If X is a normally distributed random variable and $X \sim N(\mu, \sigma)$, then the z-score is:

$$z = \frac{x - \mu}{\sigma}$$

The z-score tells you how many standard deviations the value x is above (to the right of) or below (to the left of) the mean, μ. Values of x that are larger than the mean have positive z-scores, and values of x that are smaller than the mean have negative z-scores. If x equals the mean, then x has a z-score of zero.

Example 6.1

Suppose $X \sim N(5, 6)$. This says that X is a normally distributed random variable with mean $\mu = 5$ and standard deviation $\sigma = 6$. Suppose $x = 17$. Then:

$$z = \frac{x - \mu}{\sigma} = \frac{17 - 5}{6} = 2$$

This means that $x = 17$ is **two standard deviations** (2σ) above or to the right of the mean $\mu = 5$.

Notice that: $5 + (2)(6) = 17$ (The pattern is $\mu + z\sigma = x$)

Now suppose $x = 1$. Then: $z = \frac{x - \mu}{\sigma} = \frac{1 - 5}{6} = -0.67$ (rounded to two decimal places)

This means that $x = 1$ is 0.67 standard deviations (-0.67σ) below or to the left of the mean $\mu = 5$. Notice that: $5 + (-0.67)(6)$ is approximately equal to one (This has the pattern $\mu + (-0.67)\sigma = 1$)

Summarizing, when z is positive, x is above or to the right of μ and when z is negative, x is to the left of or below μ. Or, when z is positive, x is greater than μ, and when z is negative x is less than μ.

Try It Σ

6.1 What is the z-score of x, when $x = 1$ and $X \sim N(12,3)$?

$Z = \frac{1-12}{3} = -3.67$

7

Example 6.2

Some doctors believe that a person can lose five pounds, on the average, in a month by reducing his or her fat intake and by exercising consistently. Suppose weight loss has a normal distribution. Let X = the amount of weight lost (in pounds) by a person in a month. Use a standard deviation of two pounds. $X \sim N(5, 2)$. Fill in the blanks.

$Z = \frac{10-5}{2} \Rightarrow \frac{5}{\cancel{5}}$

a. Suppose a person **lost** ten pounds in a month. The z-score when $x = 10$ pounds is $z = 2.5$ (verify). This z-score tells you that $x = 10$ is 2.5 standard deviations to the Right (right or left) of the mean five (What is the mean?).

Solution 6.2

a. This z-score tells you that $x = 10$ is **2.5** standard deviations to the **right** of the mean **five**.

b. Suppose a person **gained** three pounds (a negative weight loss). Then z = ________. This z-score tells you

$Z = \frac{-3-\cancel{10}\,5}{2}$

$= -4$

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that $x = -3$ is ________ standard deviations to the ________ (right or left) of the mean.

Solution 6.2

b. $z = -4$. This z-score tells you that $x = -3$ is **four** standard deviations to the **left** of the mean.

c. Suppose the random variables X and Y have the following normal distributions: $X \sim N(5, 6)$ and $Y \sim N(2,$
$x = 17$, then $z = 2$. (This was previously shown.) If $y = 4$, what is z?

Solution 6.2

c. $z = \frac{y-\mu}{\sigma} = \frac{4-2}{1} = 2$ where $\mu = 2$ and $\sigma = 1$.

The z-score for $y = 4$ is $z = 2$. This means that four is $z = 2$ standard deviations to the right of the mean. There
$x = 17$ and $y = 4$ are both two (of **their own**) standard deviations to the right of **their** respective means.

