Types of Euclidean Transformation

Lecture No. 7

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Presentation Overview

1. Types of Transformation

Definitions (Reflection)

Reflection is when we flip the image along a line (the mirror line). The flipped image is also called the mirror image

Reflection

Reflection along a line passing through origin making an angle θ with x-axis in anticlockwise direction is a transformation

T: $R^2 \rightarrow R^2$ defined as:

$$T(\overrightarrow{x}) = A\overrightarrow{x} + \overrightarrow{a}, \qquad \forall \vec{x} \in R^2$$

where,

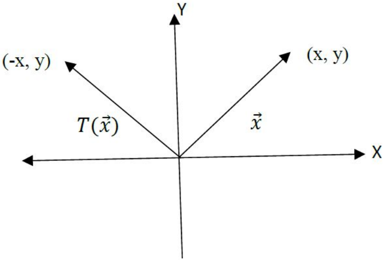
$$A=\begin{bmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{bmatrix}$$

Reflection over along y-axis

Along y-axis with $\theta = \frac{\pi}{2}$

$$A = \begin{bmatrix} cos2(\frac{\pi}{2}) & sin2(\frac{\pi}{2}) \\ sin2(\frac{\pi}{2}) & -cos2(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} cos\pi & sin\pi \\ sin\pi & -cos\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

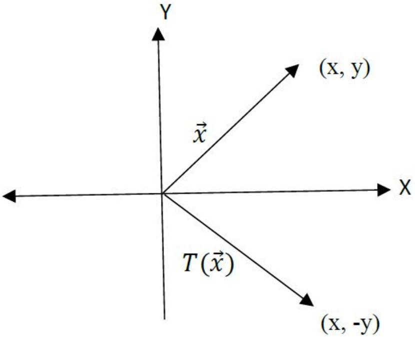


Reflection over along x-axis

Along $y = 0$ with θ =0

A= $\begin{bmatrix} cos2(0) & sin2(0) \\ sin2(0) & -cos2(0) \end{bmatrix}$ = $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

T $\begin{bmatrix} x \\ y \end{bmatrix}$ = $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix}$ = $\begin{bmatrix} x \\ -y \end{bmatrix}$



Reflection along the line y=x with $\theta=\frac{\pi}{4}$

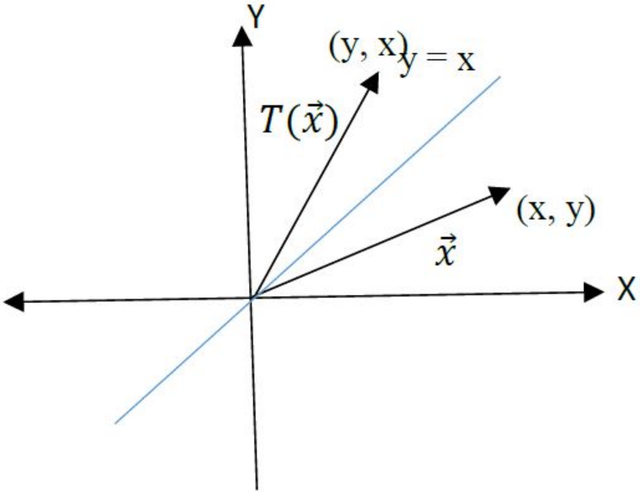
A= $\begin{bmatrix} cos2(\frac{\pi}{4}) & sin2(\frac{\pi}{4}) \\ sin2(\frac{\pi}{4}) & -cos2(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

T $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

Reflection along the line y=-x with θ=90°+45°=135°

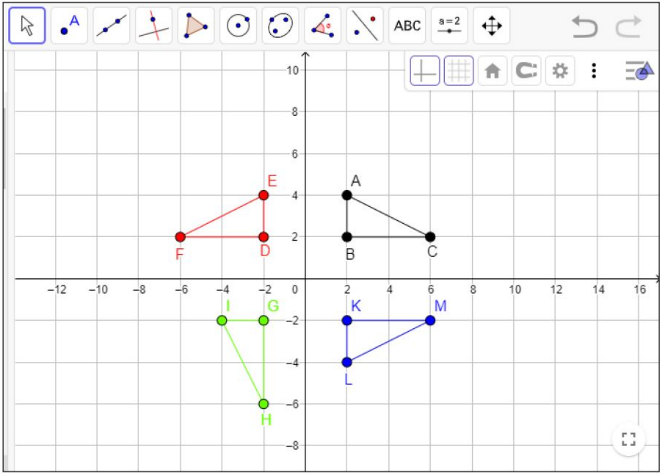
A= $\begin{bmatrix} cos2(135) & sin2(135) \\ sin2(135) & -cos2(135) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

