Calculus and Analytical Geometry

Lecture no. 18

Amina Komal

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Topic: The indefinite integral of exponential functions, logarithmic, and trigonometric functions

- Formulas
- Examples
- Practice questions

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\,dx = -\cos x + C$
5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
9. $\frac{d}{dx}[e^x] = e^x$	$\int e^x\,dx = e^x + C$
10. $\frac{d}{dx}\left[\frac{b^x}{\ln b}\right] = b^x \quad (0 < b,\ b \neq 1)$	$\int b^x\,dx = \frac{b^x}{\ln b} + C \quad (0 < b,\ b \neq 1)$
11. $\frac{d}{dx}[\ln \lvert x \rvert] = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln \lvert x \rvert + C$
12. $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}$	$\int \frac{1}{1 + x^2}\,dx = \tan^{-1} x + C$
13. $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}$	$\int \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1} x + C$
14. $\frac{d}{dx}[\sec^{-1} \lvert x \rvert] = \frac{1}{x\sqrt{x^2 - 1}}$	$\int \frac{1}{x\sqrt{x^2 - 1}}\,dx = \sec^{-1} \lvert x \rvert + C$

Some more formulas of integration:

$$\int \frac{du}{a^2 + u^2} = \frac{1}{2} tan^{-1} \frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = sin^{-1} \frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} sec^{-1} \left|\frac{u}{a}\right| + C$$

Examples:

1. Evaluate the integral $\int \left[\frac{2}{x} + 3e^x\right]\, dx$

$$\int \left[\frac{2}{x} + e^x\right]\, dx = \int \frac{2}{x}\, dx + 3\int e^x\, dx$$
$$= 2\ln x + 3\, e^x + C$$

2. Evaluate the integral $\int [csc^2 t - \sec t \tan t] dt$

$$\int [csc^2 t - \sec t \tan t] dt = \int csc^2 t\, dt - \int \sec t \tan t\, dt$$
$$= -\cot t - \sec t + C$$

3. Evaluate $\int (x^2+1)^{50}. 2x\, dx$

   Let $x^2 + 1 = u, \qquad du = 2x$

$$\int u^{50} du = \frac{u^{50+1}}{50+1} + C$$
$$= \frac{u^{51}}{51} + C$$
$$= \frac{(x^2+1)^{51}}{51} + C$$

4. Evaluate $\int \cos 5x\, dx$

Let $u = 5x \quad , du = 5\, dx$

$$dx = \frac{1}{5} du$$
$$\int \cos 5x\, dx = \int \cos u . \frac{1}{5} du$$
$$= \frac{1}{5}\int \cos u\, du$$
$$= \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C$$

5. Evaluate $\int sin^2 x \cos x \, dx$

Let $\sin x = u$, $\frac{du}{dx} = \cos x$

$$du = \cos x \, dx$$

$$\int sin^2 x \cos x \, dx = \int u^2 du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{sin^3 x}{3} + C$$

6. Evaluate $\int \frac{e^x}{\sqrt{1-e^{2x}}} \, dx$

Let $u = e^x$, $du = e^x \, dx$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}} \, dx = \int \frac{du}{\sqrt{1-u^2}}$$

$$= sin^{-1} u + C$$

$$= sin^{-1}(e^x) + C$$

7. Evaluate $\int \frac{\cos x}{sin^2 x} \, dx$

$$\int \frac{\cos x}{sin^2 x} \, dx = \int \frac{\cos x}{\sin x} . \frac{1}{\sin x} dx$$

$$= \int \cot x . \csc x \, dx$$

$$= -\csc x + C$$

8. Evaluate $\int x\sqrt{x^2 - 1} \, dx$

Let $f(x) = x^2 - 1$ $, \frac{dy}{dx} = 2x$

$$dy = 2x \, dx$$

$$\int x\sqrt{x^2 - 1} \, dx = \int \sqrt{x^2 - 1} . x \, dx$$

$$= \frac{1}{2} \int \sqrt{x^2 - 1} . 2x \, dx = \frac{1}{2} \int \sqrt{y} dy$$

$$= \frac{1}{2}\left[\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + C = \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{1}{2}\left(\frac{2}{3}\right)y^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(x^2-1)^{\frac{3}{2}} + C$$

9. Evaluate $\int \frac{4}{x\sqrt{x^2-1}} + \frac{1+x+x^3}{1+x^2}\ dx$

As,

$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}sec^{-1}\left|\frac{u}{a}\right| + C$

Here, $u = x, a = 1$

$$\int \frac{4}{x\sqrt{x^2-1}}dx = 4sec^{-1}\left|\frac{x}{1}\right| + C = 4sec^{-1}|x| + C$$

$$\int \frac{1+x+x^3}{1+x^2}\ dx = \int \left(x + \frac{1}{1+x^2}\right)\ dx$$

$$= \int x\ dx + \int \frac{1}{1+x^2}\ dx$$

$$= \frac{1}{2}x^2 + tan^{-1}x + C$$

$$\int \frac{4}{x\sqrt{x^2-1}} + \frac{1+x+x^3}{1+x^2}\ dx = 4sec^{-1}|x| + \frac{1}{2}x^2 + tan^{-1}x + C$$

10. Evaluate $\int \sec x(\sec x + \tan x)dx$

$$\int \sec x(\sec x + \tan x)dx = \int (sec^2 x + \sec x \tan x)dx$$

$$= \tan x + \sec x + C$$

11. Evaluate $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$

Let $\ln x = u,\ \ \frac{1}{x}\ dx = du$

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{du}{\sqrt{1-(u)^2}}$$

$$= sin^{-1}(u) + C$$

$$= sin^{-1}(\ln x) + C$$

Ms. Amina Komal

Practice Questions:

Evaluate the following integrals

- $\int \frac{e^{\sqrt{2t+1}}}{\sqrt{2t+1}}\, dy$
- $\int \cot x\, csc^2 x\, dx$
- $\int (1+\sin t)^9 \cot t\, dt$
- $\int \left[\frac{1}{2\sqrt{1-x^2}} - \frac{3}{1+x^2}\right] dx$
- $\int e^{\sin x + \cos x} dx$
- $\int \frac{1-\sin x}{1-sin^2 x}\, dx$
- $\int \frac{\sec x + \cos x}{2\cos x}$
- $\int [1 + sin^2\theta csc\theta] d\theta$
- $\int [3\sin x - 2sin^2 x] dx$
- $\int \left(\frac{1}{x} + sec^2 \pi x\right) dx$