4.2 REDUCTION OF ORDER

INTRODUCTION In the preceding section we saw that the general solution of a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

is a linear combination $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I. Beginning in the next section, we examine a method for determining these solutions when the coefficients of the differential equation in (1) are constants. This method, which is a straightforward exercise in algebra, breaks down in a few cases and yields only a single solution y_1 of the DE. It turns out that we can construct a second solution y_2 of a homogeneous equation (1) (even when the coefficients in (1) are variable) provided that we know a nontrivial solution y_1 of the DE. The basic idea described in this section is that *equation* (1) *can be reduced to a linear first-o der DE by means of a substitution* involving the known solution y_1. A second solution y_2 of (1) is apparent after this first-order di ferential equation is solved.

Method of Reduction of Order

Suppose that y_1 denotes a nontrivial solution of (1) and that y_1 is defined on an interval I. We seek a second solution y_2 so that the set consisting y_1 and y_2 is linearly independent on I.

General Case

Suppose we divide by $a_2(x)$ to put equation (1) in the standard form

$$y'' + P(x)y' + Q(x)y = 0. \tag{2}$$

Where $P(x)$ and $Q(x)$ are continuous on some interval I. Let us suppose further that $y_1(x)$ is a known solution of (2) on I and that $y_1(x) \neq 0$ for every x in the interval. If we define

$$y(x) = u(x)\, y_1(x),$$

it follows that

$$y'(x) = u'(x)\, y_1(x) + u(x)\, y_1'(x)$$

$$y''(x) = u''(x)\, y_1(x) + 2\, u'(x)\, y_1'(x) + u(x)\, y_1''(x)$$

So equation (2) becomes

$$\begin{aligned} y'' + P(x)y' + Q(x)y \\ &= u''(x)\, y_1(x) + 2\, u'(x)\, y_1'(x) + u(x)\, y_1''(x) \\ &+ P(x)[u'(x)\, y_1\,(x) + u(x)\, y_1'\,(x)] + Q(x)\, u(x)\, y_1(x) = 0 \end{aligned}$$

After simplification, we have

$$u(x)[y_1''(x) + P(x)y_1'\,(x) + Q(x)\;\; y_1(x)] + u''(x)\, y_1(x) + u'(x)[2\;\; y_1'(x) + P(x)y_1\,(x)] = 0, \tag{3}$$

But as $u(x) \neq 0$

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

So (3) reduces to

$$u''(x)\, y_1(x) + u'(x)[2\; y_1'(x) + P(x)y_1\,(x)] = 0,$$

Let us take $w = u'(x)$, then $w' = u''(x)$. Therefore above equation becomes

$$w'(x)\, y_1(x) + w(x)[2\; y_1'(x) + P(x)y_1\,(x)] = 0, \qquad (4)$$

Observe that equation (4) is a linear and separable. Separating variables and integrating, we obtain

$$\frac{dw}{w} + \; 2\; \frac{y_1'(x)}{y_1(x)} dx + P(x)dx = 0,$$

Integration yields

$$\ln w + 2 \ln y_1 = -\int P\; dx + \ln c$$

$$\ln|wy_1^2| = -\int P\; dx + \ln c = \ln e^{-\int P\,dx} + \ln c = \ln c\; e^{-\int P\,dx}$$

Or

$$wy_1^2 = \; c\; e^{-\int P\,dx}$$

Or

$$w = \frac{c\; e^{-\int P\,dx}}{y_1^2}$$

But $w = u'(x)$, so we have

$$w = u'(x) = \frac{c\; e^{-\int P\,dx}}{y_1^2}$$

On further solving, we have

$$u(x) = \int \frac{c\; e^{-\int P\,dx}}{y_1^2} dx + c_1$$

By choosing $c = 1$ and $c_1 = 0$, we have

$$u(x) = \int \frac{e^{-\int P\,dx}}{y_1^2} dx$$

So we find from $y(x) = u(x)\; y_1(x)$ that the second solution of equation (2) is

$$y_2(x) = \; y_1(x)\; \int \frac{e^{-\int P\,dx}}{y_1^2} dx. \qquad (5)$$

It makes a good review of differentiation to verify that the function $y_2(x)$ defined in (5) satisfies equation(2) and that y_1 and y_2 are linearly independent on any interval on which $y_1(x)$ is not zero.

