Evaluating Double Integrals over Nonrectangular Region:

In the case when the region R is a nonrectangular region, the limits of integration in the inner integral are **not constants** and the double integral can be of **two types**:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$

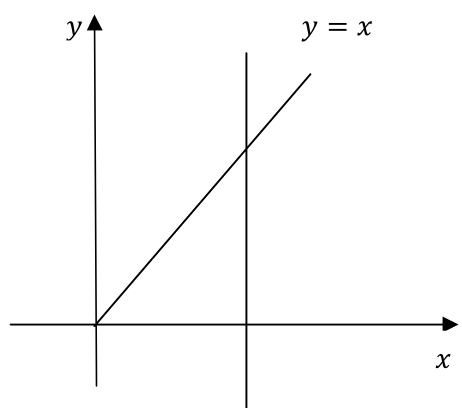
Examples:

1. Find the volume of the prism whose base is the triangle in the xy-planeformed by the x-axis and the lines$y = x$ &$x = 1$ and whose top lies in the plane

$$z = f(x,y) = 3 - x - y.$$

Solution: $\boldsymbol{Volume} = \iint_R f(x,y)dA$

$= \int_0^1 \int_0^x (3 - x - y)dydx$



$$= \int_0^1 \left(\int_0^x 3dy - x\int_0^x dy - \int_0^x ydy \right) dx$$

$= \int_0^1 \left(|3y|_0^x - x|y|_0^x - \left|\frac{y^2}{2}\right|_0^x \right) dx$

$= \int_0^1 \left([3(x) - 3(0)] - x(x - 0) - \left|\frac{x^2}{2} - \frac{0^2}{2}\right|_0^x \right) dx$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2} \right) dx = \int_0^1 \left(3x - \left[\frac{2x^2 + x^2}{2}\right] \right) dx$$

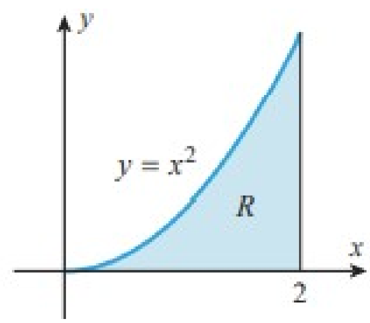
$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = 3\int_0^1 x dx - \frac{3}{2}\int_0^1 x^2\, dx$$

$$= 3\left|\frac{x^2}{2}\right|_0^1 - \frac{3}{2}\left|\frac{x^3}{3}\right|_0^1 = \frac{3}{2}(1^2 - 0) - \frac{3}{6}(1^3 - 0)$$

$$= \frac{3}{2} - \frac{1}{2} = \frac{2}{2} = 1$$

2. Evaluate $\iint_R (x + y)\, dA$ where R is the region enclosed by the parabola $y = x^2$, the line $x = 2$ and the x-axis.

Solution:



$$\iint_R (x + y)\, dA$$

$$= \int_0^2 \int_0^{x^2} (x + y)\, dy\, dx$$

$$= \int_0^2 \left(\int_0^{x^2} x dy + \int_0^{x^2} y\, dy\right) dx$$

$$= \int_0^2 \left(|xy|_0^{x^2} + \left|\frac{y^2}{2}\right|_0^{x^2}\right) dx$$

$$= \int_0^2 \left(x^3 - \frac{x^4}{2}\right) dx$$

$$= \left|\frac{x^4}{4}\right|_0^2 - \frac{1}{2}\left|\frac{x^5}{5}\right|_0^2 = \frac{4}{5}$$

Evaluate the double integrals.

1. $\int_0^1 \int_{x^2}^{x} xy^2\, dy\, dx$

2. $\int_1^{3/2} \int_y^{3-y} y\, dx\, dy$

3. $\int_0^3 \int_0^{\sqrt{9-y^2}} y\, dx\, dy$

4. $\int_{1/4}^1 \int_{x^2}^{x} \sqrt{\frac{x}{y}}\, dy\, dx$