Quantum Mechanics I - PSet 4

Telmo Cunha

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Exercise 1

I recommend solving exercise 3 before this one (unless there's some easier way that I am missing).

- (a) We have uncertainty $\Delta x = 10^{-10}m$ for a free proton which obeys the free Schrödinger's equation
$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}. \tag{1.1}$$
We want to find the time t_s such that $\Delta x(t) \approx \alpha\Delta x(t=0)$ (where α is an arbitrary spread). Let us assume that the starting wave-function is a Gaussian wave packet centered at $x = 0$, i.e.
$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\frac{1}{(\Delta x)_0}e^{-\frac{x^2}{4(\Delta x)_0^2}}, \tag{1.2}$$
where we denote $(\Delta x)_0$, the uncertainty at time $t = 0$. From the computations of exercise 3 we established that
$$\Delta x(t) = (\Delta x)_0\sqrt{1+(t/\tau)^2}, \tag{1.3}$$
where $\tau = 2m(\Delta x)_0^2/\hbar$. Therefore
$$t = \tau\sqrt{\alpha^2-1}. \tag{1.4}$$
Taking $\alpha = 2$, double the starting spread, we compute
$$t = \frac{2m_p10^{-20}}{\hbar}\sqrt{3} \approx 5.5\times10^{-13}s. \tag{1.5}$$
If $(\Delta x)_0 = 10^2$m we have
$$t = \frac{2m_p10^{-4}}{\hbar}\sqrt{3} \approx 5.5\times10^{3}s. \tag{1.6}$$
We conclude that the spread increases much faster if the starting uncertainty in the position is smaller, a localized particle, as opposed to an already large starting uncertainty in the position.

- (b) We have
$$t_s = \tau\sqrt{\alpha^2-1} = \frac{2m(\Delta x)_0^2}{\hbar}\sqrt{\alpha^2-1}, \tag{1.7}$$
using $(\Delta x)_0 \sim \frac{\hbar}{(\Delta p)_0}$ we obtain
$$t_s = \tau\sqrt{\alpha^2-1} = \frac{2m\hbar}{(\Delta p)_0^2}\sqrt{\alpha^2-1}. \tag{1.8}$$
The time to pass through a "fixed position" is $t_f \sim (\Delta x)_0/v = m\hbar/(p\Delta p)$, using again the relation above. Thus, because $p >> (\Delta p)_0$ and considering again $\alpha = 2$ we have
$$t_s >> t_f. \tag{1.9}$$

Exercise 2

- (a) The current is given by

$$\vec{J} = \frac{\hbar}{m} Im\left(\Psi^* \nabla \Psi\right) = \frac{\hbar}{m} Im\left[\left(e^{-ikz} + \frac{f^*(\theta)}{r} e^{-ikr}\right) \nabla \left(e^{ikz} + \frac{f(\theta)}{r} e^{ikr}\right)\right]. \tag{1.10}$$

We first compute $\vec{J}_1$ which corresponds to

$$\vec{J}_1 = \frac{\hbar}{m} Im\left(ik\right) \hat{e}_z = \frac{k\hbar}{m} \hat{e}_z. \tag{1.11}$$

The flux over a sphere of radius R centered at $r = 0$ is just

$$\int_{\partial B(R,0)} \vec{J}_1 \cdot \hat{n}_{ext} dS = \int_0^{2\pi} \int_0^{\pi} \cos\theta R^2 \sin\theta d\theta d\phi = 2\pi R^2 \frac{\sin^2\theta}{2}\Bigg|_{\theta=0}^{\theta=\pi} = 0, \tag{1.12}$$

where $\partial B(R,0)$ is the boundary of a ball of radius R in 3D and $\hat{e}_z \cdot \hat{n}_{ext} = \cos\theta$. Thus, we conclude that, within any sphere of radius R, we have conservation of probability, $\partial\rho/\partial t = 0$, of the incoming particles. This makes sense physically since no particles are being destroyed.

