Introduction to Geometric Quantization and Complex Hamiltonian Flows

Research Project Course - 2^{nd} Semester 19/20
Professor: João Pimentel Nunes[1]

Telmo Cunha[2] (73487)

Master's in Mathematics and Applications - Instituto Superior Técnico

Contents

---

[1] https://www.math.tecnico.ulisboa.pt/∼jpnunes/
[2] Email: telmocunha@gmail.com

1. Introduction

This project arises within the context of a research project course, part of the masters in mathematics and applications at Técnico Lisboa (IST). It serves as an introduction to the main ideas necessary for a prospective master's thesis on the topics of geometric quantization and complex Hamiltonian flows.

The current version consists mostly of independent study supervised by Professor João Pimentel Nunes, to whom I am very grateful to for the support, the several meetings through Skype and the various revisions of this document.

The topics covered in this work are mainly on symplectic geometry, geometric quantization and complex Hamiltonian flows. Although we begin with differential geometry, the exposition is very personal in the sense that it shows no particular structure and is comprised of concepts which were, at an initial stage, poorly understood by the author. In no way does it contain all the necessary background in differential geometry to understand the remaining topics.

Briefly, in each chapter we cover the following:

- **(2) Differential Geometry:** We begin by recalling the notions of **immersion, submersion, submanifolds**, **foliations**, **distributions and the Frobenius theorem**. These topics will be of particular importance when considering polarizations and the half-form correction; We then cover the basics of **de Rham Cohomology** which will be related to conditions that guarantee that a certain curvature form will be of the necessary type to obtain a prequantization map on a general symplectic manifold; We end this section on **Fiber bundles**, surely the most important section on the differential geometry chapter for our purposes. In particular, we will have to consider complex line bundles on symplectic manifolds to perform the Kostant-Souriau prequantization and later the canonical bundle, and a few related others, when considering the half-form correction. The main **references** used for this chapter were the notes by **Rui Loja Fernandes** [**17**] and the book by **Leonor Godinho and José Natário** [**12**].

- **(3) Symplectic Geometry:** In this chapter we begin by introducing the **symplectic structure on vector spaces** and how it extends to the case of a manifold structure, **symplectic manifolds**; We look at the pivotal **Darboux's theorem**, relying on **Moser's relative theorem**, which shows that dimension is the only local invariant on symplectic manifolds; We then cover **almost complex structures**, in particular the **Dolbeault splitting** on the complexified tangent and cotangent bundles, allowing us to define **forms of type** $(\mathbf{l}, \mathbf{m})$; We will then see how this splitting is reflected in the space of forms in the case of **complex manifolds** and particularly in the case when we have a triple of compatible structures, symplectic, Riemannian and complex, i.e., **Kähler manifolds**; The main **references** used for this chapter were the books by **Ana Cannas da Silva** [**3**], **Dusa McDuff and Dietmar Salamon** [**6**] and the master's thesis by **Miguel Pereira** [**14**].

- **(4) Hamiltonian Mechanics:** This chapter is a brief look at Hamiltonian mechanics where we begin by defining the concepts of **symplectic and Hamiltonian vector fields** and their relation; Next we take a look at the **Poisson and Lie brackets** and how they link through the symplectic form; We then mention **integrable systems**, in particular the notion of involution which appears explicitly on Dirac's quantization axioms; To finalize we take a look at **variational principles** in particular at how we obtain the **Euler-Lagrange equations** which give us the solutions to mechanical systems; The main **references** here were again the books [**3**] and [**12**].

- **(5) Geometric Quantization:** On the main topic of geometric quantization we begin by reviewing the **mathematical models of classical and quantum mechanics** which are mandatory to understand since we are essentially trying to turn one into the other; Next we state the **problem of quantization**, in particular the axioms defined by Dirac of what we should expect of a quantized classical system. It turns out, due to Van Hove's theorem, that a general solution to the problem of quantization does not exist; This leads us to look at less restricting conditions and the concept of **prequantization** procedures, where we will mainly focus on the Kostant-Souriau prequantization. However, this is not yet the full answer. Due to the lack of square-integrable sections it is necessary to introduce the notion of **polarizations** and the **half-form correction**. Finally, we look at how different polarizations might be connected by the idea of **pairing maps**. The main **references** consulted here were the reviews by **Alexandre Aleksandrovich Kirillov** [**2**], **A. Echeverria-Enriquez, M.C. Munoz-Lecanda, N. Roman-Roy, C. Victoria-Monge** [**1**], the textbook by **Brian C. Hall**, the paper and notes by **João Pimentel Nunes** [**10**],[**11**] and the master's thesis by **Gonçalo Oliveira** [**7**].

- **(6) Complex Hamiltonian flows:** Under development...

This barely scratches the surface of what is involved in any of these topics but hopefully it makes for a readable, slightly self-contained, introduction to the subject. It is also very much a work in progress with no shortage of concepts and results to expand on every section.

2. Differential Geometry

This chapter on differential geometry is merely a collection of basic definitions and results which were found to be necessary to understand topics in symplectic geometry and geometric quantization. We include them here only for completeness and ease of access and they are not exhaustive by any stretch of the imagination. Knowledge of fiber bundles and foliations is particularly important and should not be skipped if one has not seen it before. These concepts appear immediately when considering complex line bundles, for the Kostant-Souriau prequantization, and when choosing a polarization on the prequantum Hilbert space.

2.1. Immersions, Submersions and Submanifolds

We are interested here on the local behaviour of differentiable maps between smooth manifolds, particularly in the case where the maps are between manifolds of different dimensions. In the very important case were the dimensions do match, the maps are given as local diffeomorphisms (see further discussion of this topics in [12] chapter 1, section 5, p.24 or in [17] lecture 5, p.35).

Definition 2.1. ***Immersions and Submersions***
Let $\Psi : M \to N$ be a smooth map between smooth manifolds:
*(a) When dim $M \leq$ dim N, Ψ is called an **immersion** if $d_p\Psi : T_pM \to T_{\Psi(p)}N$ is injective, for all $p \in M$;*
*(b) When dim $M \geq$ dim N, Ψ is called a **submersion** if $d_p\Psi : T_pM \to T_{\Psi(p)}N$ is surjective, for all $p \in M$;*

The following theorem allows one to construct local canonical coordinates for immersions and submersions.

Theorem 2.2. ***Constant Rank Theorem***
Let $\Psi : M \to N$ be a smooth map and $p \in M$. If for all q in a neighbourhood of p the map $d_q\Psi : T_qM \to T_{\Psi(q)}N$ has constant rank r, then there are local coordinates $(\mathcal{U}, \phi) = (\mathcal{U}, x^1, ..., x^m)$ for M, centered at p, and local coordinates $(\mathcal{V}, \psi) = (\mathcal{V}, y^1, ..., y^n)$ for N, centered at $\Psi(p)$, such that:

$$\psi \circ \Psi \circ \phi^{-1}(x^1, ..., x^m) = (x^1, ..., x^r, 0, ..., 0).$$

Proof. The proof can be seen in [17], p.36 theorem 5.2. □

Corollary 2.3. ***Immersions - Local canonical coordinates***
*Let $f : M \to N$ be an **immersion**. For all $p \in M$ there exists local coordinates $(\mathcal{U}, \phi) = (\mathcal{U}, x^1, ..., x^m)$ for M, centered at p and local coordinates $(\mathcal{V}, \psi) = (\mathcal{V}, y^1, ..., y^n)$ for N, centered at $f(p)$, such that:*

$$\psi \circ f \circ \phi^{-1}(x^1, ..., x^m) = (x^1, ..., x^m, 0, ..., 0).$$

An immersion looks locally like the inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ where we necessarily have $m \leq n$.

Corollary 2.4. ***Submersions - Local canonical coordinates***
*Let $f : M \to N$ be a **submersion**. For all $p \in M$ there exists local coordinates $(\mathcal{U}, \phi) = (\mathcal{U}, x^1, ..., x^m)$ for M, centered at p, and local coordinates $(\mathcal{V}, \psi) = (\mathcal{V}, y^1, ..., y^n)$ for N, centered at $f(p)$, such that:*

$$\psi \circ f \circ \phi^{-1}(x^1, ..., x^m) = (x^1, ..., x^n).$$

A submersion looks locally like the projection $\mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ where we necessarily have $m \geq n$.

Definition 2.5. ***Submanifolds***
*A **submanifold** of a manifold M is a pair (N, Φ) where N is a manifold and $\Phi : N \to M$ is an injective immersion. When Φ is an homeomorphism onto its image, where on $\Phi(N)$ one takes the subspace topology, we call the pair (N, Φ) an **embedded submanifold** and Φ an **embedding**.*

Remark 1. In general it is not possible to consider submanifolds of a manifold M simply as subsets of M (see further discussion in [17], p.39, particularly example 5.10).

Definition 2.6. ***Regular Point and Regular Value***
*Let $f : M \to N$ be a smooth map between smooth manifolds of dimension m and n respectively. Given $p \in M$, if $(df)_p$ is surjective we call p a **regular point** of f. To a point $q \in N$ such that all points in $f^{-1}(q)$ are regular points, we call a **regular value** of f.*

Theorem 2.7. ***Submanifold by a Regular Value***
Consider the smooth map $f : M \to N$ and let q be a regular value of f. If $L := f^{-1}(q)$ is nonempty, then L is an embedded submanifold of M and $T_pL = ker\ (df)_p \subset T_pM$, for all $p \in M$.

Proof. Ref. [12] theorem 5.6, page 24. □

We end this section with an important theorem in differential topology by Whitney. This is not easily provable but we show a weaker result in the case of compact manifolds, the proof can be seen in appendix (7.1), where one can also recall the notion of partitions of unity.

Theorem 2.8. ***Whitney's Embedding Theorem***
Any smooth manifold M of dimension n can be embedded in $\mathbb{R}^{2n}$, therefore M is diffeomorphic to a submanifold of $\mathbb{R}^{2n}$.

2.2. Foliations

We are able to partition, or equivalently, define an equivalence relation on smooth manifolds which allows us to consider a manifold as the union of disjoint submanifolds. This is precisely the notion which we define next.

Definition 2.9. ***Foliation***
*Let M be a d dimensional manifold. A **foliation** of dimension k of the manifold M is a family $\mathcal{F} = \{L_\alpha\}_{\alpha \in A}$ of subsets of M (called **leaves**) satisfying the following properties:*

- *$M = \cup_{\alpha \in A} L_\alpha$;*
- *$L_\alpha \cap L_\beta = \emptyset$, if $\alpha \neq \beta$, i.e., we have a partition of M or equivalently an equivalence relation on M where the leaves are the equivalence classes;*
- *Each leaf L_α is **pathwise connected**, i.e., there is a continuous curve $\gamma : [0, 1] \to L_\alpha$ such that given any two points p, q in L_α we have $\gamma(0) = p$ and $\gamma(1) = q$;*
- *$\forall p \in M$, $\exists$ a smooth chart $\phi = (x^1, ..., x^k, y^1, ..., y^{d-k}) : \mathcal{U} \to \mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^{d-k}$, such that the connected components of $L_\alpha \cap \mathcal{U}$ are the sets:*

$$\{p \in \mathcal{U} : y^1(p) = const., ..., y^{d-k}(p) = const.\}.$$

Example 2.10. *Consider $\mathbb{R}^2$ and take a foliation by straight lines with a fixed slope $a \in \mathbb{R}$, i.e., $\Phi : \mathbb{R}^2 \to \mathbb{R}$, given by: $\Phi(x,y) = y - ax$. Considering now the Torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ we have an induced foliation with two possibilities. If $a \in \mathbb{Q}$ the leaves are closed curves, hence they are embedded submanifolds. On the other hand, if $a \notin \mathbb{Q}$ the leaves are dense in the torus and we only have immersed submanifolds.*

Proposition 2.11. *Let $\mathcal{F}$ be a k-dimensional foliation of a smooth manifold M. Every leaf $L \in \mathcal{F}$ is an immersed submanifold of dimension k.*

Proof. The proof can be seen [17], p.58 proposition 8.3. The proposition actually holds for a slightly better behaved notion of submanifolds called initial submanifolds, see also [17] definition 6.17 page 45. □

2.3. Distributions and Frobenius Theorem

Somewhat similarly to the concept of foliations, where we considered submanifolds of a manifold M, we can also consider subspaces of the tangent space at each point $p \in M$. These concepts are naturally related when the distribution is precisely the tangent bundle of a certain foliation.

Definition 2.12. *Distribution*
*Let M be a smooth manifold of dimension d and let $k \in \mathbb{N}$ with $1 \leq k \leq d$. A k-dimensional **distribution** D in M is a map such that $\forall p \in M$ we have:*

$$p \mapsto D_p \subset T_pM, \quad \text{where } D_p \text{ is a linear subspace of dimension } k.$$

If for any $p \in M$ there exists a neighbourhood $\mathcal{U}$ of p and smooth vector fields $X_1, ...X_k \in \mathfrak{X}(\mathcal{U})$ which spans D, i.e., such that:

$$D_q = \langle X_1|_q, ..., X_k|_q \rangle\,, \ \ \forall q \in \mathcal{U},$$

then we say that the distribution is smooth or that D is of class C^∞.

The set of **vector fields tangent to D** is naturally defined as

$$\mathfrak{X}(D) := \{X \in \mathfrak{X}(M) : X_p \in D_p, \forall p \in M\}.$$

Example 2.13. *A non-vanishing smooth vector field defines a 1-dimensional smooth distribution by:*

$$D_p := \langle X_p \rangle = \{\lambda X_p : \lambda \in \mathbb{R}\}$$

Given any $Y \in \mathfrak{X}(M)$, we have $Y \in \mathfrak{X}(D)$ if and only if $Y = fX$ for some uniquely defined $f \in C^\infty(M)$. This example shows that distributions are a way to generalize the concept of vector fields.

Example 2.14. *In general, a set of smooth linear independent vector fields $X_1, ..., X_k$ at each $p \in M$ defines a k-dimensional distribution by:*

$$D_p := \langle X_1|_p, ..., X_k|_p \rangle\,,$$

where $Y \in \mathfrak{X}(D)$ if and only if $Y = \sum_{i=1}^k f_i X_i$ for uniquely defined $f_i \in C^\infty(M)$.

Definition 2.15. ***Integral Manifold***
*Given a distribution D in M, we call a connected submanifold (N, Φ) of M an **integral manifold** of D if:*

$$d_p\Phi(T_pN) = D_{\Phi(p)}, \forall p \in N.$$

Definition 2.16. ***Integrable Smooth Distribution***
*A smooth m-dimensional distribution D on a smooth manifold M is said to be **integrable** if there exists an m-dimensional foliation $\mathcal{F} = \{L_\alpha\}_{\alpha \in A}$ on M such that:*

$$D_p = T_pL_p,$$

*for all $p \in M$. In this case the leaves of $\mathcal{F}$ are called **integral submanifolds** of the distribution D.*

Definition 2.17. ***Involutive Distribution***
*A smooth distribution D in M is said to be **involutive** if for any $X, Y \in \mathfrak{X}(D)$ one has $[X, Y] \in \mathfrak{X}(D)$.*

Theorem 2.18. ***Frobenius Theorem***
*A smooth distribution D is **integrable if and only if it is involutive**. In this case the integral foliation tangent to D is unique.*

Proof. The proof can be seen in [17], section 11, pp. 100-101. □

The concepts of foliations and distributions will be necessary when considering polarizations which are integrable distributions. They are also particularly useful when dealing with mechanical systems with constraints, see for example [12] section 5.4 on non-holonomic constraints, p.197.

2.4. de Rham Cohomology

Answering the question of whether closed forms are exact turns out to have deep topological significance. Here we turn our attention on to the basic definitions and results in de Rham cohomology. We will also state a few involved theorems, the Poincaré lemma and de Rham's theorem in particular, since these will be necessary to prove a result which guarantees the existence of the necessary structure for prequantization.

Definition 2.19. ***Closed and Exact forms***
Let $\omega \in \Omega^k(M)$, ω is called a:
*(i) **closed form** if $d\omega = 0$.*
*(ii) **exact form** if $\exists\, \eta \in \Omega^{k-1}(M)$ such that $\omega = d\eta$.*

The subspace of closed forms is represented as $Z_k(M)$ while the subspace of exact forms is represented as $B_k(M)$. That is, we have $\ker d = Z_k(M)$ while $\operatorname{Im} d = B_k(M)$. Clearly we have $B_k(M) \subset Z_k(M)$ since $d^2 = 0$.

Definition 2.20. ***de Rham Complex***
*To the pair $(\Omega(M), d)$ we call the **de Rham complex** of M which is often represented as the chain complex:*

$$0 \to \Omega^0(M) \overset{d}{\to} \Omega^1(M) \overset{d}{\to} \Omega^2(M) \overset{d}{\to} \Omega^3(M) \to \cdots$$

We can think of $(\Omega(M), d)$ as a set of differential equations associated to M. Thus, finding closed forms corresponds to solving the differential equation:

$$d\omega = 0.$$

The exact forms are then the trivial solutions to this equation which leads us to the next definition.

Definition 2.21. ***The de Rham cohomology space of order k***
The space of all solutions modulus the trivial solutions, of the equation $d\omega = 0$ on $(\Omega(M), d)$, is the vector space:

$$H^k(M) \equiv Z^k(M)/B^k(M),$$

called the ***de Rham cohomology space of order k.***

We now present several results on de Rham cohomology in the form of a solved exercise, in particular we show that:

- $H_k(M)$ has a vector space structure;
- the pull-back by a smooth map $f : N \to M$ induces a linear map on cohomology, $f^{\#} : H_k(M) \to H_k(N)$;
- the dimension of $H^0(M)$ is equal to the number of connected components of M;

Exercise (from [12], exercise 3.8.(5) p.74):
A k-form ω is called closed if $d\omega = 0$. If there exists a $(k-1)$-form β such that $\omega = d\beta$ then ω is called exact. Note that every exact form is closed. Let Z_k be the set of all closed k-forms on M and define a relation between forms on Z_k as follows: $\alpha \sim \beta$ if and only if they differ by an exact form, that is, if $\beta - \alpha = d\theta$ for some $(k-1)$-form θ.

(a) Show that this relation is an equivalence relation.

We need to check that the relation is **reflexive**, **symmetric** and **transitive**. Let $\alpha, \beta, \gamma \in Z_k$. The equivalence relation is reflexive by definition in the sense that $\alpha - \alpha = 0$ and 0 is an exact form (with 0 coefficients). If $\alpha \sim \beta$, we have $\alpha - \beta = d\theta$ and $\beta - \alpha = -d\theta = d(-\theta)$ and thus $\beta \sim \alpha$. If $\alpha \sim \beta$ and $\beta \sim \gamma$ we have $\alpha - \beta = d\theta$ and $\beta - \gamma = d\varphi$. Summing these equations we obtain $\alpha - \gamma = d(\theta + \varphi)$. We conclude then that $\sim$ is an equivalence relation on Z_k.

(b) Let $H_k(M)$ be the corresponding set of equivalence classes (called the k-dimensional de Rham cohomology space of M). Show that addition and scalar multiplication of forms define indeed a vector space structure on $H_k(M)$.

We are going to verify that the set $H_k(M)$ is closed under these operations. The remaining properties of a vector space are clear and follow from the fact that $\Omega^k(M)$ is itself a vector space. Consider the equivalence classes $[\omega]$, $[\alpha] \in H^k(M)$, $\lambda \in \mathbb{K}$ for some field $\mathbb{K}$ and the usual operations of sum and multiplication by scalars on forms.

Closure under addition: Let ξ and μ be two elements of $[\omega] + [\alpha]$. Let $\xi = \omega_1 + \alpha_1$ and $\mu = \omega_2 + \alpha_2$, where the ω_i, α_i for $i = 1, 2$, are arbitrary representatives within their respective classes. We have:

$$\xi - \mu = (\omega_1 + \alpha_1) - (\omega_2 + \alpha_2) = (\omega_1 - \omega_2) + (\alpha_1 - \alpha_2) = d\theta + d\beta = d(\theta + \beta)$$

which shows that $[\omega] + [\alpha] \equiv [\omega + \alpha] \in H^k(M)$.

Closure under scalar multiplication: Let ω_1, ω_2 be arbitrary representatives of $[\omega] \in H^k(M)$ and $\lambda \in \mathbb{K}$. We have $\omega_1 - \omega_2 = d\theta$. Consider now $\lambda[\omega] = \lambda\omega_i$ where $i = 1, 2$. We have:

$$\lambda\omega_1 - \lambda\omega_2 = \lambda(\omega_1 - \omega_2) = \lambda d\theta = d(\lambda\theta)$$

which shows that $\lambda[\omega] \equiv [\lambda\omega] \in H^k(M)$.

(c) Let $f : M \to N$ be a smooth map. Show that:

(i) the pull-back f^* carries closed forms to closed forms and exact forms to exact forms;

Let $\omega \in \Omega^k(N)$ be a closed form. Given $p \in M$ and $X_i \in \mathfrak{X}(M)$, the pull-back of ω by f is, by definition:

$$(f^*\omega)\Big|_p (X_1, ..., X_k) := \omega\Big|_{f(p)} ((df)_p X_1, ..., (df)_p X_k)$$

By the well know result of commutation between the pullback and exterior derivative we have:

$$d(f^*\omega)\Big|_p (X_1, ..., X_k) = d\omega\Big|_{f(p)} ((df)_p X_1, ..., (df)_p X_k) = f^*(d\omega) = 0,$$

since ω is a closed form ($d\omega = 0$).

From the previous result, in the case of an exact form $\xi \in \Omega^k(N)$, $\xi = d\alpha$, $\alpha \in \Omega^{k-1}(N)$, we obtain:

$$f^*(\xi) = f^*(d\alpha) = df^*(\alpha).$$

Since $f^*(\alpha)$ is a $(k-1)$-form on M, $f^*\xi$ is an exact form.

(ii) if $\alpha \sim \beta$ on N then $f^*\alpha \sim f^*\beta$ on M;

Since $\alpha \sim \beta$ on N, then $\alpha, \beta \in H^k(N)$ and $\alpha - \beta = d\theta$ where $\theta \in \Omega^{k-1}(N)$. Thus, $f^*\alpha - f^*\beta = f^*(\alpha - \beta)$, and by the previous result $f^*(\alpha - \beta) = f^*(d\theta) = df^*(\theta)$, where $f^*\theta \in \Omega^{k-1}(M)$.

Remark: $f^*\alpha - f^*\beta = f^*(\alpha - \beta)$ is true since $f^*(\alpha + \beta) = (\alpha + \beta)((df)X_1, ..., (df)X_k)$ where $\alpha, \beta \in \Omega^k(N)$. Since $\Omega^k(N)$ is a vector space it implies that $(\alpha + \beta)((df)X_1, ..., (df)X_k) = \alpha((df)X_1, ..., (df)X_k) + \beta((df)X_1, ..., (df)X_k) = f^*\alpha(X_1, ..., X_k) + f^*\beta(X_1, ..., X_k)$.

(iii) f^* induces a linear map on cohomology $f^\# : H_k(N) \to H_k(M)$ naturally defined by $f^\#[\omega] = [f^*\omega]$;

This is an immediate consequence of (ii). Given any representative α_i of $[\omega] \in H^k(N)$ we know that $f^*\alpha_i \sim f^*\alpha_j$, where i, j identify any two forms in $[\omega]$, therefore $[f^*\omega] \in H^k(M)$. It is clear by the remark on (ii) that $f^\#$ is a linear map.

(iv) if $g : L \to M$ is another smooth map, then $(f \circ g)^{\#} = g^{\#} \circ f^{\#}$.

Since $(f \circ g) : L \to N$ we have $(f \circ g)^{\#} : H^k(N) \to H^k(L)$. The desired relation is directly obtained from the fact that $g^*(f^*\alpha) = (f \circ g)^*\alpha$. To obtain forms on L we must pull them back from N to M by the function f and then from M to L by g, therefore $(f \circ g)^{\#} = g^{\#} \circ f^{\#}$.

Rigorously, for any $p' \in L$ such that $g(p') \in M$ and $\omega \in \Omega^k(N)$, given vector fields $Y_i \in \mathfrak{X}(L)$ such that $(dg)Y_i \in \mathfrak{X}(M)$, the pull-back by $f^* : \Omega^k(N) \to \Omega^k(M)$ considering the origin of points and vector fields in L is given by:

$$(f^*\omega)\Big|_{g(p')}((dg)_{p'}Y_1, ..., (dg)_{p'}Y_k) = \omega\Big|_{f(g(p'))}\big((df)_{g(p')}((dg)_{p'}Y_1), ..., (df)_{g(p')}((dg)_{p'}Y_k)\big)\,.$$

Performing the pull-back by g, i.e., $g^* : \Omega^k(M) \to \Omega^k(L)$, on $p' \in L$, we end up with:

$$g^*(f^*\omega)\Big|_{p'}(Y_1, ..., Y_k) = f^*\omega\Big|_{g(p')}((dg)_{p'}Y_1, ..., (dg)_{p'}Y_k)\,,$$

Therefore, as a pullback from $\Omega^k(N)$ to $\Omega^k(L)$, we finally have:

$$g^*(f^*\omega)\Big|_{p'}(Y_1, ..., Y_k) = \omega\Big|_{f(g(p'))}\big((df)_{g(p')}((dg)_{p'}Y_1), ..., (df)_{g(p')}((dg)_{p'}Y_k)\big)\,.$$

which gives us the desired result:

$$g^*(f^*\omega)\Big|_{p'}(Y_1, ..., Y_k) = \omega\Big|_{(f\circ g)(p')}(d(f \circ g)_{p'}Y_1, ..., d(f \circ g)_{p'}Y_k)\,.$$

(d) Show that the dimension of $H^0(M)$ is equal to the number of connected components of M.

Let $M = \cup_{i=1}^{p} U_i$, where U_i are the connected components of M. We have no k-forms with $k < 0$, therefore:

$$H^0(M) = \{f \in C^\infty(M; \mathbb{K}) : df = 0\},$$

for any field $\mathbb{K}$. That is, since the set of forms of degree (-1) is the empty set $\emptyset$, it is vacuously true that elements in $H^0(M)$ all differ in pairs by elements of the set of (-1) forms.

Let $df = 0$ on U_i, with local coordinates $(x_1, ..., x_k)$, we have:

$$df = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i} = 0 \;\Rightarrow\; \frac{\partial f}{\partial x_i} = 0,\;\; \forall i = 1, ..., k$$

which implies that f is constant on U_i. Since the function $f = 1$ on U_i is in $H^0(U_i)$, we have an isomorphism from $\mathbb{R}$ to constant functions on $H^0(U_i)$ given by $\lambda \to \lambda(1)$, $\forall \lambda \in \mathbb{R}$. Since f will in general have distinct constant values at each U_i, $i = 1, ..., p$ to identify it on M we will need an element of $\mathbb{R}^p$ which implies that $dim(H^0(M)) = p$, the number of connected components of M.

(e) Show that $H_k(M) = 0$ for every $k > dim(M)$.

There are no non-vanishing forms of degree $(k+1)$ or higher on a manifold M of dimension k, therefore $H_k(M) = 0$. This comes from the fact that differential forms are alternating tensors, i.e. $T(v_1, ..., v_i, ...v_j, ..., v_k) = -T(v_1, ..., v_j, ...v_i, ..., v_k)$. Since any form of degree higher than the dimension of the manifold would be given locally by some repeated basis element, the form would have to be zero.

We will now state the aforementioned theorems, Poincaré's lemma and a theorem by Georges de Rham. We will make use of them eventually in relation to the Kostant-Souriau prequantization (see theorem 5.10). We will not prove them, in the latter case the proof is fairly involved and requires quite a number of results in algebraic topology.

Theorem 2.22. ***Poincaré's lemma***
Let $\mathcal{U}$ be a contractible domain in $\mathbb{R}^n$ and let $k \in \mathbb{N}$. Then, for $\omega \in \Omega^k(\mathcal{U})$ such that $d\omega = 0$, there exists $\alpha \in \Omega^{k-1}(\mathcal{U})$ such that $\omega = d\alpha$. This implies that given any smooth manifold, locally all closed forms are exact.

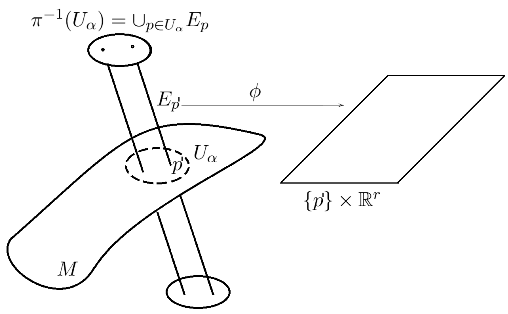
Theorem 2.23. ***de Rham's Theorem***
For any smooth manifold M, the integration map $I : H^k_{dR} \to H^k(M; \mathbb{R})$ is an isomorphism.

2.5. Fiber Bundles

2.5.1. Vector bundles

A vector bundle on a manifold M is a collection of vector spaces, $\{E_p\}_{p \in M}$, parameterized by points on M (called the **base space**) . The union of these vector spaces is itself a manifold E (called the **total space**) and is equipped with a smooth **projection** map $\pi : E \to M$ such that $\pi(E_p) = p$ and which admits a locally trivializing chart.

Formally this means that there exists an open cover of M, $M = \cup_{\alpha \in J} U_\alpha$ such that, for any U_α, the preimage $\pi^{-1}(U_\alpha)$ on E is diffeomorphic to $U_\alpha \times \mathbb{R}^n$. The diffeomorphism is given by: $\phi^p : E_p \xrightarrow{\phi} \{p\} \times \mathbb{R}^r \to \mathbb{R}^r$ such that given $\mathbf{v} \in E_p$ we have: $\phi(\mathbf{v}) = (p, \phi^p(\mathbf{v}))$.



Just as in the case of differentiable manifolds we would like that overlapping charts on M would induce the same vector space structure. This leads to the formal definition of vector bundles.

Definition 2.24. ***Vector bundle structure***
A ***vector bundle structure*** *of rank r on a differentiable manifold M is a triple $\xi(\pi, E, M)$, where $\pi : E \to M$ is a smooth map admitting a collection of trivializing charts $\mathcal{C} = \{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ of dimension r which satisfies the following properties:*

i) $\{U_\alpha : \alpha \in A\}$ is an open cover of M such that $M = \cup_{\alpha\in A} U_\alpha$;

ii) The charts are compatible, i.e., for any $\alpha, \beta \in A$, $\forall p \in U_\alpha \cap U_\beta$, the transition functions $g_{\alpha\beta}(p) \equiv \phi^p_\alpha \circ (\phi^p_\beta)^{-1} : \mathbb{R}^r \to \mathbb{R}^r$ are linear isomorphisms;

iii) The collection $\mathcal{C}$ is maximal: given any trivializing chart (U, ϕ) of dimension r satisfying conditions i) and ii), then $(U, \phi) \in \mathcal{C}$;

We call $\xi = (\pi, E, M)$ a ***vector bundle*** *of rank r.*

Definition 2.25. ***Sections***
Given $\xi = (\pi, E, M)$, a vector bundle of rank r, and an open set $U \subset M$, we call ***section*** *over U to a map $s : U \to E$ such that $\pi \circ s = id_U$. The sections over U form a real vector space denoted by $\Gamma_U(E)$.*

If *rank* ξ=r and we have a collection $s_1, ..., s_r$ such that, $\forall p \in U$ the sections $\{s_1(p), ..., s_r(p)\}$ form a basis for E_p, then we say we have a **frame** over U given by the collection $s_1, ..., s_r$.

