Calculus and Analytical Geometry

Lecture no. 5

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Topic: Limit of a Function

Outline of the lecture:

1. Introduction
   - x approaches to zero
   - x approaches to infinity
   - x approaches to a
2. Limit of a function
   - Left hand limit of a function
   - Right hand limit of a function
   - Two-sided limit of a function
3. Graphical representation of limit of a function
   - Examples
4. Practice questions

1) Introduction:

Finding limit at a point means finding height of the function near that point.

To understand the concept of limit we need to understand the meaning of following phrases:

- x approaches to zero, i.e., $x \to 0$
- x approaches to infinity, i.e., $x \to \infty$
- x approaches to a, i.e., $x \to a$

1.1) x approaches to zero, i.e., $x \to 0$

Suppose the series of values as:

$$x = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$
$$x = 1, 0.5, 0.25, 0.1250, 0.0625, \dots$$

As you can see the values are approaching towards 0 because the values are decreasing and becoming smaller and smaller. The unending decrease of x is written as $x \to 0$, which means x is approaching towards 0 but it is not exactly 0.

1.2) x approaches to infinity, i.e., $x \to \infty$

Suppose a variable x in the form of the values

$$x = 1, 10, 100, 1000, 10000, \dots$$

The values of x are increasing and these values are approaching towards ∞.

1.3) x approaches to a, i.e., $x \to a$

$x \to a$ means x approaching towards a specific number a or it is getting close to the number a, but different from a from both left and right side of a.

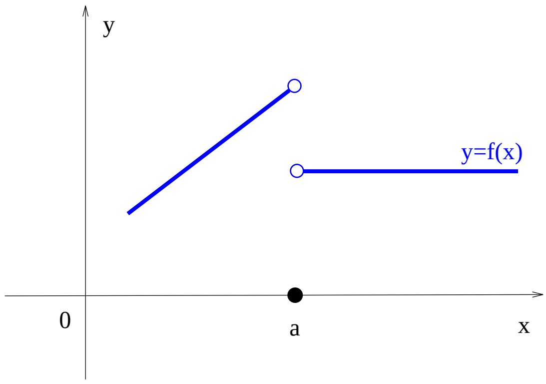
2. Limit of a function:

- **Left-Hand limit:** The height l_1 of the function f as x approaches a from the left is called the left-hand limit, and is denoted by

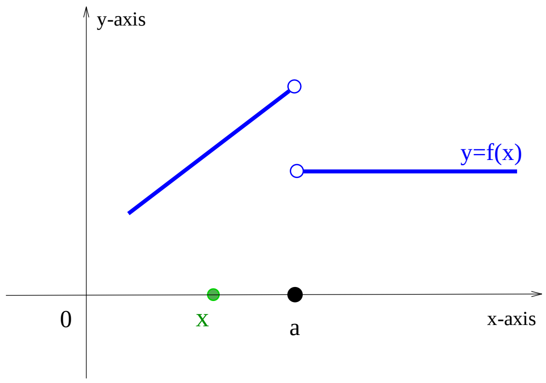
$$\lim_{x \to a^-} f(x) = L_1$$

How to find the Left-Hand Limit: The following is the geometrical approach.

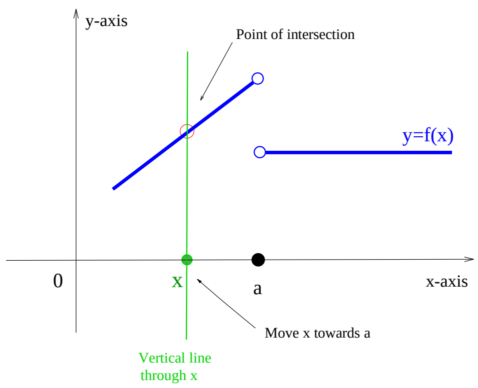
Step 1. Mark the point a at which the left-hand limit is to find.



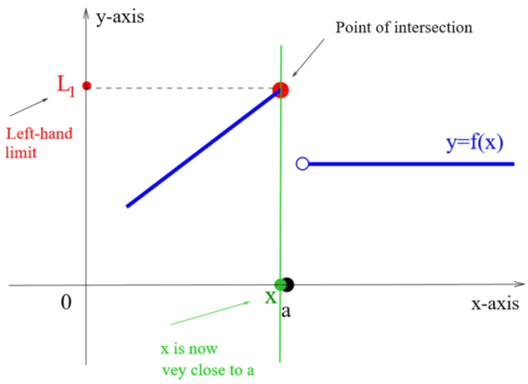
Step 2. Mark a point x on the left of a.