T $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$


Y
(y, x)
y = x
$T(\vec{x})$
(x, y)
$\vec{x}$
X

Example 1. (Reflection of Triangle)

Reflect the triangle with vertices A= (2, 4), B = (2, 2), C = (6, 2) along x-axis, y-axis and y = -x.



Example 2. (Reflection of a line)

Let y = 2x +1 be a line. Find the reflection of that line along the line y = x.

Solution. First of all we will find the angle of the line y = x with the x-axis about the origin.

y = x (General form of line is y = mx + c)

So, m =1, $\tan\theta = 1 \implies \theta = 45°$

Here, $$A = \begin{bmatrix} Cos2(45°) & Sin2(45°) \\ Sin2(45°) & -Cos2(45°) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, $$\mathbf{T}\,(\vec{x}) = \mathbf{A}\vec{x}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

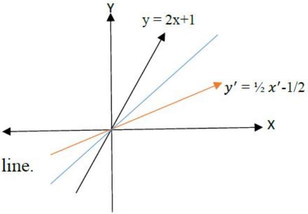
So, $x = y'$, $y = x'$

Put x and y in original line y = 2x + 1

$$x' = 2y' + 1$$

Or $2y' = x' - 1$

So $y' = 1/2\, x' - 1/2$ is the reflected line.



To draw original line y = 2x + 1 take two points on it, let A = (1, 3) and B = (2, 5).

And to draw the Reflected line $y' = 1/2\, x' - ½$, $A' = (2, 1/2)$ and $B' = (4, 3/2)$.

Reflection of circle

Let $(x-2)^2 + (y-3)^2 = 4$ be a circle. Find its reflection along the line $y = -x$

Solution.

First of all we will find the angle of line $y = -x$ with x-axis at origin line.

That is for $m = -1$, $\theta = 135°$.

The transformation of reflection is

$$\mathbf{T}(\vec{x}) = \mathbf{A}\vec{x}, \quad \forall\, \vec{x} \in \mathbf{R}^2$$

Where

$$A = \begin{bmatrix} Cos2(135°) & Sin2(135°) \\ Sin2(135°) & -\ Cos(135°) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

As, $\mathbf{T}\ (\vec{x}) = \mathbf{A}\vec{x}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x' &= -y \\ y' &= -x \end{aligned}$$

Putting $x = -y',\ y = -x'$ in the original circle $(x - 2)^2 + (y - 3)^2 = 4$, we get

$(x' + 3)^2 + (y' + 2)^2 = 4$, reflected circle.

As original circle $(x - 2)^2 + (y - 3)^2 = 4$ is with Centre = (2, 3) and Radius = 2

While Reflected circle $(x' + 3)^2 + (y' + 2)^2 = 4$ has Centre = (-3, -2), Radius = 2. We can draw both circles easily.

Example. Reflection of a line along a line which is not passing through origin.

Let y = x+2 be a line. Find its reflection along the line x =1.

Solution: Now first we will shift our line x =1 at origin. For this purpose we have to subtract vector $\begin{bmatrix}1\\0\end{bmatrix}$. Our line become x = 0. So our $\vec{x}$ vector can be written as:

$$\vec{x} = \begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x-1\\y-0\end{bmatrix}$$

Now we will find θ

$$x = 0, \quad m = \infty = \tan\theta \quad \Rightarrow \theta = \tan^{-1}\infty = 90^\circ$$

The transformation of reflection becomes

$$\mathbf{T}\,(\vec{x}) = \mathbf{A}\vec{x}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} Cos2(90°) & Sin2(90°) \\ Sin2(90°) & -\ Cos2(90°) \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix} = \begin{bmatrix} -x+1 \\ y \end{bmatrix}$$

