TÉCNICO LISBOA

Semilinear Wave Equations on Decelerated Expanding FLRW Spacetimes

Telmo André Bernardes da Cunha

Thesis to obtain the Master of Science Degree in

Applied Mathematics and Computation

Supervisors: Prof. João Lopes Costa
Prof. Pedro Alves Martins da Silva Girão

Examination Committee

Chairperson: Prof. José António Maciel Natário
Supervisor: Prof. Pedro Alves Martins da Silva Girão
Member of the Committee: Prof. Jorge Filipe Drumond Pinto da Silva

July 2025

Acknowledgments

I would like to express my sincere gratitude to both my advisors, Prof. João Costa and Prof. Pedro Girão.

To Prof. João Costa, thank you for introducing me to this problem, for several contributions to this thesis, for numerous insightful discussions and for your continuous support and motivation.

To Prof. Pedro Girão, I am grateful for many fruitful discussions and contributions and, in particular, for your encouragement during moments of uncertainty.

This thesis would not have been possible without your guidance and support, my heartfelt thanks to both of you.

Finally, to my parents... thank you, for everything.

Declaration

I declare that this document is an original work of my own authorship and that it fulfills all the requirements of the Code of Conduct and Good Practices of the Universidade de Lisboa.

Resumo

Expomos em detalhe o resultado de explosão em tempo finito para soluções de um problema de valor inicial associados a uma família de equações de onda quasilineares em $3+1$ dimensões obtido por Fritz John em [Joh85a]. Utilizando principalmente técnicas desenvolvidas na prova desse resultado, estudamos a existência global de soluções em $3+1$ associadas a uma equação de onda semilinear a um parâmetro, $\Box u = -\left[2p/\left((1-p)(t+1)\right)\right] u_t + (u_t)^2$ para $p \in (0,1)$, que surge ao considerar o operador de onda num universo em expansão desacelerada para uma métrica FLRW $\boldsymbol{g}$ a partir da equação $\Box_{\boldsymbol{g}} u = -(u_t)^2$. Pelo resultado de Fritz John mencionado acima, é sabido que, para $p = 0$, as soluções desta equação explodem em tempo finito dadas condições iniciais não triviais. Por outro lado, a partir do Teorema 2.1 de [CFO23] por Costa et al., para condições iniciais suficientemente pequenas, há existência global das soluções de $\Box_{\boldsymbol{g}} u = -(u_t)^2$, para $p > 1$, correspondente à escolha de uma métrica FLRW em expansão acelerada. Tanto quanto sabemos, permanece em aberto a questão de existência global das soluções quando $p \in (0,1)$. Expomos duas abordagens (até agora sem sucesso) ao problema. Uma primeira a partir de um argumento de continuidade nos parâmetros para p suficientemente pequeno e uma segunda utilizando o método de Klainerman e Sarnack no caso $p = 1/2$. Em ambos os casos encontramos obstruções à resolução do problema.

Palavras-chave: equações diferencias parciais, problemas de valor inicial, equações de onda não-lineares, existência global de soluções, explosão em tempo finito, espaço-tempo FLRW.

Abstract

We give in detail the proof of blow-up for solutions of a Cauchy problem associated to a family of quasilinear wave equations in $3+1$ by Fritz John found in [Joh85a]. Mainly using the techniques considered in the proof of the previous theorem we attempt to prove a blow-up result in $3+1$ for solutions of the semilinear wave equation $\Box u = -\left[2p/\left((1-p)(t+1)\right)\right] u_t + (u_t)^2$ for a fixed $p \in (0,1)$. This wave-equation is obtained by considering the wave operator in the context of a decelerated expanding FLRW spacetime for the equation $\Box_{\boldsymbol{g}} u = -(u_t)^2$. In the $p=0$ case it is known, via the aforementioned Fritz John result, that solutions of the above equation blow-up in finite time for non-trivial initial conditions. On the other hand, from Theorem 2.1 in [CFO23] by Costa et al., for sufficiently small initial data one knows there is global existence for solutions of $\Box_{\boldsymbol{g}} u = -(u_t)^2$ when $p > 1$, which corresponds to an accelerated expanding FLRW metric. To our current awareness, it is an open problem whether blow-up of solutions occur for the above equation when $p \in (0,1)$. We show two (so far unsuccessful) approaches to the problem. First we attempt a continuity argument in the parameters for sufficiently small p, while in a second attempt we use the method by Klainerman and Sarnack to try and obtain a simpler wave equation for the $p = 1/2$ case. In both approaches we find obstructions to the proof.

Keywords: partial differential equations, Cauchy problem, nonlinear wave equations, global-existence of solutions, blow-up in finite time, FLRW spacetime.

Contents

List of Figures

1

Introduction

1.1 General background

The analysis of partial differential equations (PDEs) typically begins by formulating what is known as a Cauchy problem, or initial value problem (IVP), with appropriate initial and/or boundary data. The key questions regarding such a problem are the following: does a solution to the problem exist (*existence*), is it uniquely determined by the data (*uniqueness*), and under what circumstances can additional qualitative features, such as smoothness or *regularity*, be established? In this dissertation we will be mainly concerned with a question which falls under the existence class. In particular, assuming we have local existence and uniqueness of solutions, we want to understand if they can be extended globally (*global existence*) or, to the contrary, if there is a finite time T to their existence (*blow-up in finite time*).

As stated by Lawrence Evans in [Eva10], "There is no general theory known concerning the solvability of all partial differential equations. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modelled by PDE". Thus, we will restrict ourselves to the study of a particular PDE, the *wave equation*, which enjoys almost endless application. We consider it in the following general form[1]

$$\Box u(x,t) := u_{tt}(x,t) - \Delta u(x,t) = \phi(x,t,u,u',u''), \tag{1.1}$$

where $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a solution and where u' and u'' represent, respectively, the vector of all first order derivatives of u and the vector of all second order derivatives of u. We further assume that, on linearizing (1.1) in a neighbourhood of the trivial solution $u = 0$, one recovers the classical wave equation, which implies in particular that

$$\phi = \frac{\partial \phi}{\partial u} = 0, \ \ \frac{\partial \phi}{\partial u'} = 0, \ \ \frac{\partial \phi}{\partial u''} = 0, \ \text{at } (x,t,0,0,0). \tag{1.2}$$

PDEs are classified as either linear or nonlinear, where the nonlinear ones are further classified as *semilinear*, *quasilinear* or fully nonlinear. In general, this classification reflects an increasing difficulty or

[1]Throught the thesis we consider the d'Alembert operator $\Box$ (without a lower metric index) to have the signature of (1.1), following the works by Fritz John.

complexity of the techniques involved in understanding the class of questions mentioned above [Eva10]. In equation (1.1) the function ϕ, assuming (1.2), dictates this classification. In particular, the wave equation in (1.1) is considered

- **linear**, if ϕ is of the form $\phi(x, t)$.
- **semilinear**, if ϕ is of the form $\phi(x, t, u, u')$, such that derivatives of order two in (1.1) appear linearly.
- **quasilinear**, if ϕ is of the form $\phi(x, t, u, u', u'')$ but second order derivatives appear with lower order coefficients, i.e. the second order terms in ϕ are of the form $h(x, t, u, u')u''$ for some function h.
- **fully nonlinear**, if ϕ is of the form $\phi(x, t, u, u', u'')$ without further restrictions.

In the present work we solely consider semilinear and quasilinear versions of (1.1) in the context of an initial value problem. This corresponds, for the wave equation, to setting initial conditions where one specifies the solution u and its time-derivative u_t at $t = 0$, with the possible inclusion of boundary terms depending on the domain under consideration. Throughout, the IVP we consider is of the form

$$\begin{cases} \Box u(x,t) = \phi(x, t, u, u', u''), \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x), \quad x \in \mathbb{R}^n \end{cases} \tag{1.3}$$

where $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is the solution, and f, g is the initial data over which we impose some assumptions, mainly compact support and certain regularity conditions.

For fairly general conditions on ϕ, f and g, the IVP (1.3) satisfies **local well-posedness**, i.e. it verifies local existence and uniqueness of solutions. In regards to **global existence**, besides conditions on ϕ, f and g, the spatial dimension n has a significant impact on the behaviour of solutions. For $n \geq 4$, and a large class of nonlinearities ϕ, it is known that (1.3) admits global in time unique solutions for sufficiently small initial data. We state this results for context, without proofs, in Chapter 3. Different versions can be found in [Sog95], [Rin] and [Luk], depending on dimension, type of nonlinearity and initial conditions.

This phenomena however does not extend for dimensions $n \leq 3$. In the $n = 3$ case, Fritz John shows in [Joh85a] that, for a particular class of nonlinearities ϕ, which include some of those considered in the $n \geq 4$ result mentioned above, there is no non-trivial global in time classical solution of (1.3), independently of how small the initial conditions are made to be. In particular, for that class of nonlinearities, if one considers initial conditions of the form $u(x, 0) = \epsilon f(x)$ and $u_t(x, 0) = \epsilon g(x)$, $x \in \mathbb{R}^3$, then, independently on the choice of $\epsilon > 0$, the solution u blows-up in finite time. This is quite remarkable in and of itself and is shared also by dimensions $n = 1$ and $n = 2$, see for example [Joh85b] and [Ali99] respectively. A detailed exposition of the proof of blow-up in 3+1 dimensions by Fritz John is a significant portion of this dissertation and is covered in Chapter 4.

Still, for $n = 3$, there is a class of nonlinearities ϕ, said to satisfy the **null condition**, for which global in time solutions exist for sufficiently small initial data. This is very briefly mentioned in Chapter 3 but can be seen in more detail again in [Sog95] and [Luk].

Essentially, long time behaviour of solutions of (1.3) depend heavily on dimension, type of nonlinearity and properties of the initial data.

1.2 Main problem and motivation

The main goal of this thesis is to understand the behaviour of solutions for the IVP associated to the following semi-linear wave equation in 3+1 dimensions

$$\Box u(x,t) = -\frac{2p}{(1-p)(t+1)}u_t(x,t) + (u_t(x,t))^2, \tag{1.4}$$

where p is a fixed parameter satisfying $p \in (0,1)$. In particular, we want to understand if the IVP associated to (1.4) admits non-trivial C^2 solutions u which are global in time or, rather, if solutions blow-up in finite time. Equation (1.4) follows from the well-known example

$$\Box u = (u_t)^2, \tag{1.5}$$

which belongs to the class of nonlinearities considered in Fritz John's blow-up result and features prominently in [Joh85a]. In particular, a result which establishes bounds on the time of blow-up for spherically symmetric solutions of this equation is studied in detail in Section 2 of [Joh85a].

The relation between (1.4) and (1.5) can be seen by viewing the latter equation in the form

$$\Box_{\boldsymbol{g}} u = -(u_t)^2, \tag{1.6}$$

where the wave operator (or d'Alembertian) $\Box_{\boldsymbol{g}}$, which depends on the geometry of the underlying manifold, is considered for the Minkowski metric[2]

$$\boldsymbol{g} = \boldsymbol{\eta} := -dt^2 + \sum_{i=1}^{3}(dx^i)^2. \tag{1.7}$$

Considering instead a FLRW (Friedmann–Lemaître–Robertson–Walker) metric of the form

$$\boldsymbol{g} = -dt^2 + t^{2p}\sum_{i=1}^{3}(dx^i)^2. \tag{1.8}$$

with $p \in (0,1)$, which corresponds to a decelerated expanding universe, one can obtain (1.4). This procedure can be found in detail in Section 5.1. We now make the following observations:

- Taking $p = 0$ in equation (1.4) we obtain (1.5). Thus, from the previously mentioned blow-up result by Fritz John, we know solutions of this equation blow-up in finite time for non-trivial initial data.
- From Theorem 2.1 of [CFO23] by Costa et al., it is known that (1.6) admits global in time solutions, for sufficiently small initial data, when $\boldsymbol{g}$ corresponds to a FLRW metric describing an accelerated expanding universe. This corresponds to the same metric $\boldsymbol{g}$ in (1.8) for the case where $p > 1$.

[2]Here, we consider a different metric signature compared to (1.1), following [CFO23].

To the author's current knowledge, whether (1.4) admits non-trivial classical solutions u which can be extended globally for $p \in (0, 1)$ is still an open problem. In this dissertation we attempt to prove (so far unsuccessfully) a blow-up result for solutions of (1.4) by mainly following the ideas used in Fritz John's proof of blow-up found in [Joh85a]. This is done in Section 5.4 in a few different ways.

As a final remark, in terms of application, the problem above could be viewed as a toy version mimicking phenomena of interest found in applications to cosmology, see [FMO+24] for a recent example.

