Matrices and Matrix Operations

Lecture No. 4

Aniqa Naeem

University of Central Punjab

aniqa.naeem@ucp.edu.pk

November 9, 2022

Presentation Overview

1. Introduction
2. Operations on Matrices
3. Determinants
4. Properties of Determinants
5. Inverse of a Matrix

Introduction

- Arthur Cayley, an English Lawyer who became a mathematician, was the first person to develop matrices as we know them today.
- In 1858, he published **Memoir on the Theory of Matrices** which contained the definition of a matrix.

Matrix

Definition (What does the term matrix mean?)

A matrix is an array of numbers enclosed in brackets. The numbers in the array are called the **entries** or the **elements** of the matrix.

Example:

$\begin{bmatrix} 12 & 6 \\ 3 & 7 \end{bmatrix}$ is a 2×2 matrix and is called a **square matrix**.

The **size of a matrix** is given by the number of rows and columns.

General Notation

A common notation for general matrices is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Definitions (Row Matrix)

A matrix with only one row is called a **row vector** (or a row matrix).

Example:

$\begin{bmatrix} 1 & 2 \end{bmatrix}$
is a row matrix.

Continue...

Definitions (Column Matrix)

A matrix with only one column is called a **column vector** (or a column matrix).

Example:

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

is a column matrix.

Definitions (Square Matrix)

A matrix **A** with n rows and n columns is called a **square matrix** of order n, and the shaded entries are said to be on the main diagonal of A.

Continue...

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Equal Matrices

Two matrices A and B are said to be equal if

- they have the same order.
- corresponding elements in each row and column are equal.

Example:

A= $\begin{bmatrix} 1^2 & 2^2 \\ 4 & 9 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 4 \\ 2^2 & 3^2 \end{bmatrix}$ are equal matrices.

Matrix Addition and Subtraction

If A=$\left[a_{ij}\right]$ and B=$\left[b_{ij}\right]$ are both $m \times n$ matrices, then the sum $A + B$ is an $m \times n$ matrix C=$\left[c_{ij}\right]$ defined by $c_{ij} = a_{ij} + b_{ij}$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.Thus, to obtain the sum of A and B, we merely add corresponding entries.

Example:

Let A= $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$ and B= $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -4 \end{bmatrix}$ .Then

A+B= $\begin{bmatrix} 1+0 & -2+2 & 3+1 \\ 2+1 & -1+3 & 4+(-4) \end{bmatrix}$ = $\begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & 0 \end{bmatrix}$

Continue...

Similarly, the difference $A - B$ is an $m \times n$ matrix C=$\left[c_{ij}\right]$ defined by $c_{ij} = a_{ij} - b_{ij}$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$. Thus, to obtain the difference of A and B, we merely subtract corresponding entries.

Example:

Let A= $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$ and B=$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -4 \end{bmatrix}$ .Then

A-B= $\begin{bmatrix} 1-0 & -2-2 & 3-1 \\ 2-1 & -1-3 & 4-(-4) \end{bmatrix}$ = $\begin{bmatrix} 1 & -4 & 2 \\ 1 & -4 & 8 \end{bmatrix}$

Scalar Multiplication

If A=$[a_{ij}]$ is an $m \times n$ matrix and r is a real number, then the scalar multiple of A by r , rA, is the $m \times n$ matrix C=$[c_{ij}]$, where $c_{ij} = ra_{ij}$, $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$; that is , the matrix C obtained by multiplying each entry of A by r.

Example:

Let A= $\begin{bmatrix} 4 & -2 & -3 \\ 7 & -3 & 2 \end{bmatrix}$ and $r = -2$ .Then

rA= $-2\begin{bmatrix} 4 & -2 & -3 \\ 7 & -3 & 2 \end{bmatrix}$ = $\begin{bmatrix} -8 & 4 & 6 \\ -14 & 6 & -4 \end{bmatrix}$

Matrix Multiplication

If A= $[a_{ij}]$ is an $m \times r$ matrix and B = $[b_{ij}]$ is an $r \times n$ matrix, then the product of A and B denoted by AB , is the $m \times n$ matrix C= $[c_{ij}]$ whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

Example:

Let A= $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ and B= $\begin{bmatrix} -2 & 5 \\ 1 & -3 \\ 2 & 1 \end{bmatrix}$

Continue...

