Rocket Trajectory Report - Team 2

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An analysis was performed to model the flight trajectory of a model rocket. The propulsion system and aerodynamics of the rocket were utilized to determine a more accurate trajectory. This model was then tested with a rocket built by the team. On launch day we obtained a 1.2 percent error for the altitude calculations and an 87.6 percent error for the distance calculation.

I. Introduction

Hobbyist model rocketry has existed for many decades. This hobby has inspired many prospective aerospace engineers as well as being an outlet for rocket enthusiasts. For many of the enthusiasts, precise measurements and trajectory models are outweighed by the prospect of launching larger rockets to higher altitudes. This team, also excited at the prospect watching a rocket launch, will also be performed several analyses prior to launch.

For this project, the LOC Precision Graduator rocket will be used as a test bed for the theory learned during the Fall of 2018 within the AAE 439 course offered at Purdue University. A specific model rocket engine has been specified (A G-class engine). With this specification, a model of the propulsion system will be developed to better understand how the rocket is being propelled. In tandem with this analysis an aerodynamic model is also being developed. This model will generate coefficients of lift and drag to be used to further refine the flight path. Finally, a full trajectory analysis using Huen's method will be developed to map out how and where the rocket will fly. These analyses and models are presented herein.

II. Propulsion Model

To do the propulsion analysis an experimental thrust data set was provided. This gave the thrust, as found through a test, as a function of time and a graph of this is as shown in figure 1.

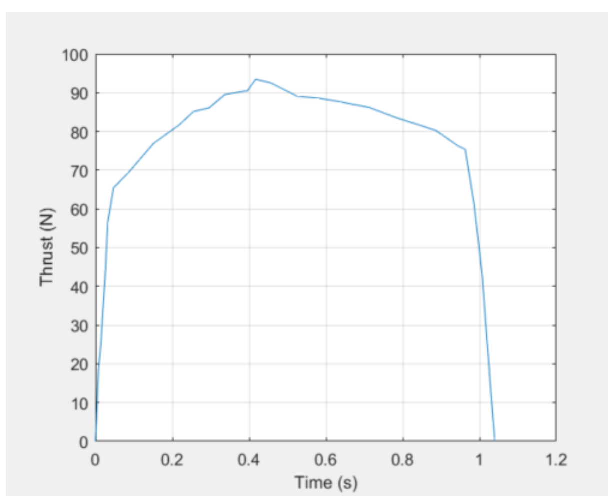

Fig. 1 Thrust of the model rocket engine over time

Using the thrust time history provided, the specific impulse was calculated using the following method. First, the total impulse of the rocket was calculated by integrating the thrust history across the burn time.

$$I = \int_0^{t_b} T dt$$

The specific impulse was then found by dividing the impulse by the total propellant mass.

$$I_{sp} = \frac{I}{m_p}$$

Next, the experimental mass flow that would need to be flowed to provide the time history was calculated. The relationship for propellant mass flow, thrust, and specific impulse was as follows.

$$\dot{m} = \frac{T}{g_e I_{sp}}$$

Using the mass flow, a graph of the mass flow over time could be created. Since the mass flow is directly proportional to the thrust the graph looks similar to the thrust data provided.

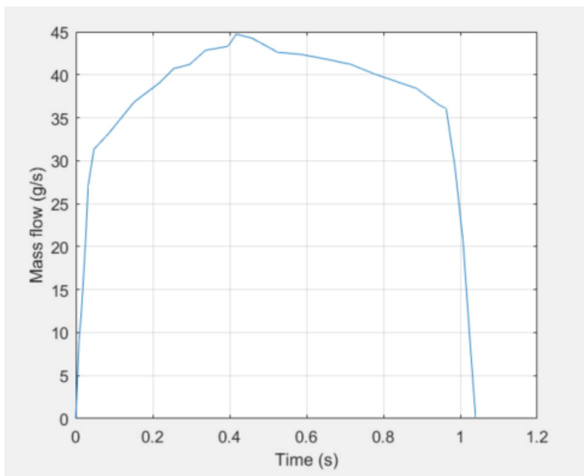

Fig. 2 Mass flow rate of propellant of the model rocket engine

Using the Rocket Motor Components propellant data for White Lightning, the c* value was found to be 1306.21 m/s. The thrust coefficient was found as follows.

$$c_f = \frac{g_e I_{sp}}{c^*}$$

As the diameter of the throat was given, the time history for the chamber pressure in Pa could be found to be given by:

$$P_c = \frac{\dot{m}c*}{A_t}$$

With this calculation a graph of the chamber pressure could be created and again since there is a linear relationship the data looks similar to the thrust data and the calculated mass flow.