1.3 Layout of the thesis

This thesis has the following structure:

- **Chapter 2 - Linear Wave Equation:** Since methods commonly used in the study of nonlinear wave equations first appear in the treatment of the linear case, we start by stating them in this context. In particular, we cover explicit solutions of the linear wave equation, for dimensions $n = 1, 2, 3$, and energy methods which establish uniqueness and properties of the solution such as domain of dependence and finite speed of propagation. For this purpose, we follow the treatment in Evans's book [Eva10].

  For the purpose of understanding the remainder of the thesis the main results of this chapter are: Section 2.1.2, where we obtain d'Alembert 's representation formula in the half-space; Section 2.1.3, particularly the Euler-Poisson-Darboux Equation (2.26) and the ideas behind the construction of Kirchoff 's representation formula via spherical averaging and the reduction to the $n = 1$ case in the half-space; Section 2.1.5 where we cover the non-homogeneous case using Duhamel's method, in particular the explicit solution of the linear non-homogeneous wave equation in 1+1 dimensions given by (2.67).

- **Chapter 3 - Nonlinear Wave Equation:** This chapter is essentially an aggregate of results providing context for the remainder of the thesis and is divided in two sections. In Section 3.1 we state a local well-posedness result for quasilinear wave equations, while in Section 3.2 we state results regarding global existence, how the problem relates to dimension and how, in the $3 + 1$ dimensional case, the null condition influences global existence. We finalize this chapter with a straightforward extension of the Euler-Poisson-Darboux equation to the semilinear case.

- **Chapter 4 - Blow-up for Quasilinear Wave Equations in 3+1 Dimensions:** In this chapter we look in detail at results by Fritz John regarding blow-up of quasi-linear wave equations in three space dimensions. In particular we look at Fritz John's publications [Joh85a] and [Joh90] where it is shown that, for a certain family of quasi-linear wave equations in 3+1 dimensions, a global in time C^2 solution must necessarily be the trivial one, assuming compact support and certain regularity over the initial conditions. This in turn implies that, given non-trivial initial conditions and independently of their size, the solutions blow-up in finite time.

We start by covering a uniqueness theorem (Theorem 4.1.1) which extends the domain of dependence result from the linear case (A.1.2) to the quasilinear case. This corresponds to Theorems 4 and 4a of [Joh85a]. In Section (4.2), as a pedagogical introduction avoiding the complexities of the general case, we apply the ideas of the general proof of blow-up result in 3+1 dimensions to the IVP associated to equation (1.5). Finally we state and prove in detail the general blow-up result divided in two parts, Theorems 4.3.1 and 4.3.2. In the former case, and under a particular assumption ($K > 0$), one can explicitly obtain an upper bound for the time of blow-up along certain characteristics. In the latter case ($K = 0$) one shows instead that a global C^2 solution must in fact be the trivial solution. These correspond to Theorems 1 and 2 of [Joh85a] and to an example starting in (45a) of [Joh90].

- **Chapter 5 - Semi-linear Wave Equations on FLRW Spacetimes:** In this last chapter we start by computing the wave operator for an underlying geometry defined by a FLRW type metric with a general expansion term. Next, we obtain (1.4) by considering (1.6) with $\Box_g$ corresponding to a FLRW metric instead of the Minskowski metric. Via a change of variable to conformal time, and by setting a specific expansion term $a(t) = t^p$, we obtain (1.4). These sections were written by mainly following [CFO23]. The remainder of the chapter is devoted to two different attempts at tackling the main problem via the ideas used in the proofs of Theorems (4.3.1) and (4.3.2).

  In Section 5.4.1, after reducing the problem to an ODE type inequality, as obtained also in Theorem (4.3.2), we try to make a continuity argument for small p. However, new obstructions in this setting prevent us from reaching the same conclusions as before.

  In Section 5.4.2, inspired by [NR23], we fix $p = 1/2$ and use the method of Klainerman and Sarnack [KS81] to construct an invertible operator transforming the solutions u of (1.4) into solutions of a simpler wave equation (5.97). However, obtaining and ODE type inequality in this setting appears to be a more challenging task.

2

Linear Wave Equation

Consider both the homogeneous and non-homogeneous linear wave equation which are given respectively by

$$\Box u(x,t) = 0, \tag{2.1}$$

and

$$\Box u(x,t) = f(x,t), \tag{2.2}$$

where $u : \overline{\mathcal{U}} \times [0,\infty) \to \mathbb{R}$ is the unknown, $f : \mathcal{U} \times [0,\infty) \to \mathbb{R}$ is given and $\mathcal{U} \subset \mathbb{R}^n$ is an open set. In the following section we obtain explicit solutions for the initial value problem associated to the above wave equations.

2.1 Representation Formulas

The representation formulas are explicit solutions of the linear wave equation, which we will present for dimensions $n = 1, 3, 2$, in this order. As we shall see, the dimension (concretely whether it is odd or even) has a significant impact on the methods applied in order to construct the solutions and in the behaviour of the solutions themselves. The structure of the present section is as follows:

- We start with d'Alembert's Formula ($n = 1,\ \mathcal{U} = \mathbb{R}$) by employing the **method of characteristics**, which is first introduced in order obtain the representation formulas for the homogeneous and non-homogeneous transport equation in Section 2.1 of [Eva10].

- Next, we obtain d'Alembert's Formula ($n = 1,\ \mathcal{U} = \mathbb{R}_+$) via the so called **reflection method**. This result plays a crucial role later when deriving the representation formula for the $n = 3$ case.

- Next, we derive Kirchoff's Formula ($n = 3,\ \mathcal{U} = \mathbb{R}^3$), obtained via **spherical means** and with recourse to d'Alembert's formula in the half-plane.

- Finally, we obtain Poisson's Formula ($n = 2,\ \mathcal{U} = \mathbb{R}^2$) via the so called **method of descent**. This is done mostly for comparison purposes since we will not rely on this result in the remainder of this thesis.

2.1.1 d'Alembert's Formula ($n = 1,\ \mathcal{U} = \mathbb{R}$)

We consider the following initial value problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,+\infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are known functions with regularity to be determined by the representation formula derived below. Let u be a sufficiently regular solution of (2.3). We can write the wave equation in (2.3) in the following way

$$(\partial_t + \partial_x)(\partial_t u - \partial_x u) = 0. \tag{2.4}$$

Defining

$$v(x,t) := (\partial_t u - \partial_x u)(x,t), \tag{2.5}$$

we can write (2.4) as

$$\partial_t v + \partial_x v = 0 \ \Leftrightarrow\ \nabla v(x,t) \cdot (1,1) = 0. \tag{2.6}$$

For a fixed $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, equation (2.6) states that v is constant along the lines parametrized by $(x+s, t+s)$, with $s \in \mathbb{R}$, obtained by integrating the vector field $(dx/ds, dt/ds)$ with tangent vector $(1,1)$ such that $(x(0), t(0)) = (x,t)$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. At $s = -t$ in particular, we have

$$v(x,t) = v(x-t, 0) =: w(x-t), \tag{2.7}$$

for some function w to be determined. From the definition of v (2.5) we now obtain for $(x,t) \in \mathbb{R} \times \mathbb{R}_+$

$$\partial_t u - \partial_x u = w(x-t) \ \Leftrightarrow\ \nabla u \cdot (-1,1) = w(x-t). \tag{2.8}$$

Again, fixing $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, along the lines parametrized by $(x-s, t+s)$, with $s \in \mathbb{R}$, we have

$$\begin{aligned} \frac{d}{ds} u(x-s, t+s) &= \nabla u(x-s,t+s) \cdot (-1,1) \\ &= \partial_t u(x-s,t+s) - \partial_x u(x-s,t+s) \\ &= w(x-s-(t+s)). \end{aligned} \tag{2.9}$$

To grab the initial data at $t = 0$, we integrate (2.9) in s from $-t$ to 0 and use the fundamental theorem of calculus in order to obtain

$$\begin{aligned} \int_{-t}^{0} \frac{d}{ds} u(x-s,t+s) ds = u(x,t) - u(x+t,0) = \int_{-t}^{0} w(x-s-(t+s)) ds \\ \Leftrightarrow u(x,t) = g(x+t) + \int_0^t w(x+s-(t-s)) ds = g(x+t) + \frac{1}{2} \int_{x-t}^{x+t} w(y) dy, \end{aligned} \tag{2.10}$$

first changing $s \to -s$, using the initial condition $u(x+t,0) = g(x+t)$ and finally the change of variable $y = x - t + 2s$. To determine w we use (2.7) together with the initial conditions of (2.3), resulting in

$$w(x) = u_t(x,0) - u_x(x,0) = h(x) - g'(x), \tag{2.11}$$

and finally leading to

$$
\begin{aligned}
u(x,t) &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} w(y)dy = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} \big(h(y) - g'(y)\big)dy \\
&= \frac{1}{2}\big(g(x+t) + g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.
\end{aligned}
\tag{2.12}
$$

One can then verify that

$$
u(x,t) = \frac{1}{2}\big(g(x+t) + g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy, \quad (x \in \mathbb{R}, t \geq 0), \tag{2.13}
$$

is indeed the solution to (2.3) known as d'Alembert's formula. In particular, one can verify the following: $u \in C^2(\mathbb{R} \times [0,\infty))$, provided $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$; the wave equation $u_{tt} - u_{xx} = 0$ is satisfied for $(x,t) \in \mathbb{R} \times (0,\infty)$; for each $x \in \mathbb{R}$ the limit $t \to 0+$ of u and u_t coincides with the initial data. Motivated by the results above, one can alternatively arrive at the same representation formula by considering the change of variables $\xi = x+t$ and $\eta = x-t$. This can be found in Appendix, see A.1.1.

Remark 2.1.1.

(i) From the representation formula we see that the value of u at (x,t) depends, through g, on the end points $x-t, x+t$ and through h over the interval $(x-t, x+t)$, i.e. the domain of dependence of the solution at (x,t) is the interval $[x-t, x+t]$.

(ii) The regularity (reg) of the solution u will be $\min\{reg(g), reg(h)+1\}$, provided that g is at least $C^2(\mathbb{R})$ and h is at least $C^1(\mathbb{R})$.

We now use the **reflection method** to obtain a representation formula of (2.3) in the half-plane, i.e. in the case where $\mathcal{U} = \mathbb{R}_+$. As mentioned, this will be useful later when considering the case $n=3$.

2.1.2 d'Alembert's Formula $(n=1,\ \mathcal{U} = \mathbb{R}_+)$

We now consider the following initial value problem with Dirichlet boundary conditions

$$
\begin{cases}
u_{tt}(x,t) - u_{xx}(x,t) = 0, \quad (x,t) \in \mathbb{R}_+ \times (0,+\infty), \\
u(x,0) = g(x), \quad u_t(x,0) = h(x), \quad x \in \mathbb{R}_+, \\
u(0,t) = 0, \quad t > 0,
\end{cases}
\tag{2.14}
$$

where again g, h are given and satisfy $g(0) = h(0) = 0$.

