Calculus and Analytical Geometry

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Area Between two Curves

Outline of the lecture:

The following topics will be discussed in this lecture

- Area between two curves
- Examples
- Practice questions

Area Between Two Curves

If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of region bounded below by $y = f(x)$, above by $y = g(x)$, on the left by the line x = a, and on the right by the line $x = b$ is

$$Area = A = \int_a^b [f(x) - g(x)]\ dx$$

If f and g are continuous functions on the interval $[a, b]$, and if $g(x) \geq f(x)$ for all x in $[a, b]$, then the area of region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$Area = A = \int_a^b [g(x) - f(x)]\ dx$$

Example: Find the area of the region bounded above by $f(x) = x + 6$, bounded below by $g(x) = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution:

The formula for area of the bounded region is given as

$$Area = A = \int_a^b [f(x) - g(x)]\ dx$$

Putting values, we have

$$A = \int_0^2 [(x + 6) - x^2]\ dx$$

$$A = \int_0^2 x\ dx + 6\int_0^2 1\ dx - \int_0^2 x^2\ dx$$

$$= \left[\frac{x^2}{2}\right]_0^2 + 6[x]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}[(2)^2 - (0)^2] + 6[2 - 0] - \frac{1}{3}[(2)^3 - (0)^3]$$

$$= \frac{1}{2}[4] + 6[2] - \frac{1}{3}[8]$$

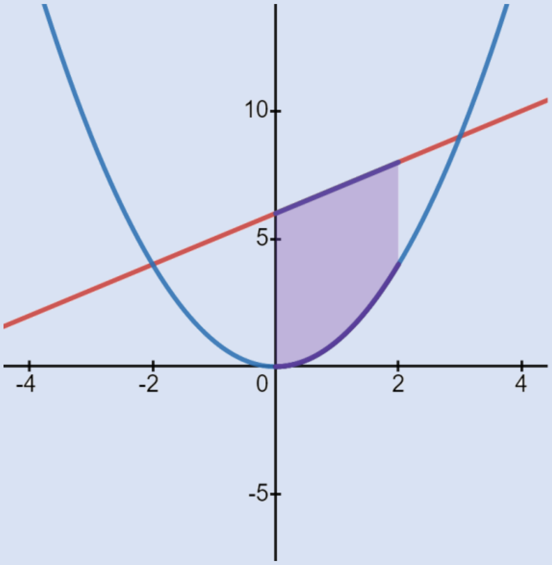
$$= 2 + 12 - \frac{8}{3}$$

$$= 14 - \frac{8}{3} = \frac{34}{3}$$

Hence, the required area is,

$$A = \frac{34}{3}$$

Sketch the area between two curves:



<u>Example 2:</u> Find the area of the region enclosed between the curves $f(x) = x + 6$ and $g(x) = x^2$.

Solution:

Step 1: Limits of integration

In order to find limits of integration, we find points of intersections of the curves $y = x + 6$ and $y = x^2$**.** So by comparing equations of both curves, we can write

$$x^2 = x + 6$$

$$x^2 - x - 6 = 0$$

$$x^2 - 3x + 2x - 6 = 0$$

$$x(x - 3) + 2(x - 3) = 0$$

$$(x + 2)(x - 3) = 0$$

$$x + 2 = 0\,, x - 3 = 0$$

From here,

$$x = -2, 3$$

The limits of integration are -2 and 3.

Step 2: Upper and lower curves

In order to decide which curve lies above the other, we take a test point in the interval $[-2, 3]$ and find values of $f(x)$ and $g(x)$ at that point. The curve with larger value will lie above the other.

For $x = 0$	$f(0) = 6$	$f(0) > g(0)$
For $x = 0$	$g(0) = 0$	

Hence, the graph of $f(x)$ lies above the graph of $g(x)$.

