Chap.6 Series Solutions of Linear Equations

Up to now we have primarily solved linear DEs of order two or higher when the equation had constant coefficients. In applications, higher-order linear equations with variable coefficients are just as important as, if not more important than, DEs with constant coefficients. Even a simple linear second-order DE with variable coefficients such as $y'' + xy = 0$ does not possess solutions that are elementary functions. But we can find two linearly independent solutions of $y'' + xy = 0$; we shall see in Section 6.1 that the solutions of this equation are defined by infinite series.

In this section we consider linear second-order DEs with variable coefficients that possess solutions in the form of *power series*.

We begin with a brief review of some of the important facts about power series.

Review of Power Series

Recall from calculus that a power series in $x - a$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

Such a series is also said to be a **power series centered at *a*.** For example, the power series $\sum_{n=0}^{\infty} c_n (x+1)^n$ is centered at *a* = -1. In this section we are concerned mainly with power series in *x*, in other words, power series such as

$$\sum_{n=1}^{\infty} 2^{n-1} x^n = x + 2x^2 + 4x^3 + \cdots.$$

that are centered at *a* = 0.

Analytic at a Point

A function *f* is analytic at a point *a* if it can be represented by a power series in *x* - *a* with a positive or infinite radius of convergence. In calculus it is seen that functions such as e^x, cos *x*, sin *x*, ln (1 - *x*), and so on can be represented by Taylor series. Recall, for example, that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

for $|x| < \infty$. These Taylor series centered at 0, called Maclaurin series, show that e^x, sin *x*, and cos *x* are analytic at *x* = 0.

Arithmetic of Power Series

Power series can be combined through the operations of addition, multiplication, and division. The procedures for power series are similar to those by which two polynomials are added, multiplied, and divided—that is, we add coefficients of like powers of x, use the distributive law and collect like terms, and perform long division. For example

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)$$
$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots$$

Since the power series for e^x and $\sin x$ converge for $|x| < \infty$, the product series converges on the same interval.

EXAMPLE 1 Adding Two Power Series

Write $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$ as a single power series whose general term involves x^k.

SOLUTION To add the two series, it is necessary that both summation indices start with the same number and that the powers of x in each series be "in phase"; that is, if one series starts with a multiple of, say, x to the first power, then we want the other series to start with the same power. Note that in the given problem the first series starts with x^0, whereas the second series starts with x^1. By writing the first term of the first series outside the summation notation,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2 \cdot 1 c_2 x^0 + \underbrace{\sum_{n=3}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{\text{series starts}\\ \text{with } x\\ \text{for } n=3}} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{\substack{\text{series starts}\\ \text{with } x\\ \text{for } n=0}},$$

we see that both series on the right-hand side start with the same power of x—namely, x^1. Now to get the same summation index, we are inspired by the exponents of x; we let $k = n - 2$ in the first series and at the same time let $k = n + 1$ in the second series. The right-hand side becomes

$$2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k. \qquad (3)$$

same

same

Remember that the summation index is a "dummy" variable; the fact that $k = n - 1$ in one case and $k = n + 1$ in the other should cause no confusion if you keep in mind that it is the *value* of the summation index that is important. In both cases k takes on the same successive values $k = 1, 2, 3, \ldots$ when n takes on the values $n = 2, 3, 4, \ldots$ for $k = n - 1$ and $n = 0, 1, 2, \ldots$ for $k = n + 1$. We are now in a position to add the series in (3) term by term:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} + c_{k-1}]x^k. \quad (4)$$ ■

If you are not convinced of the result in (4), then write out a few terms on both sides of the equality.

Power Series Solutions

Suppose the linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

is put into standard form

$$y'' + P(x)y' + Q(x)y = 0 \tag{2}$$

by dividing by the leading coefficient $a_2(x)$. We have the following definition.

DEFINITION 6.1.1 Ordinary and Singular Points

A point x_0 is said to be an **ordinary point** of the differential equation (5) if both $P(x)$ and $Q(x)$ in the standard form (6) are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

We shall be interested primarily in the case when (5) has polynomial coefficients. A polynomial is analytic at any value x, and a rational function is analytic *except* at points where its denominator is zero.

THEOREM 6.1.1 Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation (5), we can always find two linearly independent solutions in the form of a power series centered at x_0, that is, $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$. A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

A solution of the form $y = \sum_{n=0}^{\infty} c_n (x-a)^n$ is said to be a **solution about the ordinary point x_0.**

Exercise 6.1

In Problems 11 and 12 rewrite the given expression as a single power series whose general term involves x^k.

11. $\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1}$

12. $\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} nc_n x^n$

In Problems 13 and 14 verify by direct substitution that the given power series is a particular solution of the indicated DE

13. $y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad (x+1)y'' + y' = 0$

14. $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}, \quad xy'' + y' + xy = 0$