Then

$$AB=\begin{bmatrix}(1)(-2)+(2)(1)+(4)(2) & (1)(5)+(2)(-3)+(4)(1)\\(3)(-2)+(1)(1)+(2)(1) & (3)(5)+(1)(-3)+(2)(1)\end{bmatrix}$$

$$=\begin{bmatrix}-2+2+8 & 5-6+4\\-6+1+4 & 15-3+2\end{bmatrix}=\begin{bmatrix}8 & 3\\-1 & 14\end{bmatrix}$$

Augmented Matrix

Matrix multiplication has an important application to systems of linear equations. Consider a system of *m* linear equations in *n* unknowns:

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{gathered}
$$

Continue...

The matrix on the left side of this equation can be written as a product to give.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we write these matrices by A, x, and b, respectively, then we can replace the original system of *m* equations in *n* unknowns by the single matrix equation as

$$Ax = b$$

The matrix A in this equation is called the **coefficient matrix** of the system.
The **Augmented Matrix** for the system is obtained by adjoining b to A as the last column; thus the augmented matrix is

$$\left[A \,\middle|\, b\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Transpose of a Matrix

Definition

It is a new matrix which is made by returning the rows of the given matrix.

OR

Definition

If A is an $m \times n$ matrix, then **transpose of A** is denoted by A^t, is defined to be the $n \times m$ matrix that is obtained by making the rows of A into columns; that is, the first column of A^t is the first row of A, the second column of A^t is the second column of A, and so forth.

Continue...

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, A^t = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

Determinants

Definitions (Determinants)

The determinant of a matrix A is normally denoted by det(A) and is a scalar not a matrix.

The determinant of the general 2 by 2 matrix
A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as :
$det(A) = ad - bc.$
It means the determinant of a 2 by 2 matrix is the result of multiplying the entries of the leading diagonal and subtracting the product of the other diagonal.

Continue...

Example:

Find the determinant of A=$\begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$

Solution:$det(A) = 2(4) - 3(0) = 8 - 0 = 8$

Definitions (Minor of a Matrix)

Consider a square matrix A. Let a_{ij} be the entry in the i_{th} row and j_{th} column of matrix A. The minor M_{ij} of entry a_{ij} is the determinant of the remaining matrix after deleting the entries in the i_{th} row and j_{th} column.

Continue...

Definitions (Co-factor of a Matrix)

Consider a square matrix A. Let a_{ij} be the entry in the i^{th} row and j^{th} column of matrix A. The co-factor c_{ij} of the entry a_{ij} is defined as
$c_{ij} = (-1)^{i+j} M_{ij}$
where M_{ij} is the minor of entry a_{ij}.

Example:

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

Continue...

Solution:

The co-factor of $a_{11} = 3$ is

$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix}$=5 $\times$ 8 $-$ 4 $\times$ 6=40 $-$ 24=16.

The co-factor of $a_{32} = 4$ is

$C_{32} = (-1)^{3+2} M_{32} = (-1)^5 \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}$=-16 $\times 3 - 2 \times -4$=$-(18+8) = -26$.

Continue...

Definition

If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by corresponding co-factors and adding the resulting products is called the **determinant of A**, and the sums themselves are called **co-factor expansion of A**.