Definition 2.1.1. ***(Odd Extension/Reflection)***
Given a function $f : \mathbb{R}_+^0 \to \mathbb{R}$, satisfying $f(0) = 0$, we define its odd extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$
\tilde{f}(x) = \begin{cases} f(x), \quad x \geq 0, \\ -f(-x), \quad x \leq 0. \end{cases} \tag{2.15}
$$

Let u be a solution of (2.14) and consider the extensions $\tilde{u}$, $\tilde{g}$ and $\tilde{h}$. Then, one can consider an initial value problem for $\tilde{u}$ in the setting of (2.3), i.e. with $\mathcal{U} = \mathbb{R}$. Furthermore, the boundary conditions of (2.14) will be satisfied for all time $t \geq 0$ by $\tilde{u}$. In other words, the extended solution $\tilde{u}$ will be a solution of (2.14), restricted to the corresponding domain of validity, since they satisfy the same wave equation and the boundary condition $u(0,t) = \tilde{u}(0,t) = 0$ is satisfied for all $t \geq 0$ via odd extension.
To be more precise, consider the following initial value problem

$$\begin{cases} \tilde{u}_{tt}(x,t) - \tilde{u}_{xx}(x,t) = 0, \ \ (x,t) \in \mathbb{R} \times (0,+\infty), \\ \tilde{u}(x,0) = \tilde{g}(x), \ \ \tilde{u}_t(x,0) = \tilde{h}(x), \ \ x \in \mathbb{R}. \end{cases} \tag{2.16}$$

Using d'Alembert's representation formula (2.13) the solution is given by

$$\tilde{u}(x,t) = \frac{1}{2}\big[\tilde{g}(x+t) + \tilde{g}(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{h}(s)ds, \ \ x \in \mathbb{R}, \ \ t \geq 0. \tag{2.17}$$

Now, restricting to the region $x \geq 0$, $t \geq 0$, we need to be careful over the sign of the $x - t$ term. If $x - t \geq 0$ we have

$$\tilde{u}(x,t) = \frac{1}{2}\big[g(x+t) + g(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds, \ \ x \geq t \geq 0. \tag{2.18}$$

If $x - t \leq 0$, separating the integral and using the odd extension on $\tilde{h}$ we obtain

$$\tilde{u}(x,t) = \frac{1}{2}\big[g(x+t) - g(t-x)\big] - \frac{1}{2}\int_{x-t}^{0} h(-s)ds + \frac{1}{2}\int_{0}^{x+t} h(s)ds. \tag{2.19}$$

Performing the change of variable $y = -s$, and relabelling it in the next step as s, we obtain

$$\begin{aligned} \tilde{u}(x,t) &= \frac{1}{2}\big[g(x+t) - g(t-x)\big] + \frac{1}{2}\int_{t-x}^{0} h(s)ds + \frac{1}{2}\int_{0}^{x+t} h(y)dy \\ &= \frac{1}{2}\big[g(x+t) - g(t-x)\big] + \frac{1}{2}\int_{t-x}^{x+t} h(s)ds, \ \ 0 \leq x \leq t. \end{aligned} \tag{2.20}$$

Finally, if we let

$$u(x,t) = \begin{cases} \frac{1}{2}\big[g(x+t) + g(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds, \ \ x \geq t \geq 0, \\ \frac{1}{2}\big[g(x+t) - g(t-x)\big] + \frac{1}{2}\int_{t-x}^{x+t} h(s)ds, \ \ 0 \leq x \leq t, \end{cases} \tag{2.21}$$

one can check that (2.21) is indeed a solution to (2.14). It satisfies the wave equation in $x > 0$, $t > 0$ and coincides appropriately with the initial data, since $\tilde{u}$ does, and satisfies the boundary condition at $x = 0$ for $t \geq 0$ since

$$\lim_{x \to 0} u(x,t) = 0, \tag{2.22}$$

both when $x \geq t \geq 0$ and $0 \leq x \leq t$.

Remark 2.1.2.

(i) Considering $h \equiv 0$, the solution can be interpreted as an initial disturbance, given by g, which splits equally in a right moving and left moving disturbances with the latter reflecting at $x = 0$.

(ii) *The speed of propagation can be seen from the fact that the wave front, where g is constant, occurs for $x \pm t = c$, $c \in \mathbb{R}$, such that $dx/dt = \mp 1$.*

(iii) *Note that the solution $u : \mathbb{R}^0_+ \times [0, \infty) \to \mathbb{R}$ may fail to be C^2 unless $g''(0) = 0$ since*

$$\lim_{x \to 0^+} \tilde{g}''(x) = g''(0) \ \text{ and } \lim_{x \to 0^-} \tilde{g}''(x) = \lim_{y \to 0^+} -g''(y) = -g''(0). \tag{2.23}$$

We now consider the case $n = 3$. The idea is to look at spherical averages of a solution u which, as a function of time and the radius of the sphere, ends up satisfying the $n = 1$ wave equation where we can simply apply d'Alembert's representation formula, in the reflection setting we have just considered.

2.1.3 Kirchoff's Formula ($n = 3,\ \mathcal{U} = \mathbb{R}^3$)

We first consider an initial value problem in $\mathbb{R}^n$ since the next result holds for general $n \in \mathbb{N}$. Suppose that $u \in C^m(\mathbb{R}^n \times [0, \infty))$, with $m \geq 2$, is a solution of the following initial value problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & (x,t) \in \mathbb{R}^n \times (0, +\infty), \\ u(x,0) = g(x), \ \ u_t(x,0) = h(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.24}$$

We define the spherical average of an arbitrary function as follows.

Definition 2.1.2. ***(Spherical Average)***
Let $x \in \mathbb{R}^n$, $t > 0$, $r > 0$. We define the spherical average of a function $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ over the sphere centred at x of radius r by

$$\overline{f}(x; r, t) := \frac{1}{|\partial B_r|} \int_{\partial B(x,r)} f(y,t) dS(y), \tag{2.25}$$

where $|\partial B_r|$ is the measure of the sphere of radius r.

The following lemma provides the PDE satisfied by the spherical average of a solution of (2.24) for some fixed $x \in \mathbb{R}^n$.

Lemma 2.1.1. ***(Euler-Poisson-Darboux Equation)***
Fix $x \in \mathbb{R}^n$ and let u satisfy (2.24). Then, $\overline{u} \in C^m(\overline{\mathbb{R}}_+ \times [0, \infty))$ and satisfies

$$\begin{cases} \overline{u}_{tt}(r,t) - \overline{u}_{rr}(r,t) - \frac{n-1}{r} \overline{u}_r(r,t) = 0, & \text{in } \mathbb{R}_+ \times (0, \infty) \\ \overline{u}(r,0) = \overline{g}(r), \ \ \overline{u}_t(r,0) = \overline{h}(r), & \text{in } \mathbb{R}_+ \end{cases} \tag{2.26}$$

Proof. Fix $x \in \mathbb{R}^n$ and let u be a solution of (2.24). We are interested in taking the spherical average of the wave-equation in (2.24). Clearly

$$\overline{u_{tt}} = \overline{u}_{tt}, \tag{2.27}$$

since the dependence on t of (2.25) is only through u. We now want to understand the term $\overline{\Delta u}$ in terms of the spherical average $\overline{u}$. From the spherical average of u, over a sphere centred at x of radius $r > 0$, changing coordinates to $x + rz$, where $z = (y - x)/r$ is the unit normal vector on the sphere at y, allows us to change the integral to be over the unit sphere centred at the origin[1] obtaining

$$\overline{u}(x; r, t) = \frac{1}{|\partial B_r|} \int_{\partial B(x,r)} u(y, t) \, dS(y) = \frac{1}{|\partial B_1|} \int_{\partial B(0,1)} u(x + rz, t) \, dS(z). \tag{2.28}$$

Therefore we have

$$\begin{aligned} \overline{u}_r(x; r, t) &= \frac{1}{|\partial B_1|} \int_{\partial B(0,1)} \nabla u(x + rz, t) \cdot z \, dS(z) \\ &= \frac{1}{|\partial B_r|} \int_{\partial B(x,r)} \nabla u(y, t) \cdot \frac{y - x}{r} \, dS(y) \\ &= \frac{r}{n} \frac{1}{|B_r|} \int_{B(x,r)} \Delta u(y, t) \, dy, \end{aligned} \tag{2.29}$$

using the divergence theorem with unit normal $\hat{n} = (y - x)/r$ and the fact that $|\partial B_r| = (n/r)|B_r|$, with $|B_r|$ the measure of the ball of radius r.

Since u satisfies the wave-equation (2.24) we can write

$$\begin{aligned} \overline{u}_r(x; r, t) &= \frac{r}{n} \frac{1}{|B_r|} \int_{B(x,r)} u_{tt}(y, t) \, dy \\ &= \frac{r}{n} \frac{1}{|B_r|} \int_0^r \left(\int_{\partial B(x,s)} u_{tt}(y, t) \, dy \right) ds \\ &= \frac{1}{n\alpha(n) r^{n-1}} \int_0^r \left(\int_{\partial B(x,s)} u_{tt}(y, t) \, dy \right) ds, \end{aligned} \tag{2.30}$$

using the fact that $|B_r| = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} r^n =: \alpha(n) r^n$, where Γ represents the usual gamma function. Passing the r^{n-1} term to the other side and differentiating both sides of (2.30) with respect to r we obtain

$$\begin{aligned} (r^{n-1} \overline{u}_r)_r &= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}(y, t) \, dy \\ &= \frac{r^{n-1}}{|\partial B_r|} \int_{\partial B(x,r)} u_{tt}(y, t) \, dy. \end{aligned} \tag{2.31}$$

which leads to

$$(n - 1) r^{n-2} \overline{u}_r + r^{n-1} \overline{u}_{rr} = r^{n-1} \overline{u}_{tt}, \tag{2.32}$$

and finally to

$$\overline{u}_{tt} - \overline{u}_{rr} - \frac{n-1}{r} \overline{u}_r = 0, \tag{2.33}$$

for $(r, t) \in \mathbb{R}_+ \times (0, \infty)$. We omit the proof of regularity which can be seen in Lemma 1, Section 2.4 of [Eva10]. ■

We now apply this result to the $n = 3$ case. From Lemma 2.26 the PDE satisfied by the spherical average $\overline{u}$ is

$$\overline{u}_{tt} - \overline{u}_{rr} - \frac{2}{r} \overline{u}_r = 0, \tag{2.34}$$

[1] In this way, when differentiating, we only hit the integrand and not the region itself.

Let

$$v(r,t) := r\overline{u}(r,t), \tag{2.35}$$

we obtain from (2.34)

$$v_{rr} = 2\overline{u}_r + r\overline{u}_{rr} = r\overline{u}_{tt} = v_{tt}. \tag{2.36}$$

Therefore v satisfies

$$\begin{cases} v_{tt} - v_{rr} = 0, \;\; (r,t) \in \mathbb{R}_+ \times (0,+\infty), \\ v(r,0) = r\overline{g}(r), \;\; v_t(r,0) = r\overline{h}(r), \;\; r \in \mathbb{R}_+, \\ v(0,t) = 0, \;\; t \in (0,+\infty). \end{cases} \tag{2.37}$$

This initial value problem corresponds to the $n = 1$ case with $\mathcal{U} = \mathbb{R}_+$, i.e. equation (2.14), whose solution is given d'Alembert's formula via the reflection method (2.21). To obtain $u(x,t)$ we take the limit $r \to 0$ so we can simply consider the part of the solution in (2.21) where $0 \le r \le t$, i.e.

$$v(x;r,t) = \frac{r}{2}\big[\overline{g}(r+t) - \overline{g}(t-r)\big] + \frac{1}{2}\int_{t-r}^{r+t} s\overline{h}(s)ds, \;\; 0 \le r \le t, \tag{2.38}$$

keeping in mind that (2.38) depends on x through $\overline{g}$ and $\overline{h}$ which are radially defined with respect to x. Note that

$$u(x,t) = \lim_{r\to 0^+} \overline{u}(x;r,t), \tag{2.39}$$

by the continuity of u at x and the definition of the spherical average. By continuity of u, given $\epsilon > 0$ there exists $\delta > 0$ such that for all y satisfying $\|x - y\| < \delta$, equivalent to $0 < r < \delta$ for $x \neq y$, implies $|u(x,t) - u(y,t)| < \epsilon$, and therefore

$$\begin{aligned} \left| u(x,t) - \frac{1}{|\partial B_r|}\int_{\partial B(x,r)} u(y,t)\, dS(y) \right| &= \left| \frac{1}{|\partial B_r|}\int_{\partial B(x,r)} u(x,t) - u(y,t)\, dS(y) \right| \\ &\le \frac{1}{|\partial B_r|}\int_{\partial B(x,r)} |u(x,t) - u(y,t)|\, dS(y) < \epsilon. \end{aligned} \tag{2.40}$$

Defining $\tilde{g} := r\overline{g}$ and $\tilde{h} := r\overline{h}$ we obtain

$$\begin{aligned} u(x,t) &= \lim_{r\to 0^+} \overline{u}(x;r,t) = \lim_{r\to 0^+} \frac{v(x;r,t)}{r} \\ &= \lim_{r\to 0^+} \left(\frac{1}{2r}\big[\tilde{g}(r+t) - \tilde{g}(t-r)\big] + \frac{1}{2r}\int_{t-r}^{r+t} \tilde{h}(s)ds \right), \end{aligned} \tag{2.41}$$

recalling again that $\overline{g}$ and $\overline{h}$ are radially defined with respect to x. The left term of the limit we can compute as follows

$$\begin{aligned} \lim_{r\to 0^+} \frac{1}{2r}\big[\tilde{g}(r+t) - \tilde{g}(t-r)\big] &= \lim_{r\to 0^+} \frac{\tilde{g}(r+t) - \tilde{g}(t)}{2r} - \frac{\tilde{g}(t-r) - \tilde{g}(t)}{2r} \\ &= \lim_{r\to 0^+} \frac{\tilde{g}(r+t) - \tilde{g}(t)}{2r} - \lim_{r'\to 0^-} \frac{\tilde{g}(t+r') - \tilde{g}(t)}{-2r'} \\ &= \lim_{r\to 0^+} \frac{\tilde{g}(r+t) - \tilde{g}(t)}{2r} + \lim_{r'\to 0^-} \frac{\tilde{g}(t+r') - \tilde{g}(t)}{2r'} \\ &= \tilde{g}'(t). \end{aligned} \tag{2.42}$$

