Calculus and Analytical Geometry

Lecture no. 11

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Topic: Application of Derivative: Intervals of increase and decrease, Concavity

Outline of the lecture:

i. Increasing and decreasing functions
   - Definition
   - Theorem
   - Examples

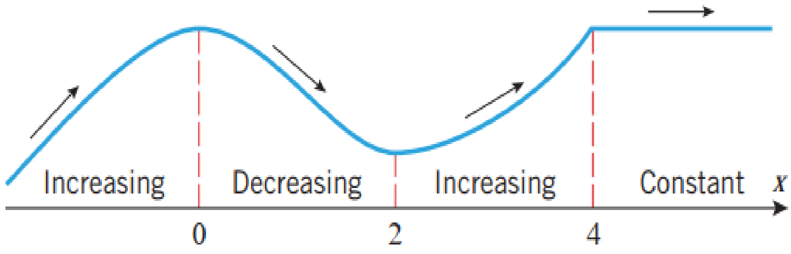
ii. Concavity
   - Definition
   - Theorem
   - Example

iii. Inflection points
   - Definition
   - Example

iv. Practice questions

➢ INCREASING AND DECREASING FUNCTIONS:

The terms increasing, decreasing and constant are used to describe the behavior of the function.



The function can be described as:

- Increasing to the left of 0
- Decreasing from the right of 0 to left of 2
- Increasing from the right of 2 to left of 4
- Constant to the right of 4

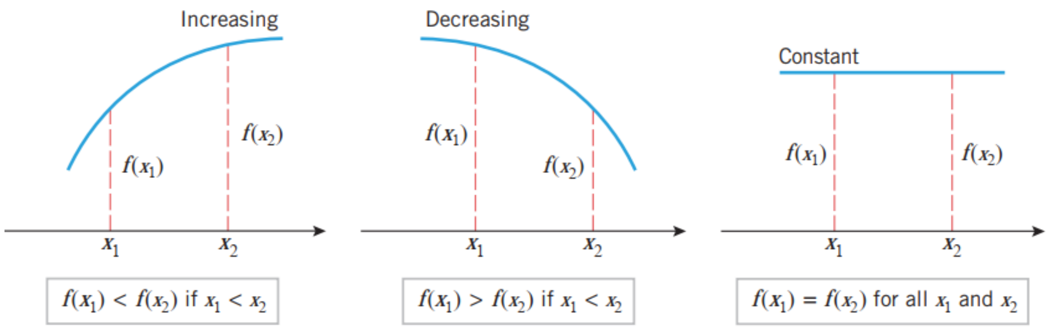
The following definition illustrate the idea precisely.

DEFINITION: Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(b) f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

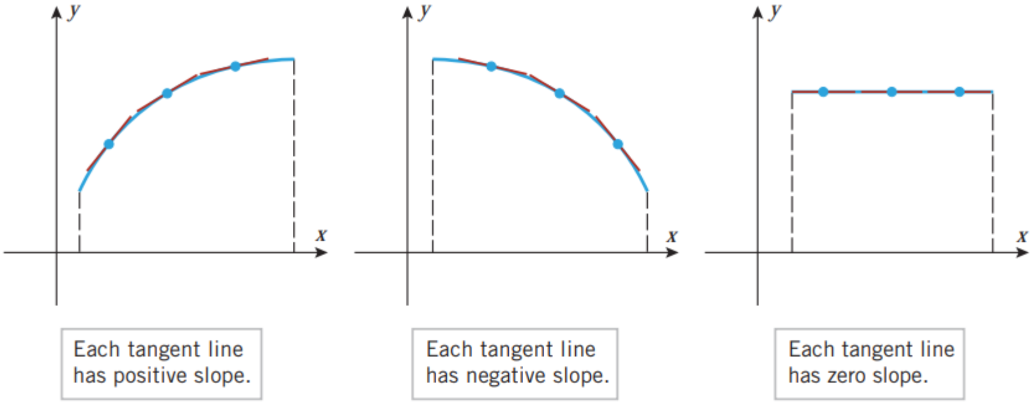
(c) f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



(a) (b) (c)

This figure also suggests:

- The function is **increasing** on any interval where each tangent line to its graph has a **positive** slope.
- The function is **decreasing** on any interval where each tangent line to its graph has a **negative** slope.
- The function is **constant** on any interval where each tangent line to its graph has a **zero** slope.