- (b) $\vec{J}_2$ corresponds to

$$\vec{J}_2 = \frac{\hbar}{m} Im\left[\frac{f^*(\theta)}{r} e^{-ikr} \nabla \left(\frac{f(\theta)}{r} e^{ikr}\right)\right] \tag{1.13}$$

The gradient in spherical coordinates is given by

$$\nabla = \frac{\partial}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}. \tag{1.14}$$

Therefore we have

$$\begin{aligned}\vec{J}_2 &= \frac{\hbar}{m} Im\left[\frac{f^*(\theta)}{r} e^{-ikr}\left[f(\theta)\left(\frac{ik}{r}e^{ikr} - \frac{e^{ikr}}{r^2}\right)\hat{r} + \left(\frac{f'(\theta)e^{ikr}}{r}\right)\hat{\theta} + 0\hat{\varphi}\right]\right] \\ &= \frac{\hbar}{m} Im\left[|f(\theta)|^2\left(\frac{ik}{r^2} - \frac{e^{-2ikr}}{r^3}\right)\hat{r} + \frac{f^*(\theta)f'(\theta)}{r^2}\hat{\theta}\right].\end{aligned} \tag{1.15}$$

Thus, the radial component is

$$\vec{J}_2 \cdot \hat{r} = \frac{\hbar}{m}|f(\theta)|^2\left(\frac{k}{r^2} - \frac{\cos(-2kr)}{r^3}\right). \tag{1.16}$$

Again, we have

$$\begin{aligned}\int_{\partial B(R,0)} \vec{J}_2 \cdot \hat{n}_{ext} dS &= \int_0^{2\pi}\int_0^{\pi} \frac{\hbar}{m}|f(\theta)|^2\left(\frac{k}{R^2} - \frac{\cos(-2kR)}{R^3}\right) R^2 \sin\theta d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} \frac{\hbar}{m}|f(\theta)|^2\left(k - \frac{\cos(-2kR)}{R}\right)\sin\theta d\theta d\phi\end{aligned} \tag{1.17}$$

So, in the limit, since $|\cos(x)| \leq 1$, we have

$$\begin{aligned}\lim_{R\to\infty}\int_{\partial B(R,0)} \vec{J}_2 \cdot \hat{n}_{ext} dS &= \int_0^{2\pi}\int_0^{\pi} \frac{\hbar k}{m}|f(\theta)|^2 \sin\theta d\theta d\phi \\ &= \frac{2\pi\hbar k}{m}\int_0^{\pi} |f(\theta)|^2 d\theta.\end{aligned} \tag{1.18}$$

- (c) We have

$$\vec{J}_{12} = \frac{\hbar}{m} Im\left(\Psi^*\nabla\Psi\right) = \frac{\hbar}{m} Im\left[e^{-ikr\cos\theta}\nabla\left(\frac{f(\theta)}{r}e^{ikr}\right) + \frac{f^*(\theta)}{r}e^{-ikr}\nabla\left(e^{ikr\cos\theta}\right)\right]. \tag{1.19}$$

Since we want only the radial component with leading terms $1/r$ we have

$$\begin{aligned}J_{12}\hat{r} &= \frac{\hbar}{m} Im\left[e^{-ikr\cos\theta}f(\theta)\left(\frac{ik}{r}e^{ikr}\right) + \frac{ik\cos\theta f^*(\theta)}{r}e^{-ikr}e^{ikr\cos\theta}\right]\hat{r} \\ &= \frac{\hbar k}{rm} Im\left[i\left(f(\theta)e^{ikr(1-\cos\theta)} + \cos\theta f^*(\theta)e^{-ikr(1-\cos\theta)}\right)\right]\hat{r}.\end{aligned} \tag{1.20}$$

