Matrix Transformation

Lecture No. 6

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Presentation Overview

1. Matrix Transformation from R^n to R^m

2. Types of Transformation

Transformation

Definitions (Matrix Transformation)

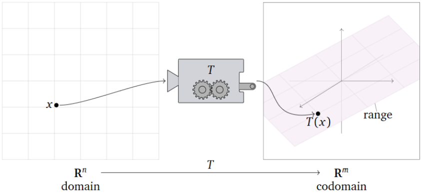
A Transformation from R^n to R^m is a rule T that assigns to each vector x in R^n a vector $T(x)$ in R^m.

- R^n is called the **domain** of T.
- R^m is called the **co-domain** of T.
- For x in R^n, the vector $T(x)$ in R^m is the **image** of x under T.
- The set of all images $\{T(x)|x\, in R^n\}$ is the **range** of T.

The notation $T : R^n \to R^m$ means **T is a transformation from R^n to R^m**.

Transformation

It may help to think of T as a “machine” that takes x as an input, and gives you $T(x)$ as the output.



The points of the domain R^n are the inputs of T:this simply means that it makes sense to evaluate T on vectors with n entries, i.e., lists of n numbers. Likewise, the points of the co-domain R^m are the outputs of T:this means that the result of evaluating T is always a vector with m entries.
The range of T is the set of all vectors in the co-domain that actually arise as outputs of the function T, for some input. In other words, the range is all vectors b in the co-domain such that $T(x) = b$ has a solution x in the domain.

Matrix Transformation

Definitions (Matrix Transformation)

Let A be an $m \times n$ matrix. The matrix transformation associated to A is the transformation
$T : R^n \to R^m$ defined by $T(\overrightarrow{x}) = A\overrightarrow{x}$
This is the transformation that takes a vector x in R^n to the vector Ax in R^m.

The matrix transformations are precisely the linear transformations from R^n to R^m, that is, the transformations with the linearity properties
$T(u + v) = T(u) + T(v)$
$T(cu) = cT(u)$ for all vectors u,v in R^n and all scalars c.
We will use these two properties as the starting point for defining more general linear transformations.

Remarks

It is important to note that a linear transformation is a special kind of function.

The input and output are both vectors.

If we denote the output vector $T(\overrightarrow{x})$ by $\overrightarrow{y}$ we can write

$$\overrightarrow{y} = A\overrightarrow{x}$$

Example 1:

Consider the letter L in figure, made up of the vectors (1, 0) or $\begin{bmatrix}1\\0\end{bmatrix}$ and (0, 2) or $\begin{bmatrix}0\\2\end{bmatrix}$, show that the effect of the linear transformation

$$\mathbf{T}\,(\vec{x}) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\vec{x}$$

on this letter, describe the transformation.

Solution: As $\mathbf{T}\,(\vec{x}) = \mathbf{A}\vec{x}$

$$\mathrm{T}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\mathrm{T}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix}$$

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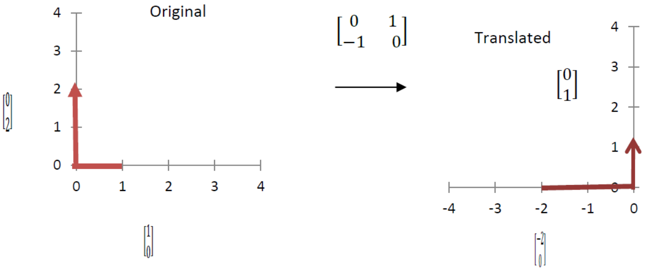


The effect of transformation on the L is rotated through an angle of 90^0 in the anticlockwise direction.

Work to do:

Q1. Consider the matrices $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

$$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Show the effect of each of these matrices on L shape in example 1 and describe each of the transformation in words.

