Quantum Mechanics I - PSet 1

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1 Exercise 1

The purpose of this exercise is to show that Maxwell's theory (energy loss by the electron via radiation) is incompatible with classical mechanics (orbital mechanics) in describing the hydrogen atom.

(a) We assume the electron is in uniform circular motion around the proton and thus has constant acceleration a. We compute the acceleration via Newton's equation to obtain

$$a = \frac{e^2}{m_e r^2}, \tag{1.1}$$

using a Coulomb force given by $F = e^2/r^2$ in cgs units. Thus

$$\frac{dE}{dt} = -\frac{2}{3}\frac{e^6}{c^3 m_e^2 r^4} \tag{1.2}$$

The energy lost in one revolution is then

$$\Delta E \approx \left|\frac{dE}{dt}\right| T, \tag{1.3}$$

where T is the period of revolution. From

$$\frac{2\pi r}{T} = v, \tag{1.4}$$

we obtain

$$T = \frac{2\pi r}{v}. \tag{1.5}$$

Now we use

$$a = \frac{v^2}{r}, \tag{1.6}$$

Auxiliary computation: This can be obtained by considering

$$\vec{r}(t) = r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j} \tag{1.7}$$

with

$$\vec{v}(t) = \dot{\vec{r}}(t) = -r\omega\sin(\omega t)\hat{i} + r\omega\cos(\omega t)\hat{j}, \tag{1.8}$$

and finally

$$\vec{a}(t) = \dot{\vec{v}}(t) = -r\omega^2\cos(\omega t)\hat{i} - r\omega^2\sin(\omega t)\hat{j}. \tag{1.9}$$

Alternatively, in polar coordinates, one has

$$\vec{r}(t) = r\hat{e_r}(t), \tag{1.10}$$

where

$$\hat{e}_r(t) = \cos\theta(t)\hat{\imath} + \sin\theta(t)\hat{\jmath}, \tag{1.11}$$

and

$$\hat{e}_\theta(t) = -\sin\theta(t)\hat{\imath} + \cos\theta(t)\hat{\jmath}, \tag{1.12}$$

with

$$\theta(t) = \omega t. \tag{1.13}$$

Therefore

$$\vec{v}(t) = r\omega\hat{e}_\theta(t), \tag{1.14}$$

and

$$\vec{a}(t) = -r\omega^2\hat{e}_r(t). \tag{1.15}$$

Using (1.6) in (1.5) we obtain for T

$$T = \frac{2\pi\sqrt{r^3 m_e}}{e}, \tag{1.16}$$

and thus

$$\Delta E = \frac{4\pi}{3}\frac{e^5}{c^3 m_e^{3/2} r^{5/2}}. \tag{1.17}$$

For the kinetic energy we have

$$K = \frac{1}{2}m_e v^2 = \frac{1}{2}m_e a r = \frac{1}{2}m_e r\frac{e^2}{m_e r^2} = \frac{e^2}{2r}. \tag{1.18}$$

Thus

$$\frac{\Delta E}{K} = \frac{8\pi}{3}\frac{e^3}{c^3 m_e^{3/2} r^{3/2}}. \tag{1.19}$$

Since

$$v = \frac{e}{\sqrt{m_e r}}, \tag{1.20}$$

we can write

$$\frac{\Delta E}{K} = \frac{8\pi}{3}\left(\frac{v}{c}\right)^3. \tag{1.21}$$

If the electron is non-relativistic $v << c$, then

$$\frac{\Delta E}{K} << 1. \tag{1.22}$$

Thus, the energy loss per orbit is negligible compared to the kinetic energy of the electron. For small time frames one can assume that the orbit is circular.

(b) We want to write the energy as a function of the radius $E(r)$. We have

$$E(r) = \frac{1}{2}m_e v^2 + V(r), \tag{1.23}$$

where

$$F = -\frac{e^2}{r^2} = -\frac{dV}{dr} \implies V(r) = -\frac{e^2}{r}, \tag{1.24}$$

and thus

$$E(r) = \frac{e^2}{2r} - \frac{e^2}{r} = -\frac{e^2}{2r}, \tag{1.25}$$

using (1.1) and (1.6). From the chain rule

$$\frac{dE}{dt} = -\frac{2}{3}\frac{e^6}{c^3 m_e^2 r^4} = \frac{dE}{dr}\frac{dr}{dt} = \frac{e^2}{2r^2}\frac{dr}{dt}, \tag{1.26}$$

