Directional Derivative (Contd.)

Example3: Find the directional derivative of the function $g(r,s) = tan^{-1}(rs)$ at the point $(1,2)$ in the direction of the vector $\vec{v} = 5\hat{\imath} + 10\hat{\jmath}$.

Solution:

We have to find

$$D_{\vec{u}}g(1,2) = \nabla g(1,2) \cdot \vec{u}$$

Step-1 Gradient of g

$$g_r = \frac{1}{1+(rs)^2} \times s$$

$$g_r = \frac{s}{1+r^2s^2}$$

$$g_s = \frac{1}{1+(rs)^2} \times r$$

$$g_s = \frac{r}{1+r^2s^2}$$

$$\nabla g(r,s) = g_r(r,s)\hat{\imath} + g_s(r,s)\hat{\jmath}$$

$$\nabla g(r,s) = \frac{s}{1+r^2s^2}\hat{\imath} + \frac{r}{1+r^2s^2}\hat{\jmath}$$

At $(1,2)$

$$\nabla g(1,2) = \frac{2}{1+4}\hat{\imath} + \frac{1}{1+4}\hat{\jmath} = \frac{2}{5}\hat{\imath} + \frac{1}{5}\hat{\jmath}$$

Step-2 Unit Vector

The unit vector $\vec{u}$ parallel to $\vec{v}$is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$|\vec{v}| = \sqrt{(5)^2 + (10)^2} = \sqrt{125} = 5\sqrt{5}$$

$$\vec{u} = \frac{5\hat{\imath}+10\hat{\jmath}}{5\sqrt{5}} = \frac{5}{5\sqrt{5}}\hat{\imath}+\frac{10}{5\sqrt{5}}\hat{\jmath} = \frac{1}{\sqrt{5}}\hat{\imath}+\frac{2}{\sqrt{5}}\hat{\jmath}$$

Step-3 Directional Derivative

$$D_{\vec{u}}g(1,2) = \nabla g(1,2) \cdot \vec{u}$$

$$D_{\vec{u}}g(1,2) = \left(\frac{2}{5}\hat{\imath} + \frac{1}{5}\hat{\jmath}\right) \cdot \left(\frac{1}{\sqrt{5}}\hat{\imath}+\frac{2}{\sqrt{5}}\hat{\jmath}\right)$$

$$D_{\vec{u}}g(1,2) = \left(\frac{2}{5}\right)\left(\frac{1}{\sqrt{5}}\right) + \left(\frac{1}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$$

$$D_{\vec{u}}g(1,2) = \frac{4}{5\sqrt{5}}$$

Example 4: Find the directional derivative of the function $h(r,s,t) = \ln(3r + 6s + 9t)$ at the point $(1,1,1)$ in the direction of the vector $\vec{v} = 4\hat{\imath} + 12\hat{\jmath} + 6\hat{k}$.

Solution: We have to find

$$D_{\vec{u}}h(1,1,1) = \nabla h(1,1,1) \cdot \vec{u}$$

Step-1 Gradient of h

$$h_r = \frac{1}{3r + 6s + 9t} \times 3 = \frac{1}{r + 2s + 3t}$$

$$h_s = \frac{1}{3r + 6s + 9t} \times 6 = \frac{2}{r + 2s + 3t}$$

$$h_t = \frac{1}{3r + 6s + 9t} \times 9 = \frac{3}{r + 2s + 3t}$$

$$\nabla h(r,s,t) = h_r(r,s,t)\hat{\imath} + h_s(r,s,t)\hat{\jmath} + h_t(r,s,t)\hat{k}$$

$$\nabla h(r,s,t) = \frac{1}{r + 2s + 3t}\hat{\imath} + \frac{2}{r + 2s + 3t}\hat{\jmath} + \frac{3}{r + 2s + 3t}\hat{k}$$

At (1,1,1)

$$\nabla h(r,s,t) = \frac{1}{1+2+3}\hat{\imath} + \frac{2}{1+2+3}\hat{\jmath} + \frac{3}{1+2+3}\hat{k}$$

$$\nabla h(r,s,t) = \frac{1}{6}\hat{\imath} + \frac{1}{3}\hat{\jmath} + \frac{1}{2}\hat{k}$$

Step-2 Unit Vector

The unit vector $\vec{u}$ parallel to $\vec{v}$is

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{\mathrm{v}} = 4\hat{\imath} + 12\hat{\jmath} + 6\hat{k}$$

$$|\vec{v}| = \sqrt{(4)^2 + (12)^2 + (6)^2} = 14$$

$$\vec{u} = \frac{4\hat{\imath} + 12\hat{\jmath} + 6\hat{k}}{14} = \frac{2}{7}\hat{\imath} + \frac{6}{7}\hat{\jmath} + \frac{3}{7}\hat{k}$$

Step-3 Directional Derivative

$$D_{\vec{u}}h(1,1,1) = \nabla h(1,1,1) \cdot \vec{u}$$

$$D_{\vec{u}}h(1,1,1) = \left(\frac{1}{6}\hat{\imath} + \frac{1}{3}\hat{\jmath} + \frac{1}{2}\hat{k}\right) \cdot \left(\frac{2}{7}\hat{\imath} + \frac{6}{7}\hat{\jmath} + \frac{3}{7}\hat{k}\right)$$

$$D_{\vec{u}}h(1,1,1) = \left(\frac{1}{6}\right)\left(\frac{2}{7}\right) + \left(\frac{1}{3}\right)\left(\frac{6}{7}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{7}\right)$$

$$D_{\vec{u}}h(1,1,1) = 0.548$$

Practice Problems

(a) Find the gradient of f.
(b) Evaluate the gradient at the point P.
(c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

7. $f(x, y) = \sin(2x + 3y)$, $P(-6, 4)$, $\mathbf{u} = \frac{1}{2}(\sqrt{3}\,\mathbf{i} - \mathbf{j})$

8. $f(x, y) = y^2/x$, $P(1, 2)$, $\mathbf{u} = \frac{1}{3}(2\mathbf{i} + \sqrt{5}\,\mathbf{j})$

9. $f(x, y, z) = x^2yz - xyz^3$, $P(2, -1, 1)$, $\mathbf{u} = \langle 0, \frac{4}{5}, -\frac{3}{5} \rangle$

10. $f(x, y, z) = y^2e^{xyz}$, $P(0, 1, -1)$, $\mathbf{u} = \langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13} \rangle$

11–17 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

11. $f(x, y) = e^x \sin y$, $(0, \pi/3)$, $\mathbf{v} = \langle -6, 8 \rangle$

12. $f(x, y) = \dfrac{x}{x^2 + y^2}$, $(1, 2)$, $\mathbf{v} = \langle 3, 5 \rangle$

13. $g(p, q) = p^4 - p^2q^3$, $(2, 1)$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

14. $g(r, s) = \tan^{-1}(rs)$, $(1, 2)$, $\mathbf{v} = 5\mathbf{i} + 10\mathbf{j}$

15. $f(x, y, z) = xe^y + ye^z + ze^x$, $(0, 0, 0)$, $\mathbf{v} = \langle 5, 1, -2 \rangle$

16. $f(x, y, z) = \sqrt{xyz}$, $(3, 2, 6)$, $\mathbf{v} = \langle -1, -2, 2 \rangle$

17. $h(r, s, t) = \ln(3r + 6s + 9t)$, $(1, 1, 1)$, $\mathbf{v} = 4\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$