Computer Vision

Geometric Primitives & Transformations

Reference: Szeliski 2.1

Images are
2D projections of
the 3D world

Simplified Image Formation

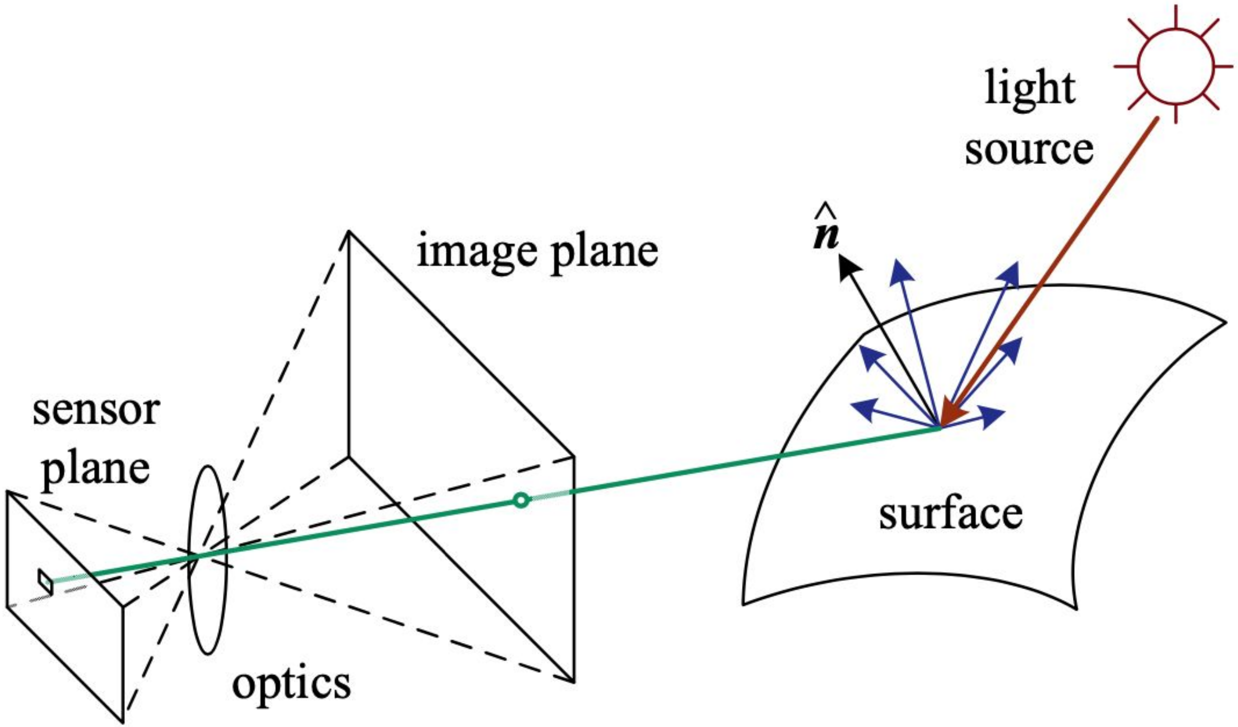
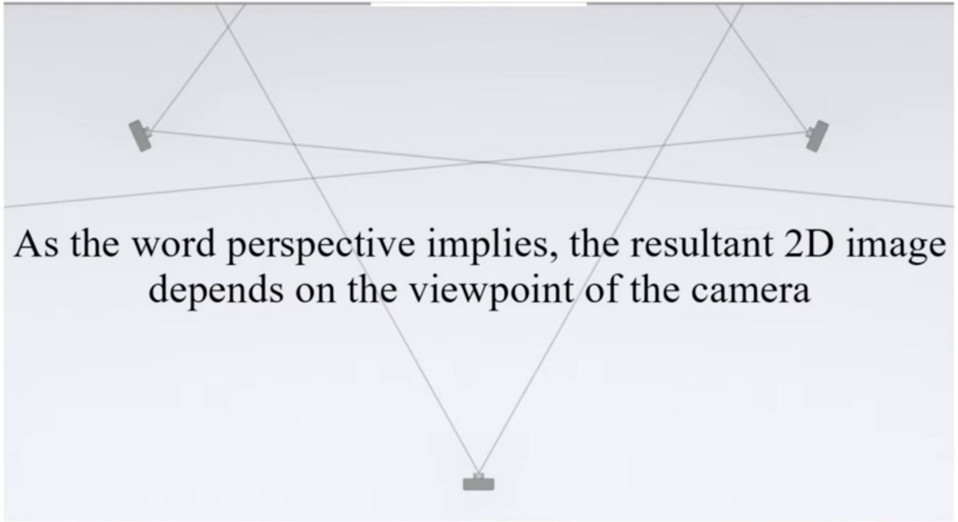


Figure: R. Szeliski

Perspective Projection



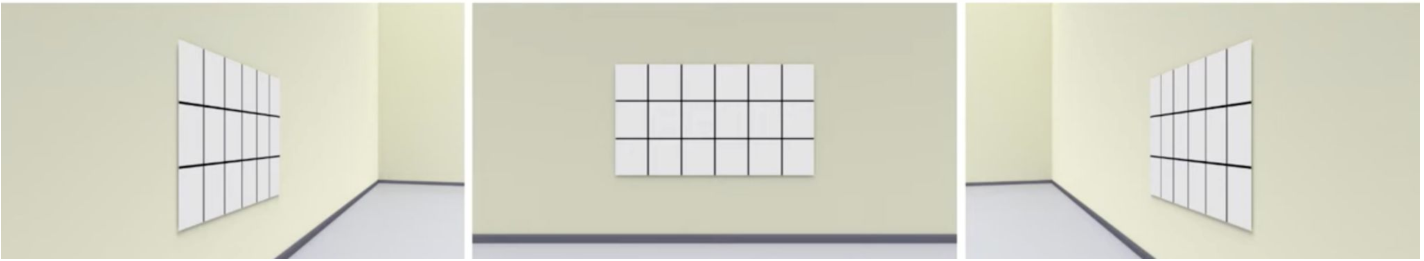
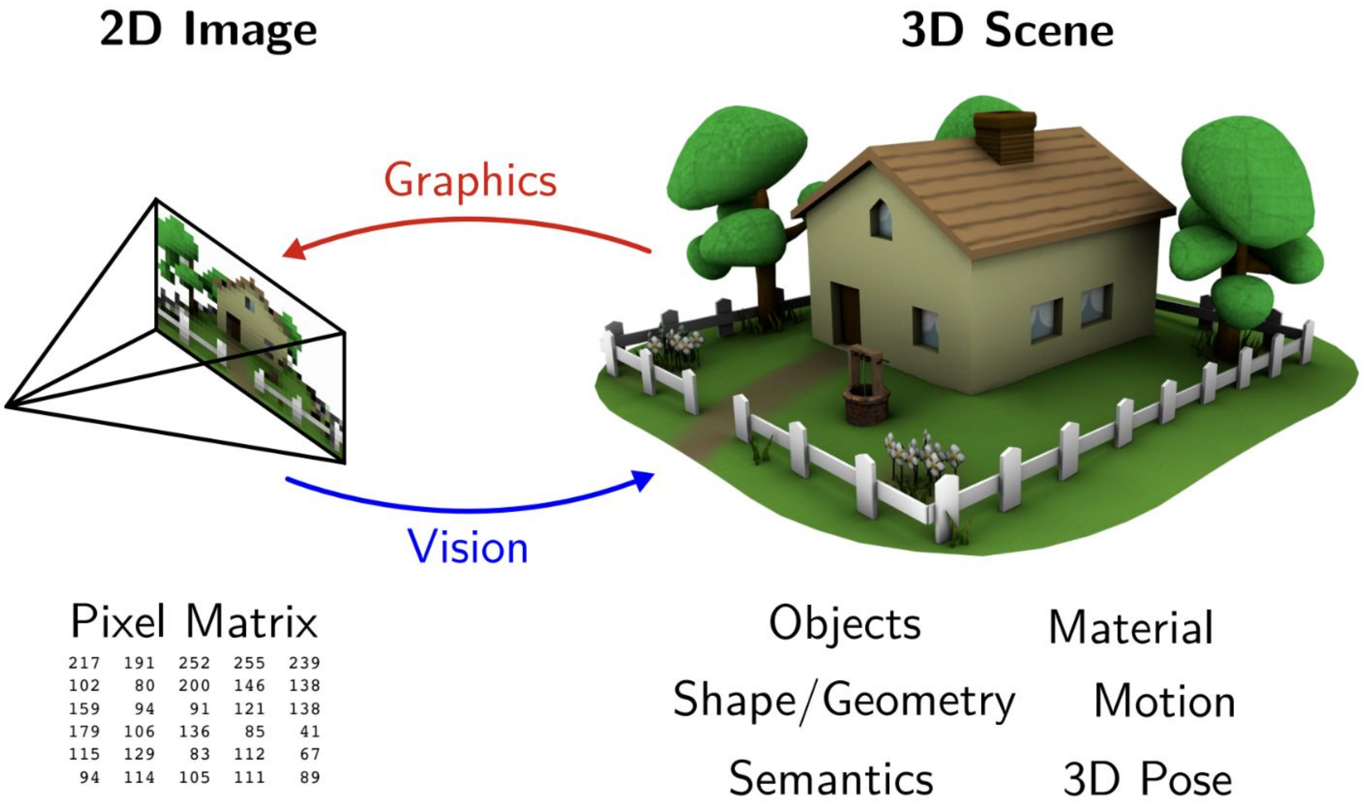


Figure: https://www.youtube.com/@huseyin_ozde...

Can we understand the 3D world from 2D images?

CV is an **ill-posed inverse problem**



Slide credit: Andreas Geiger

Geometric Primitives in 2D & 3D

2D & 3D Transformations

Points in Cartesian and Homogeneous Coordinates

2D points: $\mathbf{x} = (x, y) \in \mathcal{R}^2$ or column vect $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

3D points: $\mathbf{x} = (x, y, z) \in \mathcal{R}^3$ (often noted $\mathbf{X}$ or $\mathbf{P}$)

Homogeneous coordinates: append a 1

$$\bar{\mathbf{x}} = (x, y, 1) \qquad \bar{\mathbf{x}} = (x, y, z, 1)$$

Why?

