Imposing Radiality Constraints in Distribution System Optimization Problems

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***Abstract*—Distribution systems commonly operate with a radial topology, so all models of optimization problems in these distribution systems should consider radiality in their formulation. This work presents a literature review, a critical analysis, and a proposal for incorporating the radiality constraints in mathematical models of optimization problems for radial distribution systems. The objective is to show that the radiality constraints on such optimization problems can be considered in a simple and efficient way. The reconfiguration and expansion planning problems of distribution systems are used to test and verify the proposed radiality constraints. A generalization of radiality constraints is also examined.**

***Index Terms*—Distribution system optimization, distribution system planning, distribution system reconfiguration, mixed integer nonlinear programming, radiality constraint of the electrical distribution system.**

Notation

The notation used throughout this paper is reproduced below for quick reference.

Sets

Ω_l	Sets of branches.
Ω_b	Sets of nodes.
Ω_{b_s}	Sets of bus substation nodes ($\Omega_{b_s} \subset \Omega_b$).
Ω_{b_i}	Sets of connected nodes in the node i ($\Omega_{b_i} \subset \Omega_b$).
Ω_{b_p}	Sets of transfer nodes ($\Omega_{b_p} \subset \Omega_b$).
$\Omega_{b_{dg}}$	Sets of distributed generation nodes ($\Omega_{b_{dg}} \subset \Omega_b$).

Constants

κ_l	Capital recovery rate of circuit constructions.
κ_s	Capital recovery rate of substation reinforcement or construction.
c_{ij}	Construction cost of branch ij (US$/km).
c_{f_i}	Substation fixed cost at node i (US$).
α	Number of hours in one year (8760 h).
τ_l	Interest rate of the cost of power losses.
τ_s	Interest rate of substation operation cost.
ϕ_l	Loss factor of circuits.
ϕ_s	Loss factor of substations.
c_l	Cost per unit of energy (US$/kWh).
c_{v_i}	Substation operation cost at node i (US$/(kVA)^{2}h).
$\underline{V}$	Minimum voltage magnitude.
$\overline{V}$	Maximum voltage magnitude.
n_{ij}^0	Existent circuit in branch ij.
l_{ij}	Circuit length of branch ij.
$\bar{S}_i^0$	Maximum apparent power limit of existent substation at node i.
$\bar{S}_i$	Maximum apparent power limit of substation reinforcement or construction at node i.
$\bar{I}_{ij}$	Maximum current flow limit of branch ij.
n_b	Number of nodes ($n_b = \|\Omega_b\|$).
n_{b_s}	Number of bus substation nodes ($n_{b_s} = \|\Omega_{b_s}\|$).
n_{dg}	Number of distributed generation nodes ($n_{dg} = \|\Omega_{b_{dg}}\|$).
P_{D_i}	Active power demand at node i.
Q_{D_i}	Reactive power demand at node i.
g_{ij}	Conductance of branch ij.
b_{ij}	Susceptance of branch ij.

Functions

P_i	Active power calculated at node i.
Q_i	Reactive power calculated at node i.
P_{ij}	Active power flow that leaves node i toward node j.
Q_{ij}	Reactive power flow that leaves node i toward node j.
$I_{r_{ij}}$	Real current flow component of branch ij.

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$I_{m_{ij}}$	Imaginary current flow component of branch ij.
v	Total power losses.
f	Total investment and operation cost.

Variables

x_{ij}	Circuit that can be reconfigured on branch ij.
n_{ij}	Circuit that can be added on branch ij.
m_i	Substation number that can be added on node i.
V_i	Voltage magnitude at node i.
θ_{ij}	Difference of phase angle between nodes i and j.
P_{S_i}	Active power provided by substation at node i.
Q_{S_i}	Reactive power provided by substation at node i.

I. Introduction

ELECTRICAL distribution systems (EDS) must be adequately planned to permit an efficient and reliable operation. Although they may be found some networked systems in urban cores, the majority of EDS operate with a radial topology for various technical reasons; the two most important follow: 1) to facilitate the coordination and protection; and 2) to reduce the short-circuit current of EDS. Thus, the radiality constraint is present in almost all of the expansion and operation planning problems. The most widely known problems are the distribution system (feeder) reconfiguration problem (DSR) and the distribution system expansion planning problem (DSP).

The distribution system reconfiguration problem can be viewed as an operation planning problem of EDS. The main objective of the DSR problem is to find a radial EDS with minimum losses. This problem can be modeled as a mixed integer nonlinear programming problem (MINLP), in which case working with the radiality of the EDS has always been considered complicated, as shown in [1] and [2].

The distribution systems expansion planning problem is another MINLP problem related to the optimization of EDS. In this problem, given an initial topology of EDS, the aim is to obtain its expansion so that it can operate properly, and at the lowest investment cost, for a horizon planning in which demand is known. Thus, the DSP problem can add new substations and/or repower existing ones or change the conductors of existing circuits and/or build new circuits in branch candidates, taking into account the radiality of EDS. As in the DSR problem, the radiality in the DSP problem has also been considered complicated, as shown in [3].

