7.2.1 Inverse Laplace Transform

The Inverse Problem If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathscr{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write $f(t) = \mathscr{L}^{-1}\{F(s)\}$. For example, from Examples 1, 2, and 3 of Section 7.1 we have, respectively,

Transform	Inverse Transform
$\mathscr{L}\{1\} = \dfrac{1}{s}$	$1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$
$\mathscr{L}\{t\} = \dfrac{1}{s^2}$	$t = \mathscr{L}^{-1}\left\{\dfrac{1}{s^2}\right\}$
$\mathscr{L}\{e^{-3t}\} = \dfrac{1}{s+3}$	$e^{-3t} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+3}\right\}$

THEOREM 7.2.1 Some Inverse Transforms

$$\textbf{(a)}\ 1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\textbf{(b)}\ t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, \quad n = 1, 2, 3, \ldots \qquad \textbf{(c)}\ e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$\textbf{(d)}\ \sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \qquad \textbf{(e)}\ \cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$

$$\textbf{(f)}\ \sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} \qquad \textbf{(g)}\ \cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

EXAMPLE 1 Applying Theorem 7.2.1

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+7}\right\}$.

SOLUTION **(a)** To match the form given in part (b) of Theorem 7.2.1, we identify $n + 1 = 5$ or $n = 4$ and then multiply and divide by 4!:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$

(b) To match the form given in part (d) of Theorem 7.2.1, we identify $k^2 = 7$, so $k = \sqrt{7}$. We fix up the expression by multiplying and dividing b $\sqrt{7}$:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t.$$

EXAMPLE 2 Termwise Division and Linearity

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$.

SOLUTION We first rewrite the given function of s as two expressions by means of termwise division and then use (1):

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} \overset{\text{termwise division}}{=} \mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\} \overset{\text{linearity and fixing up constants}}{=} -2\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \quad (2)$$

$$= -2\cos 2t + 3\sin 2t. \quad \leftarrow \text{parts (e) and (d) of Theorem 7.2.1 with } k=2$$

EXAMPLE 3 Partial Fractions: Distinct Linear Factors

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$.

SOLUTION There exist unique real constants A, B, and C so that

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

$$= \frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}.$$

Since the denominators are identical, the numerators are identical:

$$s^2+6s+9 = A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2). \quad (3)$$

By comparing coefficients of powers of s on both sides of the equality, we know that (3) is equivalent to a system of three equations in the three unknowns A, B, and C. However, there is a shortcut for determining these unknowns. If we set $s=1$, $s=2$, and $s=-4$ in (3), we obtain, respectively,

$$16 = A(-1)(5), \qquad 25 = B(1)(6), \qquad \text{and} \qquad 1 = C(-5)(-6),$$

and so $A=-\frac{16}{5}$, $B=\frac{25}{6}$, and $C=\frac{1}{30}$. Hence the partial fraction decomposition is

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1}+\frac{25/6}{s-2}+\frac{1/30}{s+4}, \quad (4)$$

and thus, from the linearity of $\mathscr{L}^{-1}$ and part (c) of Theorem 7.2.1,

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}. \tag{5}$$

EXERCISES 7.2

In Problems 1–30 use appropriate algebra and Theorem 7.2.1 to find the given inverse Laplace transform

1. $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}$

2. $\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}$

3. $\mathscr{L}^{-1}\left\{\frac{1}{s^2}-\frac{48}{s^5}\right\}$

4. $\mathscr{L}^{-1}\left\{\left(\frac{2}{s}-\frac{1}{s^3}\right)^2\right\}$

5. $\mathscr{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}$

6. $\mathscr{L}^{-1}\left\{\frac{(s+2)^2}{s^3}\right\}$

7. $\mathscr{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s-2}\right\}$

8. $\mathscr{L}^{-1}\left\{\frac{4}{s}+\frac{6}{s^5}-\frac{1}{s+8}\right\}$

9. $\mathscr{L}^{-1}\left\{\frac{1}{4s+1}\right\}$

10. $\mathscr{L}^{-1}\left\{\frac{1}{5s-2}\right\}$

11. $\mathscr{L}^{-1}\left\{\frac{5}{s^2+49}\right\}$

12. $\mathscr{L}^{-1}\left\{\frac{10s}{s^2+16}\right\}$

13. $\mathscr{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\}$

14. $\mathscr{L}^{-1}\left\{\frac{1}{4s^2+1}\right\}$

15. $\mathscr{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$

16. $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\}$

17. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+3s}\right\}$

18. $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\}$

19. $\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\}$

20. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\}$