Solution: As, $\mathbf{T}\,(\vec{x}) = \mathbf{A}\vec{x}$

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

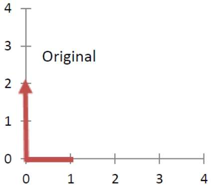
$$T\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

For $\mathbf{T}\,(\vec{x}) = \mathbf{C}\vec{x}$

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$T\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$\begin{bmatrix}0\\2\end{bmatrix}$



$\begin{bmatrix}1\\0\end{bmatrix}$

Types of Transformations

There are two types of linear transformations (defining from R^2 to R^2):
1. Euclidean Transformation.
2. Affine Transformation.

Definitions (Euclidean Transformation)

A Euclidean Transformation is a transformation $T : R^2 \to R^2$ defined by

$$T(x) = A\overrightarrow{x} + \overrightarrow{a}, \forall \overrightarrow{x} \in R^2$$

Where A is an orthogonal 2×2 matrix and $\overrightarrow{a} \in R^2$. These types of transformations always preserve distance/shape.
An orthogonal matrix holds the property $AA^T = 1$ or $A^T = A^{-1}$

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Definitions (Affine Transformation)

An Affine Transformation is a transformation $T : R^2 \rightarrow R^2$ defined by

$$T(x) = A\overrightarrow{x} + \overrightarrow{a}, \forall \overrightarrow{x} \in R^2$$

where A is a 2×2 invertible matrix and $\overrightarrow{a} \in R^2$.

Remarks:

- Every orthogonal matrix is invertible but an invertible matrix may or may not be orthogonal.
- Euclidean geometry is a subset of affine geometry or Affine transformations are the generalization of Euclidean transformation.

Types of Euclidean Transformation

- Translation.
- Reflection.
- Rotation.

Definitions (Translation)

An Affine Transformation is a transformation $T : R^2 \to R^2$ or $T : R^3 \to R^3$ defined by

$$T(x) = \overrightarrow{x} + \overrightarrow{a}, \forall \overrightarrow{x} \in R^2$$

where A is the identity matrix.

Example 2: (Translation of a triangle)

Let A = (-2, -2), B = (2, -2), C = (0, 2) form a triangle. Find the translated triangle with vector $\vec{a} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$.

Solution: As the transformation of translation is

$$\mathbf{T}\,(\vec{x}) = \vec{x} + \vec{a}\,.$$

For point A: $D = T\,(A) = \begin{bmatrix} -2 \\ -2 \end{bmatrix} + \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

For point B: $E = T\,(B) = \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

For point C: $F = T\,(C) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

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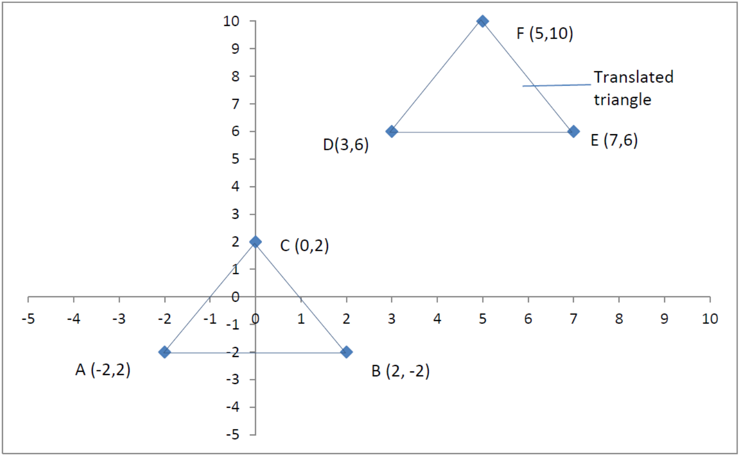


F (5,10)
Translated triangle
D(3,6)
E (7,6)
C (0,2)
A (-2,2)
B (2, -2)
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10

Example -3: (Translation of a line)

For a line 3x - 4y = 2, find the equation of line translated through vector $\vec{a} = (2, 3)$.

Solution: The transformation of translation is:

$$\mathbf{T}\,(\vec{x}) = \vec{x} + \vec{a}$$

So $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Or $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 2 \\ y + 3 \end{bmatrix}$

Which implies $x' = x + 2$ and $y' = y + 3$.