Both the DSP and DSR problems are well known, and several papers have proposed contributions to modeling and techniques to solve these problems. In the specialized literature, the optimization techniques used to solve these problems can be classified into two main groups: 1) exact optimization techniques and 2) heuristic algorithms and meta-heuristics. The first group, which includes the branch-and-bound algorithm, has been used in conjunction with relaxed models (linearized models) of the DSR and DSP problems to obtain a mixed integer linear programming problem. But when considering the more accurate models (nonlinear models) of the DSR and DSP problems, heuristic algorithms and meta-heuristics have been applied with great success in the last decades.

A. Literature Review

In the specialized literature, the radial operation constraint of an EDS appears mainly in the DSR and DSP problems. It must be pointed out that, if exact optimization techniques are employed, the radiality constraints must be explicitly represented in the mathematical modeling. This is not the case, however, when heuristic or meta-heuristic techniques are used, where the radiality constraints are controlled implicitly.

Several works propose only (1) to represent the radial operation constraint of an EDS in the DSR and DSP problems:

$$M = n_b - 1 \tag{1}$$

where M is the number of branches of the solution obtained in the DSR and DSP problems.

References [1], [4], [5], and [6] present four different heuristic algorithms to solve the DSR problem. The process begins with a completely meshed network, and in each step of the heuristic algorithm, a branch is removed; the process ends when a radial topology is encountered. The difference between each heuristic algorithm is the sensitivity analysis used to decide which branch should be removed/opened at each step. Thus, the radial operation constraint of the EDS is imposed implicitly by the heuristic algorithms, using (1) as stop criteria, and not explicitly in the DSR model. However, [1] acknowledges that (1) is often used to represent the radial operation constraint in optimization problems of EDS, but that this is a necessary but not a sufficient condition to ensure the radiality of an EDS. This study suggests that it "*would be highly desirable if the radiality constraint could be expressed in analytical form.*" If this is possible, the radiality constraints could be incorporated in a mathematical model and solved by using a precise optimization technique.

In [7], an algorithm *branch-exchange* is presented to solve the DSR problem. The algorithm starts with a feasible radial topology and then creates new radial topologies successively by implementing one *branch-exchange* at a time. Thus, the algorithm guarantees the radiality of the EDS.

Meta-heuristics have also been used to solve the DSR problem. The authors of [8] used an evolutionary algorithm; [9] and [10] used two specialized genetic algorithms. All proposals implicitly consider the radial operation constraint of the EDS. In [8], the radial operation constraint of the EDS is assured using graph theory, while that in [9] and [10] is controlled inside genetic operators.

An alternative mathematical model that allows the solution of the DSR problem using commercial software is presented in [2]. In this model, the radiality of an EDS is represented by algebraic relations using so-called *path-based connectivity modeling*. The authors of this study explicitly recognize that it is very difficult to find a mathematical model for the DSR problem and solve it with a conventional technique like the branch-to-node algorithm.

In [11]–[13] and [14], three specialized genetic algorithms and one simulated annealing are proposed to solve the DSP problem. All four proposals implicitly consider the radiality of an EDS through the operator application of each meta-heuristic. In [11], a new codification proposal is presented, using the concept of *prüfer number*; this proposal allows the genetic operators to generate a radial topology.

A mathematical model that considers the distributed generation in the DSP problem is presented in [3]. In this proposal, a constraint equivalent of (1) is shown. The authors state that the radiality cannot be guaranteed and that, "*to ensure that the network obtained is always radial, it may be necessary to add constraints with specific information about the topology of the network under analysis. Considering the diversity of situations encountered, this task can be relatively complex.*"

The authors of [15] propose the planning of the primary and secondary distribution networks in the same optimization problem; the radial operation constraint of the EDS is modeled similarly as in (1). The authors mention that "*the radial characteristic is achieved by allowing a maximum of only one positive power input at each node.*" The mathematical model presented is solved with commercial software. The same authors present in [16] an ant colony algorithm to solve the DSP problem. In this algorithm, the radial structure of the solution is controlled implicitly in the constructive phase of the meta-heuristic.

Two *branch-exchange* algorithms are presented in [17] and [18] to solve the DSP problem. The radial operation constraint of the EDS is obtained implicitly when the optimization process generates an initial radial topology; thus, the other topologies that are found, using the strategy of branch exchanging, should also be radials. In [19] and [20], some topics related to the DSP problem are presented. These authors mention that radiality requires that (1) be satisfied and, additionally, that the EDS must be connected. A mathematical model for the multistage DSP problem is also shown in [21]. In this proposal, a radial operation constraint similar to (1) is presented.

Reference [22] is one of the first papers to propose a mathematical model for the DSP problem. It recognizes the need to find optimal radial topologies, but the radiality constraints is not shown in the model. Finally, [23] presents a literature review on the DSP problem. This paper recognizes the importance of the radiality constraints but provides no detailed analysis of the subject.

B. Contributions

From the literature review above, it seems clear that the explicit representation of the radiality constraints is an issue that has not yet been appropriately solved. If this representation is possible, then the DSR and DSP problems can be solved by using integer programming techniques. The main aim of this paper is to further contribute to this subject.

The main contributions of this paper are to provide the following:

1) the sufficient conditions to ensure a radial topology in EDS optimization problems;
2) a preliminary analysis of the generalization of radiality constraints;
3) a mathematical model for the DSR and DSP problems in which the radiality constraints are represented explicitly and solved with an integer programming technique.