leading to

$$\frac{dr}{dt} = -\frac{4}{3}\frac{e^4}{c^3 m_e^2 r^2}. \tag{1.27}$$

Solving this ODE we have

$$\int_{t_0}^{t_f} r^2 \frac{dr}{dt} dt = -\frac{4}{3}\frac{e^4}{c^3 m_e^2}\int_{t_0}^{t_f} 1 dt, \tag{1.28}$$

$$\frac{1}{3}(r_f^3 - r_i^3) = -\frac{4}{3}\frac{e^4}{c^3 m_e^2}\Delta t, \tag{1.29}$$

where $r_f = r(t_f)$, $r_i = r(t_i)$ and $\Delta t = t_f - t_i$. We have finally

$$\Delta t = \frac{1}{4}\frac{c^3 m_e^2}{e^4}(r_i^3 - r_f^3) \approx 1.05 \times 10^{-11} s. \tag{1.30}$$

This clearly shows that the classical model of the atom is terribly wrong since atoms are stable.

(c) From

$$v = \frac{e}{\sqrt{m_e r}}, \tag{1.31}$$

we have for $r = 0.5A$ that $v = 2.3 \times 10^8 cm/s$. So $v/c \approx 0.0075$. The Taylor expansion of the $\gamma = 1/\sqrt{(1 - v^2/c^2)}$ factor leads to a first order relativistic correction of the order of v^2/c^2 which is of the order of 10^{-4}, not a significant alteration.

(d)

$$\lim_{r\to 0^+} E(r) = \lim_{r\to 0^+} -\frac{e^2}{2r} = -\infty. \tag{1.32}$$

There is no minimum energy.

2 Exercise 2

(a-i) We have

$$[h] = \frac{[E]}{[\nu]} = [F][d]T^{-1} = ML^2T^{-1}, \tag{2.1}$$

so

$$m^\alpha g^\beta \hbar^\gamma = M^\alpha L^\beta T^{-2\beta} M^\gamma L^{2\gamma} T^{-\gamma} = M^{\alpha+\gamma} L^{2\gamma+\beta} T^{-2\beta-\gamma}. \tag{2.2}$$

Since $E = ML^2T^{-2}$ we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \tag{2.3}$$

with unique solution given by $(\alpha, \beta, \gamma) = (1/3, 2/3, 2/3)$, i.e. the energy is given by $(mg^2\hbar^2)^{1/3}$. With only m and g we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \tag{2.4}$$

which clearly has no solutions, i.e. $rank(A) < rank(A|b)$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}. \tag{2.5}$$

(a-ii) Following (a) we need to solve

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{2.6}$$

for length, and

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \tag{2.7}$$

for speed, and

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.8}$$

for time. Solving these one obtains

$$[l] = L = \left[\left(\frac{\hbar^2}{m^2 g} \right)^{1/3} \right], \tag{2.9}$$

$$[v] = LT^{-1} = \left[\left(\frac{g\hbar}{m} \right)^{1/3} \right], \tag{2.10}$$

and

$$[t] = T = \left[\left(\frac{\hbar}{mg^2} \right)^{1/3} \right]. \tag{2.11}$$

(a-iii) With position and velocity perfectly known we have $\Delta x = 0$ and $\Delta p = 0$ which clearly violates Heisenberg's inequality $\Delta x \Delta p \geq \hbar/2$.

(a-iv) We have

$$E = \frac{1}{2}mv^2 + mgx = \frac{1}{2}m\left(\frac{g\hbar}{m}\right)^{2/3} + mg\left(\frac{\hbar^2}{m^2 g}\right)^{1/3} = \frac{3}{2}m^{1/3}g^{2/3}\hbar^{2/3}. \tag{2.12}$$

As $h \to 0$ the energy goes to zero which corresponds to the classical case where the particle sits on top of the table ($x = 0$) with zero velocity ($v = 0$).

Remark 2.1. *This argument does not show that the obtained energy corresponds to the ground state energy. This is true however, the argument can be seen by considering $xp \sim \hbar$ (minimal energy satisfying the uncertainty principle) obtaining $E(v)$. We then solve $dE/dv = 0$ to obtain v_{min}, the velocity corresponding to the minimum of the energy, this leads to the same v obtained by dimensional analysis.*

(a-v) Computations lead to a distance in the order of $10\mu m$.

(b) These results are very interesting. Basically, if we assume that the relevant lengths and masses when gravity interactions are of the same order of nuclear interactions are given by combinations of the fundamental constants governing these phenomena G, c, $\hbar$ (fair assumption) we essentially realize that gravity is mostly irrelevant in order to describe these phenomena.