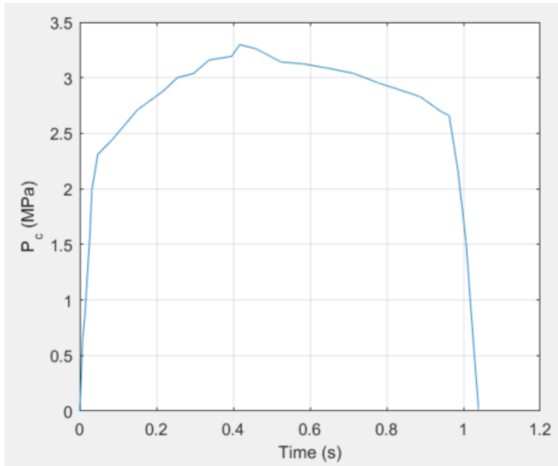

Fig. 3 Chamber pressure of the model rocket engine

The real total web distance could be estimated using the propellant density and the outside dimensions of the casing. Find values for a and n was a matter of integrating the web distance across the burn time and checking that it matched the expected web distance. The integration for web is as follows.

$$w = \int_0^{t_b} a \left(\frac{P_c}{10^6} \right)^n dt$$

The values that made this equation equal to the expected web distance were a = .2175 and n = .36.

Finally, the specific impulse was recalculated with the possibility of throat erosion. A throat erosion rate of .13 mm/s was estimated based on literature. The burn surface area of the propellant was found to be given as a function of web with the following equation, assuming the core and surfacing facing the nozzle were the only exposed surfaces.

$$A_s(w) = 2\pi \left(r_{i,i} + w\right) \left(L_i - w\right) + \pi \left(r_o^2 - \left(r_{i,i} + w\right)^2\right)$$

The instantaneous chamber pressure was then given by:

$$P_c(w) = \left(\frac{a\rho A_s(w) c^*}{10^8 \pi (r_t^2 + r_{\text{erosion}} t_{\text{elapsed}})^2} \right)^{\frac{1}{1-n}}$$

The web burn rate was then found by the following equation.

$$r_b = a \left(\frac{P_c}{10^6} \right)^n$$

The resulting specific impulse was 212.38s which was only marginally smaller than the original specific impulse of 213s. Using burn rate equation and the coefficients that were found through experimental testing a theoretical model could be made. In addition to the burn rate equation this model used the effect of throat erosion to add to its accuracy. It would calculate the pressure coefficient using these values and with that the thrust data could be calculated and compared to the experimental.