Definition 2.26. ***Vector bundle morphisms***
Given two vector bundle structures $\xi_1 = (\pi_1, E_1, M_1)$ and $\xi_2 = (\pi_2, E_2, M_2)$, we say that there is a ***vector bundle morphism*** *between them if there is a smooth map $\Psi : E_1 \to E_2$ which maps the fibers of ξ_1 linearly with the fibers of ξ_2. The map Ψ covers a smooth map $\psi : M_1 \to M_2$:*

$$\begin{tikzcd} E_1 \arrow[r, "\Psi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"'] \\ M_1 \arrow[r, "\psi"] & M_2 \end{tikzcd}$$

such that given $p \in M_1$, the map of the fibers is a linear transformation given by:

$$\Psi^p \equiv \Psi_{(E_1)_p} : (E_1)_p \to (E_2)_{\psi(p)}$$

Example 2.27. ***Examples of vector bundles***

i) *For a smooth manifold M the familiar bundles TM, T^*M and $\Lambda^k(T^*M)$ are vector bundles on M. The sections of these bundles are the vector fields and differential forms on M. Given a smooth function $\Psi : M \to N$, the differential $d\Psi : TM \to TN$ is a vector bundle morphism.*

ii) *The* ***trivial vector bundle*** *of rank r over M is the vector bundle $\epsilon^r_M = (\pi, M \times \mathbb{R}^r, M)$ where $\pi : M \times \mathbb{R}^r \to M$ is the projection on the first factor. The sections are the smooth functions on M, $\Gamma(\epsilon^r_M) = C^\infty(M; \mathbb{R}^r)$.*

iii) *A vector bundle of rank 1 is usually referred to as a* ***line bundle****. A particular example is given by a non-vanishing vector field which defines a trivial line bundle.*

2.5.2. Connections on vector bundles

We want to differentiate sections of vector bundles so that we are able to compare fibers on different points of the base manifold. However, since there is no natural way of defining this differentiation, one needs to introduce some additional structure, in particular that of a connection on a vector bundle.

Definition 2.28. ***Connection on a vector bundle***
We define a ***connection on a vector bundle****,* $\xi = (\pi, E, M)$*, as a map:*

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s$$

which satisfies the following properties:

$$\begin{aligned}
&\textit{(i)}\ \nabla_{X_1+X_2} s = \nabla_{X_1} s + \nabla_{X_2} s;\\
&\textit{(ii)}\ \nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2;\\
&\textit{(iii)}\nabla_{fX} s = f\nabla_X s;\\
&\textit{(iv)}\ \nabla_x(fs) = f\nabla_X s + X(f)s.
\end{aligned}$$

Choosing a function $f \in C^\infty(M)$ with support only on an open subset $U \subset M$, by properties (iii) and (iv) we can restrict the connection to any open subset of M.

Since the map $X \mapsto \nabla_X$ is $C^\infty(M)$-linear, property (iii), for any section s defined on a neighbourhood U of $p \in M$, given any $\boldsymbol{v} \in T_pM$ we can define:

$$\nabla_{\boldsymbol{v}} s \equiv \nabla_X s(p) \in E_p,$$

where X is any vector field defined on a neighbourhood of p such that $X_p = \boldsymbol{v}$.

Given an open set $U \subset M$ and a local trivialization of the vector bundle ξ with a basis of sections given by $\{s_1, ..., s_r\}$ on $\xi|_U$. Then, any section s on $\xi|_U$ is written as a linear combination:

$$s = f_1 s_1 + ... + f_r s_r, \tag{2.1}$$

for unique smooth functions, $\{f_i\}_{i=1}^r \in C^\infty(U)$.

We can completely determine ∇ by specifying its action on sections. Given a vector field $X \in \mathfrak{X}(M)$, applying property (iv) to the general section on $\xi|_U$, given by equation (2.1), we obtain:

$$\nabla_X s = \sum_{i=1}^r \Big((X \cdot f_i)s_i + f_i \nabla_X s_i \Big)$$

Therefore, by considering coordinates $(x^1, ..., x^d)$ on U we have a basis $\{\frac{\partial}{\partial x^j}\}_{j=1}^d$ on the corresponding tangent space which allows us to write:

$$\nabla_{\frac{\partial}{\partial x^j}} s^i = \sum_{k=1}^r \Gamma_{ji}^k s_k, \quad (j = 1, ..., d;\ i = 1, ..., r),$$

where the Γ_{ij}^k are unique smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$, called the **Christoffel symbols**, which are dependent on the local coordinate system and basis of local sections. These can also be defined by **connection 1-forms** on U given as:

$$\omega_i^j = \sum_{k=1}^{r} \Gamma_{ki}^j dx^k$$

Example 2.29. *If we have a trivial vector bundle $\xi = (\pi, E, M)$ of rank r, i.e., with a global basis of sections $\{s_1, ..., s_r\}$, we can define a connection on ξ by considering:*

$$\nabla_X s_i = 0, \;\; i = 1, .., r.$$

This implies that the Christoffel symbols vanish. Note however that the connection is dependent on the choice of trivializing sections.

Proposition 2.30. *Every vector bundle $\xi = (\pi, E, M)$ admits a connection ∇.*

Proof. (Ref. [17] prop.26.3, p.224) Given an open covering $\{U_\alpha\}$ of M such that each U_α admits a trivialization, by the previous description on each U_α we can construct a connection ∇^α which is dependent on the local coordinates and local basis of sections. Since an affine combination of connections $f\nabla_1 + (1-f)\nabla_2$, for $f \in C^\infty(M)$, is also a connection on ξ, by the transition functions on the intersections of charts for both local coordinates and local basis of sections we are able to construct a global connection ∇ in ξ. In particular, by considering a partition of unity $\{\rho_\alpha\}$ (see definition (7.3)), subordinated to the open cover $\{U_\alpha\}$, we define: $\nabla \equiv \sum_\alpha \rho_\alpha \nabla^\alpha$. $\square$

As per our initially stated objective, let see how we can compare different fibers over a vector bundle.

Definition 2.31. *The **covariant derivative** of a section along a curve $\gamma : I \subset \mathbb{R} \to M$ is the section given by:*

$$\frac{Ds}{Dt} \equiv \nabla_{\frac{d}{dt}} s.$$

*If the covariant derivative vanishes, $\frac{Ds}{Dt} = 0$, we call s a **parallel section** along γ. Here, $\frac{d}{dt}$ represents a vector field $X \in \mathfrak{X}(M)$ such that $X_{\gamma(t)} = \dot{\gamma}(t)$.*

Considering a local chart $(U, x^1, ..., x^d)$ and a basis of local sections $\{s_1, ..., s_r\}$ on U, given $\gamma(t)$ let $\gamma^i(t) := x^i(\gamma(t))$. Then we write a section along γ as:

$$s(t) = \sum_{i=1}^{r} f^i(\gamma(t)) s_i(\gamma(t)).$$

Therefore, the covariant derivative along γ has components given by:

$$\left(\frac{Ds}{Dt}\right)^i = \left(\nabla_{\frac{d}{dt}} s\right)^i = \left(\frac{df(t)^i}{dt}\right) + \left(\sum_{j,k=1}^{d,r} f^k(t) \frac{dx^j(t)}{dt} \Gamma_{jk}^i(\gamma(t))\right) \tag{2.2}$$

Relying on the theorem of existence and uniqueness of solutions to linear ordinary differential equations we can state the following lemma:

Lemma 2.32. *Given any curve $\gamma : [0, 1] \to M$ and any $v_0 \in E_{\gamma(0)}$, there exists a unique parallel section s with initial condition $s(0) = v_0$. In this case, we say that the vectors $s(t) \in E_{\gamma(t)}$ are obtained by parallel transport along γ.*

2.5.3. Curvature on vector bundles

On example 2.29 of the previous section, we saw that trivial bundles carry a natural connection such that $\nabla_X s_i = 0$, after a choice of global trivializing sections s_i.

For a general vector bundle however, we only have local trivializing sections, therefore we can not choose a basis of local sections which would allow for the definition of $\nabla s_i = 0$ globally. This obstruction arises precisely because of the **curvature** of the connection.

Definition 2.33. ***Curvature of the connection on a Vector Bundle***
Given a connection ∇ on the vector bundle $\xi = (\pi, E, M))$, the curvature of ∇ is defined as the map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(\xi) \to \Gamma(\xi)$ given by:

$$(X, Y, s) \mapsto R(X, Y)s \equiv \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) s$$

Consider a trivializing open set $U \subset M$ for ξ, where we have local coordinates $(x^1, ..., x^d)$ and a local basis of sections $\{s_1, ..., s_r\}$. On the local coordinates of U, the curvature is given by:

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) s_z = \sum_{k=1}^{r} R^k_{ijz} s_k,$$

where the components of the **curvature tensor** R can be obtained as a function of the Christoffel symbols by:

$$R^k_{ijz} = \frac{\partial \Gamma^k_{jz}}{\partial x^i} - \frac{\partial \Gamma^k_{iz}}{\partial x^j} + \Gamma^l_{iz}\Gamma^k_{jl} - \Gamma^l_{jz}\Gamma^k_{il}$$

The curvature can also be specified as matrix of differential forms called the **curvature 2-form**, $\Omega = [\Omega^j_i]$, where we have:

$$\Omega^i_j = \sum_{l<p} R^j_{lpi} dx^l \wedge dx^p$$

The relation between the **connection 1-forms** and the **curvature 2-form** is given by the so called **structure equations**. This result is given by the following theorem:

Theorem 2.34. *Given a vector bundle $\xi = (\pi, E, M)$ with connection ∇, by admitting trivializing sections, the connection 1-forms and the curvature 2-form are related by the* ***structure equations*** *given as:*

$$\Omega^i_j = d\omega^j_i + \sum_k \omega^k_i \wedge \omega^j_k,$$

where ***Bianchi's identity*** *is verified:*

$$d\Omega^i_j = \sum_k d\omega^k_i \wedge \omega^j_k - \omega^k_i \wedge \Omega^j_k.$$

3. Symplectic Geometry

When considering geometric quantization, one is interested in obtaining a quantum system out of a classical one. The mathematical machinery of classical mechanics is precisely that of symplectic geometry so it is natural we should understand it first. We begin by the very basics of symplectic linear algebra turning quickly to symplectic manifolds and eventually Kähler manifolds, which will be our main object of interest later. The main references for this chapter were [3] and [6].

3.1. Symplectic Linear Algebra

Let V be a real vector space, and $\Omega : V \times V \to \mathbb{R}$ an anti-symmetric, bilinear form.

Definition 3.1. ***Non-degenerate Form***
The form Ω is non-degenerate if $\Omega(u, v) = 0 \ \forall \ v \in V \ \Leftrightarrow \ u = 0$.

Definition 3.2. ***Symplectic Form***
A symplectic form Ω on V is an anti-symmetric non-degenerate bilinear form.

Definition 3.3. ***Symplectic Vector Space***
A symplectic vector space is a pair (V, Ω) where Ω is a sympletic form on V.

Definition 3.4. ***Symplectomorphism between vector spaces***
Two symplectic vector spaces (V_1, Ω_1) and (V_2, Ω_2) are said to be symplectomorphic if there is a linear isomorphism $T : V_1 \to V_2$ such that $T^*\Omega_2 = \Omega_1$. The linear symplectomorphisms of (V, Ω) form a group denoted by $Sp(V, \Omega)$.

A symplectic form Ω defines an isomorphism: $\Omega^\# : V \to V^*$ given by $\Omega^\#(v) = \Omega(v, .)$ where non-degeneracy implies that $\Omega^\#$ is injective.

Let us a see a few examples of symplectic vector spaces.

Example 3.5. *$V = \mathbb{R}^{2n}$ with $\Omega(u, v) = \langle u, Jv\rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner-product and J is the $2n \times 2n$ matrix given by: $J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$.*

Example 3.6. *Given a vector space W such that $V = W \bigoplus W^*$ with $\Omega((w, \xi), (w', \xi')) = \xi'(w) - \xi(w')$.*

Example 3.7. *Given, (V_1, Ω_1) and (V_2, Ω_2), the symplectic vector space $(V_1 \times V_2, \Omega_1 \bigoplus \Omega_2)$ and symplectic form given by $\Omega_1 \bigoplus \Omega_2 = \pi_1^*\Omega_1 + \pi_2^*\Omega_2$*

Example 3.8. *A complex vector space V with an Hermitian inner product $h : V \times V \to \mathbb{C}$ and symplectic form $\Omega = Im(h)$, where $Im(\cdot)$ denotes the imaginary part.*

3.1.1. Subspaces of symplectic vector spaces

Defining the symplectic orthogonal as $W^\Omega = \{v \in V : \Omega(v, w) = 0\}$, we have the following definitions:

Definition 3.9. ***Subspaces of symplectic vector spaces,*** (V, ω)***,*** $W \subset V$

- *Symplectic if $W \cap W^\Omega = \{0\}$.*
- *Isotropic if $W \subset W^\Omega$.*
- *Coisotropic if $W^\Omega \subset W$.*
- *Lagrangian if $W = W^\Omega$.*

In general $V \neq W \bigoplus W^\Omega$ but, in any case, $dimV = dimW + dimW^\Omega$.

Theorem 3.10. ***Symplectic Basis***
Let (V, Ω) be a symplectic vector space of dimension $2n$. There exists a basis $(u_1, ..., u_n, v_1, ..., v_n)$ such that $\Omega(u_j, u_k) = \Omega(v_j, v_k) = 0$ and $\Omega(u_j, v_k) = \delta_{jk}$. Such a basis is called a symplectic basis. Moreover, there exists a vector space isomorphism $\Psi : \mathbb{R}^{2n} \to V$, such that $\Psi^*\Omega = \Omega_0$.

Proof. See [6], theorem 2.1.3, pp.39-40. $\square$

3.2. Symplectic Manifolds

We now transition to the construction of a symplectic structure on a manifold. We will assume that all manifolds are connected C^∞-smooth manifolds with no boundary unless otherwise stated.

Definition 3.11. ***Symplectic form on a manifold***
A ***symplectic form*** *ω on a manifold M is a* **closed nondegenerate 2-form** *at each $p \in M$.*

If ω is symplectic then T_pM must be even dimensional and therefore M must be even dimensional. This is a consequence of ω being an anti-symmetric non-degenerate bilinear form on a vector space.

Definition 3.12. ***Symplectic Manifold***
A ***symplectic manifold*** *(M, ω) is a smooth manifold M equipped with a symplectic form ω.*

Nondegeneracy implies that there is an isomorphism between the tangent and cotangent bundles, i.e, a map $TM \to T^*M$ defined by $X \mapsto \iota(X)\omega = \omega(X, \cdot)$. We also have that $\omega^n = \omega \wedge ... \wedge \omega \neq 0$ and thus the manifold M is oriented, i.e. ω^n is a volume form.

Closedness under the exterior derivative implies that ω represents a cohomology class, $[\omega] \in H^2(M; \mathbb{R})$.

Example 3.13. *The first clear example is the euclidean space $\mathbb{R}^{2n}$ with linear coordinates $(x_1, ...x_n, y_1, ..., y_n)$, equipped with the standard symplectic form:*

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

Example 3.14. *Another typical example is that of the sphere S^2. Considering $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$, the symplectic form is just the area form given by:*

$$\omega_x(u, v) := \langle x, u \times v \rangle ,$$

where $x \in S^2$ and $u, v \in T_xS^2$.

Definition 3.15. ***Symplectomorphism between symplectic manifolds***
Let (M_1, ω_1) and (M_2, ω_2) be two 2n-dimensional symplectic manifolds. To a diffeomorphism $\varphi : M_1 \to M_2$ satisfying:

$$\varphi^*\omega_2 = \omega_1,$$

we call a ***symplectomorphism****. Considering $M = M_1 = M_2$ and $\omega = \omega_1 = \omega_2$, the group of symplectomorphisms of (M, ω) is defined as:*

$$Symp(M, \omega) := \{\phi \in Diff(M) \mid \phi^*\omega = \omega\}.$$

A major result in symplectic geometry, due to Darboux, shows that, locally, all 2n-dimensional symplectic manifolds look the same, i.e., they are locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Theorem 3.16. ***Darboux Theorem***
Given any point $p \in M$, where (M, ω) is a 2n-dimensional symplectic manifold, there is a neighbourhood coordinate chart of p, $(\mathcal{U}, x_1, ..., x_n, y_1, ..., y_n)$ such that on $\mathcal{U}$ the symplectic form ω is locally given by:

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

This shows in particular that the only local invariant of a symplectic manifold is its dimension, we will prove this result later in section 3.3. Let us look now at a particular class of examples of symplectic manifolds, cotangent bundles.

3.2.1. Cotangent bundles as symplectic manifolds

Let N be any n-dimensional manifold and consider its cotangent bundle $M = T^*N$. Given a local chart on N, $(\mathcal{U}, x_1, ..., x_n)$ and $p \in \mathcal{U}$, any element $\xi \in T_p^*N$, can be written as $\xi = \sum_{i=1}^{n} \xi_i (dx_i)_p$. Therefore, we have a coordinate chart on T^*N given by $(T^*\mathcal{U}, x_1, ..., x_n, \xi_1, ..., \xi_n)$. Next, we will construct a canonical symplectic form on M as the exterior derivative of a coordinate-free 1-form α called the **tautological 1-form**.

Definition 3.17. ***Tautological 1-form on*** M
*Let N be any n-dimensional manifold and consider $\pi : T^*N \to N$, the natural projection. Given $p \in T^*N$, $p = (x, \xi)$, where $\xi \in T_x^*N$, we have $\pi(p) = x$. Thus, we can define the **tautological 1-form** α pointwise by:*

$$\alpha_p = (d\pi_p)^* \xi, \quad \alpha_p \in T_p^* M.$$

Let us explain this definition a bit more clearly. The idea is to consider the cotangent bundle M as a manifold on its own right. From the projection map we can consider the map between its tangent bundle on p, T_pM, and the tangent bundle on N at point x, $d\pi_p : T_pM \to T_xN$, thus, to obtain a 1-form on $p \in M$ we just need to evaluate the push-forward of a vector on T_pM on a 1-form in T_x^*N. Notation wise this is equivalent to writing:

$$\alpha_p(v) = \xi\Bigg((d\pi_p)v\Bigg), \quad \text{for } v \in T_pM.$$

Now, given a chart on N, $(\mathcal{U}, x_1, ..., x_n)$ with associated coordinates on the cotangent bundle $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ the 1-form α on M can be locally written on $T^*\mathcal{U}$ as $\alpha = \sum_{i=1}^{n} \xi_i dx_i$. Therefore, we can consider M with a symplectic structure by defining the symplectic form to be

$$\omega = -d\alpha,$$

which is locally given by $\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i$.

This results are particularly useful since cotangent bundles represent the phase space of classical systems, where N is the configuration space.

3.3. Moser and Darboux Theorems

Darboux's theorem is a crucial result which will be used often. This section contains a short proof of it which relies on the stronger result by Moser, we encourage the reader to read **Part 3 - Local forms** found in [3] (pp. 35-45).

Theorem 3.18. ***Moser Theorem (Relative Version)***
Let M be a manifold, X a compact submanifold of M, $i : X \hookrightarrow M$ the inclusion map and ω_0, ω_1 symplectic forms on M. If we have $\omega_0\big|_p = \omega_1\big|_p$, $\forall p \in X$, then there exist neighbourhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X in M such that $\varphi^*\omega_1 = \omega_0$ and the following diagram commutes:

$$\begin{tikzcd} \mathcal{U}_0 \arrow[r, "\varphi"] & \mathcal{U}_1 \\ X \arrow[u, "i"] \arrow[ur, "i"'] & \end{tikzcd}$$

The proof uses the tubular neighbourhood theorem, the homotopy formula and Moser's trick, it can be seen in [3] (Section 7.3, p.45, and previous sections).

Theorem 3.19. ***Darboux Theorem***
Let (M, ω) be a symplectic manifold of dimension $2n$, and let p be any point in M. Then, we can find a coordinate system $(\mathcal{U}, x_1, ..., x_n, y_1, ..., y_n)$ centered at p such that on $\mathcal{U}$:

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Remark 2. As a consequence of the previous theorem, proving a local assertion for $\left(\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i\right)$, which is invariant under symplectomorphisms, implies it will hold for all symplectic manifolds. In particular, dimension is the the only local invariant of symplectic manifolds.

Proof. (We follow the guidelines in [3], section 8.1, p.46). Consider $X = \{p\}$, then X is clearly a compact submanifold of M. By theorem (3.10) we can consider a symplectic basis for T_pM which allows us to construct local coordinates $(x'_1, ...x'_n, y'_1, ..., y'_n)$ centered at p and valid on some neighbourhood $\mathcal{U}'$, such that:

$$\omega_p = \sum_{i=1}^{n} dx'_i \wedge dy'_i\Big|_p$$

By doing this, we can consider two symplectic forms ω_0 and ω_1 on $\mathcal{U}'$. We have the given form $\omega_0 \equiv \omega$ and $\omega_1 \equiv \sum_{i=1}^{n} dx'_i \wedge dy'_i$.

Now, by the relative version of Moser theorem, we have neighbourhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of p in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi(p) = p$ and $\varphi^*(\sum_{i=1}^{n} dx'_i \wedge dy'_i) = \omega$. By the properties of the pull-back and the exterior derivative we now have:

$$\varphi^*(\sum_{i=1}^{n} dx'_i \wedge dy'_i) = \sum_{i=1}^{n} \varphi^*(dx'_i) \wedge \varphi^*(dy'_i) = \sum_{i=1}^{n} d\varphi^*(x'_i) \wedge d\varphi^*(y'_i) = \sum_{i=1}^{n} d(x'_i \circ \varphi) \wedge d(y'_i \circ \varphi)$$

Redefining coordinates $x_i = x'_i \circ \varphi$ and $y_i = y'_i \circ \varphi$ we have then a symplectomorphism φ that allows us to write the symplectic form ω locally as $\sum_{i=1}^{n} dx_i \wedge dy_i$.

□

3.4. Compatible Almost Complex Structures

Any symplectic manifold (M, ω), after an initial choice of a Riemannian metric g on M, admits a canonical almost complex structure which is compatible. When the almost complex structure is integrable we have what is called a Kähler manifold, as we have mentioned before this is an object of extreme interest since it is the kind of structure we obtain when considering Kähler polarizations. All these terms will be defined later. We begin by introducing some basic properties of almost complex structures beginning with the construction on an arbitrary vector space. This will allow us to justify the construction of an almost complex structure on any smooth manifold M.

Definition 3.20. ***Complex Structures and Complex Vector Spaces***
*A **complex structure** on a vector space V is a linear map $J : V \to V$ such that $J^2 = -Id$. The pair (V, J) is called a **complex vector space**.*

Definition 3.21. ***Compatible Complex Structure***
*Given a symplectic vector space (V, Ω), we say that a complex structure J on V is **compatible** with Ω if $G_J(u, v) := \Omega(u, Jv)$, $\forall u, v \in V$, is a positive inner-product on V. This is equivalent to J being a symplectomorphism, i.e. $J^*\Omega(u, v) = \Omega(Ju, Jv) = \Omega(u, v)$, which satisfies a taming condition, $\Omega(u, Ju) > 0$, $\forall u \neq 0$.*

Proposition 3.22. *Given a symplectic vector space (V, Ω) there exists a compatible complex structure J on V. This construction is canonical after an initial choice of G. Note that, in general, $\Omega(u, Jv) \neq G(u, v)$, instead we have $\Omega(u, Jv) = G(\sqrt{AA^*}u, v)$.*

Proof. (Ref. [3], section 12.2, pp. 68-69). Let G be an inner product on V. Since both Ω and G are non-degenerate we have the following isomorphisms:

$$\text{isomorphisms between } V \text{ and } V^* : \begin{cases} u \in V \mapsto \Omega(u, \cdot) \in V^* \\ w \in V \mapsto G(w, \cdot) \in V^* \end{cases}$$

Therefore, there exists a linear map $A : V \to V$ such that $\Omega(u, v) = G(Au, v)$. From symmetry and antisymmetry of the metric and symplectic form respectively, and existence of the adjoint operator, we have:

$$G(A^*u, v) = G(u, Av) = G(Av, u) = \Omega(v, u) = -\Omega(u, v) = G(-Au, v),$$

which shows that the map A is anti-symmetric. Since AA^* is symmetric, $(AA^*)^* = AA^*$ and AA^* is positive, $G(AA^*u, u) = G(A^*u, A^*u) > 0$, if $u \neq 0$. These properties imply that $\sqrt{AA^*}$ is symmetric and positive-definite (since AA^* diagonalizes with positive eigenvalues). By defining:

$$J = (\sqrt{AA^*})^{-1}A,$$

we can consider the polar decomposition of A as $A = \sqrt{AA^*}J$, where J is unitary ($J^* = J$), anti-Hermitian ($J^* = -J$) and commutes with $\sqrt{AA^*}$, therefore $J^2 = -J^*J = -\text{Id}$. We just need to check that J is indeed compatible, i.e., a symplectomorphism and tame:

$$\begin{aligned} \Omega(Ju, Jv) &= G(AJu, Jv) = G(JAu, Jv) = G(Au, v) \\ &= \Omega(u, v); \\ \Omega(u, Ju) &= G(Au, Ju) = G(-JAu, u) \\ &= \Omega(\sqrt{AA^*}u, u) > 0, \text{ for } u \neq 0, \end{aligned}$$

thus J is a canonical (after a choice of g) and compatible complex structure on V. $\square$

Remark 3. Given a family of symplectic vector spaces (V_t, Ω_t) and a family G_t of positive inner-products, all smooth with respect to t, then, adapting the proof of the previous proposition we can find a smooth family J_t of compatible complex structures on V_t. This is crucial in proving the existence of a compatible canonical almost complex structure on a symplectic manifold (M, ω), where the construction is considered canonical after the choice of a metric.

Example 3.23. *The euclidean space $\mathbb{R}^{2n}$ with the standard linear coordinates $(x_1, ...x_n, y_1, ..., y_n)$, the standard symplectic structure $\omega_0 = \sum dx_j \wedge dy_j$ and the standard inner-product $g_0 = \langle \cdot, \cdot \rangle$ is a clear example of a compatible complex structure on a vector space. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$, multiplication by i induces a complex structure J_0 on the tangent spaces of $\mathbb{R}^{2n}$ (see [3], pp. 67-68).*

We now define the notion of an almost complex structure on a symplectic manifold (M, ω) which we can show always exists, by an adaption of the previous proof, remark 3 and the fact that we always have Riemanninan metrics.

Definition 3.24. ***Compatible Almost Complex Structure***
*An almost complex structure J on a symplectic manifold (M, ω) is called ω-**compatible** if the map $g_x(u, v) := \omega_x(u, J_x v)$ is a Riemannian metric on M. The triple (ω, g, J) is called a **compatible triple** when $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$.*

Proposition 3.25. *Given a symplectic manifold (M, ω) and a Riemannian metric g on M there exists a canonical way (after a choice of metric g) to define an almost complex structure J on M which is compatible. As before, in general $g_J(\cdot, \cdot) := \omega(\cdot, J\cdot) \neq g(\cdot, \cdot)$.*

As said above, the proof follows from the construction on a general vector space V and the extension to a smooth family of vector spaces which are regarded as the tangent spaces on M.

Corollary 3.26. *Since we always have Riemannian metrics on smooth manifolds, the previous proposition asserts that any symplectic manifold has a compatible almost complex structure.*

3.4.1. Dolbeault Theory - Splitting under the action of J

In this section we look at the complexified tangent $(TM \otimes \mathbb{C})$ and cotangent bundles $(T^*M \otimes \mathbb{C})$ of an almost complex manifold (M, J), in particular at how they split under the action of J. These are natural spaces on complex manifolds which we will consider later, particularly when dealing with complex polarizations.

First, we can extend the action of J linearly on each fiber $(TM \otimes \mathbb{C})_p$, $p \in M$, which is a 2n-dimensional vector space over $\mathbb{C}$, it is given by:

$$J(v \otimes c) = Jv \otimes c, \quad v \in TM, \quad c \in \mathbb{C}.$$

Since $J^2 = -\mathrm{Id}$, on $(TM \otimes \mathbb{C})_p$ J_p has eigenvalues $\pm i$. Thus, we have the following splitting $TM \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1}$:

$$\begin{aligned} T_{1,0} &= \{\omega \in TM \otimes \mathbb{C} \,|\, J\omega = +i\omega\} = \text{(+i)-eigenspace of J} \\ &= \{v \otimes 1 - Jv \otimes i \,|\, v \in TM\} \\ &= \textbf{(J-)holomorphic tangent vectors} \\ T_{0,1} &= \{\omega \in TM \otimes \mathbb{C} \,|\, J\omega = -i\omega\} = \text{(-i)-eigenspace of J} \\ &= \{v \otimes 1 + Jv \otimes i \,|\, v \in TM\} \\ &= \textbf{(J-)anti-holomorphic tangent vectors} \end{aligned}$$

The splitting of the complexified tangent bundle $(TM \otimes C)$ gives us a natural isomorphism under the projection maps: $(\pi_{1,0}, \pi_{0,1}) : TM \otimes \mathbb{C} \xrightarrow{\simeq} T_{1,0} \oplus T_{0,1}$.

$$\begin{aligned} \pi_{1,0} &: TM \to T_{1,0} & \pi_{0,1} &: TM \to T_{0,1} \\ v &\mapsto \frac{1}{2}\left(v \otimes 1 - Jv \otimes i\right) & v &\mapsto \frac{1}{2}\left(v \otimes 1 + Jv \otimes i\right) \end{aligned}$$

Similarly, the complexified cotangent bundle $(T^*M \otimes \mathbb{C})$, splits as $T^*M \otimes \mathbb{C} \xrightarrow{\simeq} T^{1,0} \oplus T^{0,1}$:

$$\begin{aligned} T^{1,0} &= (T_{1,0})^* = \{\eta \in T^*M \otimes \mathbb{C} \,|\, \eta(J\omega) = i\eta(\omega), \forall \omega \in TM \otimes \mathbb{C}\} \\ &= \{\xi \otimes 1 - (\eta \circ J) \otimes i \,|\, \xi \in T^*M\} \\ &= \textbf{complex-linear cotangent vectors} \\ T^{0,1} &= (T_{0,1})^* = \{\eta \in T^*M \otimes \mathbb{C} \,|\, \eta(J\omega) = -i\eta(\omega), \forall \omega \in TM \otimes \mathbb{C}\} \\ &= \{\xi \otimes 1 + (\eta \circ J) \otimes i \,|\, \xi \in T^*M\} \\ &= \textbf{complex-antilinear cotangent vectors} \end{aligned}$$

The projection maps are given by: $(\pi^{1,0}, \pi^{0,1}) : T^*M \otimes \mathbb{C} \xrightarrow{\simeq} T^{1,0} \oplus T^{0,1}$.

$$\begin{aligned} \pi^{1,0} &: T^*M \otimes \mathbb{C} \to T^{1,0} & \pi^{0,1} &: T^*M \otimes \mathbb{C} \to T^{0,1} \\ \eta &\mapsto \frac{1}{2}\left(\eta - i\eta \circ J\right) & \eta &\mapsto \frac{1}{2}\left(\eta + i\eta \circ J\right) \end{aligned}$$

3.4.2. Forms of type (l, m)

Given an almost complex manifold (M, J) consider the complex valued k-forms on M, $\Omega^k(M; \mathbb{C})$, which are the sections of $\Lambda^k(T^*M \otimes \mathbb{C})$. From the previous splitting we define:

$$\Lambda^k(T^*M \otimes \mathbb{C}) = \Lambda^k(T^{1,0} \oplus T^{0,1}) = \oplus_{l+m=k}(\Lambda^l T^{1,0}) \wedge (\Lambda^m T^{0,1}) := \oplus_{l+m=k}\Lambda^{l,m}.$$

Thus, the forms of type (l, m) are defined as:

Definition 3.27. ***(l,m) differential forms***
Differential forms of type (l, m) on (M, J) are the sections $\Omega^{l,m}$ of $\Lambda^{l,m}$ therefore, $\Omega^k(M; \mathbb{C}) = \oplus_{l+m=k}\Omega^{l,m}$.