Exercise 3

- (a) We want to check that

$$\int_{\mathbb{R}} \Psi^*(x,0)\Psi(x,0)dx = 1, \tag{1.21}$$

where

$$\Psi(x,0) = \frac{1}{(2\pi)^{1/4}\sqrt{a}} e^{-\frac{x^2}{4a^2}}. \tag{1.22}$$

We have

$$\int_{\mathbb{R}} \Psi^*(x,0)\Psi(x,0)dx = \frac{1}{a\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\left(\frac{x^2}{2a^2}\right)} dx = \frac{1}{a\sqrt{2\pi}} a\sqrt{2\pi} = 1, \tag{1.23}$$

where we used the fact that

$$\int_{\mathbb{R}} e^{-ax^2} = \sqrt{\frac{\pi}{a}}. \tag{1.24}$$

- (b) The wavefunction in momentum space at time $t = 0$, $\Phi(k, 0)$, is given by the Fourier transform of $\Psi(x, 0)$, i.e.

$$\begin{aligned}
\Phi(k,0) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Psi(x,0)e^{-ikx}dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi)^{1/4}\sqrt{a}} \int_{\mathbb{R}} e^{-\frac{x^2}{4a^2}-ikx}dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi)^{1/4}\sqrt{a}} 2a\sqrt{\pi}e^{-a^2k^2} \\
&= \sqrt{a}\left(\frac{2}{\pi}\right)^{1/4} e^{-a^2k^2}.
\end{aligned} \tag{1.25}$$

Remark 1.1. *We used the given integral but, to solve it, we can complete the square and go to the complex plane (verify this later).*

- (c) Assuming the particle is free we have

$$\Phi(x,t) = \sqrt{a}\left(\frac{2}{\pi}\right)^{1/4} e^{-a^2k^2} e^{-ia^2k^2\frac{t}{\tau}}, \tag{1.26}$$

which we can obtain by solving Schrödinger's equation in momentum space for a free particle (an ODE) and where $\tau = 2ma^2/\hbar$. Doing the Fourier transform again, we have

$$\begin{aligned}
\Psi_a(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi(x,t)e^{ikx}dk \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{a}\left(\frac{2}{\pi}\right)^{1/4} \int_{\mathbb{R}} e^{-a^2k^2}e^{-ia^2k^2\frac{t}{\tau}}e^{ikx}dk \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{a}\left(\frac{2}{\pi}\right)^{1/4} \int_{\mathbb{R}} e^{-(1+it/\tau)a^2k^2+ikx}dk \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{a}\left(\frac{2}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a^2(1+it/\tau)}}\exp\left(\frac{-x^2}{4a^2(1+it/\tau)}\right) \\
&= \left(\frac{a^2}{2\pi}\right)^{1/4} \sqrt{\frac{1}{a^2(1+it/\tau)}}\exp\left(\frac{-x^2}{4a^2(1+it/\tau)}\right).
\end{aligned} \tag{1.27}$$

- (d) We have

$$
\begin{aligned}
|\Psi_a(x,t)|^2 &= \left(\frac{a^2}{2\pi}\right)^{1/2} \frac{1}{a^2\sqrt{(1+(t/\tau)^2)}} \int_{\mathbb{R}} \exp\left(\frac{-x^2(1+(t/\tau)) - x^2(1-(t/\tau))}{4a^2(1+(t/\tau)^2)}\right) dx \\
&= \left(\frac{a^2}{2\pi}\right)^{1/2} \frac{1}{a^2\sqrt{(1+(t/\tau)^2)}} \int_{\mathbb{R}} \exp\left(\frac{-x^2}{2a^2(1+(t/\tau)^2)}\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{a\sqrt{(1+(t/\tau)^2)}} \int_{\mathbb{R}} \exp\left(\frac{-x^2}{2a^2(1+(t/\tau)^2)}\right) dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{a(t)} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2a^2(t)}\right) dx \equiv G(x;a(t)),
\end{aligned}
\tag{1.28}
$$

