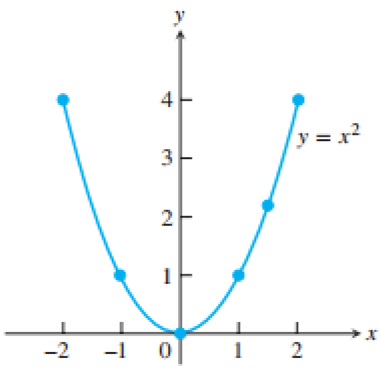
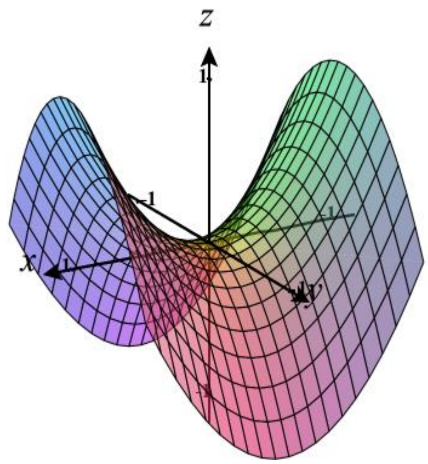
Multivariable Calculus

What do you understand from the word Multivariable Calculus?

Give some examples around you.





$$\mathbf{f}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^2 - \boldsymbol{y}^2$$

Real-Life Examples:

Example 1:

Consider an airline's ticket price pattern. To avoid flying planes with many empty seats, it sells some tickets at full price and some at a discount. For a particular route, the airline's revenue, R, earned in a given time period is determined by the number of full-price tickets, x, and the number of discount tickets, y, sold. We say that R is a function of x and y, and we write

$$\boldsymbol{R} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$$

This is just like the function notation of one-variable calculus. The variable R is the dependent variable and the variables x and y are the independent variables. The letter f stands for the *function* or rule that gives the value, or output, of R corresponding to given values of x and y.

The revenue, R, (in dollars) from a particular airline route is shown in Table 1 as a function of the number of full-price tickets and of discount tickets sold.

Revenue from ticket sales as a function of x and y

		Number of full-price tickets, x			
		100	200	300	400
Number of discount tickets, y	200	75,000	110,000	145,000	180,000
	400	115,000	150,000	185,000	220,000
	600	155,000	190,000	225,000	260,000
	800	195,000	230,000	265,000	300,000
	1000	235,000	270,000	305,000	340,000

(Table 1)

Values of x are shown across the top, values of y are down the left side, and the corresponding values of $f(x, y)$ are in the table. For example, to find the value of $f(300, 600)$, we look in the column corresponding to $x = 300$ at the row $y = 600$, where we find the number 225,000. Thus,

$$\boldsymbol{f(300, 600) = 225,000}$$

This means that the revenue from 300 full-price tickets and 600 discount tickets is $225,000.

Example 2:

Suppose you want to calculate your monthly payment on a five-year car loan; this depends on both the amount of money you borrow and the interest rate. These quantities can vary separately: the loan amount can change while the interest rate remains the same, or the interest rate can change while the loan amount remains the same. To calculate your monthly payment, you need to know both. If
the monthly payment is $m, the loan amount is $L, and the interest rate is r %, then we express the fact that m is a function of L and r by writing

$$\boldsymbol{m = f(L, r)}$$

This is just like the function notation of one-variable calculus. The variable m is called the dependent variable, and the variables L and r are called the independent variables. The letter f stands for the *function* or rule that gives the value of m corresponding to given values of L and r.

Definition:

Multivariable Calculus is the extension of calculus in one variable to calculus with functions of several variables.

Single Variable Function (one independent variable)	Multivariable Function (more than one independent variable)
$f(x) = x^2$ $y = x^2$	$R = f(x, y) = x + y$
$f\,(1) = 1$ $f\,(2) = 4$	$f\,(1,2) = 1 + 2 = 3$

Introduction to Vectors

We can observe the application of vectors in cricket when a batsman hits a shot there will be three possibilities.