Composing the exterior derivative d with the projection map $\pi^{l,m} : \Lambda^k(T^*M \otimes \mathbb{C}) \to \Lambda^{l,m}$ allows us to define the differential operators ∂ and $\overline{\partial}$ acting on forms of type (l, m):

$$\begin{aligned} \partial &:= \pi^{l+1,m} \circ d : \Omega^{l,m}(M) \to \Omega^{l+1,m}(M) \\ \overline{\partial} &:= \pi^{l,m+1} \circ d : \Omega^{l,m}(M) \to \Omega^{l,m+1}(M) \end{aligned}$$

Considering the action of this operators on 0-forms on M, i.e, the action on smooth functions $f : M \to \mathbb{C}$, since we have $df = d(\operatorname{Re} f) + i\, d(\operatorname{Im} f)$, we can define:

Definition 3.28. ***(J-)holomorphic functions***
A function f is ***(J-)holomorphic at*** $p \in M$ *if df_p is complex linear, i.e., $df_p \circ J = i\, df_p$. If f is holomorphic at all $p \in M$ we say it is* ***(J-)holomorphic****.*

Definition 3.29. ***(J-)anti-holomorphic functions***
A function f is ***(J-)anti-holomorphic at*** $p \in M$ *if df_p is complex anti-linear, i.e., $df_p \circ J = -i\, df_p$.*

Definition 3.30. *On functions, the exterior derivative acts as $d = \partial + \overline{\partial}$, where:*

$$\partial := (\pi^{1,0} \circ d) \;\; \textit{and} \;\; \overline{\partial} := (\pi^{0,1} \circ d), \textit{ then:}$$
$$f \textit{ is holomorphic} \Leftrightarrow \overline{\partial} f = 0,$$
$$f \textit{ is anti-holomorphic} \Leftrightarrow \partial f = 0.$$

We are now interested in understanding how the exterior derivative acts in the case of higher differential forms. Let us assume that $d = \partial + \overline{\partial}$, i.e. given $\beta \in \Omega^{l,m}$ we have $d\beta = \partial\beta + \overline{\partial}\beta$ where $\partial\beta \in \Omega^{l+1,m}$ and $\overline{\partial}\beta \in \Omega^{l,m+1}$.

Since for any form we have $d^2\beta = 0 = \partial^2\beta + \partial\overline{\partial}\beta + \overline{\partial}\partial\beta + \overline{\partial}^2\beta$, with these terms being elements of $\Omega^{l+2,m}$, $\Omega^{l+1,m+1}$ and $\Omega^{l,m+2}$ respectively. This implies that:

$$\begin{cases} \overline{\partial}^2 = 0 \\ \partial\overline{\partial}\beta + \overline{\partial}\partial\beta = 0 \\ \partial^2 = 0 \end{cases}$$

Thus we have the differential complex chains:

$$0 \to \Omega^{l,0} \xrightarrow{\overline{\partial}} \Omega^{l,1} \xrightarrow{\overline{\partial}} \Omega^{l,2} \xrightarrow{\overline{\partial}} \ldots$$
$$0 \to \Omega^{0,m} \xrightarrow{\partial} \Omega^{1,m} \xrightarrow{\partial} \Omega^{2,m} \xrightarrow{\partial} \ldots$$

which define the **Dolbeault cohomology groups** given by:

$$H^{l,m}_{\textit{Dolbeault}}(M) := \frac{\textit{ker } \overline{\partial} : \Omega^{l,m} \to \Omega^{l,m+1}}{\textit{im } \overline{\partial} : \Omega^{l,m-1} \to \Omega^{l,m}}$$

In words, the quotient of $\overline{\partial}$-closed forms of type $\Omega^{l,m}$ by $\overline{\partial}$-exact forms of type $\Omega^{l,m-1}$. The definitions are analogous for the complex chain given by ∂. The following diagram, taken from [3], illustrates nicely the action of this operators.

$$\begin{tikzcd}[column sep=small]
 & & \Omega^{0,0} \arrow[dl, "\partial"'] \arrow[dr, "\overline{\partial}"] & & \\
 & \Omega^{1,0} \arrow[dl, "\partial"'] \arrow[dr, "\overline{\partial}"] & & \Omega^{0,1} \arrow[dl, "\partial"'] \arrow[dr, "\overline{\partial}"] & \\
\Omega^{2,0} & & \Omega^{1,1} & & \Omega^{0,2} \\
\vdots & & \vdots & & \vdots
\end{tikzcd}$$

3.5. Complex Manifolds

We supposed initially that $d = \partial + \overline{\partial}$, we will see in this section that this is precisely the case when the manifold admits a complex structure. The idea of a complex manifold is similar with the construction of a smooth manifold we already have, the additions in this case are the structure of a complex atlas with homeomorphisms to $\mathbb{C}^n$ which on overlapping charts give rise to transition maps which are biholomorphic. Let us define this rigorously:

Definition 3.31. ***Complex Manifold***
A complex manifold M of complex dimension n is a smooth manifold together with a complex atlas $\mathcal{A} = \{(\mathcal{U}_\alpha, \mathcal{V}_\alpha, \varphi_\alpha), \alpha \in I\}$, where $M = \cup_\alpha \mathcal{U}_\alpha$, the $\mathcal{V}_\alpha$'s are open subsets taking values in $\mathbb{C}^n$ and the maps $\varphi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$ give rise to transition maps $\psi_{\alpha\beta}$ which are holomorphic bijections (biholomorphic).

$$
\begin{tikzcd}
 & \mathcal{U}_\alpha \cap \mathcal{U}_\beta \arrow[dl, "\varphi_\alpha"', "\cong"] \arrow[dr, "\varphi_\beta", "\cong"'] & \\
\mathcal{V}_{\alpha\beta} \arrow[rr, "\psi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}"] & & \mathcal{V}_{\beta\alpha}
\end{tikzcd}
$$

The following proposition shows that any complex manifold has a canonical almost complex structure which, in particular, implies that complex manifolds will have the same splitting structure we saw on the previous section.

Proposition 3.32. *Any complex manifold M admits a canonical almost complex structure J.*

Proof. (Ref. [3], proposition 15.2, pp. 83-84). The proof is done in two parts. First we construct J locally on a complex chart, and then show that it is in fact well defined globally on M by evaluating this construction on intersections of different charts.

Part 1) Let $(\mathcal{U}, z_1, ..., z_n)$ be a complex chart on M where, for each j, $z_j = x_j + iy_j$. Given $p \in \mathcal{U}$, the tangent space at p is:

$$T_pM = \mathbb{R}\text{-span of } \left\{ \left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial y_j}\right|_p : j = 1, ..., n \right\}.$$

Thus we can construct $J : T_pM \to T_pM$ on $\mathcal{U}$ by considering:

$$J_p \left(\left.\frac{\partial}{\partial x_j}\right|_p \right) = \left.\frac{\partial}{\partial y_j}\right|_p \quad \text{and} \quad J_p \left(\left.\frac{\partial}{\partial y_j}\right|_p \right) = -\left.\frac{\partial}{\partial x_j}\right|_p,$$

where $j = 1, ..., n$. Then, J_p satisfies $J_p^2 =$Id.

Part 2) Let $(\mathcal{U}_\alpha, \mathcal{V}_\alpha, \varphi_\alpha)$ and $(\mathcal{U}_\beta, \mathcal{V}_\beta, \varphi_\beta)$ be two complex charts on M such that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$. We must check that the almost complex structure J_α constructed on $\mathcal{U}_\alpha$ agrees with a similar construction J_β on $\mathcal{U}_\beta$.

We have, $\psi_{\alpha\beta} \circ \varphi_\alpha = \varphi_\beta$, defined on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathcal{V}_\alpha \cap \mathcal{V}_\beta$. Considering local coordinates $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$ on $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ respectively, we can write: $\psi_{\alpha\beta}(z_1, ..., z_n) = (w_1, ..., w_n)$. By the usual rules of differentiating compositions we obtain:

$$\begin{cases} \frac{\partial}{\partial x_k} = \sum_j \left(\frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k} \frac{\partial}{\partial v_j} \right) \\ \frac{\partial}{\partial y_k} = \sum_j \left(\frac{\partial u_j}{\partial y_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial y_k} \frac{\partial}{\partial v_j} \right) \end{cases}$$

Since we require $\psi_{\alpha\beta}$ to be biholomorphic each component of $\psi_{\alpha\beta}$ satisfies the Cauchy-Riemann equations, i.e., $\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}$ and $\frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k}$, where $j, k = 1, ..., n$. Considering these equations, under the action of J_β we obtain:

$$J_\beta \sum_j \left(\frac{\partial u_j}{\partial x_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial x_k} \frac{\partial}{\partial v_j} \right) = \sum_j \left(\frac{\partial u_j}{\partial y_k} \frac{\partial}{\partial u_j} + \frac{\partial v_j}{\partial y_k} \frac{\partial}{\partial v_j} \right)$$

Which corresponds to the equation $J_\alpha \frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}$. Performing the analogous calculation for the other term we obtain $J_\alpha \frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}$. We conclude that the construction of the almost complex structure agrees on the intersection of charts therefore it is well defined globally, in particular, $J_\alpha = J_\beta := J$.

□

3.5.1. Forms on complex manifolds

We are now interested in understanding how the Dolbeault splitting $\Omega^k(M; \mathbb{C}) = \oplus_{l+m=k} \Omega^{l,m}$, which we saw previously for an almost complex manifold, works when M is a complex manifold under the action of the canonical almost complex structure J.

Consider a coordinate neighborhood $\mathcal{U} \subset M$ with local complex complex coordinates $(z_1, ..., z_n)$, where again $z_j = x_j + iy_j$ for $x_j, y_j \in \mathbb{R}$. For any $p \in \mathcal{U}$ we consider the complexified tangent space:

$$\begin{aligned}(T_pM \otimes \mathbb{C}) &= \mathbb{C}\text{-span} \left\{ \left.\frac{\partial}{\partial x_j}\right|_p, \left.\frac{\partial}{\partial y_j}\right|_p \right\} = \\ &= \underbrace{\mathbb{C}\text{-span} \left\{ \frac{1}{2} \left(\left.\frac{\partial}{\partial x_j}\right|_p - i \left.\frac{\partial}{\partial y_j}\right|_p \right) \right\}}_{T_{1,0} = (+i)\text{-eigenspace of } J} \oplus \underbrace{\mathbb{C}\text{-span} \left\{ \frac{1}{2} \left(\left.\frac{\partial}{\partial x_j}\right|_p + i \left.\frac{\partial}{\partial y_j}\right|_p \right) \right\}}_{T_{0,1} = (-i)\text{-eigenspace of } J}\end{aligned}$$

The actions of J on this spaces are respectively:

$$J \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right) = i \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right),$$
$$J \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) = -i \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$$

To ease the notation we can define:

Definition 3.33. $\left(\frac{\partial}{\partial z_j} \right)$ ***and*** $\left(\frac{\partial}{\partial \bar{z}_j} \right)$

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left(\left.\frac{\partial}{\partial x_j}\right|_p - i \left.\frac{\partial}{\partial y_j}\right|_p \right) \quad \textit{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2} \left(\left.\frac{\partial}{\partial x_j}\right|_p + i \left.\frac{\partial}{\partial y_j}\right|_p \right)$$

Using these definition we can write:

$$(T_{1,0})_p = \mathbb{C}\text{-span}\left\{ \frac{\partial}{\partial z_j}\bigg|_p : j = 1, ..., n \right\} \text{ and } (T_{0,1})_p = \mathbb{C}\text{-span}\left\{ \frac{\partial}{\partial \overline{z}_j}\bigg|_p : j = 1, ..., n \right\}$$

We can perform the same on the complexified cotangent bundle ($T_p^*M \otimes \mathbb{C}$). Considering $dz_j = dx_j\big|_p + idy_j\big|_p$ and $d\overline{z}_j = dx_j\big|_p - idy_j\big|_p$ we obtain:

$$(T^{1,0})_p = \mathbb{C}\text{-span}\{dz_j\big|_p : j = 1, ..., n\} \text{ and } (T^{0,1})_p = \mathbb{C}\text{-span}\{d\overline{z}_j\big|_p : j = 1, ..., n\}$$

Defining a convenient new notation, a general (l, m)-form is given by:

Definition 3.34. ***General*** (l, m) ***using the multi-index notation***

$$\Omega^{l,m} = \left\{ \sum_{|J|=l,|K|=m} b_{J,K}\, dz_J \wedge d\overline{z}_K \mid b_{J,K} \in C^\infty(\mathcal{U}; \mathbb{C}) \right\}, \quad \textit{where:}$$

$J = (j_1, ..., j_m)$, with $1 \leq j_1 < ... < j_m \leq n$, $|J| = m$ and $dz_J = dz_{j_1} \wedge dz_{j_2} \wedge ... \wedge dz_{j_m}$.

As alluded earlier, the structure we considered, assuming the exterior derivative of the form $d = \partial + \overline{\partial}$, is particularly useful since this is precisely the case when the manifold is a complex manifold.

Proposition 3.35. *If M is a complex manifold then $d = \partial + \overline{\partial}$ on forms of any degree. Equivalently, given a coordinate neighbourhood $\mathcal{U}$ and a form $\beta \in \Omega^k(M; \mathbb{C})$ written as:*

$$\beta = \sum_{|J|+|K|=k} a_{J,K} dx_J \wedge dy_K, \quad \textit{with } a_{J,K} \in C^\infty(\mathcal{U}; \mathbb{C}),$$

the following equality holds:

$$d\beta = \sum (\partial a_{J,K} + \overline{\partial} a_{J,K}) dx_J \wedge dy_K = (\partial + \overline{\partial}) \sum a_{J,K} dx_J \wedge dy_K$$

Proof. (Ref [3], section 15.3, pp.86-87). From the previous definitions we have the following identity:

$$\begin{cases} dx_j = \frac{1}{2}(dz_j + d\overline{z}_j) \\ dy_j = \frac{1}{2i}(dz_j - d\overline{z}_j) \end{cases}$$

Replacing on the above expression for β results:

$$\beta = \sum_{l+m=k} \underbrace{\left(\sum_{|J|=l,|K|=m} b_{J,K} dz_J \wedge d\overline{z}_K \right)}_{\in \Omega^{l,m}}, \quad \text{where, } b_{J,K} \equiv \frac{1}{4i} a_{J,K}$$

Applying the exterior derivative and considering how it acts on functions we obtain:

$$\begin{aligned}
d\beta &= \sum_{l+m=k} \left(\sum_{|J|=l,|K|=m} db_{J,K} dz_J \wedge d\overline{z}_K \right) = \\
&= \sum_{l+m=k} \sum_{|J|=l,|K|=m} (\partial b_{J,K} + \overline{\partial} b_{J,K}) dz_J \wedge d\overline{z}_K = \\
&= \sum_{l+m=k} (\underbrace{\sum_{|J|=l,|K|=m} \partial b_{J,K} \wedge dz_J \wedge d\overline{z}_K}_{\in \Omega^{l+1,m}} + \underbrace{\sum_{|J|=l,|K|=m} \overline{\partial} b_{J,K} \wedge dz_J \wedge d\overline{z}_K}_{\in \Omega^{l,m+1}}) = \\
&= \partial\beta + \overline{\partial}\beta, \quad \text{by recovering the previous coordinates.}
\end{aligned}$$

□

Definition 3.36. ***Integrable almost complex structure*** J
An almost complex structure J on a smooth manifold M is integrable if J is induced by a structure of complex manifold on M.

The following theorem gives us a criteria to identify when an almost complex manifold (M, J) has the structure of a complex manifold.

Theorem 3.37. ***Newlander-Nirenberg***
Let (M, J) be an almost complex manifold and $\mathcal{N}$ the Nijenhuis tensor defined by $\mathcal{N}(v, w) := [Jv, Jw] - J[v, Jw] - J[Jv, w] - [v, w]$. We have the following equivalences:

$$\begin{aligned}
M \textit{ is a complex manifold} &\iff J \textit{ is integrable} \\
&\iff \mathcal{N} = 0 \\
&\iff d = \partial + \overline{\partial} \\
&\iff \overline{\partial}^2 = 0 \\
&\iff \pi^{2,0} d|_{\Omega^{0,1}} = 0
\end{aligned}$$

Proof. See [3], theorem 15.4, for several references proving this result. □

Remark 4. In general, almost complex manifolds will have $d \neq \partial + \overline{\partial}$ since we will not have local coordinates z_j to form a suitable basis of 1-forms. Despite the isomorphism of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ the coordinates will not, in general, satisfy the Cauchy-Riemann equations.

3.6. Kähler Manifolds

We arrive now at the intersection of several interesting structures, Kähler manifolds. These are smooth manifolds with three compatible structures, symplectic, complex and Riemannian. As we said before these will be useful when considering Kähler polarizations, which is often the case. Let us begin by defining the notion of compatibility and how these structures are related.

Definition 3.38. ***Compatible Triple***
We call (ω, J, g) a compatible triple of structures if we can write any one of ω, J or g in terms of the remaining two. The relations between them, and interesting follow up questions on each case are summarized in the following table (ref. [3], section 13.2, p.75):

Data	***Condition/Technique***	***Consequence***	***Question***
(ω, J)	$\omega(Ju, Jv) = \omega(u, v)$, $\omega(u, Ju) > 0$, *for* $u \neq 0$	$g(u, v) := \omega(u, Jv)$	g *flat?*
(g, J)	$g(Ju, Jv) = g(u, v)$, *i.e.,* J *is orthogonal*	$\omega(u, v) := g(Ju, v)$	ω *closed?*
(ω, g)	*polar decomposition*	*almost complex str.* J	J *integrable?*

Definition 3.39. ***Kähler Manifold***
A ***Kähler manifold*** *is a symplectic manifold (M, ω) equipped with an integrable almost complex structure, i.e., (M, ω, J, g). In this case the symplectic form ω is called a* ***Kähler form****.*

Since Kähler manifolds are complex manifolds, we have the same structure and splitting of the differential forms as seen previously. To recall:

$$\begin{cases} \Omega^k(M; \mathbb{C}) = \oplus_{l+m=k} \Omega^{l,m} \\ d = \partial + \overline{\partial} \end{cases} \qquad \begin{cases} \partial = \pi^{l+1,m} \circ d : \Omega^{l,m} \to \Omega^{l+1,m} \\ \overline{\partial} = \pi^{l,m+1} \circ d : \Omega^{l,m} \to \Omega^{l,m+1} \end{cases}$$

Thus, on a Kähler manifold M with $dim_{\mathbb{C}} M = n$, given a complex chart $(\mathcal{U}, z_1, ..., z_n)$, the differential forms on $\mathcal{U}$ are given by the following expression:

$$\Omega^{l,m} = (l, m)\text{-forms} = \left\{ \sum_{|J|=l, |K|=m} b_{J,K}\, dz_J \wedge d\bar{z}_K \mid b_{J,K} \in C^\infty(\mathcal{U}; \mathbb{C}) \right\},$$

where we used the multi-index notation defined previously.

Let us check how the above decomposition reflects the properties of the **Kähler form**, i.e, a **2-form compatible** with the complex structure which is **closed**, **real-valued** and **nondegenerate**.

- **2-form**: $\Omega^2(M; \mathbb{C}) = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$, given a local complex chart $(\mathcal{U}, z_1, ..., z_n)$ we have:

  $$\omega = \sum a_{jk} dz_j \wedge dz_k + \sum b_{jk} dz_j \wedge d\bar{z}_k + \sum c_{jk} d\bar{z}_j \wedge d\bar{z}_k$$

  for $a_{jk}, b_{jk}, c_{jk} \in C^\infty(M; \mathbb{C})$

- **Compatible**: Since we have compatibility, J is a symplectomorphism i.e. $J^*\omega = \omega$. This implies that given $\omega = \sum a_{jk} dz_j \wedge dz_k + \sum b_{jk} dz_j \wedge d\bar{z}_k + \sum c_{jk} d\bar{z}_j \wedge d\bar{z}_k$, we obtain $a_{jk} = 0 = c_{jk}$ $\forall j, k$, and thus $\omega \in \Omega^{1,1}(M)$.

- **Closed**: We have $0 = d\omega = \underbrace{\partial\omega}_{(2,1)-\text{form}} + \underbrace{\bar{\partial}\omega}_{(1,2)-\text{form}} \implies \begin{cases} \partial\omega = 0 & \omega \text{ is } \partial\text{-closed} \\ \overline{\partial}\omega = 0 & \omega \text{ is } \overline{\partial}\text{-closed} \end{cases}$

Thus ω defines a Dolbeault $(1,1)$ cohomology class, $[\omega] \in H^{1,1}_{Dolbeault}(M)$.

Replacing $b_{jk} = \frac{i}{2} h_{jk}$ we have:

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk}\ dz_j \wedge d\bar{z}_k, \qquad h_{jk} \in C^{\infty}(\mathcal{U}; \mathbb{C})$$

- **Real-valued**: ω real-valued $\iff \omega = \overline{\omega} \iff h_{jk} = \overline{h_{jk}} \iff \forall p \in \mathcal{U},\ (h_{jk}(p))$ is an Hermitian matrix.

- **Nondegeneracy**: $\omega^n = \underbrace{\omega \wedge ... \wedge \omega}_{n} \neq 0$

  By the Leibniz formula for the determinant of an $n \times n$ matrix, $\det(A_{n\times n}) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}\right)$, we can check that:

$$\omega^n = n! \left(\frac{i}{2}\right)^n \det(h_{jk})\, dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_1 \wedge d\bar{z}_n$$

  Thus we obtain ω nondegenerate $\iff \det_{\mathbb{C}}(h_{jk}) \neq 0$, which means that at every $p \in M$, $(h_{jk}(p))$ is a nonsingular matrix.

  From the positivity condition $\omega(v, Jv) > 0,\ \forall v \neq 0$, let $v \in (TM \times \mathbb{C})\Big|_p$, $v_p = \sum_{j=1}^{n} \alpha_j \left(\frac{\partial}{\partial z_j}\Big|_p + \frac{\partial}{\partial \bar{z}_j}\Big|_p \right)$, $\alpha_j \in \mathbb{C}$, we have:

$$\begin{aligned} \omega(v, Jv) = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk}(p)\ dz_j \wedge d\bar{z}_k &\left(\sum_{l=1}^{n} \alpha_l \left(\frac{\partial}{\partial z_l} + \frac{\partial}{\partial \bar{z}_l} \right), i \sum_{m=1}^{n} \alpha_m \left(\frac{\partial}{\partial z_m} - \frac{\partial}{\partial \bar{z}_m} \right) \right) \\ &= \frac{1}{2} \sum_{j,k=1}^{n} h_{jk}(p) \alpha_j \alpha_k = \frac{1}{2} \alpha^T (h_{jk}(p)) \alpha > 0,\ \forall \alpha \in \mathbb{C}^n \end{aligned}$$

  This implies that (h_{jk}) is a positive-definite matrix at each $p \in \mathcal{U}$.

In conclusion, we have that:

Kähler forms are ∂- and $\overline{\partial}$- closed $(1,1)$-forms, which, given a local complex chart $(\mathcal{U}, z_1, ..., z_n)$, take the form:

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk}\, dz_j \wedge d\bar{z}_k$$

where at every point $p \in \mathcal{U}$, $(h_{jk})_p$ is a positive-definite Hermitian matrix.

We are now interested in the following question: Given a complex manifold M can we turn it into a Kähler manifold? The answer is yes, in some cases. By the next proposition, we can construct a Kähler form if we can find a function $\rho \in \mathbb{C}^{\infty}(M; \mathbb{R})$ that satisfies the following definition:

Definition 3.40. ***Strictly plurisubharmonic function***
Given a complex manifold M, a function $\rho \in \mathbb{C}^\infty(M;\mathbb{R})$ is ***strictly plurisubharmonic*** *(s.p.s.h), if on each local complex chart $(\mathcal{U}, z_1, ..., z_n)$, the matrix $\left(\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}(p)\right)$ is positive-definite at all $p \in \mathcal{U}$.*

Proposition 3.41. *Let M be a complex manifold and $\rho \in \mathbb{C}^\infty(M;\mathbb{R})$ a (s.p.s.h). function. Then, the form ω given by:*

$$\omega = \frac{i}{2}\partial\overline{\partial}\rho,$$

is a Kähler form on M. The function ρ satisfying this proposition is called a ***global Kähler potential****.*

Proof. (Ref. [3], section 16.3, p. 93). We need to verify that this expression satisfies the properties of a Kähler form.

- ω **is a 2-form**: Clearly a 2-form by the action of $\partial\overline{\partial}$, in particular it is a form of type $(1,1)$ under Dolbeault splitting since ρ is a 0-form. That is, $\overline{\partial}\rho \to \Omega^{0,1}$ and $\partial(\overline{\partial}\rho) \to \Omega^{1,1}$.

- ω **is closed**:

$$\begin{cases} \partial\omega = \frac{i}{2}\partial^2\overline{\partial}\rho = 0 \\ \overline{\partial}\omega = \frac{i}{2}\overline{\partial}\partial\overline{\partial}\rho = -\frac{i}{2}\partial\overline{\partial}^2\rho = 0 \end{cases}$$

  where we used the fact that $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\overline{\partial}\partial = -\partial\overline{\partial}$. Thus we have $d\omega = \partial\omega + \overline{\partial}\omega = 0$.

- ω **is real-valued**:

  Since, $\overline{\omega} = -\frac{i}{2}\overline{\partial}\partial\rho = \frac{i}{2}\partial\overline{\partial}\rho = \omega$.

- ω **is compatible with** J:

  $\omega \in \Omega^{1,1} \implies J^*\omega = \omega \implies \omega(\cdot, J\cdot)$ is symmetric. We need to verify that $\omega(\cdot, J\cdot)$ is positive in order to define a Riemannian metric. First we show that for $f \in C^\infty(\mathcal{U};\mathbb{C})$ we have:

$$\partial f = \sum \frac{\partial f}{\partial z_j} dz_j \text{ and } \overline{\partial} f = \sum \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

  Let $f \in C^\infty(\mathcal{U};\mathbb{C})$, we have:

$$\begin{aligned} df &= \sum_j \left(\frac{\partial f}{\partial x_j}dx_j + \frac{\partial f}{\partial y_j}dy_j\right) = \\ &= \sum_j \left[\frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right)(dx_j + idy_j) + \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)(dx_j - idy_j)\right] = \\ &= \sum_j \left(\frac{\partial f}{\partial z_j}dz_j + \frac{\partial f}{\partial \bar{z}_j}d\bar{z}_j\right). \end{aligned}$$

Therefore we have:

$$\begin{cases} \partial f = \pi^{1,0} df = \sum_j \frac{\partial f}{\partial z_j} dz_j \\ \overline{\partial} f = \pi^{0,1} df = \sum_j \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \end{cases}$$

Using this result on the definition of ω we obtain:

$$\omega = \frac{i}{2}\partial\overline{\partial}\rho = \frac{i}{2}\sum \frac{\partial}{\partial z_j}\left(\frac{\partial \rho}{\partial \overline{z}_k}\right) dz_j \wedge d\overline{z}_k = \frac{i}{2}\sum \left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}\right) dz_j \wedge d\overline{z}_k.$$

By assumption ρ is a s.p.s.h function therefore $\left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}\right)$ is positive which implies that $\omega(\cdot, J\cdot)$ is positive.

- ω **is nondegenerate**:

  $\omega(\cdot, J\cdot)$ is positive so, in particular, ω is nondegenerate.

□

We have a local converse of the previous proposition given by the following theorem (we will not prove this result, see [3], theorem 16.5, p. 94, for more information regarding the proof).

Theorem 3.42. *Let M be a complex manifold and ω a closed, real-valued $(1,1)$-form on M. Given $p \in M$ there exists a neighbourhood $\mathcal{U}$ of p and a function $\rho \in C^\infty(M;\mathbb{C})$ such that on $\mathcal{U}$:*

$$\omega = \frac{i}{2}\partial\overline{\partial}\rho \quad \textit{is a Kähler form on } M.$$

*The function ρ is then called a **local Kähler potential**.*

Another way to obtain Kähler manifolds is by considering complex submanifolds of Kähler manifolds, this result comes as a corollary of the following proposition:

Proposition 3.43. *Let M be a complex manifold and $\rho \in \mathbb{C}^\infty(M;\mathbb{R})$ a s.p.s.h. function. Given X a complex submanifold of M, considering the inclusion map $i : X \hookrightarrow M$ then $i^*\rho$ is a s.p.s.h. function on X and thus we call $(X, i^*\omega)$ a **Kähler submanifold**.*

Proof. See [3], proposition 16.6, p.94. □

3.6.1. Compact Kähler Manifolds

Here we turn our attention to the compact case, this consideration is particularly important since compact Kähler manifolds end up describing particles internal degrees of freedom, such as spin (see [5] chapter 23). In this case, thanks to the following theorem by Hodge, we have a decomposition of the vector space of cohomology groups $H^k_{deRham}(M;\mathbb{C})$, as a direct sum of cohomology groups under Dolbeault splitting, $H^{l,m}_{Dolbeault}(M)$.

Theorem 3.44. *Hodge (complex version)*
Given a compact Kähler manifold (M,ω), we have the following isomorphism:

$$H^k_{deRham}(M;\mathbb{C}) \simeq \bigoplus_{l+m=k} H^{l,m}_{Dolbeault}(M),$$

*which is called **Hodge decomposition**, where* $H^{l,m} \simeq \overline{H^{m,l}}$.

We will not prove this theorem, but let us motivate this construction somewhat, beginning with ordinary Hodge theory (where we consider only the Riemannian structure on M).

Definition 3.45. ***Hodge-$*$ operator***
Given a smooth manifold M, on the tangent space T_pM at $p \in M$ we have a positive inner product $\langle \cdot, \cdot \rangle$ induced by a Riemannian metric g on M. Given an orthonormal basis, which is positively oriented, $(e_1, ..., e_n)$ on T_pM we define the Hodge-$*$ operator as a linear operator $* : \Lambda(T_pM) \to \Lambda(T_pM)$ satisfying:

$$\begin{aligned} *(1) &= e_1 \wedge ... \wedge_n \\ *(e_1 \wedge ... \wedge_n) &= 1 \\ *(e_1 \wedge ... \wedge_k) &= e_{k+1} \wedge ... \wedge e_n \end{aligned}$$

Thus, $* : \Lambda^k(T_pM) \to \Lambda^{n-k}(T_pM)$ and it satisfies $** = (-1)^{k(n-k)}$.

Definition 3.46. *The **Codifferential** and **Laplacian** are operators defined respectively as:*

$$\begin{aligned} \delta = (-1)^{n(k+1)+1} * d* \ &: \ \Omega^k(M) \to \Omega^{k-1}(M) \\ \Delta = d\delta + \delta d \ &: \ \Omega^k(M) \to \Omega^k(M) \end{aligned}$$

Supposing M is compact, we can define an inner-product on the space of forms by the following integral:

$$\langle \cdot, \cdot \rangle : \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}, \quad \langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta.$$

Proposition 3.47. *With respect to the above inner product we have $\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$ and thus $\delta = d^*$.*

Proof. See [4] section 14.2, p.101. □

Proposition 3.48. *The Laplacian operator is self-adjoint, $\langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle$, and we have $\langle \Delta\alpha, \alpha \rangle = ||d\alpha||^2 + ||\delta\alpha||^2 \geq 0$, where $|| \cdot ||$ represents the norm with respect to this inner-product, i.e., $|| \cdot || = \sqrt{\langle \cdot, \cdot \rangle}$.*

Proof. See [4] section 14.2, p.101. □

Definition 3.49. *The **Harmonic k-forms** are the elements of:*

$$\mathcal{H}^k := \{\alpha \in \Omega^k | \Delta\alpha = 0\}$$

Proposition 3.50. *We have $\Delta\alpha = 0 \Leftrightarrow d\alpha = d^*\alpha = 0$.*

Proof. See [14] section 1.3.5, p.8. □

And thus a harmonic form defines a de Rham cohomology class since it is d-closed. The following theorem asserts that we have an isomorphism between harmonic k-forms and de Rham cohomology.

Theorem 3.51. ***Hodge***
Every de Rham cohomology class on on a compact oriented Riemannian manifold (M, g) has a unique harmonic representative, i.e.,

$$\mathcal{H}^k \simeq H^k_{deRham}(M; \mathbb{R}).$$

Thus the spaces of harmonic k-forms $\mathcal{H}^k$ *are finite-dimensional and we have the following orthogonal decomposition with respect to* $\langle\cdot,\cdot\rangle$*, called the* ***Hodge decomposition on forms:***

$$\Omega^k \simeq \mathcal{H}^k \oplus \Delta(\Omega^k(M)) \simeq \mathcal{H}^k \oplus d\Omega^{k-1} \oplus d^*\Omega^{k+1}.$$

With the additional complex structure, and therefore when M is a Kähler manifold, we obtain the complex version of Hodge's theorem presented at the start. We have the following preliminary results.

