Polyhedral Formulations and Loop Elimination Constraints for Distribution Network Expansion Planning

Rabih A. Jabr, *Senior Member, IEEE*

***Abstract*—Distribution network expansion planning (DNEP) aims at minimizing the capital and operational cost of the expansion plan; the plan entails choosing conductor types and line construction routes together with substation installation and reinforcement that allow serving the demand while satisfying the physical and technical constraints of the expanded network. Two findings are reported in this paper. First, DNEP can be exactly formulated as a disjunctive conic program, in two equivalent formulations; both formulations admit a tight polyhedral approximation and can be solved for the globally optimal solution using software for mixed-integer linear programming (MILP). Second, the DNEP solution can be computed more efficiently when the linear relaxations of the MILP formulations are strengthened using loop elimination constraints. Numerical results on practical DNEP problems reveal that combining the parallel equivalent-circuit polyhedral formulation with the spanning tree loop elimination constraints yields MILP planning solutions with a tight relative optimality gap and within reasonable computing time. In addition, the results are at least of the same quality if not better than those reported in the recent literature.**

***Index Terms*—Nonlinear programming, optimization methods, power system planning.**

NOMENCLATURE

A. Input Data & Operators

$b_{ij}^{(k)}$	Series susceptance of line ij with type-k conductor.
$C_{ij}^{(k)}$	Cost per unit length of line ij with type-k conductor.
C_i^f	Fixed construction or reinforcement cost of the substation at node i.
C_i^v	Substation operation cost at node i.
$g_{ij}^{(k)}$	Series conductance of line ij with type-k conductor.
i	Imaginary unit.
K	Total number of conductor types.
K_ℓ	Capital recovery factor of line construction.
K_S	Capital recovery factor of substation construction or reinforcement.
ℓ_{ij}	Length of line ij.
n	Number of nodes.
n_{S0}	Number of existing substations.
n_S	Total number of existing and potential substations.
$N(i)$	Set of nodes connected to node i by a line route.
P_{Di}/Q_{Di}	Real/reactive power demand at node i.
S_i^0	Apparent power rating of an existing substation at node i.
$S_i^{\max}$	Apparent power rating of substation construction or reinforcement at node i.
ϕ_ℓ	Loss factor of lines.
ϕ_S	Utilization factor of substations.
τ_ℓ	Interest rate for the cost of power losses.
τ_S	Interest rate for the substation operation cost.
Ω_s	Set of all routes ij where i is the sending end node.
Ω_r	Set of all routes ij where i is the receiving end node.
$\Im\{\bullet\}$	Imaginary value operator.
$\Re\{\bullet\}$	Real value operator.
$\bullet^{\min/\max}$	Minimum/maximum magnitude operator.
$\lvert\bullet\rvert$	Magnitude of a number or cardinality of a set.
$\bullet^*$	Conjugate transpose operator.
$\bullet \succeq$	Matrix inequality sign in the positive semi-definite sense.
$\bar{\bullet}$	Denotes circuit quantities in the parallel equivalent-circuit formulation.

B. Decision Variables and Dependent Quantities

$I_{ij}^{(k)}$	Current magnitude in line ij with type-k conductor.
P_{Si}/Q_{Si}	Real/reactive power generation at node i.

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The author is with the Department of Electrical and Computer Engineering, American University of Beirut, Riad El-Solh/Beirut 1107 2020, Lebanon (e-mail: rabih.jabr@aub.edu.lb).

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P_{ij}/Q_{ij}	Real/reactive power flow in line ij.
$p_{ij}^{(k)}/q_{ij}^{(k)}$	Real/reactive power flow in line ij with type-k conductor.
V_i	Complex phasor voltage at node i.
x_{ij}	Binary variable which takes the value 1 if a line is installed in route ij and 0 otherwise.
X	Hermitian matrix.
X_{ii}	Element i on the diagonal of X.
$X_{i(ij)}^{(k)}$	Value of X_{ii} for the kth conductor type/parallel circuit in line ij.
X_{ij}	Element in the ith row and jth column of X.
$X_{ij}^{(k)}$	Value of X_{ij} for the kth conductor type/parallel circuit in line ij.
y_{ij}	Auxiliary variable in the single-commodity flow formulation.
y_{ij}^{w}	Auxiliary variable in the multi-commodity flow formulation.
z_{ij}^{w}	Auxiliary variable in the spanning tree formulation.
$\alpha_{ij}^{(k)}$	Binary variable which takes the value 1 if the kth conductor type/parallel circuit is installed in the line route ij and 0 otherwise.
β_i	Binary variable which takes the value 1 if a substation is constructed or reinforced at node i and 0 otherwise.

I. Introduction

DISTRIBUTION network expansion planning (DNEP) seeks to determine the minimum cost design that ensures adequate substation and line capacity to meet the forecasted load over the planning horizon. The expansion plan specifies the additional line routes, the conductor types, the substations to be reinforced and those to be constructed. DNEP is aimed at minimizing the equipment and network operational costs while satisfying the physical constraints of the network, namely Kirchhoff's voltage and current laws, and the technical constraints such as load bus voltage limits, line current-carrying capacity, and the network acyclic structure. The expansion plan can represent the planning horizon using either a single stage or multiple stages.

DNEP is a mixed-integer nonlinear optimization problem. It involves the optimization of binary variables that represent the selection of distribution equipment, and continuous variables that represent power flows and voltage magnitudes; the network behavior is mainly governed by the nonlinear power flow equations. This makes DNEP ideally suited for solution using branch-and-bound techniques for mixed-integer optimization (MIO). The majority of MIO formulations for DNEP employ linearized voltage drop constraints and are specifically solved using software for mixed-integer linear programming (MILP) [1]–[5]; in fact, even with the current status of mixed integer linear and nonlinear programming, MILP is much more developed and exhibits superior performance. The exact model of the DNEP problem, including the nonlinear power flow and radiality constraints, can be more straightforwardly handled using meta-heuristic optimization approaches such as genetic algorithms [6], [7], ant colony search [8], and simulated annealing [9]. Evolution strategies can also consider the simultaneous optimization of substation location and line construction [10]. However, unlike MILP, meta-heuristic methods cannot give assurance of the quality of the solution because they do not provide an indicator of the distance to what the global optimal solution may be. Heuristic approaches produce good solutions with a relatively modest computational effort. A popular approach from this category is the branch exchange technique which was proposed in [11] and further developed in [12]. Another promising approach, also known in the transmission planning literature, is the constructive heuristic algorithm which has been implemented in [13] with a local improvement phase and a branching technique to improve the solution quality.