The z-score allows us to compare data that are scaled differently. To understand the concept, suppose $X \sim$
6) represents weight gains for one group of people who are trying to gain weight in a six week period and
$N(2, 1)$ measures the same weight gain for a second group of people. A negative weight gain would be a we
loss. Since $x = 17$ and $y = 4$ are each two standard deviations to the right of their means, they represent the sa
standardized weight gain **relative to their means.**

Try It Σ

6.2 Fill in the blanks.

Jerome averages 16 points a game with a standard deviation of four points. $X \sim N(16,4)$. Suppose Jerome score
points in a game. The z–score when $x = 10$ is -1.5. This score tells you that $x = 10$ is 1.5 standard deviations t
left (right or left) of the mean 16 (What is the mean?).

The Empirical Rule

If X is a random variable and has a normal distribution with mean μ and standard deviation σ, then the **Empiric**
states the following:

- About 68% of the x values lie between -1σ and $+1\sigma$ of the mean μ (within one standard deviation of the mean).
- About 95% of the x values lie between -2σ and $+2\sigma$ of the mean μ (within two standard deviations of the mean
- About 99.7% of the x values lie between -3σ and $+3\sigma$ of the mean μ (within three standard deviations of the
  Notice that almost all the x values lie within three standard deviations of the mean.
- The z-scores for $+1\sigma$ and -1σ are $+1$ and -1, respectively.
- The z-scores for $+2\sigma$ and -2σ are $+2$ and -2, respectively.
- The z-scores for $+3\sigma$ and -3σ are $+3$ and -3 respectively.

The empirical rule is also known as the 68-95-99.7 rule.

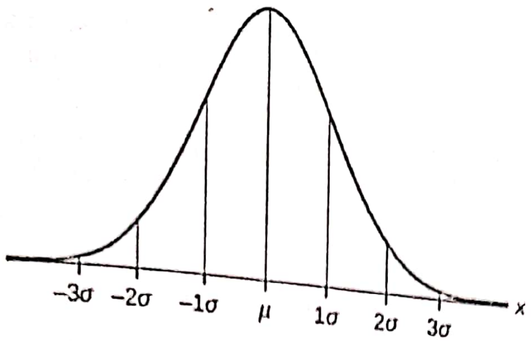


Figure 6.3

Example 6.3

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let X = the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then $X \sim N(170, 6.28)$.

μ σ 16

a. Suppose a 15 to 18-year-old male from Chile was 168 cm tall from 2009 to 2010. The z-score when x = 168 cm is z = ______. This z-score tells you that x = 168 is ______ standard deviations to the ______ (right or left) of the mean ______ (What is the mean?).

Solution 6.3

a. –0.32, 0.32, left, 170

b. Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a z-score of z = 1.27. What is the male's height? The z-score (z = 1.27) tells you that the male's height is ______ standard deviations to the ______ (right or left) of the mean.

Solution 6.3

b. 177.98 cm, 1.27, right

Try It Σ

6.3 Use the information in **Example 6.3** to answer the following questions.

$Z = \frac{x - \mu}{\sigma} = \frac{176-170}{0.96}$

a. Suppose a 15 to 18-year-old male from Chile was 176 cm tall from 2009 to 2010. The z-score when x = 176 cm is z = ______. This z-score tells you that x = 176 cm is ______ standard deviations to the ______ (right or left) of the mean ______ (What is the mean?).

b. Suppose that the height of a 15 to 18-year-old male from Chile from 2009 to 2010 has a z-score of z = –2. What is the male's height? The z-score (z = –2) tells you that the male's height is __2__ standard deviations to the __left__ (right or left) of the mean.

Example 6.4

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let Y = the height of 15 to 18-year-old males from 1984 to 1985. Then $Y \sim N(172.36, 6.34)$.

The mean height of 15 to 18-year-old males from Chile from 2009 to 2010 was 170 cm with a standard deviation of 6.28 cm. Male heights are known to follow a normal distribution. Let X = the height of a 15 to 18-year-old male from Chile in 2009 to 2010. Then $X \sim N(170, 6.28)$.

Find the z-scores for x = 160.58 cm and y = 162.85 cm. Interpret each z-score. What can you say about x = 160.58 cm and y = 162.85 cm as they compare to their respective means and standard deviations?

