# Calculus and Analytical Geometry

## Lecture no. 04


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**Topic:** Functions and graphs

**Outline of the lecture:**

i. Functions
   - Domain
   - Range

ii. Geometrical Approach
   - Onto function
   - One-to-one function

iii. Some basic functions
   - Constant function
   - Identity function
   - Linear function
   - Quadratic function
   - Rational function
   - Square root function
   - Exponential function
   - Sine function
   - Cosine function
   - Tangent function
   - Piecewise function

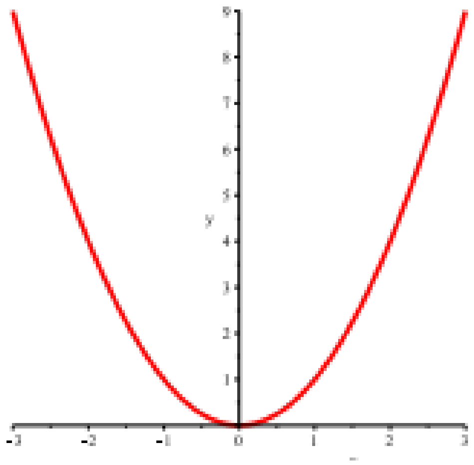
iv. Practice questions

1. **Function:**
   If a variable y **depends** on a variable x in such a way that each value of x determines **exactly one** value of y, then we say that y is a function of x.
   - **Domain of a function:**
     The domain is the set of all possible *x*-values which will make the function "work", and will output real *y*-values.
   - **Range of a function:**
     The **range** of a function is the complete set of all possible **resulting values** of the dependent variable (*y,* usually), after we have substituted the domain.
2. **Geometrical Approach:**
   Every vertical line intersects the graph of a function f exactly at one point.
   If a horizontal line intersecting the graph meets the y-axis at the point y, then y belongs to the range of f. The set of all such y points from range of f.
   - **Onto function:**
     f is onto if every horizontal line intersects the graph of f.
   - **One-to-one function:**
     f is one-to-one if every horizontal line intersects the graph of f exactly at one point.

**Example:**

Sketch the graph of the function $f(x) = x^2$ and find its domain and range. Check whether its onto and onto one or not.

**Step 1:** Graph



**Step 2:**

**Domain:** The domain of function is real numbers $\mathbb{R}$ because square of every real number is possible.

**Range:** Since all the horizontal lines that cut the graph lie above the x-axis, the range(f)=[0,+∞).

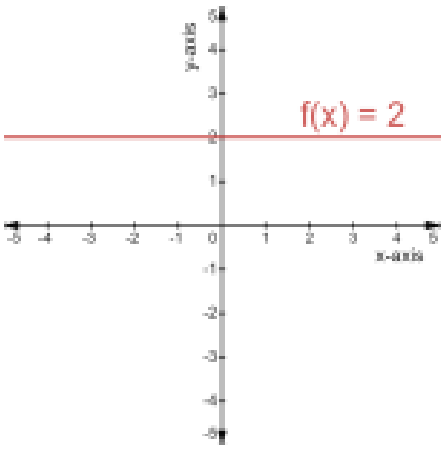
**Step 3:**

Onto function: Since the lines that lie below the x-axis do not intersect the graph, f is not onto.

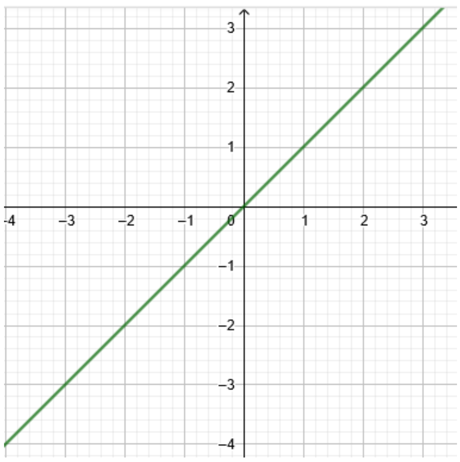
One to One function: Since each horizontal line that lies above the x-axis intersects the graph at TWO points, f is not one-to-one.