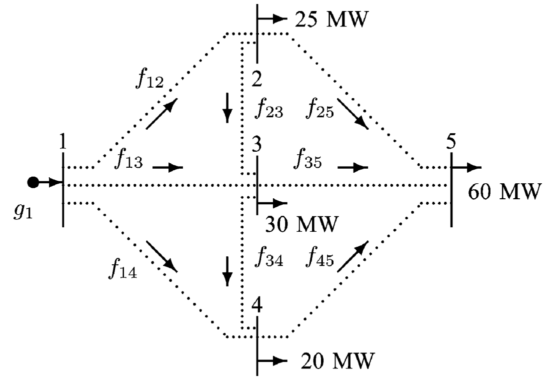

Fig. 1. Illustrative problem.

II. Radiality Constraints in the Electrical Distribution System

The EDS topology can be considered a graph consisting of n arcs and m nodes. From graph theory, it is known that a tree is a connected graph without loops; thus, it is possible to compare the radial topology of an EDS with a tree. As is shown in the Appendix and [24], the tree of a graph is a subgraph connected with $(m-1)$ arcs. Hence, one can state that the topology of an EDS with n_b nodes is radial if it satisfies the two following conditions:

- **Condition 1**—the solution must have $(n_b - 1)$ circuits;
- **Condition 2**—the solution must be connected.

Note that the radiality constraints have to be formed by **Condition 1** and **Condition 2**, and that **Condition 1** alone does not ensure the radiality of an EDS.

A. Illustrative Problem

In an EDS system of five nodes, with four load nodes and only one substation node, the aim is to minimize the sum of the squared active power flow and ensure that the final topology is radial (see Fig. 1). In this problem, only the First Kirchhoff Law (FKL) is considered; therefore, there is no need to specify branch reactances and resistances. The mathematical problem assumes the following form:

$$\min \quad v = \sum_{(ij)\in\Omega_l} f_{ij}^2 \tag{2}$$

s.t.

$$\sum_{(ji)\in\Omega_l} f_{ji} - \sum_{(ij)\in\Omega_l} f_{ij} + g_i = d_i \quad \forall i \in \Omega_b \tag{3}$$

$$|f_{ij}| \le \bar{f}_{ij} x_{ij} \quad \forall (ij) \in \Omega_l \tag{4}$$

$$0 \le g_i \le \bar{g}_i \quad \forall i \in \Omega_{b_s} \tag{5}$$

$$x_{ij} \in \{0,1\} \quad \forall (ij) \in \Omega_l \tag{6}$$

$$\sum_{(ij)\in\Omega_l} x_{ij} = n_b - 1 \tag{7}$$

where x_{ij} is a binary variable that is equal to 1 if the corresponding circuit is in operation; otherwise, it is equal to 0. The active power flow between nodes i and j is denoted by f_{ij}. The variable g_i is the active power supplied by the substation i. The parameter d_i represents the active power demand at node i. The variable $\bar{f}_{ij}$ stands for the maximum active power flow limit of branch ij. The parameter $\bar{g}_i$ denotes the maximum active power limit of the substation i. The mathematical model of this illustrative example was written in the mathematical programming language AMPL [25] and solved with CPLEX 12.1 [26] (called with default options). Considering $\bar{g}_1 = 150$ MW and $\bar{f}_{ij} = 100$ MW for all branches, the following solution was obtained:

$$\begin{aligned} v &= 11525 \text{ MW}^2,\ g_1 = 135 \text{ MW} \\ x_{12} &= x_{13} = x_{14} = x_{45} = 1 \\ x_{23} &= x_{25} = x_{34} = x_{35} = 0 \\ f_{12} &= 25 \text{ MW}, f_{13} = 30 \text{ MW} \\ f_{14} &= 80 \text{ MW}, f_{45} = 60 \text{ MW} \\ f_{23} &= f_{25} = f_{34} = f_{35} = 0 \text{ MW}. \end{aligned}$$

Note that the solution found is radial and is verifiably the optimal solution of the illustrative problem.

B. *Proof of the Radiality Condition*

In this subsection, we will prove that the feasible solution of the illustrative problem satisfies both **Condition 1** and **Condition 2**; namely, a feasible solution must be radial. The main features of this illustrative problem are:

1) only a single substation exists in the EDS (substation node);
2) all other nodes are load nodes;
3) Kirchhoff's first law must be followed; and
4) the aim is to find the best radial topology.

Proof: **Condition 1** is satisfied by (7). A solution that satisfies (3) must supply the power demand at every load bus, so that a path between the substation and each other bus exists. Therefore, every bus is linked with the substation bus, forming a connected graph, which proves **Condition 2**.

Thus, when power balance constraints (3) are combined with constraint (7), each load node is connected by a single path to the substation node; that is, the system is connected without meshes.

In the proof only features 1)–4) were used, of which hypothesis 3) is the one that guarantees that the demand in each node is fed. The proof is valid for any other optimization problem of an EDS, provided the four hypotheses are considered.

C. *Generalization of Radiality Constraint*

Three cases are presented in this section: 1) an EDS with more than one substation, 2) an EDS with distributed generation and/or reactive power sources, and 3) an EDS with zero power injection nodes (so-called transfer nodes). Note that each case is presented individually in this subsection; however, the three cases can be considered together to form a general model.

