Chapter 7: Laplace Transform

DEFINITION 7.1.1 Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt \tag{2}$$

is said to be the **Laplace transform** of f, provided that the integral converges.

EXAMPLE 1 Applying Definition 7.1.

Evaluate $\mathscr{L}\{1\}$.

SOLUTION From (2),

$$\begin{aligned}\mathscr{L}\{1\} &= \int_0^{\infty} e^{-st}(1)\, dt = \lim_{b\to\infty} \int_0^{b} e^{-st}\, dt \\ &= \lim_{b\to\infty} \frac{-e^{-st}}{s}\Bigg|_0^b = \lim_{b\to\infty} \frac{-e^{-sb}+1}{s} = \frac{1}{s}\end{aligned}$$

provided that $s \quad 0$. In other words, when $s \quad 0$, the exponent $-sb$ is negative, and $e^{-sb} \to 0$ as $b \to \infty$. The integral diverges for $s < 0$.

EXAMPLE 2 Applying Definition 7.1.

Evaluate $\mathscr{L}\{t\}$.

SOLUTION From Definition 7.1.1 we have $\mathscr{L}\{t\} = \int_0^{\infty} e^{-st}\, t\, dt$. Integrating by parts and using $\lim_{t\to\infty} te^{-st} = 0$, $s \quad 0$, along with the result from Example 1, we obtain

$$\mathscr{L}\{t\} = \frac{-te^{-st}}{s}\Bigg|_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st}\, dt = \frac{1}{s}\mathscr{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}.$$

EXAMPLE 3 Applying Definition 7.1.

Evaluate **(a)** $\mathscr{L}\{e^{-3t}\}$ **(b)** $\mathscr{L}\{e^{5t}\}$

SOLUTION In each case we use Definition 7.1.1.

(a)

$$\begin{aligned}\mathscr{L}\{e^{-3t}\} &= \int_0^\infty e^{-3t}e^{-st}\,dt = \int_0^\infty e^{-(s+3)t}\,dt\\ &= \left.\frac{-e^{-(s+3)t}}{s+3}\right|_0^\infty\\ &= \frac{1}{s+3}.\end{aligned}$$

The last result is valid for $s > -3$ because in order to have $\lim_{t\to\infty} e^{-(s+3)t} = 0$ we must require that $s + 3 > 0$ or $s > -3$.

(b)

$$\begin{aligned}\mathscr{L}\{e^{5t}\} &= \int_0^\infty e^{5t}e^{-st}\,dt = \int_0^\infty e^{-(s-5)t}\,dt\\ &= \left.\frac{-e^{-(s-5)t}}{s-5}\right|_0^\infty\\ &= \frac{1}{s-5}.\end{aligned}$$

In contrast to part (a), this result is valid for $s \quad 5$ because $\lim_{t\to\infty} e^{-(s-5)t} = 0$ demands $s - 5 > 0$ or $s \quad 5$.

EXAMPLE 4 Applying Definition 7.1.

Evaluate $\mathscr{L}\{\sin 2t\}$.

SOLUTION From Definition 7.1.1 and two applications of integration by parts we obtain

$$\begin{aligned}\mathscr{L}\{\sin 2t\} &= \int_0^\infty e^{-st}\sin 2t\,dt = \left.\frac{-e^{-st}\sin 2t}{s}\right|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt\\ &= \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt, \qquad s \quad 0\end{aligned}$$

$\lim_{t\to\infty} e^{-st}\cos 2t = 0,\ s \quad 0$ Laplace transform of $\sin 2t$

$$\begin{aligned}&= \frac{2}{s}\left[\left.\frac{-e^{-st}\cos 2t}{s}\right|_0^\infty - \frac{2}{s}\int_0^\infty e^{-st}\sin 2t\,dt\right]\\ &= \frac{2}{s^2} - \frac{4}{s^2}\mathscr{L}\{\sin 2t\}.\end{aligned}$$

At this point we have an equation with $\mathscr{L}\{\sin 2t\}$ on both sides of the equality. Solving for that quantity yields the result

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2 + 4}, \qquad s \quad 0.$$

EXAMPLE 5 Linearity of the Laplace Transform

In this example we use the results of the preceding examples to illustrate the linearity of the Laplace transform.

(a) From Examples 1 and 2 we have for $s \quad 0$,

$$\mathscr{L}\{1 + 5t\} = \mathscr{L}\{1\} + 5\mathscr{L}\{t\} = \frac{1}{s} + \frac{5}{s^2}.$$

(b) From Examples 3 and 4 we have for $s \quad 5$,

$$\mathscr{L}\left\{4e^{5t} - 10 \sin 2t\right\} = 4\mathscr{L}\left\{e^{5t}\right\} - 10\mathscr{L}\{\sin 2t\} = \frac{4}{s - 5} - \frac{20}{s^2 + 4}.$$

(c) From Examples 1, 2, and 3 we have for $s \quad 0$,

$$\begin{aligned} \mathscr{L}\{20e^{-3t} + 7t - 9\} &= 20\mathscr{L}\{e^{-3t}\} + 7\mathscr{L}\{t\} - 9\mathscr{L}\{1\} \\ &= \frac{20}{s + 3} + \frac{7}{s^2} - \frac{9}{s}. \end{aligned}$$

THEOREM 7.1.1 Transforms of Some Basic Functions

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$

(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s - a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2 + k^2}$

(e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2 + k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2 - k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2 - k^2}$

This result in (b) of Theorem 7.1.1 can be formally justified for n a positive integer using intergration by parts to first show tha

$$\mathscr{L}\{t^n\} = \frac{n}{s}\,\mathscr{L}\{t^{n-1}\}.$$

Then for $n = 1, 2,$ and 3, we have, respectively,

$$\mathscr{L}\{t\} = \frac{1}{s}\cdot\mathscr{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

$$\mathscr{L}\{t^2\} = \frac{2}{s}\cdot\mathscr{L}\{t\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2\cdot 1}{s^3}$$

$$\mathscr{L}\{t^3\} = \frac{3}{s}\cdot\mathscr{L}\{t^2\} = \frac{3}{s}\cdot\frac{2\cdot 1}{s^3} = \frac{3\cdot 2\cdot 1}{s^4}$$

If we carry on in this manner, you should be convinced that

$$\mathscr{L}\{t^n\} = \frac{n\cdots 3\cdot 2\cdot 1}{s^{n+1}} = \frac{n!}{s^{n+1}}.$$

EXERCISES 7.1

In Problems 19–36 use Theorem 7.1.1 to find $\mathscr{L}\{f(t)\}$.

19. $f(t) = 2t^4$

20. $f(t) = t^5$

21. $f(t) = 4t - 10$

22. $f(t) = 7t + 3$

23. $f(t) = t^2 + 6t - 3$

24. $f(t) = -4t^2 + 16t + 9$

25. $f(t) = (t + 1)^3$

26. $f(t) = (2t - 1)^3$

27. $f(t) = 1 + e^{4t}$

28. $f(t) = t^2 - e^{-9t} + 5$

29. $f(t) = (1 + e^{2t})^2$

30. $f(t) = (e^t - e^{-t})^2$

31. $f(t) = 4t^2 - 5\sin 3t$

32. $f(t) = \cos 5t + \sin 2t$