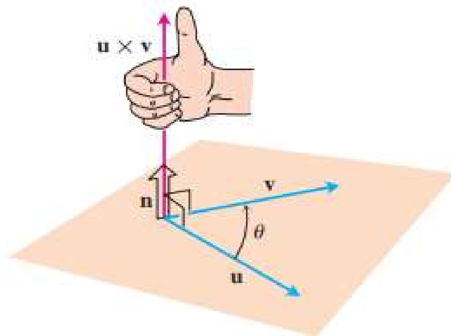
The Cross Product

Let $\vec{u}$ and $\vec{v}$ be two vectors, if $\vec{u}$ and $\vec{v}$ are not parallel, they determine a plane. We select a unit vector $\hat{n}$ perpendicular to the plane by the right-hand rule. This means that we choose $\hat{n}$ to be unit (normal) vector that points the way your right thumb points when your finger curl through the angle θ from $\vec{u}$ to $\vec{v}$. The cross-product $\vec{u}\times\vec{v}$ is a vector defined as follows:

- **Geometric Definition**

$$\vec{u}\times\ \vec{v} = (|\vec{u}||\vec{v}|Sin\ \theta)\hat{n}$$



Where $0 \le \theta \le \pi$ is the angle between $\vec{u}$&$\vec{v}$ and $\hat{n}$ is the unit vector perpendicular to $\vec{u}$&$\vec{v}$ pointing in the direction given by the right-hand rule.

- **Algebraic Definition**

$$\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$$

$$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

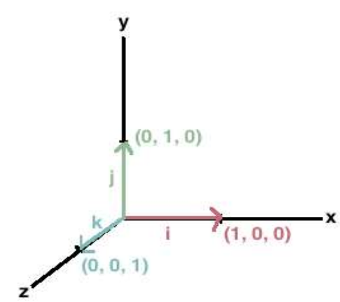
$$= \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \vec{i}(u_2v_3 - v_2u_3) - \vec{j}(u_1v_3 - v_1u_3) + \vec{k}(u_1v_2 - v_1u_2)$$

Definition:

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if

$$\vec{u}\times\vec{v} = 0$$



For Example: $\hat{\imath}\times\hat{\imath} = 0$, $\hat{\jmath}\times\hat{\jmath} = 0$, $\hat{k}\times\hat{k} = 0$

Example 1:
Find $\hat{\imath}\times\hat{\jmath}$ and $\hat{\jmath}\times\hat{\imath}$.

Solution:

For $\hat{\imath}\times\hat{\jmath}$

As $\vec{u}\times\vec{v} = (|\vec{u}||\vec{v}|Sin\ \theta)\hat{n}$

$$|\hat{\imath}| = 1$$

$$|\hat{\jmath}| = 1$$

Angle between $\hat{\imath}$ & $\hat{\jmath}$ is $\pi/2$.

By right hand rule, the vector $\hat{\imath}\times\hat{\jmath}$ is in the direction of vector $\hat{k}$ so

$$\hat{n} = \hat{k}.$$

So,

$$\hat{\imath}\times\hat{\jmath} = \left(|\hat{\imath}||\hat{\jmath}|Sin\ (\frac{\pi}{2})\right)\hat{k}$$

$$= (1.1.1)\hat{k}$$

$$\hat{\imath}\times\hat{\jmath} = \hat{k}$$

For $\vec{\jmath}\times\vec{\imath}$

The right-hand rule says that the direction of $\hat{\jmath}\times\hat{\imath}$ is $-\hat{k}$.

So, $\hat{\jmath}\times\hat{\imath} = \left(|\hat{\jmath}||\hat{\imath}|Sin\ \frac{\pi}{2}\right)(-\hat{k}) = (1.1.1)(-\hat{k})$

$$\hat{\jmath}\times\hat{\imath} = -\hat{k}$$

Similarly, we can show that

$$\hat{\jmath}\times\hat{k} = \hat{\imath},\ \hat{k}\times\hat{\imath} = \hat{\jmath}$$

Example 2: For any vector $\vec{v}$ find $\vec{v}\times\vec{v}$.