Solution 6.4

The z-score for x = -160.58 is z = -1.5.

The z-score for y = 162.85 is z = -1.5.

Both x = 160.58 and y = 162.85 deviate the same number of standard deviations from their respective means and in the same direction.

Try It Σ

$x_1 = 325$
$z_1 = -1.14$
$x_2 = 366.21$
$z_2 = -1.14$

6.4 In 2012, 1,664,479 students took the SAT exam. The distribution of scores in the verbal section of the SAT had a mean μ = 496 and a standard deviation σ = 114. Let X = a SAT exam verbal section score in 2012. Then $X \sim N(496, 114)$.

Find the z-scores for x_1 = 325 and x_2 = 366.21. Interpret each z-score. What can you say about x_1 = 325 and x_2 = 366.21 as they compare to their respective means and standard deviations?

Student 2 scored closer to the mean than student 1 and since they both had negative z-score, student 2 had better score

Example 6.5

Suppose x has a normal distribution with mean 50 and standard deviation 6.

- About 68% of the x values lie within one standard deviation of the mean. Therefore, about 68% of the x values lie between $-1\sigma = (-1)(6) = -6$ and $1\sigma = (1)(6) = 6$ of the mean 50. The values $50 - 6 = 44$ and $50 + 6 = 56$ are within one standard deviation from the mean 50. The z-scores are −1 and +1 for 44 and 56, respectively.
- About 95% of the x values lie within two standard deviations of the mean. Therefore, about 95% of the x values lie between $-2\sigma = (-2)(6) = -12$ and $2\sigma = (2)(6) = 12$. The values $50 - 12 = 38$ and $50 + 12 = 62$ are within two standard deviations from the mean 50. The z-scores are −2 and +2 for 38 and 62, respectively.
- About 99.7% of the x values lie within three standard deviations of the mean. Therefore, about 95% of the x values lie between $-3\sigma = (-3)(6) = -18$ and $3\sigma = (3)(6) = 18$ from the mean 50. The values $50 - 18 = 32$ and $50 + 18 = 68$ are within three standard deviations of the mean 50. The z-scores are −3 and +3 for 32 and 68, respectively.

Try It Σ

6.5 Suppose X has a normal distribution with mean 25 and standard deviation five. Between what values of x do 68% of the values lie?

Between 20 and 30

Example 6.6

From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was 172.36 cm, and the standard deviation was 6.34 cm. Let Y = the height of 15 to 18-year-old males in 1984 to 1985. Then $Y \sim N(172.36, 6.34)$.

a. About 68% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.

b. About 95% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.

c. About 99.7% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.

Solution 6.6

a. About 68% of the values lie between 166.02 cm and 178.7 cm. The z-scores are –1 and 1.

b. About 95% of the values lie between 159.68 cm and 185.04 cm. The z-scores are –2 and 2.

c. About 99.7% of the values lie between 153.34 cm and 191.38 cm. The z-scores are –3 and 3.

Try It Σ

6.6 The scores on a college entrance exam have an approximate normal distribution with mean, $\mu = 52$ points and a standard deviation, $\sigma = 11$ points.

a. About 68% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.
41 and 63

b. About 95% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.
30 and 74

c. About 99.7% of the y values lie between what two values? These values are ________________. The z-scores are ________________, respectively.
19 and 85

6.2 | Using the Normal Distribution

The shaded area in the following graph indicates the area to the left of x. This area is represented by the probability $P(X < x)$. Normal tables, computers, and calculators provide or calculate the probability $P(X < x)$.

