Martin-Löf's type theory

Hugo Herbelin and Nicolas Tabareau

January 10, 2018

1 Core type theory

The core infrastructure of type theory is presented on Figure 1.

2 Universes

Extension with a hierarchy of universe is obtained with the rules on Figure 2.

3 Identity type

Extension with an identity type is obtained with the rules on Figure 3.

4 Dependent function type

Extension with a dependent function type is obtained with the rules on Figure 4. One assumes given a function $\Pi(l_1, l_2)$ on universe levels.

5 Dependent sum type

Extension with a dependent sum type is obtained with the rules on Figure 5. One assumes given a function $\Sigma(l_1, l_2)$ on universe levels.

6 Natural numbers

Extension with Peano natural numbers is obtained with the rules on Figure 6. One assumes given a universe level $l_{\mathbb{N}}$ where $\mathbb{N}$ lives.

syntax of contexts

$$\Gamma \quad ::= \quad \square \mid \Gamma, a : A$$

syntax of expressions

$$t, u, A, B \quad ::= \quad a \mid \mathsf{U}_l$$

where a ranges over a set of variables
and l ranges over a set of universe levels

context formers

$$\frac{}{\square \ \mathsf{wf}}\ \mathsf{Ctx}_{\square} \qquad \frac{\Gamma \vdash A : \mathsf{U}_l}{\Gamma, a : A \ \mathsf{wf}}\ \mathsf{Ctx}_{\mathsf{cons}}$$

axiom

$$\frac{\Gamma, a : A, \Gamma' \ \mathsf{wf}}{\Gamma, a : A, \Gamma' \vdash a : A}\ \mathsf{Ax}$$

conversion rule

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \equiv B : \mathsf{U}_l}{\Gamma \vdash t : B}\ \mathsf{Conv}$$

definitional equality

$$\frac{\Gamma \vdash t \ \triangleright \ u : A}{\Gamma \vdash t \equiv u : A}\ \mathsf{Conv}_{\triangleright}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t \equiv t : A}\ \mathsf{Refl}_{\equiv} \qquad \frac{\Gamma \vdash t \equiv u : A}{\Gamma \vdash u \equiv t : A}\ \mathsf{Sym}_{\equiv} \qquad \frac{\Gamma \vdash t \equiv u : A \qquad \Gamma \vdash u \equiv v : A}{\Gamma \vdash t \equiv v : A}\ \mathsf{Trans}_{\equiv}$$

Figure 1: Core structure of type theory

type former

$$\frac{}{\Gamma \vdash \mathsf{U}_l : \mathsf{U}_{\mathcal{S}_l}}\ \mathsf{U}$$

Figure 2: Universes in type theory

extended syntax of expressions

$$t, u, v, A, B, p, q \quad ::= \quad \dots \mid t =_A u \mid \mathsf{refl}\, t \mid \mathsf{subst}\, p \,\mathsf{in}\, v$$

type former

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t =_A u : \mathsf{U}_l}$$

introduction rule

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \mathsf{refl}\, t : t =_A t}$$

elimination rule

$$\frac{\Gamma \vdash p : t =_A u \qquad \Gamma, a : A, b : t =_A a \vdash P : \mathsf{U}_l \qquad \Gamma \vdash v : P[t/a][\mathsf{refl}\, t/b]}{\Gamma \vdash \mathsf{subst}\, p \,\mathsf{in}\, v : P[u/a][p/b]}$$

reduction rule

$$\frac{\Gamma \vdash t : A \qquad \Gamma, a : A, b : t =_A a \vdash B : \mathsf{U}_l \qquad \Gamma \vdash v : B[t/a][\mathsf{refl}\, t/b]}{\Gamma \vdash \mathsf{subst}\, \mathsf{refl}\, t \,\mathsf{in}\, v \;\triangleright\; v : B[t/a][\mathsf{refl}\, t/b]}$$

congruence rules

$$\frac{\Gamma \vdash A \equiv A' \qquad \Gamma \vdash t \equiv t' : A \qquad \Gamma \vdash u \equiv u' : A}{\Gamma \vdash (t =_A u) \equiv (t' =_{A'} u') : \mathsf{U}_l}$$

$$\frac{\Gamma \vdash t \equiv t' : A}{\Gamma \vdash \mathsf{refl}\, t \equiv \mathsf{refl}\, t' : t =_A t}$$

$$\frac{\Gamma \vdash p \equiv p' : t =_A u \qquad \Gamma, a : A, q : t =_A a \vdash B : \mathsf{U}_l \qquad \Gamma \vdash v \equiv v' : B[t/a][\mathsf{refl}\, t/q]}{\Gamma \vdash \mathsf{subst}\, p \,\mathsf{in}\, v \equiv \mathsf{subst}\, p' \,\mathsf{in}\, v' : B[u/a][p/q]}$$