Solution:

As v is parallel to itself so $\vec{v}\times\vec{v} = 0$.

Example3:

Find the cross product of $\vec{u} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{v} = 3\vec{i} + \vec{k}$.
Also check that the cross product is perpendicular to both $\vec{u}$ & $\vec{v}$.

Solution:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}(1-0) - \vec{j}(2+6) + \vec{k}(0-3)$$

$$\vec{u} \times \vec{v} = \vec{i} - 8\vec{j} - 3\vec{k}$$

To check $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$, we will take dot product

$$\vec{u} \,.\, (\vec{u} \times \vec{v}) = \left(2\vec{i} + \vec{j} - 2\vec{k}\right) . (\vec{i} - 8\vec{j} - 3\vec{k})$$

$$= 2 - 8 + 6 = 0$$

Similarly,

$$\vec{v} \,.\, (\vec{u} \times \vec{v}) = \left(3\vec{i} + \vec{k}\right) . (\vec{i} - 8\vec{j} - 3\vec{k})$$

$$= 3 - 3 = 0$$

Practice Problems

Question: Find the cross product of the following vectors

1) $\vec{u} = 2\vec{i} - \vec{j} - \vec{k}$
   $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$

2) $\vec{u} = -3\vec{i} + 5\vec{j} + 4\vec{k}$

$$\vec{v} = \vec{i} - 3\vec{j} - \vec{k}$$

3) $\vec{u} = 2\vec{i} - \vec{j} - \vec{k}$

$$\vec{v} = -\vec{6}i + 3\vec{j} + 3\vec{k}$$

Ex. 12.4: 23, 24

Lines & Planes in Space

Lines in Space

Remark:

In plane, a line is determined by a point &a number giving the slope of the line.

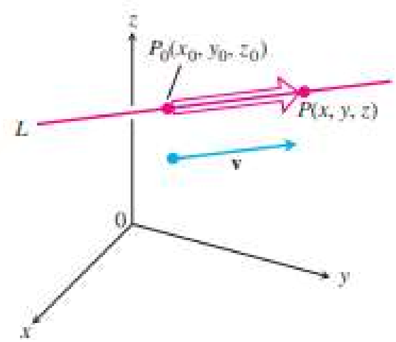
In space, a line is determined by a point through which it passes and a vector parallel to the line.

Parametric Equation & Vector Equation for a Line

Suppose that L is a line in space passing through the point $P_o(x_o, y_o, z_o)$ and parallel to a vector $\vec{v} = v_1 i + v_2 j + v_3 k$.

Then the line L is the set of all points P(x,y,z) for which $\overrightarrow{P_o P}$ is parallel to vector $\vec{v}$.

Thus $\overrightarrow{P_o P} = t\vec{v}$, for some scalar t; $t \in (-\infty, \infty)$



Note:

Two vectors $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} = t\,\vec{v}$, where t is scalar.

e.g.,

$$\vec{u} = < 1,2,3 >$$

$$\vec{v} = < 2,4,6 >$$

$$\vec{v} = 2 < 1,2,3 >$$

$$\vec{v} = 2\vec{u}$$

→ $\vec{u}$&$\vec{v}$ are parallel.

The expanded form of the equation $\overrightarrow{P_o P} = t\,\vec{v}$ is

$$(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} = t(v_1\vec{i} + v_2\vec{j} + v_3\vec{k})$$

$$x\vec{i} - x_0\vec{i} + y\vec{j} - y_0\vec{j} + z\vec{k} - z_0\vec{k} = tv_1\vec{i} + tv_2\vec{j} + tv_3\vec{k}$$

$$x\vec{i} + y\vec{j} + z\vec{k} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} + tv_1\vec{i} + tv_2\vec{j} + tv_3\vec{k}$$

$$\vec{r} = \vec{r_o} + t\vec{v}$$

$$x\vec{\imath} + y\vec{\jmath} + z\vec{k} = (x_0 + tv_1)\vec{\imath} + (y_0 + tv_2)\vec{\jmath} + (z_0 + tv_3)\vec{k}$$