The right term of the limit is handled in exactly the same way considering a primitive of $\tilde{h}$ and using the fundamental theorem of calculus. Thus, we have

$$u(x,t) = \tilde{g}'(x;t) + \tilde{h}(x;t). \tag{2.43}$$

Unrolling the definitions we have

$$\begin{aligned} \tilde{g}'(x;t) &= \frac{\partial}{\partial t}\left(\frac{t}{|\partial B_t|}\int_{\partial B(x,t)} g(y)\,dS(y)\right) = \frac{1}{|\partial B_1|}\frac{\partial}{\partial t}\left(t\int_{\partial B(0,1)} g(x+tz)\,dS(z)\right) \\ &= \frac{1}{|\partial B_t|}\int_{\partial B(x,t)} g(y)\,dS(y) + \frac{t}{|\partial B_1|}\int_{\partial B(0,1)} \nabla g(x+tz)\cdot z\,dS(z) \\ &= \frac{1}{|\partial B_t|}\int_{\partial B(x,t)} g(y)\,dS(y) + \frac{t}{|\partial B_t|}\int_{\partial B(x,t)} \nabla g(y)\cdot\left(\frac{y-x}{t}\right) dS(y). \end{aligned} \tag{2.44}$$

Replacing this result in (2.43) and recovering the definitions we obtain the solution of (2.24) for $n = 3$ which is known as *Kirchhoff's formula*

$$u(x,t) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \big(th(y) + g(y) + \nabla g(y)\cdot(y-x)\big)\,dS(y), \quad x \in \mathbb{R}^3,\ t > 0. \tag{2.45}$$

Remark 2.1.3.

(i) From (2.45), the solution u at (x,t) depends on the integral through the boundary of $B(x,t)$, i.e. through $\partial B(x,t)$. However, due to the dependence on ∇g, we actually require information of g in a spherical shell of ϵ thickness, for some $\epsilon > 0$.

(ii) The dependence of $u(x,t)$ over $\partial B(x,t)$ also implies finite speed of propagation. In particular, the speed of propagation is $v = 1$ since information of the initial conditions propagates along the surface of a cone in 3+1 dimensions, i.e. for a fixed (x,t) information reaches this point by propagating along the points (x',t') satisfying $t - t' = \|x - x'\|$.

(iii) In terms of regularity, unlike the $n = 1$ case, the solution now depends on derivatives of g. Thus, at times $t > 0$, we may lose regularity compared to the initial data. In fact, from Theorem 2 in Section 2.4 of [Eva10], one requires $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$ for u to be a C^2 solution in $\mathbb{R}^3 \times [0,\infty)$.

We now show how to obtain the representation formula for $n = 2$, known as Poisson's formula, in order to compare the domain of dependence of solutions as a function of dimension.

2.1.4 Poisson's Formula $(n = 2,\ \mathcal{U} = \mathbb{R}^2)$

Transformation (2.35) which converts the Euler-Poisson-Darboux equation (2.26) into a one-dimensional wave equation[2] no longer works for $n = 2$, so we require a different method. The idea here is to use Kirchoff's formula in $n = 3$ and simply collapse the extra dimension, i.e. have all functions involved

[2] Lemma 2 and (28) in Section 2.4 of [Eva10] shows a transformation which accomplishes this for odd $n \geq 3$.

depend only on two of the dimensions.

Suppose $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ is a solution of the initial-value problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0, & (x,t) \in \mathbb{R}^2 \times (0,+\infty), \\ u(x,0) = g(x), \ u_t(x,0) = h(x), & x \in \mathbb{R}^2. \end{cases} \tag{2.46}$$

Define $U : \mathbb{R}^3 \times [0,\infty)$ satisfying

$$U(x_1,x_2,x_3,t) := u(x_1,x_2,t), \tag{2.47}$$

for all $x_3 \in \mathbb{R}$. Then U solves the following initial-value problem

$$\begin{cases} U_{tt}(x,t) - \Delta U(x,t) = 0, & (x,t) \in \mathbb{R}^3 \times (0,+\infty), \\ U(x,0) = G(x), \ U_t(x,0) = H(x), & x \in \mathbb{R}^3, \end{cases} \tag{2.48}$$

where $G(x_1,x_2,x_3) = g(x_1,x_2)$ and $H(x_1,x_2,x_3) = h(x_1,x_2)$, for all $x_3 \in \mathbb{R}$. Recovering the intermediate solution for the case $n=3$ given by equation (2.43) and denoting $x = (x_1,x_2)$ and $\overline{x} = (x_1,x_2,0)$ (which should not cause confusion with the spherical average since x is just a coordinate) we have

$$u(x,t) = U(\overline{x},t) = \frac{\partial}{\partial t}\left(\frac{t}{|\partial \overline{B}_t|}\int_{\partial \overline{B}(\overline{x},t)} G(\overline{y})\, d\overline{S}(\overline{y})\right) + \frac{t}{|\partial \overline{B}_t|}\int_{\partial \overline{B}(\overline{x},t)} H(\overline{y})\, d\overline{S}(\overline{y}), \tag{2.49}$$

where $\overline{B}(\overline{x},t)$ denotes the ball in $\mathbb{R}^3$ and $d\overline{S}(\overline{y})$ is the measure on the two-dimensional sphere $\partial\overline{B}(\overline{x},t)$ with $\overline{y} = (y_1,y_2,y_3) \in \partial\overline{B}(\overline{x},t)$. Since G and H are constant over x_3, we can change the integral over the sphere $\partial\overline{B}(\overline{x},t)$ to twice the integral over the disk in the plane, denoted $B(x,t)$, to account for both hemispheres. Doing this for the integral in G (the integral in H changes in the same way) we obtain

$$\begin{aligned} \frac{t}{|\partial \overline{B}_t|}\int_{\partial \overline{B}(\overline{x},t)} G(\overline{y})\, d\overline{S}(\overline{y}) &= \frac{2}{4\pi t^2}\int_{B(x,t)} g(y_1,y_2) \left\| \frac{\partial \overline{y}}{\partial y_1} \times \frac{\partial \overline{y}}{\partial y_2} \right\| dy_1 dy_2 \\ &= \frac{t}{2}\frac{1}{\pi t^2}\int_{B(x,t)} \frac{g(y)}{\left(t^2 - \|y-x\|^2\right)^{1/2}}\, dy, \end{aligned} \tag{2.50}$$

with $\overline{y} = (y_1,y_2,y_3(y_1,y_2))$, where $y_3(y) = (t^2 - \|y-x\|^2)^{1/2}$, for $y = (y_1,y_2) \in B(x,t)$, and where

$$\left\| \frac{\partial \overline{y}}{\partial y_1} \times \frac{\partial \overline{y}}{\partial y_2} \right\| = \left(1 + \frac{\|y-x\|^2}{t^2 - \|y-x\|^2}\right)^{1/2} = \frac{t}{\left(t^2 - \|y-x\|^2\right)^{1/2}}. \tag{2.51}$$

Using this result in (2.49), for both the integrals of G and H, we obtain

$$\begin{aligned} u(x,t) =& \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \frac{1}{|B_t|}\int_{B(x,t)} \frac{g(y)}{(t^2 - \|y-x\|^2)^{1/2}}\, dy\right) \\ &+ \frac{t^2}{2}\frac{1}{|B_t|}\int_{B(x,t)} \frac{h(y)}{(t^2 - \|y-x\|^2)^{1/2}}\, dy. \end{aligned} \tag{2.52}$$

As before, changing coordinates to obtain an integral over the unit sphere we obtain

$$t^2 \frac{1}{|B_t|}\int_{B(x,t)} \frac{g(y)}{(t^2 - \|y-x\|^2)^{1/2}}\, dy = t\frac{1}{|B_1|}\int_{B(0,1)} \frac{g(x+zt)}{(1 - \|z\|^2)^{1/2}}\, dz, \tag{2.53}$$

where $z = (y-x)/t$ and so $dy = tdz$. The derivative term in (2.52) becomes

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(t^2 \frac{1}{|B_t|}\int_{B(x,t)} \frac{g(y)}{(t^2-\|y-x\|^2)^{1/2}}\,dy\right)\\
&= \frac{\partial}{\partial t}\left(t\frac{1}{|B_1|}\int_{B(0,1)} \frac{g(x+zt)}{(1-\|z\|^2)^{1/2}}\,dz\right)\\
&= \frac{1}{|B_1|}\int_{B(0,1)} \frac{g(x+zt)}{(1-\|z\|^2)^{1/2}}\,dz + \frac{t}{|B_1|}\int_{B(0,1)} \frac{\nabla g(x+zt)\cdot z}{(1-\|z\|^2)^{1/2}}\,dz\\
&= \frac{t}{|B_t|}\int_{B(x,t)} \frac{g(y)}{(1-\|y-x\|^2)^{1/2}}\,dy + \frac{t}{|B_t|}\int_{B(x,t)} \frac{\nabla g(y)\cdot (y-x)}{(1-\|y-x\|^2)^{1/2}}\,dy.
\end{aligned}
\tag{2.54}
$$

Finally, replacing in (2.52), we obtain a solution of (2.46) known as *Poisson's formula*

$$
u(x,t) = \frac{1}{2|B_t|}\int_{B(x,t)} \frac{tg(y)+t^2h(y)+t\nabla g(y)\cdot(y-x)}{\left(t^2-\|y-x\|^2\right)^{1/2}}dy, \quad x\in\mathbb{R}^2,\ t>0. \tag{2.55}
$$

The construction of the general representation formulas in $\mathbb{R}^n$ for $n \geq 4$ can be obtained following the same (albeit more complicated) methods. The general procedure is to first find the representation formula for odd n, by converting the Euler-Poisson-Darboux PDE into a wave equation using a particular transformation (see item d in Section 2.4 of [Eva10]), and then reduce it to the even case $n-1$ by the method of descent (see item e in Section 2.4 of [Eva10]).

Comparing (2.45) with (2.55) we notice that in the latter case, for $n=2$ or even dimensions in general, the dependence over the initial conditions is on the whole ball centred at x of radius t. As mentioned, this is opposed to the behaviour one finds in $n=3$, or for odd $n>1$ in general, where the solution u depends on the initial data g and h on the sphere centred at x of radius t. This domain of dependence distinction as a function of evenness or oddness of dimension is known as **Huygen's principle**.

2.1.5 Nonhomogeneous Wave Equation

We now treat the nonhomogeneous wave-equation (2.2) for the $n=1$ case, where the applied method generalizes for $n>1$. The goal is to obtain a solution of

$$
\begin{cases}
u_{tt}(x,t) - u_{xx}(x,t) = f(x,t), & (x,t)\in\mathbb{R}\times(0,+\infty),\\
u(x,0) = g(x),\ \ u_t(x,0) = h(x), & x\in\mathbb{R}.
\end{cases}
\tag{2.56}
$$

We divide this process into two steps.

1. First, we show that a solution of (2.56) can be given by the sum of solutions of the following two initial-value problems

$$
\begin{cases}
v_{tt}(x,t) - v_{xx}(x,t) = 0, & (x,t)\in\mathbb{R}\times(0,+\infty),\\
v(x,0) = g(x),\ \ v_t(x,0) = h(x), & x\in\mathbb{R},
\end{cases}
\tag{2.57}
$$

which is a homogenous problem and for which we have a solution given by d'Alembert's formula (2.13), and

$$
\begin{cases}
w_{tt}(x,t) - w_{xx}(x,t) = f(x,t), & (x,t)\in\mathbb{R}\times(0,+\infty),\\
w(x,0) = 0,\ \ w_t(x,0) = 0, & x\in\mathbb{R}.
\end{cases}
\tag{2.58}
$$

This follows immediately from the linearity of (2.56). Let $z(x,t) = v(x,t) + w(x,t)$ then for $(x,t) \in \mathbb{R} \times (0,\infty)$ we have

$$z_{tt} - z_{xx} = v_{tt} + w_{tt} - v_{xx} - w_{xx} = v_{tt} - v_{xx} + w_{tt} - w_{xx} = f(x,t). \tag{2.59}$$

Furthermore, $z(x,0) = v(x,0) + w(x,0) = g(x)$ and $z_t(x,0) = v_t(x,0) + w_t(x,0) = h(x)$, concluding that z is a solution of (2.56).