**3. Some basic functions:**

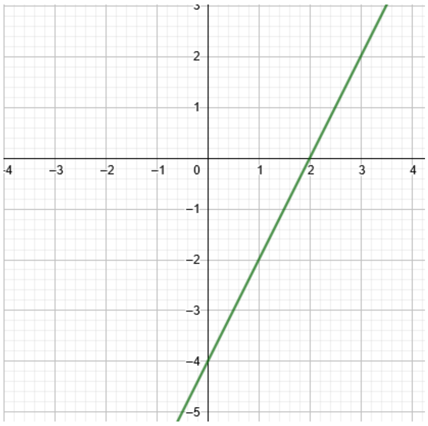
- **Constant function:** A **constant function** is a function having the same range for different values of the domain. Graphically a constant function is a straight line, which is parallel to the x-axis. Its domain is the set of all real numbers, R. So, domain = **R.** Since a constant function f(x) = k leads to only one output, which is k, its range is the set with just one element k. Range = {k}.
  **Example:** f(x)=2



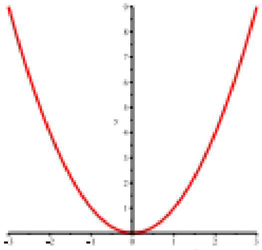

- **Identity function:** Let R be the set of real numbers. Thus, the real-valued function f : R → R by y = f(a) = a for all a ∈ R, is called the identity function. Here the domain and range (codomain) of function f are R.
  **Example:** f(x)=x



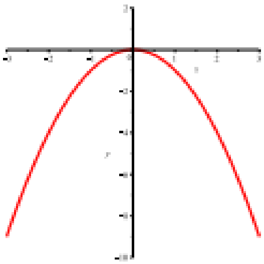
- **Linear Function:** A **linear function** is of the form f(x) = mx + b where 'm' and 'b' are real numbers.
  **Example:** 2x-4



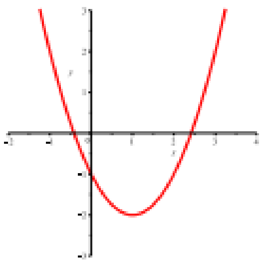

- **Quadratic function:** the function of the form $f(x) = ax^2 + bx + c, x \in R$ and a and b are fixed real numbers.



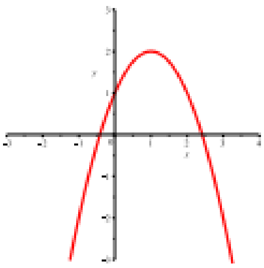
$y = x^2$



$y = -x^2$

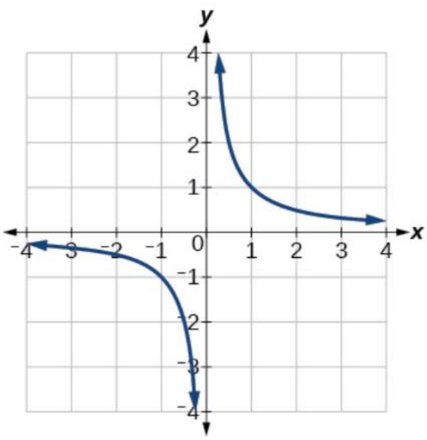


$y = x^2 - 2x - 1$

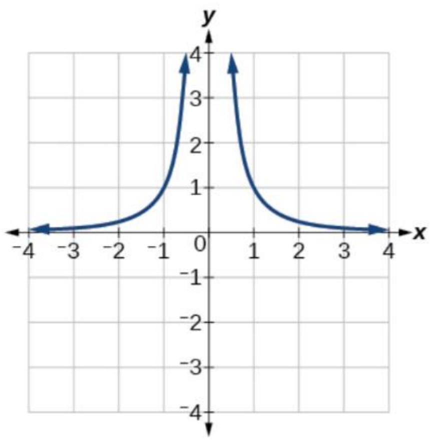


$y = -x^2 + 2x + 1$

- **Rational function:** The function of the form $\frac{P(x)}{Q(x)}$ is known as rational function where P(x) and Q(x) are polynomials.

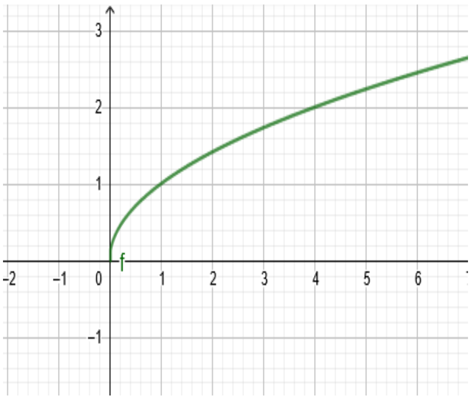



$f(x) = \frac{1}{x}$

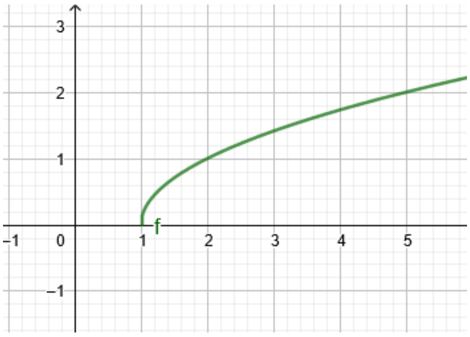



$f(x) = \frac{1}{x^2}$

- **Square root function:** The function of the form $f(x) = \sqrt{x}$. The domain and range of the function is $\geq 0$.

