Calculus and Analytical Geometry

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Definite Integral, Properties of definite integral

Outline of the lecture:

The following topics will be discussed in this lecture

- Definite integral
- Fundamental theorem of calculus and its examples
- Properties of definite integral
- Examples
- Practice questions

1. Definite integral:

A function is said to be integrable on a finite closed interval $[a, b]$ if the limit

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of points x_k^* in the subintervals. In this case we denote this limit by the symbol

$$\int_a^b f(x)\, dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

and call it the **definite integral** of f from a to b. The numbers a to b are respectively called the **lower and upper limits of integration**, and $f(x)$ is the **integrand**.

- ### Theorem:

If a function f is continuous on an interval $[a, b]$, then *f* is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is

$$A = \int_a^b f(x)\ dx$$

- ### Fundamental theorem of Integral Calculus:

If a function f is continuous on an interval $[a, b]$ and *F* is antiderivative of *f*,

$$A = \int_a^b f(x)\ dx = F(b) - F(a)$$

- ### Properties of definite integral:

1. $\int_a^a f(x) = 0$
2. $\int_a^b f(x)\, dx = -\int_b^a f(x)\ dx$
3. If *c* is any point in $[a, b]$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\ dx + \int_c^b f(x)\, dx$$

Theorem: If f and g are integrable on $[a, b]$and if c is a constant, then cf , $f+g$, and $f-g$ are integrable on $[a, b]$ such that

- $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

➢ **Example 1:** Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

$$(a)\int_1^4 2x\ dx$$

$$(b)\int_{-1}^2 (x+2)\ dx$$

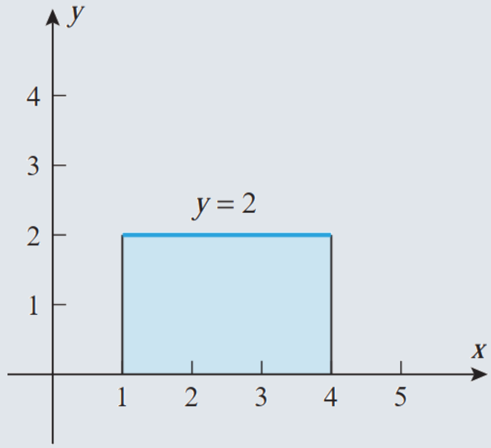
$$(c)\int_0^1 \sqrt{1-x^2}\ dx$$

Solution:

a. The graph of the integrand is the horizontal line y = 2, so the region is a rectangle of height 2 extending over the interval from 1 to 4.

$$\int_1^4 2x\ dx = length \times breadth = area\ of\ rectangle$$

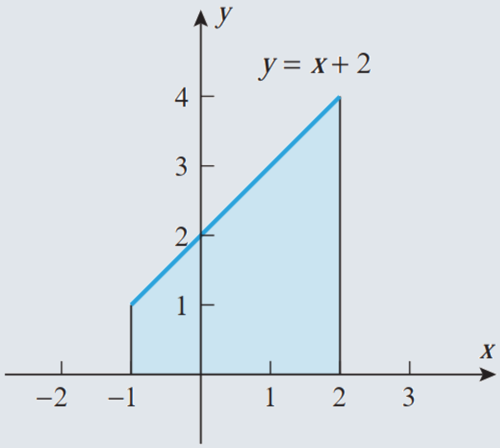
$$= 2(3) = 6$$



b. The graph of the integrand is the line $y = x + 2$, so the region is a trapezoid whose base extends from $x = -1$ to $x = 2$

$$\int_{-1}^{2} (x+2)\ dx = area\ of\ trapezoid = \frac{1}{2}(l+b)\times h$$

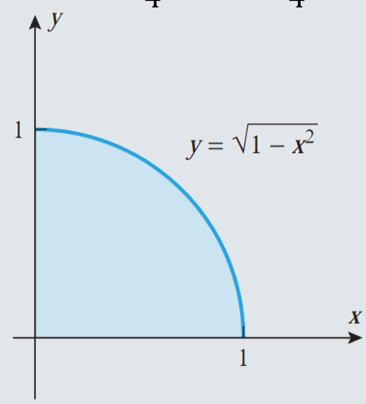
$$= \frac{1}{2}(1+4)(3) = \frac{15}{2}$$



c. The graph of $y = \sqrt{1-x^2}$ is the upper semicircle of radius 1, centered at the origin, so the region is the right quarter-circle extending from $x = \mathbf{0}$ to $x = \mathbf{1}$.