Then, $x = x' - 2$ and $y = y' - 3$.

Put these in our given equation of line that is

$3(x' - 2) - 4(y' - 3) = 2$

$3x' - 4y' = -4$ is the required translated line.

To draw original line $3x - 4y = 2$

put $\boldsymbol{x = 0}$ implies $y = -1/2$, so $\mathbf{A}\,(0, -1/2)$ is a point on this line.

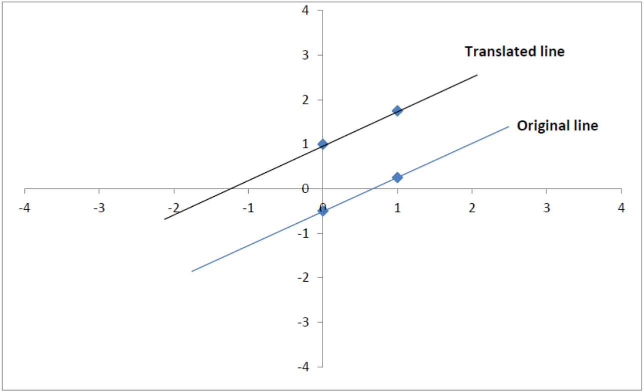
Continue...

Similarly $\boldsymbol{x} = \mathbf{1}$ implies $y = ¼$ and **B** = (1, 1/4) is another point on it.

In the same manner to draw the Translated line $3x' - 4y' = -4$

Putting $\boldsymbol{x} = \mathbf{0}$ gives $y' = 1$ and **C** = (0, 1).

Putting $\boldsymbol{x} = \mathbf{1}$ provides $y' = 7/4$ and **D**= (1, 7/4).



Example -4: (Translation of circle)

Let $(x-4)^2 + (y-3)^2 = 9$ be a circle. Find the equation of the translated circle using vector (2, 3).

Note: As equation of circle: $(x-a)^2 + (y-b)^2 = r^2$ with Centre $= (a, b)$ and Radius $= r$. While $x^2 + y^2 = r^2$ is circle with Center $= (0, 0)$ and Radius $= r$.

Solution: The transformation of translation is

$$\mathbf{T}(\vec{x}) = \vec{x} + \vec{a}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x + & 2 \\ y + & 3 \end{bmatrix}$$

$x' = x + 2$ then $x = x' - 2$ and $y' = y + 3$ then $y = y' - 3$

Putting these equations in the equation of circle

Continue...

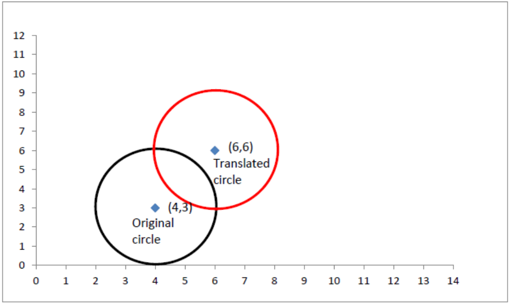
$$(x - 4)^2 + (y - 3)^2 = 9$$

$$(\boldsymbol{x'} - 2 - 4)^2 + (\boldsymbol{y'} - 3 - 3)^2 = 9$$

$$(\boldsymbol{x'} - 6)^2 + (\boldsymbol{y'} - 6)^2 = 9$$

Hence, Original circle is $(x - 4)^2 + (y - 3)^2 = 9$ with Center = (4, 3), Radius = 3.

While Translated circle is $(\boldsymbol{x'} - 6)^2 + (\boldsymbol{y'} - 6)^2 = 9$ with Center = (6, 6), Radius = 3



Note: As we said earlier that **Euclidean transformations** are distance/shape preserving. So in all above examples we can see that translation transformation being a Euclidean transformation preserves the shape of each object and just translated or moved the object.

Work to do:

Q1. Let A = (3, 4), B = (3, 2), C = (6, 2) and D = (6, 4) form a rectangle. Find its translation thorough vector (3, 5) and verify your translated rectangle from the figure below.