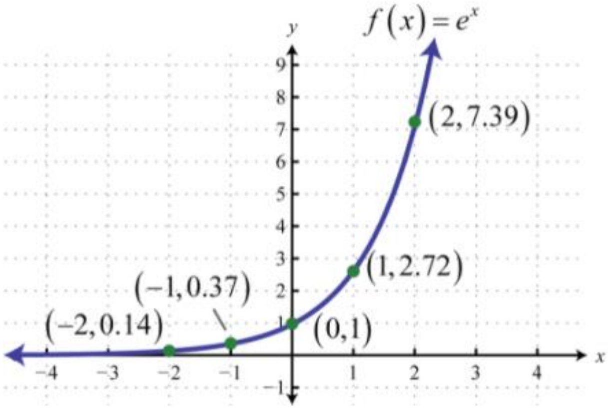



$f(x) = \sqrt{x}$

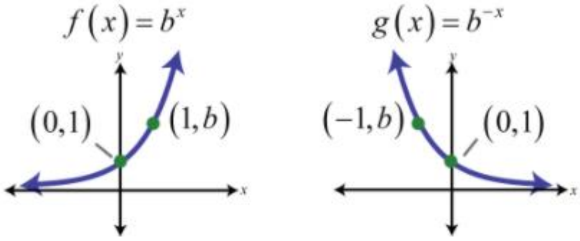



$f(x) = \sqrt{x-1}$

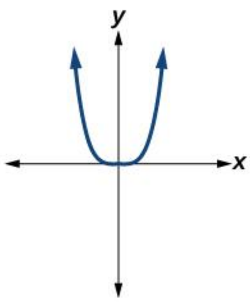
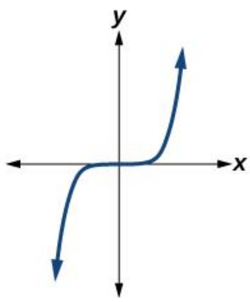
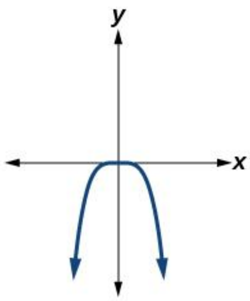
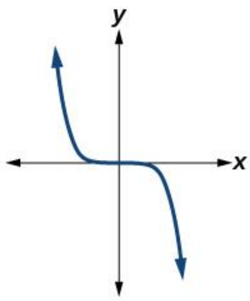
- **Exponential Function:** The function of the form $f(x) = e^x$ or $f(x) = a^x$, where a$> 0$.
  **Example:** Graph of function $f(x) = e^x$ where $e = 2.718 \approx 2.72$



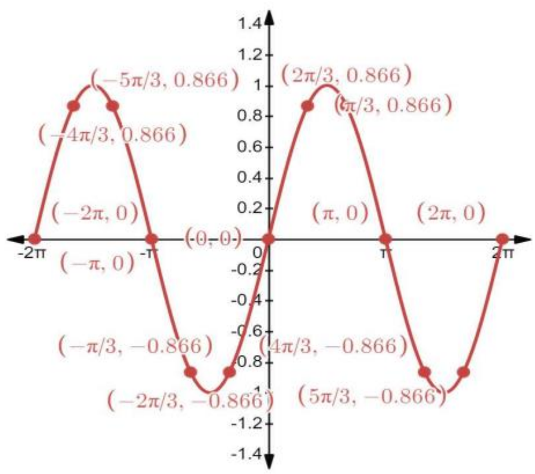

The direction of curve can change depending upon the sign with the power.



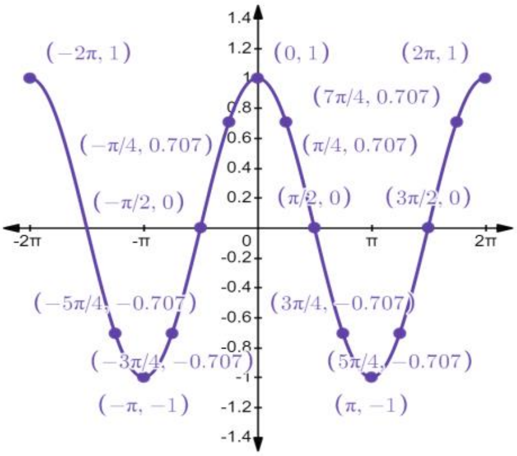

- **Power function:** The function of the form $f(x) = x^n, n \in \mathbb{Z}^+$
  The graph of power function when the power is even or odd, is given below.