$$\int_{0}^{1} \sqrt{1-x^2}\ dx = area\ of\ quarter\ circle = \frac{1}{4}\pi r^2$$

$$= \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$



- **Example 2:** Find the area under the graph of $f(x) = 3x$ over the interval $[0, 2]$, using the **Fundamental Theorem of Integral Calculus**.

Solution:

Since *a=0* and *b=2.* So by using the fundamental theorem of Integral Calculus,

$$\int_a^b f(x)\ dx = \int_0^2 3x\ dx$$

$$= 3\left[\frac{x^2}{2}\right]_0^2 = \frac{3}{2}[(2)^2 - (0)^2]$$

$$= \frac{3}{2}(4)$$

$$= 3(2)$$

$$= 6$$

- **<u>Example 3:</u>** Find the area under the graph of $f(x) = 9 - x^2$ over the interval $[0, 3]$, using the **Fundamental Theorem of Integral Calculus**.

Solution:

$$\int_a^b f(x)\ dx = \int_0^3 9 - x^2\ dx$$

$$= \int_0^3 9\ dx - \int_0^3 x^2\ dx$$

$$= 9[x]_0^3 - \left[\frac{x^3}{3}\right]_0^3$$

$$= 9[3 - 0] - \frac{1}{3}[(3)^3 - (0)^3]$$

$$= 27 - \frac{1}{3}(27)$$

$$= 27 - 9$$

$$= 18$$

- **<u>Example 4:</u>** Find the area under the graph of $f(x) = |x - 2|$ over the interval $[0, 3]$, using the **Fundamental Theorem of Integral Calculus**.

Solution:

$$\int_a^b f(x)\ dx = \int_0^3 |x - 2|\ dx$$

$$= \int_0^2 -(x-2)\ dx + \int_2^3 (x-2)\ dx$$

$$= \int_0^2 (2-x)\ dx + \int_2^3 (x-2)\ dx$$

$$= 2[x]_0^2 - \left[\frac{x^2}{2}\right]_0^2 + \left[\frac{x^2}{2}\right]_2^3 - 2[x]_2^3$$

$$= 2[2-0] - \frac{1}{2}[(2)^2 - (0)^2] + \frac{1}{2}[(3)^2 - (2)^2] - 2[3-2]$$

$$= 4 - 2 + \frac{5}{2} - 2$$

$$= \frac{8-4+5-4}{2} = \frac{5}{2}$$

$$\int_a^b f(x)\ dx = \int_0^3 |x-2|\ dx = \frac{5}{2}$$

- **Example 5:** Find the area under the graph of $f(x) = 1 - \frac{1}{2}x$ over the interval $[-1, 1]$, using the **Fundamental Theorem of Integral Calculus**.
  Solution:

$$\int_a^b f(x)\ dx = \int_{-1}^1 1 - \frac{1}{2}x\ \ dx$$

$$= \int_{-1}^1 1\ dx - \int_{-1}^1 \frac{1}{2}x\ \ dx$$

$$= \int_{-1}^1 1\ dx - \frac{1}{2}\int_{-1}^1 x\ \ dx$$

$$= [x]_{-1}^1 - \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^1$$

$$= [1-(-1)] - \frac{1}{4}[(1)^2 - (-1)^2]$$

$$= [2] - \frac{1}{4}[(1)^2 - (-1)^2] = 2 - 0$$

$$= 2$$

$$\int_a^b f(x)\ dx = \int_{-1}^1 1 - \frac{1}{2}x\ \ dx = 2$$

- **Example 6:** Find the area under the graph of $f(x) = \cos x$ over the interval $[0, \pi]$, using the **Fundamental Theorem of Integral Calculus**.