Homogeneous coordinates in 2D

2D Projective Space: $\mathcal{P}^2 = \mathcal{R}^3 - (0,0,0)$ (same story in 3D with $\mathcal{P}^3$)

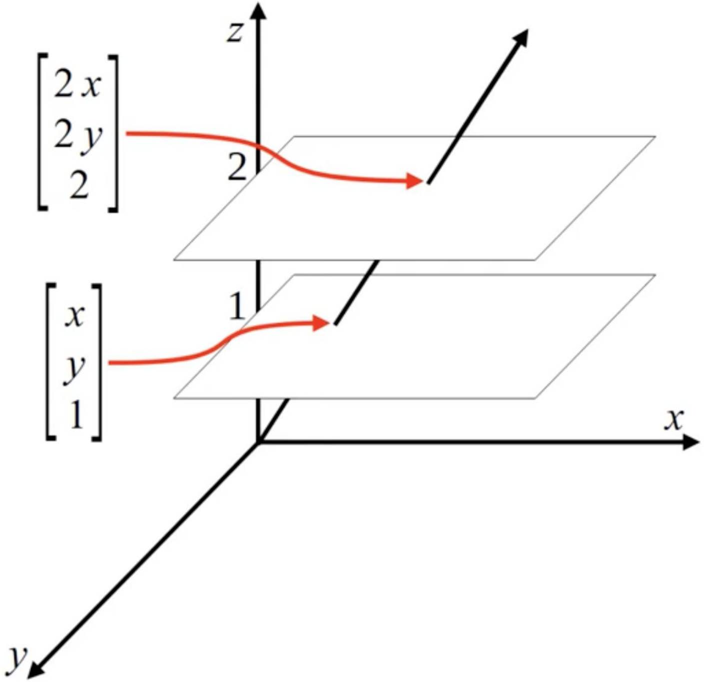
- heterogeneous → homogeneous $\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

- homogeneous → heterogeneous $\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$

- points differing only by scale are *equivalent*: $(x, y, w) \sim \lambda\,(x, y, w)$

$$\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{w}) = \tilde{w}(x, y, 1) = \tilde{w}\bar{\mathbf{x}}$$

Homogeneous coordinates in 2D



In homogeneous coordinates, a point and its scaled versions are same

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = w \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} wx \\ wy \\ w \end{bmatrix} \qquad w \neq 0$$

Figure: https://www.youtube.com/@huseyin_ozde...

Everything is easier in Projective Space

2D Lines:

Representation: $l = (a, b, c)$

Equation: $ax + by + c = 0$

In homogeneous coordinates: $\bar{x}^T l = 0$

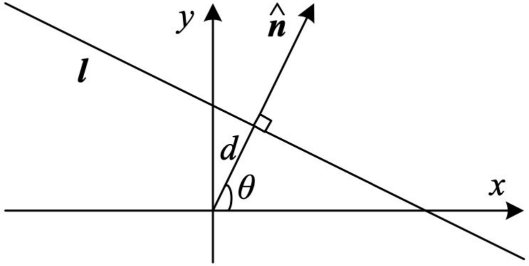
General idea: homogenous coordinates unlock the full power of linear algebra!

Everything is easier in Projective Space

2D Lines:

$$\tilde{\mathrm{x}}^{\mathrm{T}}\mathrm{l} = 0, \forall \tilde{\mathrm{x}} = (x, y, w) \in P^2$$

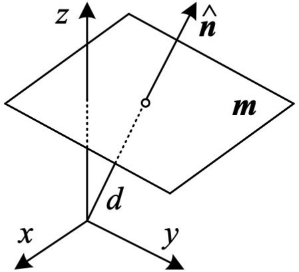
$$\mathbf{l} = (\hat{n}_x, \hat{n}_y, d) = (\hat{\mathbf{n}}, d) \text{ with } \|\hat{\mathbf{n}}\| = 1$$



3D planes: same!

$$\tilde{\mathrm{x}}^{\mathrm{T}}\mathrm{m} = 0, \forall \tilde{\mathrm{x}} = (x, y, z, w) \in P^3$$

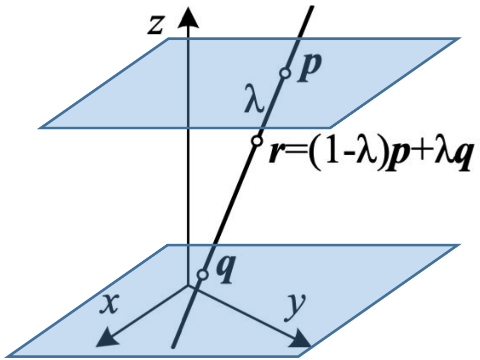
$$\mathbf{m} = (\hat{n}_x, \hat{n}_y, \hat{n}_z, d) = (\hat{\mathbf{n}}, d) \text{ with } \|\hat{\mathbf{n}}\| = 1$$



Lines in 3D

Two-point parametrization:

$$\mathbf{r} = (1-\lambda)\mathbf{p} + \lambda\mathbf{q} \qquad \tilde{\mathbf{r}} = \mu\tilde{\mathbf{p}} + \lambda\tilde{\mathbf{q}}$$



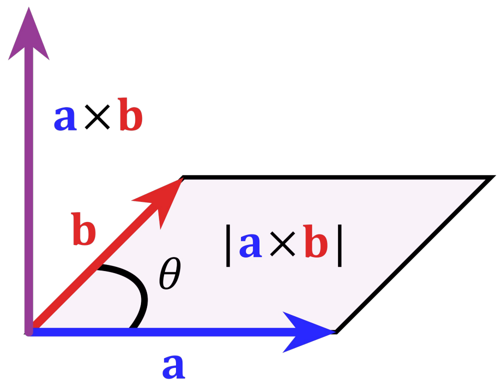
Two-plane parametrization:

coordinates (x_0, y_0) & (x_1, y_1) of intersection

with planes at $z = 0, 1$ (or other planes)

Cross-product quick reminder

$$\mathbf{a} \times \mathbf{b} = ||\mathbf{a}||\, ||\mathbf{b}|| \sin(\theta)\, \mathbf{n}$$



$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Benefits of Homogeneous Coordinates

- Line – Point duality:
  - line between two 2D points: $\tilde{\mathbf{l}} = \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2$
  - intersection of two 2D lines: $\tilde{\mathbf{x}} = \tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2$
- Representation of Infinity:
  - points at infinity: $(x, y, 0)$; line at infinity: $(0,0,1)$
- Parallel & vertical lines are easy (take-home: intersect //)
- Makes 2D & 3D transformations linear!

Questions?

Geometric Primitives in 2D & 3D

2D & 3D Transformations

The camera as a coordinate transformation

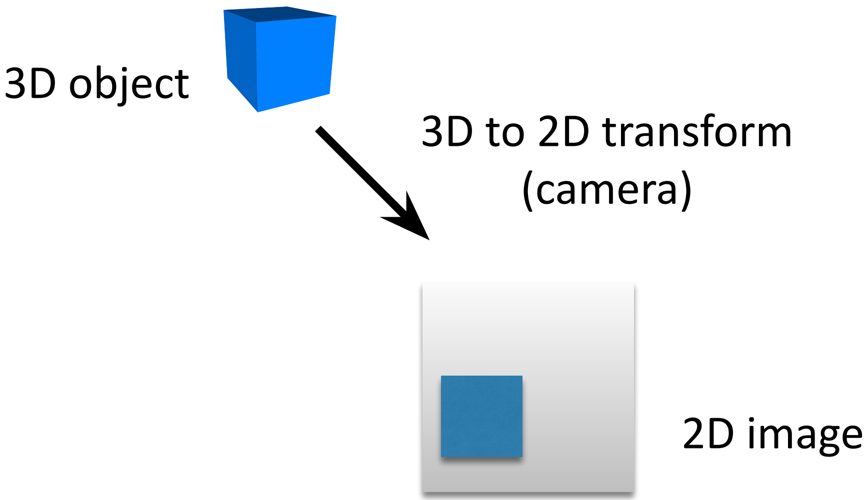
A camera is a mapping

from:

the 3D world

to:

a 2D image



Source: K. Kitani

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A camera is a mapping

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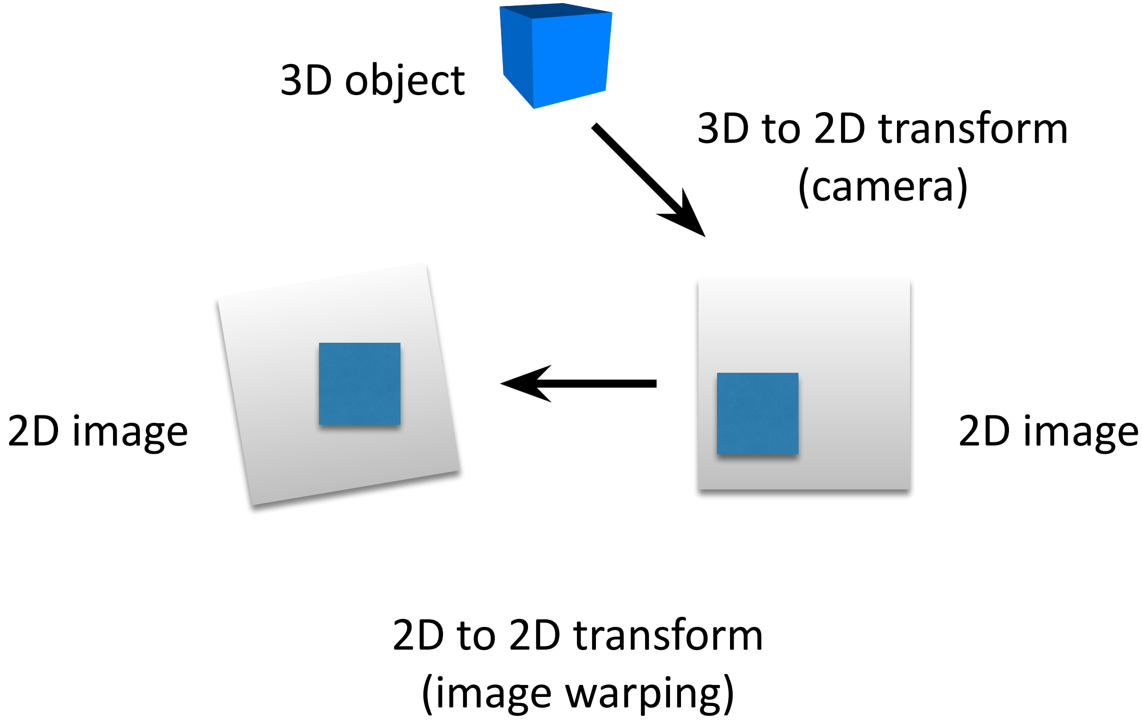


Source: K. Kitani

Cameras and objects can move!