Proposition 3.52. *When* M *is a Kähler manifold, we have* $\Delta = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$ *and* Δ *preserves the Dolbeault splitting, i.e.,* $\Delta : \Omega^{l,m} \to \Omega^{l,m}$*. Hence:*

$$\mathcal{H}^k = \bigoplus_{l+m=k} \mathcal{H}^{l,m}.$$

Proof. See [4] section 14.3, p. 103. □

Theorem 3.53. ***Hodge (Complex structure)***
Every Dolbeault cohomology class on on a compact Kähler manifold (M, ω) *has a unique harmonic representative, i.e.,*

$$\mathcal{H}^k \simeq H^{l,m}_{Dolbeault}(M).$$

Thus the spaces of harmonic (l,m)*-forms* $\mathcal{H}^{l,m}$ *are finite-dimensional and we have the following isomorphism:*

$$H^k_{deRham}(M) \simeq \mathcal{H}^k \simeq \bigoplus_{l+m=k} \mathcal{H}^{l,m} \simeq \bigoplus_{l+m=k} H^{l,m}_{Dolbeault}(M)$$

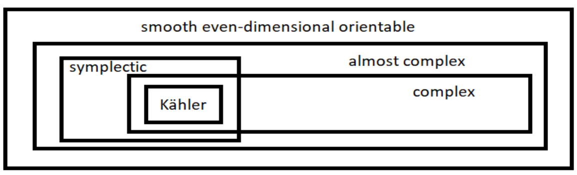
This results have topological consequences which relate, and imposes conditions on, **Betti numbers** ($b^k(M) := dim\, H^k_{deRham}(M)$) and **Hodge numbers** ($h^{l,m}(M) := dim\, H^{l,m}_{Dolbeault}(M)$). A brief discussion on this topics can be seen in [3] section 17.2, p.100. There one can also find references for the proofs of the above theorems.

For a compact Kähler manifold we have a global version of theorem 3.42, which requires the use of Hodge decomposition.

Theorem 3.54. *Let* M *be a compact Kähler manifold. Given* ω *an exact, real, type* $(1,1)$ *form. Then, there exists* $\phi \in C^\infty(M)$ *such that* $\omega = i\partial\bar{\partial}\phi$*.*

Proof. See [14] section 1.3.5, p.9. □

To finish this section on symplectic geometry, and mostly as curiosity, we exhibit the following diagram which shows how these structures are related as subsets of each other. In particular, every intersected region on the diagram is non-empty. This is taken from [3] (section 17.3, pp. 101-103) where examples can be seen within each subset.



4. Hamiltonian Mechanics

When considering classical systems, Hamiltonian mechanics is the framework responsible for the dynamics and results regarding symmetry and conservation laws. To make full use of the Hamiltonian formalism we require only the setup of a symplectic manifold. Let us begin by introducing Hamiltonian and symplectic vector fields and discussing some of their properties.

4.1. Symplectic and Hamiltonian Vector Fields

Definition 4.1. ***Hamiltonian Vector Field***
Given a symplectic manifold (M, ω) and a smooth function $H : M \to \mathbb{R}$, by the nondegeneracy of the symplectic form there is a unique vector field X_H on M such that $\iota_{X_H}\omega = dH$. The vector field X_H is called the ***Hamiltonian vector field*** *with associated* ***Hamiltonian function*** *H. The set of all smooth Hamiltonian vector fields on M is denoted by $\mathfrak{X}^{ham}(M)$.*

By integrating the Hamiltonian vector field X_H we obtain a family of diffeomorphisms $\rho_t : M \to M$ where t is defined for all time, $t \in \mathbb{R}$ (if we consider M to be compact or at least X_H to be a complete vector field) and where $\rho(0) = id_M$.

Lemma 4.2. *The family of diffeomorphisms ρ_t, obtained from X_H, preserve the symplectic form ω, i.e., $\rho_t^*\omega = \omega$, $\forall t \in \mathbb{R}$.*

Proof. (Ref. [3], section 18.1, p. 105). Using the fact that $\frac{d}{dt}\rho_* = \rho_t^*\mathcal{L}_{v_t}\omega$ and $\rho(0) = id_M$, where ρ_t is the family of diffeomorphisms generated by the time-dependent vector field X_H. By Cartan's formula we obtain:

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_H}\omega = \rho_t^*(d\iota_{X_H}\omega + \iota_{X_H}d\omega) = 0,$$

since $\iota_{X_H}\omega = dH$ and ω is closed. □

Then, given any smooth function $H : M \to \mathbb{R}$ on (M, ω), the family of diffeomorphisms ρ_t, obtained by integrating X_H, are in fact symplectomorphisms.

Example 4.3. *Consider the sphere $(M, \omega) = (S^2, d\theta \wedge dh)$ and the height function given by $H(\theta, h) = h$. We have $\iota_{X_H}(d\theta \wedge dh) = dh$ therefore $X_H = \frac{\partial}{\partial\theta}$ and integration gives us $\rho_t(\theta, h) = (\theta + t, h)$. The solution is a rotation about the h-axis and therefore H is preserved by this motion.*

In the previous example we saw that the function H is preserved by the diffeomorphisms ρ_t, however, this is a more general result. In fact, given a Hamiltonian vector field X_H, we have:

$$\mathcal{L}_{X_H}H = \iota_{X_H}dH = \iota_{X_H}\iota_{X_H}\omega = 0,$$

by the antisymmetry of the symplectic form ω. We have then $H(x) = H(\rho_t(x))$ $\forall t \in \mathbb{R}$, i.e., the integral curve $\rho(t)(x)$ is contained in a level set of H.

Definition 4.4. ***Symplectic Vector Field***
Given a symplectic manifold (M, ω), a vector field X on M which preserves the symplectic form ω, i.e., such that $\mathcal{L}_X\omega = 0$, is called a ***symplectic vector field****. The set of all smooth symplectic vector fields on M is denoted by $\mathfrak{X}^{symp}(M)$.*

By Cartan's formula, if X is a sympletic vector field on (M, ω) i.e. $\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega = 0$, then $\iota_X\omega$ is a closed 1-form on M, since ω is closed.

Thus, by Poincaré's lemma 2.22, locally on every contractible set every symplectic vector field is Hamiltonian. Globally, if $H^1_{deRham}(M) = 0$, then every symplectic vector field is Hamiltonian, i.e. given X such that $d(\iota_X\omega) = 0$, since locally any such form is exact and $[\Omega^1_{closed}(M)] = H^1_{deRham}(M) = [0]$, then they all are. As a summary of these results we have:

$$\begin{cases} X \text{ is } \textbf{symplectic} \Leftrightarrow \iota_X\omega \text{ is } \textbf{closed}; \\ X \text{ is } \textbf{Hamiltonian} \Leftrightarrow \iota_X\omega \text{ is } \textbf{exact}; \\ \text{If } H^1_{deRham}(M) = 0, \quad symplectic \implies Hamiltonian; \end{cases}$$

We now give an example of a symplectic vector field which is not Hamiltonian.

Example 4.5. *Consider the 2-torus $(M, \omega) = (T^2, d\theta_1 \wedge d\theta_2)$ and the vector fields $X_1 = \frac{\partial}{\partial\theta_1}$ and $X_2 = \frac{\partial}{\partial\theta_2}$. They are both symplectic but not Hamiltonian vector fields. Despite deceivingly having $\iota_{X_1}\omega = d\theta_2$ the coordinates (θ_1, θ_2) are not globally defined on T^2.*

4.2. Lie and Poisson Brackets

In this section we introduce the Lie and Poisson Brackets and see how they are related through the symplectic structure.

Let us begin by recalling that given a vector field $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, with df the corresponding 1-form, we have:

$$X \cdot f := df(X) = \mathcal{L}_X f.$$

Definition 4.6. ***Lie Bracket***
*Given a smooth manifold M, the **Lie Bracket** is an operator, $[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. It is also denoted $[X, Y] = \mathcal{L}_X Y$, which conceptually means the derivative of Y along the flow of X.*

Proposition 4.7. ***Properties of the Lie Bracket***
*The Lie bracket satisfies **antisymmetry**, **bilinearity**, the **Jacobi identity** and **Leibniz's rule**. Given $X, Y, Z \in \mathfrak{X}(M)$:*

Jacobi identity: $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$;

Leibniz's Rule: *Given $f, g \in C^\infty(M)$, we have:*

$$[fX, gY] = fg[X, Y] + f(X \cdot g)Y - g(Y \cdot f)X$$

Definition 4.8. ***Lie Algebra***
*The Lie bracket equips vector spaces $V = \Gamma(TM)$ of all vector fields on M (i.e., smooth sections of the tangent bundle $TM \to M$) with the structure of a **Lie algebra**.*

Proposition 4.9. *Given a symplectic manifold (M, ω) and two symplectic vector fields X and Y then $[X, Y]$ is Hamiltonian with Hamiltonian function $\omega(Y, X)$.*

Proof. (Ref. [3], proposition 18.3, p.108). We have:

$$\begin{aligned}\iota_{[X,Y]}\omega &= \mathcal{L}_X\iota_Y\omega - \iota_Y\mathcal{L}_X\omega \\ &= d\iota_X\iota_Y\omega + \iota_X + d\iota_Y\omega - \iota_Y + d\iota_X\omega - \iota_Y\iota_X d\omega \\ &= d(\omega(Y,X)),\end{aligned}$$

where we used the result $\iota_{[X,Y]}\alpha = [\mathcal{L}_X, \iota_Y]\alpha$ and Cartan's formula. □

Corollary 4.10. *We have the following inclusions of Lie algebras* $(\mathfrak{X}^{ham}(M), [\cdot,\cdot]) \subseteq (\mathfrak{X}^{symp}(M), [\cdot,\cdot]) \subseteq (\mathfrak{X}(M), [\cdot,\cdot])$

Definition 4.11. ***Poisson Bracket***
Given a symplectic manifold (M,ω)*, the* ***Poisson Bracket*** *is an operator,* $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ *defined as* $\{f,g\} := \omega(X_f, X_g)$*.*

Proposition 4.12. $X_{\{f,g\}} = -[X_f, X_g]$

Proof. If $X_{\omega(X_f,X_g)}$ is an Hamiltonian vector field, with Hamiltonian function $\omega(X_f, X_g)$, then we have:

$$\iota\left(X_{\omega(X_f,X_g)}\right)\omega = d(\omega(X_f, X_g)).$$

By proposition [4.9], $d(\omega(X_f, X_g)) = \iota_{[X_g,X_f]}\omega$ and thus $X_{\omega(X_f,X_g)} = [X_g, X_f]$. Now, by the definition of the Poisson bracket, we obtain $X_{\{f,g\}} = -[X_f, X_g]$. □

We also have the following useful result:

Proposition 4.13. $\{f,g\} = X_g f$

Proof. By the nondegeneracy of ω there is an isomorphism between vector fields and forms thus we may define the unique vector field $X_f = \Omega_{df} = \omega^{-1}(df)$ such that:

$$\begin{aligned}\{f,g\} &= \omega(X_f, X_g) = \omega(\Omega_{df}, X_g) \\ &= (\iota_{\Omega_{df}}\omega)(X_g) = df(X_g) \\ &= X_g f\end{aligned}$$

□

Proposition 4.14. ***Properties of the Poisson Bracket***
The Poisson bracket satisfies ***antisymmetry, bilinearity****, the* ***Jacobi identity*** *and* ***Leibniz's rule****.*

Definition 4.15. ***Poisson Algebra***
The action of the Poisson bracket on smooth functions $C^\infty(M)$ *of a sympletic manifold* (M,ω) *defines a* ***Poisson algebra****, i.e. it is a commutative associative algebra with a Lie bracket* $\{\cdot,\cdot\}$ *satisfying the Leibniz rule.*

In conclusion, given a symplectic manifold (M,ω) and the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$, by proposition 4.12, we have the following Lie algebra anti-homomorphism:

$$\begin{aligned} C^\infty(M) &\to \mathfrak{X}(M) \\ H &\to X_H \\ \{\cdot,\cdot\} &\to -[\cdot,\cdot] \end{aligned}$$

4.3. Integrable Systems

For the triple (M, ω, H), called a Hamiltonian system, we have a notion of integrability, i.e. we are able to obtain solutions for this system under certain initial conditions.

Theorem 4.16. *A function f is constant along integral curves of X_H if and only if $\{f, H\} = 0$, i.e., if f is in **involution** with H.*

Proof. (Ref. [3], theorem 18.9, p.109). Let ρ_t be the flow of the Hamiltonian vector field X_H. We have:

$$\begin{aligned} \frac{d}{dt}(f \circ \rho_t) &= \rho_t^* \mathcal{L}_{X_H} f = \rho_t^* \iota_{X_H} df = \rho_t^* \iota_{X_H} \iota_{X_f} \omega = \\ &= \rho_t^* \omega(X_f, X_H) = \rho_t^* \{f, H\} \end{aligned}$$

□

Definition 4.17. ***First Integral/Integral of Motion/Constant of Motion***
*A function f which is constant along the curves of X_H, as in the above theorem, is called a **first integral**.*

We say that functions $f_1, ..., f_n$ on M are **independent** if their differentials $(df_1)_p, ..., (df_n)_p$ are linearly independent at all points p on some open dense subset of M, i.e., they are enough to construct coordinates on M. In general, Hamiltonian systems do not admit first integrals which are independent of the Hamiltonian function.

The Hamiltonian vector fields of commuting first integrals, with respect to the Poisson bracket, generate an isotropic subspace of T_pM, i.e., $\omega(X_{f_i}, X_{f_j}) = \{f_i, f_j\} = 0$. Therefore, the maximum number of independent first integrals is half the dimension of M, which is then called a Lagrangian subspace. This is true by symplectic linear algebra since given a symplectic basis $e_1, ..., e_n, v_1, ...v_n$ we have $\omega(e_i, e_j) = 0 = \omega(v_i, v_j)$ and $\omega(e_i, v_j) = \delta_{ij}$.

Definition 4.18. ***(Completely) Integrable Hamiltonian system***
*A Hamiltonian system (M, ω, H) is **(completely) integrable** if it possesses $n = \frac{1}{2} dim\, M$ independent first integrals, $f_1 = H, f_2, ..., f_n$ which are pairwise in involution with respect to the Poisson bracket, i.e., $\{f_i, f_j\} = 0$ for all i, j.*

For a regular value, $c \in \mathbb{R}^n$, of $f := (f_1, ..., f_n)$ independent first-integrals, the level set $f^{-1}(c)$ is a Lagrangian submanifold since it is n-dimensional and its tangent bundle is isotropic.

4.4. Variational Principles

The motions of the mechanical systems arise as solutions to a variational problem concerned with minimizing the mean value of kinetic minus potential energy. By the correspondence of Newton's equations with Hamilton equations, solving the latter in phase space for the integral curve of the Hamiltonian vector field with Hamiltonian function H gives us the solution to Newton's second law in configuration space.

Let us consider a mechanical system whose configuration space is $\mathbb{R}^n$ with coordinates $q = (q_1, ..., q_n)$ and a corresponding phase space in $\mathbb{R}^{2n}$ given by coordinates (q, v). We assume the system has a smooth function $L : \mathbb{R}^{2n} \to \mathbb{R}$, called the Lagrangian, given by $L = L(q, \dot{q}, t) = T - V$. Given a differentiable curve $\gamma : [0, 1] \to \mathbb{R}^n$ we define the action of this Lagrangian along γ by:

$$A_L(\gamma) = \int_0^T L(\gamma(t), \dot{\gamma}(t))dt$$

Given x and y in $\mathbb{R}^n$, let $C([0, T], x, y)$ denote the family of differentiable curves $\gamma : [0, T] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(T) = y$. We now look for critical points of A_L on $C([0, T], x, y)$, that is, curves γ such that for any $\Gamma_s \in C([0, T], x, y)$ we have, $s \in (-\epsilon, \epsilon)$ with $\Gamma_0 = \gamma$ and:

$$\left.\frac{dA_L(\Gamma_s)}{ds}\right|_{s=0} = 0$$

The principle of least action states that the path γ_0 for which the action $\mathcal{A}_\gamma$ is minimized is the actual physical path of the system.

Proposition 4.19. ***Euler-Lagrange equations***
The curve γ is a critical point of A_L if and only if it satisfies the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(\gamma(t), \dot{\gamma}(t))\right) = \frac{\partial L}{\partial q}(\gamma(t), \dot{\gamma(t)})$$

Proof. Let $c_i : [0, T] \to \mathbb{R}$, $i \in 1, ..., n$ be smooth functions such that $c_i(0) = c_i(T) = 0$. Define $\Gamma_s(t) = (\gamma_1(t) + sc_1(t), ..., \gamma_n(t) + sc_n(t))$. Then, we have that:

$$\frac{dA_L(\Gamma_s)}{ds} \mid_{s=0} = \int_0^T \sum_i \frac{d}{ds} L\left(\gamma_i(t) + sc_i(t), \dot{\gamma}_i(t) + s\dot{c}_i(t)\right) =$$

where $q_i = \gamma_i(t) + sc_i(t)$ and $\dot{q}_i = \dot{\gamma}_i(t) + s\dot{c}_i(t)$.

$$= \int_0^T \sum_i \left(\frac{\partial L}{\partial q_i}(\gamma, \dot{\gamma})c_i + \frac{\partial L}{\partial v_i}\dot{c}_i\right) dt$$

using integration by parts, i.e, $\int_0^T \frac{\partial L}{\partial v_i}(t)\dot{c}_i(t)dt = \frac{\partial L}{\partial v_i}c_i(t)|_0^T - \int_0^T \frac{d}{dt}(\frac{\partial L}{\partial v_i}(t))c_i(t)dt$, where $\frac{\partial L}{\partial v_i}c_i(t)|_0^T = 0$ since there is no variation on the endpoints, we obtain:

$$= \int_0^T \sum_i \left(\frac{\partial L}{\partial q_i}(\gamma, \dot{\gamma}) - \frac{d}{dt}\frac{\partial L}{\partial v_i}(\gamma, \dot{\gamma})\right) c_i\, dt$$

setting equal to zero we obtain then $\frac{d}{dt}\left(\frac{\partial L}{\partial v}(\gamma(t),\dot{\gamma}(t))\right) = \frac{\partial L}{\partial q}(\gamma(t),\dot{\gamma(t)})$.

$\square$

Assuming the Legendre condition: $\left(\frac{\partial^2 L}{\partial v_i \partial v_j}(x,v)\right) > 0$ for all (x,v) we can guarantee that the solution of the Euler-Lagrange equation is at least locally minimizing. For the theorem and further discussion on this topic see section 19.5, p.117 of [3].

5. Geometric Quantization

Our best approaches to describing natural phenomena, at least on the sub-atomic level, led us to the quantum theory. So, as far as we understand it, reality is essentially quantum in nature with our usual classical descriptions being merely approximations in certain limiting situations.

However, as stated in [5], it is often the case that the problem of determining the quantum behaviour of a particular system is far more addressable if one considers first its classical limit. One such example is given by the study of electromagnetic radiation, where one quantizes the classical Maxwell's equations.

It is also the case that a lot of interesting physical systems need to be represented by more than just $\mathbb{R}^{2n}$. Often, one takes the phase space to be a cotangent bundle of some smooth manifold but in general one works with an arbitrary symplectic manifold as the space of classical states.

There are several approaches to what is understood by quantization, here we are solely concerned with geometric quantization. The objective is to obtain a typical quantum structure by expanding on the geometric description of classical systems. Both theories, be it classical or quantum, have a comparable underlying structure. In both cases we have a description of states of the system, which we call the phase space, and observables which consist of "measurable" properties of the system. The following table gives a brief presentation of the mathematical objects involved in each theory.

	Classical Theory	Quantum Theory
Space of states	Symplectic manifold (M, ω)	Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$
Observables	$f \in C^\infty(M; \mathbb{R})$	Linear self-adjoint operators on $\mathcal{H}$ $(\mathcal{L}_{sa}(\mathcal{H}))$

Very briefly then, the goal is to associate to our symplectic manifold an Hilbert space representing our quantum states and promote functions on the symplectic manifold to operators acting on this Hilbert space. As we shall see later, there are quite a number of requirements to be met when performing this transition.

On the first sections, where we cover the process of **prequantization**, we mainly follow [2] and [1]. On the sections after, we will essentially turn to [5]. Here we will consider the concept of a **polarization**, the **half-form correction** and **pairing maps**. This sections are also heavily influenced by [10], [11] and [7].

5.1. Mathematical Model of Classical Mechanics

For simple cases we might have an accurate description of the phase space of a certain classical system by considering our manifold as $\mathbb{R}^{2n}$, with the usual pair of coordinates $(q_1, ..., q_n, p_1, ..., p_n)$, representing position and momentum. Note that this is also a symplectic manifold with a canonical symplectic form given by $\omega_0 = \sum_i^n dq_i \wedge dp_i$.

In general however, we consider the phase space M to be the cotangent bundle of some smooth manifold N called the configuration space, thus $M = T^*N$. In this case, the coordinates, q_i with $i = 1, .., n$, are defined only locally on some chart $(\mathcal{U}, \varphi)$, with φ an homeomorphism to an open set $\mathcal{V} \subset \mathbb{R}^n$. The corresponding coordinates, p_j with $j = 1, .., n$, are functionals on the dual space $(\mathbb{R}^n)^*$ which give us trivializing coordinates on the cotangent bundle over $\mathcal{U}$ by identifying $T^*\mathcal{U}$ with $\mathcal{V} \times (\mathbb{R}^n)^*$.

On $M = T^*N$ we have a well-defined 1-form given by $\theta = \sum_{k=1}^{n} p_k dq_k$ therefore its differential is also well-defined and given by:

$$\omega = d\theta = \sum_{k=1}^{n} dp_k \wedge dq_k \tag{5.1}$$

This gives us a 2-form form which is obviously closed and nondegenerate on M, i.e., a symplectic form.

However, we should not restrict ourselves to considering only cotangent bundles of some smooth manifold. In general, to obtain the Hamiltonian formalism, and thus a classical system, one needs only to consider a **symplectic manifold** (M, ω) which, by **Darboux's theorem** (3.19), as a symplectic form given locally as in the form of equation (5.1) by some suitable choice of coordinates. This choice of coordinates is not uniquely defined therefore considering q as the "position" and p as the "momentum" is purely conventional.

The **states of the system** are identified with the coordinates on the phase space M, which physically corresponds to the specification of the position and momentum of all particles present in the system.

The **observables (or physical quantities)** acting on these states are identified with smooth functions on M, i.e, the space $C^\infty(M)$. On this space, we introduce an operator, the **Poisson bracket** (def. 4.11), which gives it a structure of a **Lie Algebra** (def. 4.8).

The time-evolution of a system, its dynamics, is obtained by choosing an arbitrary function $H \in C^\infty(M)$ called the Hamiltonian. In physical terms this function usually corresponds to the energy of the system. There are two equivalent ways of describing the dynamics:

- **Hamiltonian picture**: The states are time-independent while observables are time-dependent, represented as smooth functions on $M \times \mathbb{R}$. The equations of motion are given by Hamilton's equation:

  $$\dot{F} = \{H, F\} = \omega(X_H, X_F)$$

  for an arbitrary observable $F \in C^\infty(M \times \mathbb{R})$ where X_H, X_F are the corresponding Hamiltonian vector fields (see def. 4.1). By applying this equation to the canonical variables p_k and q_k one obtains Hamilton's equations:

  $$\dot{q_k} = \frac{\partial H}{\partial p_k}, \quad \dot{p_k} = -\frac{\partial H}{\partial q_k}$$

- **Liouville picture**: The states are time-dependent while the observables are now time-independent. This description is mostly used in statistical mechanics where we consider the density of states $\rho(q, p, t)$ with the time dependency on $q(t)$ and $p(t)$. In this picture, time evolution is given by the Liouville equation:

  $$\dot{\rho}(p, q, t) = \{\rho, H\} = -\{H, \rho\}$$

Although both descriptions can be shown to be equivalent, here we are concerned solely with the Hamiltonian formulation, since it is the one used in the geometric quantization procedure which we will cover.

A **conserved quantity** in a classical system corresponds to a function F which is in involution with the Hamiltonian, i.e. $\{F, H\} = 0$, we call such a function F a **first integral**. A finite dimensional Lie algebra $\mathfrak{g}$ generated by several first integrals corresponds to a Lie group $G = exp\ \mathfrak{g}$ of symmetries which acts on the system. We expect these symmetries to be manifest on the corresponding quantized system which would be consistent with a group of unitary operators acting on the Hilbert space.

Definition 5.1. ***Complete set of classical observables***
Let (M, ω) be a symplectic manifold representing a classical system. A set of observables $\{f_j\}$ is said to be a complete set of classical observables if and only if, for every other observable g such that $\{f_j, g\} = 0$, for all j, implies g is constant.

If a set $\{f_j\}$ is complete then it serves as local coordinates around the neighbourhood of any $p \in M$. A trivial example of a complete set is the coordinates $\{q_1, ..., q_n, p_1, ...p_n\}$ on the phase space $T^*\mathbb{R}^n$. It is also our expectation that complete sets in a classical system would manifest as a complete set of operators acting on the quantum Hilbert space.

5.2. Mathematical Model of Quantum Mechanics

In the case of a quantum system the **phase space** can be characterized by a complex Hilbert space $\mathcal{H}$, usually taken to be the complex Hilbert space of square-integrable functions $L^2(\mathbb{R}^n)$. A state of the system corresponds to a function $\psi \in L^2(\mathbb{R}^n)$ with unit-norm, which we call a wave-function.

Remark 5. We probably should state that individual points on this Hilbert space do not necessarily correspond to different states of the system. Since the dynamical equations in quantum mechanics are linear, we can not distinguish between a state ψ or $\lambda\psi$, $\lambda \in \mathbb{C}$. Thus, a single state is actually represented by a ray in this space. This redundancy can be taken care of by considering the projective Hilbert space (see for example [1], section 2.1.1, p. 7). Here however, we shall not be too concerned with this.

The **observables** are given by self-adjoint linear operators with domain and range on the Hilbert space, $\mathcal{L}_{sa}(\mathcal{H})$. We take operators to be self-adjoint to assure that their eigenvalues are real since they represent measurable physical quantities of the state of the system.

The vector space of observables in a quantum system is equipped with two bilinear operations corresponding to the usual multiplication of functions and the Poisson bracket on the classical system. These are defined respectively as:

- **Jordan multiplication:**

  $$\hat{A} \circ \hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})$$

  With this operation the set of observables forms a commutative non-associative algebra.

- **Commutator:**

  $$[\hat{A}, \hat{B}]_h = \frac{2\pi i}{h}(\hat{A}\hat{B} - \hat{B}\hat{A}).$$

  With respect to this operation the set of observables forms a Lie Algebra.

Observables in quantum mechanics do not necessarily have an exactly defined value. In this case, the inner product on the space of states is given by an integral which we can interpret in terms of a probability distribution.

The dynamics of the system is given by the operator $\hat{H}$, called the Hamiltonian or energy operator. Just as in the classical case we have two descriptions of the dynamics:

- **Heisenberg Picture**: We have time independent-states with time-dependent observables. The motion of the system is described by Heisenberg's equation:

  $$\dot{\hat{A}} = [\hat{H}, \hat{A}]_h,$$

  which is the analog of Hamilton's equations. The integrals of the system are all the operators which commute with the Hamiltonian operator $\hat{H}$. The law of conservation of energy for example, is manifest from the fact that $\hat{H}$ commutes with itself.

- **Schrödinger Picture**: In this picture states are time-dependent while the observables are time-independent. The evolution of the system is given by Schrödinger's equation:

$$\dot{\psi} = \frac{2\pi i}{h}\hat{H}\psi$$

The Heisenberg picture is more convenient for the purpose of geometric quantization. This is clear from the immediate resemblance between the equations of motion with those of the Hamiltonian picture in the classical case. This difference is captured by the brackets, the quantum case given by the commutator while in the classical case it is given by the Poisson bracket.

Definition 5.2. ***Complete set of quantum observables***
Let $\mathcal{H}$ be a Hilbert space representing a quantum system. A set of operators $\{O_j(\mathcal{H})\}$ is said to be a complete set if and only if, for every other operator O such that $[O_j, O] = 0$ for all j, implies that $O = \lambda I$ for some constant $\lambda \in \mathbb{C}$.

The following proposition relates the notion of a complete set of quantum observables with the irreducibility of the Hilbert space $\mathcal{H}$.

Proposition 5.3. *If a set of self-adjoint operators $\{O_j\}$ on $\mathcal{H}$ is a complete set of operators, then $\mathcal{H}$ is irreducible under the action of $\{O_j\}$. This is equivalent to stating that every closed subspace of $\mathcal{H}$, which is invariant under the action of $\{O_j\}$, is either equal to $\{0\}$ or $\mathcal{H}$.*

Proof. (Ref. [1] proposition 1, p.10). □

That complete sets in the classical sense should give rise to complete sets in the corresponding quantized system will be a major requirement for the process of quantization. This concept is intimately related with the correspondence of symmetries in the classical and quantum systems (see further discussion on this in [1], section 2.2.2. pp. 10-13).

A major result in quantum mechanics is the **Heisenberg uncertainty principle**. In regards to the position and momentum operators in $L^2(-\infty, +\infty)$ it states that $[\hat{p}, \hat{x}] \geq \frac{h}{4\pi}$. This requirement of the quantum theory has major implications in the mathematical construction of the quantization procedure. In particular, it leads to the necessary structure of a **polarization**.

As a guiding example will see how the classical and quantum descriptions are related in the case of $\mathbb{R}^{2n}$.

Example 5.4. ***Classical and quantum descriptions on*** $M = \mathbb{R}^{2n}$
Let us consider the configuration space given by $\mathbb{R}^n$ with coordinates $\{q^i\}_{i=1}^n$. Then, the symplectic manifold $M = \mathbb{R}^{2n}$ with the canonical symplectic form given by $\omega = \sum_{i=1}^n dq^i \wedge dp^i$ and Euclidean coordinates $\{q^i, p^i\}_{i=1}^n$ corresponds to the phase space (the cotangent bundle) of the system.

A wave function, of the position coordinates, is a map $\psi : \mathbb{R}^n \mapsto \mathbb{C}$ which lives on the complex Hilbert space of square-integrable functions $\mathcal{H} = L^2(\mathbb{R}^n, dq)$, where dq denotes a Lebesgue measure in $\mathbb{R}^n$. The usual inner-product in $\mathcal{H} = L^2(\mathbb{R}^n, dq)$ is given by:

$$\langle \psi, \phi \rangle = \int_{\mathbb{R}^n} \overline{\psi}\phi \, dq.$$

Where we interpret $|\psi|^2 = \langle \psi, \psi \rangle$ as a probability density.

Given an Hamiltonian function $H \in C^\infty(\mathbb{R}^{2n})$ *with the following form:*

$$H = \sum_{i=1}^{n} \frac{(p^i)^2}{2} + V(q),$$

where $V : \mathbb{R}^n \mapsto \mathbb{R}$ *represents the potential, the basic physical quantities or observables are:*

$$\text{the position operator: } q^i \mapsto q^i;$$
$$\text{the momentum operator: } p^i \mapsto -i\hbar \frac{\partial}{\partial q^i};$$
$$\text{the energy operator: } \hat{H} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial {q^i}^2} + V(q).$$

Note that it is the relation $\{q^i, p^j\} = \delta^{ij}$ *which leads to Heisenberg's uncertainty principle,* $\frac{i}{\hbar}[\hat{q}^i, \hat{p}^j]$.

5.3. Statement of the quantization problem

Having roughly described the basics of the mathematical aspects of classical mechanics and quantum mechanics, we are now ready to state the general problem of quantization. The idea is to construct a quantum mechanical system which is the analog of a classical one. Mathematically, this consists of starting with the geometry of a symplectic manifold (M, ω), associated to a classical system, and from this structure build an Hilbert space $\mathcal{H}$ with a set of self-adjoint operators acting on it which would represent the quantum analog of this system.

Remark 6. It is possible in general for a quantum system to have several distinct classical systems as limiting cases.

As said before, it is natural to expect that the obtained quantized system should manifest the same symmetries as the starting classical version. If we have a Lie group G of symmetries on the classical phase space, which can be represented by the group of symplectomorphisms acting on it, on the corresponding quantum system these symmetries should have a representation as unitary transformations acting on the Hilbert space $\mathcal{H}$.

However, the maximal symmetry group on (M, ω) is the infinite dimensional group of all symplectomorphisms (i.e. transformations on M which preserve the symplectic form ω). While on a quantum system the maximal symmetry group is the infinite dimensional group $PU(\mathcal{H})$, the group of all projective unitary transformations. Since these two groups are not isomorphic (see [2]) we have no reason to expect a quantum symmetry for each classical one. What happens is that some finite dimensional symmetry groups might be preserved while others are broken, these are called "quantum anomalies" in the commutation relations.