Step 3: Area Between two curves

The formula for area of the bounded region between two curves is given as

$$Area = A = \int_a^b [f(x) - g(x)]\ dx$$

Putting values, we have

$$A = \int_{-2}^{3} [(x+6) - x^2]\ dx$$

$$A = \int_{-2}^{3} x\ dx + 6\int_{-2}^{3} 1\ dx - \int_{-2}^{3} x^2\ dx$$

$$= \left[\frac{x^2}{2}\right]_{-2}^{3} + 6[x]_{-2}^{3} - \left[\frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{1}{2}[(3)^2 - (-2)^2] + 6[3 - (-2)] - \frac{1}{3}[(3)^3 - (-2)^3]$$

$$= \frac{1}{2}[9 - 4] + 6[3 + 2] - \frac{1}{3}[27 + 8]$$

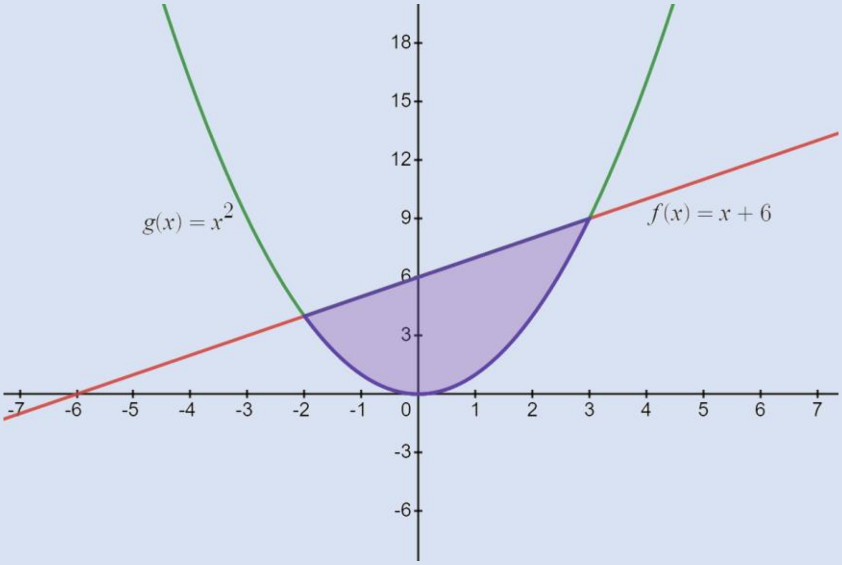
$$= \frac{5}{2} + 30 - \frac{35}{3}$$

$$= \frac{15 + 180 - 70}{3} = \frac{125}{3}$$

Hence, the required area is,

$$A = \frac{125}{3}$$

Sketch the area between two curves:



<u>Example 3:</u> Find the area of the region enclosed between the curves $f(x) = 2x$ and $g(x) = x^2$.
Solution:
Step 1: Limits of integration
In order to find limits of integration, we find points of intersections of the curves $y = 2x$ and $y = x^2$. So by comparing equations of both curves, we can write

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0\,, x = 2$$

Step 2: Upper and lower curves
In order to decide which curve lies above the other, we take a test point in the interval [0; 2] and find values of $f(x)$ and $g(x)$ at that point. The curve with larger value will lie above the other.

$$For\ x = 1\ f(1) = 2(1) = 2$$

$$For\ x = 1\ g(1) = (1)2 = 1 \qquad f(1) > g(1)$$

Hence, the graph of $f(x)$ lies above the graph of $g(x)$

Step 3: Area Between two curves
The formula for area of the bounded region between two curves is given as

$$Area = A = \int_a^b [f(x) - g(x)]\ dx$$

By Substituting the values,

$$A = \int_0^2 [2x - x^2]\ dx$$

$$= \int_0^2 [2x]\ dx - \int_0^2 [x^2]\ dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= [x^2]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= [(2)^2 - (0)^2] - \frac{1}{3}[(2)^3 - (0)^3]$$