The solution space and objective function of the DNEP problem are non-convex (and also disconnected); in addition, the number of possible network topologies grows exponentially with the number of buses [14]. This result is expected given that the number of binary variables in DNEP increases with the problem size. In fact, the unit commitment problem is similarly characterized by an exponential increase in the number of possible commitment schedules in function of the number of generating units. Nevertheless, independent system operators have switched to MILP for unit commitment scheduling because it enhances operational efficiency as compared to traditional approaches such as Lagrangian relaxation [15]. This has motivated investigating in this research the solution of the exact DNEP formulation via MILP. Although [16] has shown that commercial software for MIO can solve certain instances of DNEP modeled with nonlinear constraints, it is shown herein that the MILP formulation can solve larger practical problems and can also handle optimization over different conductor types. The MILP formulation is brought on by 1) the disjunctive programming technique [17] combined with the conic relaxation of the power flow equations in an acyclic network [18] and 2) the linear representation of conic constraints by tight polyhedral approximations [19]. Recent research suggests that the performance of contemporary mixed-integer programming solvers may be affected by even minor changes in the problem formulation [20]; this has led to formulating two equivalent representations for modeling the choice between conductor types. In the first representation, the disjunctive model chooses one of several circuits each representing one conductor type, whereas in the second, each of the line conductor types is modeled as the parallel equivalent of circuits which are switched sequentially.

The modeling of radiality constraints is of particular importance for DNEP problems solved via MIO. Another related constraint is that a load node cannot be fed by two different substations. The radiality constraints have been recently studied in [16] where Lavorato *et al.* prove that for a n-node network having n_S installed substations and load nodes with nonzero demand, the following constraints guarantee that the network is acyclic: 1) Kirchhoff's current law is enforced at all nodes and 2) the summation of all binary variables representing line connection is equal to $(n - n_S)$. In case a node practically has zero loads connected to it, then it is assumed to carry a fictitious load of small value, for instance 0.001 pu; alternatively, it can be modeled as a transfer node [16]. Reference [16] also discusses additional constraints that preclude the formation of isolated islands fed by distributed generation. Although [16] focuses on enforcing radiality with the minimum number of additional constraints and variables, it is known that some MILP formulations can have their linear relaxations strengthened by introducing auxiliary variables [21], [22]. In fact, the solution time using MILP depends on the strength of the formulation's relaxation, i.e., tighter formulations produce better (larger) lower bounds making it more likely that the MILP solver closes the optimality gap within shorter time. This has motivated the investigation of embedding additional constraints that aim to strengthen the relaxations of the MIO formulations for DNEP; three types of loop elimination constraints are studied in this work. These constraints have been originally proposed in the context of the traveling salesman [22] and the minimum spanning tree problems [21] and are extended herein to DNEP with multiple substations; they are referred to as the single-commodity flow constraints, the multi-commodity flow constraints, and the spanning tree constraints.

The rest of this paper is organized as follows. Section II presents the mathematical formulation of the DNEP problem. Conic relaxations using the single-circuit and parallel equivalent-circuit formulations are presented in Section III. Section IV discusses loop elimination constraints that aim to strengthen the relaxations of the MILP formulations. Numerical results are presented in Section V on a 23-node system, a 54-node system, and a 136-node real distribution system; comparisons are reported with the constructive heuristic algorithm in [13] and the mixed-integer nonlinear optimization method in [16]. The paper is concluded in Section VI. The Appendix summarizes the technique employed for constructing polyhedral approximations of conic constraints.

II. Distribution Network Expansion Planning

Consider a distribution network having n nodes, each having a real/reactive load (P_{Di}/Q_{Di}) or a substation (P_{Si}/Q_{Si}) connected to it. Every distribution route between nodes i and j can have one line installed with a conductor of type-k. There are a total of n_S substations that can be reinforced or installed; they are numbered to correspond to the first n_S nodes in the network. The objective of DNEP is to produce a design which minimizes the yearly energy loss and production costs in addition to the equipment installation cost discounted for one year of the planning horizon [13], [16], i.e., to minimize

$$\begin{aligned} K_\ell \sum_{ij\in\Omega_s}\sum_{k=1}^{K}\alpha_{ij}^{(k)} C_{ij}^{(k)}\ell_{ij} + K_S\sum_{i=1}^{n_S}\beta_i C_i^f \\ + 8760(1+\tau_\ell)\phi_\ell c_\ell \sum_{i=1}^{n}(P_{Si}-P_{Di}) \\ + 8760(1+\tau_S)\phi_S\sum_{i=1}^{n_S} C_i^v(P_{Si}^2+Q_{Si}^2). \end{aligned} \tag{1}$$

The binary variable $\alpha_{ij}^{(k)}$ takes the value 1 if a line with type-k conductor is installed in the distribution route ij and 0 otherwise $(\alpha_{ij}^{(k)} = \alpha_{ij}^{(k)})$:

$$\alpha_{ij}^{(k)} \in \{0,1\}, \quad ij\in\Omega_s, \quad k=1,\ldots,K \tag{2}$$

$$x_{ij} = \sum_{k=1}^{K}\alpha_{ij}^{(k)} \le 1, \quad ij\in\Omega_s. \tag{3}$$

Equation (3) can account for the case when a line with a specific conductor type is already installed by setting the corresponding binary variable to 1, or the case when a line is already installed but can have its conductor type changed by replacing the inequality sign in the summation of (3) with an equality. The binary variable β_i is related to the substation capacity limit as follows:

$$\sqrt{P_{Si}^2+Q_{Si}^2} \le S_i^0 + \beta_i S_i^{\max}, \beta_i\in\{0,1\}, i=1,\ldots,n_S \tag{4}$$

where $i=1,\ldots,n_{S0}$ corresponds to substation reinforcement ($S_i^0\neq 0$ and $S_i^{\max}\neq 0$), and $i=n_{S0}+1,\ldots,n_S$ corresponds to substation installation ($S_i^0 = 0$ and $S_i^{\max}\neq 0$). The real and reactive power balance constraints at each node are given by

$$\begin{aligned} P_{Si}-P_{Di} &= \sum_{j\in N(i)} P_{ij} \\ &= \sum_{j\in N(i)}\sum_{k=1}^{K}\alpha_{ij}^{(k)}p_{ij}^{(k)}, \quad i=1,\ldots,n \end{aligned} \tag{5}$$

$$\begin{aligned} Q_{Si}-Q_{Di} &= \sum_{j\in N(i)} Q_{ij} \\ &= \sum_{j\in N(i)}\sum_{k=1}^{K}\alpha_{ij}^{(k)}q_{ij}^{(k)}, \quad i=1,\ldots,n \end{aligned} \tag{6}$$

where

$$\begin{aligned} p_{ij}^{(k)} + \mathrm{i}q_{ij}^{(k)} = \left(g_{ij}^{(k)} - \mathrm{i}b_{ij}^{(k)}\right)(|V_i|^2 - V_iV_j^*) \\ ij\in\Omega_s\cup\Omega_r \quad k=1,\ldots,K \end{aligned} \tag{7}$$

and $P_{Si} = Q_{Si} = 0$ for $i=n_S+1,\ldots,n$. DNEP requires limiting the line current magnitude to its maximum permissible capacity, which depends on the conductor type, and can be expressed as [23]

$$\alpha_{ij}^{(k)}(I_{ij}^{(k)})^2 \le (I_{ij\max}^{(k)})^2 \quad ij\in\Omega_s \quad k=1,\ldots,K \tag{8}$$

where

$$\begin{aligned} \left(I_{ij}^{(k)}\right)^2 = \left((g_{ij}^{(k)})^2 + (b_{ij}^{(k)})^2\right)\left(|V_i|^2+|V_j|^2 - 2\Re\{V_iV_j^*\}\right), \\ ij\in\Omega_s,\ k=1,\ldots,K. \end{aligned} \tag{9}$$

