Chain Rule (Contd.)

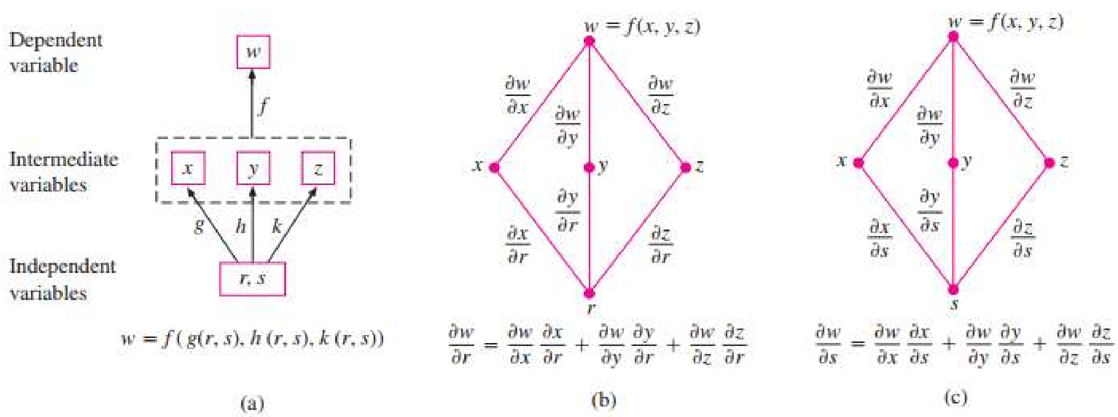
Functions Defined on Surfaces:

If we are interested in the temperature $w = w(x, y, z)$ on a globe in space, we might prefer to think of x, y and z as functions of variables r and s that gives points' longitudes & latitudes. If $x = x(r, s), y = y(r, s)$ and $z = z(r, s)$, we could then express the temperature as a function of r and s with the composite function.

$$w = w(x(r, s), y(r, s), z(r, s))$$

Under the right conditions, w could have partial derivatives with respect to both r and s that could be calculated in the following way.

Chain Rule for Two independent and Three Intermediate Variables



Suppose that $w = w(x, y, z)$; $x = x(r, s), y = y(r, s)$ and $z = z(r, s)$. If all the four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example 3

Express $\frac{\partial w}{\partial r}$and $\frac{\partial w}{\partial s}$ in terms of r and s if

$$w = x + 2y + z^2;\ x = \frac{r}{s},\ \ y = r^2 + \ln s\,,\ \ z = 2r$$

Solution:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)\,(2r) + (2z)\,(2)$$

$$= \left(\frac{1}{s}\right) + 4r + (4r)\,(2)$$

$$= \left(\frac{1}{s}\right) + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)\,(0)$$

$$= \left(\frac{2}{s}\right) - \left(\frac{r}{s^2}\right)$$

Remark 1:

If f is a function of two variables instead of three, each equation becomes correspondingly one term shorter.

If $w = f(x, y);\ x = g(r, s), y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Example 4:

Express $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial r}$ in terms of s and r

$$w = x^2 + y^2, \qquad x = r - s, \quad y = r + s$$

Solution: $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$

$$= (2x)(1) + (2y)(1)$$

$$= 2(r - s) + 2(r + s) = 4r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$= (2x)(-1) + (2y)(1)$$

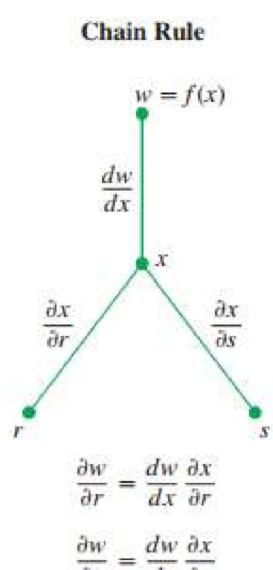
$$= -2(r - s) + 2(r + s)$$

$$= 4s$$

Remark 2: If f is a function of x alone, our equations become even simpler.

If $w = w(x);\ x = x(r, s)$, then

$\frac{\partial w}{\partial r} = \frac{dw}{dx}.\frac{\partial x}{\partial r}$ and $\frac{\partial w}{\partial s} = \frac{dw}{dx}.\frac{\partial x}{\partial s}$



Chain Rule: Two Independent Variables

In exercise 7 and 8, (a) express $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as a function of u and vby using chain rule. Then (b) evaluate $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ at the given point (u, v).

1. $z = 4e^x \ln y, \; x = \ln(u \cos v), \; y = u \sin v; \; (u, v) = \left(2, \frac{\pi}{4}\right)$
2. $z = \tan^{-1}\left(\frac{x}{y}\right), \; x = u \cos v, \; y = u \sin v; \; (u, v) = \left(1.3, \frac{\pi}{6}\right)$