| | Even power | Odd power |
|---|---|---|
| Positive constant $k > 0$ |  $x \to -\infty,\ f(x) \to \infty$ and $x \to \infty,\ f(x) \to \infty$ |  $x \to -\infty,\ f(x) \to -\infty$ and $x \to \infty,\ f(x) \to \infty$ |
| Negative constant $k < 0$ |  $x \to -\infty,\ f(x) \to -\infty$ and $x \to \infty,\ f(x) \to -\infty$ |  $x \to -\infty,\ f(x) \to \infty$ and $x \to \infty,\ f(x) \to -\infty$ |

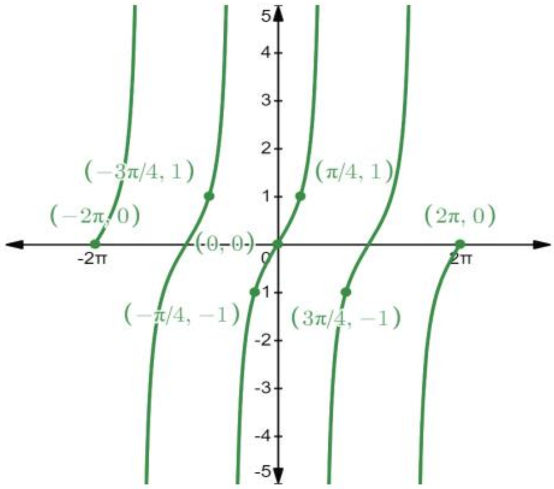
- **Sine function:** A function of the form $f(x) = \sin x$ is known as sine function. It is represented in the interval $[-2\pi, 2\pi]$



- **Cosine function:** A function of the form $f(x) = \cos x$ is known as sine function. It is represented in the interval $[-2\pi, 2\pi]$




- **Tangent function:** A function of the form $f(x) = \tan x$ is known as sine function. It is represented in the interval $[-2\pi, 2\pi]$




- **Piece-wise function:**

  A piecewise function is a function built from pieces of different functions over different intervals.

  **Example 1:** Absolute value function.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x > 0, \end{cases}$$

**Graph of piece wise function:**

Draw the graph of the following function:

$$f(x) = \begin{cases} x^2 & if\ x \leq 1 \\ 3 & if\ 1 < x \leq 2 \\ x & if\ x > 2 \end{cases}$$

**Solution:**

Let's divide the function into 3 subfunctions: $f_1(x) = x^2$, $f_2(x) = 3$, $f_3(x) = x$

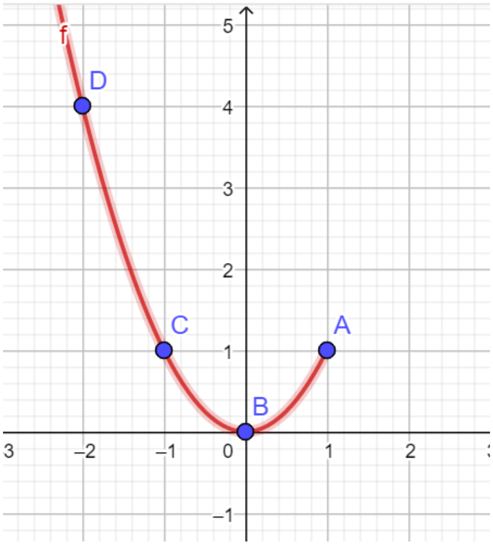
**Step 1:** Completing the table

| $X$ | $1$ | $0$ | $-1$ | $-2$ |
|---|---|---|---|---|
| $f_1(x)$ | $1$ | $0$ | $1$ | $4$ |

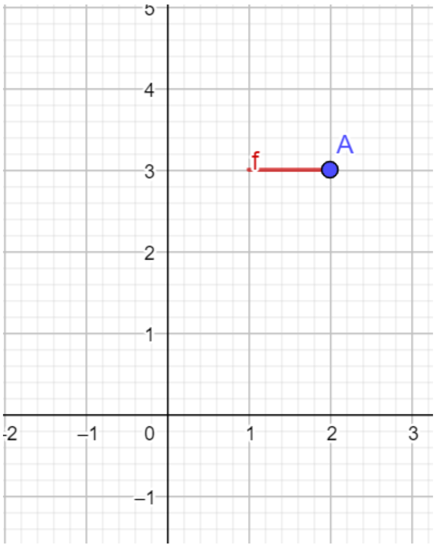
| $X$ | $2$ |
|---|---|
| $f_2(x)$ | $3$ |

| $X$ | $3$ | $4$ | $5$ | $6$ |
|---|---|---|---|---|
| $f_3(x)$ | $3$ | $4$ | $5$ | $6$ |