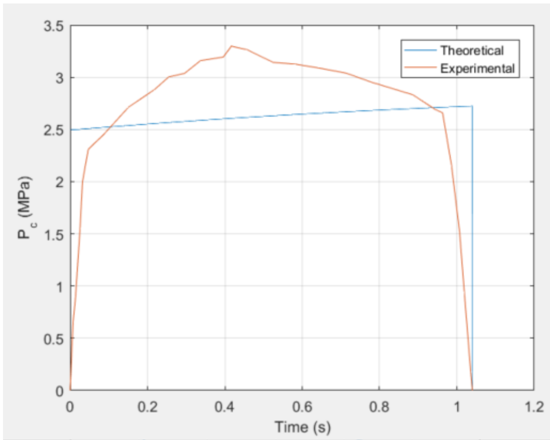

Fig. 4 Chamber Pressure Comparison

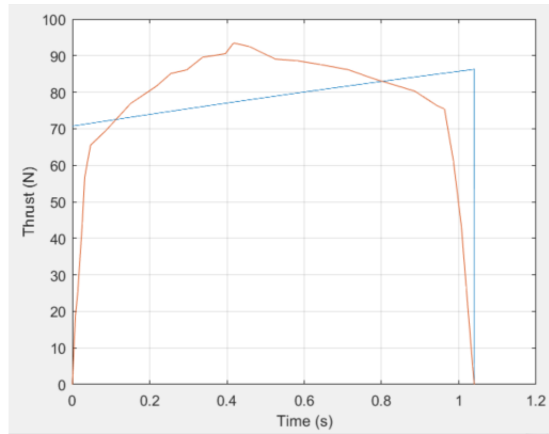

Fig. 5 Thrust Comparison

As seen in the graphs above the theoretical thrust data shows an increasing thrust value over the duration of flight. This is not exactly matched with the experimental data provided as it has a definitive peak that occurs before the end of the engine's primary burn and decreases from there. However the theoretical values can be used as a good substitute for the experimental as they both have a similar total impulse, which can be seen by the area that each graph covers.

III. Aerodynamic Model

The aerodynamic model for our rocket engine was determined using two different methods for the drag and lift respectively. These different methods are based on prior physics knowledge as well as TR-11, a model rocket technical report. [1] The values derived herein were then used in the trajectory analysis to determine height and distance.

A. Drag Model

The coefficient of drag had to be modeled both for the ascent of the vehicle and for the descent. These two stages differ due to the deployment of a parachute once the included charge is set off within the engine. For this reason, the drag analysis will be split into two parts.

1. Coefficient of Drag: Ascent

For the ascent of the model rocket, the process of analysis within TR-11 [1] was followed fairly rigorously. Herein will be included a short summary of the steps within that document.

The coefficient of drag for a rocket at a zero angle of attack can be found by summing up the coefficient of drag for each individual component. These components include: the body tube, the nose cone, the fins, the interference between the fins and the body tube, and the launch lugs. Before this can be completed, a method for determining Reynolds number will need to be created.

There are several regressions and models used to create the functions herein. To start, the atmospheric pressure and density will need to be found as a function of altitude. To this end, NASA's atmospheric model was utilized [2].

$$P = 101.29 * \left(\frac{T + 273.1}{288.08}\right)^{5.256}$$

$$\rho = \frac{P}{0.2869 * T}$$

where P is the pressure in KPa, Temperature in is Kelvin, and density is in kg/m^3. One major assumption made with this model is that the temperature will not vary noticeably between the ground and the altitude of our rocket launch.

Once we determine the density of the air at the launch site temperature, we next will need to determine the dynamic viscosity of the air. To this end, Sutherland's law will be used.[3]

$$\mu = \frac{C_1 T^{3/2}}{T + S}$$

where C_1 for air is the constant 1.458×10^{-6}, T is the temperature in Kelvin, and S is the reference temperature of 110.4 K. This relation was used to determine the dynamic viscosity of the air at a given temperature.

From the dynamic viscosity and the density, the kinematic viscosity can be calculated.

$$\nu = \frac{\mu}{\rho}$$

This is the kinematic viscosity that will be used to determine the Reynolds number for our aerodynamic model. The other two values required are the characteristic length and the velocity. Velocity will be a variable that the trajectory team will use to determine a variable Reynolds number.

Characteristic length will depend upon another simplification. The model rocket will be approximated as a cylinder with a total length of 39 inches (0.9906 meters). Furthermore, the fins will be modeled separately and will have a characteristic length of 6 inches (0.1524 meters). With this simplification, the Reynolds number calculation becomes:

$$Re_L = \frac{L^* \cdot u}{\nu}$$

where u will act as a variable for the trajectory analysis.

With the Reynolds number determined, the next step is to determine the coefficient of friction for the different pieces. The following relations were used:

For laminar flow: Blausius solution:

$$c_f = \frac{0.664}{\sqrt{Re_x}}$$

For turbulant flow: Prandtl's one-seventh power law

$$c_f = \frac{0.027}{Re_x^{1/7}}$$

The fins will be modeled as a laminar flow and the body will be modeled as a turbulent flow. There will be few more parameters that will need to be determined before the final coefficient calculations. These are the cross sectional surface areas and the wetted surface areas. Simplifications have been made for these computations. The ratio of wetted surface area to cross sectional surface area of a cylinder will be approximated as four times the length divided by the diameter.

$$\left[\frac{S_w}{S_{cyl}}\right]_{Bt} = 4\frac{L_{Bt}}{d}$$

For the ogive nose cone, the approximation will be $\frac{8}{6}$ times the length/diameter ratio subtracted by $\frac{2}{15}$.