That is:
Co-factor expansion along the j_{th} column.
$det(A) = a_{1j}c_{1j} + a_{2j}c_{2j} + ... + a_{nj}c_{nj}$
Co-factor expansion along the i_{th} row.
$det(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + ... + a_{in}c_{in}$

Example

Example: Find the determinant of the matrix.

$$A=\begin{bmatrix} -7 & -14 & 21 \\ 5 & 10 & 1 \\ -52 & 8 & 12 \end{bmatrix}$$

Solution:

$$det(A)=\begin{vmatrix} -7 & -14 & 21 \\ 5 & 10 & 1 \\ -52 & 8 & 12 \end{vmatrix}$$

$$=-7\begin{vmatrix} 10 & 1 \\ 8 & 12 \end{vmatrix} - (-14)\begin{vmatrix} 5 & 1 \\ -52 & 12 \end{vmatrix} + 21\begin{vmatrix} 5 & 10 \\ -52 & 8 \end{vmatrix}$$

$$=-7(10\times 12 - 8\times 1) + 14(5\times 12 - (-52\times 1)) + 21(5\times 8 - (-52\times 10))$$

$$=-7(120-8) + 14(60+52) + 21(40+520)$$

$$=-7(112) + 14(112) + 21(560)$$

$$=-784 + 1568 + 11760 = 12544$$

Remark:
If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $det(A)$ is the product of the entries on the main diagonal of the matrix, that is, $det(A)= a_{11}a_{22}...a_{nn}$.

Basic Properties of Determinants

Suppose that A and B are $n \times n$ matrices and k is any scalar. Then:

1. $det(A + B) \neq det(A) + det(B)$

Example:

$A = \begin{pmatrix} 5 & -6 \\ 0 & -12 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 0 \\ 1 & 9 \end{pmatrix}$

$A + B = \begin{pmatrix} 5 + (-3) & -6 + 0 \\ 0 + 1 & 12 + 9 \end{pmatrix} = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}$

$det(A + B) = (2)(-3) = (1)(-6) = -6 + 6 = 0$

$det(A) = (5)(-12) - (0)(-6) = -60 - 0 = -60$

$det(B) = (-3)(9) - (1)(0) = -27 - 0 = -27$

$det(A) + det(B) = -60 + (-27) = -60 - 27 = -87$

Clearly,

$det(A + B) \neq det(A) + det(B)$

Continue...

2. $det(kA) = k^n det(A)$

Example: $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $k = 2$

$$kA = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$det(kA) = 2 \times 8 - 6 \times 4 = 16 - 24 = -8$$

$$k^2 det(A) = 2^2 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4(1 \times 4 - 3 \times 6) = 4(4 - 6) = 4(-2) = -8$$

$$det(kA) = k^2 det(A)$$

Continue...

3. $det(AB) = det(A)det(B)$

Example: $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}5 & 6\\ 7 & 6\end{bmatrix}$

$det(A) = 1 \times 4 - 3 \times 2 = 4 - 6 = -2$
$det(B) = 5 \times 6 - 7 \times 6 = 30 - 42 = -12$
$AB =$
$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \times \begin{bmatrix}5 & 6\\ 7 & 6\end{bmatrix} = \begin{bmatrix}1(5)+7(2) & 1(6)+2(6)\\ 3(5)+4(7) & 3(6)+4(6)\end{bmatrix} = \begin{bmatrix}5+14 & 6+12\\ 15+28 & 18+24\end{bmatrix}$$
$$= \begin{bmatrix}19 & 18\\ 43 & 42\end{bmatrix}$$
$$det(AB) = \begin{vmatrix}19 & 18\\ 43 & 42\end{vmatrix} = 19 \times 42 - 43 \times 18 = 798 - 774 = 24$$
$det(AB) = det(A)det(B)$

Continue...

4. A square matrix A is invertible if and only if $det(A) \neq 0$.
5. $det(A^{-1} = \frac{1}{a_{11}}\frac{1}{a_{22}}...\frac{1}{a_{nn}})$
6. The points $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2), P_3(x_3, y_3, z_3), P_4(x_4, y_4, z_4)$ are not co-planar if and only if the determinant.

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \neq 0$$

Remarks:

If A is an $n \times n$ matrix, then the following statements are equivalent.