- Parametric Equation

$$x = x_0 + tv_1$$

$$y = y_0 + tv_2$$

$$z = z_o + tv_3$$

Where t is any scalar and $t \in (-\infty, \infty)$

- Vector Equation

$$\vec{r} = \vec{r_o} + t\,\vec{v}$$

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

$$\vec{r}_0 = x_0\vec{\imath} + y_0\vec{\jmath} + z_0\vec{k}$$

$$\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$$

Example 1:

Find parametric equation & vector equation for the line through (-2,0,4) and parallel to the vector

$$\vec{v} = 2\vec{\imath} + 4\vec{\jmath} - 2\vec{k}$$

Solution:

$$P = (x_o, y_o, z_o) = (-2,0,4)$$

$$\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k} = 2\vec{\imath} + 4\vec{\jmath} - 2\vec{k}$$

Parametric equation of line:

$$x = x_o + t\,v_1 = -2 + 2t$$

$$y = y_o + t\,v_2 = 4t$$

$$z = z_o + t\,v_3 = 4 - 2t$$

$$-\infty < t < \infty$$

Vector Equation of Line

$$\vec{r} = \vec{r}_0 + t\,\vec{v}$$

$$x\vec{\imath} + y\vec{\jmath} + z\vec{k} = (-2\vec{\imath} + 0\vec{\jmath} + 4\vec{k}) + t\,(2\vec{\imath} + 4\vec{\jmath} - 2\vec{k})\,;\ -\infty < t < \infty$$

Example 2:

Find the parametric equation and vector equation of the line through $P(-3,2,-3)$ and $Q(1,-1,4)$.

Solution:

$$P = (x_o, y_o, z_o) = (-3,2,-3)$$

The vector parallel to line is

$$\vec{v} = \overrightarrow{PQ} = (1-(-3))\vec{\imath} + (-1-2)\vec{\jmath} + (4-(-3))\vec{k}$$

$$\vec{v} = 4\vec{\imath} - 3\vec{\jmath} + 7\vec{k}$$

Parametric equation of line:

$$x = x_o + t\, v_1 = -3 + 4t$$

$$y = y_o + t\, v_2 = 2 - 3t$$

$$z = z_o + t\, v_3 = -3 + 7t$$

$$-\infty < t < \infty$$

Vector Equation:

$$xi + yj + zk = (-3i + 2j - 3k) + t\,(4i - 3j + 7k)$$

where $-\infty < t < \infty$

Exercise 12.5 (Thomas Calculus): Q 1-20

Question 6: Find the Parametric equation of the line through the point (3, -2,1) and parallel to the line

$$x = 2 - 2t$$

$$y = 2 - t$$

$$z = 3t$$

Solution:

$$P = (3,-2,1)$$

$$\vec{v} = < 2,-1,3 >$$

$$x = 3 + 2t$$

$$y = -2 - t$$

$$z = 1 + 3t$$

Question 5: Find the Parametric equation of the line through origin and parallel to the vector

$$\vec{v} = 2\vec{j} + \vec{k}$$

Solution:

$$P = (0,0,0)$$

$$\vec{v} = 2\vec{j} + \vec{k}$$

$$x = 0 + 0t$$

$$y = 0 + 2t$$

$$z = 0 + t$$

Question 10: Find the Parametric equation of the line through the point (2, 3, 0) perpendicular to the vector$\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$ and$\vec{v} = 3\vec{i} + 4\vec{j} + 5\vec{k}$.

Solution:

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \vec{i}(10 - 12) - \vec{j}(5 - 9) + \vec{k}(4 - 6)$$

$$= -2\vec{i} + 4\vec{j} - 2\vec{k}$$

$$P = (2,3,0)$$

$$x = 2 - 2t$$

$$y = 3 + 4t$$

$$z = 0 - 2t$$

Ex. 12.5: 1-7, 10.