In addition, the node voltage magnitudes are restricted within their standard operating ranges:

$$V_i^{\min} \le |V_i| \le V_i^{\max}, i=1,\ldots,n. \tag{10}$$

The last constraint dictates the number of lines in the network:

$$\sum_{ij\in\Omega_s} x_{ij} = n - n_{S0} - \sum_{i=n_{S0}+1}^{n_S}\beta_i. \tag{11}$$

As noted in [16], (11) together with the power balance constraints (5)–(6) ensure that the network has a number of trees equal to the number of (existing and installed) substations; each load node is therefore connected to one of the substations. This conclusion is true provided that each load node has nonzero power demand, which can be a small artificial value. The above formulation can be also modified to include transfer nodes, which have no generation or demand, and can be potentially disconnected from the network. The modeling of transfer nodes requires the introduction of additional binary variables and linear constraints; details are given in [16]. The DNEP problem is therefore expressed by minimizing (1) subject to (2)–(11); it is an MIO problem with a nonlinear objective function and nonlinear constraints. Its relaxation, which is obtained by relaxing the integrality constraints on the binary variables, is a non-convex program. The solution to a similarly formulated problem was reported in [16] using the commercial solver KNITRO, however, multiple conductor types were not accounted for.

III. Conic and Polyhedral Formulations

This section derives two alternative MILP formulations of the DNEP problem. The approach followed to obtain these formulations comprises three steps. The first step makes use of the conic power flow formulation that renders the power flow problem in acyclic (radial) networks into a convex program [18]. The second step is based on the disjunctive approach for eliminating the product of binary and continuous variables [17]. The third step employs a technique to accurately model conic constraints via linear ones through the use of polyhedral approximations [19]; this technique is reviewed in the Appendix of the paper. The end result is an MILP formulation of the DNEP problem.

In [24], Lavaei and Low proved that the optimal power flow problem can be reformulated as a rank-constrained semi-definite program, and provided evidence that for many practical networks the solution of a semi-definite program (primal or convex Lagrangian dual) can be used to recover an optimal power flow (OPF) solution satisfying the rank-1 constraint. For acyclic networks, [25] showed that the rank-1 constraint will always be satisfied provided that load over-satisfaction is allowed. Let X denote an $n \times n$ Hermitian matrix ($X = X^*$) with X_{ij} being an element in the ith row and jth column. If X is given by

$$X = \begin{bmatrix} |V_1|^2 & \cdots & V_1V_i^* & \ldots & V_1V_n^* \\ \vdots & \ddots & \vdots & & \vdots \\ V_iV_1^* & \cdots & |V_i|^2 & \cdots & V_iV_n^* \\ \vdots & & \vdots & \ddots & \vdots \\ V_nV_1^* & \cdots & V_nV_i^* & \cdots & |V_n|^2 \end{bmatrix} \qquad (12)$$

then (7), (9), and (10) can be written in terms of the elements of X to give new expressions for the real line power flow (13), the reactive line power flow (14), the current magnitude squared (15), and the voltage magnitude squared (16):

$$p_{ij}^{(k)} = g_{ij}^{(k)}(X_{ii} - \Re\{X_{ij}\}) - b_{ij}^{(k)}\Im\{X_{ij}\} \qquad (13)$$

$$q_{ij}^{(k)} = -b_{ij}^{(k)}(X_{ii} - \Re\{X_{ij}\}) - g_{ij}^{(k)}\Im\{X_{ij}\} \qquad (14)$$

$$(I_{ij}^{(k)})^2 = ((g_{ij}^{(k)})^2 + (b_{ij}^{(k)})^2) \times (X_{ii} + X_{jj} - 2\Re\{X_{ij}\}) \qquad (15)$$

$$(V_i^{\min})^2 \le X_{ii} \le (V_i^{\max})^2 \qquad (16)$$

where the range of indices remains as previously defined. Therefore, by using the result in [25], it is possible to ascertain that if matrix X is positive semi-definite ($X = X^* \succeq 0$) and load over-satisfaction is allowed, then the modified problem given by minimizing (1) subject to constraints (2)–(6), (8), (11), (13)–(16) is equivalent to the original problem of minimizing (1) subject to (2)–(11). The semi-definite constraint in the modified problem can be further replaced with a set of rotated conic quadratic constraints [18]:

$$X_{ii}X_{jj} \ge \Re\{X_{ij}\}^2 + \Im\{X_{ij}\}^2, \quad ij \in \Omega_s \qquad (17)$$

where it is understood that $\Re\{X_{ij}\} = \Re\{X_{ji}\}$ and $\Im\{X_{ij}\} = -\Im\{X_{ji}\}$. It is important to note at this stage that the load over-satisfaction assumption is required in the formal proofs of the formulations involving either positive semi-definite [25] or conic constraints [18]. However, this assumption is sufficient but not necessary in the sense that even if there are upper bounds on loads, the solution to the modified problem can be the same as the original one. Numerical results in this paper suggest that this assumption is not required in practice.

The modified problem is a first step towards convexifying the relaxation of the DNEP problem. In the second step, the binary variables are removed from the product terms in (5), (6), and (8). This can be achieved using either the single-circuit formulation or the parallel equivalent-circuit formulation; both approaches are discussed below. Although these formulations are mathematically equivalent, the performance of CPLEX [26], which is a state-of-the-art solver for MILP, is different when applied to them.