Figure 3: Identity type

extended syntax of expressions

$$t, u, v, A, B, p, q \quad ::= \quad \ldots \mid \Pi a\!:\!A.\,B \mid \lambda a\!:\!A.\,u \mid v\,t$$

type former

$$\frac{\Gamma \vdash A : \mathsf{U}_{l_1} \qquad \Gamma, a : A \vdash B : \mathsf{U}_{l_2}}{\Gamma \vdash \Pi a\!:\!A.\,B : \mathsf{U}_{\Pi(l_1,l_2)}}$$

introduction rule

$$\frac{\Gamma, a : A \vdash u : B}{\Gamma \vdash \lambda a\!:\!A.\,u : \Pi a\!:\!A.\,B}$$

elimination rule

$$\frac{\Gamma \vdash v : \Pi a\!:\!A.\,B \qquad \Gamma \vdash t : A}{\Gamma \vdash v\,t : B[t/a]}$$

reduction rule

$$\frac{\Gamma, a : A \vdash u : B \qquad \Gamma \vdash t : A}{\Gamma \vdash (\lambda a\!:\!A.\,u)\,t \triangleright u[t/a] : B[t/a]}\ \beta_\Pi$$

observational rule

$$\frac{\Gamma \vdash v : \Pi a\!:\!A.\,B}{\Gamma \vdash \lambda a\!:\!A.\,v\,a \triangleright v : \Pi a\!:\!A.\,B}\ \eta_\Pi$$

congruence rules

$$\frac{\Gamma \vdash A \equiv A' : \mathsf{U}_{l_1} \qquad \Gamma, a : A \vdash B \equiv B' : \mathsf{U}_{l_2}}{\Gamma \vdash \Pi a\!:\!A.\,B \equiv \Pi a\!:\!A'.\,B' : \mathsf{U}_{\Pi(l_1,l_2)}}$$

$$\frac{\Gamma \vdash A \equiv A' : \mathsf{U}_{l} \qquad \Gamma, a : A \vdash u \equiv u' : B}{\Gamma \vdash \lambda a\!:\!A.\,u \equiv \lambda a\!:\!A'.\,u' : \Pi a\!:\!A.\,B}$$

$$\frac{\Gamma \vdash v \equiv v' : \Pi a\!:\!A.\,B \qquad \Gamma \vdash t \equiv t' : A}{\Gamma \vdash v\,t \equiv v'\,t' : B[t/a]}$$

Figure 4: Typing and computational rules for Π

extended syntax of expressions

$$t, u, v, A, B, p, q \quad ::= \quad \dots \mid \Sigma a\!:\!A.\,B \mid \langle t, u\rangle \mid v.1 \mid v.2$$

type former

$$\frac{\Gamma \vdash A : \mathsf{U}_{l_1} \qquad \Gamma, a : A \vdash B : \mathsf{U}_{l_2}}{\Gamma \vdash \Sigma a\!:\!A.\,B : \mathsf{U}_{\Sigma(l_1,l_2)}}$$

introduction rule

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B[t/a]}{\Gamma \vdash \langle t, u\rangle : \Sigma a\!:\!A.\,B}$$

elimination rules

$$\frac{\Gamma \vdash v : \Sigma a\!:\!A.\,B}{\Gamma \vdash v.1 : A} \qquad \frac{\Gamma \vdash v : \Sigma a\!:\!A.\,B}{\Gamma \vdash v.2 : B[v.1/a]}$$

reduction rules

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B[t/a]}{\Gamma \vdash (\langle t, u\rangle).1 \,\triangleright\, t : A}\,\beta^1_\Sigma \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B[t/a]}{\Gamma \vdash (\langle t, u\rangle).2 \,\triangleright\, u : B[t/a]}\,\beta^2_\Sigma$$

observational rule

$$\frac{\Gamma \vdash v : \Sigma a\!:\!A.\,B}{\Gamma \vdash \langle v.1, v.2\rangle \,\triangleright\, v : \Sigma a\!:\!A.\,B}\,\eta_\Sigma$$

congruence rules

$$\frac{\Gamma \vdash A \equiv A' : \mathsf{U}_{l_1} \qquad \Gamma, a : A \vdash B \equiv B' : \mathsf{U}_{l_2}}{\Gamma \vdash \Sigma a\!:\!A.\,B \equiv \Sigma a\!:\!A'.\,B' : \mathsf{U}_{\Sigma(l_1,l_2)}}$$

$$\frac{\Gamma \vdash t \equiv t' : A \qquad \Gamma, a : A \vdash u \equiv u' : B}{\Gamma \vdash \langle t, u\rangle \equiv \langle t', u'\rangle : \Sigma a\!:\!A.\,B}$$

$$\frac{\Gamma \vdash v \equiv v' : \Sigma a\!:\!A.\,B}{\Gamma \vdash v.1 \equiv v'.1 : A} \qquad \frac{\Gamma \vdash v \equiv v' : \Sigma a\!:\!A.\,B}{\Gamma \vdash v.2 \equiv v'.2 : B[v.1/a]}$$