- A is invertible
- $Ax = 0$ has only the trivial solution.
- The reduced row echelon form of A is I_n.
- A can be expressed as a product of elementary matrices.
- $Ax = b$ is consistent for every $n \times 1$ matrix b.
- $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.
- $det(A) \neq 0$

Definition (Inverse of a Matrix)

If A is an $n \times n$ matrix, A matrix C of order $n \times n$ is called multiplicative inverse of A if $AC = I = CA$ where I is the $n \times n$ identity matrix.

Definitions (Invertible Matrix)

If the inverse of a square matrix exists, it is called an invertible matrix.
In this case, we say that A is invertible and we call C an inverse of A.

Example:

Example:

$A=\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

Then

$AB=\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$

$BA=\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$

Thus, A and B are invertible and each is the inverse of other.

Remarks:

- In general, a square matrix with a row or column of zeros is singular.
- An invertible matrix has exactly one inverse.

Formula for the Inverse of a Matrix:

If A is an $n \times n$ matrix and $det(A) \neq 0$, then

$$A^{-1} = \frac{1}{det(A)}(adjA) = \begin{bmatrix} \frac{A_{11}}{detA} & \frac{A_{21}}{detA} & \cdots & \frac{A_{n1}}{detA} \\ \frac{A_{12}}{detA} & \frac{A_{22}}{detA} & \cdots & \frac{A_{n2}}{detA} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \frac{A_{1n}}{detA} & \frac{A_{2n}}{detA} & \cdots & \frac{A_{nn}}{detA} \end{bmatrix}$$

For a 2×2 matrix this formula is easy to use.
For example:

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$ in which case the inverse is given by the formula

$$A_{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Examples:

Example 1: Determine whether the matrix is invertible. If so, find its inverse.

(a) A= $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

Solution:

$det(A)$=3 $\times$ 6 - 5 $\times$ 4=18 $-$ 20=$-$2$\neq$ 0. Thus A is invertible and its inverse is

$$A^{-1}= \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{6}{-2} & \frac{-4}{-2} \\ \frac{-5}{-2} & \frac{3}{-2} \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & \frac{3}{-2} \end{bmatrix}$$

(b) A= $\begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}$

Solution:

$det(A)$=1 $\times$ 10 - 5 $\times$ 2=10 $-$ 10=0

Thus the matrix is not invertible since $det(A) = 0$

Continue...

Example 2:

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

We leave it for you to show that

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

and also that

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}, \quad B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Remark:

If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

For a matrix of order higher then 2×2 we use the inversion algorithm to find its inverse(based on steps used for the reduced row echelon form)

Inversion Algorithm

To find the inverse of an invertible matrix A, find the sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

Example 3: Find the inverse of

$$A=\begin{bmatrix} 4 & -2 & 1 \\ 5 & 0 & 3 \\ -1 & 2 & 6 \end{bmatrix}$$

Solution: $\left[A \,\middle|\, b\right]=\left[\begin{array}{ccc|ccc} 4 & -2 & 1 & 1 & 0 & 0 \\ 5 & 0 & 3 & 0 & 1 & 0 \\ -1 & 2 & 6 & 0 & 0 & 1 \end{array}\right]$

Continue...

$$\left[\begin{array}{ccc|ccc} 1 & \frac{-1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 5 & 0 & 3 & 0 & 1 & 0 \\ -1 & 2 & 6 & 0 & 0 & 1 \end{array}\right] \qquad R_1 = \frac{R_1}{4}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{-1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{5}{2} & \frac{7}{4} & \frac{-5}{4} & 1 & 0 \\ -1 & 2 & 6 & 0 & 0 & 1 \end{array}\right] \qquad R_2 = R_2 - 5R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{-1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{5}{2} & \frac{7}{4} & \frac{-5}{4} & 1 & 0 \\ 0 & \frac{3}{2} & \frac{25}{4} & \frac{1}{4} & 0 & 1 \end{array}\right] \qquad R_3 = R_3 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{-1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{7}{10} & \frac{-1}{2} & \frac{2}{5} & 0 \\ 0 & \frac{3}{2} & \frac{25}{4} & \frac{1}{4} & 0 & 1 \end{array}\right] \qquad R_2 = \frac{2}{5}R_2$$

Continue...