A. Single-Circuit Formulation

In the single-circuit formulation, the choice of a particular conductor for a line ij is modeled by connecting a single circuit that forms this line. In other words, the choice of each conductor corresponds to the choice of one of K circuits to be connected between the two nodes i and j; this is what is modeled in (3). To remove the product of binary and continuous variables, (5), (6), and (8) are written as (18), (19), and (20), respectively:

$$P_{Si} - P_{Di} = \sum_{j \in N(i)} P_{ij} = \sum_{j \in N(i)} \sum_{k=1}^{K} p_{ij}^{(k)}, \quad i = 1, \ldots, n \qquad (18)$$

$$Q_{Si} - Q_{Di} = \sum_{j \in N(i)} Q_{ij} = \sum_{j \in N(i)} \sum_{k=1}^{K} q_{ij}^{(k)}, \quad i = 1, \ldots, n \qquad (19)$$

$$(I_{ij}^{(k)})^2 \le \alpha_{ij}^{(k)}(I_{ij\,\max}^{(k)})^2, \ \ ij \in \Omega_s, \ k = 1, \ldots, K \qquad (20)$$

where

$$p_{ij}^{(k)} = g_{ij}^{(k)}\left(X_{i(ij)}^{(k)} - \Re\{X_{ij}^{(k)}\}\right) - b_{ij}^{(k)}\Im\{X_{ij}^{(k)}\} \qquad (21)$$

$$q_{ij}^{(k)} = -b_{ij}^{(k)}\left(X_{i(ij)}^{(k)} - \Re\{X_{ij}^{(k)}\}\right) - g_{ij}^{(k)}\Im\{X_{ij}^{(k)}\} \qquad (22)$$

$$(I_{ij}^{(k)})^2 = \left((g_{ij}^{(k)})^2 + (b_{ij}^{(k)})^2\right) \times \left(X_{i(ij)}^{(k)} + X_{j(ji)}^{(k)} - 2\Re\{X_{ij}^{(k)}\}\right) \qquad (23)$$

and $ij \in \Omega_s \cup \Omega_r$ in (21)–(22), $ij \in \Omega_s$ in (23), $k = 1, \ldots, K$. In (21)–(23), the elements of X are now associated with a particular circuit (i.e., conductor type) as indicated by the index k; in particular, $X_{i(ij)}^{(k)}$ and $X_{ij}^{(k)}$ are set to zero if circuit k is not installed ($\alpha_{ij}^{(k)} = 0$) using

$$V_i^{\min} V_j^{\max} \alpha_{ij}^{(k)} \le X_{i(ij)}^{(k)} \le V_i^{\max} V_j^{\max} \alpha_{ij}^{(k)}, \quad ij \in \Omega_s \cup \Omega_r \tag{24}$$

$$0 \le \Re\{X_{ij}^{(k)}\} \le V_i^{\max} V_j^{\max} \alpha_{ij}^{(k)}, \quad ij \in \Omega_s \tag{25}$$

$$-V_i^{\max} V_j^{\max} \alpha_{ij}^{(k)} \le \Im\{X_{ij}^{(k)}\} \le V_i^{\max} V_j^{\max} \alpha_{ij}^{(k)}, \quad ij \in \Omega_s \tag{26}$$

where $k = 1, \ldots, K$. Otherwise, if circuit k is installed ($\alpha_{ij}^{(k)} = 1$), $X_{i(ij)}^{(k)}$ is set equal to the squared voltage magnitude at node i:

$$(V_i^{\min})^2 (1 - \alpha_{ij}^{(k)}) \le |V_i|^2 - X_{i(ij)}^{(k)} \le (V_i^{\max})^2 (1 - \alpha_{ij}^{(k)}), \quad ij \in \Omega_s \cup \Omega_r, \ \ k = 1, \ldots, K. \tag{27}$$

In summary, $p_{ij}^{(k)} q_{ij}^{(k)}$, and $I_{ij}^{(k)}$ [given by (21)–(23)] are all zero when the k^{th} circuit is not installed in the line route ij; this necessitates setting the corresponding $X_{i(ij)}^{(k)}$ [and $X_{j(ji)}^{(k)}$] together with the real and imaginary parts of $X_{ij}^{(k)}$ all to zero. Equation (24)–(26) force these quantities to be zero whenever $\alpha_{ij}^{(k)} = 0$, i.e., when the circuit is not installed. However, when the circuit is installed, i.e., $\alpha_{ij}^{(k)} = 1$, (27) forces $X_{i(ij)}^{(k)}$ to take the value of $|V_i|^2$. Similarly, (24)–(26) constrain the real and imaginary parts of $X_{ij}^{(k)}$ within the bounds set by the definition of X in (12). To complete the formulation, the rotated conic quadratic constraint (17) is rewritten in terms of the newly indexed variables of X:

$$X_{i(ij)}^{(k)} X_{j(ji)}^{(k)} \ge \Re\{X_{ij}^{(k)}\}^2 + \Im\{X_{ij}^{(k)}\}^2, \quad ij \in \Omega_s, \quad k = 1, \ldots, K. \tag{28}$$

The disjunctive single-circuit formulation is therefore a mixed-integer conic program formed by minimizing (1) subject to (2)–(4), (10), (11), (18)–(28); in contrast to the model in Section II, the mixed-integer conic program has a convex continuous relaxation.

B. Parallel Equivalent-Circuit Formulation

Consider a model where a line ij that can be constructed using one of K possible conductor types is represented using up to K different parallel circuits connected between the nodes i and j. In the parallel equivalent-circuit formulation, the choice of the first conductor type for a line is modeled by connecting the first circuit between the corresponding nodes, the choice of the second conductor type is modeled by connecting the first two circuits in parallel, and the choice of the mth conductor type is modeled by connecting the first m circuits in parallel. The conductors are indexed by $k = 1, \ldots, K$ in increasing order of current carrying capacity, and the sequential order of circuit connections is given by

$$\alpha_{ij}^{(k)} \le \alpha_{ij}^{(k-1)}, \ \ ij \in \Omega_s, \ k = 2, \ldots, K, \ \ x_{ij} = \alpha_{ij}^{(1)}. \tag{29}$$

Assume that the line connected between nodes i and j is to be built using a conductor of type-m and whose admittance is given by $g_{ij}^{(m)} + \mathrm{i} b_{ij}^{(m)}$. According to the parallel equivalent-circuit model, this line is equivalently formed by the parallel combination of m circuits (indexed 1 to m). Let the quantities with the bar sign denote the parameters of the parallel circuits, it then follows from the rule for combining admittances in parallel that

$$\bar{g}_{ij}^{(1)} = g_{ij}^{(1)}, \quad \bar{b}_{ij}^{(1)} = b_{ij}^{(1)} \tag{30}$$

$$\bar{g}_{ij}^{(k)} = g_{ij}^{(k)} - g_{ij}^{(k-1)}, \quad \bar{b}_{ij}^{(k)} = b_{ij}^{(k)} - b_{ij}^{(k-1)}, \quad k = 2, \ldots, m. \tag{31}$$

The real and reactive power injection (18) and (19) would still hold in this formulation together with (21)–(23) modified such that the series conductance and susceptance symbols with the bar signs are employed, thus denoting that these equations hold for each of the parallel circuits:

$$p_{ij}^{(k)} = \bar{g}_{ij}^{(k)} (X_{i(ij)}^{(k)} - \Re\{X_{ij}^{(k)}\}) - \bar{b}_{ij}^{(k)} \Im\{X_{ij}^{(k)}\} \tag{32}$$

$$q_{ij}^{(k)} = -\bar{b}_{ij}^{(k)} \left(X_{i(ij)}^{(k)} - \Re\{X_{ij}^{(k)}\} \right) - \bar{g}_{ij}^{(k)} \Im\{X_{ij}^{(k)}\} \tag{33}$$

$$(I_{ij}^{(k)})^2 = \left((\bar{g}_{ij}^{(k)})^2 + (\bar{b}_{ij}^{(k)})^2 \right) \times \left(X_{i(ij)}^{(k)} + X_{j(ji)}^{(k)} - 2\Re\{X_{ij}^{(k)}\} \right). \tag{34}$$