1. One is, dropping the ball just before the fielder,
2. Second is catching out
3. the third one is reaching the maximum score for a shot that is sixer.

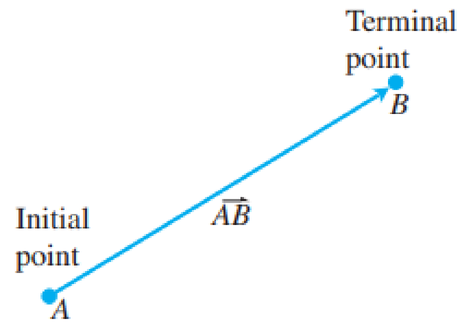
All these possibilities depend on some factors like in which direction the ball has been hit by the batsman, angle between bat and the direction line, and how much force is applied for that particular shot.

Definition:

Physical quantities that have both magnitude and direction.

Examples: Displacement, Force, Acceleration, Velocity

- A vector in plane/space is directed line segment. The directed line segment $\overrightarrow{AB}$ has initial point A and the terminal point B.



Examples:

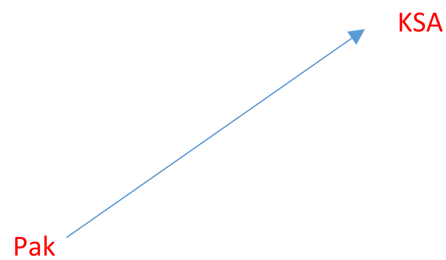
A= (x_1, y_1, z_1)
B= (x_2, y_2, z_2)
$\overrightarrow{AB}$= $< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$

Displacement vector

Suppose you are a pilot and you are planning a flight from Pakistan to KSA.

You must know two things:

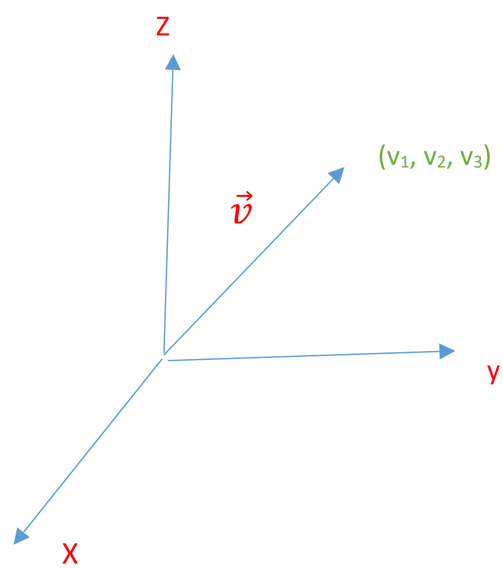
I. The distance to be travelled.
II. The direction to go.



Both these quantities together specify the displacement or displacement vector between the two countries.

Note: The magnitude (length) of the displacement vector is the distance between the points and is represented by the length of the arrow.

Component and Standard form of Vectors



If $\vec{v}$ is a three-dimensional vector with initial point at the origin and terminal point (v_1, v_2, v_3) then the component form of $\vec{v}$ is

$$\vec{v} = < v_1, v_2, v_3 >$$

and the standard form is

$$\vec{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$$

Magnitude or Length of the Vector

The length or magnitude of the vector

$\vec{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$ is $|\vec{v}| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$

Note: It is a non-negative number.

EXAMPLE 1 Component Form and Length of a Vector

Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution

(a) The standard position vector $\mathbf{v}$ representing $\overrightarrow{PQ}$ has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \qquad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

and

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

(b) The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

So, the length of vector $\overrightarrow{PQ}$ is 3.

Vector Algebra

Two principal operations involving vectors are vector addition and scalar multiplication.