$$\left[\frac{S_w}{S_{ogive}}\right]_{N} = \frac{8}{6}\frac{L_N}{d} - \frac{2}{15}$$

For the fins, the wetted surface area will be approximated as two times the surface area of each side. This value was calculated to be 35.074 in^2 (0.0226 m^2) per fin. The launch lugs, as well, we calculated to have a cross sectional area of 0.196 in^2 ($1.2645 \cdot 10^{-4}$ m^2) and a wetted surface area of 17.67 in^2 ($1.14 \cdot 10^{-2}$ m^2) .

From this point, the following relations were used to determine the coefficient of drag for each of the different components. All equations are from TR-11 [1].

Overall drag coefficient:

$$C_{D,O} = C_{D,N} + C_{D,Bt} + C_{D,B} + C_{D,0F} + C_{D,int} + C_{D,LL} \tag{1}$$

Body tube and nose cone drag:

$$C_{D,N} + C_{D,Bt} = 1.02 C_f \left[1 + \frac{1.5}{(L/D)^{3/2}}\right] \frac{S_w}{S_{Bt}} \tag{2}$$

Base drag:

$$C_{D,B} = \frac{0.029}{\sqrt{C_{D,N} + C_{D,Bt}}} \tag{3}$$

Characteristic zero degree fin drag:

$$C^*_{D,0F} = 2C_f \left(1 + 2\frac{t}{c}\right) \tag{4}$$

Zero degree fin drag:

$$C_{D,0F} = C^*_{D,0F} \frac{S_f}{S_{Bt}} \tag{5}$$

Interference drag:

$$C_{D,int} = C^*_{D,0F} \frac{\text{Chord}}{S_{Bt}} \frac{d}{2} \times \text{ No. of fins} \tag{6}$$

Launch lug drag:

$$C_{D,LL} = \frac{1.2S_{LL} + 0.045S_{LL,w}}{S_{Bt}} \tag{7}$$

These computations were performed in a code file and used in the trajectory analysis for our ascent. Notice that a simplification of utilizing only a zero degree drag coefficient for the fin drag was used. This approximation was carried through to the trajectory analysis despite a non-zero angle of attack from vertical.

2. Coefficient of Drag: Parachute Guided Descent

Once the rocket was assembled, parachute deployment tests were completed. An indoor drop test was performed from the second floor of Armstrong Hall of Engineering. One person stood on the bridge overlooking the main atrium and a second person stood on the first floor to record measurements, take a video, and make sure the rocket was not in danger of breaking. This person would hold the rocket by the parachute to ensure complete parachute deployment. The bottom half of the rocket, attached to the top of the rocket and the parachute by an elastic cord, was held on the second floor so that the rocket would not fall all the way to the floor. The test was performed with different lengths of cord to ensure the rocket did not hit the floor. The full length of cord was eventually used as the height was sufficient to prevent the rocket's full contact with the ground.

Several drop tests of the rocket were performed. This ensured that the analysis of the drop would be accurate. For example, the first few times, either the parachute did not open, or the rocket hit the wall of the bridge. Once a good method of dropping the rocket was decided upon, a video was taken of the drop in order to measure the time and distance the rocket dropped. From the drop test, a velocity of about 6 feet per second was measured. The area of the parachute was measure to be 380 square inches. Physical data of the rocket was then measured (and received from the TA). The mass was 1.8 lbm, diameter was 2.71 inches, length of the body tube was 30 inches, length of the nose was 9 inches and the surface area of the fin to be 17.537 square inches.

From this data we were able to determine the drag force the parachute enacted upon the bottom of the rocket. This came out to be 0.8164 lbf and from this the calculated coefficient of drag for the parachute was approximately 1.6.

B. Lift Model

The lift model was performed in a much different fashion from the drag model. This is due, in part, to the time constraints of the project. A full viscous boundary layer analysis was unable to be performed within the time-table. Thus, an approximation was determined from observations during the drop test.

To find the lift coefficient from the drop test, we measured the torque exerted to move the rocket to the angle we observed in the video. This angle was calculated using pixel measurements. A circle was centered at the bottom of the rocket with radius being the length of the rocket. Straight lines were then drawn and measured to find relative distances and thus an angle.

