Basis

If V is a vector space and $S = \{v_1, v_2, \dots, v_n\}$ is a finite set of vectors in V, then S is called basis for V if the following two conditions hold:

(i) S is linearly independent
(ii) S spans V

Note: If $v_1, v_2, \dots, v_n$ form basis for a vector space V, then they must be distinct and non-zero.

Example 1: $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ forms a basis for R^3 because S spans R^3 and S is linearly independent.

Example 2: Standard basis for R^n are

$$S = \{e_1 = (1,0,0, \dots, 0), e_2 = (0,1,0, \dots, 0), \dots, e_n = (0,0,0, \dots, 1)\}$$

as they span R^n and are also linearly independent.

Example 3: Show that the vectors $v_1 = (1,2,1), v_2 = (2,9,0), v_3 = (3,3,4)$ form basis for R^3.

Solution: For this we must show that the vectors are linearly independent and span R^3.

Linearly independent:

$$k_1v_1 + k_2v_2 + k_3v_3 = 0$$

$$k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4) = (0,0,0)$$

$$(k_1 + 2k_2 + 3k_{3,}2k_1 + 9k_2 + 3k_3, k_1 + 4k_3) = (0,0,0)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 9 & 3 & 0 \\ 1 & 0 & 4 & 0 \end{array}\right)$$

By using Gaussian elimination technique, we come up with

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 0 & -3/5 & 0 \end{array}\right)$$

Which implies that

$$k_1 + 2k_2 + 3k_3 = 0 \qquad (1)$$

$$k_2 - \frac{3}{5}k_3 = 0 \qquad (2)$$

$$-\frac{3}{5}k_3 = 0 \qquad (3)$$

Eq. (3) gives $k_3 = 0$

Eq. (2) gives $k_2 = 0$ by inserting value of k_3 while Eq. (1) implies that$k_1 = 0$

As all $k's$ are zero. Hence vector are linearly independent.

Now, we prove that $Span\{(v_1, v_2, v_3)\} = R^3$

$$(a, b, c) = k_1v_1 + k_2v_2 + k_3v_3$$

$$(a, b, c) = k_1(1,2,1) + k_2(2,9,0) + k_3(3,3,4)$$

$$(a, b, c) = (k_1 + 2k_2 + 3k_{3,}2k_1 + 9k_2 + 3k_3, k_1 + 4k_3)$$

$$k_1 + 2k_2 + 3 = a$$

$$2k_1 + 9k_2 + 3k_3 = b \qquad (A)$$

$$k_1 + 4k_3 = c$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 2 & 9 & 3 & b \\ 1 & 0 & 4 & c \end{array}\right)$$

Firs, we check that weather the inverse of the above system exists or not. For this,

$$\det\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1 \neq 0$$

$\Rightarrow$ Span exists.

By using the following row operations:

$$R_2 - 2R_1, R_3 - R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 5 & -3 & b-2a \\ 0 & -2 & 1 & c-a \end{array}\right)$$

By using the following row operations:

$$R_2/5$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & -3/5 & (b-2a)/5 \\ 0 & -2 & 1 & c-a \end{array}\right)$$

By using the following row operations:

$$R_3 + 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & -\frac{3}{5} & \frac{(b-2a)}{5} \\ 0 & 0 & -\frac{1}{5} & \frac{-9a+2b+5c}{5} \end{array}\right)$$

$$k_1 + 2k_2 + 3k_3 = a \qquad (1)$$

$$k_2 - \frac{3}{5}k_3 = \frac{(b-2a)}{5} \qquad (2)$$

$$\frac{1}{5}k_3 = \frac{-9a+2b+5c}{5} \qquad (3)$$

From (3)

$$k_3 = 9a - 2b - 5c$$

Put this into (2)

$$k_2 = 5a - b - 3c$$

Using values of k_2, k_3 in (1)

$$k_1 = -36a + 8b + 21c$$

As the system (A) has solution. So, v_1, v_2, v_3 spans R^3 and are linearly independent.

$\Rightarrow v_1, v_2, v_3$ forms basis for R^3.

Example 4: Let $\boldsymbol{v_1} = (\mathbf{1}, \mathbf{1}), \boldsymbol{v_2} = (\mathbf{3}, \mathbf{5}), \boldsymbol{v_3} = (\mathbf{4}, \mathbf{2})$.Check whether v_1, v_2, v_3 form basis for R^2 or not?

Solution:

Linearly independent:

$$k_1v_1 + k_2v_2 + k_3v_3 = (0,0)$$

$$k_1(1,1) + k_2(3,5) + k_3(4,2) = (0,0)$$

$$(k_1 + 3k_2 + 4k_3, k_1 + 5k_2 + 2k_3) = (0,0)$$

$\Rightarrow\ k_1 + 3k_2 + 4k_3 = 0 \qquad (1)$

$k_1 + 5k_2 + 2k_3 = 0 \qquad (2)$

Subtract (1) and (2)

$$-2k_2 + 2k_3 = 0$$

$\Rightarrow$
$$k_2 = k_3$$

Put in (1)

$$k_1 + 3k_3 + 4k_3 = 0$$

$$k_1 + 7k_3 = 0$$

$$k_1 = -7k_3$$

Let

$$k_3 = t$$

$$k_1 = -7t$$

$$k_2 = t$$

As k_1, k_2, k_3 are not zero. So, v_1, v_2, v_3 are linearly dependent. So, v_1, v_2, v_3 does not form basis for R^2.

Example 5: Check whether following sets form basis for R^2 or not?