EXAMPLE 1: Find the general solution to

$$y'' + 4y' + 4y = 0 \qquad (1)$$

Given that $y_1(x) = e^{-2x}$ is a solution.

SOLUTION: Let

$$y_2 = u(x)y_1(x) = u(x)e^{-2x}$$

be the solution of (1), then it must satisfies (1).

$$y'_2 = u'e^{-2x} - 2\,u\,e^{-2x}$$

$$y''_2 = u''e^{-2x} - 4u'e^{-2x} + 4\,u\,e^{-2x}$$

Substituting the values of y_2' and y''_2 in (1).

$$y_2'' + 4y_2' + 4y_2 = (u''e^{-2x} - 4u'e^{-2x} + 4\,u\,e^{-2x}) + 4(u'e^{-2x} - 2\,u\,e^{-2x}) + 4(u(x)e^{-2x}) = 0$$

After simplification, we have

$$u''(x)e^{-2x} = 0$$

This tells us either $u''(x) = 0$ or $e^{-2x} = 0$.

But $e^{-2x} \neq 0$, so $\qquad u''(x) = 0$

Integration yields

$$u'(x) = c$$

Again integrating, we have

$$u(x) = c\,x + d$$

If we take $c = 1$ and $d = 0$, we have

$$u(x) = x$$

So we have obtained the second solution as $y_2 = u(x)e^{-2x} = xe^{-2x}$.

Method II. We can also use the formula derived in (5) as

$$y_2(x) = y_1(x)\int \frac{e^{-\int P\,dx}}{y_1^2}dx$$

$$= e^{-2x}\int \frac{e^{-\int 4\,dx}}{e^{-4x}}dx = e^{-2x}\int \frac{e^{-4x}}{e^{-4x}}dx = e^{-2x}\int 1\,dx = xe^{-2x}.$$

To see whether they linearly independent or not, we use Wronskian of the two function $y_1(x) = e^{-2x}$ and $y_2 = xe^{-2x}$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x} \neq 0.$$

So y_1 and y_2 are linearly independent. Hence the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{-2x} + c_2 x e^{-2x}.$$

EXAMPLE 2 A Second Solution by Formula (5)

The function $y_1 = x^2$ is a solution of $x^2 y'' - 3xy' + 4y = 0$. Find the general solution of the differential equation on the interval $(0, \infty)$.

SOLUTION From the standard form of the equation,

$$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0,$$

we find from (5)

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx \qquad \leftarrow e^{3\int dx/x} = e^{\ln x^3} = x^3$$

$$= x^2 \int \frac{dx}{x} = x^2 \ln x.$$

The general solution on the interval $(0, \infty)$ is given by $y = c_1 y_1 + c_2 y_2$; that is, $y = c_1 x^2 + c_2 x^2 \ln x$. ■

Exercise 4.2

In Problems 1–16 the indicated function $y_1(x)$ is a solution of the given DE. Use reduction of order or formula (7) as instructed, to find the 2nd solution $y_2(x)$.

1. $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}$
2. $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$
3. $y'' + 16y = 0; \quad y_1 = \cos 4x$
4. $y'' + 9y = 0; \quad y_1 = \sin 3x$
5. $y'' - y = 0; \quad y_1 = \cosh x$
6. $y'' - 25y = 0; \quad y_1 = e^{5x}$

7. $9y'' - 12y' + 4y = 0; \quad y_1 = e^{2x/3}$

8. $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$

9. $x^2y'' - 7xy' + 16y = 0; \quad y_1 = x^4$

10. $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$

11. $xy'' + y' = 0; \quad y_1 = \ln x$

12. $4x^2y'' + y = 0; \quad y_1 = x^{1/2}\ln x$

13. $x^2y'' - xy' + 2y = 0; \quad y_1 = x\sin(\ln x)$

14. $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2\cos(\ln x)$

15. $(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0; \quad y_1 = x + 1$

16. $(1 - x^2)y'' + 2xy' = 0; \quad y_1 = 1$