From this angle, and the rotational acceleration determined in a similar manner, the next parameter to calculate was the moment of inertia. Here, a previous simplification of treating the body as a cylinder was used. The moment of inertia for a 39 in cylinder was calculated. Thus, two expressions could be used to determine lifting force from torque:

$$\tau = I \cdot \alpha \tag{8}$$

$$\tau = \vec{r} \times \vec{F} \tag{9}$$

From measurements: the distance from the center of lift and the center of gravity was 13.5 inches (0.3429 meters), the angular acceleration of the rocket was 0.882 $\frac{rad}{s^2}$. The moment of inertia of a cylinder of the same length was 0.067399 $kg\ m^2$, and we calculated the torque on the body to be 0.05945 Nm. Therefore, the lifting force would be 0.1733 N. Taking this lift force and utilizing the drag equation [4] we determined the coefficient of lift as a function of velocity.

$$C_l = \frac{2L}{\rho \cdot V^2 \cdot A} \tag{10}$$

Where L is the force of lift, ρ is the density of the air, V is the velocity of the craft, and A is the surface area of the wings. This coefficient was then fed to the trajectory analysis, both for the ascent and the descent of the rocket.

IV. Model of the Trajectory

The code to model the trajectory is condensed into two MATLAB files. All vehicle data and methods are condensed into a class titled ModelRocket.m. This class is referenced in the main simulation code and contains any data or function that relates directly to the vehicle itself (ie. computeThrust). Using a class allows us to keep the simulation code short and clean. The second file, "BallisticSimulation2D.m" contains any variable or function necessary to complete the simulation. The functions are mainly numerical methods. For the initial version of our trajectory analysis, our team settled on using Euler's method to simulate the sample trajectories that would help verify our code. We used Euler's Method to calculate all the 2-dimensional ballistics for our rocket parameters. Euler's Method was utilized to solve the first order, first-degree differential equation given to us that simulates the Force generated by the rocket motor. We tracked the following parameters to extract the trajectory using Euler's Method:

1) Horizontal Position
2) Vertical Position
3) Horizontal Velocity
4) Vertical Velocity

As the local error in the code is proportional to the square of the step size, we selected the smallest possible step size with the data provided to us, which we determined to be 0.1 seconds to minimize this error.

Euler's Method in our code works by iterating a simple equation until the end of the boundary condition given to it. It starts by first adding the parameter's initial value to the product of the change in slope for that parameter with the time step to result in the next iteration of that particular parameter and continues iterating the condition is returned as false.

We decided to keep Euler's Method in our final trajectory code as it served 2 vital purposes, it helped us confirm our values using Huen's Method and it is also the first step in the process of calculating trajectories using Huen's Method.

Heun's method, which improves on the accuracy of our values, is detailed in the pseudocode below.

```
Algorithm: HeunsMethod(dt,t,ax0,ay0,vx0,vy0,x0,y0,m,rho,wv)
    INPUT: dt, t, ax0, ay0, vx0, vy0, x0, y0, m, rho, wv
    OUTPUT: vx, vy, x, y
    //Description of input/output
    dt is the time step.
    t is the current time.
    ax0, ay0, vx0, vy0, x0, y0 are motion values at the beginning of the time step.
    m is the mass of the model rocket.
    rho is the atmospheric density assumed to be constant.
    wv is the input wind attacking the vehicle's front.
    //Calculations
    t1 ← time at the end of the step (future).
    vx1, vy1, x1, y1 ← initial estimates of future values.
    T1 ← future thrust value.
    mFlow1 ← future mass flow rate value.
    m1 ← future mass value.
    ax1, ay1 ← F_net / m1, initial estimates of future acceleration values.
    ax_avg, ay_avg ←  average current and future estimates.
    vx, vy, x, y ← make better estimates of these values.
    return vx, vy, x, y
```

V. Launch Performance

For the day of the launch, the team was tasked with selecting a launch angle to have the rocket return to the launch pad during its descent. In order to calculate the landing location, the code developed by the trajectory team was used. The conditions during the day of the launch had winds from 15-17 mph and gusts of around 23 mph. As seen in figure 6, multiple-launch angles were run to determine the launch angle. The angle of 85 degrees from the horizontal was selected since it returned a distance of 6 m in front of the launch pad and a max altitude of 338 m. In order to achieve the desired angle, the legs of the launch pad were rotated to tilt the entire structure so that the tip of the launch rail was 6.8 ft off the ground.