Solution:

$$\int_a^b f(x)\ dx = \int_0^{\pi} \cos x\ \ dx$$
$$= [\sin x]_0^{\pi}$$
$$= [\sin \pi - \sin 0]$$
$$= [0 - 0] = 0$$
$$\int_a^b f(x)\ dx = \int_0^{\pi} \cos x\ \ dx = 0$$

- **Example 7:** Evaluate $\int_0^3 f(x)\ dx,$ if

$$f(x) = \begin{cases} x^2 & , if\ x \le 2 \\ 3x - 2 & , if\ x \ge 2 \end{cases}$$

Solution:

According to the intervals, $a = 0, b = 3, c = 2$. According to the properties of definite integral

$$\int_0^3 f(x)\ dx = \int_0^2 f(x)\ dx + \int_2^3 f(x)\ dx$$
$$\int_0^3 f(x)\ dx = \int_0^2 x^2\ dx + \int_2^3 3x - 2\ dx$$
$$= \int_0^2 x^2\ dx + 3\int_2^3 x\ dx - 2\int_2^3 1\ dx$$
$$= \left[\frac{x^3}{3}\right]_0^2 + 3\left[\frac{x^2}{2}\right]_2^3 - 2[x]_2^3$$
$$= \frac{1}{3}[(2)^3 - (0)^3] + \frac{3}{2}[(3)^2 - (2)^2] - 2[3 - 2]$$
$$= \frac{8}{3} + \frac{3}{2}[5] - 2$$
$$= \frac{8}{3} + \frac{15}{2} - 2$$
$$= \frac{16 + 45 - 12}{6} = \frac{49}{6}$$
$$\int_0^3 f(x)\ dx = \frac{49}{6}$$

- **<u>Example 8:</u>** In each part evaluate the integral, if

$$f(x) = \begin{cases} 2x & , if\ x \leq 2 \\ 2 & if\ x \geq 2 \end{cases}$$

$$\boldsymbol{a}. \int_0^1 f(x)\,dx \qquad \boldsymbol{b}. \int_{-1}^1 f(x)\,dx \qquad \boldsymbol{c}. \int_1^{10} f(x)\,dx \qquad \boldsymbol{d}. \int_{1/2}^5 f(x)\,dx$$

Solution:

a)

$$\int_0^1 f(x)\,dx = \int_0^1 2x\,dx = [x^2]$$

$$= [(1)^2 - (0)^2]_0^1 = 1$$

b)

$$\int_{-1}^1 f(x)\,dx = \int_{-1}^1 2x\,dx = [x^2]$$

$$= [(1)^2 - (-1)^2]_{-1}^1 = 0$$

c)

$$\int_1^{10} f(x)\,dx = \int_1^{10} 2\ \,dx = 2[x]_1^{10}$$

$$= 2[10 - 1]$$

$$= 2(9) = 18$$

d)

$$\int_{\frac{1}{2}}^5 f(x)\,dx = \int_{\frac{1}{2}}^5 2\,dx = 2[x]_{\frac{1}{2}}^5$$

$$= 2[5 - 1/2] = 2\left[\frac{9}{2}\right] = 9$$

Practice Question:

- Find the area under the graph of $y = 1 - x^2$. Over the interval $[0,2]$
- Find the area under the graph of $y = x^3$. Over the interval $[0,3]$
- Find the area under the graph of $y = x^2$. Over the interval $[-1,3]$
- Evaluate $\int_{-1}^{2} f(x)\ dx,$ if

$$f(x) = \begin{cases} |x+2| & ,if\ x \le 0 \\ x+2 & ,if\ x \ge 0 \end{cases}$$

- Evaluate $\int_{-\pi/3}^{\pi/3} \sin x\ \ dx$
- Evaluate $\int_{0}^{3} (1 - \frac{1}{2}x)\ \ dx$