

Martin-Löf's type theory

Hugo Herbelin and Nicolas Tabareau

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1 Core type theory

The core infrastructure of type theory is presented on Figure 1.

2 Universes

Extension with a hierarchy of universe is obtained with the rules on Figure 2.

3 Identity type

Extension with an identity type is obtained with the rules on Figure 3.

4 Dependent function type

Extension with a dependent function type is obtained with the rules on Figure 4. One assumes given a function $\Pi(l_1, l_2)$ on universe levels.

5 Dependent sum type

Extension with a dependent sum type is obtained with the rules on Figure 5. One assumes given a function $\Sigma(l_1, l_2)$ on universe levels.

6 Natural numbers

Extension with Peano natural numbers is obtained with the rules on Figure 6. One assumes given a universe level $l_{\mathbb{N}}$ where \mathbb{N} lives.

<i>syntax of contexts</i>		<i>syntax of expressions</i>	
$\Gamma ::= \square \mid \Gamma, a : A$		$t, u, A, B ::= a \mid \mathsf{U}_l$	
where a ranges over a set of variables and l ranges over a set of universe levels			
<i>context formers</i>		<i>axiom</i>	
$\frac{}{\square \text{ wf}} \text{Ctx}_{\square}$	$\frac{\Gamma \vdash A : \mathsf{U}_l}{\Gamma, a : A \text{ wf}} \text{Ctx}_{\text{cons}}$	$\frac{\Gamma, a : A, \Gamma' \text{ wf}}{\Gamma, a : A, \Gamma' \vdash a : A} \text{Ax}$	
<i>conversion rule</i>			
$\frac{\Gamma \vdash t : A \quad \Gamma \vdash A \equiv B : \mathsf{U}_l}{\Gamma \vdash t : B} \text{Conv}$			
<i>definitional equality</i>			
$\frac{\Gamma \vdash t \triangleright u : A}{\Gamma \vdash t \equiv u : A} \text{Conv}_{\triangleright}$			
$\frac{\Gamma \vdash t : A}{\Gamma \vdash t \equiv t : A} \text{Refl}_{\equiv}$	$\frac{\Gamma \vdash t \equiv u : A}{\Gamma \vdash u \equiv t : A} \text{Sym}_{\equiv}$	$\frac{\Gamma \vdash t \equiv u : A \quad \Gamma \vdash u \equiv v : A}{\Gamma \vdash t \equiv v : A} \text{Trans}_{\equiv}$	

Figure 1: Core structure of type theory

<i>type former</i>
$\frac{}{\Gamma \vdash \mathsf{U}_l : \mathsf{U}_{S_l}} \mathsf{U}$

Figure 2: Universes in type theory

extended syntax of expressions

$t, u, v, A, B, p, q ::= \dots \mid t =_A u \mid \text{refl } t \mid \text{subst } p \text{ in } v$

type former

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash t =_A u : \mathbb{U}_l}$$

introduction rule

elimination rule

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \text{refl } t : t =_A t} \quad \frac{\Gamma \vdash p : t =_A u \quad \Gamma, a : A, b : t =_A a \vdash P : \mathbb{U}_l \quad \Gamma \vdash v : P[t/a][\text{refl } t/b]}{\Gamma \vdash \text{subst } p \text{ in } v : P[u/a][p/b]}$$

reduction rule

$$\frac{\Gamma \vdash t : A \quad \Gamma, a : A, b : t =_A a \vdash B : \mathbb{U}_l \quad \Gamma \vdash v : B[t/a][\text{refl } t/b]}{\Gamma \vdash \text{subst refl } t \text{ in } v \triangleright v : B[t/a][\text{refl } t/b]}$$

congruence rules

$$\frac{\Gamma \vdash A \equiv A' \quad \Gamma \vdash t \equiv t' : A \quad \Gamma \vdash u \equiv u' : A}{\Gamma \vdash (t =_A u) \equiv (t' =_{A'} u') : \mathbb{U}_l}$$

$$\frac{\Gamma \vdash t \equiv t' : A}{\Gamma \vdash \text{refl } t \equiv \text{refl } t' : t =_A t} \quad \frac{\Gamma \vdash p \equiv p' : t =_A u \quad \Gamma, a : A, q : t =_A a \vdash B : \mathbb{U}_l \quad \Gamma \vdash v \equiv v' : B[t/a][\text{refl } t/q]}{\Gamma \vdash \text{subst } p \text{ in } v \equiv \text{subst } p' \text{ in } v' : B[u/a][p/q]}$$