1) More Than One Substation: This case appears frequently in the DSP problem. Thus, **Condition 1** or constraint (1) must be modified to

$$M = n_b - n_{b_s} \tag{8}$$

where n_{b_s} is the total number of substation nodes in the EDS. Constraint (8) plus the power balance constraints guarantee that $n_b - n_{b_s}$ circuits will be constructed to feed the $n_b - n_{b_s}$ load nodes and n_{b_s} radial topologies will be constructed in the final solution.

Proof: It must be proved that there are n_{b_s} trees (each tree with one and only one substation) if and only if both $\sum_{ij\in\Omega_l} x_{ij} = n_b - n_{b_s}$ and FKL are satisfied.

First we will prove that n_{b_s} trees implies that both $\sum_{ij\in\Omega_l} x_{ij} = n_b - n_{b_s}$ and FKL are satisfied. Let k_i be the number of nodes of the ith subgraph. The number of branches of the ith tree is $k_i - 1$, according to (1). Therefore, the number of branches of the n_{b_s} trees is $\sum_{i=1}^{n_{b_s}}(k_i - 1)$, which is equal to $n_b - n_{b_s}$. Moreover, as each tree has a substation, FKL is satisfied.

Now, it must be proved that if both $\sum_{ij\in\Omega_l} x_{ij} = n_b - n_{b_s}$ and FKL are satisfied, it implies that there are n_{b_s} trees. Let S be the number of subgraphs. It must be proved that $S = n_{b_s}$ and that all the subgraphs are trees, each tree with one and only one substation.

- Suppose that $S = n_{b_s} + \beta > n_{b_s}$, with $\beta > 0$; so there is at least one subgraph without substation and its load nodes cannot be supplied, which contradicts that FKL is satisfied. As the supposition is not true, therefore $S \le n_{b_s}$.
- Suppose that $S = n_{b_s} - \beta < n_{b_s}$, with $\beta > 0$. As the minimum number of arcs of the ith graph is $k_i - 1$, the total number of arcs is $\sum_{ij\in\Omega_l} x_{ij} \ge \sum_{i=1}^{n_{b_s}-\beta}(k_i - 1) = n_b - n_{b_s} + \beta > n_b - n_{b_s}$, which contradicts $\sum_{ij\in\Omega_l} x_{ij} = n_b - n_{b_s}$. As the supposition is not true, therefore $S \ge n_{b_s}$.
- As $S \le n_{b_s}$ and $S \ge n_{b_s}$, then $S = n_{b_s}$, which means that there are n_{b_s} subgraphs.
- Let C_i be the minimum number of arcs to take off from the ith subgraph to obtain a tree. The total number of arcs is $\sum_{ij\in\Omega_l} x_{ij} = \sum_{i=1}^{n_{b_s}}(k_i - 1 + C_i) = n_b - n_{b_s} + \sum_{i=1}^{n_{b_s}} C_i$. Thus, to obtain $\sum_{ij\in\Omega_l} x_{ij} = n_b - n_{b_s}$, it is necessary that $\sum_{i=1}^{n_{b_s}} C_i = 0$, which implies that $C_i = 0$. Then all n_{b_s} subgraphs are trees.
- Since FKL must be satisfied, each tree has at least one substation. As there are n_{b_s} trees and substations, there is one and only one substation for each tree.

2) Distributed Generation and/or Reactive Power Sources: This case also appears frequently in the DSP problem. Note that hypotheses 1)–4) are still valid since reactive power sources are included in the load nodes. As the distributed generators can feed some loads independently, a part of the EDS can operate isolated. In this case, **Condition 2** is not guaranteed by the power balance constraints (3), and new constraints should be added to ensure that a distributed generator is not isolated from the substation, as shown in (9)–(12):

$$\sum_{(ji)\in\Omega_l} k_{ji} - \sum_{(ij)\in\Omega_l} k_{ij} = K_i \quad \forall i \in \Omega_b \tag{9}$$

$$K_i = 1 \quad \forall i \in \Omega_{dg} \tag{10}$$

$$K_i = 0 \quad \forall i \notin \Omega_{dg} \cup \Omega_{b_s} \tag{11}$$

$$|k_{ij}| \le n_{dg} x_{ij} \quad \forall (ij) \in \Omega_l \tag{12}$$

where K_i represented a fictitious load of each distributed generator that only can be fed by the substation. Variable k_{ij} denotes the fictitious flow associated with branch ij. If it is allowed that the distributed generator feeds some loads independently, then (9)–(12) are not considered in the model.

3) Transfer Nodes: A transfer node is simply one node with no generation or demand. Transfer nodes are not so frequent in an EDS; they are used to connect a load node to other load nodes. A transfer node is not a terminal node (this is the main condition about using the transfer nodes); thus, there are at least two more circuits "leaving" the transfer node.

To model the use of a transfer node, one must define the binary variable y_i such that y_i is equal to 1 if the transfer node is used; otherwise, y_i is equal to 0. To consider transfer nodes in the EDS, (8) is replaced by (13):

$$\sum_{(ij)\in\Omega_l} x_{ij} = n_b - n_{b_s} - \sum_{j\in\Omega_{b_p}} (1-y_j) \quad (13)$$
$$x_{ij} \le y_j \quad \forall (ij)\in\Omega_l, \forall j\in\Omega_{b_p} \quad (14)$$
$$x_{ji} \le y_j \quad \forall (ji)\in\Omega_l, \forall j\in\Omega_{b_p} \quad (15)$$
$$\sum_{(ij)\in\Omega_l} x_{ij} + \sum_{(ji)\in\Omega_l} x_{ji} \ge 2y_j \quad \forall j\in\Omega_{b_p} \quad (16)$$
$$y_j \in \{0,1\} \quad \forall j\in\Omega_{b_p}. \quad (17)$$

Constraints (13)–(17) avoid loop generation due to the presence of transfer nodes in the EDS and also prevent the appearance of a terminal transfer node (with only one connected circuit). On the other hand, if it is known that all transfer nodes are part of the radial topology of the EDS, a simple method is to assume a small value of load (for example, 0.001 pu) in all transfer nodes to ensure that all nodes are connected.