2. The initial-value problem (2.58) is also nonhomogeneous and we solve it using what is know as *Duhamel's method*. First one constructs a parameterized family of solutions of the following homogeneous initial-value problem

$$\begin{cases} \tilde{w}_{tt}(x,t;s) - \tilde{w}_{xx}(x,t;s) = 0, & (x,t) \in \mathbb{R} \times (s,+\infty), \\ \tilde{w}(x,s;s) = 0, \ \ \tilde{w}_t(x,s;s) = f(x,s) \ \ x \in \mathbb{R}, \end{cases} \tag{2.60}$$

with parameter $s \in [0,t]$. Then one shows that a solution of (2.58) is given by

$$w(x,t) = \int_0^t \tilde{w}(x,t;s) ds. \tag{2.61}$$

To show this we first use d'Alembert's solution (2.13) for (2.60) obtaining

$$\tilde{w}(x,t;s) = \frac{1}{2} \int_{x-(t-s)}^{x+t-s} f(y,s) dy. \tag{2.62}$$

Now, following (2.61), we consider $w(x,t)$ given by

$$w(x,t) = \int_0^t w(x,t;s) ds = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+t-s} f(y,s) dy ds = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y,t-s) dy ds. \tag{2.63}$$

Clearly $w(x,0) = 0$ and by Leibniz's integral rule

$$\begin{aligned} w_t(x,t) &= \frac{1}{2} \int_{x-t}^{x+t} f(y,0) dy + \frac{1}{2} \int_0^t \int_{x-s}^{x+s} \partial_t f(y,t-s) dy ds \\ &= \tilde{w}(x,t;t) + \int_0^t \tilde{w}_t(x,t;s) ds \\ &= \int_0^t \tilde{w}_t(x,t;s) ds, \end{aligned} \tag{2.64}$$

where $\tilde{w}(x,t;t) = 0$ by (2.62). From 2.64 we obtain $w_t(x,0) = 0$ which confirms the initial conditions of (2.58) are satisfied by w. Differentiating, using Leibniz's integral rule again and also (2.60) we have

$$\begin{aligned} w_{tt}(x,t) &= \tilde{w}_t(x,t;t) + \int_0^t \tilde{w}_{tt}(x,t;s) ds \\ &= f(x,t) + \int_0^t \tilde{w}_{xx}(x,t;s) ds \\ &= f(x,t) + w_{xx}(x,t), \end{aligned} \tag{2.65}$$

confirming that (2.61) is indeed a solution of (2.58).

Writing explicitly a solution of (2.56) using (2.13) , (2.61) and (2.62) we obtain

$$u(x,t) = \frac{1}{2}\big[g(x+t) + g(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds + \frac{1}{2}\int_0^t \left(\int_{x-(t-s)}^{x+t-s} f(y,s)dy\right)ds, \tag{2.66}$$

for all $x \in \mathbb{R}$ and $t \geq 0$. Equivalently we will frequently also write

$$u(x,t) = \frac{1}{2}\big[g(x+t) + g(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds + \frac{1}{2}\int\int_{T_{x,t}} f(y,s)dy\,ds, \tag{2.67}$$

where $T_{x,t}$ denotes the triangle with vertices $\{(x,t),(x-t,0),(x+t,0)\}$.

Remark 2.1.4.

(i) Via (2.66), to have a classical solution $u \in C^2(\mathbb{R} \times \mathbb{R}_+)$ we require that $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, as in d'Alembert's formula, and also that $f \in C^1(\mathbb{R} \times \mathbb{R}_+)$.

(ii) The introduction of the nonhomogenous term, or forcing term, $f(x,t)$ increases the domain of dependence to the whole characteristic triangle $T_{x,t}$ when compared to d'Alembert's formula. At any point in the domain of dependence the forcing term could introduce a disturbance which propagates to (x,t).

Remark 2.1.5. *A certain integral norm, which usually corresponds to energy in physical systems, is extremely useful to prove properties of solutions, such as uniqueness, domain of dependence and finite speed of propagation, bypassing the laborious construction of representation formulas. For a brief overview of some basic results in this regard see (A.1.2) in Appendix.*

3

Nonlinear Wave Equation

Unlike the linear case of the previous chapter, there are no representation formulas for nonlinear wave equations in any generality. The behaviour of solutions of the IVP (1.3) in the nonlinear case can be extremely sensitive to initial conditions, the type of nonlinearity and dimension. Depending on these choices, solutions may exhibit very different phenomena, blow-up in finite time in some settings and global existence in others.

In this chapter, we briefly overview some basic results regarding local well-posedness and global existence of solutions to the following IVP for semilinear and quasilinear wave equations

$$\begin{cases} \Box u(x,t) = \phi(x,t,u,u',u''), \\ u(x,0) = f(x),\ u_t(x,0) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

For fairly general conditions on ϕ, f and g, the IVP (3.1) satisfies local well-posedness in any dimension n.

Global existence however, besides conditions on ϕ, f and g, remarkably depends on the dimension as well. For $n \leq 3$, there are classes of nonlinearities ϕ which do not admit global in time solutions, independently of how small the initial data is and despite it having compact support. The case $n = 3$ will be the main topic of Chapter 4. In contrast, for $n \geq 4$, semilinear wave equations exhibit small data global existence for a larger class of nonlinearities.

For $n = 3$ there is a class of nonlinearities ϕ, said to satisfy the null condition, for which global in time solutions exist for sufficiently small initial data.

Results stated in this chapter mainly follow [Sog95], [Rin] and [Luk].

3.1 Local well-posedness

The following theorem establishes well-posedness of the IVP (3.1) for quasilinear wave equations under certain assumptions (Theorems 4.1 and 4.3 in Chapter I of [Sog95]).

Theorem 3.1.1. *Consider the following initial value problem*

$$\begin{cases} \sum_{\alpha,\beta=0}^{n} g^{\alpha\beta}(u,u')\partial_\alpha\partial_\beta u = F(u,u'), \\ u(x,0) = f(x),\ u_t(x,0) = g(x), \quad x \in \mathbb{R}^n \end{cases} \tag{3.2}$$

for $0 \le \alpha, \beta \le n$, *where* $g^{\alpha\beta}$ *and* F *are* C^∞ *with all derivatives* $O(1)$ *and* $f, g \in C_0^\infty(\mathbb{R}^n)$. *Assume also that* $F(0,0) = 0$ *and*

$$\sum_{\alpha,\beta} |g^{\alpha\beta} - g_0^{\alpha\beta}| < \frac{1}{2}, \tag{3.3}$$

where $g_0^{\alpha\beta} = diag(1, -1, ..., -1)$. *Then there is a* $T > 0$ *such that* (3.2) *has a unique solution* $u \in C^\infty([0,T] \times \mathbb{R}^n)$. *If* T_* *denotes the supremum of such times* T, *then either* $T_* = \infty$ *or*

$$\sum_{|k| \le (n+6)/2} \left|\partial^k u(x,t)\right| \notin L^\infty([0,T_*) \times \mathbb{R}^n). \tag{3.4}$$

Proof. See Theorems 4.1 and 4.3 in Chapter I of [Sog95]. ∎

Remark 3.1.1. *Condition* (3.3) *allows for small disturbances of the wave operator close to the Minkowski metric. Equivalently, the second order terms are only slightly perturbed compared to* $\Box u$.

This result can be generalized for initial conditions in certain Sobolev spaces, in particular for $(f, g) \in H^{s+1} \times H^s$ with $s \ge n+2$, which is the version of Theorem 4.1 in Chapter I of [Sog95]. In any case, in the present work, we will restrict ourselves to sufficiently regular initial conditions with compact support.

Interestingly enough, the following proposition from [Rin] in 1+1 dimensions shows a counterexample to local existence.

Proposition 3.1.2. *Consider the following IVP in 1+1 dimensions*

$$\begin{cases} u_{tt} - u_{xx} = u_t^2, \\ u(0,x) = f(x),\ u_t(0,x) = g(x) \ \ x \in \mathbb{R}. \end{cases} \tag{3.5}$$

There is initial data $f, g \in C^\infty(\mathbb{R})$ *such that for any* $\epsilon > 0$, *there is no* $u \in C^\infty([-\epsilon, \epsilon] \times \mathbb{R})$ *solving* (3.5).

Proof. First, we construct a solution of (3.5) which does not depend on x, i.e. such that $u_{tt} = (u_t)^2$. Let $u_t(0) = k > 0$ then

$$\int_0^t \frac{u_{ss}}{u_s^2}\, ds = \int_0^t 1\, ds, \tag{3.6}$$

and we have

$$\frac{1}{k} - \frac{1}{u_t} = t, \ \Leftrightarrow\ u_t = \frac{k}{1-kt}. \tag{3.7}$$

Integrating again, we obtain

$$u(x,t) = u(x,0) - \log(1-kt). \tag{3.8}$$

Thus, if we set $f(x) = 0$, (3.8) gives a solution of (3.5) which blows-up at $t = 1/k$.

Let $a \in \mathbb{R}$, if we set $f(x) = 0$ and $g(x) = k$ in $[a-1/k, a+1/k]$, then $u(x,t) = -\log(1-kt)$ is the unique solution of (3.5) within the cone $K(a, 1/k)$ (see the uniqueness Theorem 14 of [Rin]), which blows-up at $t = 1/k$. Let $\psi \in C_0^\infty(\mathbb{R})$ be a mollifier satisfying

$$\psi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases} \tag{3.9}$$

Let $k \in \mathbb{N}$, and define

$$g_k(x) := k\psi(x - 4k). \tag{3.10}$$

Then, $g_k(x) = k$ for $x \in [4k - 1/k, 4k + 1/k]$. Letting $f(x) = 0$ and

$$g(x) = \sum_{k=1}^{\infty} g_k(x), \tag{3.11}$$

then $g \in C^\infty(\mathbb{R})$ with $g(x) = k$ for $x \in [4k - 1/k, 4k + 1/k]$. In particular, g is smooth since, fixing $x \in \mathbb{R}$, $g(x) \neq 0$ only within the support of g_k for a unique k. With this initial data, any solution u of (3.5) must blow-up in time $1/k$ for any $k \geq 1$, proving there is no local solution. ■

The following figure (Fig. 3.1) shows a visual idea of the proof. At the basis of each cone, centred at $4k$ with radius $1/k$, the solution satisfies (3.8), with $u(x,0) = f(x) = 0$, and blows-up in time $t = 1/k$ for any $k \geq 1$.

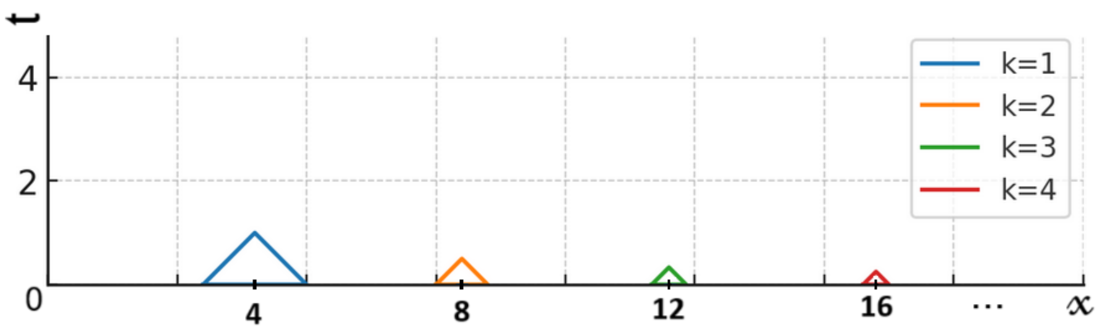

Figure 3.1: Idea behind the proof of Proposition 3.1.2 with $k = 1, 2, 3, 4$.

Naturally, this result does not contradict Theorem 3.1.1 since in this case $F'(u_t) = 2u_t$ is not bounded, i.e. $F' \notin O(1)$, and the initial data does not have compact support. In particular, both conditions contribute to the obstruction to local existence in this example.

3.2 Global existence

The following theorem states that solutions to certain quasilinear wave equations in $n+1$ dimensions can be extended globally for $n \geq 4$ (Theorem 2.1 in Chapter II of [Sog95]).