Step 3. Draw a vertical line through x so that it intersects the graph of f.



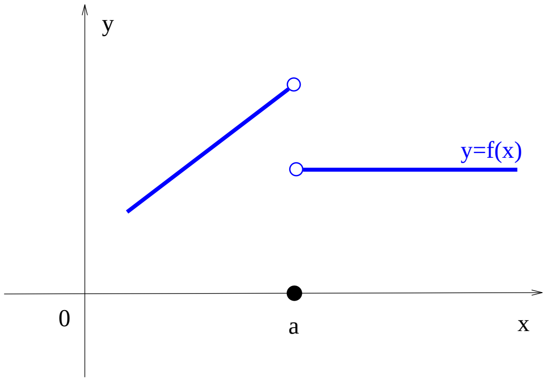
Step 4. Move x towards a. As x gets very close to a, the height of the point of intersection of the vertical line (through x) and the graph is the left-hand limit. In the following figure it is L_1.



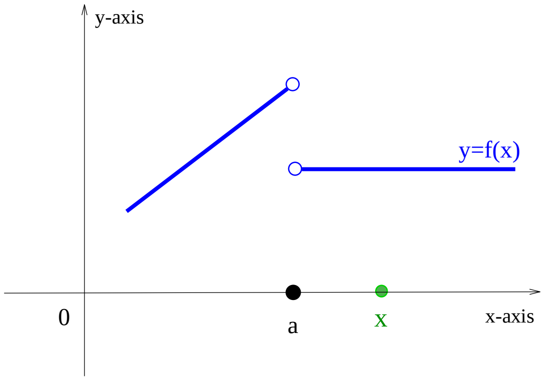
• **Right-Hand Limit:** The height l_2 of the function *f* as *x* approaches *a* from the right is called the right-hand limit, and is denoted by $\lim_{x \to a^+} f(x) = L_2$

How to find the Left-Hand Limit: The following is the geometrical approach.

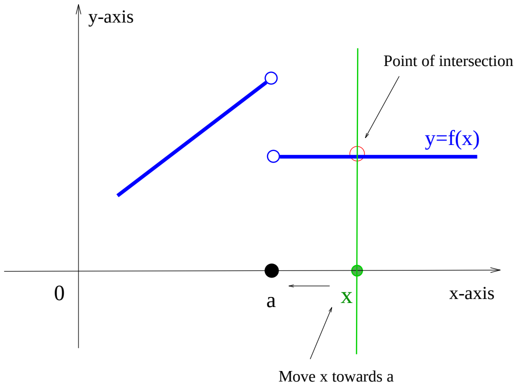
Step 1. Mark the point *a* at which the right-hand limit is to find



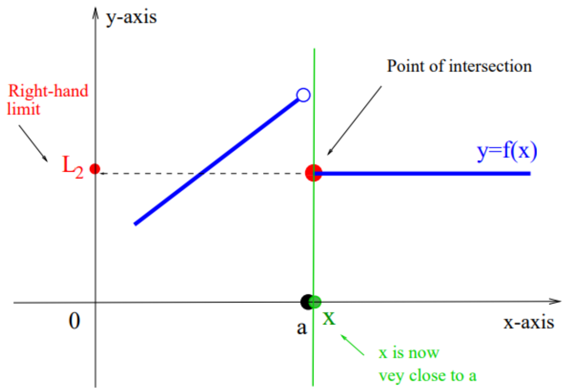
Step 2. Mark a point *x* on the right of *a*.



Step 3. Draw a vertical line through *x* so that it intersects the graph of *f*.



Step 4. Move *x* towards *a*. As *x* gets very close to *a*, the height of the point of intersection of the vertical line (through *x*) and the graph is the left-hand limit. In the following figure it is L_2.