Proof: Consider a transfer node at bus j. Two cases must be considered.

Case 1) The transfer node is not used. In this case, using the main condition of a transfer node, no branches connect to bus j, and we have

$$x_{ij} = 0, \forall (ij)\in\Omega_l, \forall j\in\Omega_{b_p}$$
$$x_{ji} = 0, \forall (ji)\in\Omega_l, \forall j\in\Omega_{b_p}.$$

As (16) guarantees that $y_j = 0$, that implies that bus j is disconnected. Then the number of nodes that remains connected must be $n_b - 1$, requiring $n_b - n_{b_s} - 1$ branches, which agrees with (13).

Case 2) The transfer node is used. First, considering that only one branch connect to bus j, then $y_j = 1$ according to (14) and (15), which means that the right member of (16) equals 2; however, the left member of (16) is one, which turns this possibility unfeasible. Now, assuming that at least two branches connect to bus j, (14) and (15) force $y_j = 1$, so (13) turns into (8), satisfying the radiality conditions.

III. Optimization Problem of Radial Distribution Systems

In this section, the DSR and DSP problems are modeled as mixed integer nonlinear programming problems using the radiality constraints described in Section II.

A. Distribution System Reconfiguration Problem

The DSR problem is modeled as follows:

$$\min \quad v = \sum_{(ij)\in\Omega_l} \left(g_{ij}x_{ij}(V_i^2 + V_j^2 - 2V_iV_j\cos\theta_{ij})\right) \quad (18)$$

s.t.

$$P_{S_i} - P_{D_i} - \sum_{j\in\Omega_{b_i}} (x_{ij}P_{ij}) = 0 \quad \forall i\in\Omega_b \quad (19)$$
$$Q_{S_i} - Q_{D_i} - \sum_{j\in\Omega_{b_i}} (x_{ij}Q_{ij}) = 0 \quad \forall i\in\Omega_b \quad (20)$$
$$\underline{V} \le V_i \le \overline{V} \quad \forall i\in\Omega_b \quad (21)$$
$$x_{ij}(I_{r_{ij}}^2 + I_{m_{ij}}^2) \le \overline{I}_{ij}^2 \quad \forall (ij)\in\Omega_l \quad (22)$$
$$x_{ij}\in\{0,1\} \quad \forall (ij)\in\Omega_l \quad (23)$$
$$\text{Eqs. } (14) - (17)$$
$$\sum_{(ij)\in\Omega_l} x_{ij} = n_b - 1 - \sum_{j\in\Omega_{b_p}} (1-y_j). \quad (24)$$

The objective function (18) represents the power losses of the distribution system operation. Equations (19) and (20) represent the conventional equations of load balance; the elements of P_{ij} and Q_{ij} are given by (25) and (26), respectively:

$$P_{ij} = V_i^2 g_{ij} - V_iV_j(g_{ij}\cos\theta_{ij} + b_{ij}\sin\theta_{ij}) \quad (25)$$
$$Q_{ij} = -V_i^2 b_{ij} - V_iV_j(g_{ij}\sin\theta_{ij} - b_{ij}\cos\theta_{ij}). \quad (26)$$

Equation (21) represents the constraints on the voltage magnitude of nodes. The elements of current flow in branch ij of (22) are given by (27) and (28):

$$I_{r_{ij}} = g_{ij}(V_i\cos\theta_i - V_j\cos\theta_j) - b_{ij}(V_i\sin\theta_i - V_j\sin\theta_j) \quad (27)$$
$$I_{m_{ij}} = g_{ij}(V_i\sin\theta_i - V_j\sin\theta_j) + b_{ij}(V_i\cos\theta_i - V_j\cos\theta_j). \quad (28)$$

Equation (23) represents the binary nature of x_{ij}. The circuit between buses $i-j$ is connected if the corresponding value is equal to one and is not connected if it is equal to zero. In the proposed model of the DSR problem, transfer nodes and a single substation are considered. Thus, P_{S_i} and Q_{S_i} have nonzero values only at the substation node. Note that the existence of this unique substation node is represented by (24). The presence of transfer nodes in the DSR problem is modeled by constraints (14)–(17) and (24), as shown in Section II-C.