Figure 5: Typing and computational rules for Σ

extended syntax of expressions

$$t, u, v, A, B, p, q \quad ::= \quad \dots \mid \mathbb{N} \mid 0 \mid \mathsf{succ}\, t \mid \mathsf{rec}\, [0 \,\mapsto\, t \,|\, \mathsf{succ}\, a \,\mapsto_b\, u]\, v$$

type former

$$\frac{}{\Gamma \vdash \mathbb{N} : \mathsf{U}_{l_{\mathbb{N}}}}$$

introduction rules

$$\frac{}{\Gamma \vdash 0 : \mathbb{N}} \qquad \frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}\, t : \mathbb{N}}$$

elimination rule

$$\frac{\Gamma, a : \mathbb{N} \vdash B : \mathsf{U}_l \qquad \Gamma \vdash t : B[0/a] \qquad \Gamma, a : \mathbb{N}, b : B[n/a] \vdash u : B[\mathsf{succ}\, n/a] \qquad \Gamma \vdash v : \mathbb{N}}{\Gamma \vdash \mathsf{rec}\, [0 \,\mapsto\, t \,|\, \mathsf{succ}\, a \,\mapsto_b\, u]\, v : B[v/a]}$$

reduction rules

$$\frac{\Gamma, a : \mathbb{N} \vdash B : \mathsf{U}_l \qquad \Gamma \vdash t : B[0/a] \qquad \Gamma, a : \mathbb{N}, b : B[n/a] \vdash u : B[\mathsf{succ}\, n/a]}{\Gamma \vdash \mathsf{rec}\, [0 \,\mapsto\, t \,|\, \mathsf{succ}\, a \,\mapsto_b\, u]\, 0 \,\rhd\, t : B[0/a]} \; \beta^0_{\mathbb{N}}$$

$$\frac{\Gamma, a : \mathbb{N} \vdash B : \mathsf{U}_l \qquad \Gamma \vdash t : B[0/a] \qquad \Gamma, a : \mathbb{N}, b : B[n/a] \vdash u : B[\mathsf{succ}\, n/a] \qquad \Gamma \vdash v : \mathbb{N}}{\Gamma \vdash \mathsf{rec}\, [0 \,\mapsto\, t \,|\, \mathsf{succ}\, a \,\mapsto_b\, u]\, \mathsf{succ}\, v \,\rhd\, u[v/a][\mathsf{rec}\, [0 \,\mapsto\, t \,|\, \mathsf{succ}\, a \,\mapsto_b\, u]\, v/b] : B[\mathsf{succ}\, v/a]} \; \beta^{\mathsf{succ}}_{\mathbb{N}}$$

observational rule

$$\frac{\Gamma, a : \mathbb{N} \vdash E[a] : A \qquad \Gamma \vdash v : \mathbb{N}}{\Gamma \vdash \mathsf{rec}\, [0 \,\mapsto\, E[0] \,|\, \mathsf{succ}\, a \,\mapsto_b\, E[\succ a]]\, v \,\rhd\, E[v] : A} \; \eta_{\mathbb{N}}$$

where $E[a]$ is made only from elimination rules applied to a

congruence rules

$$\frac{\Gamma \vdash A \equiv A' : \mathsf{U}_{l_1} \qquad \Gamma, a : A \vdash B \equiv B' : \mathsf{U}_{l_2}}{\Gamma \vdash \mathsf{succ}\, t \equiv \mathsf{succ}\, t' : \mathbb{N}}$$

$$\frac{\Gamma \vdash t \equiv t' : B[0/a] \qquad \Gamma, a : \mathbb{N}, b : B[n/a] \vdash u \equiv u' : B[\mathsf{succ}\, n/a] \qquad \Gamma \vdash v \equiv v' : \mathbb{N}}{\Gamma \vdash \mathsf{rec}\, [0 \,\mapsto\, t \,|\, \mathsf{succ}\, a \,\mapsto_b\, u]\, v \equiv \mathsf{rec}\, [0 \,\mapsto\, t' \,|\, \mathsf{succ}\, a \,\mapsto_b\, u']\, v' : B[v/a]}$$

Figure 6: Typing and computational rules for $\mathbb{N}$