Chap. 4 HIGHER-ORDER DIFFERENTIAL EQUATIONS

In Chapter 2 we saw that we could solve a few first-order differential equations by recognizing them as separable, linear, exact equations. We turn now to the solution of ordinary differential equations of order two or higher.

INITIAL-VALUE PROBLEM

For a linear differential equation an ***n*th-order initial-value problem** is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

Subject to: $y(x_0) = y_0,\ y'(x_0) = y_1, \dots,\ y^{(n-1)}(x_0) = y_{n-1}.$

Recall that for a problem such as this one we seek a function defined on some interval I, containing x_0, that satisfies the differential equation and the n initial conditions specified at x_0: $y(x_0) = y_0$, $y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$.

HOMOGENEOUS EQUATIONS

A linear nth-order differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (6)$$

is said to be **homogeneous,** whereas an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (7)$$

with $g(x)$ not identically zero, is said to be **nonhomogeneous.** For example, $2y'' - 3y' - 5y = 0$ is a homogeneous linear second-order differential equation, whereas $x^3y''' + 6y' + 10y = e^x$ is a nonhomogeneous linear third-order differential equation. The word *homogeneous* in this context does not refer to coefficients that are homogeneous functions.

We shall see that to solve a nonhomogeneous linear equation (7), we must first be able to solve the **associated homogeneous equation** (6).

Superposition Principle—Homogeneous Equations

Let $y_1, y_2, \dots, y_k$ be solutions of the homogeneous nth-order differential equation (6) on an interval I. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x),$$

where the $c_i, i = 1, 2, \dots, k$ are arbitrary constants, is also a solution on the interval.

EXAMPLE 4 The functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear $x^3y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$. By the superposition principle the linear combination

$$y = c_1x^2 + c_2x^2 \ln x,$$

is also a solution of the equation on the interval.
The function $y = e^{7x}$ is a solution of $y'' - 9y' + 14y = 0$. Because the differential equation is linear and homogeneous, the constant multiple $y = ce^{7x}$ is also a solution.

Linear Dependence/Independence

A set of functions $f_1(x),\ f_2(x), \ldots,\ f_n(x)$ is said to be **linearly dependent** on an interval *I* if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x) = 0$$

for every *x* in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent.**

DEFINITION 4.1.2 Wronskian

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n - 1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

THEOREM 4.1.3 Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (6) on an interval *I*. Then the set of solutions is **linearly independent** on *I* if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

DEFINITION 4.1.3 Fundamental Set of Solutions

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear nth-order differential equation (6) on an interval *I* is said to be a **fundamental set of solutions** on the interval.

THEOREM 4.1.5 General Solution—Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x),$$

where c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

EXAMPLE 7 General Solution of a Homogeneous DE

The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$. By inspection the solutions are linearly independent on the x-axis. This fact can be corroborated by observing that the Wronskian

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every x. We conclude that y_1 and y_2 form a fundamental set of solutions, and consequently, $y = c_1e^{3x} + c_2e^{-3x}$ is the general solution of the equation on the interval. ■

THEOREM 4.1.6 General Solution—Nonhomogeneous Equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (7) on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (6) on I. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p,$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Exercise 4.1

In Problems 1–4 the given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family that is a solution of the initial-value problem.

1. $y = c_1e^x + c_2e^{-x}$, $(-\infty, \infty)$;
$y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$

2. $y = c_1e^{4x} + c_2e^{-x}$, $(-\infty, \infty)$;
$y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$

3. $y = c_1x + c_2x \ln x$, $(0, \infty)$;
$x^2y'' - xy' + y = 0$, $y(1) = 3$, $y'(1) = -1$

4. $y = c_1 + c_2 \cos x + c_3 \sin x$, $(-\infty, \infty)$;
$y''' + y' = 0$, $y(\pi) = 0$, $y'(\pi) = 2$, $y''(\pi) = -1$

In Problems 23–30 verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

23. $y'' - y' - 12y = 0$; $e^{-3x}, e^{4x}, (-\infty, \infty)$

24. $y'' - 4y = 0$; $\cosh 2x, \sinh 2x, (-\infty, \infty)$

25. $y'' - 2y' + 5y = 0$; $e^x \cos 2x, e^x \sin 2x, (-\infty, \infty)$

26. $4y'' - 4y' + y = 0$; $e^{x/2}, xe^{x/2}, (-\infty, \infty)$

27. $x^2y'' - 6xy' + 12y = 0$; $x^3, x^4, (0, \infty)$

28. $x^2y'' + xy' + y = 0$; $\cos(\ln x), \sin(\ln x), (0, \infty)$

29. $x^3y''' + 6x^2y'' + 4xy' - 4y = 0$; $x, x^{-2}, x^{-2}\ln x, (0, \infty)$

30. $y^{(4)} + y'' = 0$; $1, x, \cos x, \sin x, (-\infty, \infty)$

In Problems 31–34 verify that the given two-parameter family of functions is the general solution of the nonhomogeneous differential equation on the indicated interval.

31. $y'' - 7y' + 10y = 24e^x$;
$y = c_1e^{2x} + c_2e^{5x} + 6e^x, (-\infty, \infty)$

32. $y'' + y = \sec x$;
$y = c_1 \cos x + c_2 \sin x + x \sin x + (\cos x) \ln(\cos x)$,
$(-\pi/2, \pi/2)$

33. $y'' - 4y' + 4y = 2e^{2x} + 4x - 12$;
$y = c_1e^{2x} + c_2xe^{2x} + x^2e^{2x} + x - 2, (-\infty, \infty)$

34. $2x^2y'' + 5xy' + y = x^2 - x$;
$y = c_1x^{-1/2} + c_2x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x, (0, \infty)$