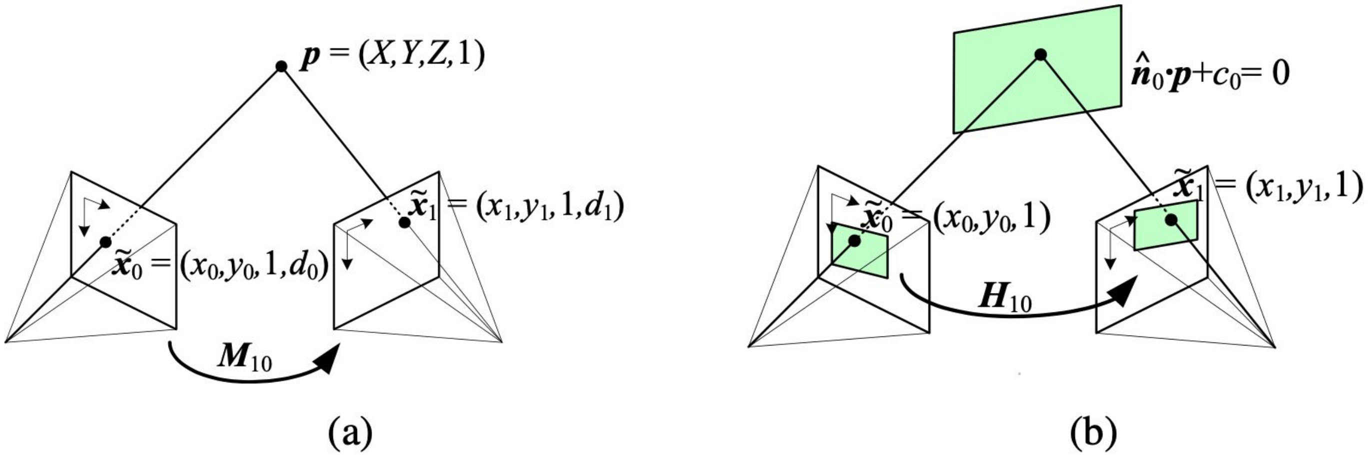


Figure 2.12 *A point is projected into two images: (a) relationship between the 3D point coordinate* $(X, Y, Z, 1)$ *and the 2D projected point* $(x, y, 1, d)$*; (b) planar homography induced by points all lying on a common plane* $\hat{\mathbf{n}}_0 \cdot \mathbf{p} + c_0 = 0$.

Figure: R. Szeliski

2D Transformations Zoo

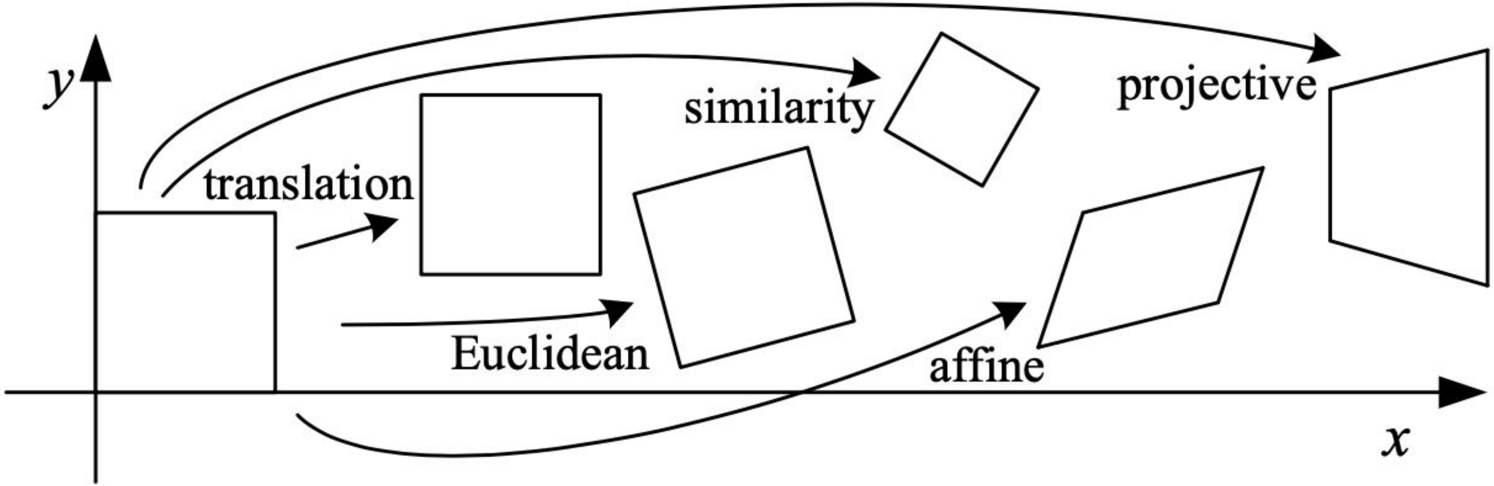


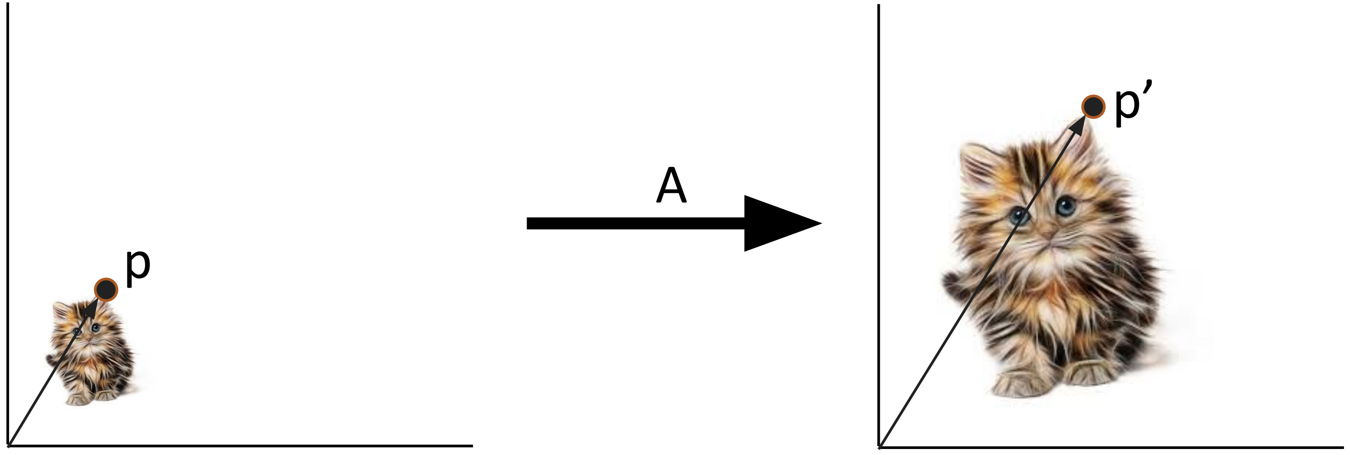
Figure: R. Szeliski

Transformation = Matrix Multiplication

Scale $$\mathbf{M} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$	Flip across y $$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
Rotate $$\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$	Flip across origin $$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
Shear $$\mathbf{M} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$$	Identity $$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

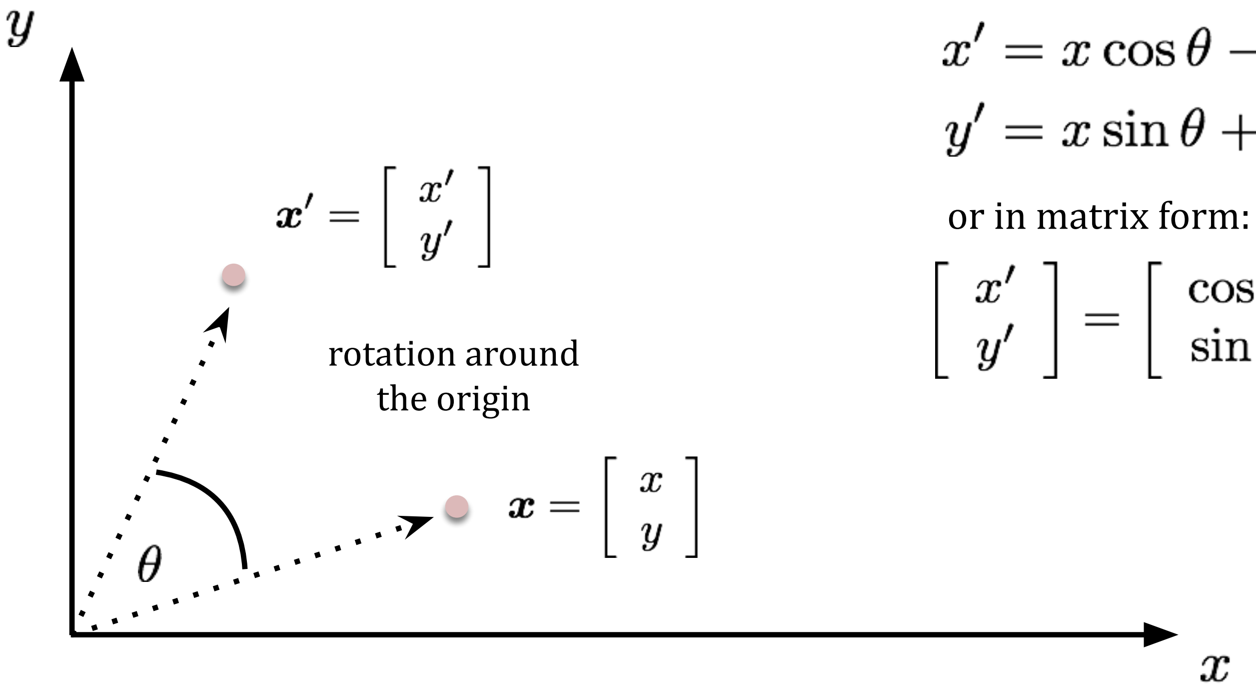
Scaling

$$\underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\text{A}} \times \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{p}} = \underbrace{\begin{bmatrix} s_x x \\ s_y y \end{bmatrix}}_{\text{p'}}$$



Slide: JC. Niebles

Rotation



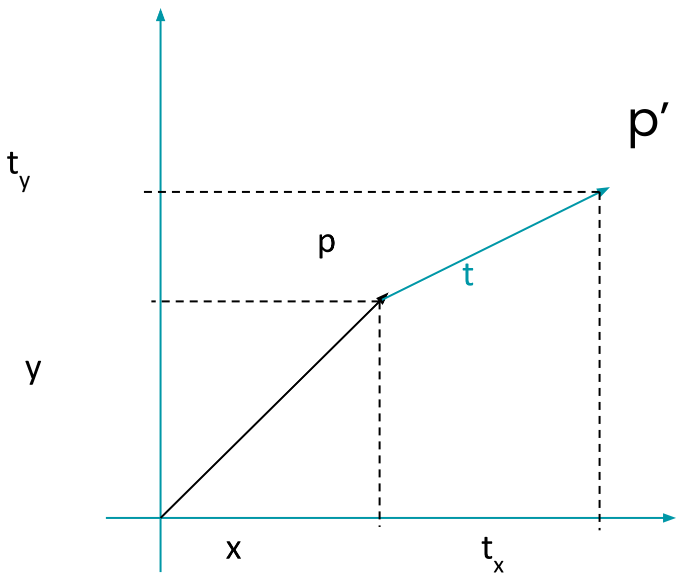
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

or in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Slide: K. Kitani

Translation

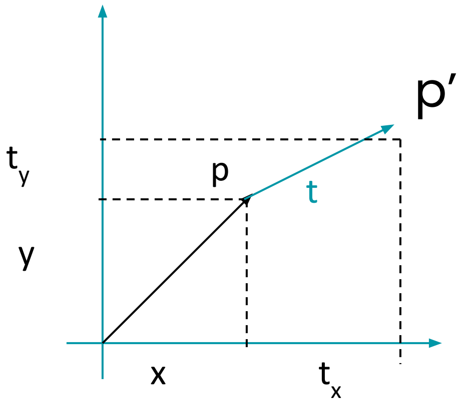


$$x' = x + t_x$$
$$y' = y + t_y$$

As a matrix?