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{7}{10} & \frac{-1}{2} & \frac{2}{5} & 0 \\ 0 & \frac{3}{2} & \frac{25}{4} & \frac{1}{4} & 0 & 1 \end{array}\right] \qquad R_1 = R_1 + \frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{7}{10} & \frac{-1}{2} & \frac{2}{5} & 0 \\ 0 & 0 & \frac{26}{4} & 1 & \frac{-3}{5} & 1 \end{array}\right] \qquad R_3 = R_3 - \frac{3}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{7}{10} & \frac{-1}{2} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & \frac{5}{26} & \frac{-3}{26} & \frac{5}{26} \end{array}\right] \qquad R_3 = \frac{5}{26}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{26} & \frac{7}{26} & \frac{-3}{26} \\ 0 & 1 & \frac{7}{10} & \frac{-1}{2} & \frac{2}{5} & 0 \\ 0 & 0 & 1 & \frac{5}{26} & \frac{-3}{26} & \frac{5}{26} \end{array}\right] \qquad R_1 = R_1 - \frac{3}{5}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{26} & \frac{7}{26} & \frac{-3}{26} \\ 0 & 1 & 0 & \frac{-33}{52} & \frac{25}{52} & \frac{-7}{52} \\ 0 & 0 & 1 & \frac{5}{26} & \frac{-3}{26} & \frac{5}{26} \end{array}\right]$$

$$R_2 = R_2 - \frac{7}{10}R_3$$

$$A^{-1} = \begin{bmatrix} \frac{-3}{26} & \frac{7}{26} & \frac{-3}{26} \\ \frac{-33}{52} & \frac{25}{52} & \frac{-7}{52} \\ \frac{5}{26} & \frac{-3}{26} & \frac{5}{26} \end{bmatrix}$$

Continue...

Example 4: A= $\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

Solution: $\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \qquad R_3 = R_3 + R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \qquad R_2 = R_2 - 2R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \qquad R_3 = R_3 + R_2$$

Since the last row is zero on left side, so A is not invertible.

Exercise

Q1.

a. Compute

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$$

b. Check whether the points (0,4,1), (4,0,0), (3,5,2) and (2,2,5) lie in the same plane or not.

Q2. For any square matrices A, B , P with P invertible, Complete the following

det A^{-1}= .... det(AB)=................ det(PAP^{-1}) =...........

Q3. Let A and B be 3 × 3 matrices with detA= 4 and det B = -3. Use properties of determinants to compute:

a. det(AB)
b. det(5A)
c. detB^T
d. detA^{-1}
e. detA^3

Q4. Determine whether the statement is true or false, and justify your answer.

a. For all square matrices A and B of the same size

$$(A+B)^2 = A^2 + 2AB + B^2.$$

b. The product of two elementary matrices of the same size must be an elementary matrix.

c. If A is an $n \times n$ matrix that is not invertible, then the linear system Ax=0 has infinitely many solutions.

d. It is impossible for a linear system of linear equations to have exactly two solutions.

e. If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = A^{-1}B^{-1}.$$

f. Every elementary matrix is invertible.

g. If A is invertible and a multiple of the first row of A is added to the second row, then the resulting matrix is invertible.

h. If the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution, then the linear system $\boldsymbol{Ax} = \boldsymbol{c}$ also must have a unique solution.

Q5. Use the inversion algorithm to find the inverses of the given matrices, if exist.

(iii) $\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \\ 5 & 5 & 1 \end{bmatrix}$

Q6. Find all values of c, if any, for which the given matrix is invertible.

(iii) $\begin{bmatrix} c & -c & 1 \\ 1 & 1 & c \\ 0 & 1 & c \end{bmatrix}$

(iv) $\begin{bmatrix} c & c & -1 \\ 1 & 1 & 2c \\ 0 & 1 & c \end{bmatrix}$

Continue...