THEOREM: Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on an open interval (a, b)

(a) *If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.*

(b) *If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.*

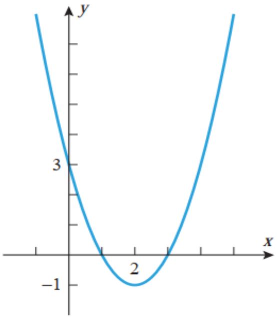
(c) *If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.*

Examples:

1. Find the intervals on which $f(x) = x^2 - 4x + 3$ the function is increasing and the intervals on which the function is decreasing.

Solution:

- Consider the graph of function:



The graph of the functions suggest that the function is decreasing for $x \leq 2$ and increasing for $x \geq 2$.

- Differentiate the given function w.r.t x

$$f'(x) = \frac{d}{dx}(x^2 - 4x + 3)$$
$$= 2x - 4$$

It follows that,

$$f'(x) < 0 \quad if\ x < 2$$

$$f'(x) > 0 \quad if\ x > 2$$

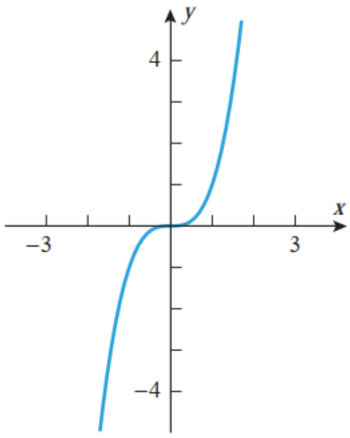
Since the functions is continuous at every point of x, then according to the theorem

- f is decreasing on the interval $(-\infty, 2]$
- f is increasing on the interval $[2, +\infty)$

2. Find the intervals on which $f(x) = x^3$ the function is increasing and the intervals on which the function is decreasing.

Solution:

- Consider the graph of function:



The graph of the functions suggest that the function is decreasing for $x \leq 2$ and increasing for $x \geq 2$.

- Differentiate the given function w.r.t x

$$f'(x) = \frac{d}{dx}(x^3)$$
$$= 3x^2$$

It follows that,

$$f'(x) > 0 \quad if\ x < 0$$
$$f'(x) > 0 \quad if\ x > 0$$

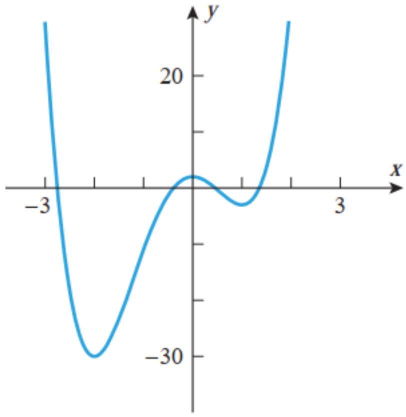
Since the functions is continuous at every point of x, then according to the theorem

- f is increasing on the interval $(-\infty, 0]$
- f is increasing on the interval $[0, +\infty)$
- We can conclude that the function is increasing on the whole real line $(-\infty, +\infty)$

3. Find the intervals on which $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ the function is increasing and the intervals on which the function is decreasing.

Solution:

- Consider the graph of function:



The graph of the functions suggests that:

- ➢ The function is decreasing at $x \le -2$
- ➢ The function is increasing from $-2 \le x \le 0$
- ➢ The function is decreasing from $0 \le x \le 1$
- ➢ The function is decreasing from $x \ge 1$

- Differentiate the given function w.r.t x

$$f'(x) = \frac{d}{dx}(3x^4 + 4x^3 - 12x^2 + 2\,)$$
$$= 12x^3 + 12x^2 - 24x$$

Now we'll analysis the signs of $f'(x)$ on each interval

Interval	$f'(x)$	Conclusion
$x < -2$	$-$	Increasing on $(-\infty, -2]$
$-2 < x < 0$	$+$	Increasing on $[-2,0]$
$0 < x < 1$	$-$	Decreasing on $[0,1]$
$1 < x$	$+$	Increasing on $[1,+\infty)$

➢ CONCAVITY

Concavity is the rate of change of function's derivative. Although the derivative reveals where the function is increasing or decreasing it does not reveal the direction of curvature.