Equation (24)–(28) also still hold in addition to constraints requiring circuits which are installed in parallel to have the same value for $X_{ij}^{(k)}$ as $X_{ij}^{(l)}$:

$$0 \le \Re\{X_{ij}^{(1)}\} - \Re\{X_{ij}^{(k)}\} \le V_i^{\max} V_j^{\max} (1 - \alpha_{ij}^{(k)}), \quad ij \in \Omega_s, \ \ k = 2, \ldots, K \tag{35}$$

$$-V_i^{\max} V_j^{\max} (1 - \alpha_{ij}^{(k)}) \le \Im\{X_{ij}^{(1)}\} - \Im\{X_{ij}^{(k)}\} \le V_i^{\max} V_j^{\max} (1 - \alpha_{ij}^{(k)}), \quad ij \in \Omega_s, k = 2, \ldots, K. \tag{36}$$

A particular complication in the parallel equivalent-circuit formulation is the setting of the maximum current carrying capacities for the parallel circuits given the maximum conductor currents in the original data. Assume that the line connected between nodes i and j has a conductor of type-m with current carrying capacity $I_{ij\max}^{(m)}$. By the current-divider rule, the maximum currents in the parallel circuits 1 to m are

$$\bar{I}_{ij\max}^{(k,m)} = I_{ij\max}^{(m)} \frac{\left| \bar{g}_{ij}^{(k)} + \mathrm{i} \bar{b}_{ij}^{(k)} \right|}{\left| g_{ij}^{(m)} + \mathrm{i} b_{ij}^{(m)} \right|}, \quad k = 1, \ldots, m. \tag{37}$$

Note that (37) gives the current ratings of the first m circuits, such that equivalent line current carrying capacity is $I_{ij\max}^{(m)}$. If the chosen line conductor were of a different type, say $(m-1)$,

TABLE I
ADDITIONAL NUMBER OF CONTINUOUS VARIABLES, LINEAR CONSTRAINTS, AND CONIC CONSTRAINTS IN THE LIFTED FORMULATIONS

formulation	# of variables	# of linear constraints	# of conic constraints
Single-	$4\|\Omega_s\|(K-1)$	$11\|\Omega_s\|K$	$\|\Omega_s\|(K-1)$
Equivalent-	$4\|\Omega_s\|(K-1)$	$\|\Omega_s\|(15K-4)$	$\|\Omega_s\|(K-1)$

then the current carrying capacities of the first $(m-1)$ circuits would still be given by an equation similar to (37), however, their values would be different than those used in forming the equivalent of the line with type-m conductor. Therefore, (20) is equivalent to (38) with $ij \in \Omega$:

$$\begin{aligned}\left(I_{ij}^{(1)}\right)^2 &\le \alpha_{ij}^{(1)}\left(\bar{I}_{ij\max}^{(1,1)}\right)^2 \\ &\quad + \sum_{k=2}^{n}\alpha_{ij}^{(k)}[(\bar{I}_{ij\max}^{(1,k)})^2-(\bar{I}_{ij\max}^{(1,k-1)})^2] \\ &\vdots \\ (I_{ij}^{(m)})^2 &\le \alpha_{ij}^{(m)}(\bar{I}_{ij\max}^{(m,m)})^2 \\ &\quad + \sum_{k=m+1}^{n}\alpha_{ij}^{(k)}[(\bar{I}_{ij\max}^{(m,k)})^2-(\bar{I}_{ij\max}^{(m,k-1)})^2] \\ &\vdots \\ (I_{ij}^{(K)})^2 &\le \alpha_{ij}^{(K)}(\bar{I}_{ij\max}^{(K,K)})^2. \end{aligned} \tag{38}$$

This equivalence is easily established by noting that a conductor of type-m is chosen by setting $\alpha_{ij}^{(k)}$ to 1 for $k=1,\ldots,m$, and to 0 for $k=m+1,\ldots,K$. It follows from (38) that the right hand side bounding quantities reduce to the maximum circuit current limits (squared) which are in accordance with the current-divider rule in (37).

The disjunctive parallel equivalent-circuit formulation is therefore also a mixed-integer conic program formed by minimizing (39) subject to (2), (4), (10), (11), (18), (19), (24)–(38), where (39) is the modified version of the objective function (1) and accounts for the fact that circuits in this model are added sequentially in parallel:

$$\begin{aligned}&K_\ell \sum_{ij\in\Omega_s}\left[\alpha_{ij}^{(1)}C_{ij}^{(1)}\ell_{ij} + \sum_{k=2}^{K}\alpha_{ij}^{(k)}(C_{ij}^{(k)}-C_{ij}^{(k-1)})\ell_{ij}\right] \\ &\quad + K_S\sum_{i=1}^{n_S}\beta_i C_i^f + 8760(1+\tau_\ell)\phi_\ell c_\ell \sum_{i=1}^{n}(P_{Si}-P_{Di}) \\ &\quad + 8760(1+\tau_S)\phi_S\sum_{i=1}^{n_S}C_i^v(P_{Si}^2+Q_{Si}^2). \end{aligned} \tag{39}$$

C. Complexity of the Formulations

The single- and parallel equivalent-circuit formulations result in convex relaxations of the DNEP problem. This however comes at the price of lifting the problem to a higher-dimensional space, i.e., introducing additional continuous variables and their corresponding constraints. Table I shows the number of additional continuous variables, linear constraints, and conic constraints for each of the lifted formulations as compared to the base formulation. The base formulation employs the conic network model but has terms involving the product of continuous and binary variables; it is the result of the first step described at the beginning of Section IV. To enable the use of MILP, the conic constraints are approximated by polyhedra (step 3). The size of the polyhedral approximation of each rotated conic constraint is a function of a parameter κ , as described in the Appendix. Therefore, a total of $|\Omega_s|K$ rotated conic quadratic constraints are replaced in the linearized formulation by a set of $4\kappa|\Omega_s|K$ linear inequalities that involve $(2\kappa-1)|\Omega_s|K$ additional continuous variables.

Although the lifted and linearized formulations entail an increase in the problem size, experience with state-of-the-art MILP solvers, such as CPLEX [26], shows that these formulations are suitable for solving several DNEP problems reported in the recent literature. The scope of use of the lifted formulations would continue to improve with further enhancements in solver technology and computing hardware.

IV. LOOP ELIMINATION CONSTRAINTS

Reference [16] gives a formal proof that (11) together with the power balance constraints guarantee that the DNEP solution will be an acyclic network when all load nodes have nonzero demand. The structure of the network is given by the binary variables x_{ij} which are defined by either (3) or (29), depending on the formulation. This section discusses the use of auxiliary variables and loop elimination constraints that aim at tightening the relaxations of the original formulations. The use of tightened formulations is known to make the solution by branch-and-bound significantly easier [21], [22]. The resulting time required to solve the MILP problem with a small relative optimality gap is the most important measure of the quality of the DNEP formulations. Three types of constraints are discussed below; with these constraints, there is no longer a need to introduce small fictitious loads at the zero injection nodes.