Having understood the mathematical descriptions of both theories, we are in a good position to state the requirements we expect a quantization procedure should obey. The following definition, due to Paul Dirac, states these requirements in the form of axioms.

Definition 5.5. ***Dirac Axioms of Quantization***
A quantization of a classical system, on a symplectic manifold (M, ω), is a linear map $F \mapsto \hat{F}$ which associates elements of the Poisson algebra $C^\infty(M)$ to the set of operators $\mathcal{L}_{sa}(\mathcal{H})$, acting on the Hilbert space $\mathcal{H}$. This map should satisfy the following:

(i) *$\hat{1} = I_{\mathcal{H}}$, the 1 on the left represents the function $f \in C^\infty(M)$ which is identically equal to 1 while $I_{\mathcal{H}}$ is the identity operator acting on $\mathcal{H}$.*

(ii) *$\widehat{\{F, G\}} = [\hat{F}, \hat{G}]_h = i\hbar(\hat{F}\hat{G} - \hat{G}\hat{F})$, the map is a Lie Algebra morphism;*

(iii) *$\widehat{F^*} = (\hat{F})^*$, complex conjugation transforms to the adjoint operator;*

(iv) *Given a complete set of classical observables $\{f_j\}$ on (M, ω) then the set $\{O_{f_j}\}$ of operators on $\mathcal{H}$ should also form a complete set.*

It turns out, by expanding on Van Hove's theorem (see [13]), that it is not possible to find a general method of quantization obeying all the requirements of the previous definition. There are however many interesting mathematical results arising from the framework of geometric quantization.

We now introduce the concept of a prequantization which is a map of the type $F \mapsto \hat{F}$ discussed above.

5.4. Prequantization

The Koopman-Van Hove-Segal Representation

In the context of geometric quantization, if a linear map $F \mapsto \hat{F}$ satisfies the first three properties of definition (5.5), we call it a prequantization. Such a map was constructed by Irving Segal (by generalizing results of Bernard Koopman and Léon Van Hove) for the case when $M = T^*N$, $\omega = d\theta$, i.e., when the symplectic manifold is the cotangent bundle of some configuration space. The map is given by:

$$\hat{f} = f - i\hbar X_f + \theta(X_f), \tag{5.2}$$

where X_f is the Hamiltonian vector field on M generated by the function $f \in C^\infty(M)$.

The motivation behind this particular map can be seen as follows. Let $f, g \in C^\infty(M)$ and consider the corresponding Hamiltonian vector fields X_f, X_g. By proposition 4.12 we have:

$$X_{\{f,g\}} = -[X_f, X_g] \;\Rightarrow\; \widehat{\{f,g\}} = -i\hbar X_{\{f,g\}} \;\Leftrightarrow \widehat{\{f,g\}} = i\hbar[X_f, X_g].$$

It is tempting then to choose $\hat{f} = -i\hbar X_f$ as a prequantization map, since it satisfies the desired commutation relations. However condition (i), i.e. $\hat{1} = I_{\mathcal{H}}$ is clearly not true. To change this we add a term such that the prequantization map now becomes:

$$\hat{f} = -i\hbar X_f + f,$$

this solves the problem of condition (i) but ends up breaking the commutation relations we had previously. To obtain the relations required of a prequantization we add a term which is related to the symplectic potential $d\theta = \omega$ thus ending up with equation (5.2). That it does indeed satisfy the requirements for a prequantization will be shown in the next section on the more general prequantization procedure by Kostant and Souriau.

However, this map is not a full quantization, it could never be considering the statement we made above regarding Van Hove's theorem. Let us see an example, in $\mathbb{R}^2$, where it fails to satisfy the 4^{th} axiom.

Example 5.6. *Consider $M = T^*\mathbb{R}$. Let the canonical symplectic form be given by $\theta = pdq$, and thus $\omega = dp \wedge dq$, the coordinate and momentum operators are mapped as:*

$$\widehat{f(p,q)} = \hat{q} = q - i\frac{h}{2\pi}\frac{\partial}{\partial p}, \quad \widehat{g(p,q)} = \hat{p} = i\frac{h}{2\pi}\frac{\partial}{\partial q},$$

where one obtains X_f, X_g by definition 4.1. If we consider the operators given by:

$$\left(\frac{\partial}{\partial p}\right) \text{ and } \left(\frac{\partial}{\partial q} + \frac{2\pi i}{h}p\right).$$

We can easily see that they commute with $\hat{q}$ and $\hat{p}$:

$$
\begin{aligned}
\left[\frac{\partial}{\partial p},\hat{q}\right] &= \frac{\partial}{\partial p}\left(q+\frac{h}{2\pi i}\frac{\partial}{\partial p}\right)-\left(q+\frac{h}{2\pi i}\frac{\partial}{\partial p}\right)\left(\frac{\partial}{\partial p}\right)=\frac{\partial q}{\partial p}+\frac{h}{2\pi i}\frac{\partial^2}{\partial p^2}-q\frac{\partial}{\partial p}-\frac{h}{2\pi i}\frac{\partial^2}{\partial p^2}=0\\
\left[\frac{\partial}{\partial p},\hat{p}\right] &= -\frac{h}{2\pi i}\frac{\partial^2}{\partial p\partial q}+\frac{h}{2\pi i}\frac{\partial^2}{\partial q\partial p}=0\\
\left[\left(\frac{\partial}{\partial q}+\frac{2\pi i}{h}p\right),\hat{q}\right] &= 1+\frac{h}{2\pi i}\frac{\partial^2}{\partial q\partial p}+\frac{2\pi i}{h}pq+p\frac{\partial}{\partial p}-q\frac{\partial}{\partial q}-\frac{2\pi i}{h}qp-\frac{h}{2\pi i}\frac{\partial^2}{\partial p\partial q}-1=0\\
\left[\left(\frac{\partial}{\partial q}+\frac{2\pi i}{h}p\right),\hat{p}\right] &= -\frac{h}{2\pi i}\frac{\partial^2}{\partial q^2}-p\frac{\partial}{\partial q}+\frac{h}{2\pi i}\frac{\partial^2}{\partial q^2}+\frac{\partial}{\partial q}p=0
\end{aligned}
$$

Since these operators are clearly not multiples of the identity operator, then $\hat{q}$ and $\hat{p}$ do not form a complete set as operators acting on $\mathcal{H}$, while they did form a complete set on $M = T^*\mathbb{R}$ trivially. In conclusion, condition (iv) for a full quantization is not satisfied.

5.4.1. The Kostant-Souriau Prequantization

The generalization of the previous prequantization, from cotangent bundles to general symplectic manifolds leads to an operator that acts on **complex line bundles over** M, where the fibers are equipped with a connection and an Hermitian structure which are compatible.

The fundamental ingredient for the previous prequantization was that $\omega = d\theta$. If it is not the case that ω is exact, the map in (5.2) will not work, in particular it will not work on arbitrary compact manifolds (the volume form is given by ω^n and thus can not be exact due to Stoke's theorem and positivity of the volume integral). However, since a closed form ω is always exact locally (by Poincaré's lemma, theorem 2.22), we can cover M by open sets U_α such that in each U_α the form ω is given locally by $\omega = d\theta_\alpha$, for some suitable 1-form θ_α.

It follows from equation (5.2), that on each U_α, we have the prequantization map $\hat{F}_\alpha \in C^\infty(U_\alpha)$ given by:

$$\hat{F}_\alpha = F - i\hbar X_F + \theta_\alpha(X_F),$$

It so happens that this maps can be "glued together" in a way which allows us to obtain a global operator $\hat{F}$, that acts on the sections of a line bundle L over M. For this to be the case, it is necessary that the cohomology class given by the form ω be integral, i.e., the integral of the form ω over an arbitrary 2-cycle in M must be an integer. Let us construct this formally.

Let L be a complex vector bundle over M with a one-dimensional fibre, where we assume that L is equipped with an Hermitian structure $\langle\cdot,\cdot\rangle$ and a connection ∇ which are compatible, that is:

$$X \cdot \langle s_1, s_2\rangle = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle\,,$$

where X is a vector field on M and ∇_X is an operator corresponding to covariant differentiation of sections along X.

If the bundle L over U_α admits non-vanishing sections s_α, we have an isomorphism between smooth functions on U_α and the space of sections $\Gamma(L, U_\alpha)$, given by:

$$C^\infty(U_\alpha) \ni \phi \leftrightarrow \phi \cdot s_\alpha \in \Gamma(L, U_\alpha)$$

Under this isomorphism the operator ∇_X takes the form:

$$\nabla_X(\phi \cdot s_\alpha) = (X \cdot \phi)s_\alpha + \phi \nabla_X s_\alpha$$

We can define $\nabla_X s_\alpha = \psi s_\alpha$, for a certain $\psi \in C^\infty(\mathcal{U}_\alpha)$, specifying ψ from θ_α (this is an assumption[2]) according to:

$$\nabla_X s_\alpha = \frac{i}{\hbar}\theta_\alpha(X) \cdot s_\alpha,$$

and thus we obtain, for a general section $s = \sum_i f^i s_i$ on U_α, with $f^i \in C^\infty(M)$:

$$\nabla_X s = \sum_i X \cdot f^i + \frac{i}{\hbar}\theta_\alpha(X)s_i \tag{5.3}$$

Comparing this result with equation (5.2) suggests that the prequantization map should be:

$$\hat{F} = F - i\hbar X_F + \theta_\alpha(X_F) = F - i\hbar\left(X_F + \frac{i}{\hbar}\theta_\alpha(X_F)\right) = F - i\hbar\nabla_{X_F} \tag{5.4}$$

Let us confirm that the prequantization conditions are indeed satisfied by this map:

- **Linearity:** Given F and G generating functions for the Hamiltonian fields X_F and X_G we have:

  $$\widehat{F+G} = (F+G) - i\hbar\nabla_{X_{F+G}} = \hat{F} + \hat{G},$$

  since the map $F \mapsto X_F$ is a Lie algebra homomorphism, in particular it satisfies $X_{\alpha F+\beta G} = \alpha X_F + \beta X_G$. Linearity is then verified.

- **Condition 1:** Clearly true by plugging the function, $F = 1$ on M, in equation (5.4).

- **Condition 3:** (Ref. [1], theorem 8, pp.33-34). The operators $\hat{F}$ corresponding to $F \in C^\infty(M)$ should be self-adjoint, i.e., we want to check that:

  $$\left\langle \hat{F}s_\alpha, s_\beta \right\rangle = \left\langle s_\alpha, \hat{F}s_\beta \right\rangle,$$

  where we define the scalar product on the space of sections of L by:

  $$(s_1, s_2) = \int_M \langle s_1, s_2\rangle \, dv,$$

  with $dv = \omega^n$. We then have:

$$\left(\hat{F}s_\alpha, s_\beta\right) = \int_M \left\langle \hat{F}s_\alpha, s_\beta \right\rangle dv = \int_M \langle Fs_\alpha - i\hbar\nabla_{X_F}s_\alpha, s_\beta\rangle \, dv = \int_M \langle Fs_\alpha, s_\beta\rangle - i\hbar\,\langle \nabla_{X_F}s_\alpha, s_\beta\rangle \, dv =$$
$$= \int_M \langle Fs_\alpha, s_\beta\rangle \, dv - i\hbar\int_M \nabla_{X_F}\langle s_\alpha, s_\beta\rangle \, dv + i\hbar\int_M \langle s_\alpha, \nabla_{X_F}s_\beta\rangle \, dv,$$

[2]This assumption leads to the correct prequantization procedure and manifests a relation between the curvature of the connection with the symplectic form.

where we used the assumption of metric compatibility on the last line. Proceeding with the calculations:

$$\int_M \langle F s_\alpha, s_\beta \rangle \, dv - i\hbar \int_M \nabla_{X_F} \langle s_\alpha, s_\beta \rangle \, dv + i\hbar \int_M \langle s_\alpha, \nabla_{X_F} s_\beta \rangle \, dv =$$
$$= \int_M \langle s_\alpha, F s_\beta \rangle \, dv + \int_M \langle s_\alpha, -i\hbar \nabla_{X_F} s_\beta \rangle \, dv - i\hbar \int_M \nabla_{X_F} \langle s_\alpha, s_\beta \rangle \, dv =$$
$$= \int_M \left\langle s_\alpha, \hat{F} s_\beta \right\rangle dv - i\hbar \int_M \nabla_{X_F} \langle s_\alpha, s_\beta \rangle \, dv,$$

where we used equation (5.4) on s_β when transitioning to the last expression. If condition 3 is to be verified then the remaining integral on the right must vanish. The integral is of the type:

$$\int_M X_F \cdot h \, dv,$$

where $h \in C^\infty(M)$ is a function with compact support (see definition (7.2)) representing $\langle s_\alpha, s_\beta \rangle$. We have the following relations:

$$(X_F \cdot h) \, dv = \mathcal{L}_{X_F}(h \, dv) - h \mathcal{L}_{X_F}(dv) = \mathcal{L}_{X_F}(h \, dv),$$

where on the second equality we used the fact that $\mathcal{L}_{X_F}(dv) = 0$ since the symplectic form is preserved by the Hamiltonian vector field. Now, using Cartan's formula we obtain:

$$\mathcal{L}_{X_F}(h \, dv) = \iota_{X_F} d(h \, dv) + d(\iota_{X_F}(h \, dv)) = d(\iota_{X_F}(h \, dv)),$$

where $d(h \, dv) = 0$, since it is a maximum rank form on M. We finally end up with:

$$\int_M d(\iota_{X_F}(h \, dv))$$

This last integral vanishes since h is a function with compact support, by Stoke's theorem:

$$\int_M d(\iota_{X_F}(h \, dv)) = \int_{\partial M} \iota_{X_F}(h dv).$$

Since h is a smooth function and has compact support that means $h = 0$ on ∂M, thus **condition 3 is satisfied**.

To verify **condition 2** it is necessary to look at a few results first.

Definition 5.7. ***Curvature form of the connection*** ∇ ***on sections*** $s \in \Gamma(L)$
The curvature of a connection ∇ *acting on sections* $s \in \Gamma(L)$ *is defined as a 2-form* Ω *on* M*, which associates to a pair of vector fields* $X, Y \in \mathfrak{X}(M)$ *the map:*

$$\Omega(X,Y)s = \frac{1}{2\pi i}\bigg([\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \bigg) s \tag{5.5}$$

Proposition 5.8. *Locally, on each* U_α*, the curvature form* Ω *is given by the following expression:*

$$\Omega = \frac{1}{h} d\theta_\alpha \tag{5.6}$$

Proof. Applying equation (5.3) to the definition of curvature, equation (5.5), on a general section basis element s, we obtain:

$$
\begin{aligned}
2\pi i\Omega(X,Y)s =& \Big([\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \Big) s = \nabla_X \left(\frac{i}{\hbar}\theta_\alpha(Y)\cdot s \right) - \nabla_Y \left(\frac{i}{\hbar}\theta_\alpha(X)\cdot s \right) \\
& - \frac{i}{\hbar}\theta_\alpha([X,Y])\cdot s = \\
=& \left(\frac{i}{\hbar}\right)\Big((X\cdot\theta_\alpha(Y))\cdot s + \theta_\alpha(Y)\nabla_X s - (Y\cdot\theta_\alpha(X))\cdot s - \theta_\alpha(X)\nabla_Y s \Big) + \\
& - \frac{i}{\hbar}\theta_\alpha([X,Y])\cdot s.
\end{aligned}
$$

Now, making use of the following relation (for an arbitrary smooth 1-form β):

$$
d\beta(X,Y) = X\cdot\beta(Y) - Y\cdot\beta(X) - \beta([X,Y]), \quad \text{we obtain:}
$$

$$
\begin{aligned}
\Omega(X,Y)s =& \frac{1}{\hbar}\Big((X\cdot\theta_\alpha(Y)) + \theta_\alpha(Y)\nabla_X - (Y\cdot\theta_\alpha(X)) - \theta_\alpha(X)\nabla_Y \Big)\cdot s + \\
& - \frac{1}{\hbar}\Big((X\cdot\theta_\alpha(Y)) - (Y\cdot\theta_\alpha(X)) - d\theta_\alpha(X,Y) \Big)\cdot s = \\
=& \frac{1}{\hbar} d\theta_\alpha(X,Y)s,
\end{aligned}
$$

where the terms of the form $\theta_\alpha(X_i)\nabla_{X_j} \propto \theta_\alpha(X_i)\theta_\alpha(X_j)$, cancel since they are just products of functions. $\square$

Using these results let us check that **condition 2** is verified, i.e. the relation $\widehat{\{f,g\}} = [\hat{f},\hat{g}]$ is satisfied (ref. [1], theorem 8, pp. 33-34). Consider $s \in \Gamma(L)$, then:

$$
\begin{aligned}
[\hat{f},\hat{g}]s =& [f - i\hbar\nabla_{X_f}, g - i\hbar\nabla_{X_g}]s = (f - i\hbar\nabla_{X_f})(g - i\hbar\nabla_{X_g})s - (g - i\hbar\nabla_{X_g})(f - i\hbar\nabla_{X_f})s = \\
&= (fg)s - (i\hbar f\nabla_{X_g})s - i\hbar\nabla_{X_f}(gs) - \hbar^2\nabla_{X_f}\nabla_{X_g}(s) \\
&\quad - (gf)s - (i\hbar g\nabla_{X_f})s - i\hbar\nabla_{X_g}(fs) - \hbar^2\nabla_{X_g}\nabla_{X_f}(s) = \\
&= -\hbar^2[\nabla_{X_f},\nabla_{X_g}](s) - i\hbar\left(f\nabla_{X_g}s - g\nabla_{X_f}s + (X_f\cdot g)s + g\nabla_{X_f}s - (X_g\cdot f)s - f\nabla_{X_g}s \right) = \\
&= -\hbar^2[\nabla_{X_f},\nabla_{X_g}](s) - i\hbar\left((X_f\cdot g)s - (X_g\cdot f)s \right).
\end{aligned}
$$

Now, since $\iota_{X_f}\omega = df$, we have the following intermediate result:

$$
\{f,g\} = \omega(X_f,X_g) = \iota_{X_g}\iota_{X_f}\omega = df(X_g) = X_g\cdot f
$$

Thus, replacing the terms $(X_g\cdot f)$ and $(X_f\cdot g)$ on the original equation, we obtain:

$$
\begin{aligned}
&- \hbar^2[\nabla_{X_f},\nabla_{X_g}](s) - i\hbar\left((X_f\cdot g)s - (X_g\cdot f) \right)(s) = -\hbar^2[\nabla_{X_f},\nabla_{X_g}](s) + 2i\hbar\{f,g\}(s) = \\
&= -\hbar^2\Big(\nabla_{[X_f,X_g]} + 2\pi i\Omega(X_f,X_g) \Big)(s) + 2i\hbar\{f,g\}(s) = \left(\hbar^2\nabla_{-[X_f,X_g]} - 2\pi i\hbar^2\frac{1}{h}\omega(X_f,X_g) + 2i\hbar\{f,g\} \right)(s) = \\
&= \Big(\hbar^2\nabla_{-[X_f,X_g]} - i\hbar\{f,g\} + 2i\hbar\{f,g\} \Big)(s) = \Big(\hbar^2\nabla_{-[X_f,X_g]} + i\hbar\{f,g\} \Big)(s) = i\hbar\Big(- i\hbar\nabla_{X_{\{f,g\}}} + \{f,g\} \Big)(s) \\
&= i\hbar\widehat{\{f,g\}}(s)
\end{aligned}
$$

In order, we used equation (5.5) then equation (5.6) (with the assumption that $d\theta_\alpha = \omega$) and finally proposition (4.12).

By assuming the global result, $\Omega = \frac{1}{h}\omega$, we obtain a prequantization therefore we have the following theorem:

Theorem 5.9. *The Kostant-Souriau formula (5.4) gives a prequantization on (M, ω) if and only if the curvature form Ω of the connection ∇ coincides with $\frac{1}{h}\omega$.*

We assumed the existence of an Hermitian structure and a connection which are compatible on the line bundle L over M. Therefore, to obtain a prequantization in this manner, one must know when it is possible to define such a structure. On the other hand, from the previous theorem, if we have such a structure, it is enough to determine 2-forms on M that will serve as curvature forms for this connection, i.e., satisfying proposition (5.8). The following theorem gives an answer to both of these problems.

Theorem 5.10. *A form Ω is the curvature form of some line bundle L over M with connection ∇, if and only if the cohomology class defined by the form Ω is integral (that is, the integral of the form Ω over an arbitrary 2-cycle in M is an integer). An Hermitian structure on L compatible with ∇ exists if and only if the form Ω is real.*

Proof. (Refs. [2] theorem 2.23 (pp.151-152) and [7] section 5.2.2 (pp. 67-68)). □

Although we will not dwell on this topic, an interesting question to pose is whether the symplectic manifold (M, ω) uniquely determines the Kostant-Souriau prequantization, i.e., if it is defined solely by equivalent line bundles $(L_i, \langle \cdot, \cdot \rangle_i, \nabla_i)$ with suitable diffeomorphisms between them. This turns out to be the case when the manifold is simply-connected. A general theorem and further discussion can be seen in [2], theorem 2.3, pp.152-153.

Summary:, the Kostant-Souriau prequantization picks up a classical system, represented by the symplectic manifold (M, ω), and on it constructs a complex line bundle (L, π, M) equipped with an Hermitian metric and a compatible connection ∇, such that the curvature $\Omega = \frac{1}{2\pi i}\mathrm{curv}_\nabla = \frac{1}{h}\omega$, where curv_∇ is the usual curvature on vector bundles (see def. 2.33). Thus, we have a map between classical and quantum observables acting on their respective spaces. The conditions necessary for this construction are given by theorem (5.10).

5.5. Naive quantization of $\mathbb{R}^{2n}$

Let us attempt to quantize the familiar case of $\mathbb{R}^{2n}$ with canonical symplectic form $\omega_0 = \sum_j dq_j \wedge dp_j$, and fix eventual issues in a straight-forward approach. The main goal here is to look at a clear example which precisely motivates the introduction of the general setting of geometric quantization, i.e. the introduction of polarizations and the half-form correction. The discussion here is based on chapter 22 of [5], particularly on sections 22.3 to 22.5.

5.5.1. Prequantization of $\mathbb{R}^{2n}$

We start with a classical system which consists of the space of states given as pairs $(q_1, ...q_n, p_1, ...p_n) \in \mathbb{R}^{2n}$, and the set of classical observables given by smooth functions on $\mathbb{R}^{2n}$, i.e., $f \in C^\infty(\mathbb{R}^{2n})$.

By the Kostant-Souriau prequantization which we saw in the last section, let us consider a complex line bundle L over M, equipped with a compatible Hermitian structure and connection ∇. We can make sense of the smooth sections on L, $\Gamma(L, \mathbb{R}^{2n})$ as our Hilbert space of (pre)quantum states $\mathcal{H}_{preQ}$. In this case this space is just the space of complex valued functions $f \in C^\infty(\mathbb{R}^{2n})$.

In regards to the space of classical observables, the prequantization map promotes the functions $f \in C^\infty(\mathbb{R}^{2n})$ to (pre)quantum operators which will now act on the space of sections $\mathcal{O}_{preQ}(\Gamma(L, \mathbb{R}^{2n}))$. The explicit form of this map, which we saw on the previous section, is given by:

$$\hat{f} = f - i\hbar X_f + \theta(X_f) = f - i\hbar \nabla_{X_f},$$

where X_f is the Hamiltonian vector field corresponding to the function f and $\omega = d\theta$.

We have by now stated a few reasons why the prequantization map is not satisfactory as a full quantization. A particular example, which illustrates in physical terms why this can not be the case, is given by the harmonic oscillator. Here we solely comment on the result, for the full discussion see [5], proposition 22.6, pp. 473-474.

The idea is the following, if we take the Hamiltonian of the harmonic oscillator (which corresponds to the energy of the system), $H(q,p) = \frac{1}{2m}(p^2 + (m\omega q)^2)$, the solutions to Schrödinger's equation, in polar coordinates, are given by the eigenvectors $\psi_n(r, \phi) = f(r)e^{-in\phi}$, with corresponding energy eigenvalues $E_n = n\hbar\omega$. In these solutions n is any integer, i.e. $n \in \mathbb{Z}$, which means that the energy is not bounded from below. This result clearly shows that this is not an acceptable solution if one wishes to obtain a physically relevant quantization procedure.

5.5.2. Hilbert space of quantum states

To surpass this issues we must restrict to a subspace of $\mathcal{H}_{preQ}$, i.e. a subspace of the prequantum Hilbert space . Motivated by the usual Schrödinger's quantization, where the wave functions either depend fully on position or momentum coordinates, we attempt to somehow allow only half of the coordinates of $\mathbb{R}^{2n}$.

Before we do this however let us introduce the notion of a gauge transformation.

Definition 5.11. ***Gauge transformation***
Let $U : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$ be a unitary map. Given two affine connections ∇^1, ∇^2 (or equivalently two symplectic potentials) which define the prequantum operators $Q^1_{pre}(f), Q^2_{pre}(f)$ of a function $f \in C^\infty(\mathbb{R}^{2n})$, we call U a gauge transformation if:

$$U\nabla^1_X U^{-1} = \nabla^2_X \Rightarrow UQ^1_{pre}(f)U^{-1} = Q^2_{pre}(f),$$

for every vector field $X \in \mathfrak{X}(\mathbb{R}^{2n})$.

The choice of coordinates depend on this notion of gauge transformation. One can not simply consider functions of the form $\psi(q,p) = \psi(q,p')$ where $p \neq p'$ (as an example of a state dependent only on position coordinates) since these turn out to not be gauge-invariant, i.e. by applying a unitary transformation on these states we end up changing the values of observables. In general, the correct way of doing this is by considering covariantly constant functions along a set of n distinct directions, which corresponds to the notion of independence for gauge-invariant transformations. This is precisely the idea when one chooses a polarization $\mathcal{P}$ on the prequantum Hilbert space which we will cover in some detail on the next section.

There are three typically considered subspaces of the prequantum Hilbert space $C^\infty_{\mathbb{C}}(\mathbb{R}^{2n})$:

- **Position subspace:** where one takes functions ψ such that $\nabla_{\partial/\partial p_j}\psi = 0$, for all j.
- **Momentum subspace:** where one takes functions ψ such that $\nabla_{\partial/\partial q_j}\psi = 0$, for all j.
- **Holomorphic subspace:** which is given by functions ψ such that $\nabla_{\partial/\partial \bar{z}_j}\psi = 0$, for all j, where $z_j = x_j - i\alpha p_j$, $\alpha > 0$ and $\partial/\partial\bar{z}_j = \partial/\partial q_j + i\alpha\partial/\partial p_j$.

Fixing a symplectic potential θ (see the following remark on why this is not an issue), the explicit form of the functions in each of the resulting subspaces is given by the following proposition:

Proposition 5.12. *Considering the convenient symplectic potential $\theta = p_j dq_j$ we have the following subspaces of complex valued smooth functions on $\mathbb{R}^{2n}$:*

- ***Position subspace:*** $\psi(\mathbf{q},\mathbf{p}) = \phi(\mathbf{q})$.
- ***Momentum subspace:*** $\psi(\mathbf{q},\mathbf{p}) = e^{i\mathbf{q}\cdot\mathbf{p}/\hbar}\phi(\mathbf{p})$.
- ***Holomorphic subspace:*** $\psi(\mathbf{q},\mathbf{p}) = F(z_1,...,z_n)e^{-\alpha|\mathbf{p}|^2/(2\hbar)}$*, where F is an arbitrary holomorphic function on $\mathbb{C}^n$ and as before $z_j = q_j - i\alpha p_j$.*

Proof. (Ref. [5] proposition 22.8, pp.475-476). □

Remark 7. The explicit form of these subspaces is dependent on the choice of symplectic potential. However, given two symplectic potentials θ_1, θ_2 for the canonical symplectic form ω they turn out to be unitarily equivalent, i.e. they are obtained from each other by gauge transformations and therefore lead to the same results (see proposition 22.5 on [5], pp. 471-472).

In the pursuit of physically relevant solutions on the previous subspaces, we are interested only in those functions which are square-integrable, i.e., those which belong to $L^2(\mathbb{R}^{2n})$. In this regard, there are immediate issues with the subspaces we have identified so far, on the position and momentum subspaces there are no non-vanishing functions which are square-integrable. Using the position space as an example, the norm is given by:

$$|\psi(\mathbf{q},\mathbf{p})|^2 = \int_{\mathbb{R}^{2n}} \psi\overline{\psi} d\mathbf{p} d\mathbf{q} = \int_{\mathbb{R}^n} d\mathbf{p} \int_{\mathbb{R}^n} \phi\overline{\phi} d\mathbf{q},$$

since $\mathbb{R}^n$ is not bounded, the norm is infinite unless $\psi(\mathbf{q}) = 0$ almost everywhere. The situation is exactly the same if we consider the momentum subspace.

In general this issue is solved by considering a correction based on the concept of half-forms which we will introduce later when quantizing a generic symplectic manifold. For now, we proceed naively by integrating in the regions where one obtains sensible results, i.e., only on $\mathbb{R}^n$ in the variables for which the function depends on. By doing this we finally promote our space to an actual Hilbert space of quantum states. In each of the previous cases these spaces are given by:

- **Position Hilbert Space:** Space of functions on $\mathbb{R}^{2n}$ with corresponding norm given by:

$$\psi(\mathbf{q},\mathbf{p}) = \phi(\mathbf{q}), \qquad ||\psi||^2 = \int_{\mathbb{R}^n} |\phi(\mathbf{q})|^2 d\mathbf{q}$$

- **Momentum Hilbert Space:** Space of functions on $\mathbb{R}^{2n}$ with corresponding norm given by:

$$\psi(\mathbf{q},\mathbf{p}) = e^{i\mathbf{q}\cdot\mathbf{p}/\hbar}\phi(\mathbf{p}), \qquad ||\psi||^2 = \int_{\mathbb{R}^n} |\phi(\mathbf{p})|^2 d\mathbf{p}$$

- **Holomorphic Hilbert Space:** This space turns out to have non-zero square integrable functions over $\mathbb{R}^{2n}$, and forms a closed subspace of $L^2(\mathbb{R}^{2n})$. By identifying the states ψ with the holomorphic functions F in:

$$\Psi(\mathbf{q},\mathbf{p}) = F(z_1, ..., z_n)e^{-\alpha|\mathbf{p}|^2/(2\hbar)}, \quad \text{with norm } ||\Psi(F)||^2_{L^2(\mathbb{C}^n,\nu)} = \int_{\mathbb{C}^n} |F(\mathbf{z})|^2 \mu_{2\alpha\hbar}(\mathbf{z}) d\mathbf{z}.$$

  This space may be unitarily identified with a Segal-Bargmann space (see 7.2 in appendix). It is worth noting here that the choice of $z_j = q_j - i\alpha p_j$, $\alpha > 0$ is important since in the case of $z_j = q_j + i\alpha p_j$ the resulting functions are not square-integrable in $\mathbb{R}^{2n}$. This is a particular case of a more general positivity condition which divides the complex polarizations in those called Kähler polarizations, which are the well-behaved, and pseudo-Kähler polarizations which are not.

Despite the fact that we considered only three possibilities for the n directions, one could in general take any other combination to obtain different quantum Hilbert spaces.

5.5.3. Quantum observables

Having defined a new Hilbert space of quantum states we must, in general, change the prequantum operators such that they act invariantly on these spaces. By acting invariantly we mean such that $\hat{f}$ maintains us on the space of polarized sections. In the quantum spaces we are considering however, we have the following result:

Proposition 5.13. *The three Hilbert spaces of quantum states defined above, position, momentum and holomorphic spaces, are invariant under the action of the prequantum operators $\hat{q}_j$ and $\hat{p}_j$.*

Proof. Applying the prequantization formula (5.4), considering $\theta = \sum_j p_j dq_j$, we obtain:

$$\hat{q_j} = q_j - i\hbar\frac{\partial}{\partial p_j} + \underbrace{\theta\left(\frac{\partial}{\partial p_j}\right)}_{=0}, \qquad \hat{p_j} = p_j + i\hbar\frac{\partial}{\partial q_j} + \underbrace{\theta\left(-\frac{\partial}{\partial q_j}\right)}_{(=-p_j)\vee(=0)}.$$

Restricting on the position space, i.e. removing the dependence on the momentum, we end up with $\hat{q_j} = q_j$ and $\hat{p_j} = i\hbar\frac{\partial}{\partial q_j}$. Similarly on the momentum space we obtain $\hat{q_j} = -i\hbar\frac{\partial}{\partial p_j}$ and $\hat{p_j} = p_j$.