Theorem 3.2.1. *Let $n \geq 4$ and consider the following initial value problem*

$$\begin{cases} \sum_{\alpha,\beta=0}^{n} g^{\alpha\beta}(u')\partial_\alpha\partial_\beta u = F(u'), \\ u(x,0) = \epsilon f(x),\ u_t(x,0) = \epsilon g(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{3.12}$$

for $0 \leq \alpha, \beta \leq n$, where $g^{\alpha\beta}$ and F are C^∞ and $f, g \in C_0^\infty(\mathbb{R}^n)$. Assume also that

$$\sum_{\alpha,\beta=0}^{n} g^{\alpha\beta}(0)\partial_\alpha\partial_\beta = \Box, \quad F(0) = 0, \quad F'(0) = 0. \tag{3.13}$$

Then, there exists sufficiently small $\epsilon > 0$ such that (3.12) admits a global C^∞ solution.

Proof. See Theorem 2.1 of Chapter II in [Sog95]. ■

For $n = 3$, for the nonlinearities considered in (3.12), there is no such result. In particular, for $n = 3$, Fritz John proves in [Joh85a] that, for a large class of nonlinearities, solutions of (3.1) blow-up in finite time, see Theorems 4.3.1 and 4.3.2. A well known example of blow-up in finite time in $3+1$ dimensions is given by the following equation

$$\Box u = (u_t)^2, \tag{3.14}$$

for non-trivial initial data of compact support. We cover this example in detail in Proposition 4.2.1 which motivates the study of (1.4) in Chapter 5.

The following theorem, Theorem 2.2 in Chapter II of [Sog95], provides lower bounds for the time of existence for solutions of (3.12) when $n \leq 3$.

Theorem 3.2.2. *There exists a constant $c > 0$, depending only on $(f, g) \in C_0^\infty(\mathbb{R}^n)$, such that for small $\epsilon > 0$ the IVP (3.12) has a smooth solution for $0 \leq t \leq T_\epsilon$, where*

$$T_\epsilon = \begin{cases} e^{c/\epsilon}, & n = 3 \\ (c/\epsilon)^2, & n = 2 \\ c/\varepsilon, & n = 1. \end{cases} \tag{3.15}$$

Proof. See Theorem 2.2 of Chapter II in [Sog95]. ■

This is coined as an almost global existence result for $n = 3$ since the lower bound T_ϵ grows exponentially as the size of the initial data decreases by taking smaller ϵ.

In $3+1$ dimensions however there are certain nonlinearities ϕ, those said to satisfy the null-condition, for which solutions of (3.1) can be extended globally. This idea was introduced by Klainerman in [Kla80] and leads to the following definition

Definition 3.2.1. ***(Null Condition)***

Consider the quadratic form

$$Q(\phi, \psi) = \sum_{\alpha,\beta} q^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi, \tag{3.16}$$

for fixed constants $q^{\alpha\beta} \in \mathbb{R}$ and $\phi, \psi \in \mathbb{R}^4$. We say that Q satisfies the null condition if

$$Q(\xi, \xi) = q^{\alpha\beta}\xi_\alpha\xi_\beta = 0, \text{ whenever } m^{\alpha\beta}\xi_\alpha\xi_\beta = -\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 0, \tag{3.17}$$

where $m^{\alpha\beta}$ corresponds to the Minkowski metric. Equivalently, Q vanishes when evaluated on null vectors with respect to the Minkowski metric.

We have the following theorem from [Sog95]

Theorem 3.2.3. *Let $n = 3$ and consider the following IVP*

$$\begin{cases} \Box u = Q(u', u'), \\ u(x, 0) = \epsilon f(x),\ u_t(x, 0) = \epsilon g(x), \quad x \in \mathbb{R}^3 \end{cases} \tag{3.18}$$

where Q is a quadratic form satisfying the null condition. Then, for fixed $f, g \in C^\infty(\mathbb{R}^3)$, there exists sufficiently small $\epsilon > 0$ such that (3.17) admits a global smooth solution.

Proof. See Theorem 5.1 of Chapter II in [Sog95]. ■

As a trivial check notice that $F = (u_t)^2$ does not satisfy the null condition since we can write it as a quadratic form Q corresponding to the matrix $\text{diag}(1, 0, 0, 0)$ which fails to satisfy (3.17), consider for example the vector $(1, 1, 0, 0)$.

To finalize this Chapter, we do a straightforward extension of the Euler-Poisson-Darboux Lemma 2.26 for (3.1), which will be required in the blow-up proof of the following Chapter.

3.3 Euler-Poisson-Darboux Equation (Quasilinear case)

Corollary 3.3.1. ***(Euler-Poisson-Darboux Equation (Quasilinear case))***
Fix $x \in \mathbb{R}^n$ and let u satisfy (3.1) globally. Then, $\overline{u}$ satisfies

$$\begin{cases} \overline{u}_{tt}(r, t) - \overline{u}_{rr}(r, t) - \frac{n-1}{r}\overline{u}_r(r, t) = \overline{\phi(x, t, u, u', u'')}, & \text{in } \mathbb{R}_+ \times (0, \infty) \\ \overline{u}(r, 0) = \overline{g}(r),\ \ \overline{u}_t(r, 0) = \overline{h}(r), & \text{in } \mathbb{R}_+ \end{cases} \tag{3.19}$$

Proof. Fix $x \in \mathbb{R}^n$ and let u be a globally defined solution of (3.1). Taking the spherical average of the wave equation in (3.1) we have as before $\overline{u_{tt}} = \overline{u}_{tt}$ and, from (2.29) and (2.30), we now have

$$\begin{aligned} \overline{u}_r(x; r, t) &= \frac{r}{n}\frac{1}{|B_r|}\int_{B(x,r)} \Delta u(y, t)\, dy \\ &= \frac{r}{n}\frac{1}{|B_r|}\int_{B(x,r)} \left(u_{tt}(y, t) - \phi(y, t, u, u', u'')\right)\, dy \\ &= \frac{1}{n\alpha(n)r^{n-1}}\int_0^r \left(\int_{\partial B(x,s)} \left(u_{tt}(y, t) - \phi(y, t, u, u', u'')\right)\, dy\right) ds \end{aligned} \tag{3.20}$$

As before, passing the r^{n-1} term to the other side and differentiating both sides of (3.20) with respect to r, we obtain

$$(r^{n-1}\overline{u}_r)_r = \frac{r^{n-1}}{|\partial B_r|}\int_{\partial B(x,r)} \left(u_{tt}(y, t) - \phi(y, t, u, u', u'')\right)\, dy. \tag{3.21}$$

which leads to

$$(n-1)r^{n-2}\overline{u}_r + r^{n-1}\overline{u}_{rr} = r^{n-1}\left(\overline{u}_{tt} - \overline{\phi(x,t,u,u',u'')}\right), \tag{3.22}$$

and finally to

$$\overline{u}_{tt} - \overline{u}_{rr} - \frac{n-1}{r}\overline{u}_r = \overline{\phi(x,t,u,u',u'')}, \tag{3.23}$$

for $(r,t) \in \mathbb{R}_+ \times (0,\infty)$. ■

Consider now $n = 3$ and let $v(r,t) := r\overline{u}(r,t)$. Then, v satisfies

$$\begin{cases} v_{tt} - v_{rr} = r\overline{\phi}, \ \ (r,t) \in \mathbb{R}_+ \times (0,+\infty), \\ v(r,0) = r\overline{g}(r), \ \ v_t(r,0) = r\overline{h}(r), \ \ r \in \mathbb{R}_+, \\ v(0,t) = 0, \ \ t \in (0,+\infty). \end{cases} \tag{3.24}$$

Assuming u is a given solution of (3.1), we can view $\overline{\phi}$ as a forcing term $\overline{\phi}(r,t)$, making (3.24) a linear nonhomogeneous wave equation with solution given by (2.67), i.e.

$$v(r,t) = \frac{1}{2}\big[g(r+t) + g(r-t)\big] + \frac{1}{2}\int_{r-t}^{r+t} h(s)ds + \frac{1}{2}\int\int_{T_{r,t}} \rho\overline{\phi}(\rho,\tau)d\rho\, d\tau, \tag{3.25}$$

resulting in

$$\overline{u}(r,t) = \frac{1}{2r}\big[g(r+t) + g(r-t)\big] + \frac{1}{2r}\int_{r-t}^{r+t} h(s)ds + \frac{1}{2r}\int\int_{T_{r,t}} \rho\overline{\phi}(\rho,\tau)d\rho\, d\tau. \tag{3.26}$$

We will make use of this observation in the proof of blow-up of the following chapter.

4

Blow-up for Quasilinear Wave Equations in 3+1 Dimensions

In this chapter we consider the following initial value problem for a quasi-linear wave equation

$$\begin{cases} \Box u = \phi(x,t,u,u',u''), \\ u(x,0) = f(x),\ u_t(x,0) = g(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{4.1}$$

with solution $u(x,t) := u(x_1,x_2,x_3,t)$ and where u', u'' represents, respectively, the vector of all first order derivatives of u and the vector of all second order derivatives of u.

Before looking at the proof of the main theorem, regarding blow-up, we first establish a uniqueness result, in $\mathbb{R}^n$, which will be required in its proof. For this next section we follow Theorems 4 and 4a from [Joh85a] and also a set of notes by Peter Constantin [Con] covering the same results.

4.1 Uniqueness Theorem for nonlinear wave equations (Theorems 4 and 4a of [Joh85a])

The following theorem states that, if the initial data in a ball centred at $x_0 \in \mathbb{R}^n$ of radius $a \in \mathbb{R}_+$ is trivial and the non-linearity ϕ satisfies (4.2) then the solution will remain trivial within the cone $K(x_0,a)$, i.e. $u = 0$ is the only possible solution within the cone. This is essentially an extension of Theorem A.1.2 for the quasilinear case.

Theorem 4.1.1. *Let $\phi(x,t,u,u',u'')$ be a C^2 function of its arguments satisfying*

$$\phi(x,t,0,0,u'') = 0,\ \forall\, x,t,u''. \tag{4.2}$$

Let u be a C^2 solution of the equation

$$\Box u = \phi(x,t,u,u',u''), \tag{4.3}$$

in the cone

$$K(x_0,a) = \{(x,t)\,|\,\|x-x_0\| + t \le a;\ t \ge 0\}, \tag{4.4}$$

for some $x_0 \in \mathbb{R}^n$, $a \in \mathbb{R}_+$, with initial conditions

$$u(x,0) = u_t(x,0) = 0, \text{ for } \|x - x_0\| < a. \tag{4.5}$$

Then $u(x,t) = 0$ in the cone $K(x_0, a)$.

Proof. The main idea of the proof is to show that $u' = 0$ in $K(x_0, a)$ by checking that this is so over each leaf of a hyperboloidal foliation of $K(x_0, a)$, concluding then that $u = 0$ in the cone via the initial conditions (4.5).

Consider a foliation of the interior of the cone $K(x_0, a)$ by a parametrized family of hyperboloid surfaces S_λ, radially symmetric about x_0, given by

$$S_\lambda := \{(x,t) \mid \|x - x_0\| < a;\, t = \psi(x,\lambda)\}, \tag{4.6}$$

where

$$\psi(x,\lambda) := \frac{2\lambda(a^2 - \|x - x_0\|^2)}{a^2 + \lambda^2 + \sqrt{(a^2-\lambda^2)^2 + 4\lambda^2\|x - x_0\|^2}}, \quad \text{with } 0 \le \lambda < a. \tag{4.7}$$

Notice that $\psi(x,\lambda) \ge 0$ and that it shares, for all λ, the boundary of

$$S_0 := \{(x,t) \mid \|x - x_0\| < a;\, t = 0\}. \tag{4.8}$$

See Figure 4.1 for an example in the $n = 1$ case for two different values of λ.

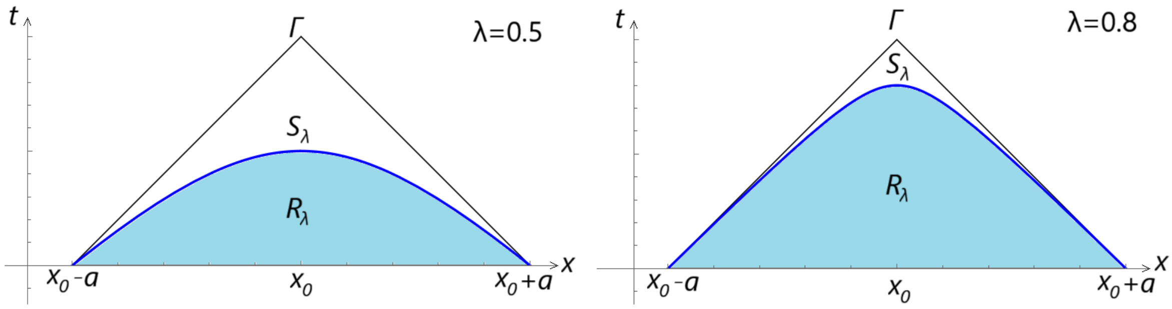

Figure 4.1: The cone $K(x_0, a)$, S_λ and R_λ for $\lambda = 0.5$ (left) and $\lambda = 0.8$ (right) in 1+1 dimensions.