Now we will add vector

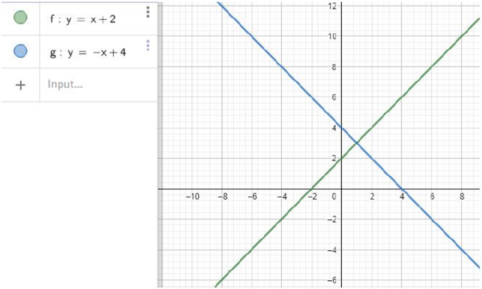
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x+1 \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -x+2 \\ y \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} x' = -x+2 \\ y' = y \end{array}$$

$$\text{Or} \quad \begin{array}{l} x = -x'+2 \\ y = y' \end{array}$$

Now substitute in



<u>Original line</u> y = x+2

having points, A= (0, 2)

and B= (-1, 1) getting

<u>Reflected line</u> $y' = -x' + 4$ having $A' = (2,2)$ and $B' = (3,1)$.

Work to do:

Q. Let $y=x+3$ be a line. Find its reflection along the line $x=-1$

3-Rotation

Rotation about origin through an angle θ is a transformation $T: R^2 \to R^2$ defined as:

$$T(\vec{x}) = A\vec{x} \quad : \forall \vec{x} \in R^2$$

Where

$$A = \begin{bmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix}$$

- If the direction of θ is not defined, then it is understood to be in anticlockwise direction.
- If θ is in clockwise direction, then replace θ by $-\theta$ in the above definition as:

$$A = \begin{bmatrix} Cos(\theta) & Sin(\theta) \\ -Sin(\theta) & Cos(\theta) \end{bmatrix}$$

Example 1: Sketch the image of given rectangle with vertices A(0,0), B(3,0), C(3,2), D(0,2) under the rotation of 30^0 (anticlockwise).

Solution: As the transformation of rotation is

$$T(\vec{x}) = A\vec{x} \quad : \forall \vec{x} \in R^2$$

Where

$$A = \begin{bmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix}$$

As $\theta = 30^0$, so

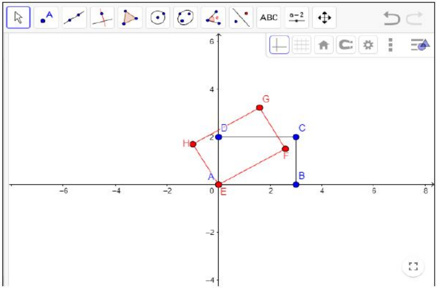
$$A = \begin{bmatrix} Cos(30^0) & -Sin(30^0) \\ Sin(30^0) & Cos(30^0) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

For point A: $$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For point B: $$T\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2.598 \\ 1.5 \end{bmatrix}$$

For point C: $$T\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}\frac{3\sqrt{3}}{2}-1\\ \frac{3}{2}+\sqrt{3}\end{bmatrix}=\begin{bmatrix}1.599\\3.23\end{bmatrix}$$

For point D: $$T\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}-1\\1.732\end{bmatrix}$$



Work to do:

Q1. Sketch the image of given parallelogram with vertices A(0,1), B(3,0), C(5,-2), D(2,-1) under the rotation of 90^0 (anticlockwise) .

Q2. Sketch the image of given triangle with vertices A(2,4), B(2,2), C(4,2) under the rotation of 90^0 (clockwise) .

Example 2. Let y = 2x+5 be a line. Find the equation of line after rotating it through an angle of $\frac{\pi}{2}$ clockwise direction about origin.

Solution: The matrix of rotation in clockwise direction is

$$A = \begin{bmatrix} Cos(\theta) & Sin(\theta) \\ -Sin(\theta) & Cos(\theta) \end{bmatrix}$$

As $\theta = \frac{\pi}{2}$, so

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

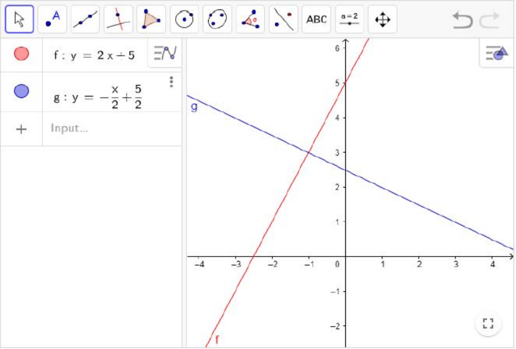
or $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

or $$\begin{cases} x' = y \\ y' = -x \end{cases} \Rightarrow \begin{cases} x = -y' \\ y = x' \end{cases}$$

Put value of x and y in original equation of line $y = 2x + 5$ and obtain

$$y' = -\frac{x'}{2} + \frac{5}{2}$$

This is the rotated line with angle $\frac{\pi}{2}$ in clockwise direction.



