# Calculus and Analytical Geometry

**Lecture no. 19**

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## Topic: Integration by Parts

- Integration by parts
- Definition
- LIATE
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## 1. INTEGRATION BY PARTS:

Integration by Parts is a special method of integration that is often useful when two functions are multiplied together, but is also helpful in other ways.

**RULE:**

$$\int u\,dv = uv - \int v du$$

Where, $u = f(x)\,, dv = g(x)\,dx$

$$du = f'(x)\,dx\,,\ \ v = G(x)$$

**LIATE:**

LIATE is a useful strategy for choosing $u$ and $dv$ that can be applied when the integrand is a product of two functions from *different* categories in the list:
**L**ogarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential

**EXAMPLES:**

i. **Use integration by parts to evaluate $\int x^3 lnx\,dx$.**

According to LIATE,

$$u = \ln x\ \ ,\ \ dv = x^3 dx$$

$$du = \frac{1}{x}dx\ \ and\ v = \int x^3 dx = \frac{x^4}{4}$$

Now by using the formula,

$$\int u\,dv = uv - \int v du$$

$$\int x^3 lnx\,dx = \tfrac{x^4}{4}\ln x - \int \left(\tfrac{x^4}{4}\right)\tfrac{1}{x}dx$$

$$= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx$$

$$= \frac{x^4}{4}\ln x - \frac{1}{4}\left(\frac{x^4}{4}\right) + C$$

$$= \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$$

**ii. Use integration by parts to evaluate $\int \boldsymbol{\sin x \ ln(cosx)dx}$.**

$u = ln(cosx)$ (logarithmic function)

$dv = sin\ x\ dx$ (Trigonometric function)

$$du = \frac{1}{\cos x}(-\sin x)\ dx\,, \quad v = \int \sin x\ dx = -\cos x$$

Now, by using the formula

$$\int u\ dv = uv - \int vdu$$

$$\int sin\ x\ ln(cosx)dx = uv - \int vdu$$

$$= (\ln(cosx))(-cosx) - \int (-cosx)(-tanx)dx$$

$$= -cosx \ln(cosx) - \int (\cos x)\frac{\sin x}{\cos x} dx$$

$$= -\cos x \ln(cosx) - \int \sin x\ \ dx$$

$$= -\cos x \ln(cosx) + \cos x + C$$

**iii. Evaluate $\int \boldsymbol{x^2 e^x\ dx}$**

According to LIATE,

$$u = x^2\ ,\ \ dv = e^x dx$$

$$du = 2x\ dx\ \ and\ v = \int e^x dx = e^x$$

Now by using the formula,

$$\int u\ dv = uv - \int vdu$$

$$\int x^2 e^x\ dx = x^2 e^x - \int e^x . 2x\ dx$$

$$= x^2 e^x - 2\int xe^x\ dx \qquad (1)$$

Note that, to compute $\int xe^x\ dx$ we again need to use the formula.

$$u = x \ , \ \ dv = e^x dx$$

$$du = \ dx \ \ and \ v = \int e^x dx = e^x$$

$$\int u \, dv = uv - \int v du$$

$$\int xe^x \, dx = xe^x - \int e^x . \ dx$$

$$= xe^x - e^x$$

Substitute in equation (1)

$$\int x^2 e^x \, dx = x^2 e^x - 2[xe^x - e^x] + C$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

iv. **Integrate $\int e^x \cos x \, dx$.**

According to LIATE,

$$u = \cos x \, , \ \ dv = e^x \ \ dx$$

$$du = -\sin x \ \ dx \ \ and \ v = \int e^x \ \ dx = e^x$$

Now by using the formula,

$$\int u \, dv = uv - \int v du$$

$$\int e^x \cos x \, dx = e^x \cos x - \int e^x . -\sin x \, dx$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x . \sin x \, dx \qquad (1)$$

To integrate $\int e^x \sin x \, dx$

$$u = -\sin x \ \ , \ \ dv = e^x dx$$

$$du = -\cos x \ \ dx \ \ and \ v = \int e^x dx = e^x$$

Now by using the formula,

$$\int u\,dv = uv - \int v du$$

$$\int e^x \sin x\,dx = -e^x \sin x - \int (-\cos x)\, e^x\,dx$$

$$= -e^x \sin x + \int \cos x\, e^x\,dx$$

Substitute in equation (1)

$$\int e^x \cos x\,dx = e^x \cos x - \left(-e^x \sin x + \int \cos x\, e^x\,dx\right)$$

$$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int \cos x\, e^x\,dx$$

Now, by adding the constant of integral

$$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x + C_1$$

$$\int e^x \cos x\,dx = \frac{e^x \cos x + e^x \sin x}{2} + C$$

## 2. TABULAR INTEGRATION:

Integrals of the form

$$\int p(x) f(x)\,dx$$

where $p(x)$ is a polynomial, can sometimes be evaluated using repeated integration by parts in which $u$ is taken to be $p(x)$ or one of its derivatives at each stage. Since $du$ is computed by differentiating $u$, the repeated differentiation of $\boldsymbol{p(x)}$ will eventually produce 0, at which point you may be left with a simplified integration problem. A convenient method for organizing the computations into two columns is called ***tabular integration by parts***.

**RULES:**

***Tabular Integration by Parts***

**Step 1.** Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

**Step 2.** Integrate $f(x)$ repeatedly and list the results in the second column.
**Step 3.** Draw an arrow from each entry in the first column to the entry that is one row down in the second column.
**Step 4.** Label the arrows with alternating + and - signs, starting with a +.
**Step 5.** For each arrow, form the product of the expressions at its tip and tail and then
multiply that product by +1 or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

**Examples:**

**1) Integrate $\int (x^2 - x)\cos x\, dx$**

| REPEATED DIFFERENTIATION | | REPEATED INTEGRATION |
|---|---|---|
| $x^2 - x$ | $+$ | $\cos x$ |
| $2x - 1$ | $-$ | $\sin x$ |
| $2$ | $+$ | $-\cos x$ |
| $0$ | | $-\sin x$ |

Combine the product of functions connected by arrows according to the operation signs above the arrow to obtain

$$\int (x^2 - x)\cos x\, dx = (x^2 - x)\sin x - (2x - 1)(-\cos x) + 2(-\sin x) + C$$

$$= (x^2 - x)\sin x + (2x - 1)(\cos x) - 2\sin x + C$$
$$= (x^2 - x - 2)\sin x + (2x - 1)(\cos x) + C$$

**2) Integrate $\int x^2\sqrt{x-1}\,dx$**

| REPEATED DIFFERENTIATION | | REPEATED INTEGRATION |
|---|---|---|
| $x^2$ | $+$ | $(x-1)^{1/2}$ |
| $2x$ | $-$ | $\frac{2}{3}(x-1)^{3/2}$ |
| $2$ | $+$ | $\frac{4}{15}(x-1)^{5/2}$ |
| $0$ | | $\frac{8}{105}(x-1)^{7/2}$ |

Combine the product of functions connected by arrows according to the operation signs above the arrow to obtain

$$\int x^2\sqrt{x-1}\,dx = x^2.\frac{2}{3}(x-1)^{\frac{3}{2}} - 2x.\frac{2}{3}(x-1)^{\frac{5}{2}} + 2.\frac{8}{105}(x-1)^{\frac{7}{2}} + C$$

**Practice Questions:**

- $\int xe^{-2x}$
- $\int x^2 \sin x \, dx$
- $\int y\sqrt{y+3}\, dy$
- $\int x\, e^{-x}\, dx$
- $\int x \sin 3x \, dx$
- $\int x \ln x \, dx$
- $\int x^2 e^x \, dx$