(a) $\{(2,1),(3,0)\}$
(b) $\{(0,0),(1,3)\}$

Example 6: $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is the basis for M_{22}.

Solution:

Linearly independent:

$$k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$k_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow k_1 = k_2 = k_3 = k_4 = 0$$

Spanning:

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} = \mathrm{k}_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \mathrm{k}_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \mathrm{k}_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \mathrm{k}_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} = \begin{bmatrix} \mathrm{k}_1 & \mathrm{k}_2 \\ \mathrm{k}_3 & \mathrm{k}_4 \end{bmatrix}$$

$$k_1 = a, k_2 = b, k_3 = c, k_4 = d$$

As it spans and are linearly independent. So, these form basis for M_{22}

Example 7: Show that the set

$$\boldsymbol{S = \{t^2 + 1, t - 1, 2t + 2\}}$$

is a basis for the vector space P_2.

Solution:

Linearly Independent:

$$k_1v_1 + k_2v_2 + k_3v_3 = \vec{0}$$

$$k_1(t^2 + 1) + k_2(t - 1) + k_3(2t + 2) = 0t^2 + 0t + 0$$

$$k_1t^2 + k_1 + k_2t - k_2 + 2k_3t + 2k_3 = 0t^2 + 0t + 0$$

$$k_1t^2 + k_2t + 2k_3t + k_1 - k_2 + 2k_3 = 0t^2 + 0t + 0$$

$$k_1t^2 + (k_2 + 2k_3)t + (k_1 - k_2 + 2k_3) = 0t^2 + 0t + 0$$

Equating corresponding components:

$$\begin{cases} k_1 = 0 \ldots (1) \\ k_2 + 2k_3 = 0 \ldots (2) \\ k_1 - k_2 + 2k_3 = 0 \ldots (3) \end{cases}$$

Put $k_1 = 0$ in equation (3), we get:

$$-k_2 + 2k_3 = 0 \ldots (4)$$

Add (2) and (4)

$$k_2 + 2k_3 = 0$$

$$-k_2 + 2k_3 = 0$$

$$4k_3 = 0$$

$$k_3 = 0$$

Put $k_3 = 0$, put in (2), we get

$$k_2 = 0$$

As k_1, k_2, k_3 are all zero. So S is linearly independent.

Spanning:

$$p = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$at^2 + bt + c = k_1(t^2 + 1) + k_2(t - 1) + k_3(2t + 2)$$

$$at^2 + bt + c = k_1 t^2 + k_1 + k_2 t - k_2 + 2k_3 t + 2k_3$$

$$at^2 + bt + c = k_1 t^2 + k_2 t + 2k_3 t + k_1 - k_2 + 2k_3$$

$$at^2 + bt + c = k_1 t^2 + (k_2 + 2k_3)t + (k_1 - k_2 + 2k_3)$$

$$\begin{cases} a = k_1 \ldots \ldots \ldots (1) \\ b = k_2 + 2k_3 \ldots \ldots \ldots (2) \\ c = k_1 - k_2 + 2k_3 \ldots \ldots \ldots (3) \end{cases}$$

Put $k_1 = a$ in equation (3)

$$c = a - k_2 + 2k_3$$

$$-k_2 + 2k_3 = c - a \ldots \ldots \ldots (4)$$

Add (2) and (4)

$$k_2 + 2k_3 = b$$

$$-k_2 + 2k_3 = c - a$$

$$4k_3 = b + c - a$$

$$k_3 = \frac{b+c-a}{4}$$

Put value of k_2 in equation (2)

$$k_2 + 2k_3 = b$$

$$k_2 + 2\left(\frac{b+c-a}{4}\right) = b$$

$$k_2 + \frac{b+c-a}{2} = b$$

$$k_2 = b - \frac{b+c-a}{2}$$

$$k_2 = \frac{2b-b-c+a}{2}$$

$$k_2 = \frac{b-c+a}{2}$$

So, $k_1 = a, k_2 = \frac{b-c+a}{2}, k_3 = \frac{b+c-a}{4}$

It means S spans V.

So, S forms basis for P_2.

Example 8: Show that the set $S = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = (1,0,0,0)$$

$$v_2 = (0,1,0,0)$$

$$v_3 = (0,0,1,0)$$

$$v_4 = (0,0,0,1)$$

Example 9: Which of the following sets of vectors are bases for R^2.

(a) $\{(1,3),(1,-1)\}$
(b) $\{(0,0),(1,2),(2,4)\}$
(c) $\{(1,2),(2,-3),(3,2)\}$
(d) $\{(1,3),(-2,6)\}$

Example 10: Which of the following sets of vectors are bases for P_3

(a) $\{t^3+2t^2+3t, 2t^3+1, 6t^3+8t^2+6t+4, t^3+2t^2+t+1\}$

(b) $\{t^3 - t, t^3 + t^2 + 1, t - 1\}$

Dimension:

The dimension of a non-zero vector space V is the number of vectors in a basis for V.

Example 1:

$dim(R^2) = 2$ standard basis are $\{(1,0), (0,1)\}$

$dim(R^3) = 3$ standard basis are $\{(1,0,0), (0,1,0), (0,0,1)\}$

$dim(R^n) = n$ standard basis are $\{(1,0, \ldots ,0), (0,1,0,0, \ldots 0), \ldots , (0,0,0, \ldots ,1)\}$

Example 2:

$$dim(M_{mn}) = mn$$

Where M_{mn} is a vector space of matrices of order $m \times n$

Example 3:

$$\mathbf{dim(P_n) = n + 1}$$