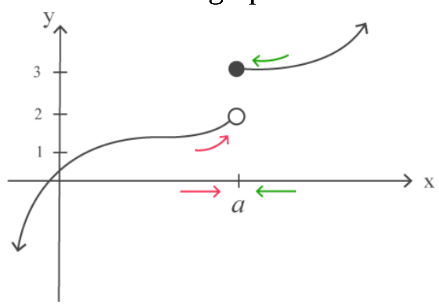
- **Two-sided limit of the function:** A function f is said to have a two-sided limit at a point $x = a$, if both the left-hand limit and right-hand limit are same, say L. i.e.,

$$\lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L \quad \text{then } \lim_{x\to a} f(x) = L$$

3. <u>Graphical concept of a limit:</u>

After the understanding of basic concept of limit, now we'll understand that if a graph of a function is given then how can we find the limit of that function at any point a.

Example 3.1: Consider the graph of a function

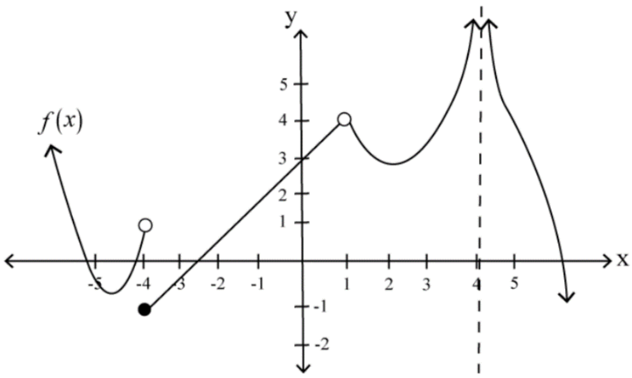


Solution:

$\lim_{x\to a^-} f(x) = 2$ and $\lim_{x\to a^+} f(x) = 3$

As left and right-hand side is not equal $\lim_{x\to a} f(x) =$ does not exist.

Example 3.2: Consider another function and find the limit at $\lim_{x\to -4} f(x)$, $\lim_{x\to 1} \text{f(x)}$, $\lim_{x\to 4} f(x)$



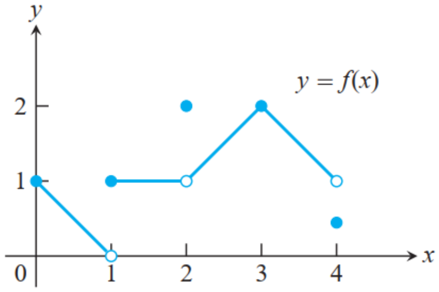
Solution:

a) $\lim_{x\to -4^-} f(x) = 1$, $\lim_{x\to -4^+} f(x) = -1$, $\lim_{x\to -4} f(x) = does$ not exist

b) $\lim_{x\to 1^-} f(x) = 4$, $\lim_{x\to 1^+} f(x) = 4$, $\lim_{x\to 1} f(x) = 4$

c) $\lim_{x\to 4^-} f(x) = \infty$, $\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to 4} f(x) = \infty$

Example 3.2: find the limit at $x = 0,1,2,3$ and 4 of the following function



Solution:

a) $\lim_{x\to 0^-} f(x) = does\ not\ exist$, $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0} f(x) = does$ not exist

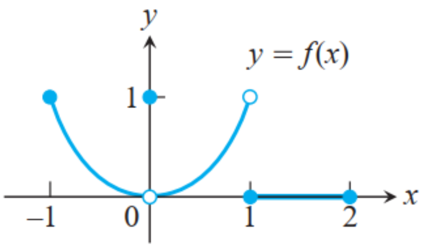
b) $\lim_{x\to 1^-} f(x) = 0$, $\lim_{x\to 1^+} f(x) = 1$, $\lim_{x\to 1} f(x) = does$ not exist

c) $\lim_{x\to 2^-} f(x) = 1$, $\lim_{x\to 2^+} f(x) = 1$, $\lim_{x\to 2} f(x) = 1$ even though f(2)=2

d) $\lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to 3^+} f(x) = 2$, $\lim_{x\to 3} f(x) = 2$

e) $\lim_{x\to 4^-} f(x) = 1$ even though f(4)$\neq$ 1 , $\lim_{x\to 4^+} f(x) = does\ not\ exist$,
$\lim_{x\to 4} f(x) = does$ not exist

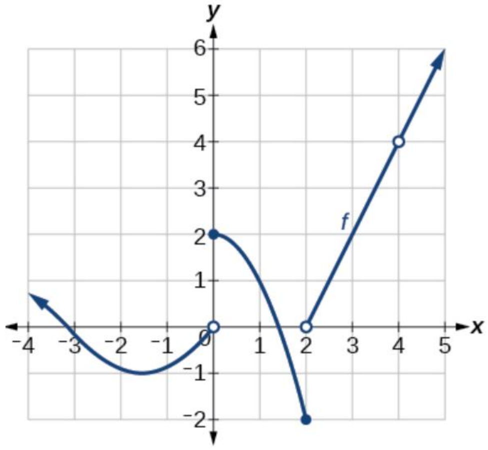
Example 3.4: which of the following statements about the graph y=f(x) are true?