where

$$
a(t) = a\sqrt{1+(t/\tau)^2}. \tag{1.29}
$$

Exercise 4

- (a) In 1D, using Fourier's transform we compute

$$
\begin{aligned}
\int_{\mathbb{R}} |\Phi(p)|^2 dp &= \frac{1}{\hbar}\int_{\mathbb{R}} \left[\frac{1}{2\pi}\left(\int_{\mathbb{R}} \Psi^*(x') e^{i\frac{p}{\hbar}x'} dx'\right)\left(\int_{\mathbb{R}} \Psi(x) e^{-i\frac{p}{\hbar}x} dx\right)\right] dp \\
&= \frac{1}{\hbar}\int_{\mathbb{R}} \Psi(x) \left(\int_{\mathbb{R}} \Psi^*(x') \underbrace{\frac{1}{2\pi}\left(\int_{\mathbb{R}} e^{i\frac{p}{\hbar}(x'-x)} dp\right)}_{\hbar\delta(x'-x)} dx'\right) dx \\
&= \frac{1}{\hbar}\int_{\mathbb{R}} \Psi(x)\hbar\Psi^*(x) dx \\
&= \int_{\mathbb{R}} |\Psi(x)|^2 dx,
\end{aligned}
\tag{1.30}
$$

where we used the change of variable $p' = p/\hbar$ to compute the Dirac delta. In 3D, using Fourier's transform we compute

$$
\begin{aligned}
\int_{\mathbb{R}^3} |\Phi(\vec{p})|^2 d^3p &= \int_{\mathbb{R}^3} \left[\frac{1}{(2\pi\hbar)^3}\left(\int_{\mathbb{R}^3} \Psi^*(\vec{x}') e^{i\frac{\vec{p}}{\hbar}\cdot\vec{x}'} d^3x'\right)\left(\int_{\mathbb{R}^3} \Psi(\vec{x}) e^{-i\frac{\vec{p}}{\hbar}\cdot\vec{x}} d^3x\right)\right] d^3p \\
&= \frac{1}{\hbar^3}\int_{\mathbb{R}^3} \Psi(\vec{x}) \left(\int_{\mathbb{R}^3} \Psi^*(\vec{x}') \underbrace{\frac{1}{(2\pi)^3}\left(\int_{\mathbb{R}^3} e^{i\frac{\vec{p}}{\hbar}(\vec{x}'-\vec{x})} d^3p\right)}_{\hbar^3\delta^3(\vec{x}'-\vec{x})} dx'\right) d^3x \\
&= \int_{\mathbb{R}^3} |\Psi(\vec{x})|^2 d^3x,
\end{aligned}
\tag{1.31}
$$

where we used the change of variable $\vec{p}^{\,'} = \vec{p}/\hbar$, such that $d^3p = \hbar^3 d^3p'$, to compute the Dirac delta.

- (b) The ground state wavefunction of the hydrogen atom is given by

$$
\Psi(\vec{x}) = Ne^{-r/a_0}, \tag{1.32}
$$

where $r = |\vec{x}|$, a_0 is the Bohr radius and N is a normalization constant. To find N we solve

$$
\int_{\mathbb{R}^3} \Psi^2(\vec{x})\Psi(\vec{x}) d^3x = 1, \tag{1.33}
$$

for N. Thus, we have

$$\int_{\mathbb{R}^3} \Psi^*(\vec{x})\Psi(\vec{x})d^3x = 1, \tag{1.34}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty} |N|^2 e^{-2r/a_0} r^2 \sin(\theta) dr d\theta d\varphi = 4\pi \int_0^{+\infty} r^2 |N|^2 e^{-2r/a_0} dr = 1. \tag{1.35}$$

Integrating by parts twice we obtain

$$\int_0^{+\infty} r^2 e^{-2r/a_0} = \frac{a_0^3}{4}. \tag{1.36}$$

using the fact that the exponential term decays much faster than quadratic and linear terms in the limits. So, we obtain

$$4\pi |N|^2 \frac{a_0^3}{4} = 1, \tag{1.37}$$

which means that $N = 1/\sqrt{\pi a_0^3}$, taking the real positive normalization. To find N' we compute

$$\int_{\mathbb{R}^3} \Phi^*(\vec{p})\Phi(\vec{p})d^3p = 1, \tag{1.38}$$

where $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$ is the radial variable, and leads to

$$4\pi |N'|^2 \int_0^{+\infty} \frac{p^2}{\left(1 + \frac{a_0^2 p^2}{\hbar^2}\right)^4} dp = 1. \tag{1.39}$$