A. Single-Commodity Flow Constraints

The single-commodity flow (SCF) model was originally discussed for breaking sub-tours in the traveling salesman problem (TSP) [22]. For DNEP, this formulation has a continuous variable $y_{ij}(y_{ji}=-y_{ij})$ that represents a possible fictitious flow along each line route; the flow constraints require that each load node is connected to a substation:

$$0 \le \sum_{j\in N(i)} y_{ij} \le n - n_{s0}, \quad i=1,\ldots,n_{S0} \tag{40}$$

$$\begin{aligned}-1+\beta_i \le \sum_{j\in N(i)} y_{ij} &\le -1+(n-n_{S0})\beta_i, \\ &\quad i=n_{S0}+1,\ldots,n_S\end{aligned} \tag{41}$$

$$\sum_{j\in N(i)} y_{ij} = -1, \quad i=n_S+1,\ldots,n \tag{42}$$

$$|y_{ij}| \le (n-n_{S0})x_{ij}, ij\in\Omega_s. \tag{43}$$

Note that (41) reduces to a substation node equation, (40), if a substation is installed ($\beta_i=1$) and to a load node equation, (42), if the substation is not installed ($\beta_i=0$). Reference [16] proposes a model which is a particular case of the one above to avoid having isolated parts of the network due to the presence of distributed generation (DG).

B. Multi-Commodity Flow Constraints

This model is similar to the previous one except that it uses an embedded multi-commodity flow (MCF) problem; it was shown to strengthen the linear relaxation of the TSP formulation [22]. The multi-commodity flow formulation uses a set of possible fictitious flows $y_{ij}^w (y_{ji}^w = -y_{ij}^w)$ for each load (or commodity) taken separately:

$$0 \leq \sum_{j \in N(i)} y_{ij}^w \leq 1,\ i = 1, \ldots, n_{S0},\ w = n_{S0} + 1, \ldots, n \quad (44)$$

$$-1 + \beta_i \leq \sum_{j \in N(i)} y_{ij}^i \leq -1 + 2\beta_i, \quad i = n_{S0} + 1, \ldots, n_S \quad (45)$$

$$0 \leq \sum_{j \in N(i)} y_{ij}^w \leq \beta_i, \quad i = n_{S0} + 1, \ldots, n_S \quad w = n_{S0} + 1, \ldots, n, \quad w \neq i \quad (46)$$

$$\sum_{j \in N(i)} y_{ij}^i = -1, \quad i = n_S + 1, \ldots, n \quad (47)$$

$$\sum_{j \in N(i)} y_{ij}^w = 0, \quad i = n_S + 1, \ldots, n, \quad w = n_{S0} + 1, \ldots, n, \quad w \neq i \quad (48)$$

$$|y_{ij}^w| \leq x_{ij}, \quad y_{ij}^w \in \mathrm{Z}, \quad w = n_{S0} + 1, \ldots, n. \quad (49)$$

Equation (45) and (46) which correspond to nodes for potential substation installation reduce to either the substation node equation [(44) for $\beta_i = 1$] or the load node equations [(47) and (48) for $\beta_i = 0$]. The above formulation basically requires having a different flow pattern for serving a fictitious load at each of the load nodes, taken one at a time. The fact that a load node should not be connected to more than one substation can be built in the model by requiring y_{ij}^w to be integer, i.e., if the fictitious demand is 1 and the flows can only take values from the set $\{-1, 0, 1\}$, then the load node cannot be served from more than one substation. The integrality of y_{ij}^w allows the solver to exploit the above fact.

C. Spanning Tree Constraints

The spanning tree (ST) constraints were proposed in [21] to formulate a linear program of the minimum spanning tree problem. The constraints can be extended to DNEP by encoding every branch as a directed arc with respect to an acyclic network rooted at node w:

$$x_{ij} = z_{ij}^w + z_{ji}^w, \quad z_{ij}^w \in \{0, 1\}, \quad z_{ji}^w \in \{0, 1\} \quad ij \in \Omega_s, \quad w = 1, \ldots, n \quad (50)$$

$$0 \leq \sum_{j \in N(i)} z_{ij}^w \leq 1, \quad i = 1, \ldots, n_{S0}, \quad w = 1, \ldots, n, \quad w \neq i \quad (51)$$

$$1 - \beta_i \leq \sum_{j \in N(i)} z_{ij}^w \leq 1, \quad i = n_{S0} + 1, \ldots, n_S, \quad w = 1, \ldots, n,\ w \neq i \quad (52)$$

$$\sum_{j \in N(i)} z_{ij}^w = 1,\ i = n_S + 1, \ldots, n,\ w = 1, \ldots, n,\ w \neq i \quad (53)$$

$$\sum_{j \in N(w)} z_{wj}^w = 0,\ w = 1, \ldots, n. \quad (54)$$

Equation (51) is for existing substation nodes, (52) for potential substation nodes, and (53) for load nodes; (54) sets the root node for each acyclic network. A load node, not located at the root node w of the network, cannot be isolated because it has one arc directed out of it. The constraint set also does not allow the formation of cycles. If there is a cycle of branches containing the root node w, then by (50) there is a corresponding cycle of directed arcs. However, there cannot be such a cycle containing the root node w because: 1) if the cycle exists and is directed, then it has at least one arc directed out of node w which is impossible by (54) and the cycle is not directed; 2) if the cycle exists and is not directed, then there exists a node with two arcs directed out of it which is impossible by (51)–(53). Therefore, there are no cycles in the DNEP solution.

V. Numerical Results

The DNEP problems were solved using the CPLEX API for MATLAB [26]. The OPF function in MATPOWER [27] was used to compute the exact value of the operational cost. The simulations were carried out on an Intel Core I7 CPU running at 1.73 GHz with 6 GB of RAM. CPLEX was run with default parameters; in particular, the relative optimality gap tolerance was kept at 0.01% which is considered a very high accuracy. However, for large problems, such a low tolerance value may require an excessive computation time to satisfy. Therefore, a time limit of 4 hours (14 400 s) was set, i.e., the solver will terminate its search and report the best integer solution found in 4 hours even if it does not manage to close the relative optimality gap to within 0.01%.