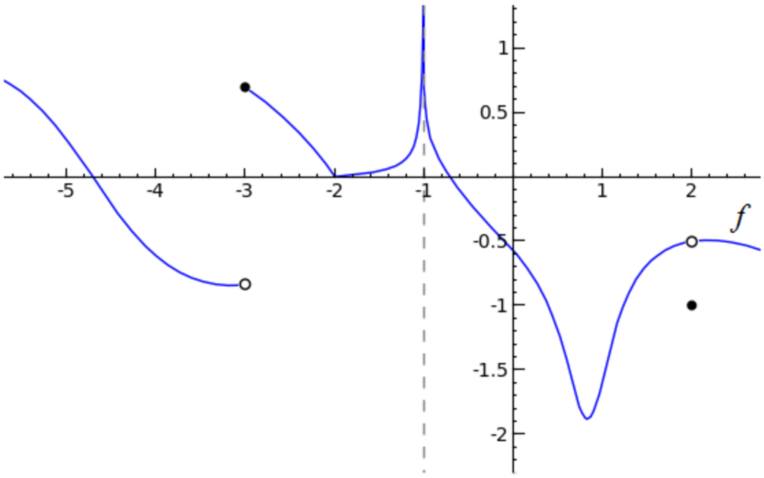
a. $\lim_{x\to -1^+} f(x) = 1$	**b.** $\lim_{x\to 0^-} f(x) = 0$
c. $\lim_{x\to 0^-} f(x) = 1$	**d.** $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$
e. $\lim_{x\to 0} f(x)$ exists	**f.** $\lim_{x\to 0} f(x) = 0$
g. $\lim_{x\to 0} f(x) = 1$	**h.** $\lim_{x\to 1} f(x) = 1$
i. $\lim_{x\to 1} f(x) = 0$	**j.** $\lim_{x\to 2^-} f(x) = 2$
k. $\lim_{x\to -1^-} f(x)$ does not exist.	**l.** $\lim_{x\to 2^+} f(x) = 0$

Practice Questions:

1. Find the limit of the following function at x=-3,0,2,4

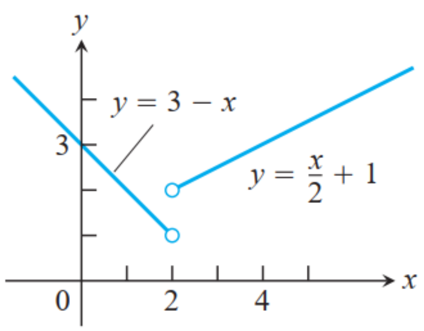


2. Find the limit of the following function at x=-3,-2,-1,0,1,2



3. Determine the following statements for the graph are true or false?

$$\text{Let } f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2. \end{cases}$$



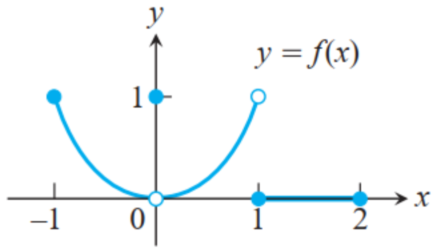
a. Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

d. Does $\lim_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Determine the given statements are true or false for the graph?



a. $\lim\limits_{x\to -1^+} f(x) = 1$

b. $\lim\limits_{x\to 0^-} f(x) = 0$

c. $\lim\limits_{x\to 0^-} f(x) = 1$

d. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$

e. $\lim\limits_{x\to 0} f(x)$ exists

f. $\lim\limits_{x\to 0} f(x) = 0$

g. $\lim\limits_{x\to 0} f(x) = 1$

h. $\lim\limits_{x\to 1} f(x) = 1$

i. $\lim\limits_{x\to 1} f(x) = 0$

j. $\lim\limits_{x\to 2^-} f(x) = 2$

k. $\lim\limits_{x\to -1^-} f(x)$ does not exist.

l. $\lim\limits_{x\to 2^+} f(x) = 0$