Critical Points and Optimization

(Section 9.5 Applied Calculus)

To optimize a function means to find the largest or smallest value of the function. If the function represents profit, we may want to find the conditions that maximize profit. On the other hand, if the function represents cost we may we may want to find the conditions that minimize cost.

Local Maxima and Minima for Functions of Two Variables

- f has a local maximum at point P_o if $f(P_o) \geq f(P)$ for all points P nearP_o
- f has a local minimum at point P_o if $f(P_o) \leq f(P)$ for all points P nearP_o

Critical Points

If a function $f(x, y)$ has a local maximum or minimum at a point (x_o, y_o) not on the boundary of the domain of$f$, then either

$$\boldsymbol{f_x(x_o, y_o) = 0 \& f_y(x_o, y_o) = 0}$$

or (at least) one partial derivative is undefined at(x_o, y_o). Points where each of the partial derivative is either zero or undefined are called critical points.

Is a critical point a local maxima or local minima?

Classification of Critical Points:

<u>SecondDerivative Test for Functions of Two Variables:</u>

Suppose (x_o, y_o) is a critical point where$f_x(x_o, y_o) = f_y(x_o, y_o) = 0$.

Let

$$\boldsymbol{D = f_{xx}(x_o, y_o) f_{yy}(x_o, y_o) - \left(f_{xy}(x_o, y_o)\right)^2}$$

- If$\boldsymbol{D} > 0 \& \boldsymbol{f_{xx}(x_o, y_o)} > 0$, then f has a local minimum at $(\boldsymbol{x_o, y_o})$.
- If $\boldsymbol{D} > 0 \& \boldsymbol{f_{xx}(x_o, y_o)} < 0$, then f has a local maximum at $(\boldsymbol{x_o, y_o})$.
- If $\boldsymbol{D} < 0$then $\boldsymbol{f}$ hasneithermaximum nor minimum at $(\boldsymbol{x_o, y_o})$.
- If$\boldsymbol{D} = \boldsymbol{0}$then test is conclusive.

Question:Find all the critical points and determine whether each is a local maximum, local minimum or neither.

a. $f(x, y) = x^2 - 2x + y^2 - 4y + 5$
b. $f(x, y) = x^3 + y^3 - 3x^2 - 3y + 10$
c. $f(x, y) = y^3 - 3xy + 6x$

Solution:

a. $f(x, y) = x^2 - 2x + y^2 - 4y + 5$

$$f_x = 2x - 2 + 0 - 0 + 0 = 2x - 2$$

$$f_y = 0 - 0 + 2y - 4 + 0 = 2y - 4$$

To find the critical point, put

$$f_x = f_y = 0$$

$$f_x = 2x - 2 = 0 \qquad => x = 1$$

$$f_y = 2y - 4 = 0 => y = 2$$

(1,2) is the critical point.

Now we check whether (1,2)is a local maximum, local minimum or neither. For this, we have to use the second derivative test.

$f_x = 2x - 2$

$f_{xx} = 2,\ f_{xx}(1,2) = 2$

$f_{xy} = 0\ ,\ f_{xy}(1,2) = 0$

$f_y = 2y - 4$

$f_{yy} = 2,\ f_{yy}(1,2) = 2$

$$\boldsymbol{D = f_{xx}(1,2) f_{yy}(1,2) - \left(f_{xy}(1,2)\right)^2}$$

$$\boldsymbol{D} = 2 \text{ x } 2 - 0^2 = 4 > 0$$

$$\boldsymbol{D} > 0 \qquad \& \qquad \boldsymbol{f_{xx} = 2} > 0$$

$=> f$ has a local minimum at (1,2)

b. $f(x,y) = x^3 + y^3 - 3x^2 - 3y + 10$

$$f_x = 3x^2 + 0 - 6x - 0 + 0 = 3x^2 - 6x$$

$$f_y = 0 + 3y^2 - 0 - 3 + 0 = 3y^2 - 3$$

To find the critical points, put

$$f_x = f_y = 0$$

$f_x = 3x^2 - 6x = 0 \qquad => 3x(x-2) = 0 \qquad => x = 0, x = 2$

$f_y = 3y^2 - 3 = 0 \quad => 3(y^2 - 1) = 0 \quad => y^2 - 1 = 0 \quad => y = \pm 1$

So the critical points are

$$(\mathbf{0}, \mathbf{1}), (\mathbf{0}, -\mathbf{1}), (\mathbf{2}, \mathbf{1}), (\mathbf{2}, -\mathbf{1})$$