Vector Addition:

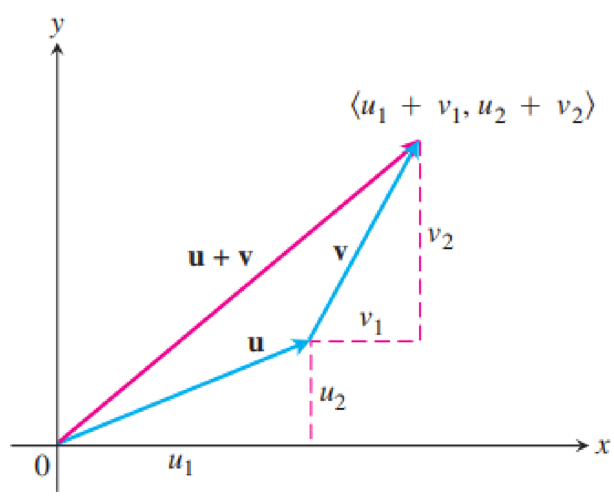
1) Let $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$

$\vec{u} + \vec{v} = =< u_1 + v_1, u_2 + v_2, u_3 + v_3 >$

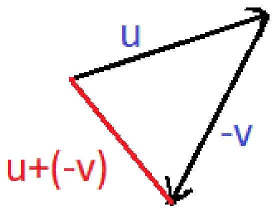
Example:

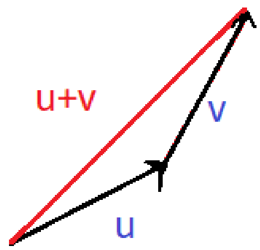
Let $\vec{u} =< u_1, u_2 >$ and $\vec{v} =< v_1, v_2 >$

Vectors can be added geometrically using head to tail rule.



2) **u** - **v** = **u** + (-**v**)





Question: Find **u**-**w**?

Scalar Multiplication of Vectors

A scalar is simply a real number, it can be positive, negative or zero.

Let $\vec{v} =< v_1, v_2, v_3 >$

then $k\vec{v} = k < v_1, v_2, v_3 > =< kv_1, kv_2, kv_3 >$ where k is any scalar.

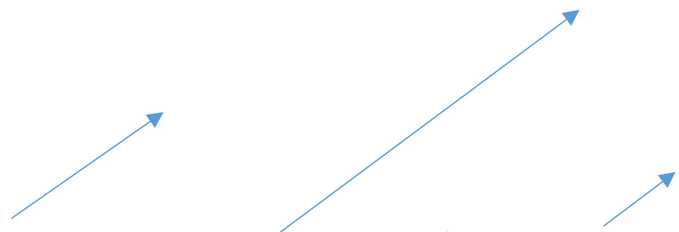
Note: If k > 0 then k$\vec{v}$ has the same direction as $\vec{v}$; if k < 0 then the direction of k$\vec{v}$ is opposite to that of $\vec{v}$.

Examples:

$\vec{v} =< 1, 2, 3 >$

$2\vec{v} =< 2, 4, 6 >$

$\frac{1}{2}\vec{v} =< \frac{1}{2}, \frac{2}{2}, \frac{3}{2} >$



$\vec{v}$ $\quad 2\vec{v}$ $\quad \frac{1}{2}\vec{v}$

Question: Let $\vec{u} =< -1, 3, 1 >, \vec{v} =< 4, 7, 0 >$

a. Find $2\vec{u} + 3\vec{v}$
b. $\vec{u} - \vec{v}$
c. $|\frac{1}{2}\vec{v}|$

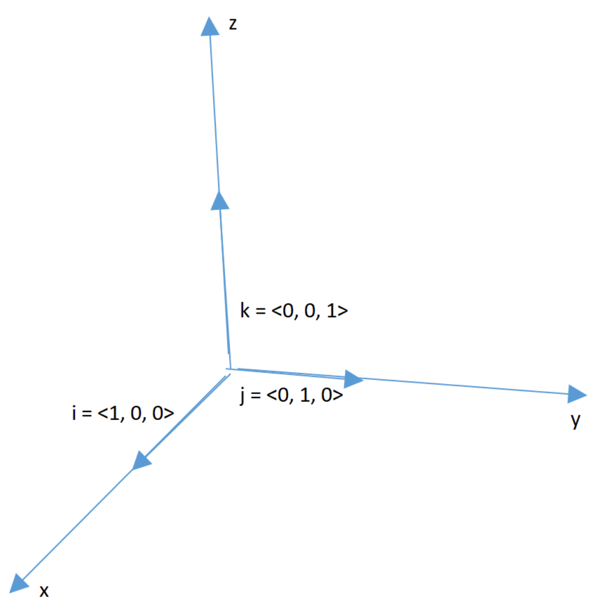
Unit Vector

A vector $\vec{v}$ of length 1 is called unit vector.

