 3z - 2w = 0 \dashrightarrow (4) \end{cases}$$

Solution: Do yourself

Continue...

Howard Anton (Exercise 1.2)

Q1. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

Q3. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Continue...

Solution ©:

$$x_4 + 3x_5 = 9 \text{---------------} x_4 = 9 - 3x_5 \text{-}$$

$$x_3 + x_4 + 6x_5 = 5$$

$$x_1 + 7x_2 - 2x_3 - 8x_5 = -3$$

Put value of x_4 in eq 2

$$x_3 + 9 - 3x_5 + 6x_5 = 5$$

$$x_3 = -4 - 3x_5$$

Put value of x_3 in eq 3

$$x_1 + 7x_2 - 2(-4 - 3x_5) - 8x_5 = -3$$

Continue...

$$x_1 + 7x_2 + 8 + 6x_5 - 8x_5 = -3$$

$$x_1 + 7x_2 = -11 + 2x_5$$

$$x_1 = 11 + 2x_5 - 7x_2$$

Let $x_2 = t, x_5 = s$

$$x_1 = 11 + 2s - 7t$$

$$x_3 = -4 - 3s$$

$$x_4 = 9 - 3s$$

Continue...

Work to do:

Q5 Solve the linear system of equation by using Gauss elimination method

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Howard Anton (Exercise 1.2)

21.
$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

Answer:

$w = t,\ x = -t,\ y = t,\ z = 0$

22.
$$\begin{aligned} x_1 + 3x_2 + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

23.
$$\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$$

Continue...

Determine the values of a for which the system has no solutions, exactly one solution, or infinite many solutions.

25.
$$\begin{aligned} x + 2y - \quad 3z &= 4 \\ 3x - y + \quad 5z &= 2 \\ 4x + y + \left(a^2 - 14\right)z &= a + 2 \end{aligned}$$

Answer:

If $a = 4$, there are infinitely many solutions; if $a = -4$, there are no solutions; if $a \neq \pm 4$, there is exactly one solution.


Thank you