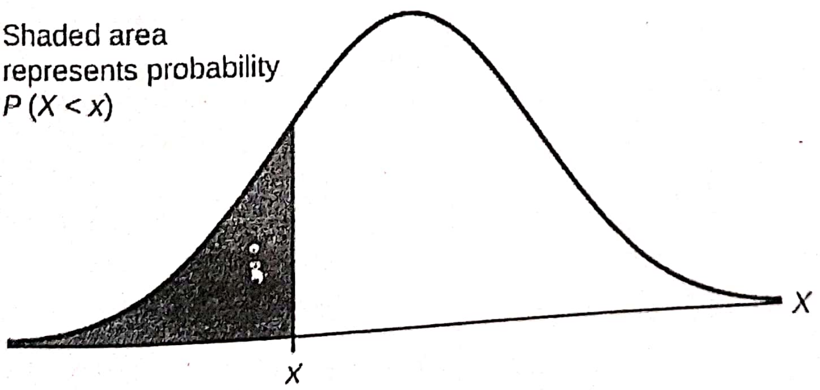


Figure 6.4

The area to the right is then $P(X > x) = 1 - P(X < x)$. Remember, $P(X < x)$ = **Area to the left** of the vertical line through x. $P(X < x) = 1 - P(X < x)$ = **Area to the right** of the vertical line through x. $P(X < x)$ is the same as $P(X \leq x)$ and $P(X > x)$ is the same as $P(X \geq x)$ for continuous distributions.

Calculations of Probabilities

Probabilities are calculated using technology. There are instructions given as necessary for the TI-83+ and TI-84 calculators.

NOTE

To calculate the probability, use the probability tables provided in **Appendix H** without the use of technology. Th tables include instructions for how to use them.

Example 6.7

If the area to the left is 0.0228, then the area to the right is 1 – 0.0228 = 0.9772.

Try It Σ

6.7 If the area to the left of *x* is 0.012, then what is the area to the right?

1 – 0.0228 = 0.9772

Example 6.8

The final exam scores in a statistics class were normally distributed with a mean of 63 and a standard deviation of five.

a. Find the probability that a randomly selected student scored more than 65 on the exam.

Solution 6.8

a. Let X = a score on the final exam. $X \sim N(63, 5)$, where $\mu = 63$ and $\sigma = 5$.

Draw a graph.

Then, find $P(x > 65)$.

$P(x > 65) = 0.3446$

$\frac{65-63}{5} = 0.4$

$1-(0.6554)$

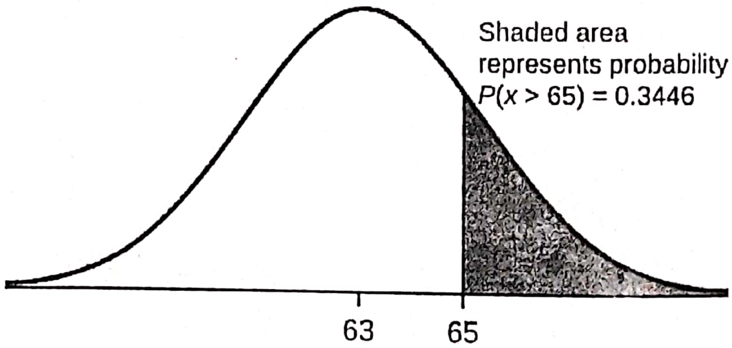


Figure 6.5

The probability that any student selected at random scores more than 65 is 0.3446.

Using the TI-83, 83+, 84, 84+ Calculator

Go into 2nd DISTR.

Solution 6.8

c. Find the 90th percentile. For each problem or part of a problem, draw a new graph. Draw the x-axis. S area that corresponds to the 90th percentile.