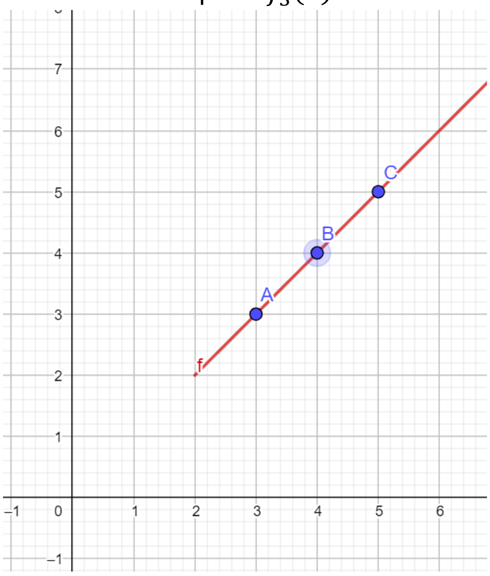
Graph of $f_1(x)$:



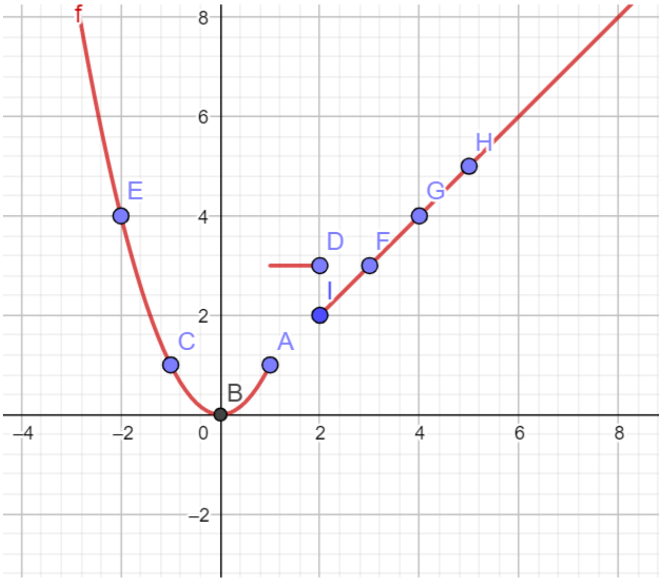

Graph of $f_2(x)$:




Graph of $f_3(x)$:



**Graph of complete piece wise function:**

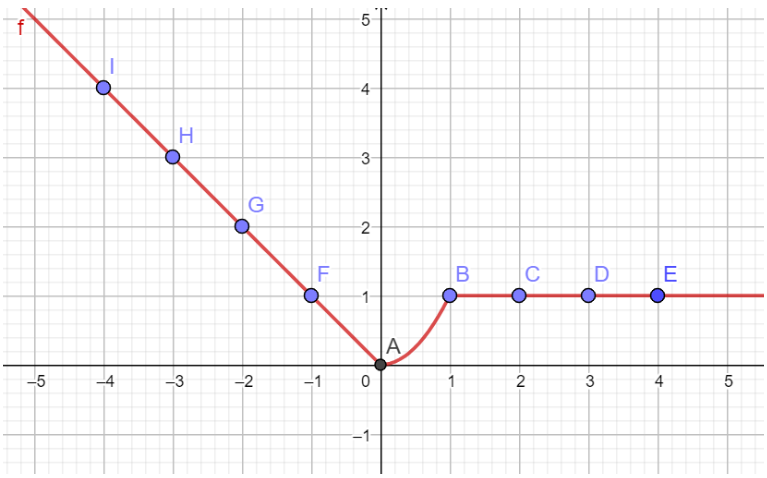



**Example 2:**

Draw graph of the following piece-wise function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

**Graph:**



**Practice Questions:**

Draw the graphs of following equations:

1) $f(x) = 3\sin x$
2) $f(x) = 2x - 5$
3) $f(x) = |x + 3|$
4) $f(x) = \frac{-2}{x-3}$
5) $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \leq x \end{cases}$
6) $f(x) = \begin{cases} -3 & x < -2 \\ \frac{3}{2}x & -2 \leq x < 0 \\ -x & 0 \leq x < 1 \\ 2 & 1 \leq 3 \\ 0 & 3 < x \end{cases}$
7) $f(x) = -2x^2 + 1$