$f_x = 3x^2 - 6x$	$f_y = 3y^2 - 3$
$f_{xx} = 6x - 6, f_{xx}(0,1) = -6$	$f_{yy} = 6y, f_{yy}(0,1) = 6$
$f_{xy} = 0, f_{xy}(0,1) = 0$	

$$\boldsymbol{D} = \boldsymbol{f_{xx}}(\mathbf{0}, \mathbf{1}) \boldsymbol{f_{yy}}(\mathbf{0}, \mathbf{1}) - \left(\boldsymbol{f_{xy}}(\mathbf{0}, \mathbf{1})\right)^{\mathbf{2}}$$

$$\boldsymbol{D} = -\mathbf{6} \text{ x } \mathbf{6} - \mathbf{0}^{\mathbf{2}} = -36 < 0$$

So, $(\mathbf{0}, \mathbf{1})$ is neither maximum nor minimum.

Now for$(\mathbf{0}, -\mathbf{1})$,

$f_{xx} = 6x - 6, f_{xx}(0, -1) = -6$	$f_{yy} = 6y, f_{yy}(0, -1) = -6$
$f_{xy} = 0, f_{xy}(0, -1) = 0$	

$$\boldsymbol{D} = \boldsymbol{f_{xx}}(\mathbf{0}, -\mathbf{1}) \boldsymbol{f_{yy}}(\mathbf{0}, -\mathbf{1}) - \left(\boldsymbol{f_{xy}}(\mathbf{0}, -\mathbf{1})\right)^{\mathbf{2}}$$

$$\boldsymbol{D} = (-\mathbf{6} \text{ x} - \mathbf{6}) - \mathbf{0}^{\mathbf{2}} = 36 > 0$$

As $\boldsymbol{D} > 0 \qquad \& \qquad f_{xx}(0, -1) = -6 < 0$

So f has local maximum at $(0, -1)$.

Now for$(\mathbf{2},\mathbf{1})$,

$f_{xx} = 6x - 6, f_{xx}(2,1) = 6$

$f_{xy} = 0, f_{xy}(2,1) = 0$

$f_{yy} = 6y, f_{yy}(2,1) = 6$

$$\boldsymbol{D} = \boldsymbol{f}_{\boldsymbol{xx}}(\mathbf{2},\mathbf{1})\boldsymbol{f}_{\boldsymbol{yy}}(\mathbf{2},\mathbf{1}) - \left(\boldsymbol{f}_{\boldsymbol{xy}}(\mathbf{2},\mathbf{1})\right)^{\mathbf{2}}$$

$$\boldsymbol{D} = \mathbf{6}\ \mathbf{x}\ \mathbf{6} - \mathbf{0}^{\mathbf{2}} = 36 > 0$$

As $\boldsymbol{D} > 0$ & $f_{xx}(2,1) = 6 > 0$

So f has local minimum at $(2,1)$.

For $(\mathbf{2}, -\mathbf{1})$,

$f_{xx} = 6x - 6, f_{xx}(2,-1) = 6$

$f_{xy} = 0, f_{xy}(2,-1) = 0$

$f_{yx} = 0, f_{yx}(2,-1) = 0$

$f_{yy} = 6y, f_{yy}(2,-1) = -6$

$$\boldsymbol{D} = \boldsymbol{f}_{\boldsymbol{xx}}(\mathbf{2},-\mathbf{1})\boldsymbol{f}_{\boldsymbol{yy}}(\mathbf{2},-\mathbf{1}) - \left(\boldsymbol{f}_{\boldsymbol{xy}}(\mathbf{2},-\mathbf{1})\right)^{\mathbf{2}}$$

$$\boldsymbol{D} = (\mathbf{6}\ \mathbf{x} - \mathbf{6}) - \mathbf{0}^{\mathbf{2}} = -36 > 0$$

As $\boldsymbol{D} < 0$

So f has neither local minima nor local maxima at $(2,-1)$.

c. Do it yourself.

In Problems 6–15, find all the critical points and determine whether each is a local maximum, local minimum, or neither.

6. $f(x, y) = x^2 + 4x + y^2$

7. $f(x, y) = x^2 + xy + 3y$

8. $f(x, y) = x^2 + y^2 + 6x - 10y + 8$

9. $f(x, y) = y^3 - 3xy + 6x$

10. $f(x, y) = x^2 - 2xy + 3y^2 - 8y$

11. $f(x, y) = x^3 - 3x + y^3 - 3y$

12. $f(x, y) = x^3 + y^2 - 3x^2 + 10y + 6$

13. $f(x, y) = x^3 + y^3 - 6y^2 - 3x + 9$

14. $f(x, y) = x^3 + y^3 - 3x^2 - 3y + 10$

15. $f(x, y) = 400 - 3x^2 - 4x + 2xy - 5y^2 + 48y$