Let k = the 90th percentile. The variable k is located on the x-axis. $P(x < k)$ is the area to the left of k. percentile k separates the exam scores into those that are the same or lower than k and those that are the higher. Ninety percent of the test scores are the same or lower than k, and ten percent are the same or hi; variable k is often called a **critical value.**

$k = 69.4$

$P(X<k) = 0.90$

$P\left(\frac{X-\mu}{\sigma} < \frac{k-63}{5}\right) = 0.90$

$P\left(Z < \frac{k-63}{5}\right) = 0.90$

$\Phi\left(\frac{k-63}{5}\right) = 0.90$

$\frac{k-63}{5} = \Phi^{-1} 0.90$

Now search the value of z corresponding to prob 0.90 (Reverse)

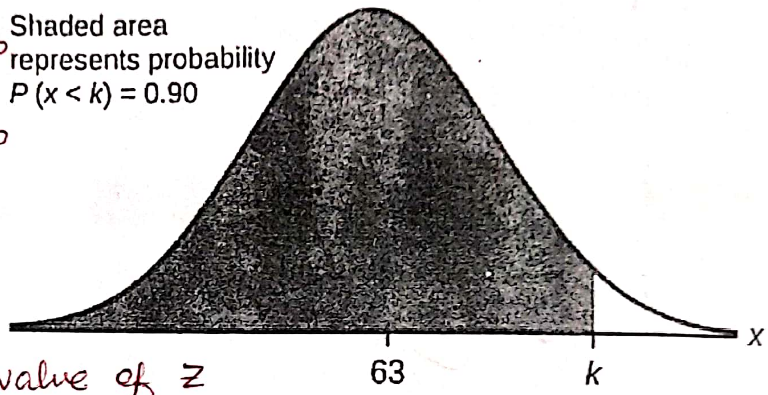


Figure 6.6

$\frac{k-63}{5} = 1.29$

$k-63 = 1.29 \times 5$

$k = 6.45 + 63$

$\boxed{k = 69.45}$

The 90th percentile is 69.4. This means that 90% of the test scores fall at or below 69.4 and 10% fall at c To get this answer on the calculator, follow this step:

Using the TI-83, 83+, 84, 84+ Calculator

invNorm in 2nd DISTR. invNorm(area to the left, mean, standard deviation)
For this problem, invNorm(0.90,63,5) = 69.4

d. Find the 70th percentile (that is, find the score k such that 70% of scores are below k and 30% of the sc above k).

Solution 6.8

d. Find the 70th percentile.

Draw a new graph and label it appropriately. $k = 65.6$

The 70th percentile is 65.6. This means that 70% of the test scores fall at or below 65.5 and 30% fall at o

invNorm(0.70,63,5) = 65.6

Try It Σ

6.8 The golf scores for a school team were normally distributed with a mean of 68 and a standard deviatior Find the probability that a randomly selected golfer scored less than 65.

$P(X<65) = P\left(\frac{X-\mu}{\sigma} < \frac{65-68}{3}\right) =$ ~~P(X<65) = 0.1587~~

$P(Z<-1) = 0.1587$

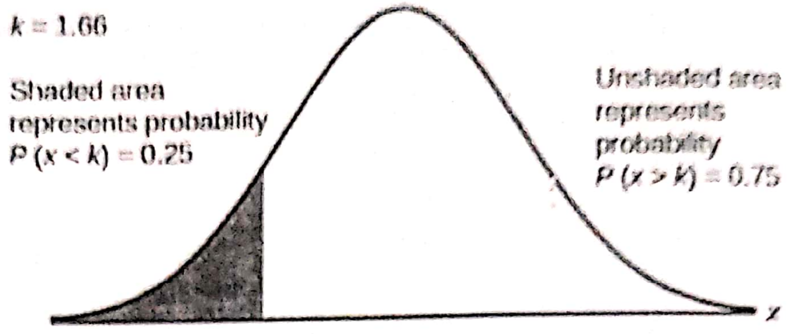


Figure 6.8

invNorm(0.25,2,0.5) = 1.66

The maximum number of hours per day that the bottom quartile of households uses a personal computer for entertainment is 1.66 hours.

Try It Σ

6.9 The golf scores for a school team were normally distributed with a mean of 68 and a standard deviation of three. Find the probability that a golfer scored between 66 and 70.