In the case of the holomorphic space, since $z_j = q_j - i\alpha p_j$ and $\bar{z}_j = q_j + i\alpha p_j$, we obtain the following relations:

$$\frac{\partial}{\partial q_j} = \frac{1}{2}\left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j}\right), \quad \frac{\partial}{\partial p_j} = -\frac{1}{2i\alpha}\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \bar{z}_j}\right).$$

Thus, restricting on the holomorphic space, i.e. in the the direction of z_j, we have:

$$\hat{q_j} = \frac{z_j}{2} + \frac{\hbar}{2\alpha}\frac{\partial}{\partial z_j}, \quad \text{and} \quad \hat{p_j} = \frac{i\hbar}{2}\frac{\partial}{\partial z_j}.$$

These subspaces are all invariant under the prequantized position and momentum operators. This is due to the fact that the chosen polarizations are translation invariant, i.e. the set of directions is the same at each point.

□

We will see later that this result is general, i.e. given a prequantum operator $\hat{f}$ obtained from a function f, this operator preserves the quantum space if and only if the Hamiltonian flow X_f preserves the polarization $\mathcal{P}$, where the space of quantum states is defined by $\mathcal{P}$. We should note also that in each space, the "coordinate operator", i.e. q_j in the position space, p_j in the momentum space and z_j in the holomorphic space, acts simply by multiplication.

In general, the restriction of prequantum operators to the quantum Hilbert space will not be invariant. Furthermore, even when this is the case, they are not necessarily self-adjoint or even symmetric when acting on $L^2(\mathbb{R}^n)$. For an example of this see [5] proposition 22.12, p.479.

Let us consider again the prequantum operator corresponding to the Hamiltonian of the harmonic oscillator. This time however we will be considering the holomorphic polarization, i.e. the Hilbert quantum space of holomorphic functions presented above. The energy eigenvalues are of the form $E_n = n\hbar\omega$ with $n \in \mathbb{N}_0$, already we see, despite the eigenvalues not matching the known result of $E_n = (n + 1/2)\hbar\omega$, that this result is physically more reasonable than considering the full prequantum Hilbert space. The expected result is obtained once the half-form correction is introduced. The full treatment of this example can be seen in [5] proposition 22.14, pp.480-481.

By this treatment of $\mathbb{R}^{2n}$ we see a clear motivation to further explore the notions of polarizations and the half-form correction.

5.6. Polarizations

The Kostant-Souriau prequantization setup will not, in general, satisfy the irreducibility requirement (condition four of Dirac's quantization axioms). In other terms, the canonical Darboux coordinates are not uniquely defined and will depend in general on all $2n$ coordinates on M. This means that, in general, the uncertainty principle (which comes from $\{x_i, p_j\} = \delta_{ij}$) obtained by the Schrödinger quantization will not be satisfied.

In the physical construction of quantum mechanics, the states do not usually depend simultaneously on both coordinates and momentum, x_j, p_j. Thus, the space of pre-quantum states is too large for a physical quantum theory. We must therefore introduce some extra structure, the concept of a polarization. This will allow us to define a smaller space, corresponding to the actual quantum states, by picking a Lagrangian sub-bundle selecting only half of the possible directions. We should note that, since we are interested in allowing complex variables, as in the case of the holomorphic quantum space of the previous section, we should consider the complexified tangent bundle for general polarizations.

Definition 5.14. ***Polarization***
*Let (M, ω) be a symplectic manifold and consider the complexified tangent bundle over M, i.e., $(TM \otimes \mathbb{C})$. A subbundle $\mathcal{P} \subset (TM \otimes \mathbb{C})$ is called a **polarization** if it fulfills the following conditions:*

1) *The space $\mathcal{P}(x)$ is a Lagrangian subspace of $(TM \otimes \mathbb{C})$ for each $x \in M$. Equivalently, $\mathcal{P}$ is a maximal isotropic subspace of $(TM \otimes \mathbb{C})$, i.e., rank$\mathcal{P} = \frac{1}{2}$dim $M = n$ and $\omega|_{\mathcal{P}\times\mathcal{P}} = 0$.*

2) *The distribution $x \mapsto \mathcal{P}(x)$ is integrable. By Frobenius's theorem (2.18) we can restate this condition, that is, given $X, Y \in \mathfrak{X}(\mathcal{P}(x))$ we have $[X, Y] \in \mathfrak{X}(\mathcal{P}(x))$.*

In the **general case**, one can define the following sub-bundles of the tangent bundle:

$$E = (\mathcal{P} + \overline{\mathcal{P}}) \cap TM \quad \text{and} \quad D = \mathcal{P} \cap \overline{\mathcal{P}} \cap TM.$$

In general, the dimensions of E and D can change, at each point p of the base manifold M, in such a way that $\dim E_p + \dim D_p = 2n$. Therefore, if $\dim E_p = n + k$ then $\dim D_p = n - k$, where k can take integer values from 0 to n. To turn this definition into a more manageable concept it is also typical to require of a polarization that dim $(\mathcal{P} \cap \overline{\mathcal{P}} \cap TM) =$ constant.

When E is a nonintegrable distribution it is not known *at present* ([2] - 1990, p. 156) how the local description of the polarization is constructed. One then works with what are called **strongly integrable polarizations**, which allows for the introduction of local coordinates as shown by proposition 5.16 (the full treatment of this can be seen in [15]).

Definition 5.15. ***Strongly integrable/Reducible Polarizations***
*A polarization $\mathcal{P}$ on a symplectic manifold (M, ω) is called **strongly integrable or reducible** if and only if the following are satisfied:*
(i) *E is involutive (which implies that D is also involutive);*
(ii) *The integral manifolds $\mathcal{E} = M/E$ and $\mathcal{D} = M/D$ (def. 2.15) are differentiable manifolds such that the canonical projection $\pi : \mathcal{D} \to \mathcal{E}$ is a submersion.*

From now on we consider all polarizations $\mathcal{P}$ to be fall under the category of strongly integrable polarizations. This allows for the construction of local coordinates as shown by the following proposition:

Proposition 5.16. *Given a polarization $\mathcal{P}$ on (M,ω), in every subset $\mathcal{U} \subset M$, we have a set of local coordinates $\{x_1, ..., x_{n-k}, y_1, ..., y_{n-k}, u_1, ..., u_k, v_1, ..., v_k\}$ such that on $\mathcal{U}$ we have:*

$$D = span\left\{\frac{\partial}{\partial x_j}\bigg|_{j=1}^{n-k}\right\}$$
$$\mathcal{P} = span\left\{\frac{\partial}{\partial x_j}\bigg|_{j=1}^{n-k}, \frac{\partial}{\partial \bar{z}_i}\bigg|_{l=1}^{k}\right\},\ \textit{where } z_l := u_l + iv_l.$$

Proof. (Ref. [15], proposition 5.4.7, pp. 97-98). □

If a subbundle $\mathcal{P}$ is a polarization then the complex conjugate subbundle $\overline{\mathcal{P}}$ is also a polarization. In general one can consider mixed states of real and complex directions, however here we will focus solely on the following particular cases:

Definition 5.17. *Real polarization*
*When $\mathcal{P} = \overline{\mathcal{P}}$, $\mathcal{P}$ is a **real polarization**. In this case, $\mathcal{P}$ is the complexification of some real integrable subbundle $\mathcal{P}^{\mathbb{R}} \subset TM$. The manifold M admits a foliation of half its dimension such that the space $\mathcal{P}^{\mathbb{R}}(p)$ coincides with the tangent space of the leaf passing through the point p. Each leaf is a Lagrangian submanifold of M equipped with a canonical affine structure. Locally, we have coordinates $\{x_1, ...x_n, y_1, ..., y_n\}$ such that, on a neighbourhood at each $p \in M$, we have $\mathcal{P} = span\{\frac{\partial}{\partial x_i}\}_{i=1...,n}$.*

Definition 5.18. *Complex polarization*
*When $\mathcal{P} \cap \overline{\mathcal{P}} = \{0\}$ on M, we call the polarization $\mathcal{P}$ **pseudo-Kähler**. In this case $i\omega(\cdot,\cdot) : \overline{\mathcal{P}} \times \mathcal{P} \to C^\infty_{\mathbb{C}}(M)$ is a nondegenerate Hermitian form on $\mathcal{P}(x)$. If this form is also positive definite, we call it a **Kähler polarization** (or holomorphic polarization). Having a pseudo-Kählerian polarization, by Newlander-Nirenberg's theorem [3.37], we can introduce local coordinates $\{z_i\}_{i=1,...,n}$ such that on a neighbourhood of an arbitrary point $x \in M$, $\mathcal{P}$ is spanned by $\{\partial/\partial \bar{z}^i\}_{i=1,...,n}$.*

Definition 5.19. *Polarized section*
Given a section $s \in C^\infty(L)$, we say s is $\mathcal{P}$-polarized if it is covariantly constant along $\overline{\mathcal{P}}$, that is:

$$\nabla_{\overline{\mathcal{P}}} s = 0.$$

Our new Hilbert space of quantum states associated to a polarization $\mathcal{P}$ *could* be defined by the closure of square-integrable polarized sections of L, i.e.:

$$\mathcal{H}_{\mathcal{P}} = \overline{\left\{s \in \Gamma_{L^2}\left(L, \frac{\omega^n}{n!}\right) : \nabla_{\overline{\mathcal{P}}} s = 0\right\}}. \tag{5.7}$$

The subtle *could* with this definition comes from the fact that there are quite undesirable results, particularly in the case of real polarizations. In this situation, it is often the case that we can not find non-vanishing square-integrable sections satisfying the condition of covariant constancy (this is already the case in $\mathbb{R}^{2n}$ as we have seen). This motivates the introduction of a correction factor by half-forms which allows one to obtain more interesting results. In particular, it gives the correct energy eigenvalues for the harmonic oscillator. But, before that, let us see first how the process of quantization works without introducing this correction terms.

5.7. Quantization without half-forms

Our main goal now is to understand how to reduce the space of prequantum operators to those which preserve the space of square-integrable polarized sections given by (5.7). Here we follow chapters 23.4 to 23.7 of [5] as our guideline.

Working with arbitrary polarizations we define first the notion of a quantizable function.

Definition 5.20. ***Quantizable function***
Given a symplectic manifold (M, ω) and a polarization $\mathcal{P}$ on M, the complex valued functions $f \in C^{\infty}(M)$ are said to be ***quantizable*** *if the prequantized operator $\hat{f}$ preserves the polarized smooth sections with respect to $\mathcal{P}$.*

We see next which conditions are necessary to guarantee that a complex valued function $f \in C^{\infty}(M)$ is quantizable.

Definition 5.21. ***Polarization preserving vector field***
A vector field X, real or complex, for which the commutator $[X, Y] \in \mathcal{P}$, $\forall Y \in \mathcal{P}$ is called a ***polarization preserving*** *vector field for the polarization $\mathcal{P}$.*

Remark 8. If $X \in \mathcal{P}$, X trivially preserves $\mathcal{P}$ since $\mathcal{P}$ is integrable. However, we may have a $\mathcal{P}$-preserving vector field X where $X \notin \mathcal{P}$.

Theorem 5.22. *Given a complex valued function $f \in C^{\infty}(M)$, if the Hamiltonian vector field X_f preserves $\overline{\mathcal{P}}$ then f is quantizable.*

Proof. (Ref. [5], theorem 23.24 pp.497-498). □

Remark 9. The converse of the previous theorem is false in general. It happens, that for a given polarization $\mathcal{P}$, we might have no non-zero globally defined polarized sections. This implies that every function is quantizable.

A particular case of the previous theorem occurs for functions which satisfy $X(f) = 0$ for all $X \in \overline{\mathcal{P}}$. In this case we have:

Proposition 5.23. *Given a complex valued function $f \in C^{\infty}(M)$, such that $df = 0$ along $\overline{\mathcal{P}}$, then the prequantized operator $\hat{f}$ is quantizable with respect to $\mathcal{P}$ and it acts simply by multiplication.*

Proof. (Ref. [5], theorem 23.25, p.498). □

We saw particular instances of these results when quantizing $\mathbb{R}^{2n}$ in section 5.5.

5.7.1. Real polarizations

Considering now the case of real polarizations we see first a result which guarantees the existence of local polarized sections.

Proposition 5.24. ***Polarized sections - Local existence (Real case)***
Given a real polarization $\mathcal{P}$ on a symplectic manifold M, $\forall p \in M$ there exists a neighbourhood $\mathcal{U}$ of p and a $\mathcal{P}$-polarized section s of the complex line bundle L over U such that $s(p) \neq 0$.

Proof. (Ref. [5], proposition 23.26, p.499). □

For real polarizations however, the global existence of polarized sections presents a number of difficulties. According to [5] (p.499) if the leaves of $\mathcal{P}$ are not embedded it is unlikely one can find global polarized sections. Furthermore, even when this is the case, there are possible hurdles. If the leaves of the polarization are embedded submanifolds, there are no non-vanishing polarized sections of L on every leaf R of $\mathcal{P}$ for which there exists a loop γ that has non-trivial **holonomy** (see 7.3 in appendix and [5] section 23.5.2 for further details). To a leaf R of $\mathcal{P}$ for which every loop has trivial holonomy we call a **Bohr-Sommerfeld leaf**.

In summary, the case of real polarizations presents a number of challenges which limits the appearance of interesting results in the current state. As seen before, even in the simpler case of $\mathbb{R}^{2n}$ or in the case of vertical polarizations for arbitrary cotangent bundles $M = T^*N$ we have no non-vanishing polarized sections which are square-integrable. Again, this motivates the introduction of some type of correction which turns out to be the half-form correction.

5.7.2. Complex polarizations

In the case of complex polarizations, we will be concerned only with **Kähler polarizations** since these are the ones leading to the existence of non-vanishing square-integrable polarized sections.

In the complex case there is a natural choice for a polarization. By Newlander-Nirenberg's theorem (3.37) we can construct a unique complex structure J on M such that $\mathcal{P}_p = \pi_{1,0}(TM \otimes \mathbb{C})_p = {T^{1,0}}_p$ at each $p \in M$. Just as in the real case, we have a local existence result in the complex case.

Proposition 5.25. ***Polarized sections - Local existence (Complex case)***
Given a complex polarization $\mathcal{P}$ on a symplectic manifold M, $\forall p \in M$ there exists a neighbourhood $\mathcal{U}$ of p and a $\mathcal{P}$-polarized section s, of the complex line bundle L over $\mathcal{U}$, such that $s(p) \neq 0$.

Proof. (Ref. [5], pp.503-505). □

Now, let s and s' be two locally $\mathcal{P}$-polarized sections of L on a neighbourhood $\mathcal{U}$ of $p \in M$. There is a unique complex-valued function $f \in C^\infty(M)$ such that $s' = fs$ on $\mathcal{U}$. Then, given $X \in \overline{\mathcal{P}}$ we have:

$$\nabla_X s' = \nabla_X fs = X \cdot f + f\nabla_X s = 0,$$

therefore $df(X) = X \cdot f = 0$, $\forall X \in \overline{\mathcal{P}}_p$. This implies that f is a J-holomorphic function (see def. (3.28)), for the complex structure J above where $\mathcal{P} = T^{1,0}$. Then, we can define a family of local trivializations of L which are related by holomorphic functions such that the $\mathcal{P}$-polarized sections are precisely the holomorphically related sections of L.

It's possible to show (see [5], proposition 14.15) that in the case of a complex polarization, the space of square-integrable polarized sections of L forms a closed subspace of the prequantum Hilbert space. Given $p \in M$ the map $s \mapsto s(p) \in \mathbb{C}$ is a continuous linear functional which, by Riesz's theorem, can be represented by an inner-product with a unique element of the quantum Hilbert space.

Definition 5.26. ***Coherent States***
*Given a complex polarization $\mathcal{P}$ on M. For each $p \in M$, we define the **coherent state** χ_p as the unique element of the quantum Hilbert space such that:*

$$s(p) = \langle \chi_p, s \rangle, \quad \textit{for all } s.$$

There is a natural identification with this space and the Segal-Bargmann space, and thus also between the coherent states (see the discussion in [5], p. 502).

5.8. Quantization with half-forms

As we discussed before, in an attempt to obtain non-vanishing square-integrable polarized sections, in the particular case of real polarizations, we introduce a correction term known as **half-forms**. In the case of complex polarizations, even though we can already find non-zero square-integrable sections, it turns out that the introduction of the half-form correction has some advantageous properties. A particular case is that of the harmonic oscillator, where one obtains the correct energy eigenvalues.

5.8.1. Real polarizations

Let us consider the Euclidean plane $\mathbb{R}^2$ as a guiding example of what we expect of the half-form correction, in the case of a real polarization. Consider the vertical polarization on $\mathbb{R}^2$, i.e. only coordinate dependent wave-functions $\psi(q)$. We have seen that the polarized sections will not have finite norm since the integration is performed over the entire manifold, that is both in q and p.

Introduce then the half-form Hilbert space represented by $s \otimes \sqrt{dq}$, where s is a polarized section of L and $\sqrt{dq}$ is thought of as a "section of the square root of the canonical bundle". Then, to compute the norm of this new state, we consider its square, $|s|^2 dq$, where s is only a function of q, and we think of it as 1-form in $\mathbb{R}$ rather than $\mathbb{R}^2$. We obtain:

$$||s||^2 := \int_{\mathbb{R}} |s|^2(q) dq.$$

Although it looks similar to the naive reduction we performed in the case of $\mathbb{R}^{2n}$, this procedure works on general symplectic manifolds and the quantized observables will turn out to be self-adjoint, which was not the case previously.

Let us see how the space of half-forms is constructed rigorously beginning with the necessary definitions of the space of leaves and the canonical bundle. Throughout the discussion consider the symplectic manifold (M, ω) and the prequantum line bundle L over M, with compatible Hermitian structure and connection, and $\mathcal{P}$ an arbitrary real polarization.

Definition 5.27. ***Space of leaves of a polarization*** $\mathcal{P}$
The ***leaf space*** Ξ *of a polarization* $\mathcal{P}$ *is the set of all leaves, i.e. maximal connected integral submanifolds of* $\mathcal{P}$*, equipped with a quotient map* $q : M \to \Xi$*, at each* $p \in M$*, given by* $p \mapsto \Xi$*, i.e., the map takes* p *to the unique leaf which contains* p*. The topology on this space is the usual quotient topology, i.e., a set* U *is open in* Ξ *if* $q^{-1}(U)$ *is open in* M*.*

To perform geometric quantization we must assume that Ξ has the structure of an n-dimensional smooth manifold. Since there is no canonical volume measure on Ξ, we define it in such a way that, at each point, we obtain an n-form which we can then integrate on Ξ.

By introducing the canonical bundle, we obtain sections on M mapping to n-forms which are related in a particular way with the chosen polarization. Considering the leaf space Ξ a smooth n-manifold, we are able to identify the space of polarized sections of the canonical bundle with the space of all n-forms on the space of leaves Ξ. In other words, we are able to bring a volume measure to the n-dimensional space of leaves Ξ, having the notion of a "polarized volume measure".

Definition 5.28. ***Canonical bundle***
The ***canonical bundle*** $\mathcal{K}_\mathcal{P}$ *of* $\mathcal{P}$ *is the real line bundle with sections mapping to* n*-forms* α *satisfying the following properties:*

$$\iota_X \alpha = 0, \qquad \text{for any vector field } X \in \mathcal{P}.$$

We say that a section α *of* $\mathcal{K}_\mathcal{P}$ *is* ***polarized*** *if:*

$$\iota_X (d\alpha) = 0, \qquad \text{for any vector field } X \in \mathcal{P}.$$

Equivalently, the complexified canonical bundle $\mathcal{K}_\mathcal{P}^\mathbb{C}$ *has sections which are complex valued* n*-forms which satisfy the same relations (in this case we consider vector fields along* $\overline{\mathcal{P}}$*).*

Example 5.29. *Consider* $\mathbb{R}^{2n}$ *with vertical polarization* $\mathcal{P}$ *on* M*. An* n*-form* α *on* $\mathbb{R}^{2n}$ *is a section of* $\mathcal{K}_\mathcal{P}$ *if and only if* α *is of the form:*

$$\alpha = f(\mathbf{q}, \mathbf{p}) dq_1 \wedge ... \wedge dq_n.$$

If α *had terms of the form* dp_j *then, for any vector field* X *containing a term* $\partial/\partial p_j$ *we would not have* $\iota_X \alpha = 0$*. The* n*-form* α *is a polarized section of* $\mathcal{K}_\mathcal{P}$ *if and only if it is of the form:*

$$\alpha = g(\mathbf{q}) dq_1 \wedge ... \wedge dq_n,$$

where f *and* g *are smooth functions on* $\mathbb{R}^{2n}$ *and* $\mathbb{R}^n$ *respectively. If the function* f *were dependent on* p_j *we would have* $d\alpha$ *as a function of* $dp_j \wedge dq_1 \wedge ... \wedge dq_n$ *and we would have* $\iota_X(d\alpha) \neq 0$*.*

Proposition 5.30. *Considering the leaf space* Ξ *of* $\mathcal{P}$ *a smooth manifold and* α *a polarized section of* $\mathcal{K}_\mathcal{P}$*, there exists a unique* n*-form* $\tilde{\alpha}$ *on* Ξ *satisfying:*

$$\alpha = q^* \tilde{\alpha},$$

where q *is the quotient map. Conversely, given* β *an* n*-form on* Ξ*, then* $\alpha := q^*(\beta)$ *is a polarized section of* $\mathcal{K}_\mathcal{P}$*.*

Proof. (Ref. [5], proposition 23.37 pp. 507-508). □

5.8.2. Square roots of the canonical bundle

We now assume that the leaf space Ξ of $\mathcal{P}$ is orientable with a particular choice of orientation.

Definition 5.31. *Choose an oriented non-vanishing* n*-form* β *on* Ξ *such that* $\alpha = q^*(\beta)$ *is a non-vanishing section of* $\mathcal{K}_\mathcal{P}$*. We say that a section of* $\mathcal{K}_\mathcal{P}$ *is* ***non-negative*** *if it is, at each point, a non-negative multiple of* α*.*

By selecting an oriented form in Ξ this allows to construct a trivialization of the canonical bundle by having α as a basis section defined globally. Then, we are able to define the notion of the square-root of the canonical bundle.

Definition 5.32. ***Square-root of the canonical bundle*** $\delta_\mathcal{P}$
By the ***square-root of the canonical bundle*** $\delta_\mathcal{P}$*, we mean a line bundle* $\delta_\mathcal{P}$ *on* M *together with a particular isomorphism from* $\delta_\mathcal{P} \otimes \delta_\mathcal{P}$ *to* $\mathcal{K}_\mathcal{P}$*. That is, given* $s_1, s_2 \in \delta_\mathcal{P}$*, we think of* $s_1 \otimes s_2$ *as a section on the canonical bundle* $\mathcal{K}_\mathcal{P}$*. We also assume that the isomorphism is such that given* $s \in \delta_\mathcal{P}$ *we have* $s \otimes s$ *non-negative on* $\mathcal{K}_\mathcal{P}$ *(if this is not the case one can compose with* $-I$ *on the fibers of* $\mathcal{K}_\mathcal{P}$*).*

We can also consider the complexification of this line bundle, $\delta_{\mathcal{P}}^{\mathbb{C}}$, where one then thinks of $s_1 \otimes s_2$ as a section on the complexified canonical bundle $\mathcal{K}_{\mathcal{P}}^{\mathbb{C}}$ for $s_1, s_2 \in \delta_{\mathcal{P}}^{\mathbb{C}}$.

Given a section α on $\mathcal{K}_{\mathcal{P}}$ and a vector field X in $\mathcal{P}$, we can define the n-form $\tilde{\nabla}_X \alpha$ by:

$$\tilde{\nabla}_X \alpha = \iota_X(d\alpha). \tag{5.8}$$

We call $\tilde{\nabla}$ the partial connection on $\mathcal{K}_{\mathcal{P}}$. Since we have $\iota_X \alpha = 0$, then $\tilde{\nabla}_X \alpha = \tilde{\mathcal{L}}_X \alpha$. By the following proposition, $\tilde{\mathcal{L}}_X \alpha$ is again a section on $\mathcal{K}_{\mathcal{P}}$.

Proposition 5.33. *If $X \in \mathfrak{X}(M)$ is a polarization ($\mathcal{P}$) preserving vector field and α a section on the canonical bundle $\mathcal{K}_{\mathcal{P}}$ then $\tilde{\mathcal{L}}_X \alpha$ is also in $\mathcal{K}_{\mathcal{P}}$. Furthermore, if α is polarized then so is $\tilde{\mathcal{L}}_X \alpha$.*

Proof. (Ref. [5] proposition 23.38, p. 508). □

By definition 5.28 a section α of $\mathcal{K}_{\mathcal{P}}$ is polarized if and only if $\tilde{\nabla}_X \alpha = \tilde{\mathcal{L}}_X \alpha = 0$.

By the following proposition, the partial connection can be "projected" on the square root of the canonical bundle $\delta_{\mathcal{P}}$.

Proposition 5.34. *Fixing a square root $\delta_{\mathcal{P}}$ of $\mathcal{K}_{\mathcal{P}}$, for any vector field X in $\mathcal{P}$ there is a unique linear operator $\overline{\nabla}_X : \delta_{\mathcal{P}} \to \delta_{\mathcal{P}}$ satisfying:*

$$\begin{aligned}
&\overline{\nabla}_X(f s_1) = (X \cdot f) s_1 + f \overline{\nabla}_X s_1 \\
&\tilde{\nabla}_X(s_1 \otimes s_2) = (\overline{\nabla}_X s_1) \otimes s_2 + s_1 \otimes (\overline{\nabla}_X s_2),
\end{aligned}$$

for any smooth function f and sections s_1, s_2 on $\delta_{\mathcal{P}}$.

If X is a polarization preserving vector field then there is a unique linear operator $\overline{\mathcal{L}} : \delta_{\mathcal{P}} \to \delta_{\mathcal{P}}$ satisfying:

$$\begin{aligned}
&\bar{\mathcal{L}}_X(f s_1) = (X \cdot f) s_1 + f \bar{\mathcal{L}}_X s_1 \\
&\tilde{\mathcal{L}}_X(s_1 \otimes s_2) = (\bar{\mathcal{L}}_X s_1) \otimes s_2 + s_1 \otimes (\bar{\mathcal{L}}_X s_2),
\end{aligned}$$

for any smooth function f and sections s_1, s_2 on $\delta_{\mathcal{P}}$.

Proof. (Ref. [5], proposition 23.41 pp.510-511) □

These constructions extend naturally when considering sections on $\delta_{\mathcal{P}}^{\mathbb{C}}$ instead of $\delta_{\mathcal{P}}$. We say then that a section $s \in \delta_{\mathcal{P}}^{\mathbb{C}}$ is $\mathcal{P}$-polarized if $\overline{\nabla}_X s = 0$ for all $X \in \mathcal{P}$.

5.8.3. Half-form Hilbert space

Given the prequantum line bundle L over M, we consider now the tensor product bundle $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$. Locally, we decompose the sections s_j of $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$ as $s_j = \mu_j \otimes \nu_j$, where μ_j is a section on L and ν_j a section on $\delta_{\mathcal{P}}^{\mathbb{C}}$. Furthermore, given two sections s_1 and s_2 on $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$ we have a globally defined section on $\mathcal{K}_{\mathcal{P}}^{\mathbb{C}}$ given by:

$$(s_1, s_2) := (\mu_1, \mu_2) \overline{\nu_1} \otimes \nu_2, \tag{5.9}$$

where (μ_1, μ_2) is a scalar defined by the inner-product which comes from the Hermitian structure on L.

By the connection ∇ on L and the partial connection $\overline{\nabla}$ defined on $\delta_{\mathcal{P}}^{\mathbb{C}}$ one can define a a global connection $\widehat{\nabla}$ on $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$ given by:

$$\widehat{\nabla}_X s = (\nabla_X \mu) \otimes \nu + \mu \otimes (\overline{\nabla}_X \nu) \tag{5.10}$$

We are then able to define a polarized section of $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$, i.e. a section s is polarized if $\widehat{\nabla}_X s = 0$ for any vector field X in $\mathcal{P}$. One can also see that if $s_1, s_2 \in L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$ are polarized sections then (s_1, s_2) is a polarized section of $\mathcal{K}_{\mathcal{P}}$.

In general we do not always have globally defined polarized sections of $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$. Similarly to the case without the half-form correction, we say that a leaf R is **Bohr-Sommerfeld** if there exists a nonzero section s of $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$ over R, such that $\widehat{\nabla}_X s = 0$ for all tangent vectors to R. We now finally define the **half-form space** which will essentially be our Hilbert space of quantum states.

Definition 5.35. ***Half-form space***
*Given a real polarization $\mathcal{P}$ and any square root $\delta_{\mathcal{P}}$ of $\mathcal{K}_{\mathcal{P}}$, we call the space of smooth polarized sections of $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$ the **half-form space**. For a polarized section s of $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$, we define the norm of s by:*

$$\langle s, s\rangle = ||s||^2 = \int_{\Xi} \widetilde{(s, s)},$$

where $\langle \cdot, \cdot \rangle$ is given by equation (5.9) and where $\widetilde{(s_1, s_2)}$ is the n-form on Ξ given by proposition 5.30, i.e. the pull-back of the n-form on the canonical bundle $\mathcal{K}_{\mathcal{P}}$ given by the isomorphism $\delta_{\mathcal{P}} \otimes \delta_{\mathcal{P}} \simeq \mathcal{K}_{\mathcal{P}}$. Given s_1 and s_2 in $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$ we define their inner-product by (assuming $||s_1|| < \infty$ and $||s_2|| < \infty$):

$$\langle s_1, s_2\rangle = \int_{\Xi} \widetilde{(s_1, s_2)}, \tag{5.11}$$

Definition 5.36. ***Half-form Hilbert space***
*The **half-form Hilbert space** is the completion of the space of all square-integrable polarized sections s with respect to the norm defined above.*

Example 5.37. *Let $M = \mathbb{R}^2$ and consider L the trivial bundle on $\mathbb{R}^2$ with connection $\nabla_X = X - (i/\hbar)\theta(X)$ with $\theta = pdq$. Consider the vertical polarization in $\mathbb{R}^2$ and orient $\mathbb{R}$ such that the 1-forms are positive multiples of dq. Let $\sqrt{dq}$ be a trivializing section of $\delta_{\mathcal{P}}$ such that $\sqrt{dq} \otimes \sqrt{dq} = dq$. Then, every polarized section s of $L \otimes \delta_{\mathcal{P}}$ has the form:*

$$s = \psi(q) \otimes \sqrt{q},$$

where ψ is a function on $\mathbb{R}$. The norm of s is computed by:

$$||s||^2 = \int_{\mathbb{R}} |\psi(q)|^2 dq.$$

The 1-form dq is a globally defined polarized section of the canonical bundle $\mathcal{K}_{\mathcal{P}}$ with $\sqrt{dq}$ a polarized section of $\delta_{\mathcal{P}}$. Polarized sections on the half-form space $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$ are given as $s = \psi(q) \otimes \sqrt{dq}$. In this case $\widetilde{(s, s)} = (s, s) = |\psi(q)|^2 dq$, which is a $1-$form on the space of leaves $\Xi \simeq \mathbb{R}$ instead of $\mathbb{R}^2$.