B. Distribution System Planning Problem

The DSP problem is modeled as follows:

$$\begin{aligned}\min \quad f = {} & \kappa_l \sum_{(ij)\in\Omega_l} (c_{ij}\, n_{ij}\, l_{ij}) + \kappa_s \sum_{i\in\Omega_{b_s}} (c_{f_i}\, m_i) \\ & + \delta_l \sum_{(ij)\in\Omega_l} (g_{ij}(n^0_{ij}+n_{ij})(V_i^2+V_j^2-2V_iV_j\cos\theta_{ij})) \\ & + \delta_s \sum_{i\in\Omega_{b_s}} (c_{v_i}(P^2_{S_i}+Q^2_{S_i})) \end{aligned} \tag{29}$$

s.t.

$$P_{S_i} - P_{D_i} - \sum_{j\in\Omega_b} ((n^0_{ij}+n_{ij})P_{ij}) = 0 \; \forall i \in \Omega_b \tag{30}$$

$$Q_{S_i} - Q_{D_i} - \sum_{j\in\Omega_b} ((n^0_{ij}+n_{ij})Q_{ij}) = 0 \; \forall i \in \Omega_b \tag{31}$$

$$\underline{V} \le V_i \le \overline{V} \quad \forall\, i \in \Omega_b \tag{32}$$

$$P^2_{S_i} + Q^2_{S_i} \le (\bar{S}^0_i + m_i \bar{S}_i)^2 \quad \forall i \in \Omega_{b_s} \tag{33}$$

$$(n^0_{ij}+n_{ij})(I^2_{r_{ij}} + I^2_{m_{ij}}) \le \bar{I}^2_{ij} \quad \forall (ij) \in \Omega_l \tag{34}$$

$$n^0_{ij} + n_{ij} \le 1 \quad \forall (ij) \in \Omega_l \tag{35}$$

$$n_{ij} \in \{0,1\} \quad \forall (ij) \in \Omega_l \tag{36}$$

$$m_i \in \{0,1\} \quad \forall i \in \Omega_{b_s} \tag{37}$$

$$\text{Eqs. } (9) - (11)$$

$$|k_{ij}| \le n_{dg} n_{ij} \quad \forall (ij) \in \Omega_l \tag{38}$$

$$\sum_{(ij)\in\Omega_l} (n^0_{ij}+n_{ij}) = n_b - n_{b_s} \tag{39}$$

where $\delta_l = \alpha\tau_l\phi_l c_l$ and $\delta_s = \alpha\tau_s\phi_s$. The objective function (29) is the annualized investment and operation cost based on [27] and [28]. The first part represents the investment cost (construction of circuits and construction/reinforcement of substations); the second and third parts represent the annual cost of power losses and substation operation, respectively. Equations (30) and (31) represent the conventional equations of load balance, and the elements of P_{ij} and Q_{ij} are given by (25) and (26), respectively. Equation (32) represents the constraints on voltage magnitude of nodes, while (33) represents the maximum capacity of substation i. Note that, in (33), both the reinforcement of the existing substation ($\bar{S}^0_i \neq 0$) and the construction of a new substation ($\bar{S}^0_i = 0$) are modeled. The elements of current flow in branch ij of (34) are given by (27) and (28). Equation (35) ensures that duplication of circuits (existing and proposed) is not allowed. Equations (36) and (37) represent the binary nature of the circuits and substations that can be added to the distribution system, respectively. The element (circuit or substation) is constructed in the EDS if the corresponding value is equal to one and is not constructed if it is equal to zero.

The presence of distributed generation in the DSP problem is modeled by constraints (9)–(11) and (38), as shown in Section II-C. In the presented DSP model, consider all load nodes to have nonzero power injection (no transfer nodes present) and n_{b_s} substations. This condition is represented by (39). Note that the existent circuits plus the added circuits must be equal to $n_b - n_{b_s}$.

TABLE I
RESULTS SUMMARY OF THE DSR PROBLEM

Systems	33-nodes	84-nodes	119-nodes	136-nodes	417-nodes
Losses (kW)	139.55	469.88	853.58	280.19	685.88
Times (s)	19	3030	4007	4473	14256
Binary variables	37	96	133	156	474

C. Comments

The binary variables n_{ij} and m_i, of the DSP problem, and x_{ij}, of the DSR problem, are decision variables, and a feasible operation solution for the distribution system depends on their values. The remaining variables represent the operating state of a feasible solution. For a feasible investment proposal, defined through specified values of n_{ij}, m_i, and x_{ij}, several feasible operation states are possible.

The fact that the solutions of DSP and DSR problems are radial solutions is a natural consequence of the proof presented in Section II. Note that the power balance constraints (19), (20), (30), and (31) guarantee that the solution is a connected one and that (1), (24), and (39) are equivalent constraints.

The presence of transfer nodes, n_{b_s} substations, distributed generation, and/or reactive power sources can be included in the DSP and DSR models shown above; thus, a more general formulation of each optimization problem is obtained.

IV. Tests and Results

The objective of this section is to present an analysis of the radiality constraints shown in Section II. Thus, to exemplify the two optimization models previously presented, seven test systems were used. For the DSR problem tests, the systems of 33, 84, 119, 136, and 417 nodes were used, while the 23- and 54-node systems were used for the DSP problem tests. The DSR and DSP problems were solved using a nonlinear B&B algorithm based on [29] and coded in AMPL language, while the nonlinear programming (NLP) problems were solved through the commercial solver KNITRO [30]. The numerical results have been obtained using a PC Intel Core 2 Duo 6700, 2 GB RAM.