It is represented by $\hat{v}$.

The standard unit vectors in R^3 (space) are

i = <1,0,0>, **j** = < 0,1,0>, **k** = <0,0,1>



Any vector $\vec{v} =< v_1, v_2, v_3 >$ can be written as a linear combination of the standard unit vectors as follows:

$$\vec{v} =< v_1, v_2, v_3 >= v_1 < 1, 0, 0 > + v_2 < 0, 1, 0 > + v_3 < 0, 0, 1 >$$

$$= v_1\mathbf{i} + v_2\,\mathbf{j} + v_3\mathbf{k}$$

Example:

$$\vec{v} =< 2, 4, 6 >= 2 < 1, 0, 0 > +4 < 0, 1, 0 > +6 < 0, 0, 1 >$$

$$=< 2, 0, 0 > +< 0, 4, 0 > +< 0, 0, 6 >$$

$$= < 2, 4, 6 >$$

$$= 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}.$$

Remark:

If a vector is not unit vector, then we can make it unit vector by dividing it by its magnitude i.e.

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

Example:

$$\vec{v} =< 1, 2, 3 >$$

$$|\vec{v}\,| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

Vector $\vec{v}$ is not unit vector because its magnitude is not 1.

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{< 1,2,3 >}{\sqrt{14}} = < \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} >$$

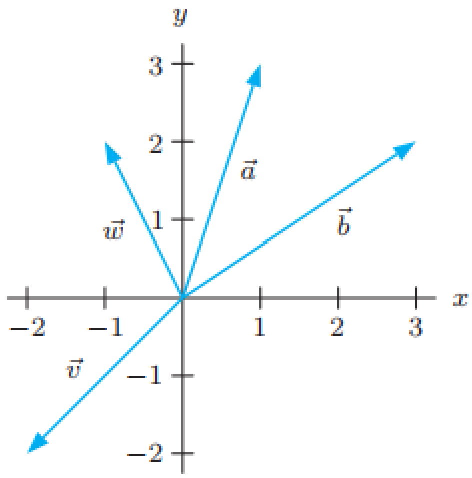
Now it is unit vector in the direction of vector $\vec{v}$.

Practice Problems

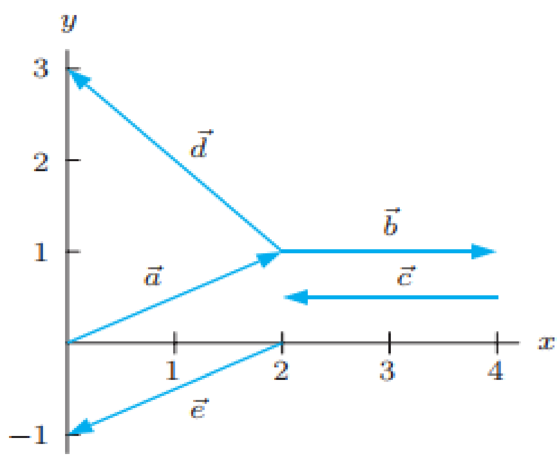
Question 1: Find the unit vector in the opposite direction to **i** – **j** + **k.**

Question 2: Resolve the vectors into components:

1.



2.



3. A vector starting at the point $Q = (4, 6)$ and ending at the point $P = (1, 2)$.

4. A vector starting at the point $P = (1, 2)$ and ending at the point $Q = (4, 6)$.

Ex. 12.2: 1-9,11,12,17-22