Work to do:

Q3. Let $y = -2x + 7$ be a line. Find the equation of line after rotating it through an angle of $180^0, 270^0$ clockwise direction about origin.

Example 3. Let $(x-4)^2 + (y-3)^2 = 9$ be a circle. Find the equation of circle after rotating it through an angle of 90^0 in anticlockwise direction about origin.

Solution: The matrix of rotation in anticlockwise direction is

$$A = \begin{bmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix}$$

As $\theta = 90^0$, so

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Therefore, for $T(\vec{x}) = A\vec{x} \, : \forall \, \vec{x} \in R^2$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

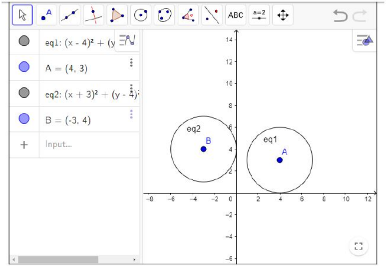
Or $\begin{cases} x' = -y \\ y' = x \end{cases} \Rightarrow \begin{cases} x = y' \\ y = -x' \end{cases}$

Putting these values of x and y in original equation of circle

$$(x-4)^2 + (y-3)^2 = 9$$

We get $(x'+3)^2 + (y'-4)^2 = 9$

This is the equation of rotated circle with angle $\frac{\pi}{2}$ in anticlockwise direction.



Work to do:

Q4. Let $(x-4)^2+(y-3)^2=9$ be a circle. Find the equation of circle after rotating it through an angle of $180^0, 270^0$ in clockwise direction about origin.

Example 4: Let $\frac{x^2}{(4)^2}+\frac{y^2}{(3)^2}=1$ be an ellipse. Find the equation of ellipse after rotating it through an angle of 90^0 in anticlockwise direction about origin.

Solution: The transformation of rotation in anticlockwise direction is

$$T(\vec{x}) = A\vec{x} \quad : \forall \vec{x} \in R^2$$

$$T(\vec{x}) = \begin{bmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{bmatrix} \vec{x}$$

As $\theta = 90^0$, $\qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$

Hence, $\qquad \begin{cases} x' = -y \\ y' = x \end{cases}$ or $\begin{cases} x = y' \\ y = -x' \end{cases}$

Put these values of x and y in original equation of ellipse, we get the rotated ellipse with angle $\frac{\pi}{2}$ in anticlockwise direction as

$$\frac{x'^2}{9} + \frac{y'^2}{16} = 1$$

For plotting we can neglect the dash (‘) from our rotated equation of ellipse.

Original Ellipse

$$\frac{x^2}{16}+\frac{y^2}{9}=1$$

Major axis is along x-axis

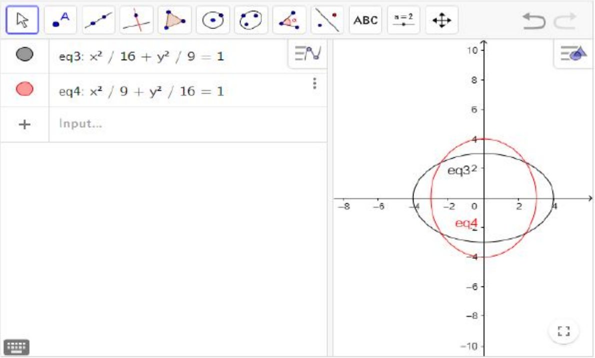
$$a=\pm 4, \quad b=\pm 3$$

Rotated Ellipse

$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

Major axis is along- yaxis

$$a=\pm 3, \quad b=\pm 4$$



Q 5. Let $\frac{x^2}{9}+\frac{y^2}{16}=1$ be an ellipse. Find the equation of ellipse after rotating it through an angle of 180^0 in anticlockwise direction about origin.