Slide: JC. Niebles

Translation with homogeneous coordinates



$$p = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

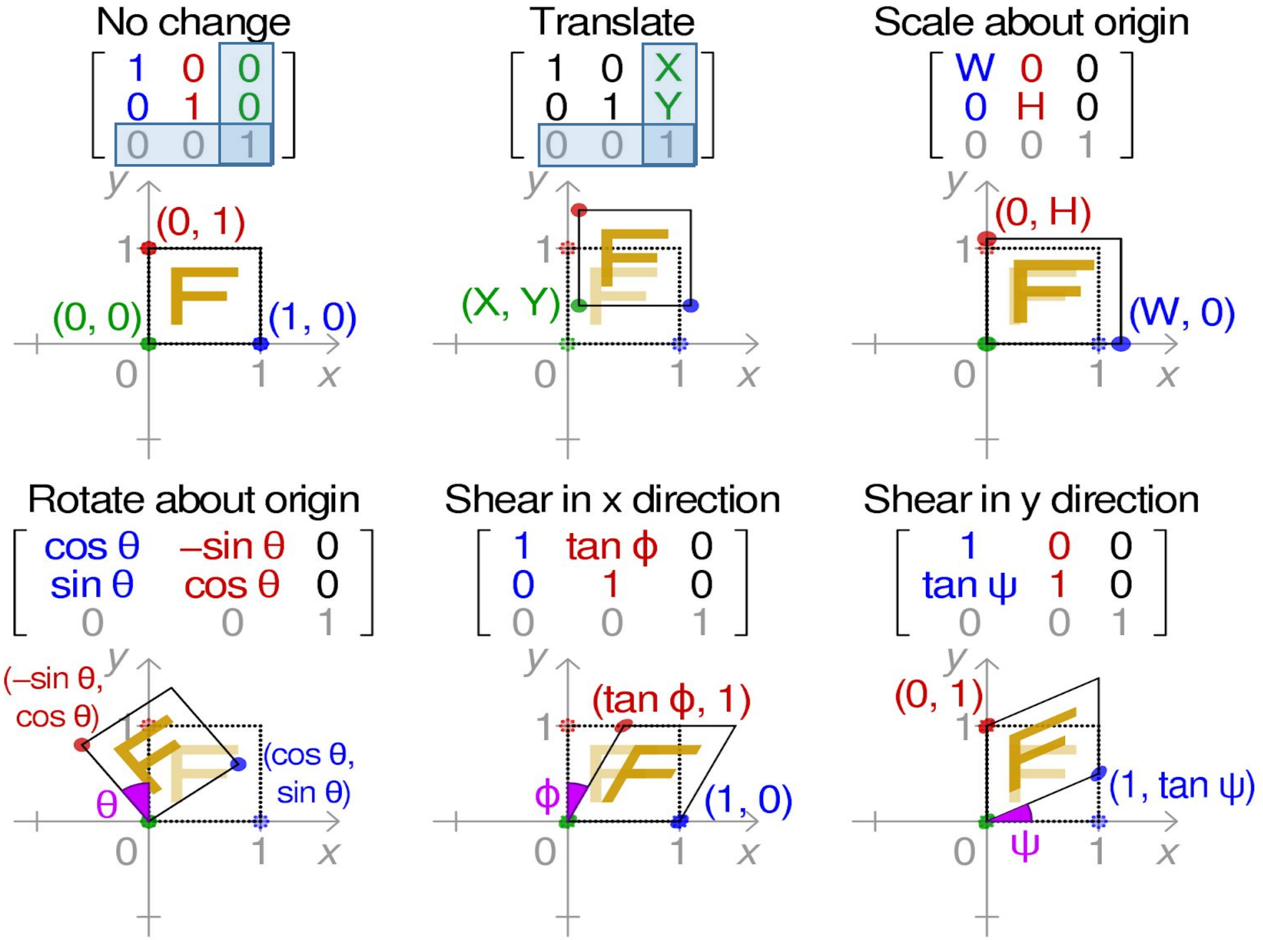
$$t = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \rightarrow \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix}$$

$$p' = Tp$$

$$p' \rightarrow \begin{bmatrix} x + tx \\ y + ty \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{bmatrix} p = Tp$$

Slide: JC. Niebles

2D Transformations with homogeneous coordinates



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & X \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} W & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \tan\phi & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ \tan\psi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure: Wikipedia

Questions?

2D Transformations Zoo

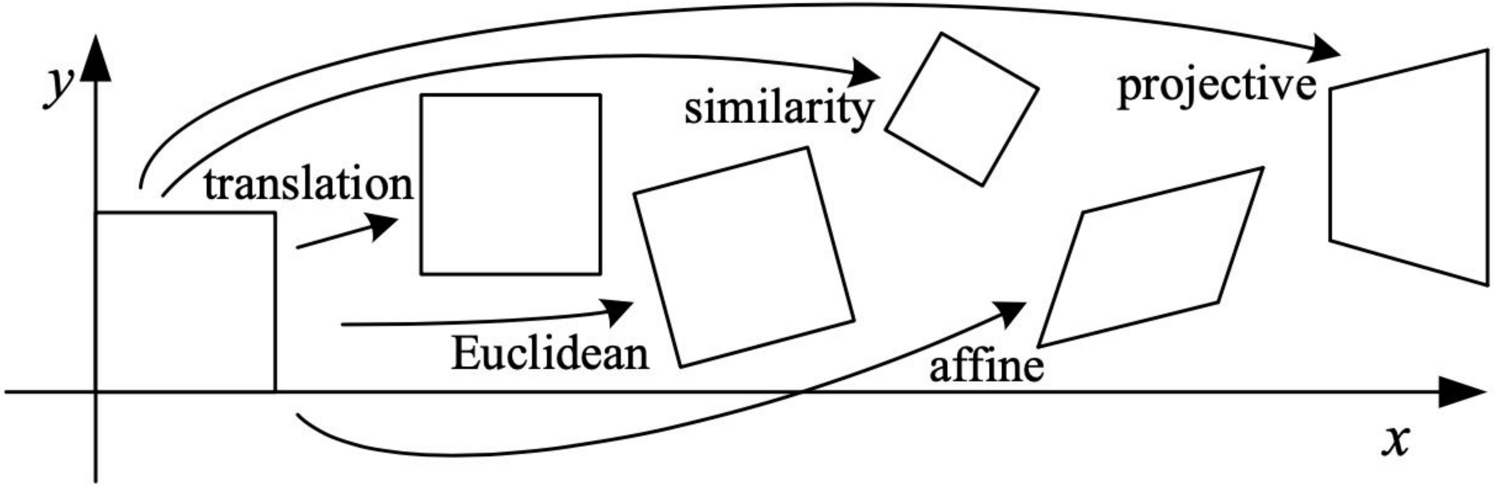


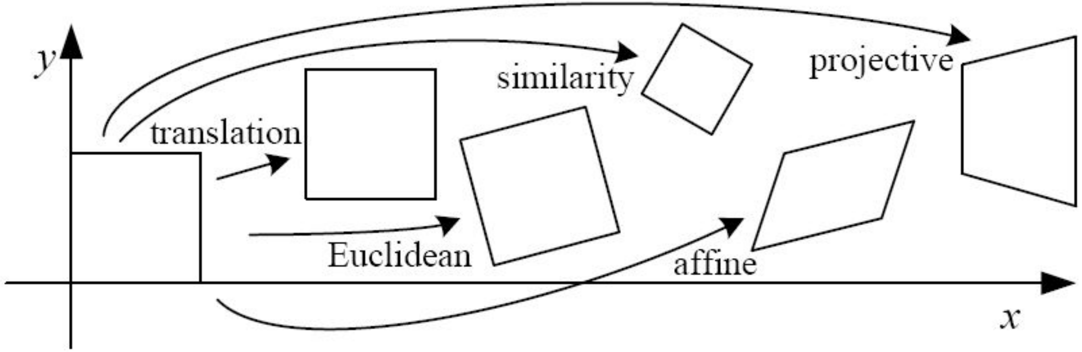
Figure: R. Szeliski

Euclidean / Rigid Transformation

Euclidean (rigid): rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

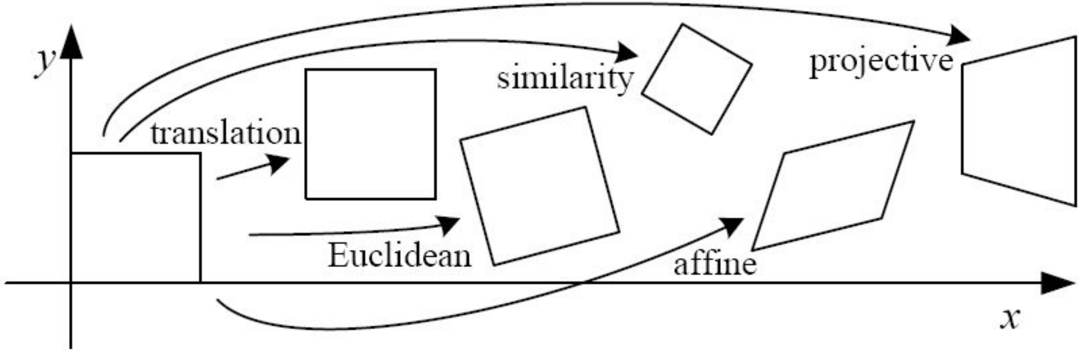
How many degrees of freedom?



Similarity Transformation

Similarity: Scaling + rotation + translation
$$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?