0.4950

Example 6.10

In the United States the ages 13 to 55+ of smartphone users approximately follow a normal distribution with approximate mean and standard deviation of 36.9 years and 13.9 years, respectively.

a. Determine the probability that a random smartphone user in the age range 13 to 55+ is between 23 and 64.7 years old. $P(23<x<64.7)$ $\Phi(b)-\Phi(a)$

Solution 6.10

a. normalcdf(23,64.7,36.9,13.9) = 0.8186

$P\left(\frac{23-36.9}{13.9}<Z<\frac{64.7-36.9}{13.9}\right) \Rightarrow \Phi(2)-\Phi(-1)$

$= 0.8186$

b. Determine the probability that a randomly selected smartphone user in the age range 13 to 55+ is at most 50.8 years old.

$P(X>50.8)$

Solution 6.10

b. normalcdf(-10^{99},50.8,36.9,13.9) = 0.8413

c. Find the 80th percentile of this distribution, and interpret it in a complete sentence.

Solution 6.10

c.

invNorm(0.80,36.9,13.9) = 48.6

The 80th percentile is 48.6 years.

80% of the smartphone users in the age range 13 – 55+ are 48.6 years old or less.

Try It Σ

6.10 Use the information in **Example 6.10** to answer the following questions.

a. Find the 30th percentile, and interpret it in a complete sentence.

b. What is the probability that the age of a randomly selected smartphone user in the range 13 to 55+ is less than 27 years old.

v. imp 14th Sep

Example 6.11

In the United States the ages 13 to 55+ of smartphone users approximately follow a normal distribution with approximate mean and standard deviation of 36.9 years and 13.9 years respectively. Using this information, answer the following questions (round answers to one decimal place).

a. Calculate the interquartile range (IQR).

Solution 6.11

a.

$IQR = Q_3 - Q_1$

Calculate $Q_3 = 75^{th}$ percentile and $Q_1 = 25^{th}$ percentile.

invNorm(0.75,36.9,13.9) = Q_3 = 46.2754

invNorm(0.25,36.9,13.9) = Q_1 = 27.5246

$IQR = Q_3 - Q_1 = 18.8$

b. Forty percent of the ages that range from 13 to 55+ are at least what age?

Solution 6.11

b.

Find k where $P(x \geq k) = 0.40$ ("At least" translates to "greater than or equal to.")

0.40 = the area to the right.

Area to the left = 1 – 0.40 = 0.60.

The area to the left of k = 0.60.

invNorm(0.60,36.9,13.9) = 40.4215.

k = 40.4.

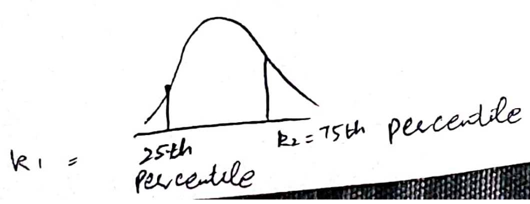
Forty percent of the ages that range from 13 to 55+ are at least 40.4 years.

Try It Σ

6.11 Two thousand students took an exam. The scores on the exam have an approximate normal distribution with a mean $\mu = 81$ points and standard deviation $\sigma = 15$ points.

a. Calculate the first- and third-quartile scores for this exam.

b. The middle 50% of the exam scores are between what two values? (follow Example 6.12 b part)



14th Sep
V.V. imp

Example 6.12

A citrus farmer who grows mandarin oranges finds that the diameters of mandarin oranges harvested on follow a normal distribution with a mean diameter of 5.85 cm and a standard deviation of 0.24 cm.

a. Find the probability that a randomly selected mandarin orange from this farm has a diameter larger cm. Sketch the graph.

