# Variation of Parameters…..Continued

**EXAMPLE 2** **General Solution Using Variation of Parameters**

Solve $4y'' + 36y = \csc 3x$.

**SOLUTION** We first put the equation in the standard form (6) by dividing by 4

$$y'' + 9y = \frac{1}{4}\csc 3x.$$

Because the roots of the auxiliary equation $m^2 + 9 = 0$ are $m_1 = 3i$ and $m_2 = -3i$, the complementary function is $y_c = c_1 \cos 3x + c_2 \sin 3x$. Using $y_1 = \cos 3x$, $y_2 = \sin 3x$, and $f(x) = \frac{1}{4}\csc 3x$, we obtain

$$W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3,$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}, \qquad W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}.$$

Integrating $\quad u_1' = \dfrac{W_1}{W} = -\dfrac{1}{12} \quad$ and $\quad u_2' = \dfrac{W_2}{W} = \dfrac{1}{12}\dfrac{\cos 3x}{\sin 3x}$

gives $u_1 = -\frac{1}{12}x$ and $u_2 = \frac{1}{36}\ln|\sin 3x|$. Thus a particular solution is

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|.$$

The general solution of the equation is

$$y = y_c + y_p = c_1\cos 3x + c_2 \sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|. \quad (11)$$

## EXAMPLE 3 General Solution Using Variation of Parameters

Solve $y'' - y = \dfrac{1}{x}$.

**SOLUTION** The auxiliary equation $m^2 - 1 = 0$ yields $m_1 = -1$ and $m_2 = 1$. Therefore $y_c = c_1 e^x + c_2 e^{-x}$. Now $W(e^x, e^{-x}) = -2$, and

$$u_1' = -\frac{e^{-x}(1/x)}{-2}, \qquad u_1 = \frac{1}{2}\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt,$$

$$u_2' = \frac{e^{x}(1/x)}{-2}, \qquad u_2 = -\frac{1}{2}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt.$$

Since the foregoing integrals are nonelementary, we are forced to write

$$y_p = \frac{1}{2}e^{x}\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt,$$

and so $$y = y_c + y_p = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2}e^{x}\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt. \quad (12)$$

# Exercise 4.6

**11.** $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$

**12.** $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

**13.** $y'' + 3y' + 2y = \sin e^x$

**14.** $y'' - 2y' + y = e^t \arctan t$

**15.** $y'' + 2y' + y = e^{-t} \ln t$

**16.** $2y'' + 2y' + y = 4\sqrt{x}$

**17.** $3y'' - 6y' + 6y = e^x \sec x$

**18.** $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

In Problems 19–22 solve each differential equation by variation of parameters, subject to the initial conditions $y(0) = 1, y'(0) = 0$.

**19.** $4y'' - y = xe^{x/2}$

**20.** $2y'' + y' - y = x + 1$

**21.** $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$

**22.** $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$