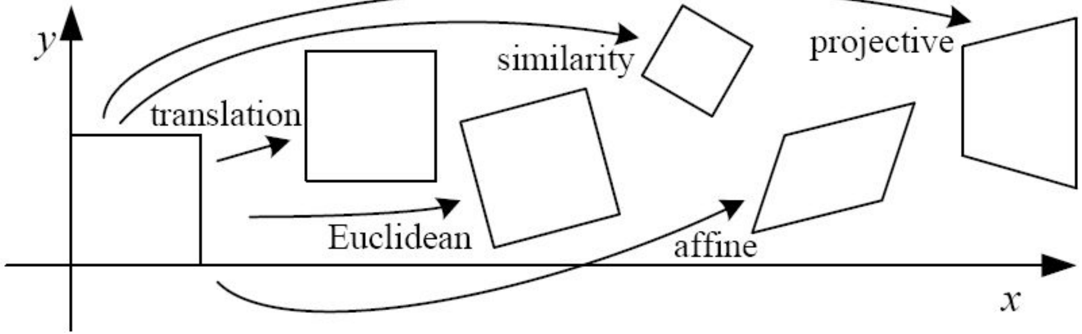


Similarity Transformation

Similarity: Scaling + rotation + translation

$$\begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha\cos\theta & -\alpha\sin\theta & b_0 \\ \alpha\sin\theta & \alpha\cos\theta & b_1 \\ 0 & 0 & 1 \end{bmatrix}$$

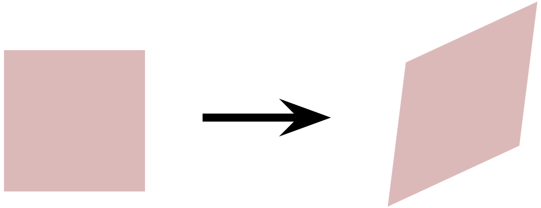
How many degrees of freedom?



Affine Transformation

Affine transformations are combinations of

- Arbitrary (4-DOF) linear transformations + translations



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Cartesian coordinates

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

Homogeneous coordinates

$$\longrightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & b_0 \\ A_{10} & A_{11} & b_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Source: K. Kitani

Affine Transformation

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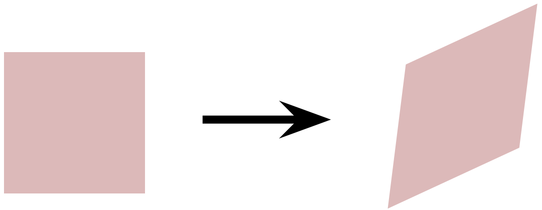

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$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Cartesian coordinates

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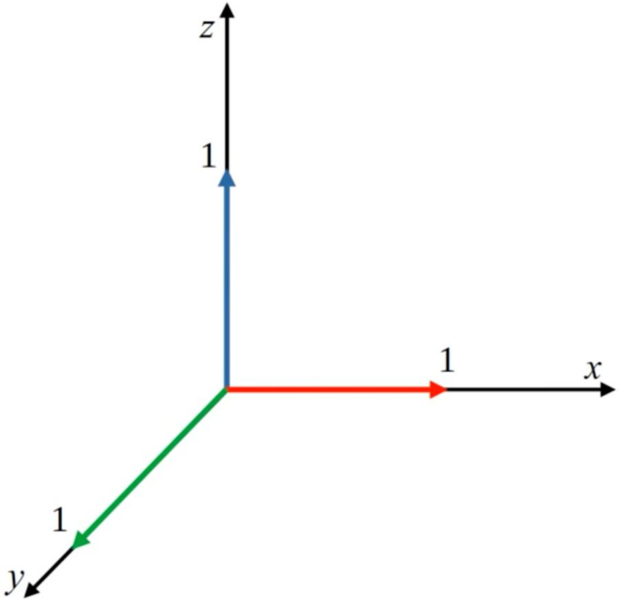
How many degrees of freedom?

Source: K. Kitani

Affine Transformation

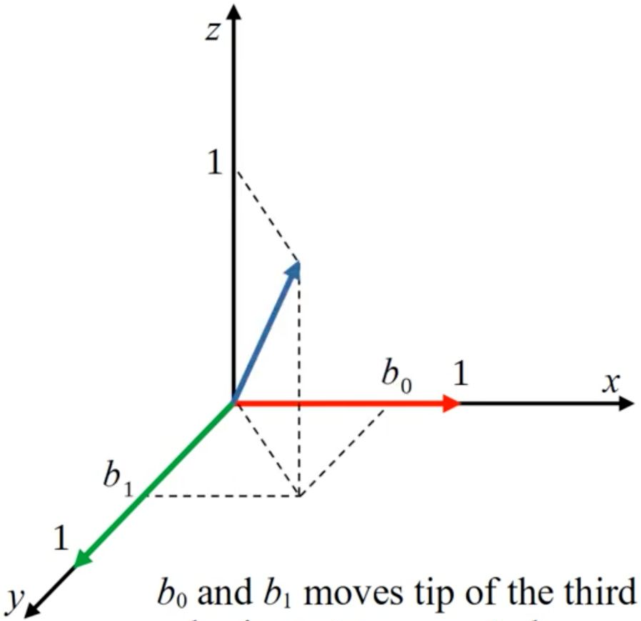


This matrix is a linear transformation matrix in 3D

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} A_{00} & A_{01} & b_0 \\ A_{10} & A_{11} & b_1 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Figure: https://www.youtube.com/@huseyin_ozde...

Affine Transformation



b_0 and b_1 moves tip of the third basis vector on $z=1$ plane

This matrix is a linear transformation matrix in 3D

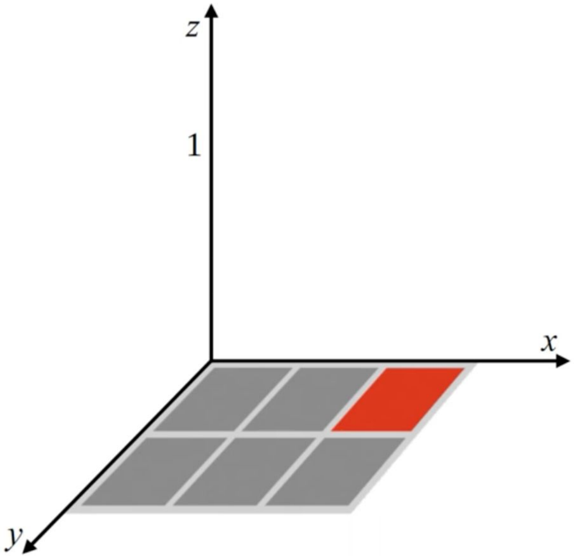
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & b_0 \\ A_{10} & A_{11} & b_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Then this column is the third basis vector of transformed vector space

And what b_0 and b_1 do is to change the orientation of that basis vector

Figure: https://www.youtube.com/@huseyin_ozde...

Affine Transformation



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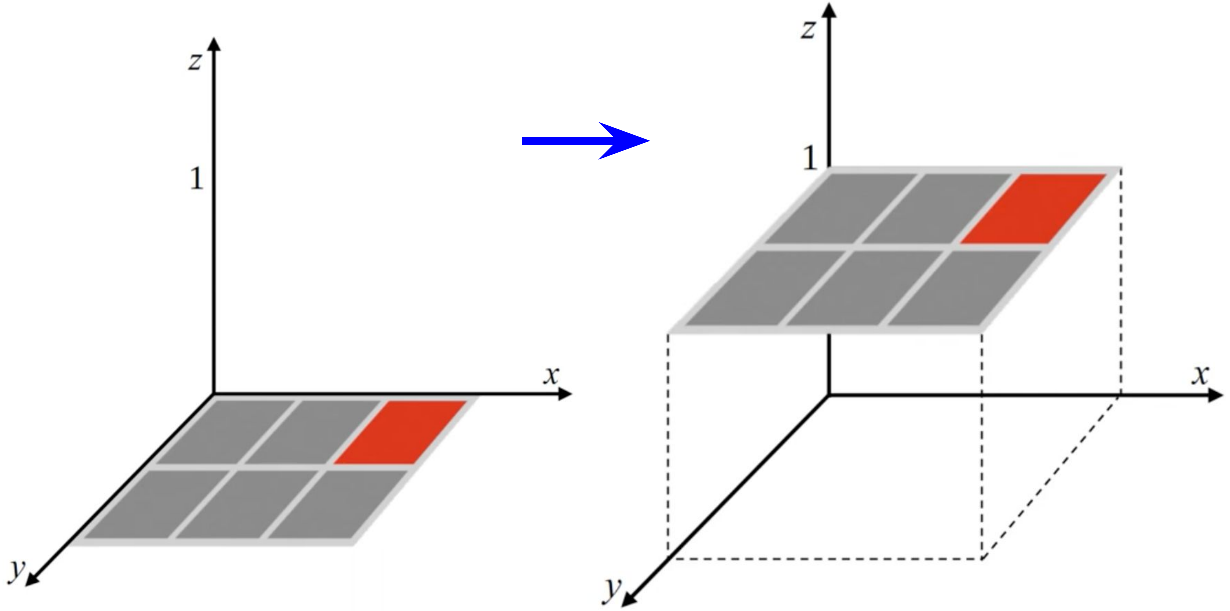

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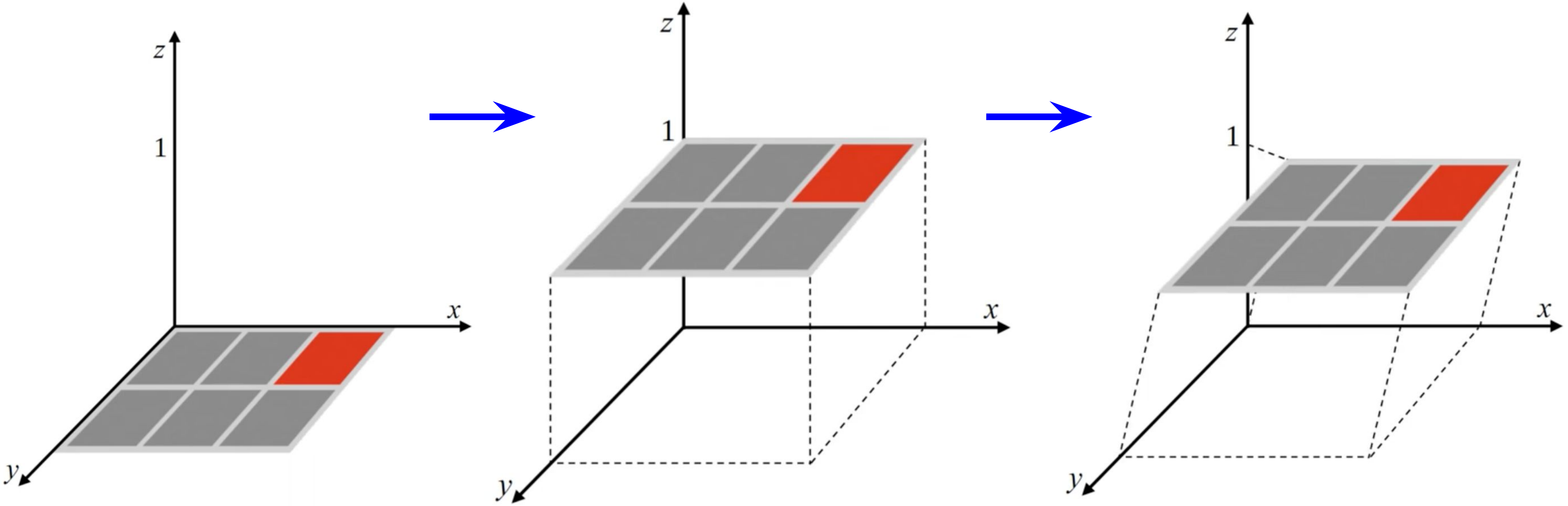