A. Distribution System Reconfiguration

The test systems data can be found in [7], [10], [27], [31], and [32]. The 33-, 119-, and 417-node systems are 12.66-kV, 11-kV, and 10-kV systems, respectively, without transfer nodes, while the 84- and 136-node systems are 11.4-kV and 13.8-kV systems with 17 and 28 transfer nodes, respectively. The data of the real 417-node system were adapted to be used in the DSR problem. Table I presents a summary of the obtained results as well as the total CPU time used by the nonlinear B&B algorithm and the number of binary variables for each test system. For the 119-node system, the B&B algorithm obtained a solution better than the one presented in [31], while the results obtained for the 33-, 84-, and 136-node systems are equal to those presented in [10]. Table II shows the open circuits in all systems. In the 84- and 136-node systems, all transfer nodes are used. The radial final topology of the 33-node system is shown in Fig. 2 and

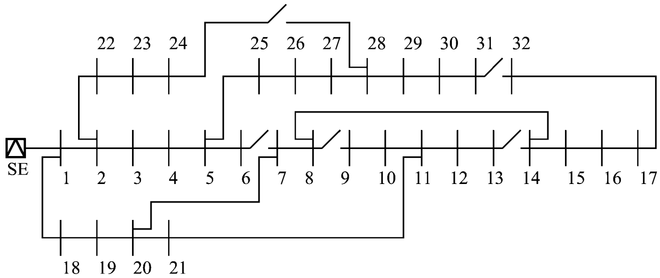

Fig. 2. Distribution system reconfiguration for the original 33-node system (without transfer nodes).

TABLE II
OPEN CIRCUITS OF THE DSR PROBLEM

Systems	
33-nodes	$6-7, 8-9, 13-14, 24-28, 31-32.$
84-nodes	$6-7, 11-43, 12-13, 14-18, 16-26, 28-32, 33-34, 38-39, 61-62, 71-72.$
119-nodes	$23-24, 25-26, 35-36, 41-42, 44-45, 52-53, 61-62, 74-75, 77-78, 95-100, 101-102, 114-115, 56-45, 113-86, 110-89.$
136-nodes	$6-7, 9-24, 15-83, 31-35, 48-51, 50-96, 55-98, 66-79, 79-131, 84-135, 89-90, 90-129, 91-104, 92-104, 92-132, 95-96, 105-106, 108-114, 125-126, 128-77, 134-135.$
417-nodes	$2-6, 2-9, 19-32, 20-71, 45-49, 59-57, 74-80, 83-98, 48-38, 66-67, 24-34, 57-19, 23-28, 39-46, 76-77, 48-44, 67-383, 93-112, 56-146, 65-251, 147-56, 220-219, 274-208, 262-263, 382-250, 279-280, 280-281, 234-235, 104-107, 106-102, 303-305, 313-317, 128-345, 146-141, 159-180, 207-208, 374-201, 133-138, 175-178, 172-178, 368-369, 322-321, 229-230, 295-294, 134-144, 137-154, 265-266, 287-299, 242-248, 313-308, 308-312, 316-318, 302-303, 310-303, 311-306, 259-323, 333-335, 380-267, 335-322.$

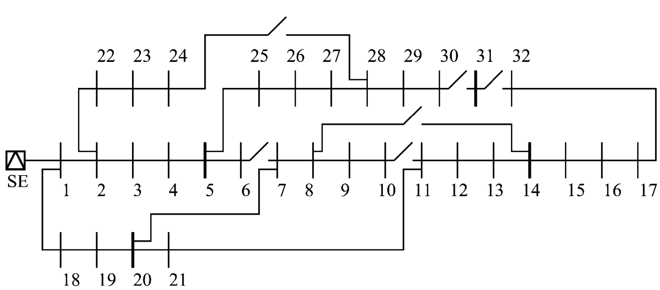

Fig. 3. Distribution system reconfiguration for the 33-node system with four transfer nodes (nodes 5, 14, 20, and 31).

the final topologies obtained for the other four test systems are also verifiably radial. Furthermore, due to the use of constraints (14)–(17) and (24) in the DSR problem, the presence of transfer nodes does not result in loops in the systems.

To show the efficiency of the proposed DSR model, a second test is performed in the 33-node system. In this test, the nodes 5, 14, 20, and 31 were replaced by transfer nodes. The radial final topology obtained by the DSR model is shown in Fig. 3. Note that the 5, 14, and 20 nodes are used in the final topology, while the node 31 is not connected because it became a terminal node. The total losses for this test were equal to 107.79 kW.

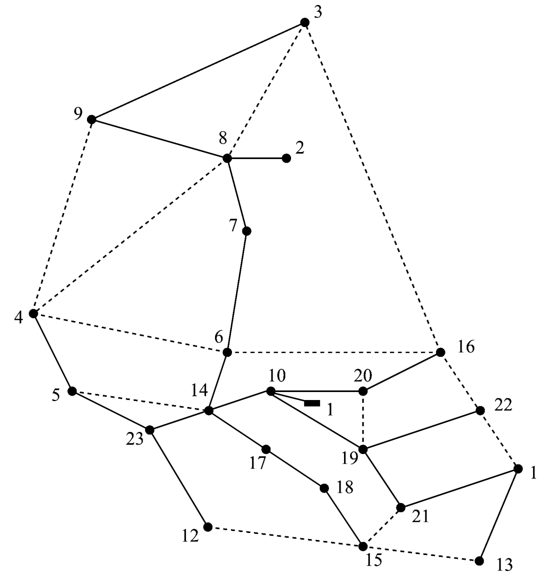

Fig. 4. Distribution system expansion plan for the 23-node system.