A. DNEP With One Conductor Type

DNEP was initially tested with one conductor type using a 23-bus system [16] and a 136-bus realistic distribution system [13]; the complete test system data and planning solutions were obtained from [28]. With one conductor type, the single-circuit and parallel equivalent-circuit formulations are identical; the MILP solutions were obtained with the three types of loop elimination constrains (LEC): SCF, MCF, ST, and then without these constraints. The results are summarized in Tables II and III for the 23- and 136-bus systems, respectively; the tables include the CPLEX solution time, the relative optimality gap, the line construction cost, the substation reinforcement/installation cost, and the total cost that comprises both the installation and operational components. It is evident that for the smaller system, all formulations have comparable performance and produce a global solution with zero relative optimality gap. The obtained solution in this case is identical to that in [16] and [28]; however, the solution in [16] using a mixed-integer nonlinear programming solver (KNITRO) required 1706 s on a similar PC, while the MILP solution required a maximum of 13 s. For the larger 136-bus system, Table III shows that the loop elimination constraints play an important role in closing the relative optimality gap to within 0.01% in less than 4 hours. In fact, without these additional constrains, CPLEX could not close the gap to less than 0.01% in 4 hours of computation. The solution is obtained most quickly when employing the spanning tree constraints, and produces a plan which is $14,978 cheaper than the constructive heuristic algorithm (CHA) design in [13], [28]. For completeness, Table VIII in the Appendix shows the new design for the 136-bus system.

TABLE II
RESULTS FOR THE 23-BUS SYSTEM WITH ONE CONDUCTOR TYPE

constr.	time (s)	rel-gap (%)	cost ($) line	substation	total
SCF	4	0.00	151892	0	172110
MCF	13	0.00	151892	0	172110
ST	5	0.00	151892	0	172110
w/o LEC	9	0.00	151892	0	172110
[16]/[28]	-	-	151892	0	172110

TABLE III
RESULTS FOR THE 136-BUS SYSTEM WITH ONE CONDUCTOR TYPE

constr.	time (s)	rel-gap (%)	cost ($) line	substation	total
SCF	2219	1.00E-02	4760	0	5458582
MCF	3560	9.99E-03	4760	0	5458582
ST	1660	1.00E-02	4760	0	5458582
w/o LEC	14400	1.61E-02	3600	0	5458927
[13]/[28]	-	-	5360	0	5473560

TABLE IV
RESULTS FOR THE 23-BUS SYSTEM WITH TWO CONDUCTOR TYPES

constr.	time (s)	rel-gap (%)	cost ($) line	substation	total
SINGLE-CIRCUIT FORMULATION					
SCF	110	0.00	153913	0	170969
MCF	33	0.00	153913	0	170969
ST	15	0.00	153913	0	170969
w/o LEC	350	9.23E-03	153913	0	170969
PARALLEL EQUIVALENT-CIRCUIT FORMULATION					
SCF	53	0.00	153913	0	170969
MCF	47	8.50E-03	153913	0	170969
ST	13	1.86E-04	153913	0	170969
w/o LEC	457	0.00	153913	0	170969
[13]/[28]	-	-	151892	0	172110

B. DNEP With Two Conductor Types

DNEP was also tested on a 23-bus and a 54-bus system, where each line can be constructed using one of two conductor types. The 54-bus system has the option of installing 2 new substations and reinforcing the 2 existing ones. In this part of the study, the simulations were carried out with and without the loop elimination constraints in the two formulations of Section III: the single-circuit formulation (III-A) and the parallel equivalent-circuit formulation (III-B). The results are summarized in Tables IV and V. For the 23-bus system (Table IV), the solution is most efficiently obtained (in 13 s) using the parallel equivalent-circuit formulation with the spanning tree constraints. For all formulations and constraint types, the cost is the same and is $1141 less than that of the CHA design [13], [28]; this new design is given in Table IX in the Appendix.

For the 54-bus system (Table V), the CPLEX gap remained more than 5% after 4 hours when solving the single-circuit formulation. However, for the parallel equivalent-circuit formulation, the gap was closed to within 0.01% when employing the loop elimination constraints. In this case, the fastest solution is again obtained using the spanning tree constraints. Although all designs in Table V, except that corresponding to the single-circuit formulation without LEC, correspond to the optimal solution, only those that have a small relative optimality gap give the designer an assurance of the quality of the expansion plan.

TABLE V
RESULTS FOR THE 54-BUS SYSTEM WITH TWO CONDUCTOR TYPES

constr.	time (s)	rel-gap (%)	cost ($) line	substation	total
SINGLE-CIRCUIT FORMULATION					
SCF	14400	5.34	39576	540000	3515431
MCF	14400	5.32	39576	540000	3515431
ST	14400	5.04	39576	540000	3515431
w/o LEC	14400	5.38	40701	540000	3516551
PARALLEL EQUIVALENT-CIRCUIT FORMULATION					
SCF	6099	8.02E-03	39576	540000	3515431
MCF	8324	1.46E-03	39576	540000	3515431
ST	4045	9.78E-03	39576	540000	3515431
w/o LEC	14400	5.26	39576	540000	3515431
[13]/[28]	-	-	39576	540000	3515431

TABLE VI
NUMBER OF VARIABLES FOR THE DIFFERENT TEST SYSTEMS

system	K	$\|\Omega_s\|$/ effective #	n_s/ effective #	# of auxiliary variables SCF	MCF	ST
23-bus	1	35/35	1/0	35	770	1610
136-bus	1	149/87	2/0	149	20115	40826
23-bus	2	35/35	1/0	35	770	1610
54-bus	2	61/44	4/4	61	3172	6588

TABLE VII
NUMBER OF VARIABLES AND CONSTRAINTS BEFORE AND AFTER CPLEX PRE-SOLVE (WITH ST CONSTRAINTS)

system	before pre-solve rows	columns	binary	after pre-solve rows	columns	binary
23-bus	4296	3000	1646	3620	2484	1245
136-bus	51128	46435	40977	13152	8665	3592
SINGLE-CIRCUIT FORMULATION (K=2)						
23-bus	6956	4260	1681	6312	3838	1389
54-bus	16154	11250	6714	10214	6266	2497
PARALLEL EQUIVALENT-CIRCUIT FORMULATION (K=2)						
23-bus	7096	4260	1681	6365	3723	1275
54-bus	16398	11250	6714	9236	5136	1378

C. MILP Problem Size

Table VI shows statistics relating to the number of variables for the test systems with one and two conductor types. Column 3 shows the total/effective number of line routes; the effective number of line routes is less than the total number for the 136- and 54-bus systems because these networks have some existing lines whose routes are not considered in the expansion (the binary variables corresponding to the routes with existing lines are already known). Similarly, column 4 shows the total number of existing substations and the effective number that can be expanded or installed.