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Affine Transformation

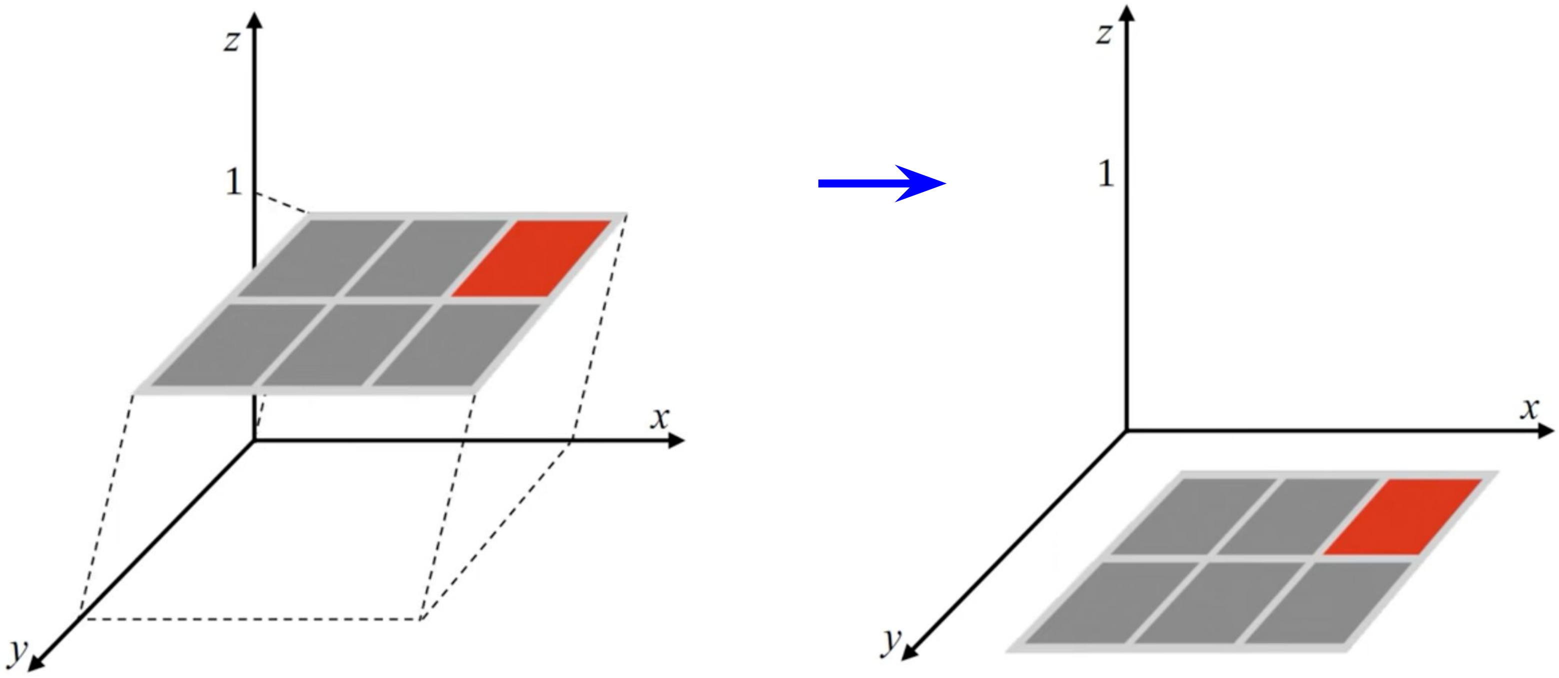
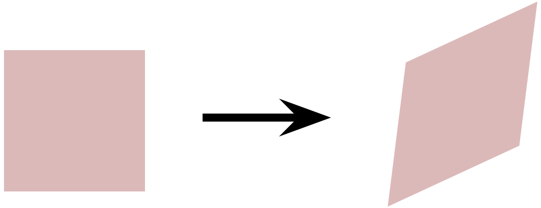


Figure: https://www.youtube.com/@huseyin_ozde...

Affine Transformation

Affine transformations are combinations of

- Arbitrary (4-DOF) linear transformations + translations



Properties of affine transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines map to parallel lines
- ratios are preserved

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & b_0 \\ A_{10} & A_{11} & b_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Source: K. Kitani

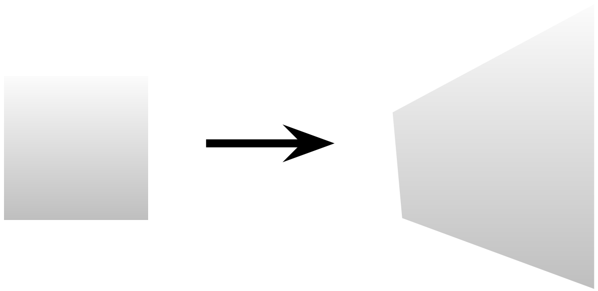
Projective Transformation (homography)

Projective transformations are combinations of

- Affine transformations + projective warps

$$w\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & b_0 \\ A_{10} & A_{11} & b_1 \\ h_0 & h_1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

How many degrees of freedom?



Source: K. Kitani

Projective Transformation (homography)

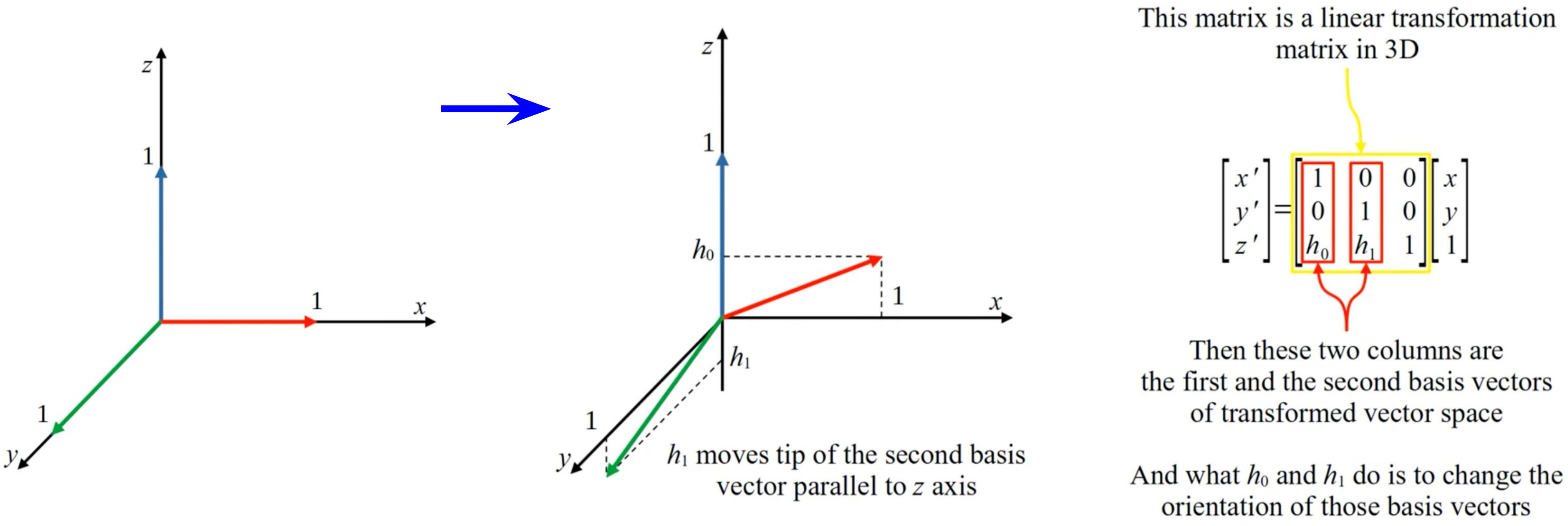
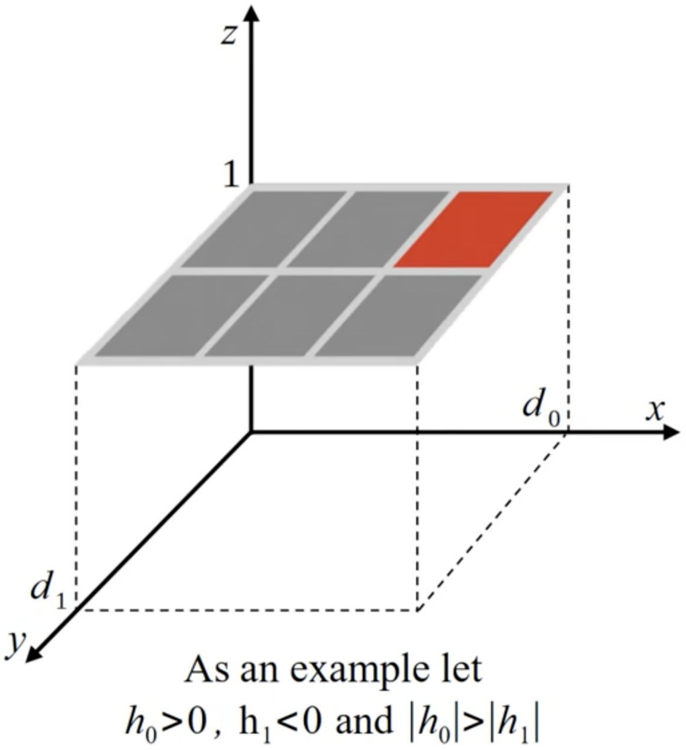


Figure: https://www.youtube.com/@huseyin_ozde...

