# Chain Rule

In single variablecalculus when $w = w(x)$is a differentiable function of $x$ and $x = x(t)$is differentiable function of $t$, $w$ becomes a differentiable function of t and $\frac{dw}{dt}$can be calculated with the help of formula

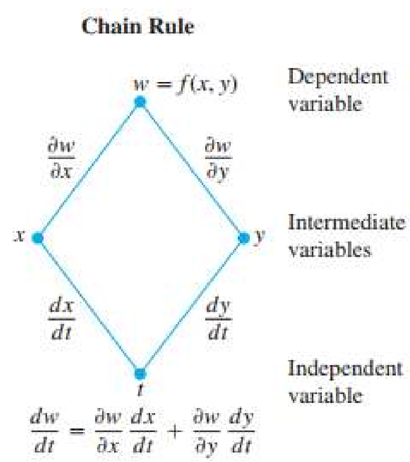
$$\frac{dw}{dt} = \frac{dw}{dx} . \frac{dx}{dt}$$

## Functions of Two Variables

If $w = w(x, y)$has continuous partial derivatives $f_x$and $f_y$and if $x = x\ (t)$ and $y = y\ (t)$ are differentiable functions of $t$, then the composite function $w = f(x(t)\ , y(t))$ is a differentiable function of $t$ and

$$\frac{dw}{dt} = w_x\big(x(t), y(t)\big) * \ x'(t) + \ w_y\big(x(t), y(t)\big) * y'(t)$$

or $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$




**Example 1:**

Use chain rule to find the derivative of $w = xy$with respect to $t$ along the path $x = \cos t$,$y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

**Solution:**We apply chain rule to find $\frac{dw}{dt}$as follows:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} . \frac{dx}{dt} + \frac{\partial w}{\partial y} . \frac{dy}{dt}$$

$$= \left[\frac{\partial (xy)}{\partial x}\right]\left[\frac{d(\cos\ )}{dt}\right] + \left[\frac{\partial (xy)}{\partial y}\right]\left[\frac{d(\sin t)}{dt}\right]$$

$$= \ (y)\ (-\ sin\ t)\ +\ (x)\ (cos\ t) = \ (sin\ t)\ (-\ sin\ t)\ +\ (cos\ t)\ (cos\ t)$$

$$= \ -\sin^2 t\ + \cos^2 t = \cos(2t)$$

In this example we can check the result with direct calculation. As a function of t,

$$w = xy = \cos t * \sin t = \frac{sin\ 2t}{2}$$

$$\frac{dw}{dt} = \frac{d}{dt}\left[\frac{\sin 2t}{2}\right] = \frac{1}{2}\ (2\cos 2t) = \cos 2t$$

in either case, at the given value of $t$,

$$\left[\frac{dw}{dt}\right]_{t=\frac{\pi}{2}} = \cos(2.\tfrac{\pi}{2})) = \cos \pi = -1$$

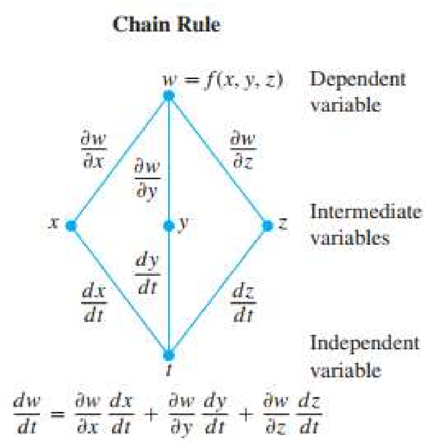
## Functions of Three Variables

You can probably predict the Chain Rule for functions of three variables, as it only involves adding the expected third term to the two-variable formula.

**Chain Rule for Three Independent Variables**

If $w = w(x, y, z)$ is differentiable and $x, y, z$ are differentiable functions of $t$, then $w$ is a differentiable function of $t$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}$$




**Example 2:**Changes in Function's values along a Helix
Find $\frac{dw}{dt}$ where$w = xy + z$; $x = \cos t$ , $y = \sin t$ , $z = t$
In this example the values of w are changing along the path of a helix. What is the derivative's value at t = 0?

**Solution:**

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (y)\ (-\sin t) + (x)\ (\cos t) + (1)\ (1)$$

$$= (\sin t)\ (-\sin t) + (\cos t)\ (\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$\left[\frac{dw}{dt}\right]_{t=0} = 0 + \cos^2(0) + 1 = 2$$

Ex. 14.4: 1 – 6.

**Chain Rule: One Independent Variable**

(a) Express $\frac{dw}{dt}$ as a function of $t$by using chain rule. Then (b) evaluate $\frac{dw}{dt}$ at the given value of $t$.

1. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t; \quad t = \pi$
2. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t; \quad t = 0$
3. $w = \frac{x}{z} + \frac{y}{z}, \quad x = \cos^2 t, \quad y = \sin^2 t, \quad z = 1/t; \quad t = 3$
4. $w = \ln(x^2 + y^2 + z^2), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t};$ $t = 3$
5. $w = 2ye^x - \ln z, \quad x = \ln(t^2 + 1), \quad y = \tan^{-1} t, \quad z = e^t;$ $t = 1$
6. $w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}; \quad t = 1$