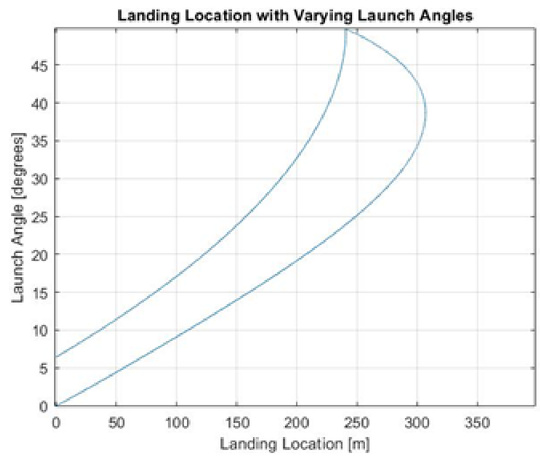

Fig. 6 Landing Distances and Altitudes with Varying Launch Angle

With the launch angle set the rocket was pointed into the direction of the wind. Figure 7 shows the final set up of the launch pad right before launch. The rocket achieved a max altitude of 334 m and landed 161 m from the launch pad. The percent error for the altitude of the rocket is 1.2% and the percent error for the landing location is 96.3%. The model's altitude predicted by the model was close to the actual altitude. This shows that the model is capable of accurately predicting the max altitude the model rocket was able to achieve. The range predicted by the rocket was found to be larger than the predicted value. This discrepancy is due to a multitude of factors. The assumption with the model has the wind to be a constant in terms of speed throughout all its flight. Wind speed picks up as altitude increases due to the lack of friction from the surface of the earth. The higher winds speeds caused the rocket during its descent to obtain a larger horizontal velocity pushing it further from the launch pad.

Fig. 7 Launchpad with Armed Rocket

VI. Conclusions

There were many sources of error that led to the inaccuracies of of this model. Firstly among these was the aerodynamic model. Many, **many**, simplifications and assumptions were applied during the aerodynamic analysis. These have all been labeled within that section of the report and will not be revisited here. However, performing a more thorough boundary layer analysis using viscous flow theory would be able to provide far more accurate results than the values used. One possible avenue for future work would be the development of a viscous model using either Euler's method or Huen's method to map boundary layer development across all surfaces. Such model would greatly improve the accuracy of the calculated drag and lift.

The trajectory model, too, had several sources of error. The model oscillated the values it was returning for our landing location. When using Heun's method, too small of a time step can return incorrect values that cause the oscillating returns. This caused the lower launch angles to return similar distances to the smaller ones. To improve the model in the future a small time step should be used and in addition, a model should be developed to help compensate the increased wind speeds at altitude.

The propulsion model utilized experimental data and as such was not directly impacted by the shortcomings of simplified numerical methods. Rather, errors within the propulsion model can be derived from the lack of data. Solid rocket motors have the inconvenience of being re-testable once fired. Thus, only one set of test data was provided to the propulsion team. Any differences between that set of data, that rocket, and the engine used on the day of launch give room for error in the calculations defined by the team.

Despite the multiple sources of error, and the very windy launch day, the team was able to accomplish an almost exact match for trajectory altitude and measured altitude. With only a 1.2 percent error, the simplifications have still given reasonable results. The 87.6 percent error in the distance of the rocket, however, reveals just how much these simplifications can impact the trajectory of a rocket. The insights garnered by this report, and our errors within, will be taken forward to future projects and, eventually, industry.

References

[1] Gregorek, G. M., "Aerodynamic Drag of Model Rockets," Tech. rep., Estes Industries, 1970. URL https://www.oldrocketplans.com/pubs/Estes/estTR-11/TR-11.pdf.

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[3] Sutherland, W., "The Viscosity of Gases and Molecular Force," *Philosophical Magazine*, 1893, pp. 507–531.

[4] "The Lift Equation," , 2019. URL https://www.grc.nasa.gov/www/k-12/VirtualAero/BottleRocket/airplane/lifteq.html.