Projective Transformation (homography)



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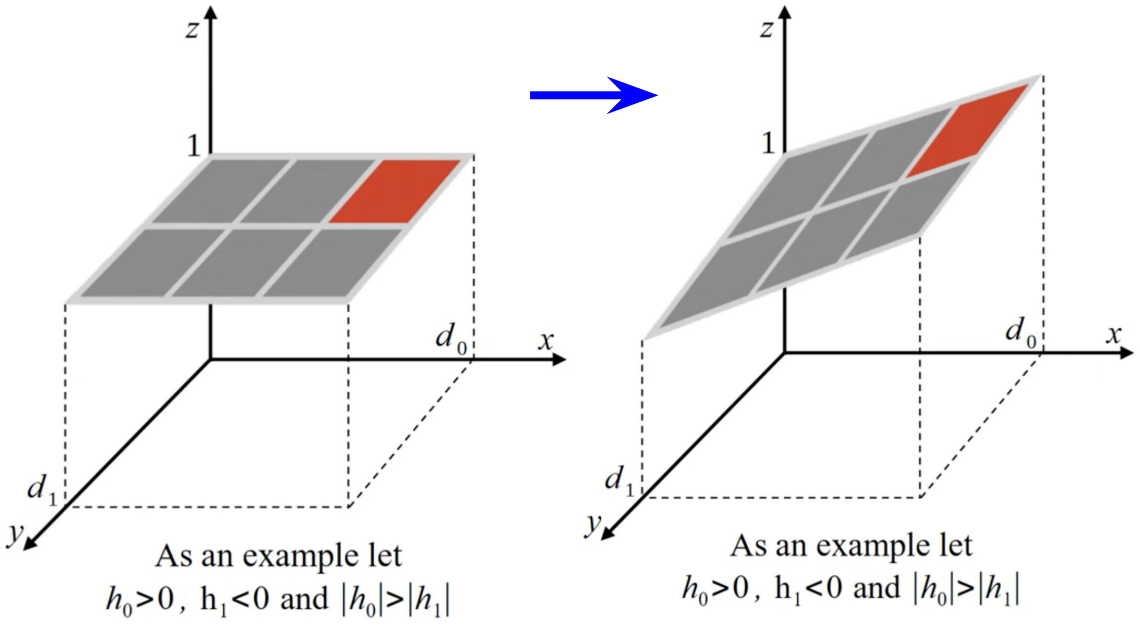
Projective Transformation (homography)



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Projective Transformation (homography)

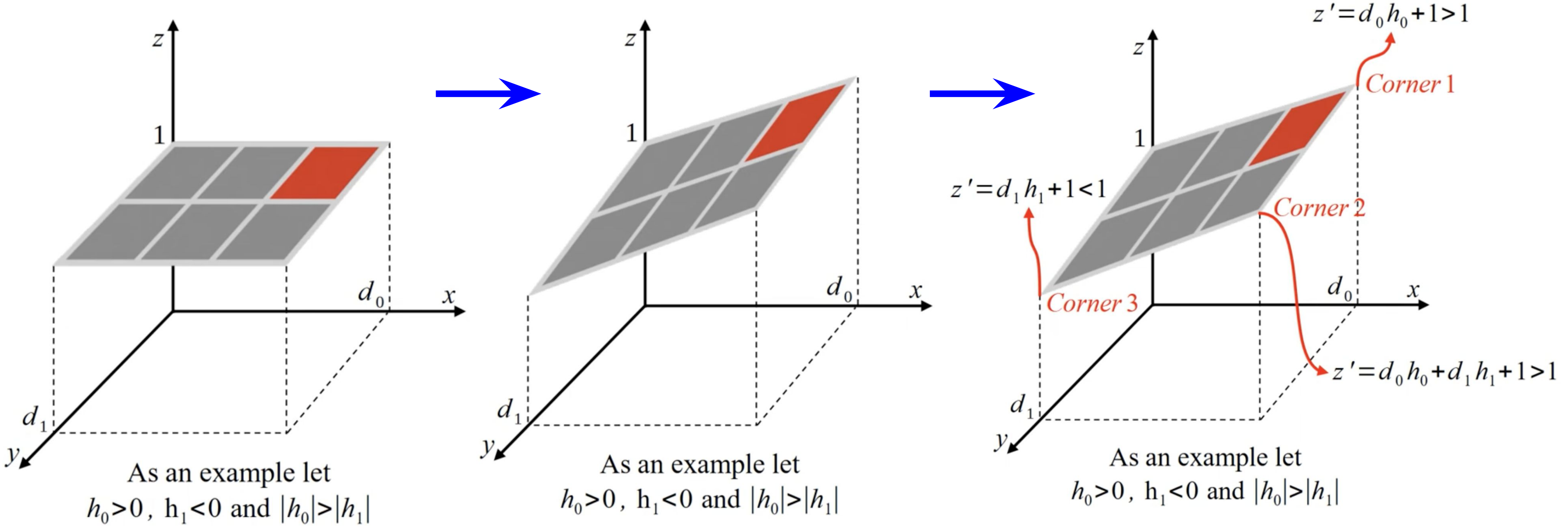


Figure: https://www.youtube.com/@huseyin_ozde...

Projective Transformation (homography)

When going back to Cartesian coordinates

$$\frac{x'}{z'}$$

$$\frac{y'}{z'}$$

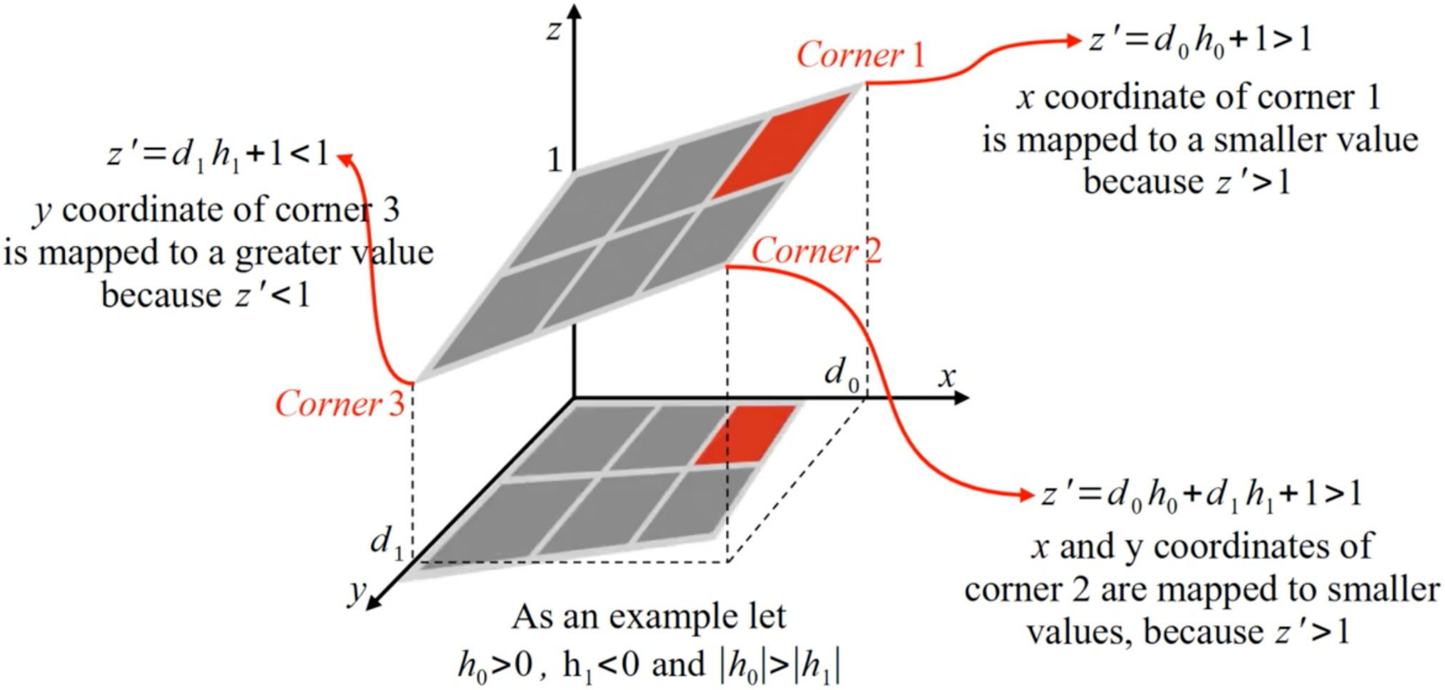
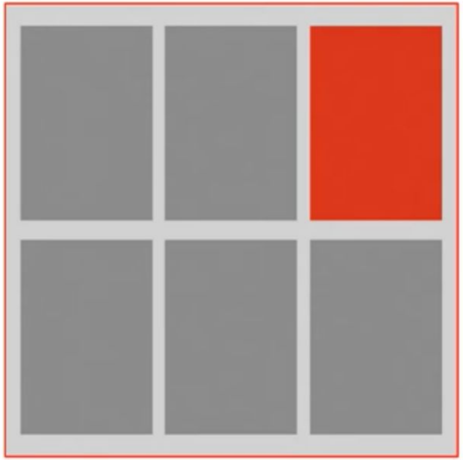


Figure: https://www.youtube.com/@huseyin_ozde...

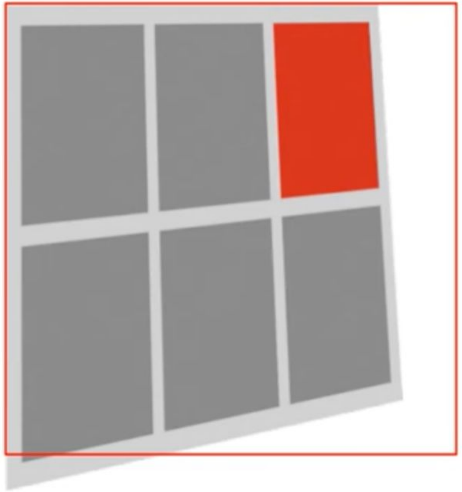
Projective Transformation (homography)



Original Image

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h_0 & h_1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$h_0 > 0$, $\mathrm{h}_1 < 0$ and $|h_0| > |h_1|$



Warped Image

Figure: https://www.youtube.com/@huseyin_ozde...

Projective Transformation (homography)

Projective transformations are combinations of

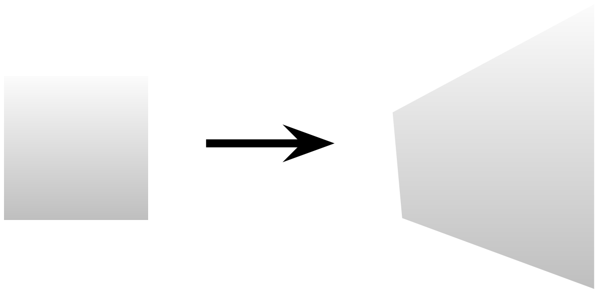
- Affine transformations + projective warps

$$w\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & b_0 \\ A_{10} & A_{11} & b_1 \\ h_0 & h_1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

How many degrees of freedom?

Properties of projective transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines do not necessarily map to parallel lines
- ratios are not necessarily preserved



Source: K. Kitani

Questions?

Composing Transformations

Transformations = Matrices => Composition by Multiplication!

$$p' = R_2 R_1 S p$$

In the example above, the result is equivalent to

$$p' = R_2(R_1(Sp))$$

Equivalent to multiply the matrices into single transformation matrix:

$$p' = (R_2 R_1 S)p$$

Order Matters! Transformations from ***right to left***.