The basic number of binary variables for DNEP is given by: $K|\Omega_s| + n_s$. Additional auxiliary variables, as shown in Table VI, are employed when using SCF (continuous variables), MCF (integer variables), and ST (binary variables). However, the total number of variables is not directly passed to the branch-and-cut algorithm. CPLEX starts with a pre-solve stage in which it tries to reduce the size of the problem using preprocessing techniques and also attempts to strengthen the continuous relaxation; this preprocessing is effective for DNEP because the embedded power flow equations are sufficient to get an acyclic network under certain conditions [16]. Table VII shows the original and reduced size of the

MILP problem as produced by the CPLEX pre-solve for the formulation with ST constraints (that requires the largest number of auxiliary variables). The problem size is given by the number of rows and columns of the linear constraint matrix together with the number of binary variables. The statistics in Table VII indicate that pre-solve significantly reduces the number of binary variables, and this is particularly true for the 136- and 54-bus systems in which part of the network structure is already known. For example, the number of binary variables for the 136-bus system prior to pre-solve is $40\,977$ $(=\ 1 \times 149 + 2 + 40\,826)$, whereas after pre-solve it drops to 3592. For the 54-bus system, the number of binary variables is originally 6714 $(=\ 2 \times 61 + 4 + 6588)$. After pre-solve, it is reduced to 2497 for the single-circuit formulation, and 1378 for the parallel equivalent-circuit formulation. This difference is attributed to the manner in which x_{ij} is related to $\alpha_{ij}^{(k)}$ [(3) versus (29)] and corroborates with the superior computational performance of the parallel-equivalent circuit formulation. The other constraint types (SCF and MCF) also benefit from a similar reduction in problem size.

VI. Conclusion

This paper presents and compares different MILP formulations of the DNEP problem. The formulations employ tight polyhedral approximations of mixed-integer conic programs and differ in the manner by which the choice of conductor types is represented. The strengthening of the linear relaxations of the MILP models is also investigated by embedding within the DNEP problem auxiliary variables and constraints. The numerical results show that 1) solution using CPLEX with a tight relative optimality gap tolerance is made easier by the introduction of auxiliary variables, and 2) the preferred formulation employs the more involved parallel equivalent-circuit representation of conductors together with the spanning tree constraints. Comparisons with methods in the recent literature suggest that MILP is faster than a mixed-integer non-linear programming approach and tends to give slightly better solutions than the constructive heuristic algorithm.

Appendix

A. Polyhedral Approximation of Conic Constraints

Each of the constraints in (28) is in the form of a rotated quadratic cone:

$$r_1 r_2 \geq x_0^2 + y_0^2, \quad r_1, r_2 \geq 0. \tag{55}$$

The quadratic term in the objective function (1) and (39) can be brought into the constraints also using a rotated quadratic cone. This Appendix summarizes the approach in [19] to tightly approximate the conic constraint in (55) using a linear set. Towards this end, (55) is expressed as

$$r \geq \sqrt{x_0^2 + y_0^2} \tag{56}$$

$$r' \geq \sqrt{(x_0')^2 + (y_0')^2} \tag{57}$$

$$r' = \frac{(r_1 + r_2)}{2}, \;\; x_0' = \frac{(r_1 - r_2)}{2}, \;\; y_0' = r. \tag{58}$$

The approximation of (56) [and similarly (57)] using linear constraints requires introducing 2κ variables (x_i for $i = 1, \ldots, \kappa$ and y_i for $i = 1, \ldots, \kappa$) to construct the following polyhedron:

$$x_{i+1} - x_i \cos\frac{\pi}{2^i} - y_i \sin\frac{\pi}{2^i} = 0, \;\; i = 0, \ldots, \kappa - 1 \tag{59}$$

$$y_{i+1} - y_i \cos\frac{\pi}{2^i} + x_i \sin\frac{\pi}{2^i} \geq 0, \quad i = 0, \ldots, \kappa - 1 \tag{60}$$

$$y_{i+1} + y_i \cos\frac{\pi}{2^i} - x_i \sin\frac{\pi}{2^i} \geq 0, \quad i = 0, \ldots, \kappa - 1 \tag{61}$$

$$x_\kappa \cos\frac{\pi}{2^\kappa} + y_\kappa \sin\frac{\pi}{2^\kappa} - r = 0. \tag{62}$$

Note that (59) and (62) can be used to substitute the variables x_i for $i = 1, \ldots, \kappa$ and y_κ out of (59)–(62); the result is a

TABLE VIII
New Design of the 136-Bus System

from	to	from	to	from	to	from	to
201	2	32	36	69	70	105	106
2	3	36	37	69	71	107	108
3	4	37	38	71	72	109	110
4	5	36	39	72	73	111	112
5	6	201	40	71	74	112	113
6	7	40	41	74	75	113	114
7	8	41	42	202	76	109	115
7	9	41	43	76	77	115	116
9	10	43	44	77	78	110	117
9	11	44	45	78	79	117	118
11	12	44	46	79	80	105	119
11	13	46	47	80	81	119	120
11	14	47	48	81	82	120	121
14	15	48	49	82	83	202	122
14	16	49	50	82	84	122	123
16	17	50	51	202	86	123	124
201	18	49	52	86	87	124	125
18	19	52	53	87	88	124	126
19	20	53	54	87	89	126	127
20	21	54	55	89	90	126	128
21	22	55	56	90	91	128	129
21	23	53	57	91	92	128	130
23	24	57	58	92	93	130	131
23	25	58	59	94	95	131	132
25	26	59	60	95	96	132	133
26	27	60	61	96	97	133	134
27	28	61	62	94	98	134	135
28	29	48	63	98	99	135	136
29	30	202	64	202	100	16	85
30	31	64	65	100	101	38	99
29	32	65	66	101	102	45	114
32	33	66	67	102	103	45	118
33	34	67	68	102	104	63	108
34	35	68	69	104	105	-	-

TABLE IX
New Design of the 23-Bus System With Two Conductor Types

from	to	conductor	from	to	conductor
1	10	2	10	20	1
2	8	1	11	13	1
3	9	1	11	21	1
4	5	1	12	23	1
5	23	1	14	17	1
6	7	1	14	23	1
6	14	1	15	18	1
7	8	1	16	20	1
8	9	1	17	18	1
10	14	1	19	21	1
10	19	1	19	22	1

set of constraints comprising 2κ linear inequalities in $\kappa + 2$ variables:

$$F.[r, x_0, y_0, y_1, \ldots, y_{\kappa-1}] \geq 0. \tag{63}$$

Reference [19] shows that (63) approximates (56) with accuracy $\varepsilon = \cos(\pi/2^{\kappa})^{-1} - 1$, such that

$$(1 + \varepsilon) r \geq \sqrt{x_0^2 + y_0^2}. \tag{64}$$

Therefore with $\kappa = 16$, the accuracy $\epsilon = 1.15 \times 10^{-9}$.

B. New DNEP Designs Obtained by MILP

Tables VIII and IX show the MILP designs for the 136-bus and 23-bus system with 2 conductor types, respectively.

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Rabih A. Jabr (M'02–SM'09) was born in Lebanon. He received the B.E. degree in electrical engineering (with high distinction) from the American University of Beirut, Beirut, Lebanon, in 1997 and the Ph.D. degree in electrical engineering from Imperial College London, London, U.K., in 2000.

Currently, he is an Associate Professor in the Department of Electrical and Computer Engineering at the American University of Beirut. His research interests are in mathematical optimization techniques and power system analysis and computing.