Exact Equations – IVPs

Example 3.

Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2.$

SOLUTION By writing the differential equation in the form

$$(\cos x \sin x - xy^2)\, dx + y(1 - x^2)\, dy = 0,$$

we recognize that the equation is exact because

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}.$$

Putting values of M = $\cos x \sin x - xy^2$and N = $y - yx^2$ in the formula:

$$\int \boldsymbol{M\,dx} + \int (\boldsymbol{Terms\ of\ N\ without\ x})\,\boldsymbol{dy = c}$$

$$\int (\cos x \sin x - xy^2)\,dx + \int y\,dy = c$$

$$y^2(1 - x^2) - cos^2 x = c$$

$$\text{Using} \quad y(0) = 2 \quad \Longrightarrow \quad c = 3$$

Hence, solution of given IVP is

$$y^2(1 - x^2) - cos^2 x = 3$$

In Problems 21–26 solve the given initial-value problem.

21. $(x + y)^2\, dx + (2xy + x^2 - 1)\, dy = 0, \quad y(1) = 1$

22. $(e^x + y)\, dx + (2 + x + ye^y)\, dy = 0, \quad y(0) = 1$

23. $(4y + 2t - 5)\, dt + (6y + 4t - 1)\, dy = 0, \quad y(-1) = 2$

24. $\left(\dfrac{3y^2 - t^2}{y^5}\right)\dfrac{dy}{dt} + \dfrac{t}{2y^4} = 0, \quad y(1) = 1$

25. $(y^2 \cos x - 3x^2 y - 2x)\, dx$
$+ (2y \sin x - x^3 + \ln y)\, dy = 0, \quad y(0) = e$

26. $\left(\dfrac{1}{1 + y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y + \sin x), \; y(0) = 1$

Making Non-exact Equations Exact

Method

For the non-exact ODE

$$M(x,y)dx + N(x,y)dy = 0 \qquad (\mathbf{1})$$

1. Evaluate $M_y - N_x$.

2. Check:
   - If $\frac{M_y - N_x}{N}$ is a function of x alone, then integrating factor for (1) is:

$$\boldsymbol{I.F = e^{\int \frac{M_y - N_x}{N} dx}}$$

   - If $\frac{M_y - N_x}{M}$ is a function of y alone, then integrating factor for (1) is:

$$\boldsymbol{I.F = e^{-\int \frac{M_y - N_x}{M} dy}}$$

3. Multiply the integrating factor with (1) to make it exact.

Example 4.

The nonlinear first-order differential equation

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

is not exact. With the identifications $M = xy$, $N = 2x^2 + 3y^2 - 20$, we find the partial derivatives $M_y = x$ and $N_x = 4x$. The first quotient from (13) gets us nowhere, since

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

Whereas,

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}.$$

The integrating factor is then $e^{\int 3dy/y} = e^{3\ln y} = e^{\ln y^3} = y^3$. After we multiply the given DE by $\mu(y) = y^3$, the resulting equation is

$$xy^4\, dx + (2x^2y^3 + 3y^5 - 20y^3)\, dy = 0.$$

Which is an exact DE. Solving this equation using the formula for exact equations, we get the solution as

$$\tfrac{1}{2}x^2y^4 + \tfrac{1}{2}y^6 - 5y^4 = c.$$

Solve the given ODEs by finding an appropriate integrating factor.

$(2y^2 + 3x)\, dx + 2xy\, dy = 0$

$6xy\, dx + (4y + 9x^2)\, dy = 0$

$\cos x\, dx + \left(1 + \frac{2}{y}\right) \sin x\, dy = 0$

$(10 - 6y + e^{-3x})\, dx - 2\, dy = 0$

$(y^2 + xy^3)\, dx + (5y^2 - xy + y^3 \sin y)\, dy = 0$