Figure 3: Identity type

<i>extended syntax of expressions</i>	
$t, u, v, A, B, p, q ::= \dots \mid \Pi a : A. B \mid \lambda a : A. u \mid v t$	
<i>type former</i>	
$\frac{\Gamma \vdash A : \mathbb{U}_{l_1} \quad \Gamma, a : A \vdash B : \mathbb{U}_{l_2}}{\Gamma \vdash \Pi a : A. B : \mathbb{U}_{\Pi(l_1, l_2)}}$	
<i>introduction rule</i>	<i>elimination rule</i>
$\frac{\Gamma, a : A \vdash u : B}{\Gamma \vdash \lambda a : A. u : \Pi a : A. B}$	$\frac{\Gamma \vdash v : \Pi a : A. B \quad \Gamma \vdash t : A}{\Gamma \vdash v t : B[t/a]}$
<i>reduction rule</i>	<i>observational rule</i>
$\frac{\Gamma, a : A \vdash u : B \quad \Gamma \vdash t : A}{\Gamma \vdash (\lambda a : A. u) t \triangleright u[t/a] : B[t/a]} \beta_{\Pi}$	$\frac{\Gamma \vdash v : \Pi a : A. B}{\Gamma \vdash \lambda a : A. v a \triangleright v : \Pi a : A. B} \eta_{\Pi}$
<i>congruence rules</i>	
$\frac{\Gamma \vdash A \equiv A' : \mathbb{U}_{l_1} \quad \Gamma, a : A \vdash B \equiv B' : \mathbb{U}_{l_2}}{\Gamma \vdash \Pi a : A. B \equiv \Pi a : A'. B' : \mathbb{U}_{\Pi(l_1, l_2)}}$	
$\frac{\Gamma \vdash A \equiv A' : \mathbb{U}_l \quad \Gamma, a : A \vdash u \equiv u' : B}{\Gamma \vdash \lambda a : A. u \equiv \lambda a : A'. u' : \Pi a : A. B}$	
$\frac{\Gamma \vdash v \equiv v' : \Pi a : A. B \quad \Gamma \vdash t \equiv t' : A}{\Gamma \vdash v t \equiv v' t' : B[t/a]}$	

Figure 4: Typing and computational rules for Π

<i>extended syntax of expressions</i>		
$t, u, v, A, B, p, q ::= \dots \mid \Sigma a : A. B \mid \langle t, u \rangle \mid v.1 \mid v.2$		
<i>type former</i>		
$\frac{\Gamma \vdash A : \mathbb{U}_{l_1} \quad \Gamma, a : A \vdash B : \mathbb{U}_{l_2}}{\Gamma \vdash \Sigma a : A. B : \mathbb{U}_{\Sigma(l_1, l_2)}}$		
<i>introduction rule</i>	<i>elimination rules</i>	
$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B[t/a]}{\Gamma \vdash \langle t, u \rangle : \Sigma a : A. B}$	$\frac{\Gamma \vdash v : \Sigma a : A. B}{\Gamma \vdash v.1 : A}$	$\frac{\Gamma \vdash v : \Sigma a : A. B}{\Gamma \vdash v.2 : B[v.1/a]}$
<i>reduction rules</i>		<i>observational rule</i>
$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B[t/a]}{\Gamma \vdash (\langle t, u \rangle).1 \triangleright t : A} \beta_{\Sigma}^1$	$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B[t/a]}{\Gamma \vdash (\langle t, u \rangle).2 \triangleright u : B[t/a]} \beta_{\Sigma}^2$	$\frac{\Gamma \vdash v : \Sigma a : A. B}{\Gamma \vdash \langle v.1, v.2 \rangle \triangleright v : \Sigma a : A. B} \eta_{\Sigma}$
<i>congruence rules</i>		
$\frac{\Gamma \vdash A \equiv A' : \mathbb{U}_{l_1} \quad \Gamma, a : A \vdash B \equiv B' : \mathbb{U}_{l_2}}{\Gamma \vdash \Sigma a : A. B \equiv \Sigma a : A'. B' : \mathbb{U}_{\Sigma(l_1, l_2)}}$		
$\frac{\Gamma \vdash t \equiv t' : A \quad \Gamma, a : A \vdash u \equiv u' : B}{\Gamma \vdash \langle t, u \rangle \equiv \langle t', u' \rangle : \Sigma a : A. B}$		
$\frac{\Gamma \vdash v \equiv v' : \Sigma a : A. B}{\Gamma \vdash v.1 \equiv v'.1 : A}$	$\frac{\Gamma \vdash v \equiv v' : \Sigma a : A. B}{\Gamma \vdash v.2 \equiv v'.2 : B[v.1/a]}$	