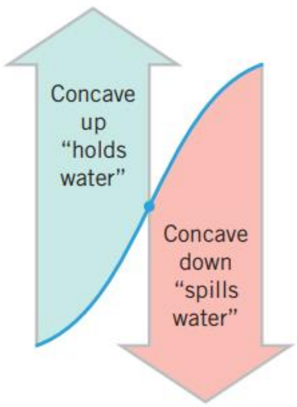
▪ CURVATURE:

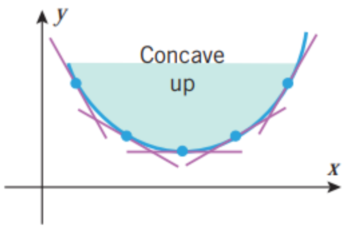
Curvature is the amount by which a curve deviates from being a straight line.

▪ CONCAVE UP OR CONCAVE DOWN:

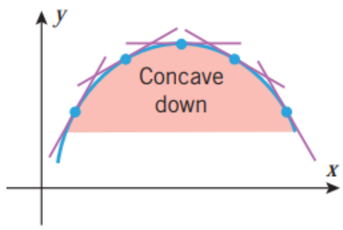
DEFINITION: If f is differentiable on an open interval, then f is said to be

- ***concave up*** on the open interval if f is increasing on that interval,
- f is said to be ***concave down*** on the open interval if f is decreasing on that interval.





Increasing slopes



Decreasing slopes

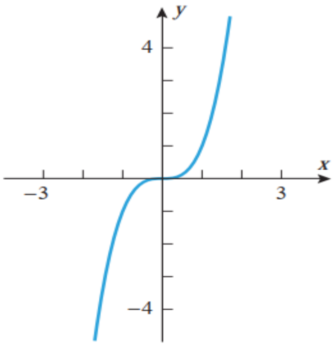
THEOREM: Let f be twice differentiable on an open interval.
(a) If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.
(b) If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval

Example:

Suggest the function $f(x) = x^3$ is concave up or concave down.

Solution:

- Consider the graph of function:



The graph of the functions suggest that the function is decreasing for $x \leq 2$ and increasing for $x \geq 2$.

- Differentiate the given function w.r.t x

$$f'(x) = \frac{d}{dx}(x^3)$$
$$= 3x^2$$
$$f''(x) = \frac{d}{dx}(3x^2)$$
$$= 6x$$

It follows that,

$$f''(x) < 0 \quad if\ x < 0$$

$$f''(x) > 0 \quad if\ x > 0$$

So, function is concave up for $x > 0$ and concave down for $x < 0$.

➢ INFLECTION POINT

DEFINITION: If f is continuous on an open interval containing a value $x0$, and if f changes the direction of its concavity at the point $(x0, f(x0))$, then we say that f has an ***inflection point at $x0$***, and we call the point $(x0, f(x0))$ on the graph of f an ***inflection point*** of f

Example: Consider the function $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivative to find on which intervals the function is increasing, decreasing, concave up and concave down. Locate the inflection points.

Solution:

$$f'(x) = \frac{d}{dx}(x^3 - 3x^2 + 1)$$
$$= 3x^2 - 6x$$
$$f''(x) = 6x - 6 = 6(x - 1)$$

Interval	$f'(x)$	Conclusion
$x < 0$	$+$	Increasing on $(-\infty, 0]$
$0 < x < 2$	$-$	Decreasing on $[0,2]$
$x < 2$	$+$	Increasing on $[2, +\infty)$

Interval	$f''(x)$	Conclusion
$x < 1$	$-$	Decreasing on $(-\infty, 1]$
$x < 1$	$+$	Increasing on $[1, +\infty)$

➢ PRACTICE QUESTIONS:

Find the intervals on which the function is increasing, decreasing, concave up and concave down. Locate the Inflection points.

1. $f(x) = x^2 - 3x + 8$
2. $f(x) = (2x + 1)^3$
3. $f(x) = \frac{x}{x^2+2}$