TABLE III
RESULTS SUMMARY FOR THE 23-NODE SYSTEM (US$)

Circuit cost	Losses cost	Total cost
151892	20227	172119

B. Distribution System Planning

1) The 23-Node Distribution System: The data for this system are available in [14]. This is a 34.5-kV distribution system, supplied by a 10-MVA substation, which feeds an oil production area with 22 load nodes, without distributed generation. For this test, the voltage threshold is 3%, the average power factor is equal to 0.9, the cost of energy losses is 0.05 US$/kWh, the loss factor equals 0.35, the interest rate is 10%, and the planning period extends to 20 years.

The radial distribution system expansion plan is depicted in Fig. 4, and Table III shows a summary of the obtained results. In Fig. 1, the dashed lines represent the proposed routes, and continuous lines show the circuits constructed. In this case, the investment and operation costs of substations are equal to zero. Note that (39) guarantees that only 22 circuits can be constructed and (30) and (31) guarantee that all 22 load nodes are fed by the existent substation (node 1) in one radial system. The total CPU time is equal to 1706 s. The topology obtained by the proposed methodology is equal to the one found by [14] and [28].

2) The 54-Node Distribution System: This is a 15-kV system with 107.8 MVA to feed 50 load nodes. The objective of this test is planning the distribution system considering four substations, two existing substations and two candidate substations (see [13]). There are 17 existent circuits and 44 paths to new circuits (candidates). Three distributed generators are installed in the nodes 10, 16, and 33. The maximum voltage deviation considered is 5%, the average power factor is equal to 0.9, the cost of energy losses is 0.05 US$/kWh, the loss factor is equal to 0.35, the interest rate is 10%, the cost of operation of these

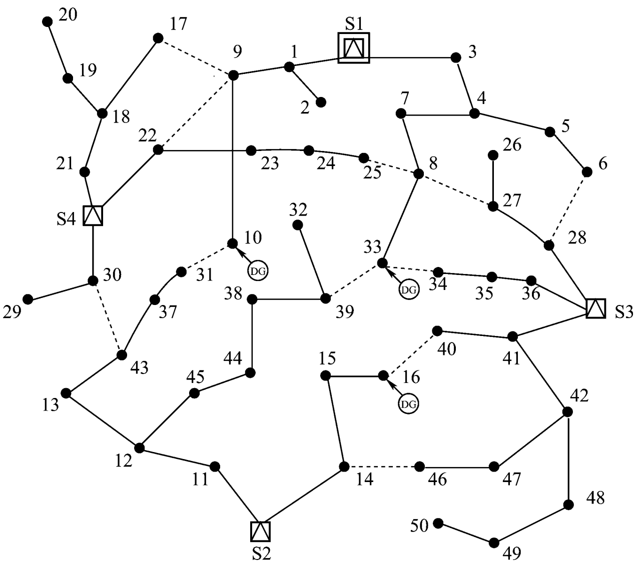

Fig. 5. Distribution system expansion plan for the 54-node system.

TABLE IV
RESULTS SUMMARY FOR THE 54-NODE SYSTEM (US$)

Circuit cost	Loss cost	Substation cost	Operation cost	Total cost
39452	2923	540000	2640891	3223266

substations is $0.001\ \mathrm{US\$/(kVA)^2h}$, and the planning period is 20 years.

Table IV presents a summary of the obtained results; the solution is presented in Fig. 5. Four radial systems were built (one for each substation), and two new substations ($S3$ and $S4$) were necessary to feed the load nodes, while substation $S1$ was expanded. Note that (39) guarantees that only 50 circuits are constructed, and (30) and (31) guarantee that all 50 load nodes are fed in four radial systems. Additionally, constraints (9)–(11) and (38) prevent that the distributed generators can feed some loads independently. The total CPU time is equal to 6872 s.

V. CONCLUSIONS

This paper presented a literature review, a critical analysis, and a proposal to incorporate the radiality constraints into distribution system optimization problems more simply and efficiently. It also presented a preliminary analysis of the generalization of radiality constraints.

The DSR and DSP problems were modeled as mixed integer nonlinear programming problems with radial constraints, and the solution was obtained through the use of a nonlinear branch-and-bound algorithm. The radiality constraints model presented can be extended to other distribution system optimization problems.

The analysis and the assumptions made in this paper are important for the understanding and appropriate use of the radiality constraints, which can be included explicitly in the optimization model of radial distribution systems.

APPENDIX

The following properties were obtained from [24].

Property 1: "Let T be a proper tree graph with $m(\geq 2)$ nodes, and let $(i,j) \in T$. Then disconnecting (i,j) from T, that is, removing the arc (i,j) from T but leaving the nodes i and j in T, decomposes T into two trees T_1 and T_2."

1) Property 2: "A proper tree graph T has at least two end nodes."

2) Property 3: "A tree with m nodes has $(m-1)$ arcs. This is clearly true for $m=1$ or $m=2$. By induction, assume that this property holds for a tree with $(m-1)$ nodes and consider a tree with m nodes, $m \geq 3$. By Property 2, an end node i exists. Disconnect the (unique) arc incident at the end node and obtain two trees T_1 and T_2 (by Property 1), where $T_1 = \{i\}$. Hence, T_1 has zero arcs, and T_2 has $(m-1)$ nodes. By the induction hypothesis, it has $(m-2)$ arcs. Therefore, T has $(m-2)+1 = (m-1)$ arcs."

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