Changing variable to $p' = a_0 p/\hbar$ we get

$$4\pi |N'|^2 \frac{\hbar^3}{a_0^3} \int_0^{+\infty} \frac{p'^2}{(1 + p'^2)^4} dp' = 1. \tag{1.40}$$

The rational integral gives

$$\int_0^{+\infty} \frac{p'^2}{(1 + p'^2)^4} dp' = \frac{\pi}{32}, \tag{1.41}$$

so we conclude that

$$|N'|^2 = \frac{8a_0^3}{\pi^2 \hbar^3}, \tag{1.42}$$

and therefore $N' = 2\frac{\sqrt{2}a_0^{3/2}}{\pi \hbar^{3/2}}$, taking the real positive normalization.

The probability that the electron may be found with momentum $|\vec{p}| = p \geq \hbar/a_0$ is given by

$$\begin{aligned} P(p > \hbar/a_0) &= 4\pi |N'|^2 \frac{\hbar^3}{a_0^3} \int_1^{+\infty} \frac{p'^2}{(1 + p'^2)^4} dp' \\ &= \frac{32}{\pi} \int_1^{+\infty} \frac{p'^2}{(1 + p'^2)^4} dp' \\ &\approx 0.2878, \end{aligned} \tag{1.43}$$

using Mathematica. Thus, the probability is around 28.8%.

Exercise 5

Consider the Hamiltonian

$$H = \frac{p^2}{2m} + V(x). \tag{1.44}$$

We want to compute $d\langle x\rangle/dt$ and $d\langle p\rangle/dt$ and show they satisfy the given relations (Ehrenfest Theorem). We have

$$\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{d}{dt}\left(\int_{\mathbb{R}} \Psi^*(x,t)x\Psi(x,t)dx\right) \\
&= \int_{\mathbb{R}} \left(\Psi_t^*(x,t)x\Psi(x,t) + \Psi^*(x,t)x\Psi_t(x,t)\right)dx \\
&= \frac{i}{\hbar}\int_{\mathbb{R}} (\hat{H}\Psi)^* x\Psi - \Psi^* x\hat{H}\Psi dx \\
&= \frac{i}{\hbar}\int_{\mathbb{R}} \Psi^*\hat{H}x\Psi - \Psi^* x\hat{H}\Psi dx \\
&= \frac{i}{\hbar}\int_{\mathbb{R}} \Psi^*[\hat{H},x]\Psi dx.
\end{aligned} \tag{1.45}$$

Where we used the Hermiticity of $\hat{H}$. Now, observe that

$$[\hat{H},x] = [\hat{p}^2/2m + V(x), x] = \frac{1}{2m}[\hat{p}^2, x] = -\frac{i\hbar\hat{p}}{m}, \tag{1.46}$$

since x and $V(x)$ commute and using the commutation relations we computed in problem set 3. Therefore

$$\frac{d\langle x\rangle}{dt} = \frac{1}{m}\int_{\mathbb{R}} \Psi^*\hat{p}\Psi dx = \frac{1}{m}\langle\hat{p}\rangle. \tag{1.47}$$