Remark 4.1.1. *The hyperboloid surfaces S_λ are, for all λ, non-characteristic with respect to $\Box$. We will comment after the proof on why this is required.*

For this particular choice of the surface $\psi(x,\lambda)$ we have the following properties

$$|\psi(x,\lambda)| \le \lambda < a, \text{ for } \|x - x_0\| < a, \tag{4.9}$$

$$|\partial_\lambda \psi(x,\lambda)| \le 1, \text{ for } \|x - x_0\| < a, \tag{4.10}$$

$$\theta := \sum_{i=1}^{n} (\partial_{x_i}\psi)^2 \le \left(\frac{2\lambda a}{a^2 + \lambda^2}\right)^2 < 1. \tag{4.11}$$

The proof of these properties can be found in Appendix A, see (A.2.1), (A.2.2) and (A.2.3).

Consider now the region

$$R_\lambda := \{(x,t) \mid \|x - x_0\| < a, 0 \le t \le \psi(x;\lambda)\}, \tag{4.12}$$

which corresponds to the region below S_λ down to S_0 (see again Fig. 4.1) and define the vector field X given by

$$X_{(x,t)} := \left(-2u_t \nabla_x u, u_t^2 + \|\nabla_x u\|^2\right), \tag{4.13}$$

with divergence given by

$$\begin{aligned} \nabla \cdot X &= -2\sum_i \partial_i(u_t \partial_i u) + \partial_t(u_t^2 + \|\nabla_x u\|^2) \\ &= 2u_t u_{tt} + 2\sum_i u_i u_{ti} - 2\sum_i u_{it} u_i - 2u_t \sum_i u_{ii} \\ &= 2u_t(u_{tt} - \sum_i u_{ii}) \\ &= 2u_t \Box u, \end{aligned} \tag{4.14}$$

where we use the fact that u is C^2 and thus $u_{it} = u_{ti}$. Integrating on R_λ and using the divergence theorem we obtain

$$\begin{aligned} 2\int_{R_\lambda} u_t \Box u \, dxdt &= 2\int_{R_\lambda} \nabla \cdot X \, dxdt \\ &= \int_{S_0} X \cdot \hat{n}_0 \, dS_0 + \int_{S_\lambda} X \cdot \hat{n}_\lambda \, dS_\lambda. \end{aligned} \tag{4.15}$$

From (4.5) we have $X = 0$ at S_0 therefore the first integral vanishes. The unit normal $\hat{n}_\lambda$ is the unit normal to the surface given by the level curve $F(x,t;\lambda) = t - \psi(x;\lambda) = 0$ and thus

$$\hat{n}_\lambda = \frac{\nabla_{(x,t)} F}{\|\nabla_{(x,t)} F\|} = \frac{(-\nabla_x \psi(x;\lambda), 1)}{\sqrt{1 + \sum_i (\partial_i \psi(x;\lambda))^2}}. \tag{4.16}$$

Therefore, from (4.15), we have

$$\begin{aligned} 2\int_{R_\lambda} u_t \Box u \, dxdt &= \int_{S_\lambda} X \cdot \hat{n}_\lambda \, dS_\lambda \\ &= \int_{S_\lambda} X \cdot \frac{(-\nabla_x \psi(x;\lambda), 1)}{\sqrt{1 + \sum_i (\partial_i \psi(x;\lambda))^2}} \, dS_\lambda \\ &= \int_{S_\lambda} \left(u_t^2 + \|\nabla_x u\|^2 + 2\sum_i (\partial_i \psi) u_t \partial_i u\right) \frac{dS_\lambda}{\sqrt{1 + \sum_i (\partial_i \psi(x;\lambda))^2}} \\ &= \int_{S_\lambda} \left(u_t^2 + \|\nabla_x u\|^2 + 2\sum_i (\partial_i \psi) u_t \partial_i u\right) dS'_\lambda, \end{aligned} \tag{4.17}$$

where we define

$$dS'_\lambda = \frac{dS_\lambda}{\sqrt{1 + \sum_i (\partial_i \psi(x;\lambda))^2}}. \tag{4.18}$$

Since $\sum_i (\partial_i \psi) u_t \partial_i u = \nabla_x \psi \cdot (u_t \nabla_x u)$ by Cauchy-Schwartz's inequality we obtain

$$\nabla_x \psi \cdot (u_t \nabla_x u) \le |\nabla_x \psi \cdot (u_t \nabla_x u)| \le \|\nabla_x \psi\| \|u_t \nabla_x u\| = |u_t| \|\nabla_x \psi\| \|\nabla_x u\|, \tag{4.19}$$

and thus

$$\nabla_x\psi \cdot (u_t \nabla_x u) \geq -|\nabla_x\psi \cdot (u_t\nabla_x u)| \geq -|u_t|\|\nabla_x\psi\|\|\nabla_x u\|, \tag{4.20}$$

which, from (4.17), leads to

$$\begin{aligned}
2\int_{R_\lambda} u_t \Box u \, dxdt &= \int_{S_\lambda} \left(u_t^2 + \|\nabla_x u\|^2 + 2\sum_i (\partial_i\psi) u_t \partial_i u \right) dS'_\lambda \\
&\geq \int_{S_\lambda} \left(u_t^2 + \|\nabla_x u\|^2 - 2|u_t|\|\nabla_x\psi\|\|\nabla_x u\| \right) dS'_\lambda .
\end{aligned} \tag{4.21}$$

We further have

$$\begin{aligned}
(|u_t| - \|\nabla_x u\|)^2 = u_t^2 + \|\nabla_x u\|^2 - 2|u_t|\|\nabla_x u\| \geq 0 \\
\Leftrightarrow \;\; u_t^2 + \|\nabla_x u\|^2 \geq 2|u_t|\|\nabla_x u\| \\
\Leftrightarrow \;\; -(u_t^2 + \|\nabla_x u\|^2) \leq -2|u_t|\|\nabla_x u\| \\
\Leftrightarrow \;\; -\sqrt{\theta}(u_t^2 + \|\nabla_x u\|^2) \leq -2\sqrt{\theta}|u_t|\|\nabla_x u\| = -2|u_t|\|\nabla_x\psi\|\|\nabla_x u\|,
\end{aligned} \tag{4.22}$$

for θ as defined in (4.11). Thus, from (4.21) and (4.22) we obtain

$$\begin{aligned}
\int_{S_\lambda} (u_t^2 + \|\nabla_x u\|^2 - 2|u_t|\|\nabla_x\psi\|\|\nabla_x u\|) \, dS'_\lambda &\geq \int_{S_\lambda} \|u'\|^2 (1-\sqrt{\theta}) dS'_\lambda \\
&\geq \frac{(a-\lambda)^2}{a^2+\lambda^2} \int_{S_\lambda} \|u'\|^2 dS'_\lambda
\end{aligned} \tag{4.23}$$

where we used (4.11) to compute

$$1 - \sqrt{\theta} \geq 1 - \frac{2\lambda a}{a^2+\lambda^2} = \frac{(a-\lambda)^2}{a^2+\lambda^2}. \tag{4.24}$$

Returning to (4.17) and using (4.3) we conclude that

$$\begin{aligned}
2\int_{R_\lambda} u_t \phi \, dxdt = 2\int_{R_\lambda} u_t \Box u \, dxdt \\
&\geq \frac{(a-\lambda)^2}{a^2+\lambda^2} \int_{S_\lambda} \|u'\|^2 dS'_\lambda \\
&\geq \frac{(a-\lambda)^2}{\sqrt{2}(a^2+\lambda^2)} \int_{S_\lambda} \|u'\|^2 dS_\lambda ,
\end{aligned} \tag{4.25}$$

using again (4.11) to obtain $dS'_\lambda \geq (1/\sqrt{2}) dS_\lambda$, defined in (4.18).

To proceed we use the fact that there exists a constant M, depending on u, s.t.

$$|\phi(x,t,u,u',u'')| \leq M\sqrt{|u|^2 + a^2\|u'\|^2}, \; \forall \, (x,t) \in K(x_0,a). \tag{4.26}$$

This follows from (4.2), the proof of this fact can be seen in Appendix A, see (A.2.4). Defining

$$I(\lambda) := \int_{S_\lambda} \|u'\|^2 dS_\lambda , \tag{4.27}$$

by (4.25) and (4.26) we have

$$
\begin{aligned}
I(\lambda) = \int_{S_\lambda} \|u'\|^2 dS_\lambda &\leq \frac{2\sqrt{2}(a^2+\lambda^2)}{(a-\lambda)^2} \int_{R_\lambda} u_t \phi \, dxdt \\
&\leq \frac{2\sqrt{2}(a^2+\lambda^2)}{(a-\lambda)^2} \int_{R_\lambda} |u_t||\phi| dxdt \\
&\leq \frac{2\sqrt{2}M(a^2+\lambda^2)}{(a-\lambda)^2} \int_{R_\lambda} |u_t| \sqrt{|u|^2 + a^2\|u'\|^2} \, dxdt.
\end{aligned} \tag{4.28}
$$

Now, since

$$
\begin{aligned}
a^2 u_t^2 &\leq u^2 + a^2(u_t^2 + \|\nabla_x u\|^2) = u^2 + a^2\|u'\|^2 \\
\Leftrightarrow |u_t| &\leq \frac{1}{a}\sqrt{u^2 + a^2\|u'\|^2},
\end{aligned} \tag{4.29}
$$

we further obtain

$$
\begin{aligned}
I(\lambda) &\leq \frac{2\sqrt{2}M(a^2+\lambda^2)}{(a-\lambda)^2} \int_{R_\lambda} |u_t| \sqrt{|u|^2 + a^2\|u'\|^2} \, dxdt \\
&\leq \frac{2\sqrt{2}M(a^2+\lambda^2)}{a(a-\lambda)^2} \int_{R_\lambda} u^2 + a^2\|u'\|^2 \, dxdt.
\end{aligned} \tag{4.30}
$$

From the fundamental theorem of calculus and Cauchy-Schwartz's inequality we have

$$
u^2(x,t) = 2\int_0^t u(x,s)u_s(x,s)ds \leq 2\Big(\int_0^t u^2 ds\Big)^{1/2}\Big(\int_0^t u_s^2 ds\Big)^{1/2}. \tag{4.31}
$$

Integrating once more from 0 to ψ and considering the integrals on the right hand side up to time $t = \psi(x;\lambda)$, since the integrand is non-negative, we obtain

$$
\begin{aligned}
\int_0^\psi u^2 dt &\leq \int_0^\psi \left[2\Big(\int_0^t u^2 ds\Big)^{1/2}\Big(\int_0^t u_t^2 ds\Big)^{1/2}\right] dt \\
&\leq \int_0^\psi \left[2\Big(\int_0^\psi u^2 ds\Big)^{1/2}\Big(\int_0^\psi u_t^2 ds\Big)^{1/2}\right] dt \\
&= 2\psi\Big(\int_0^\psi u^2 ds\Big)^{1/2}\Big(\int_0^\psi u_t^2 ds\Big)^{1/2}.
\end{aligned} \tag{4.32}
$$

Squaring both sides and assuming $(\int_0^\psi u^2 dt) \neq 0$ we obtain

$$
\int_0^\psi u^2 dt \leq 4\psi^2 \int_0^\psi u_t^2 dt. \tag{4.33}
$$

If however $(\int_0^\psi u^2 dt) = 0$ the relation remains valid. Thus, since $\psi < a$ from (4.9),

$$
\int_0^\psi u^2 dt \leq 4\psi^2 \int_0^\psi u_t^2 dt \leq 4a^2 \int_0^\psi u_t^2 dt. \tag{4.34}
$$

Returning to (4.30) and using (4.34) we have

$$
\begin{aligned}
I(\lambda) &\leq \frac{2\sqrt{2}M(a^2+\lambda^2)}{a(a-\lambda)^2} \int_{R_\lambda} u^2 + a^2\|u'\|^2 dxdt \\
&\leq \frac{2\sqrt{2}M(a^2+\lambda^2)}{a(a-\lambda)^2} \left[\int_{\|x-x_0\|<a} \Big(\int_0^\psi u^2 dt\Big) dx + \int_{R_\lambda} a^2\|u'\|^2 dxdt\right] \\
&\leq \frac{2\sqrt{2}M(a^2+\lambda^2)}{a(a-\lambda)^2} \left[\int_{\|x-x_0\|<a} \Big(4a^2 \int_0^\psi u_t^2 dt\Big) dx + \int_{R_\lambda} a^2\|u'\|^2 dxdt\right],
\end{aligned} \tag{4.35}
$$

