Calculus and Analytical Geometry

Lecture no. 20

Amina Komal

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Topic: Integration by Parts

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1. INTEGRATION BY PARTS:

Integration by Parts is a special method of integration that is often useful when two functions are multiplied together, but is also helpful in other ways.

RULE:

$$\int u\,dv = uv - \int v du$$

Where, $u = f(x)\,, dv = g(x)\,dx$

$$du = f'(x)\,dx\,, \quad v = G(x)$$

LIATE:

LIATE is a useful strategy for choosing *u* and *dv* that can be applied when the integrand is a product of two functions from *different* categories in the list: **L**ogarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential

EXAMPLES:

i. **Integrate $\int (\boldsymbol{ln}\ \boldsymbol{x})^2\ \boldsymbol{dx}$**

Solution:

$$u = (\ln x)^2\ , \qquad dv = dx$$

$$du = 2\frac{\ln x}{x}\,dx \quad and \quad v = x$$

$$\int (ln\ x)^2\ dx = x(\ln x)^2 - 2\int x\frac{\ln x}{x}\,dx$$

$$= x(\ln x)^2 - 2\int \ln x\ \ dx \qquad (1)$$

To integrate $\int \ln x\ \ dx$

$$u = \ln x\ , \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \quad and \quad v = x$$

$$\int u\,dv = uv - \int v du$$

$$\int \ln x\ \ dx = \ln x . x - \int x . \frac{1}{x}\,dx$$

$$= x \ln x - \int 1 \; dx$$

$$= x\, ln\, x - x + C_1$$

$$\int (ln\, x)^2 \, dx = x(\ln x)^2 - 2(x\, ln\, x - x) + C$$

$$= x(\ln x)^2 - 2x\, ln\, x + 2x + C$$

ii. **Integrate $\int \ln(x^2+4)\, dx$**

Solution:

$$u = \ln(x^2+4)\,, \qquad dv = dx$$

$$du = \frac{2x}{x^2+4}\, dx \qquad and \qquad v = x$$

$$\int u\, dv = uv - \int v du$$

$$\int \ln(x^2+4)\, dx = \ln(x^2+4)\,.\, x - \int x\,.\frac{2x}{x^2+4}\, dx$$

$$= \ln(x^2+4)\,.\, x - 2\int \frac{x^2}{x^2+4}\, dx \qquad (1)$$

By dividing,

$$\frac{x^2}{x^2+4} = 1 - \frac{4}{x^2+4}$$

$$\int \frac{x^2}{x^2+4}\, dx = \int \left(1 - \frac{4}{x^2+4}\right) dx$$

$$= x - 4\int \frac{1}{x^2+4}\, dx$$

$$= x - 4.\frac{1}{2} tan^{-1}\frac{x}{2} + C_1 = x - 2tan^{-1}\frac{x}{2} + C_1$$

Substitute in equation (1)

$$\int \ln(x^2+4)\, dx = x\ln(x^2+4) - 2\left(x - 2tan^{-1}\frac{x}{2}\right) + C$$

$$= x\ln(x^2+4) - 2x + 4tan^{-1}\frac{x}{2} + C$$

iii. Integrate $\int \boldsymbol{cos^{-1}(2x)dx}$

Solution:

$$u = cos^{-1}(2x), \qquad dv = dx$$
$$du = -\frac{2}{\sqrt{1-4x^2}}\,dx, \qquad v = x$$
$$\int cos^{-1}(2x)dx = xcos^{-1}(2x) - \int -\frac{2x}{\sqrt{1-4x^2}}\,dx \qquad (1)$$
$$= xcos^{-1}(2x) - \int -\frac{2x}{\sqrt{1-4x^2}}\,dx$$
$$\int \frac{2x}{\sqrt{1-4x^2}}\,dx = \frac{1}{4}\int -\frac{8x}{\sqrt{1-4x^2}}\,dx$$
$$= \frac{1}{4}\left(\frac{(1-4x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) = \frac{1}{4}\left(\frac{(1-4x^2)^{-\frac{1}{2}+1}}{\frac{1}{2}}\right) + C$$
$$= \frac{1}{2}\sqrt{1-4x^2} + C$$

Substituting the value in equation (1) again,

$$\int cos^{-1}(2x)dx = xcos^{-1}(2x) - \frac{1}{2}\sqrt{1-4x^2} + C$$

iv. Integrate $\int \frac{\ln x}{\sqrt{x}}\,\boldsymbol{dx}$.

Solution:

$$u = \ln x\,, \qquad dv = \frac{1}{x}\,dx$$
$$du = \frac{1}{x}\,dx, \qquad v = 2\sqrt{x}$$
$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}.\ln x - \int 2\sqrt{x}.\frac{1}{x}\,dx$$
$$= 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}\,dx$$
$$= 2\sqrt{x}\ln x - 2\left(\frac{(x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + C$$
$$= 2\sqrt{x}\ln x - 2\left(\frac{(x)^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$
$$= 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

v. **Integrate $\int \boldsymbol{tan^{-1}(3x)dx}$**

Solution:

$$u = tan^{-1}(3x), \qquad dv = dx$$
$$du = \frac{3}{1+9x^2}\,dx, \qquad v = x$$
$$\int tan^{-1}(3x)dx = xtan^{-1}(3x) - \int \frac{3x}{1+9x^2}\,dx \qquad (1)$$
$$\int \frac{3x}{1+9x^2}\,dx = \frac{1}{6}\int \frac{18x}{1+9x^2}\,dx$$
$$= \frac{1}{6}\ln(1+9x^2) + C$$

Substitute value in equation (1)

$$\int tan^{-1}(3x)dx = xtan^{-1}(3x) - \frac{1}{6}\ln(1+9x^2) + C$$

vi. **Integrate $\int \boldsymbol{x\, tan^2 x\ dx}$**

Solution:

$$u = x, \qquad dv = tan^2 x\, dx = (sec^2 x - 1)dx$$
$$du = dx, \qquad v = \tan x - x$$
$$\int x\, tan^2 x\ dx = xtan\, x - \int (\tan x - x)\ dx \qquad (1)$$
$$\int (\tan x - x)\ dx = \int \tan x\, dx - \int x\ dx$$
$$= \int \frac{\sin x}{\cos x}\ dx - \frac{x^2}{2} = \ln(\cos x) - \frac{x^2}{2} + C$$

Substitute the value in equation (1)

$$\int x\, tan^2 x\ dx = xtan\, x - \ln(\cos x) + \frac{x^2}{2} + C$$

Practice Questions:

- $\int sin^{-1}xdx$
- $\int cos\ (lnx)\ dx$
- $\int x\ sec^2x\ dx$
- $\int sin^{-1}x\ dx$
- $\int ln\ (x^2+4)dx$
- $\int \sqrt{x}\ \ ln\ x\ dx$
- $\int ln(3x-2)dx$