Alternatively, we have

$$\begin{aligned}
\frac{d\langle x\rangle}{dt} &= \frac{d}{dt}\left(\int_{\mathbb{R}} \Psi^*(x,t)x\Psi(x,t)dx\right) \\
&= \int_{\mathbb{R}} \left(\Psi_t^*(x,t)x\Psi(x,t) + \Psi^*(x,t)x\Psi_t(x,t)\right)dx \\
&= \int_{\mathbb{R}} \left[\left(-i\frac{\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + iV\Psi^*\right)x\Psi(x,t) + \Psi^*(x,t)x\left(i\frac{\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - iV\Psi\right)\right]dx \\
&= \int_{\mathbb{R}} \left[\left(-i\frac{\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\right)x\Psi(x,t) + \Psi^*(x,t)x\left(i\frac{\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2}\right)\right]dx \\
&= \frac{i\hbar}{2m}\int_{\mathbb{R}} \left[\Psi^*(x,t)x\left(\frac{\partial^2\Psi}{\partial x^2}\right) - \left(\frac{\partial^2\Psi^*}{\partial x^2}\right)x\Psi(x,t)\right]dx \\
&= \ldots
\end{aligned} \tag{1.48}$$

We could now use integration by parts but this form is way more time consuming. We now show the second relation, we have

$$\begin{aligned}
\frac{d}{dt}\langle\hat{p}\rangle &= \frac{d}{dt}\left(\int_{\mathbb{R}} \Psi^*\hat{p}\Psi dx\right) \\
&= \int_{\mathbb{R}} \left(\Psi_t^*\hat{p}\Psi + \Psi^*\frac{\partial\hat{p}}{\partial t}\Psi + \Psi^*\hat{p}\Psi_t\right)dx \\
&= \left\langle\frac{\partial\hat{p}}{\partial t}\right\rangle + \frac{i}{\hbar}\int_{\mathbb{R}} \Psi^*\hat{H}\hat{p}\Psi - \Psi^*\hat{p}\hat{H}\Psi dx \\
&= \frac{i}{\hbar}\int_{\mathbb{R}} \Psi^*[\hat{H},\hat{p}]\Psi dx,
\end{aligned} \tag{1.49}$$

where the time derivative of the momentum operator vanishes since operators in Schrödinger's picture are time-independent.

$$\begin{aligned}\frac{d}{dt}\langle\hat{p}\rangle &= \int_{\mathbb{R}}\Psi^*(V(x)\hat{p}-\hat{p}V(x))\Psi dx \\ &= \frac{i}{\hbar}\int_{\mathbb{R}}\Psi^*\left(\frac{\hbar}{i}V(x)\frac{\partial\Psi}{\partial x}-\frac{\hbar}{i}\frac{\partial V}{\partial x}\Psi-\frac{\hbar}{i}V(x)\frac{\partial\Psi}{\partial x}\right)dx \\ &= -\left\langle\frac{\partial V}{\partial x}\right\rangle.\end{aligned} \tag{1.50}$$

Exercise 6

We want to show that

$$\frac{d}{dt}\Delta p(t)=0, \tag{1.51}$$

for a free particle wave packet. We have then

$$\begin{aligned}\frac{d}{dt}\Delta p(t) &= \frac{d}{dt}\left[\int_{\mathbb{R}}\Psi^*(x,t)\left(\hat{p}-\langle\hat{p}\rangle\right)^2\Psi(x,t)dx\right] \\ &= \frac{d}{dt}\left[\int_{\mathbb{R}}\Psi^*\hat{p}^2\Psi dx-2\langle\hat{p}\rangle\int_{\mathbb{R}}\Psi^*\hat{p}\Psi dx+\langle\hat{p}\rangle^2\int_{\mathbb{R}}\Psi^*\Psi dx\right].\end{aligned} \tag{1.52}$$