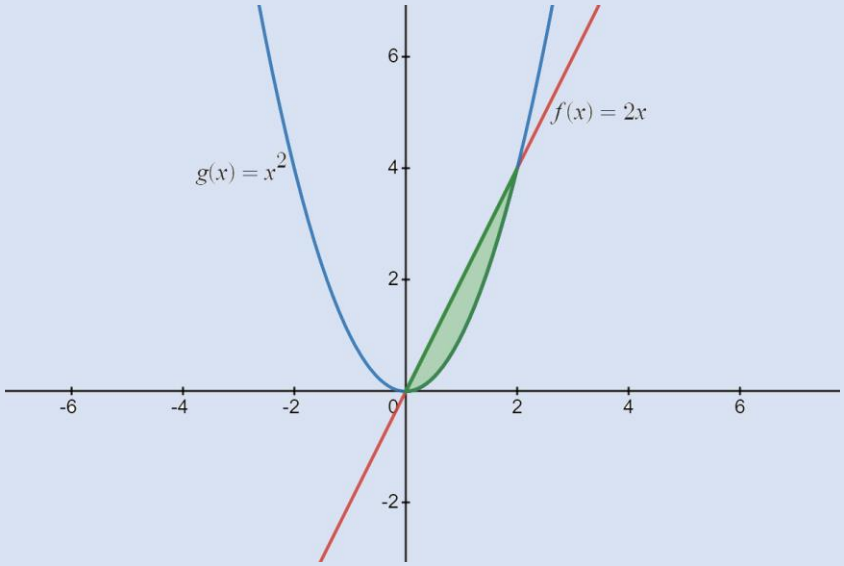
$$= 4 - \frac{8}{3}$$

$$= \frac{12-8}{3} = \frac{4}{3}$$

Hence, the required area

$$A = \frac{4}{3}$$

Sketch the area between two curves:



Example 4: Find the area of the region enclosed between the curves $y = x^3 - 4x$ and $y = 0$, $x = 0, x = 2$.

Solution:

$$Area = A = \int_a^b [f(x) - g(x)]\ dx$$

By Substituting the values,

$$A = \int_0^2 [0 - (x^3 - 4x)]\ dx$$

$$= \int_0^2 [4x - x^3]\ dx$$

$$= 4\left[\frac{x^2}{2}\right] - \left[\frac{x^4}{4}\right]$$

$$= 2[(2)^2 - (0)^2] - \frac{1}{4}[(2)^4 - (0)^4]$$

$$= 2(4) - \frac{1}{4}(16)$$

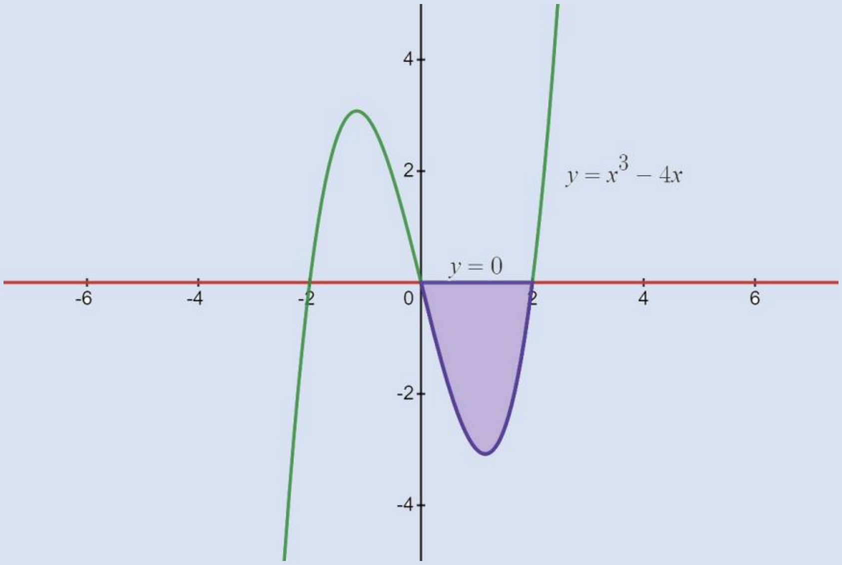
$$= 8 - 4$$

$$= 4$$

So, the required area is

$$A = 4$$

Sketch the area between two curves:



Practice Questions:

1. Find the area of region enclosed between $f(x) = \sqrt{x}, and\ g(x) = \frac{1}{4}\ x$.
2. Sketch the region enclosed by the curves $y = x^2$ and $y = \sqrt{x}\ , x = 0, x = 2$.
3. Sketch the region enclosed by the curves $y = \cos 2x, y = 0, x = -\pi/4,\ x = \frac{\pi}{4}$ and its area.