Solution 6.12

a. normalcdf(6,10^99,5.85,0.24) = 0.2660

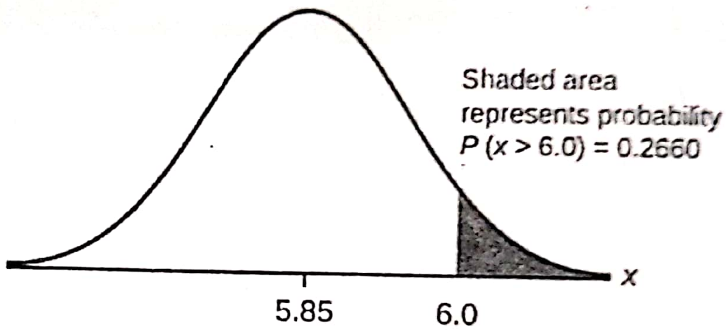


Figure 6.9

b. The middle 20% of mandarin oranges from this farm have diameters between ______ and ______.

Solution 6.12

b.

$1 - 0.20 = 0.80$

The tails of the graph of the normal distribution each have an area of 0.40.

Find $k1$, the 40th percentile, and $k2$, the 60th percentile ($0.40 + 0.20 = 0.60$).

$k1$ = invNorm(0.40,5.85,0.24) = 5.79 cm

$k2$ = invNorm(0.60,5.85,0.24) = 5.91 cm

c. Find the 90th percentile for the diameters of mandarin oranges, and interpret it in a complete sentence.

Solution 6.12

c. 6.16: Ninety percent of the diameter of the mandarin oranges is at most 6.16 cm.

14th Sep
V.V. imp

Try It Σ

6.12 Using the information from **Example 6.12**, answer the following:

a. The middle 40% of mandarin oranges from this farm are between ______ and ______.

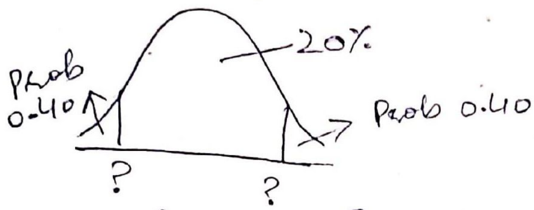
b. Find the 16th percentile and interpret it in a complete sentence.

Follow V. Example 6.12 b part

6.3 | Normal Distribution (Lap Times)

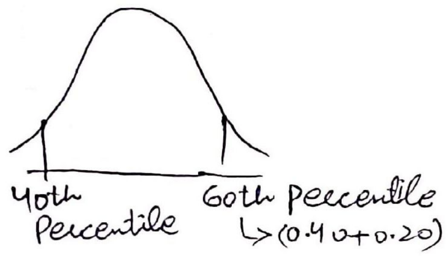
Example 6-12 (Part b)

Sol



$1 - 0.20 = 0.80 \Rightarrow \frac{0.80}{2} = 0.40$

Now find k_1 "The 40th percentile and k_2 "The 60th percentile" because $0.40 + 0.20$



Now

$P(X \le k_1) = 0.40$

$P\left(\frac{X-\mu}{\sigma} \le \frac{k_1 - 5.85}{0.24}\right) = 0.40$

$P\left(Z \le \frac{k_1 - 5.85}{0.24}\right) = 0.40$

$\phi\left(\frac{k_1 - 5.85}{0.24}\right) = 0.40$

$\frac{k_1 - 5.85}{0.24} = \phi^{-1}(0.40)$

$\frac{k_1 - 5.85}{0.24} = -0.25$

$\boxed{k_1 = 5.79}$

Now for 60th percentile

$P(X \le k_2) = 0.60$

$P\left(\frac{X-\mu}{\sigma} \le \frac{k_2 - 5.85}{0.24}\right) = 0.60$

$P\left(Z \le \frac{k_2 - 5.85}{0.24}\right) = 0.60$

$\phi\left(\frac{k_2 - 5.85}{0.24}\right) = 0.60$

$\frac{k_2 - 5.85}{0.24} = \phi^{-1}(0.60)$

$\frac{k_2 - 5.85}{0.24} = 0.26$

$\boxed{k_2 = 5.91}$

* Do Try it 6.12 by following same steps.