Figure 5: Typing and computational rules for Σ

<i>extended syntax of expressions</i>	
$t, u, v, A, B, p, q ::= \dots \mid \mathbb{N} \mid 0 \mid \text{succ } t \mid \text{rec}[0 \mapsto t \mid \text{succ } a \mapsto_b u] v$	
<i>type former</i>	
$\frac{}{\Gamma \vdash \mathbb{N} : \mathbf{U}_{l_{\mathbb{N}}}}$	
<i>introduction rules</i>	
$\frac{}{\Gamma \vdash 0 : \mathbb{N}}$	$\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \text{succ } t : \mathbb{N}}$
<i>elimination rule</i>	
$\frac{\Gamma, a : \mathbb{N} \vdash B : \mathbf{U}_l \quad \Gamma \vdash t : B[0/a] \quad \Gamma, a : \mathbb{N}, b : B[n/a] \vdash u : B[\text{succ } n/a] \quad \Gamma \vdash v : \mathbb{N}}{\Gamma \vdash \text{rec}[0 \mapsto t \mid \text{succ } a \mapsto_b u] v : B[v/a]}$	
<i>reduction rules</i>	
$\frac{\Gamma, a : \mathbb{N} \vdash B : \mathbf{U}_l \quad \Gamma \vdash t : B[0/a] \quad \Gamma, a : \mathbb{N}, b : B[n/a] \vdash u : B[\text{succ } n/a]}{\Gamma \vdash \text{rec}[0 \mapsto t \mid \text{succ } a \mapsto_b u] 0 \triangleright t : B[0/a]} \beta_{\mathbb{N}}^0$	
$\frac{\Gamma, a : \mathbb{N} \vdash B : \mathbf{U}_l \quad \Gamma \vdash t : B[0/a] \quad \Gamma, a : \mathbb{N}, b : B[n/a] \vdash u : B[\text{succ } n/a] \quad \Gamma \vdash v : \mathbb{N}}{\Gamma \vdash \text{rec}[0 \mapsto t \mid \text{succ } a \mapsto_b u] \text{succ } v \triangleright u[v/a][\text{rec}[0 \mapsto t \mid \text{succ } a \mapsto_b u] v/b] : B[\text{succ } v/a]} \beta_{\mathbb{N}}^{\text{succ}}$	
<i>congruence rules</i>	
$\frac{\Gamma \vdash A \equiv A' : \mathbf{U}_{l_1} \quad \Gamma, a : A \vdash B \equiv B' : \mathbf{U}_{l_2}}{\Gamma \vdash \text{succ } t \equiv \text{succ } t' : \mathbb{N}}$	
$\frac{\Gamma \vdash t \equiv t' : B[0/a] \quad \Gamma, a : \mathbb{N}, b : B[n/a] \vdash u \equiv u' : B[\text{succ } n/a] \quad \Gamma \vdash v \equiv v' : \mathbb{N}}{\Gamma \vdash \text{rec}[0 \mapsto t \mid \text{succ } a \mapsto_b u] v \equiv \text{rec}[0 \mapsto t' \mid \text{succ } a \mapsto_b u'] v' : B[v/a]}$	

Figure 6: Typing and computational rules for \mathbb{N}