Summary: It is quite difficult to motivate the construction of the half-form correction besides the fact that it allows one to obtain square-integrable sections on real polarizations and that it gives us the correct quantization of the harmonic oscillator for example. In a simplified fashion this are the steps we took:

- We begin with a symplectic manifold (M, ω) and with the structure of a complex line bundle L over M, equipped with a connection and Hermitian structure which are compatible. Then, we make a choice of a real polarization $\mathcal{P}$ on M, which generates a "sub-foliation" Ξ on M given by the space of leaves of $\mathcal{P}$.
- Instead of integrating on the entire manifold M, we intend to integrate solely on the space of leaves Ξ of our polarization, considering Ξ to be a smooth n-dimensional manifold with an orientation. Thus, since Ξ has no canonical volume measure we intend to construct one that, essentially, suits our needs.
- The canonical bundle, $\mathcal{K}_{\mathcal{P}}$, is a bundle on M where the sections are n-forms which vanish on any vector field lying in $\mathcal{P}$. We can then define the notion of a polarized section to be given as $\iota_X(d\alpha) = 0$, for $\alpha \in \mathcal{K}_{\mathcal{P}}$.
- By the pull-back through the quotient map we can turn a polarized section on the canonical bundle $\mathcal{K}_{\mathcal{P}}$ into a unique n-form on Ξ. Likewise, given a form $\beta \in \Xi$ we obtain a polarized section on $\mathcal{K}_{\mathcal{P}}$ (see proposition 5.30).
- Since we have an orientation on Ξ, we can trivialize $\mathcal{K}_{\mathcal{P}}$, with a global section $\alpha = q^*\beta$, which allows one to define the "square root of the canonical bundle" $\delta_{\mathcal{P}}$.
- We think of sections $s_1, s_2 \in \delta_{\mathcal{P}}$ as sections in $\delta_{\mathcal{P}} \otimes \delta_{\mathcal{P}} \simeq \mathcal{K}_{\mathcal{P}}$, where we make a choice of isomorphism.
- Then, we can construct a partial connection on $\mathcal{K}_{\mathcal{P}}$, which we are able to "project" on sections of $\delta_{\mathcal{P}}$, allowing us to define polarized sections of $\delta_{\mathcal{P}}$.
- Next, we construct the space $L \otimes \delta_{\mathcal{P}}^{\mathbb{C}}$ where we can again define a notion of polarized sections and introduce an inner-product.

Thus we finally consider our Hilbert space of quantum states $\mathcal{H}$, associated to the polarization $\mathcal{P}$ together with the half-form correction, to be the completion of:

$$\mathcal{H} = \Big\{ \psi \otimes \sqrt{\alpha} \in \Gamma(L \otimes \delta_{\mathcal{P}}) \;\Big|\; \widehat{\nabla}_X(\psi \otimes \sqrt{\alpha}) = 0,\ \forall X \in \mathcal{P}, \\ \text{and } ||\psi \otimes \sqrt{\alpha}||^2 = \int_{\Xi} \langle \psi, \psi \rangle \, q^*(\sqrt{\alpha} \otimes \sqrt{\alpha}) < \infty \Big\},$$

with inner-product on $L \otimes \delta_{\mathcal{P}}$ given by:

$$\Big(\psi \otimes \sqrt{\alpha}, \phi \otimes \sqrt{\beta} \Big) = \int_{\Xi} \langle \psi, \phi \rangle \, q^*(\sqrt{\alpha} \otimes \sqrt{\beta}),$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian structure on L and $q^*(\sqrt{\alpha} \otimes \sqrt{\beta})$ is the pull-back of the chosen isomorphism to the canonical bundle $\mathcal{K}_{\mathcal{P}}$ to an n-form on the space of leaves Ξ.

5.8.4. Quantum observables on the half-form space

Considering the new half-form Hilbert space, the space of prequantum operators must change accordingly so that they preserve $\mathcal{P}$. The following definition gives us a way to obtain polarization preserving self-adjoint, or at least symmetric, operators on the half-form Hilbert space.

Definition 5.38. ***Quantum operators acting on the half-form Hilbert space*** *Given any function f on M which preserves the polarization $\mathcal{P}$ (def. 5.21), the operator $Q(f)$ acting on the half-form Hilbert space of $\mathcal{P}$ is defined as;*

$$Q(f)s = (\hat{f}\mu) \otimes \nu + i\hbar\mu \otimes \mathcal{L}_{X_f}\nu,$$

where $\hat{f}$ represents the prequantum operator of f and where s decomposes locally as $s = \mu \otimes \nu$, with μ a section of L and ν a section of $\delta^{\mathbb{C}}_{\mathcal{P}}$.

By the following theorems, the quantized observables $Q(f)$ satisfy the desired commutation relations and are self-adjoint, or at least symmetric operators.

Theorem 5.39. *Let f and g be functions on M with corresponding Hamiltonian vector fields X_f and X_g which preserve the polarization $\mathcal{P}$. Then, the operators $Q(f)$ and $Q(g)$ satisfy:*

$$\frac{1}{i\hbar}[Q(f), Q(g)] = Q(\{f, g\}).$$

on the space of smooth polarized sections of $L \otimes \delta^{\mathbb{C}}_{\mathcal{P}}$.

Proof. (Ref. [5] theorem 23.46, p. 515). □

Theorem 5.40. *Given a real valued function $f \in C^{\infty}(M)$ such that X_f preserves $\mathcal{P}$, then the operator $Q(f)$ is symmetric on the space of smooth sections of $L \otimes \delta^{\mathbb{C}}_{\mathcal{P}}$ for which $\widetilde{s, s}$ has compact support on Ξ.*

Proof. (Ref. [5] theorem 23.47, pp. 516-517). □

5.8.5. Kähler polarizations

In the case of Kähler polarizations, the half-form correction leads to better results. In the particular case of the quantum harmonic oscillator for example, the corrected version results precisely in the energy spectrum given by $(n + 1/2)\hbar\omega$, $n = 0, 1, 2, ...$, while the non-corrected version gives us $n\hbar\omega$ (see example 23.53 in [5], p.520 and the previous discussion on p. 518).

We now simply state the general results of the complex case since the construction is pretty similar to the real case. Let us consider a quantizable $2n$-dimensional symplectic manifold (M, ω) with prequantum line bundle (L, ∇) over M and with a Kähler polarization $\mathcal{P}$ on M. First we should note that there is a relation between Kähler polarizations and Kähler manifolds.

Proposition 5.41. *A Kähler polarization $\mathcal{P}$ on a symplectic manifold M induces an almost complex structure J on M which is integrable and therefore (M, ω, J) is a Kähler manifold. Conversely, on a Kähler manifold there are always two polarizations, the holomorphic $\mathcal{P}$ and anti-holomorphic $\overline{\mathcal{P}}$.*

Proof. (Ref. Theorem 96, [7] p. 70). □

In the complex case, where $\mathcal{P} \neq \overline{\mathcal{P}}$, the **canonical bundle** $\mathcal{K}_{\mathcal{P}}$ of $\mathcal{P}$ is the complex line bundle where the sections are the n-forms α satisfying $\iota_X \alpha = 0$. The sections of $\mathcal{K}_{\mathcal{P}}$, which correspond to the $(n,0)$-forms on M, are said to be polarized if $\iota_X(d\alpha) = 0$ for each $X \in \overline{\mathcal{P}}$.

We define again the notion of a **square root of the canonical bundle** $\delta^{\mathbb{C}}_{\mathcal{P}}$ which consists of a complex line bundle on M and a particular choice of isomorphim $\delta^{\mathbb{C}}_{\mathcal{P}} \otimes \delta^{\mathbb{C}}_{\mathcal{P}} \simeq \mathcal{K}^{\mathbb{C}}_{\mathcal{P}}$. With the same notion of **partial connection** $\widehat{\nabla}$ we consider sections on $L \otimes \delta_{\mathcal{P}}$ to be **polarized** if $\widehat{\nabla}_X s = 0$.

There is now however a natural Hermitian structure on the complex canonical bundle which we did not have in the real case.

Proposition 5.42. ***Hermitian structure on*** $\delta^{\mathbb{C}}_{\mathcal{P}}$
If α is an $(n,0)$-form on M then, at each point $p \in M$, the $2n$-form given by:

$$(-1)^{n(n-1)/2}(-i)^n \bar{\alpha} \wedge \alpha,$$

is a non negative multiple of the Liouville form $\lambda = \frac{1}{n!}\omega^n$. This allows to define a unique Hermitian structure on $\delta^{\mathbb{C}}_{\mathcal{P}}$ such that, for each section s of $\delta^{\mathbb{C}}_{\mathcal{P}}$, we have:

$$|s|^2 = \left(\frac{(-1)^{n(n-1)/2}(-i)^n}{2^n} \frac{\overline{(s \otimes s)} \wedge (s \otimes s)}{\lambda} \right)^{1/2},$$

where given an arbitrary $2n$-form β, $\frac{\beta}{\lambda}$ denotes the unique function on M such that $\beta = (\frac{\beta}{\lambda})\lambda$.

With the Hermitian structure on L and an Hermitian structure on $\delta_{\mathcal{P}}$ one can define an Hermitian structure on $L \otimes \delta_{\mathcal{P}}$.

Definition 5.43. ***Half-form Hilbert space***
*The **half-form Hilbert space** for a Kähler polarization $\mathcal{P}$ on M, is the space of square-integrable sections of $L \otimes \delta^{\mathbb{C}}_{\mathcal{P}}$*

The quantized observables acting on the half-form Hilbert space are given by:

Definition 5.44. ***Quantum operators acting on the half-form Hilbert space***
Given any function f on M which preserves the polarization $\mathcal{P}$, the operator $Q(f)$ acting on the half-form Hilbert space of $\mathcal{P}$ is defined as;

$$Q(f)s = (\hat{f}\mu) \otimes \nu + i\hbar\mu \otimes \mathcal{L}_{X_f}\nu,$$

where $\hat{f}$ represents the prequantum operator of f and where s decomposes locally as $s = \mu \otimes \nu$, with μ a section of L and ν a section of $\delta^{\mathbb{C}}_{\mathcal{P}}$.

As we had previously, the quantized observables $Q(f)$ satisfy $\frac{1}{i\hbar}[Q(f), Q(g)] = Q(\{f,g\})$ on the space of smooth polarized sections of $L \otimes \delta_{\mathcal{P}}$.

Proposition 5.45. ***Quantum harmonic oscillator***
Let $M = \mathbb{R}^{2n}$ with Kähler polarization $\mathcal{P}$ given by the complex coordinate $z = (x - \frac{ip}{m\omega})$, where ω is the frequency. We take $\delta_{\mathcal{P}} = \sqrt{dz}$ and consider the Hamiltonian of the harmonic oscillator $H = \frac{1}{2m}(p^2 + (m\omega q)^2)$. Then,$X_H$ preserves the polarization $\mathcal{P}$ and the operator $Q(H)$ has the energy eigenvalues given by $E_n = (n + 1/2)\hbar\omega$, for $n \in \mathbb{N}_0$.

Proof. (Ref. [5], example 23.53, p.520). □

5.9. Pairing Maps

The space of quantum states we have constructed is dependent on the choice of a polarization. The concept of a pairing map is then defined as a way to compare the resulting quantum states obtained by distinct polarizations. We will see how this works for the case of two **transverse real polarizations** with the other possible combinations being treated similarly with slight modifications. The discussion here is based on chapter 23.8 of [5].

Consider two polarizations $\mathcal{P}^1$ and $\mathcal{P}^2$ with corresponding leaf spaces Ξ_1 and Ξ_2, which we assume to be n-dimensional oriented manifolds. By transverse we mean that, at each point $p \in M$, $\mathcal{P}^1_p \cap \mathcal{P}^2_p = \{0\}$. This implies that, given polarized sections s_1', s_2' in $\mathcal{K}_{\mathcal{P}^1}$ and $\mathcal{K}_{\mathcal{P}^2}$ respectively, the $2n$-form $s_1' \wedge s_2'$ is non-vanishing on M. We can define the following map:

Definition 5.46. *Given any $p \in M$ we define a bilinear pairing from $\delta_{\mathcal{P}^1_p} \times \delta_{\mathcal{P}^2_p} \to \mathbb{R}$ by:*

$$(\nu_1, \nu_2)_p = \left(\frac{(\nu_1 \otimes \nu_1) \wedge (\nu_2 \otimes \nu_2)}{\lambda} \right)^{1/2}. \tag{5.12}$$

Here we use again the previous notation, the numerator is a $2n$-form and $\lambda = \omega^n/n!$ is a $2n$-form (the Liouville form) such that (ν_1, ν_2) is a function on M.

Similarly, we can extend to the map $\delta^{\mathbb{C}}_{\mathcal{P}^1_p} \times \delta^{\mathbb{C}}_{\mathcal{P}^2_p} \to \mathbb{C}$, which is taken to be sesquilinear (conjugate linear in the first factor and linear in the second).

By considering the Hermitian structure on L we can extend to the following map:

Definition 5.47. *Given any $p \in M$ we define a pairing map $(L_p \otimes \delta^{\mathbb{C}}_{\mathcal{P}^1_p}) \times (L_p \otimes \delta^{\mathbb{C}}_{\mathcal{P}^2_p}) \to \mathbb{C}$ by:*

$$(\mu_1 \otimes \nu_1, \mu_2 \otimes \nu_2)_p = \langle \mu_1, \mu_2 \rangle_p \, (\nu_1, \nu_2)_p, \tag{5.13}$$

where $\langle \cdot, \cdot \rangle$ corresponds to the Hermitian structure on L and $(\cdot, \cdot)$ is given by definition 5.46.

Definition 5.48. *Let $\mathcal{H}^1 = (L \otimes \delta^{\mathbb{C}}_{\mathcal{P}^1})$ and $\mathcal{H}^2 = (L \otimes \delta^{\mathbb{C}}_{\mathcal{P}^2})$ be the half-form Hilbert spaces corresponding to polarizations $\mathcal{P}^1$ and $\mathcal{P}^2$. Assuming the integral is absolutely convergent we define the pairing between sections $s_1 \in \mathcal{H}^1$ and $s_2 \in \mathcal{H}^2$ by:*

$$\langle s_1, s_2 \rangle_{\mathcal{P}^1, \mathcal{P}^2} := c \int_M (s_1, s_2) \lambda \tag{5.14}$$

where $(\cdot, \cdot)$ is given by definition 5.47 and c is a constant which depends on $\hbar$ and the dimension of M.

Definition 5.49. ***Pairing map***
*Let $s_1 \in \mathcal{H}^1$ and $s_2 \in \mathcal{H}^2$. A **pairing map** is a map of the form $\Lambda_{\mathcal{P}^1, \mathcal{P}^2} : \mathcal{H}^1 \to \mathcal{H}^2$ which satisfies:*

$$\langle s_1, s_2 \rangle_{\mathcal{P}^1, \mathcal{P}^2} = \langle \Lambda_{\mathcal{P}^1, \mathcal{P}^2} \, s_1, s_2 \rangle_{\mathcal{H}_2} \tag{5.15}$$

If we can find a constant C such that $| \langle s_1, s_2 \rangle_{\mathcal{P}^1, \mathcal{P}^2} | \leq C||s_1|| \, ||s_2||$ then, there is a unique bounded, and therefore continuous, operator $\Lambda_{\mathcal{P}^1, \mathcal{P}^2}$ which satisfies (5.15).

Definition 5.50. ***Equivalent Polarizations***
Given transverse real polarizations $\mathcal{P}^1, \mathcal{P}^2$ and any two sections s_1, s_2 in $\mathcal{H}^1, \mathcal{H}^2$ respectively, if the pairing map $\langle \Lambda_{\mathcal{P}^1,\mathcal{P}^2} s_1, s_2 \rangle$ is unitary, or at least a constant multiple of a unitary map, we say that the polarizations $\mathcal{P}^1, \mathcal{P}^2$ are equivalent.

This leads naturally to the concept of a quantization which is independent of the choice of polarization.

Definition 5.51. ***Polarization independent quantization***
If the pairing map of definition 5.49 is unitary, or a constant multiple of a unitary map, for any two polarizations on the prequantum Hilbert space, we call the quantization ***polarization independent****.*

One example where we indeed have a unitary map is the case of $\mathbb{R}^2$ with vertical polarization $\mathcal{P}^V$ and horizontal polarization $\mathcal{P}^H$. In this case, the map $\Lambda_{\mathcal{P}^V,\mathcal{P}^H}$ is a scaled version of the Fourier transform and in particular a constant multiple of a unitary map.

Example 5.52. *(Ref. [5], example 23.54, p. 522) Let $M = \mathbb{R}^2$ and consider L to be trivial with canonical 1-form $\theta = pdq$. Let $\mathcal{P}^V$ be the vertical polarization, covariantly constant along $\partial/\partial p$ and $\mathcal{P}^H$ be the horizontal polarization, covariantly constant along $\partial/\partial q$. The sections on the half-form space are given respectively by:*

$$s_1(q,p) = \phi(q) \otimes \sqrt{dq}, \quad \text{and} \quad s_2(q,p) = \psi(p)e^{iqp/\hbar} \otimes \sqrt{dp},$$

where ϕ, ψ are functions on $\mathbb{R}$ which we have obtained in proposition 5.12. The pairing map is then given by:

$$\langle s_1, s_2 \rangle_{\mathcal{P}^V,\mathcal{P}^H} = c \int_{R^2} \left\langle \phi(q), \psi(p)e^{iqp/\hbar} \right\rangle (\sqrt{dq}, \sqrt{dp})\lambda,$$

where $\langle \cdot, \cdot \rangle$ corresponds to the Hermitian structure on L and $(\cdot, \cdot)$ is given by definition 5.46. Here the Liouville form is just $\lambda = \omega = dp \wedge dq$ so, considering $c = 1$, we obtain:

$$\langle s_1, s_2 \rangle_{\mathcal{P}^V,\mathcal{P}^H} = -\int_{\mathbb{R}^2} \overline{\phi(q)}\psi(p)e^{iqp/\hbar} dqdp. \tag{5.16}$$

The term $(\sqrt{dq}, \sqrt{dp})$ is calculated as follows:

$$(\sqrt{dq}, \sqrt{dp}) = \left(\frac{\sqrt{dq} \otimes \sqrt{dq} \wedge \sqrt{dp} \otimes \sqrt{dp}}{dq \wedge dp} \right)^{1/2} = 1$$

By looking at equation (5.16), we are then able to construct a pairing map $\Lambda_{\mathcal{P}^1,\mathcal{P}^2}$ which is scaled version of the Fourier transform. Considering $\langle \Lambda_{\mathcal{P}^1,\mathcal{P}^2} s_1, s_2 \rangle = ||s_2||^2$ we obtain:

$$\Lambda_{\mathcal{P}^1,\mathcal{P}^2}(\phi(q) \otimes \sqrt{dq}) = \psi(p)e^{iqp/\hbar} \otimes \sqrt{dp} \;\Rightarrow\; \psi(p) = -\int_{\mathbb{R}} \phi(q)e^{-iqp/\hbar} dq$$

Since there are currently no general results, it is uncommon to find pairing maps which are unitary up to a constant. Therefore, constructing examples where this is the case is of particular importance.

6. Complex Hamiltonian Flows

After reviewing a few basic definitions and results, we begin by introducing the concept of a Lie Series, due to Gröbner, which leads to complex time evolution and complex Hamiltonian flows. This section will mainly follow the work by Gröbner [18], in particular sections I.1,I.2, the paper on complex Hamiltonian flows by José M. Mourão and João P. Nunes [9], sections 2 to 4 of [16] and section 2 of [14].

6.1. Preliminaries

Let us begin by reviewing some basic concepts regarding analytic functions, vector fields and flows. In the following discussion consider M a general smooth manifold of dimension n.

Definition 6.1. ***Real analytic function***
*A function $f : \mathbb{R} \to \mathbb{R}$ is **real analytic** on an open set $D \subset \mathbb{R}$ if, given any $x_0 \in D$ the following series:*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \ \textit{where } a_n \in \mathbb{R}, \ \forall n \in \mathbb{N}_0,$$

converges, for x in a neighbourhood of x_0.

Definition 6.2. ***Real analytic manifold***
A smooth manifold M with an atlas $\mathcal{A}$ is real analytic if all transition charts on $\mathcal{A}$ are real analytic functions.

Definition 6.3. ***Vector field on a smooth manifold*** M
A smooth vector field X on M is a smooth map of the form:

$$\begin{aligned} X : &M \to TM \\ &p \mapsto X(p) := X_p \in T_pM. \end{aligned}$$

The set of all differentiable vector fields on M is denoted by $\mathfrak{X}(M)$ and we will denote the set of all real analytic vector fields as $\mathfrak{X}^\omega(M)$.

A smooth vector field $X \in \mathfrak{X}(M)$ acts on smooth functions $f : M \to \mathbb{R}$ in the following way:

$$\begin{aligned} X \cdot f : &M \to \mathbb{R} \\ &p \mapsto X_p \cdot f := X_p(f). \end{aligned}$$

Thus, it represents a linear operator $X : C^\infty(M) \to C^\infty(M)$.

Regarding smooth vector fields we have the following results.

Proposition 6.4. *Given $X, Y \in \mathfrak{X}(M)$ there exists $Z \in \mathfrak{X}(M)$ such that $Z = [X, Y]$.*

Theorem 6.5. ***Local flow***
Let $X \in \mathfrak{X}(M)$, for each $p \in M$ there is a neighbourhood W of p, an interval $I = (-\epsilon, \epsilon)$ and a map $F : W \times I \to M$ such that:

(i) *given $q \in W$, the curve $F(q,t)$, $t \in I$, is an integral curve of X at q, i.e., $F(q, 0) = q$ and $\frac{\partial F}{\partial t}(q,t) = X_{F(q,t)}$;*

(ii) *the map F is differentiable and we call F the local flow of X at point p. Fixing t we consider:*

$$\begin{aligned}\psi_t :&W \to M\\ &q \mapsto F(q,t) = c_q(t).\end{aligned}$$

Proposition 6.6. *The maps defined by the local flow $\psi_t : W \to M$ are local diffeomorphisms which satisfy:*

$$(\psi_t \circ \psi_s)(q) = \psi_{t+s}(q), \quad \text{when } t, s, t+s \in I \text{ and } \psi_s(q), \psi_t(q) \in W.$$

Theorem 6.7. *If $X \in \mathfrak{X}(M)$ has compact support then X is complete. In particular, if M is a compact manifold then all $X \in \mathfrak{X}(M)$ are complete vector fields.*

$$(\psi_t \circ \psi_s)(q) = \psi_{t+s}(q), \quad \text{when } t, s, t+s \in I \text{ and } \psi_s(q), \psi_t(q) \in W.$$

Theorem 6.8. *The flows ψ_1 and ψ_2 of two complete smooth vector fields X_1, X_2 commute (i.e. $\psi_{1,t} \circ \psi_{2,s} = \psi_{2,s} \circ \psi_{1,t}$ for all $s, t \in \mathbb{R}$) if and only if $[X_1, X_2] = 0$.*

6.2. Introduction to Lie Series

We begin this section by presenting the essential results of Gröbner's General Theory of the Lie Series [18], in regards to obtaining the concept of a complex Hamiltonian flow. We will do so in the context of differential geometry and particularly in the case of real analytic compact manifolds.

Let us consider a compact real analytic manifold M of dimension n, i.e. with charts on M which have have real analytic transition functions. We denote by $\mathfrak{X}^\omega(M;\mathbb{C})$ the space of real analytic vector fields on M, that is if we consider a neighbourhood $\mathcal{U}$ of $p \in M$ with local coordinates $(x^1, ..., x^n)$ then $X_p \in \mathfrak{X}^\omega(M;\mathbb{C})$ is given by:

$$X_p = X^1(p)\left(\frac{\partial}{\partial x_1}\right)_p + ... + X^n(p)\left(\frac{\partial}{\partial x_n}\right)_p,$$

where each $X^i(p)$ is real analytic in $\mathcal{U}$.

This vector field acts on real analytic complex-valued functions $f : M \to \mathbb{C}$ in the usual way:

$$\begin{aligned}X \cdot f :& M \to \mathbb{C}\\ &p \mapsto X_p \cdot f := X_p(f).\end{aligned}$$

Considering $C^\omega(M;\mathbb{C})$ to be the algebra of real analytic complex-valued functions on M, the vector field X acts as a linear operator $X : C^\omega(M;\mathbb{C}) \to C^\omega(M;\mathbb{C})$.

We can formally define a series, called a **Lie Series**, which is shown in [18] to be absolutely and uniformly convergent on a sufficiently small compact domain $K \subset M$.

Theorem 6.9. ***Convergence of the Lie Series***
Consider a complex-valued real analytic function $f \in C^\omega(M;\mathbb{C})$ and a real analytic vector field $X \in \mathfrak{X}^\omega(M;\mathbb{C})$. There exists $T_f \in \mathbb{R}^+$, defining the radius of an open disk centered at the origin

$D_0 \subset \mathbb{C}$, such that for $\tau \in D_0 = \{\tau \in \mathbb{C} : |\tau| < T_f\}$ the Lie series:

$$\sum_{k=0}^{\infty} \frac{\tau^k}{k!} X^k(f) =: e^{\tau X} \cdot f(x_1, ..., x_n),$$

converges absolutely and uniformly, defining a real analytic function on $M \times D_0$, i.e. a function of the $n+1$ variables $(x_1, ..., x_n, \tau)$. We call this series the Lie series for X and f.

Proof. Ref. theorems 1,2,3 of [18]. □

Remark 10. By the nature of the Lie Series we are interested solely on its action on complex-valued functions. If it were the case that $f \in \mathbb{R}$ then, in general, we will have $e^{\tau X} \cdot f \in \mathbb{C}$. This is not particularly useful, especially when considering the functions f as the underlying real coordinates on the manifold.

We shall now explore the properties of the Lie Series, we assume throughout that both the vector fields and functions are real analytic on the domains under consideration such that the previous theorem holds true. We have the following well known results:

Proposition 6.10. *Let $k \in \mathbb{N}$ and $\mathbf{x} = (x_1, ..., x_n)$, we have the following known results:*

$$X^k \cdot \Big(\sum_{j=1}^{n} f_j(\mathbf{x}) \Big) = \sum_{j=1}^{n} X^k \cdot f_j(\mathbf{x}),$$

$$X^k \cdot \Big(f_1(\mathbf{x}) f_2(\mathbf{x}) \Big) = \sum_{j=0}^{k} \binom{k}{j} \Big(X^j \cdot f_1(\mathbf{x}) \Big) \Big(X^{k-j} \cdot f_2(\mathbf{x}) \Big)$$

From this proposition it is clear that we have:

$$e^{\tau X} \cdot \Big(\sum_{j=1}^{n} f_j(\mathbf{x}) \Big) = \Big(\sum_{j=1}^{n} e^{\tau X} \cdot f_j(\mathbf{x}) \Big).$$

We can also show from this proposition that $e^{\tau X}$ is a (local) automorphism in τ of the algebra $C^\omega(M; \mathbb{C})$.

Theorem 6.11. *Automorphism of the algebra* $C^\omega(M)$
The operator $e^{\tau X}$ corresponding to the Lie Series is an automorphism of the algebra $C^\omega(M)$ for $|\tau| < min\{T_{f_1}, T_{f_2}\}$. Given $f_1, f_2 \in C^\omega(M; \mathbb{C})$ we have:

$$e^{\tau X} \cdot (f_1 f_2) = (e^{\tau X} \cdot f_1)(e^{\tau X} \cdot f_2).$$

Proof. Ref. theorem 5 of [18]. By the second expression of proposition 6.10 we have:

$$e^{\tau X} \cdot \Big(f_1(\mathbf{x}) f_2(\mathbf{x}) \Big) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X^k \cdot \Big(f_1(\mathbf{x}) f_2(\mathbf{z}) \Big) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\tau^k}{k!} \binom{k}{j} \Big(X^j \cdot f_1(\mathbf{x}) \Big) \Big(X^{k-j} \cdot f_2(\mathbf{x}) \Big) =$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\tau^k}{j!(k-j)!} \Big(X^j \cdot f_1(\mathbf{x}) \Big) \Big(X^{k-j} \cdot f_2(\mathbf{x}) \Big)$$

If we define $k-j=p$ we see that both indices k and p run independently from 0 to ∞ and thus we obtain:

$$\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\frac{\tau^j\tau^p}{j!p!}\Big(X^j\cdot f_1(\mathbf{x})\Big)\Big(X^p\cdot f_2(\mathbf{x})\Big)=\Big(\sum_{j=0}^{\infty}\frac{\tau^j}{j!}X^j\cdot f_1(\mathbf{x})\Big)\Big(\sum_{p=0}^{\infty}\frac{\tau^p}{p!}X^p\cdot f_2(\mathbf{x})\Big)=$$
$$=\Big(e^{\tau X}f_1(\mathbf{x})\Big)\Big(e^{\tau X}f_2(\mathbf{x})\Big).$$

This result can be generalized by induction to any finite product. □

Theorem 6.12. ***Commutation theorem***
Let (U,φ) be a chart on M with local coordinates $(x^1,x^2,...,x^n)$. Consider a real analytic function $f\in C^\omega(U;\mathbb{C})$ and a real analytic vector field $X\in\mathfrak{X}^\omega(U;\mathbb{C})$. Considering the local representation of f, $\hat{f}:\hat{U}\to\mathbb{C}$, there exists $T>0$ such that for $|\tau|<T$ the action of $e^{\tau X}$ on $\hat{f},x^1,x^2,...,x^n$ defines real analytic functions on $\hat{U}\times D_T$ where $D_T=\{\tau\in\mathbb{C}:|\tau|<T, T=min\{T_f,T_{x_1},...,T_{x_n}\}\}$. Then, we can find a small enough open set $\hat{V}\subset\hat{U}$ such that:

$$\widehat{e^{\tau X}\cdot f}(x^1,...,x^n)=\sum_{k=0}^{\infty}\frac{\tau^k}{k!}X^k\cdot\hat{f}(x^1,...,x^n)=\hat{f}(e^{\tau X}x^1,...,e^{\tau X}x^n).$$

Proof. Ref. theorem 6 of [18].

□

Considering X a real vector field and $\tau=t$ with $t\in\mathbb{R}$, from the previous theorem we have $e^{tX}\cdot f=f\circ\varphi_t^X$. In this way we see that the action of the operator e^{tX} is just the evolution through the flow of X. If we consider a given coordinate x^j, $j=1,2,...,n$ we have:

$$\frac{d}{dt}(e^{tX}\cdot x^j)=\sum_{k=1}^{\infty}\frac{t^{k-1}}{(k-1)!}X\cdot(X^{k-1}(x^j))=X\cdot(e^{tX}\cdot x^j).$$

Thus, by considering the action on $\mathbf{x}=(x^1,x^2,...,x^n)$, the expression $e^{tX}\cdot\mathbf{x}$, satisfies the flow equation of X, $\frac{d\hat{c}^i}{dt}=\hat{X}^i(\hat{c}(t))$ where $c(t)$ is the integral curve of X. They are the same differential equation and thus we have a unique solution by the Picard-Lindelöf theorem of ordinary differential equations.

6.3. Lie Series on Complex Manifolds

Let us now consider a compact manifold M, with dim(M)= $2n$, equipped with an integrable almost complex structure J_0.

Theorem 6.13. *Let $U\subset M$ be an open neighbourhood of a point $p\in M$ with local J_0-holomorphic coordinates given by $\{z^i\}_{j=1,...,n}$. There exists $T\in\mathbb{R}^+$ such that for any $\tau\in D_T=\{\tau\in\mathbb{C}:|\tau|<T\}$ the action of the Lie series operator on the J_0-holomorphic coordinates is well defined,*

$$z^i_\tau:=e^{\tau X}\cdot z^i, \textit{for } i=1,...,n.$$

This generates new coordinates $\{z^i_\tau\}_{i=1}^n$ on a neighbourhood $V\subset U$ of p which in turn define a new integrable almost complex structure J_τ where the coordinates $\{z^i_\tau\}_{i=1}^n$ are $J_\tau-$holomorphic.

Proof. Ref. theorem 2.5 [9]. Let $p \in M$ and consider an open neighbourhood U of p with local J_0-holomorphic coordinates $\{z_i\}_{i=1,\dots,n}$. By theorem 6.9, we can consider a smaller neighbourhood $V' \subset U$ of p such that $\overline{V'} \subset U$ with $\overline{V'}$ compact, i.e. V' is relatively compact in U and $T > 0$ such that for $|\tau| < T$ the functions $\{z^i_\tau\}_{i=1,\dots,n}$ are well-defined on V'. Since $dz^1 \wedge ... \wedge dz^n \wedge d\bar{z}^1 \wedge ... \wedge d\bar{z}^n \neq 0$ at all points in V', by continuity in τ and taking a smaller T if necessary, we have $dz^1_\tau \wedge ... \wedge dz^n_\tau \wedge d\bar{z}^1_\tau \wedge ... \wedge d\bar{z}^n_\tau \neq 0$ for all points in V'. Thus, by applying the inverse function theorem, we obtain a new system of coordinates $\{z^i_\tau\}_{i=1,\dots,n}$ on a, possibly smaller open neighbourhood $V \subset V'$. □

Remark 11. Just a few comments regarding the proof above:

i) We always have a V' satisfying the specified conditions since any manifold has a basis given by regular coordinate balls.

ii) We take a possibly smaller T since a coordinate could go to zero in time T.

iii) The possibly smaller V comes as a direct consequence of using the inverse function theorem.