(b-i) Doing computations similar to $(a-i)$ we obtain

$$L_p := \left(\frac{G\hbar}{c^3}\right)^{1/2.} \tag{2.13}$$

(b-ii) We have

$$L_p \approx 1.62 \times 10^{-35} m. \tag{2.14}$$

So, twenty orders of magnitude smaller than the size of the nucleus $\sim 1fm = 10 \times^{-15} m$.
(b-iii) Similarly we have

$$M_p := \left(\frac{c\hbar}{G}\right)^{1/2} \approx 2.18 \times 10^{-8} kg. \tag{2.15}$$

This corresponds roughly to 19 times the order of magnitude of the proton mass, $m_p \approx 1.67 \times 10^{-27} kg$.

Since the Planck length is much much smaller than the size of the nucleus and the mass of the proton is much much smaller than the Planck mass this essentially means that gravity has barely no effect on nuclear interactions since its force is proportional to mass and inversely proportional to the square of the distance.

3 Exercise 3

(a-i) The frequency is given by

$$\nu = \frac{c}{\lambda}, \tag{3.1}$$

therefore the frequency is in the range $[4.28, 7.69] \times 10^{14} Hz$. The photon energy is given by

$$E = h\nu, \tag{3.2}$$

therefore the energy is in the range $[2.84, 4.97] \times 10^{-19} eV$.

(a-ii) We just need to compute

$$\frac{P}{h\nu}, \tag{3.3}$$

to obtain the number of photons emitted per second. The results are $4.53 \times 10^2 6$ photons/s, $3.19 \times 10^1 5$ photons/s, and $7.10 \times 10^2 3$ photons/s for the microwave, laser and cell phone respectively.

(a-iii) The energy required to change the temperate of a certain mass of water by ΔT is given by

$$Q = m(c_v)_{water} \Delta T = 8.36 \times 10^{10} ergs, \tag{3.4}$$

where $m = 200g$, $\Delta T = 10K$ and $(c_v)_{water} = 4.18 \times 10^7 erg/gK$. The numbers of photons is given by

$$\frac{Q}{h\nu} = \frac{8.36 \times 10^{10}}{1.66 \times 10^{-17}} = 5.02 \times 10^{27}. \tag{3.5}$$

(a-iv) Since the frequency of radio waves is smaller than X-rays, for the same amount of carried energy, we have many more photons in radio waves than X-rays. This means that radio waves are more tractable by a statistic treatment (classical electromagnetic wave treatment) compared to X-rays.

The purpose of this exercise is to show that de Broglie's wavelength for everyday objects is extremely small and basically unnoticeable. Only at the atomic scale do we see reasonably detectable wave behavior.

(b-i) We need to compute

$$\lambda = \frac{h}{p} = \frac{h}{mv}. \tag{3.6}$$

This gives $1.48 \times 10^{-38} m$.
(b-ii) This gives $6.63 \times 10^{-31} m$.
(b-iii) Treating each particle has a sphere, since we are given a diameter, we get

$$m = \frac{4}{3}\pi \left(\frac{d}{2}\right)^3 \cdot \rho = 1.05 \times 10^{-16} g. \tag{3.7}$$

Extracting v from

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T, \tag{3.8}$$

we obtain

$$v = \sqrt{3\frac{k_B T}{m}}, \tag{3.9}$$

which gives $v = 0.34 m/s$ and therefore $\lambda = h/(mv) = 1.84 \times 10^{-14} m$.
(b-iv) Similarly, we obtain $\lambda = h/(mv) = 27nm$.

4 Exercise 4

(a) The relation for constructive interference in the double slit experiment is given by

$$d \sin\theta = \Delta r = m\lambda, \tag{4.1}$$

assuming $D >> d$, where m is the order of the maximum. Since

$$\sin\theta \approx \tan\theta \approx \frac{y_m}{D}, \tag{4.2}$$

where y_m is the distance from the center of the target, we have

$$d\frac{y_m}{D} = m\lambda. \tag{4.3}$$

The distance between consecutive maxima is $w = y_{m+1} - y_m$ therefore

$$w\frac{d}{D} = \lambda, \tag{4.4}$$

and

$$w = \frac{D}{d}\lambda. \tag{4.5}$$

From de Broglie's relation $\lambda = h/p$ we finally obtain

$$w = \frac{Dh}{dp}. \tag{4.6}$$