Applications of Multiple Integrals

Area of Bounded Regions in the Plane:

The area of closed bounded region R is

$$A = \iint_R dA.$$

Examples:

Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution:

$$\text{Area} = \iint_R dA = \int_0^4 \int_{\frac{x^2}{2}}^{2x} dydx$$

$$= \int_0^4 |y|_{\frac{x^2}{2}}^{2x} dx = \int_0^4 \left(2x - \frac{x^2}{2}\right) dx$$

$$= \left| x^2 - \frac{x^3}{3} \right|_0^4 = 16 - \frac{16}{3} = \frac{16}{3}$$

Practice Problems:

Sketch the region bounded by the given lines and curves and then find its area.

1. The coordinate axes and the line $x + y = 2$
2. The lines $x = 0, y = 2x$, and $y = 4$

Density of Thin Plate:

Definition: Suppose that we have a thin plate, so thin that it's practically 2-dimensional. **The density of this plate is defined as the mass per unit area.**

So, $$\boldsymbol{mass = density \times area}$$

Examples:

1. A thin plate covers the triangular region bounded by x-axis& the lines$x = 1$ &$y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $f(x, y) = 6x + 6y + 6$. Find the plate's mass.

$$\boldsymbol{Mass} = \iint_{\boldsymbol{R}} \boldsymbol{f(x, y)dA}$$

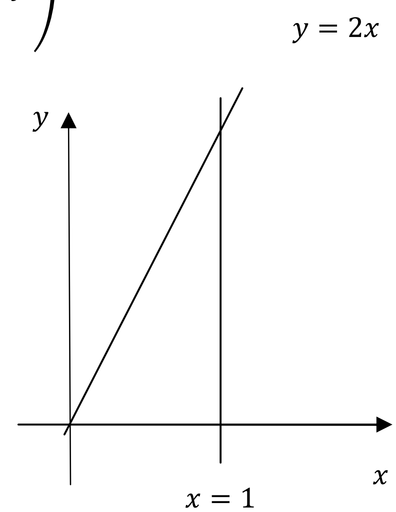
$$= \int_0^1 \int_0^{2x} (6x + 6y + 6)dydx = \int_0^1 \left(\int_0^{2x} 6xdy + \int_0^{2x} 6ydy + \int_0^{2x} 6dy \right) dx$$

$$= \int_0^1 |6xy + 6y^2 + 6y|_0^{2x} dx = \int_0^1 (12x^2 + 12x^2 + 12x)dx$$

$$= \int_0^1 (24x^2 + 12x)dx = \int_0^1 24x^2dx + \int_0^1 12xdx$$

$$= 24 \left|\frac{x^3}{3}\right|_0^1 + 12 \left|\frac{x^2}{2}\right|_0^1 = 24\left(\frac{1^3}{3} - \frac{0^3}{3}\right) + 12\left(\frac{1^2}{2} - \frac{0^2}{2}\right)$$

$$= \frac{24}{3} + \frac{12}{2} = 8 + 6 = 14$$



2. Find the mass M of a metal plate R bounded by $y = x$ & $y = x^2$ with density given by $f(x, y) = 1 + xy$

Solution: $M = \int_0^1 \int_{x^2}^{x} f(x, y)dydx$

$$M = \int_0^1 \int_{x^2}^{x} (1 + xy)dydx$$

$$= \int_0^1 \left(\int_{x^2}^{x} 1dy + \int_{x^2}^{x} xydy \right) dx = \int_0^1 \left[|y|_{x^2}^{x} + x \left| \frac{y^2}{2} \right|_{x^2}^{x} \right] dx$$

$$= \int_0^1 \left[(x - x^2) + x \left(\frac{x^2}{2} - \frac{x^4}{2} \right) \right] dx = \int_0^1 \left[(x - x^2) + \left(\frac{x^3}{2} - \frac{x^5}{2} \right) \right] dx$$

$$= \left| \frac{x^2}{2} \right|_0^1 - \left| \frac{x^3}{3} \right|_0^1 + \frac{1}{2} \left| \frac{x^4}{4} \right|_0^1 - \frac{1}{2} \left| \frac{x^6}{6} \right|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{8} - \frac{1}{12}$$

$$= \frac{5}{24}$$