Theorem 6.14. *There exists $T \in \mathbb{R}^+$ such that for $\tau \in D_T = \{\tau \in \mathbb{C} : |\tau| < T\}$ there exists a global complex structure J_τ on M which extends the beginning complex structure J_0 by the unique biholomorphism*

$$\varphi_\tau : (M, J_\tau) \to (M, J_0),$$

which acts on the J_0-holomorphic coordinates by the operator $e^{\tau X}$.

Proof. Ref. theorem 2.6 [9]. Since M is compact we can consider a finite atlas of M given by J_0-holomorphic charts $\{(U_\alpha, z_\alpha)\}_{\alpha=1,\dots,N}$. From the previous theorem, for $p \in U_\alpha$ we have an open set $V_{\alpha,p} \subset U$ and $T_{\alpha,p} > 0$ such that the functions,

$$z_{\alpha,p,\tau} = e^{\tau X} \cdot z_{\alpha,p} = \left(e^{\tau X} \cdot z^1_{\alpha,p}, ..., e^{\tau X} \cdot z^n_{\alpha,p}\right),$$

are well-defined holomorphic functions of τ with $z_{\alpha,p,\tau}(V_{\alpha,p}) \subset z_\alpha(U_\alpha)$ for $\tau \in D_{T_{\alpha,p}}$. Here we take $z_{\alpha,p}$ to be the restriction of z_α to $V_{\alpha,p}$, i.e. $z_{\alpha,p} = z_\alpha\big|_{V_{\alpha,p}}$.

Consider now a cover of M given by $\{V_{\alpha,p}\}_{\alpha=1,\dots,N}$, for all $p \in M$. Again we can take a finite subcover $\{V_{\alpha_j,p_j}\}_{\alpha=1,\dots,K}$ and consider $T' = min_j\{T_{\alpha_j,p_j}\}$. Let $\phi_{\alpha_j\alpha_k}$ be the transitions functions between these charts. From the commutation theorem, there exists T, $T \in (0, T']$ such that:

$$z_{\alpha_j,p_j,\tau} = e^{\tau X} \cdot z_{\alpha_j,p_j} = e^{\tau X} \cdot (\phi_{\alpha_j\alpha_k} \circ z_{\alpha_k,p_k}) = \phi_{\alpha_j\alpha_k} \circ z_{\alpha_k,p_k,\tau}, \quad \forall \tau \in D_T. \tag{6.1}$$

Thus, the same transition functions work for the complex-time evolved coordinates on the intersections $V_{\alpha_j,p_j} \cap V_{\alpha_k,p_k}$. We can define then a new atlas on M given by $\{V_{\alpha_j,p_j}, z_{\alpha_j,p_j,\tau}\}_{j=1,\dots K}$ which defines a new complex structure J_τ on M that is equivalent to the starting one, J_0. Recall that every complex manifold has a canonical complex structure.

For each of the starting charts U_α, $\alpha = 1, .., N$, denote by $\phi_\alpha : z_\alpha(U_\alpha) \subset \mathbb{C}^n \to U_\alpha$ the inverse of the coordinate function z_α. For $\tau \in D_T$ we define:

$$\varphi_{\tau,j} = \phi_{\alpha_j} \circ z_{\alpha_j,p_j,\tau} : V_{\alpha_j,p_j} \to U_\alpha$$

From (6.1) we know that the maps $\{\varphi_{\tau,j}\}_{j=1,\dots,K}$ agree on the overlapping charts thus they define a global bijective map $\varphi_\tau : M \to M$. This map is surjective since it is a local diffeomorphism at each point $p \in M$ and is homotopic to the identity, mapping each connected component on M to itself. The inverse map $\varphi_{-\tau}$ is also well defined and is given on the charts $\varphi_\tau(V_{\alpha_j,p_j})$ by:

$$e^{-\tau\cdot X}\cdot z_{\alpha_j,p_j,\tau} = z_{\alpha_j,p_j}.$$

The map φ_τ is then a biholomorphism from (M, J_τ) to (M, J_0), being the unique one which satisfies $z_{\alpha_j,p_j,\tau} = z_{\alpha_j,p_j}\circ\varphi_\tau$.

$\square$

Remark 12. Note that J_τ is a $(1,1)-$tensor field and thus we have $J_\tau = \varphi_{\tau^*}^{-1}J_0$ or $J_\tau = \varphi_\tau^* J_0$. We define then $J_\tau := \varphi_{\tau^*}^{-1}\circ J_0\circ\varphi_{\tau^*}$.

Corollary 6.15. *Consider $f \in C^\infty(M)$ a not necessarily J_0-holomorphic function on (M, J_0) with local J_0-holomorphic coordinates $\{z^i\}_{i=1,...,n}$. Given $X \in \mathfrak{X}^\omega(M)$ and the corresponding complex flow φ_τ, guaranteed by theorem 6.14 for a certain D_T, locally we have:*

$$\widehat{\varphi_\tau^* f}(z^1,...,z^n,\bar{z}^1,...,\bar{z}^n) = \hat{f}(z^1_\tau,...,z^n_\tau,\bar{z}^1_\tau,...,\bar{z}^n_\tau),$$

where, for each $i = 1,...,n$,

$$z^i_\tau = e^{\tau X}\cdot z^i \qquad \text{and} \qquad \bar{z}^i_\tau = e^{\bar{\tau}X}\cdot\bar{z}^i.$$

For real time $t \in \mathbb{R}$, the biholomorphism φ_τ is the usual flow of X in time t. For complex time $t = \tau \notin \mathbb{R}$, φ_τ is called the complex flow of $X \in \mathfrak{X}^\omega(M)$ and is, in general, dependent on the starting complex structure J_0. It should also be noted that, for $\tau \notin \mathbb{R}$, φ_τ does not necessarily satisfy the usual property of a flow $\varphi_{\tau+\sigma} \neq \varphi_\tau\circ\varphi_\sigma$ since these maps are dependent on the starting complex structure.

Writing the dependence on the starting complex structure explicitly as $\varphi_\tau^{J_0}$, we have the following commutative diagram for some $T > 0$ with $|\tau|, |\sigma|, |\tau+\sigma| < T$ (Ref. remark 2.12 page 22 [16]).

$$\begin{tikzcd}
& (M, J_{\tau+\sigma}) \arrow[dl, "\varphi_\sigma^{J_\tau}"'] \arrow[dr, "\varphi_\tau^{J_\sigma}"] \arrow[dd, "\varphi_{\sigma+\tau}^{J_0}"] & \\
(M, J_\tau) \arrow[dr, "\varphi_\tau^{J_0}"'] & & (M, J_\sigma) \arrow[dl, "\varphi_\sigma^{J_0}"] \\
& (M, J_0) &
\end{tikzcd}$$

6.4. Lie Series on Kähler Manifolds

We now extend these results to the case of Kähler manifolds. By introducing a compatible real analytic symplectic form ω, which results in the compatible triple (M, ω, J_0), immediate questions arise in the context of a complex time evolution, particularly whether the action of the biholomorphism φ_τ on the complex structure J_0, i.e J_τ, is still compatible with the symplectic form ω. As we have seen, an Hamiltonian vector field keeps the symplectic form fixed and we will consider precisely this case.

Consider a Kähler manifold (U, ω, J_0) where $U \subset \mathbb{R}^{2n}$ is an open set, J_0 is the standard complex structure on $\mathbb{R}^{2n} \subset \mathbb{C}^n$ with J_0-holomorphic coordinates $\{z_i\}_{i=1,\dots,n}$ and ω is a real-analytic symplectic form.

Theorem 6.16. *Let $f \in C^\omega(U)$ with real-analytic Hamiltonian vector field X_f with flow φ_t. By virtue of theorem 6.9 there is open $V \subset U$ and $T > 0$ such that the real-analytic Hamiltonian vector field of $e^{\tau X_h} \cdot f$ relative to ω is given by:*

$$X_{e^{\tau X_h} \cdot f} = \sum_{k=0}^{\infty} \frac{\tau^k \mathcal{L}^k_{X_h}(X_f)}{k!} =: e^{\tau \mathcal{L}_{X_h}} X_f, \quad |\tau| < T.$$

Proof. Ref. proposition 2.3.1 [14]. First, we prove by induction that:

$$\mathcal{L}^k_{X_h} X_f = X_{X^k_h(f)}.$$

For $k = 1$ we have:

$$\mathcal{L}_{X_h} X_f = [X_h, X_f] = X_{\{h,f\}} = X_{X_h(f)},$$

by proposition (4.2). Assuming this holds for k we have:

$$\mathcal{L}^{k+1}_{X_h} X_f = \mathcal{L}_{X_h}\left(\mathcal{L}^k_{X_h} X_f\right) = \mathcal{L}_{X_h} X_{X^k_h(f)} = [X_h, X_{X^k_h(f)}] = X_{\{h, X_{X^k_h(f)}\}} = X_{X^{k+1}_h(f)}.$$

We have then:

$$X_{e^{\tau X_h} \cdot f} = X_{\sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_{X^k_h(f)}} = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_{X^k_h(f)} = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \mathcal{L}^k_{X_h} X_f := e^{\tau \mathcal{L}_{X_h}} X_f.$$

□

Remark 13. For $t \in \mathbb{R}$ this result reflects that φ_t of theorem 6.14 is a symplectomorphism where $(\varphi_t^{-1})_* X_f = X_{\varphi_t^* f}$. In general however, for $t = \tau \notin \mathbb{R}$, φ_τ will not be a symplectomorphism.

Lemma 6.17. *Under the conditions of the previous theorem one has:*

$$e^{\tau \mathcal{L}_{X_h}} X_{z^i} = X_{z^i_\tau} \quad \textit{and} \quad e^{\bar{\tau} \mathcal{L}_{X_h}} X_{\bar{z}^i} = X_{\bar{z}^i_\tau}$$

Proof. Ref. lemma 3.4 [9]. The results follows from the previous proposition. □

The following theorem answers our initial question, in particular it shows that (M, ω, J_τ) is a Kähler manifold with $g_\tau = \omega(\cdot, J_\tau$ defining a Riemannian metric on M.

Theorem 6.18. *Let $(M, \omega, J_0, \gamma_0)$ be a real-analytic compact Kähler manifold with Kähler form ω, complex structure J_0 and Riemannian metric γ_0 which are all real-analytic. Given $h \in C^\omega(M)$, $\exists T > 0$ defining $D_T := \{\tau \in \mathbb{C} : |\tau| < T\}$ such that:*

(i) (M, ω, J_τ) is a Kähler manifold where $J_\tau = \varphi_\tau^*$ and φ_τ is the biholomorphism of theorem 6.14.

(ii) Given a local analytic Kähler potential κ_0 for (M, ω, J_0), a local Kähler potential for (M, ω, J_τ) is given by:

$$\kappa_\tau = -2Im\,\psi_\tau, \textit{where} \qquad \psi_\tau = -\frac{i}{2} e^{\tau X_h} \cdot \kappa_0 + \tau h - \alpha_\tau,$$

where α_τ is the analytic continuation in t of

$$\alpha_\tau = \int_0^t e^{t' X_h} \cdot \theta(X_h) dt',$$

where $\theta = \frac{i}{2}(\partial_0 - \bar{\partial}_0)\kappa_0$, with ∂_0 and $\bar{\partial}_0$ are the Dolbeault operators relative to the initial complex structure J_0.

Proof. Ref. theorem 4.1 [9] for the original proof or, for a step by step approach, theorem 2.4.1 from [14]. □

In general, for $\tau \notin \mathbb{R}$, φ_τ will not be a symplectomorphism, i.e. $\varphi_\tau^*(\omega) \neq \omega$. This implies that the resulting Kähler structure $(M, \omega, J_\tau, \gamma_\tau)$ will not be equivalent to $(M, \omega, J_0, \gamma_0)$. However, when $\tau = t \in \mathbb{R}$, φ_t is indeed a symplectomorphism.

6.5. Example: Kähler Plane

Let us look at the particular case of the Kähler structure on the plane, i.e. $(\mathbb{R}^2 \cong \mathbb{C}, \omega, J_0, \gamma)$, where $\omega = dx \wedge dy$ is the canonical symplectic form, J_0 is the canonical complex structure with (global) holomorphic coordinate $z = x + iy$ and γ is the usual Euclidean metric. We have the following useful results:

$$\begin{cases} x = \dfrac{z+\bar{z}}{2} \\ y = \dfrac{z-\bar{z}}{2i} \end{cases} \Rightarrow \begin{cases} dx = \dfrac{dz+d\bar{z}}{2} \\ dy = \dfrac{dz-d\bar{z}}{2i} \end{cases} \Rightarrow \quad \omega = \left(\frac{dz+d\bar{z}}{2}\right) \wedge \left(\frac{dz-d\bar{z}}{2i}\right) = \frac{i}{2}\, dz \wedge d\bar{z} = \frac{i}{2}\partial\bar{\partial}(z\bar{z})$$

We can easily check that $y^2 = \frac{1}{2}z\bar{z}$ and thus y^2 is a (global) Kähler potential.

Let us consider now the Hamiltonian function $h = h(y) \in C^\omega(\mathbb{R})$, which we require to be strongly convex, i.e. $h''(y) \geq c > 0 \; \forall y \in \mathbb{R}$. The requirement of strong convexity is imposed to ensure that we obtain a Riemannian metric later on. However, the choice of this particular Hamiltonian is simply to ensure an easily solvable differential equations which still maintains the illustrative purposes of the example.

The Hamiltonian vector field X_h corresponding to $h(y)$ is obtained by the following equation:

$$\iota_{X_h}\omega = dh(y) = h'(y)dy$$

Replacing the data for calculation we obtain:

$$\frac{i}{2}dz \wedge d\bar{z}\,(X_h, \cdot) = h'(y)\left(\frac{dz-d\bar{z}}{2i}\right) \Rightarrow X_h = h'(y)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) = h'(y)\frac{\partial}{\partial x}$$

The flow parameterized by real time $t \in \mathbb{R}$ is the solution of the following system of ODEs:

$$\begin{cases} \dot{x} = h'(y) \\ \dot{y} = 0 \end{cases} \Rightarrow \; \phi_t^{X_h} = \Big(x + h'(y)t\,,\, y\Big).$$

Considering $t = \tau \in \mathbb{C}$, i.e. the flow parameterized by complex-valued time τ, results on the real coordinate x turning into a complex value, something which we can not interpret easily. However, if we instead consider the complex-time evolution from the perspective of the holomorphic coordinate z, we can make sense of this complex flow.

Since we consider everything to be real-analytic, by applying the theory of Lie Series we have seen before with complex time-evolution parameterized by $\tau = t + is$, $t, s \in \mathbb{R}$ we obtain:

$$z_\tau = e^{\tau X_h} \cdot z = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X_h^k \cdot z = z + \tau h'(y) = x + th'(y) + i(y + sh'(y)),$$

since $X_h^k \cdot z = 0$ for any $k > 1$. Thus, if we have $s \in \mathbb{R}$ such that $\forall y$, $y + sh'(y) \neq 0$, we obtain a (global) diffeomorphism on the plane given by $\varphi_\tau(x, y) = (x + th'(y), y + sh'(y))$.

Let us consider now the simpler case of a purely complex parameterization $\tau = is$, $s \in \mathbb{R}^+$. The choice of positive s is to, again, ensure a positive metric which will be of the form $g_s(u, v)$.

The complex-time evolution of the coordinate is given by $z_\tau = x + \tau h'(y) + iy$ and thus we have:

$$z_\tau = x + ish'(y) + iy = x + i\left(sh'(y) + y\right).$$

In this case, the diffeomorphism on the plane is given by:

$$\varphi_s(x, y) = (x, y + sh'(y)).$$

The symplectic form ω is kept fixed by the Hamiltonian flow, recall that $\mathcal{L}_{X_h}\omega = d\iota_{X_h}\omega + \iota_{X_h}d\omega = 0$, due to the fact that X_h is Hamiltonian and ω is a closed form.

This new coordinate z_τ is the holomorphic coordinate of a new complex structure J_τ which is not holomorphic on the canonical complex structure J_0 we started with. We can easily see that $\varphi_s = \big(u(x, y), v(x, y)\big)$ does not satisfy the Cauchy-Riemann equations.

$$\frac{\partial u(x, y)}{\partial x} = 1 \neq \frac{\partial v(x, y)}{\partial y} = 1 + sh''(y), \quad \text{for} \quad s \neq 0$$

By theorem 2.6 in [9] one can obtain the new complex structure J_τ which turns φ_τ into a biholomorphic map $\varphi_\tau : (\mathbb{R}^2, J_\tau) \to (\mathbb{R}^2, J_0)$. The biholomorphism condition is given by $d\varphi_\tau \circ J_\tau = J \circ d\varphi_\tau$ and thus:

$$J_\tau = d\varphi_\tau^{-1} \circ J \circ d\varphi_\tau.$$

Performing the calculations one obtains:

$$J_\tau = \frac{1}{1 + sh''(y)} \begin{bmatrix} 1 + sh''(y) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 + sh''(y) \end{bmatrix} = \begin{bmatrix} 0 & -(1 + sh''(y)) \\ \frac{1}{1+sh''(y)} & 0 \end{bmatrix}$$

Having a new complex structure J_τ bears the question, are the symplectic and complex structures still compatible? Let us see how the symplectic form looks as a function of the new coordinate z_τ, we have:

$$dz_\tau = dx + i\big(1 + sh''(y)\big)\, dy \iff dz_\tau \wedge d\bar{z}_\tau = -2i\big(1 + sh''(y)\big)dx \wedge dy,$$

the symplectic form is then given by:

$$\omega = dx \wedge dy = \frac{i}{2}\frac{dz_\tau \wedge d\bar{z}_\tau}{1 + sh''(y)},$$

which is a form of type $(1, 1)$ in the coordinate z_τ and thus the symplectic structure is still compatible with the new complex structure J_τ, i.e. $J_\tau^*\omega = \omega$ (see just below definition 3.39 for the compatibility criteria on a Kähler form).

For this to remain a Kähler manifold we need to check that we still have a Riemannian metric obtainable through the symplectic and complex structures. Let us see how the metric $g_\tau(u, v) = \omega(u, J_\tau v)$ changes. Considering general vectors $u = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ and $v = c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}$ and the J_τ given above we have:

$$
\begin{aligned}
g_s(u,v) =& \omega(u, J_\tau v) = dx \wedge dy\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}, J_\tau\left(c\frac{\partial}{\partial x}, d\frac{\partial}{\partial y}\right)\right) = \\
&= \left(\frac{dx \otimes dx}{1+sh''(y)} + (1+sh''(y))dy \otimes dy\right)(u,v)
\end{aligned}
$$

In the limit where $s \to \infty$ we see that there is a collapse of the plane into the $y-$axis.

Consider now the family of Kähler polarizations $P_{J_\tau} = \left\langle \frac{\partial}{\partial \bar{z}_\tau} \right\rangle$ and see what happens in the limiting case where $s \to \infty$.

Considering $\frac{\partial}{\partial \bar{z}_\tau} = \alpha(x,y)\frac{\partial}{\partial x} + \beta(x,y)\frac{\partial}{\partial y}$, we have:

$$
\begin{cases}
d\bar{z}_\tau\left(\dfrac{\partial}{\partial \bar{z}_\tau}\right) = 1 \\
dz_\tau\left(\dfrac{\partial}{\partial \bar{z}_\tau}\right) = 0
\end{cases}
\Leftrightarrow \quad \frac{\partial}{\partial \bar{z}_\tau} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{i}{1+sh''(y)}\frac{\partial}{\partial y}\right)
$$

Thus, when we have $s \to \infty$ the polarizations converge to real polarizations.

This particular example shows firstly that this method works despite the fact that the Kähler plane is not compact and secondly, that the region of convergence extends to ∞, i.e., s is allowed to go to infinity.

7. Appendix

7.1. Embeddings of compact m-dimensional manifolds in $\mathbb{R}^N$

We are interested in showing the result that a compact m-manifold can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$. For this, the main idea is that of finite partitions of unity, we will prove that they exist and their utility comes from the fact that a compact m-manifold is a normal space.

Definition 7.1. ***m-Manifold***
A topological m-manifold is a topological space X which is Hausdorff and second countable such that at each point x of X there is a neighborhood which is homeomorphic to $\mathbb{R}^m$.

Definition 7.2. ***Support of a Function***
The support of a real valued function $\phi : X \to \mathbb{R}$, is defined to be the closure of the set $\phi^{-1}(\mathbb{R}-\{0\})$.

Definition 7.3. ***Partition of Unity***
Given a space X and a finite indexed open covering of X given by $\{U_1, ..., U_n\}$. A partition of unity dominated by $\{U_1, ..., U_n\}$ is an indexed family of continuous functions, $\phi_i : X \to [0, 1]$ for i=1,...,n, such that:
(1) (support ϕ_i) $\subset U_i$ for each i.
(2) $\sum_{i=1}^{n} \phi_i(x) = 1$ for each x.

Theorem 7.4. ***Existence of finite partitions of unity***
Given a finite open covering $\{U_1, ..., U_n\}$ of the normal space X then there exists a partition of unity dominated by $\{U_1, ..., U_n\}$.

Proof. Step 1: We begin by showing that we can construct a "smaller" covering of X, $\{V_i\}$ from $\{U_i\}$, such that $\overline{V_i} \subset U_i$ for each i using an induction argument.

Let $A = X - (U_2 \cup ... \cup U_n)$, A is a closed subset of X since it is the complement in X of a union of open sets. Besides, since the U_i are an open cover of X then A must be contained in the open set U_1. Now, since X is normal and A is a closed set contained in U_1 then there is an open set V_1 containing A such that $\overline{V_1} \subset U_1$ ([8], lemma 31.1). Thus, the new collection $\{V_1, U_2, ..., U_n\}$ is also an open cover of X.

Assuming then that the collection given by $\{V_1, ..., V_{k-1}, U_k, U_{k+1}, ..., U_n\}$ is an open cover of X let $A = X - (V_1 \cup ... \cup V_{k-1}) - (U_{k+1} \cup ... \cup U_n)$. As before, A is a closed set that must be contained in U_k. Using the same argument, there is an open set V_k containing A such that $\overline{V_k} \subset U_k$ thus $\{V_1, ..., V_{k-1}, V_k, U_{k+1}, ..., U_n\}$ covers X. Performing the n steps of the induction we obtain the open covering of X,$\{V_1, V_2, ..., V_n\}$ satisfying $\overline{V_i} \subset U_i$.

Step 2: Starting from the open covering of X, $\{U_1, U_2, ..., U_n\}$ construct $\{V_1, V_2, ..., V_n\}$ satisfying $\overline{V_i} \subset U_i$ for each i and, using the same procedure, construct an open covering $\{W_1, W_2, ..., W_n\}$ such that $\overline{W_i} \subset V_i$ for each i. Using Urysohn's lemma ([8], theorem 33.1), choose for each i a continuous function $\psi_i : X \to [0, 1]$ such that $\psi_i(\overline{W_i}) = 1$ and $\psi_i(X - V_i) = 0$. Since $\psi_i^{-1}(\mathbb{R} - 0)$ is contained in V_i we have (*support* $\psi_i) \subset \overline{V_i} \subset U_i$. Since the collection $\{W_i\}$ covers X, the sum $\Psi(x) = \sum_{i=1}^{n} \psi_i(x)$ is positive for each x, Therefore we can define, for each j, $\phi_j(x) = \frac{\psi_j(x)}{\Psi(x)}$ obtaining the desired partition of unity. □

Finally, we have all the necessary tools to prove the result we were interested in. The statement is the following:

Theorem 7.5. *If X is a compact m-manifold, then X can be embedded in $\mathbb{R}^N$ for some positive integer N.*

Proof. Since X is locally homeomorphic to $\mathbb{R}^m$ we can consider an open covering of X given by $U = \cup_{x \in X} U_x$ where U_x is a locally homeomorphic neighborhood of x in X. Since X is compact we can choose a finite subcover of U given by $\{U_1, ..., U_n\}$ where each U_i can be embedded in $\mathbb{R}^m$. Let the embeddings be given by $g_i : U_i \to \mathbb{R}^m$ for each i.

Since X is compact and Hausdorff, X is a normal space ([8], theorem 32.3) therefore, by theorem 7.4, there is a finite partition of unity dominated by $\{U_1, ..., U_n\}$ given by the functions $\phi_1, ..., \phi_n$.

Now, for each $i = 1, ..., n$, define a function $h_i : X \to \mathbb{R}^m$ given by:

$$h_i(x) = \begin{cases} \phi_i(x) \cdot g_i(x) & for \;\; x \in U_i \\ \mathbf{0} = (0, ..., 0) & for \;\; x \in X - (support \;\phi_i) \end{cases}$$

We now check that $h_i(x)$ is well defined. On the intersection of the two domains the definition of h_i agrees since $\phi_i(x) = 0$ when x is outside the support of ϕ_i. Each h_i is a continuous function since the restriction to the open sets U_i and $X - (support \;\phi_i)$ are continuous.

We now define a new function $F : X \to (\underbrace{\mathbb{R} \times ... \times \mathbb{R}}_{\texttt{n times}} \times \underbrace{\mathbb{R}^m \times ... \times \mathbb{R}^m}_{\text{n times}})$ given by:

$$F(x) = (\phi_1(x), ..., \phi_n(x), h_1(x), ..., h_n(x))$$

F is clearly a continuous function since each coordinate function is continuous ([8], theorem 18.4).

Finally, to show that F is an embedding it is enough to show that F is injective since X is compact and $\mathbb{R} \times ... \times \mathbb{R} \times \mathbb{R}^m \times ... \times \mathbb{R}^m$ is Hausdorff ([8], theorem 26.6) (i.e., F is a surjective function, restricting the codomain to the image of the domain, since the domain is compact and the codomain is Hausdorff if we show that F is injective than it is an homeomorphism by the cited theorem and therefore an embedding).

Let $F(x) = F(y)$, then $\phi_i(x) = \phi_i(y)$ and $h_i(x) = h_i(y)$ for all i. We have that $\phi_i(x) > 0$ for some i (since $\sum_i = 1^n \phi_i(x) = 1$ therefore $\phi_i(y) > 0$ and both $x, y \in U_i$.We have then:

$$\phi_i(x) \cdot g_i(x) = h_i(x) = h_i(y) = \phi_i(y) \cdot g_i(y)$$

Since $\phi_i(x) = \phi_i(y) > 0$ then $g_i(x) = g_i(y)$. But since the g_i's are embeddings they are injective therefore $x = y$ and F is an embedding for some $N = 2n$ (n is not related to the dimension of the manifold). $\square$

7.2. Segal-Bargmann Space

Here we simply state the definition and a single result regarding the Segal-Bargmann space. A full treatment of this can be seen in [5] section 14.4, pp. 292-301.

Definition 7.6. ***Segal-Bargmann space*** $\mathcal{H}L^2(\mathbb{C}^n, \mu_\hbar)$
Consider the space of holomorphic functions f on $\mathbb{C}^n$ which satisfy the following:

$$||f||_\hbar := \left(\int_{\mathbb{C}^n} |f(\boldsymbol{z})|^2 \mu_\hbar(\boldsymbol{z}) d\boldsymbol{z}\right)^{1/2} < \infty,$$

where $\mu_\hbar$ is a density given by $\mu_\hbar = \frac{1}{(\pi\hbar)^n} e^{-|\boldsymbol{z}|^2/\hbar}$. Acting on this space we define creation (a_j^*) and annihilation (a_j) operators given by:

$$a_j^* = z_j, \quad a_j = \hbar \frac{\partial}{\partial z_j},$$

which satisfy $[a_j^*, a_k^*] = [a_j, a_k] = 0$ and $[a_j, a_k^*] = \hbar \delta_{jk} I$.

Proposition 7.7. *The Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}^n, \mu_\hbar)$ is complete with respect to the norm $||\cdot||_\hbar$ and forms a Hilbert space with associated inner-product:*

$$\langle f, g \rangle_\hbar := \int_{\mathbb{C}^n} \overline{f(\boldsymbol{z})} g(\boldsymbol{z}) \mu_\hbar(\boldsymbol{z}) d\boldsymbol{z}.$$

The space of holomorphic polynomials forms a dense subspace of $\mathcal{H}L^2(\mathbb{C}^n, \mu_\hbar)$.

Proof. (Ref. [5] proposition 14.5, pp. 296-297). □

7.3. Holonomy

Consider a symplectic manifold (M, ω) with an Hermitian line bundle L with connection ∇ and curvature 2-form $\omega = d\theta$. Given a loop $\gamma : [a, b] \to M$ we can construct a section $s \in \Gamma(L, M)$ which is covariantly constant along γ. This construction is given by:

$$s(\gamma(T)) = exp\left\{ i \int_{\gamma(a)}^{\gamma(T)} \theta(\dot{\gamma}(t)) dt \right\},$$

where θ is the local symplectic potential. The values of the section at the start and ending point will in general differ by a constant of absolute value 1.

Definition 7.8. ***Holonomy***
*The **holonomy** of a loop $\gamma : [a, b] \to M$ is the unique constant α with unit absolute value such that $s(\gamma(b)) = \alpha s(\gamma(a))$, where s is the covariantly constant section along the directions of γ.*

Proposition 7.9. *If S is a compact oriented surface with boundary ∂S in M which is a loop, we have:*

$$holonomy(\partial S) = exp\left\{ i \int_S \omega \right\}, \quad and \quad exp\left\{ i \int_S \omega \right\} = 1,$$

if S is a closed surface, where the boundary $\partial S = \emptyset$.

Proof. (Ref. [5] proposition 23.6, pp. 487-489). □

8. Bibliography

[1] A. Echeverria-Enriquez, M.C. Munoz-Lecanda, N. Roman-Roy, C. Victoria-Monge: Mathematical Foundations of Geometric Quantization. In: *Extracta Mathematicae 13(2) (1998) 135-238*

[2] Alexandre Aleksandrovich Kirillov: Geometric Quantization. In: *Encyclopedia of Mathematical Sciences, Dynamical Systems* Vol. 4 - translated by G. Wassermann (Springer-Verlag, 1990), pp. 139–176

[3] Ana Cannas da Silva: *Lectures on Symplectic Geometry.* Springer Verlag Berlin and Heidelberg GmbH Co KG, 2008

[4] Andrei Moroianu: *Lectures on Kähler Geometry.* Cambridge University Press, 2007

[5] Brian C. Hall: *Quantum Theory for Mathematicians.* Springer, 2013

[6] Dusa McDuff and Dietmar Salamon: *Introduction to Symplectic Topology.* Oxford University Press, 2017

[7] Gonçalo Oliveira: *Geometry and Quantization*, Instituto Superior Técnico, Master's Thesis, 2010

[8] James R. Munkres: *Topology* 2^{nd} *edition.* Pearson, 2000

[9] José M. Mourão and João P. Nunes: On complexified analytic Hamiltonian flows and geodesics on the space of Kähler metrics. (2018)

[10] João Pimentel Nunes: Degenerating Kähler structures and geometric quantization. In: *World Scientific - Reviews in Mathematical Physics* (2014)

[11] João Pimentel Nunes: Handouts on Quantization and Kähler Geometry. In: *Conference on Dynamical Aspects of Pseudo-Riemannian Geometry* (2020)

[12] Leonor Godinho and José Natário: *An Introduction to Riemannian Geometry with applications to Mechanics and Relativity.* Springer, 2014

[13] Mark J. Gotay: Functorial Geometric Quantization and Van Hove's Theorem. In: *International Journal of Theoretical Physics, Vol. 19, No. 2* (1980)

[14] Miguel Pereira: *Complexified Symplectomorphisms and Geometric Quantization*, Instituto Superior Técnico, Master's Thesis, 2017

[15] N.M.J. Woodhouse: *Geometric Quantization* 2^{nd} *edition.* Oxford Science Publications, 1991

[16] Pedro Miguel Silva: *Complex flows and geodesics in the space of Kähler metrics*, Instituto Superior Técnico, Master's Thesis, 2018

[17] Rui Loja Fernandes: *Differential Geometry.* Notes - 2018

[18] W. Gröbner: General theory of the Lie series, in Contributions to the method of Lie series. In: *Bibliographisches Institut Manheim* (1967)