Scaling & Translating != Translating & Scaling

$$p'' = TSp = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$p''' = STp = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

Similarity: Translation + Rotation + Scaling

p'= (T R S) p

$$p' = TRSp = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} RS & t \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is the form of the general-purpose transformation matrix

2D Transforms = Matrix Multiplication

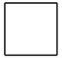
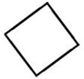
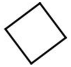
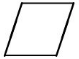
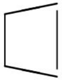
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

Table 2.1 *Hierarchy of 2D coordinate transformations, listing the transformation name, its matrix form, the number of degrees of freedom, what geometric properties it preserves, and a mnemonic icon. Each transformation also preserves the properties listed in the rows below it, i.e., similarity preserves not only angles but also parallelism and straight lines. The 2 $\times$ 3 matrices are extended with a third $[\mathbf{0}^T \; 1]$ row to form a full 3 $\times$ 3 matrix for homogeneous coordinate transformations.*

Figure: R. Szeliski

3D Transforms = Matrix Multiplication

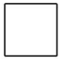
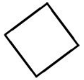
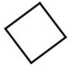
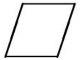
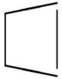
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{3\times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{3\times 4}$	7	angles	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{3\times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{4\times 4}$	15	straight lines	

Table 2.2 *Hierarchy of 3D coordinate transformations. Each transformation also preserves the properties listed in the rows below it, i.e., similarity preserves not only angles but also parallelism and straight lines. The 3 $\times$ 4 matrices are extended with a fourth* $[\mathbf{0}^T \; 1]$ *row to form a full 4 $\times$ 4 matrix for homogeneous coordinate transformations. The mnemonic icons are drawn in 2D but are meant to suggest transformations occurring in a full 3D cube.*

Figure: R. Szeliski

Questions?

3D Rotations: SO(3) representations

Euler Angles: yaw, pitch, roll (α, β, γ)
→ compose $R(\gamma)R(\beta)R(\alpha)$ (order, axes!)

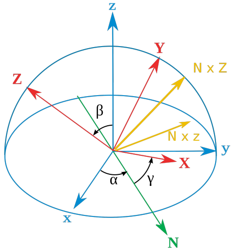
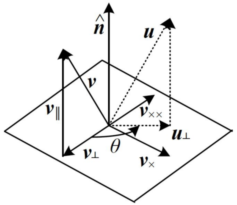


Figure: Wikipedia

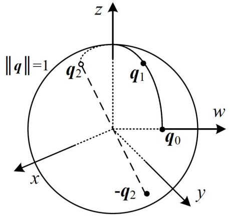
Axis-angle: $(\hat{n}, \theta)$ or $\omega = \theta\hat{n}$
→ matrix via Rodrigues formula (simple for small θ)

$$\mathbf{R}(\hat{\mathbf{n}}, \theta) = \mathbf{I} + \sin\theta[\hat{\mathbf{n}}]_\times + (1 - \cos\theta)[\hat{\mathbf{n}}]^2_\times \approx \mathbf{I} + [\theta\hat{\mathbf{n}}]_\times$$



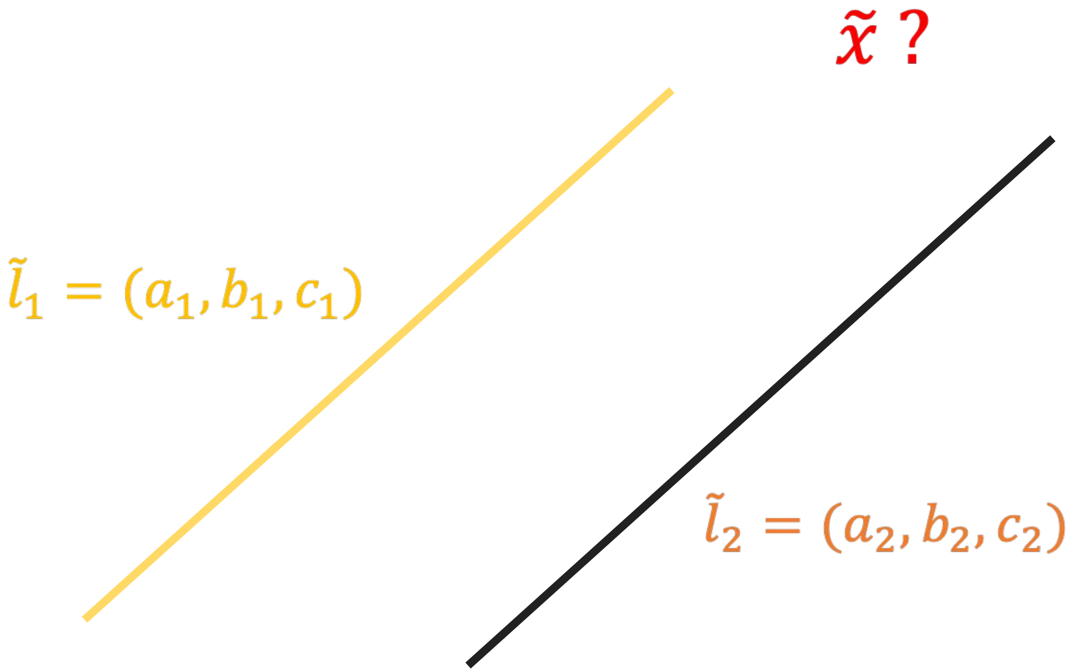
Unit Quaternions: $q = (\overbrace{x, y, z}, w) = (\sin\frac{\theta}{2}\,\hat{n}, \cos\frac{\theta}{2})$, $\|q\| = 1$
→ continuous, nice algebraic properties, matrix via Rodrigues

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - zw) & 2(xz + yw) \\ 2(xy + zw) & 1 - 2(x^2 + z^2) & 2(yz - xw) \\ 2(xz - yw) & 2(yz + xw) & 1 - 2(x^2 + y^2) \end{bmatrix}$$



See Szeliski 2.1.3 for more

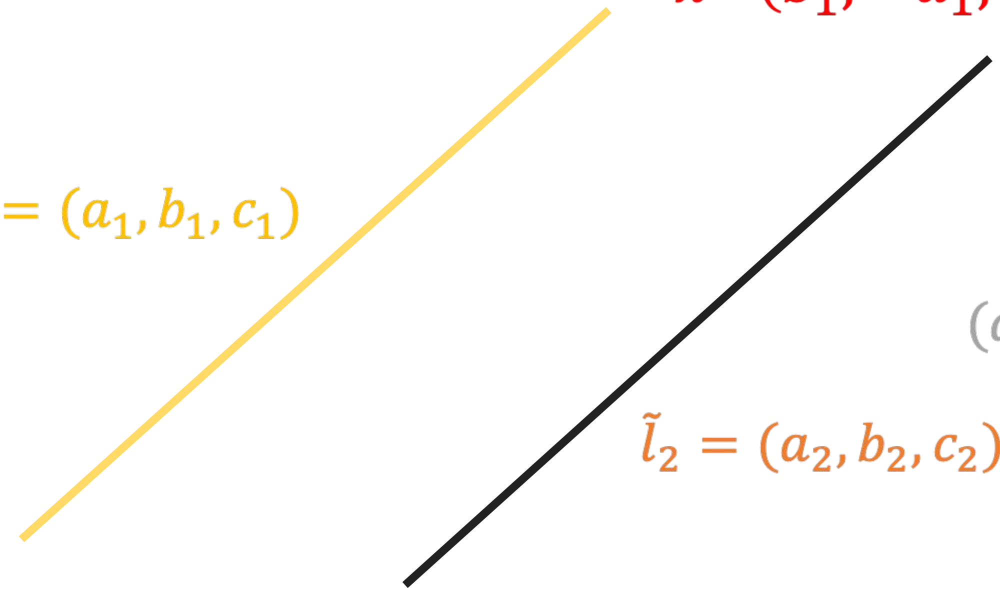
Intersecting Parallel Lines



Intersecting Parallel Lines

$$\tilde{x} = \tilde{l}_1 \times \tilde{l}_2$$

$$\tilde{x} \sim (b_1, -a_1, 0)$$



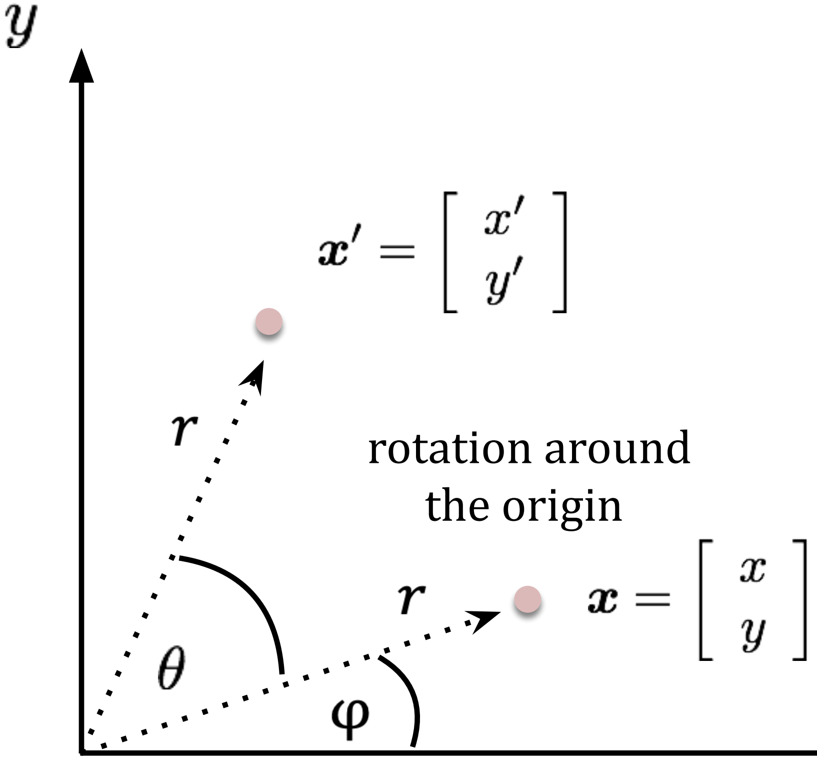
$$\tilde{l}_1 = (a_1, b_1, c_1)$$

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

$$(a_2, b_2) = w(a_1, b_1)$$

$$\tilde{l}_2 = (a_2, b_2, c_2)$$

2D planar transformations



Polar coordinates…

x = r cos (φ)

y = r sin (φ)

x’ = r cos (φ + θ)

y’ = r sin (φ + θ)

Trigonometric Identity…

x’ = r cos(φ) cos(θ) – r sin(φ) sin(θ)

y’ = r sin(φ) cos(θ) + r cos(φ) sin(θ)

Substitute…

x’ = x cos(θ) - y sin(θ)

y’ = x sin(θ) + y cos(θ)