We now have

$$\begin{aligned}\frac{d}{dt}\Delta p(t) &= \int_{\mathbb{R}}\left(\Psi_t^*\hat{p}^2\Psi+\Psi^*\hat{p}^2\Psi_t\right)dx-2\langle\hat{p}\rangle\left[\int_{\mathbb{R}}\left(\Psi_t^*\hat{p}\Psi dx-\Psi^*\hat{p}\Psi_t\right)dx\right] \\ &\quad+\frac{d\langle\hat{p}\rangle^2}{dt}\int_{\mathbb{R}}\Psi^*\Psi dx+\langle\hat{p}\rangle^2\frac{d}{dt}\left(\int_{\mathbb{R}}\Psi^*\Psi dx\right) \\ &= \frac{i}{\hbar}\int_{\mathbb{R}}\left(\Psi^*\hat{H}\hat{p}^2\Psi-\Psi^*\hat{p}^2\hat{H}\Psi\right)dx+\frac{2i}{\hbar}\langle\hat{p}\rangle\int_{\mathbb{R}}\left(\Psi^*\hat{p}\hat{H}\Psi-\Psi^*\hat{H}\hat{p}\Psi\right) \\ &\quad+2\langle\hat{p}\rangle\frac{d\hat{p}}{dt} \\ &= \frac{i}{\hbar}\left[\langle[\hat{H},\hat{p}^2]\rangle+2\langle\hat{p}\rangle\langle[\hat{p},\hat{H}]\rangle+2\langle\hat{p}\rangle\langle[\hat{H},\hat{p}]\rangle\right] \\ &= \frac{i}{\hbar}\left[\langle[\hat{H},\hat{p}^2]\rangle-2\langle\hat{p}\rangle\langle[\hat{H},\hat{p}]\rangle+2\langle\hat{p}\rangle\langle[\hat{H},\hat{p}]\rangle\right] \\ &= \frac{i}{\hbar}\langle[\hat{H},\hat{p}^2]\rangle=0\end{aligned} \tag{1.53}$$

were we used the fact that the wavefunction is normalized and that $[\hat{H},\hat{p}^2]=0$, since we are considering a free particle.

Exercise 7

We are given $\Psi_0(x)$, normalized, and satisfying the following

$$\begin{cases}\langle x\rangle_{\Psi_0}=x_0, \\ \langle p\rangle_{\Psi_0}=p_0.\end{cases} \tag{1.54}$$

Define the boost operator $\hat{B}_q$, depending on $q\in\mathbb{R}$ and acting on functions of the position x, to be given by

$$\hat{B}_q f(x)=e^{iqx/\hbar}f(x). \tag{1.55}$$

We consider a new wavefunction

$$\Psi_{\text{new}} = \hat{B}_q \Psi_0(x). \tag{1.56}$$

- (a)

$$\langle \hat{x} \rangle_{\Psi_{\text{new}}} = \int_{\mathbb{R}} e^{-i\frac{qx}{\hbar}} \Psi_0^* x e^{i\frac{qx}{\hbar}} \Psi_0 dx = \langle x \rangle_{\Psi_0} = x_0. \tag{1.57}$$

- (b)

$$\begin{aligned} \langle \hat{p} \rangle_{\Psi_{\text{new}}} &= \int_{\mathbb{R}} e^{-i\frac{qx}{\hbar}} \Psi_0^* \frac{\hbar}{i} \frac{\partial}{\partial x} \left(e^{i\frac{qx}{\hbar}} \Psi_0 \right) dx \\ &= \frac{\hbar}{i} \int_{\mathbb{R}} e^{-i\frac{qx}{\hbar}} \Psi_0^* \left(\frac{iq}{\hbar} e^{i\frac{qx}{\hbar}} \Psi_0 + e^{i\frac{qx}{\hbar}} \frac{\partial \Psi_0}{\partial x} \right) dx \\ &= q + \langle \hat{p} \rangle_{\Psi_0} \\ &= q + p_0. \end{aligned} \tag{1.58}$$

- (c) The phase factor adds a momentum boost q to the expected momentum of the initial wavefunction.

- (d) We compute

$$\begin{aligned} [\hat{p}, \hat{B}_q] f &= \frac{\hbar}{i} \left[\frac{\partial}{\partial x} \left(e^{iqx/\hbar} f(x) \right) - \left(e^{iqx/\hbar} \frac{\partial f}{\partial x} \right) \right] \\ &= q e^{iqx/\hbar} f(x) \\ &= q \hat{B}_q f. \end{aligned} \tag{1.59}$$

  And, similarly

$$[\hat{x}, \hat{B}_q] f = x e^{iqx/\hbar} f(x) - e^{iqx/\hbar} x f(x) = 0. \tag{1.60}$$