Now, from $u_t^2 \leq \|u'\|^2$, we obtain

$$\begin{aligned} I(\lambda) &\leq \frac{2\sqrt{2}M(a^2+\lambda^2)}{a(a-\lambda)^2}\left[5a^2 \int_{R_\lambda} \|u'\|^2 dxdt\right] \\ &= \frac{10\sqrt{2}Ma(a^2+\lambda^2)}{(a-\lambda)^2} \int_{R_\lambda} \|u'\|^2 dxdt \\ &= \frac{10\sqrt{2}Ma(a^2+\lambda^2)}{(a-\lambda)^2} \int_{\|x-x_0\|<a} \left(\int_0^{\psi(x;\lambda)} \|u'\|^2 dt \right) dx. \end{aligned} \tag{4.36}$$

Define

$$c(\lambda) := \frac{10\sqrt{2}Ma(a^2+\lambda^2)}{(a-\lambda)^2}. \tag{4.37}$$

For fixed x and considering $t = \psi(x,\mu)$, we proceed from (4.36) by integrating instead along μ up to λ by taking $dt = \psi_\mu d\mu$ which gives

$$\begin{aligned} I(\lambda) &\leq c(\lambda) \int_{\|x-x_0\|<a} \left(\int_0^{\psi(x;\lambda)} \|u'\|^2 dt \right) dx \\ &= c(\lambda) \int_{\|x-x_0\|<a} \left(\int_0^{\lambda} \psi_\mu \|u'(x,\psi(x,\mu))\|^2 d\mu \right) dx \\ &= c(\lambda) \int_0^{\lambda} \left(\int_{\|x-x_0\|<a} \psi_\mu \|u'(x,\psi(x,\mu))\|^2 dx \right) d\mu \\ &= c(\lambda) \int_0^{\lambda} \left(\int_{S_0} \psi_\mu \|u'(x,\psi(x,\mu))\|^2 dx \right) d\mu. \end{aligned} \tag{4.38}$$

Using (4.10) we have

$$\begin{aligned} I(\lambda) &\leq c(\lambda) \int_0^{\lambda} \left(\int_{S_0} \psi_\mu \|u'(x,\psi(x,\mu))\|^2 dx \right) d\mu \\ &\leq c(\lambda) \int_0^{\lambda} \left(\int_{S_0} \|u'(x,\psi(x,\mu))\|^2 dx \right) d\mu \\ &\leq c(\lambda) \int_0^{\lambda} \left(\int_{S_0} \|u'(x,\psi(x,\mu))\|^2 \left(\sqrt{1+\sum_i (\partial_i \psi(x;\lambda))^2} \right) dx \right) d\mu \\ &= c(\lambda) \int_0^{\lambda} \left(\int_{S_\mu} \|u'(x,\psi(x,\mu))\|^2 dS_\mu \right) d\mu \\ &\leq c(\lambda) \int_0^{\lambda} I(\mu) d\mu, \end{aligned} \tag{4.39}$$

recalling the definition of $I(\lambda)$. We have then

$$I(\lambda) \leq c(\lambda) \int_0^{\lambda} I(\mu) d\mu. \tag{4.40}$$

Let us suppose that $I(\mu)$ does not vanish on some open set contained in $[0,\lambda]$. Since $I(0) = 0$, using the fundamental theorem of calculus we can write

$$\begin{aligned} &\frac{\partial_\lambda\left(\int_0^\lambda I(\mu)d\mu\right)}{\int_0^\lambda I(\mu)d\mu} \leq c(\lambda) \\ \Leftrightarrow\quad &\partial_\lambda\left(log\Big(\int_0^\lambda I(\mu)d\mu\Big)\right) \leq c(\lambda). \end{aligned} \tag{4.41}$$

Integrating in λ from ϵ to λ s.t. $0<\epsilon<\lambda<a$ we have

$$\begin{aligned} &\int_\epsilon^\lambda \partial_\alpha log\Big(\int_0^\alpha I(\mu)d\mu\Big)d\alpha \leq \int_\epsilon^\lambda c(\alpha)d\alpha \\ \Leftrightarrow\quad & log\Big(\int_0^\lambda I(\mu)d\mu\Big) \leq \int_\epsilon^\lambda c(\alpha)d\alpha + log\Big(\int_0^\epsilon I(\mu)d\mu\Big) \\ \Leftrightarrow\quad & \int_0^\lambda I(\mu)d\mu \leq \Big(\int_0^\epsilon I(\mu)d\mu\Big)exp\Big(\int_\epsilon^\lambda c(\alpha)d\alpha\Big). \end{aligned} \tag{4.42}$$

Taking the limit $\epsilon \to 0$ we obtain $\int_0^\lambda I(\mu)d\mu \leq 0$, since the exponential part is bounded. Furthermore, because $I(\mu)\geq 0$, we conclude that $I(\mu)=0$ for $0\leq\mu\leq\lambda$ and thus $\|u'\|=0$ in S_λ for any $\lambda\in[0,a)$. Again, from the continuity of u we have $\|u'\|=0$ in the whole cone $K(x_0,a)$ and thus u is constant in $K(x_0,a)$. By the initial condition $u(x,0)=0$, we finally conclude that $u(x,t)=0$ in $K(x_0,a)$. ■

Remark 4.1.2. *Fritz John refers to Theorem 4.1.1 as a uniqueness result throughout [Joh85a]. As far as we understand this is specifically for trivial initial conditions in the base of the cone, implying that $u\equiv 0$ is the only possible solution in the cone, i.e. no outside information reaches inside the cone ($v\leq 1$). As far as we can tell this does not extend immediately for non-trivial conditions since, given two solutions u and v inside the cone, it is not clear one can write (4.3) in $w:=u-v$. In particular, we would need to write (4.26) in w.*

Regarding Remark 4.1.1: Suppose we have a foliation of $K(x_0,a)$ by characteristics of $\Box$, i.e. in equation (4.15) the term given by

$$\int_{S_\lambda} X\cdot\hat{n}_\lambda dS_\lambda, \tag{4.43}$$

where $X=(-2u_t\nabla_x u, u_t^2+\|\nabla_x u\|^2)$, is now over an S_λ which is given by

$$t=\psi(x;\lambda)=\lambda-\|x-x_0\|. \tag{4.44}$$

Consider the function $z(x,t;\lambda)$ and a level surface of it, corresponding to S_λ, given by

$$z(x,t;\lambda)=t+\|x-x_0\|-\lambda=0. \tag{4.45}$$

The exterior normal vector to this surface is given by

$$\hat{n}=\frac{\nabla_{(x,t)}z}{\|\nabla_{(x,t)}z\|}=\frac{1}{\sqrt{2}}\left(\frac{x_i-x_{0,i}}{\|x-x_0\|},1\right),\quad i=1,...,n. \tag{4.46}$$

Then (4.43) becomes

$$\begin{aligned}\int_{S_\lambda} X\cdot\hat{n}_\lambda dS_\lambda =&\frac{1}{\sqrt{2}}\int_{S_\lambda} u_t^2+\|\nabla_x u\|^2-2u_t\sum_i u_i\left(\frac{x_i-x_{0,i}}{\|x-x_0\|}\right)dS_\lambda\\ =&\frac{1}{\sqrt{2}}\int_{S_\lambda} u_t^2+\|\nabla_x u\|^2+2u_t\sum_i u_i\partial_i\psi(x;\lambda)dS_\lambda\\ \geq&\frac{1}{\sqrt{2}}\int_{S_\lambda}\|u'\|^2(1-\sqrt{\theta})dS_\lambda,\end{aligned}\tag{4.47}$$

using the exact same computations applied before in order to obtain (4.23). However, for this new ψ, we have

$$\theta=\sum_i(\partial_i\psi)^2=1,\tag{4.48}$$

instead of (4.11). The integral on $\|u'\|^2$ now vanishes and we no longer have control over it.

Reversing condition (4.5), i.e. placing a trivial initial condition outside a ball of finite radius $R>0$, or rather having initial conditions with compact support, leads immediately to the following corollary to Theorem 4.1.1.

Corollary 4.1.2. *Assume ϕ satisfies the conditions of Theorem 4.1.1 and let u be a C^2 solution of*

$$\Box u=\phi(x,t,u,u',u''),\tag{4.49}$$

in the slab

$$x\in\mathbb{R}^n,\ 0\leq t<T.\tag{4.50}$$

Assume further that

$$u(x,0)=u_t(x,0)=0,\ \textit{for}\ \|x\|\geq R.\tag{4.51}$$

Then

$$u(x,t)=0,\ \textit{for}\ \ 0\leq t<T,\ \|x\|\geq R+t.\tag{4.52}$$

The following figure, Figure 4.2, shows the idea of Corollary 4.1.2.

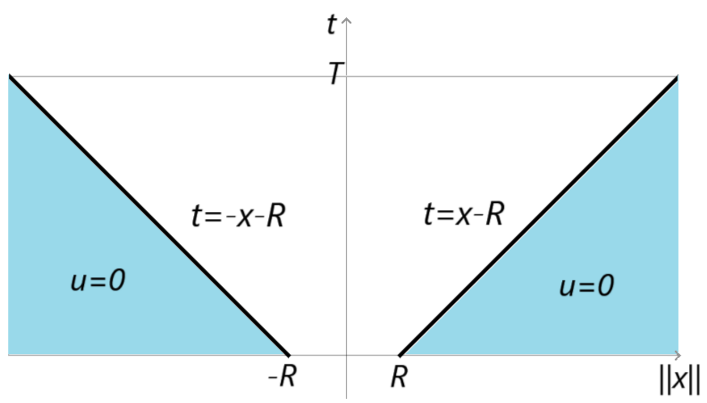

Figure 4.2: Picture of Corollary 4.1.2 for initial conditions with compact support in $B(0,R)$.

Theorem 4.1.1 states that information outside $B(x_0, a)$, the base of the cone, does not reach inside the cone, i.e. the speed of propagation v satisfies $v \leq 1$. Thus, assuming the existence of a solution $u \in C^2$ of (4.3), and provided the non-linearity ϕ satisfies (4.2), we have finite speed of propagation for quasilinear wave equations.

In the next section, before looking at the proof of blow-up in 3+1 dimensions by Fritz John in its general form, we first study a particular example, equation (1.5), which falls in the setting of that result. This serves as a an introduction to the main ideas of the proof without the extra details of the general case.

For the remainder of this Chapter, we will further assume that, on linearizing (4.1) in a neighbourhood of the trivial solution $u = 0$, one recovers the classical wave equation, i.e.

$$\Box u = 0. \tag{4.53}$$

Expanding ϕ up to linear order at $u = 0$, $u' = 0$ and $u'' = 0$, we have

$$\phi(x, t, u, u', u'') \approx \phi(x, t, 0, 0, 0) + \left.\frac{\partial \phi}{\partial u}\right|_{(x,t,0,0,0)} \cdot u + \left.\frac{\partial \phi}{\partial u'}\right|_{(x,t,0,0,0)} \cdot u' + \left.\frac{\partial \phi}{\partial u''}\right|_{(x,t,0,0,0)} \cdot u''. \tag{4.54}$$

Thus, to verify (4.53), we assume that ϕ satisfies the following

$$\phi = \frac{\partial \phi}{\partial u} = 0, \quad \frac{\partial \phi}{\partial u'} = 0, \quad \frac{\partial \phi}{\partial u''} = 0, \text{ at } (x, t, 0, 0, 0), \tag{4.55}$$

i.e. that ϕ is at least quadratic in u, u' and u''. Additionally, this allows $u \equiv 0$ to be a solution of (4.1).

4.2 Blow-up of solutions of $\Box v = v_t^2$ in 3+1 dimensions

We are interested in understanding blow-up phenomena of solutions to the initial value problem associated to the following semilinear wave equation in 3+1 dimensions

$$\Box v = (v_t)^2, \tag{4.56}$$

for initial data $u|_{t=0}$ and $u_t|_{t=0}$ of compact support. We first provide a brief outline, without details, of the main ideas in the proof of this particular example. To this end we follow the general proof of Theorem 1 of [Joh85a] by Fritz John. The general case is treated in Section 4.3.

1. Let v be a global C^3 solution of the IVP associated to equation (4.56). This IVP falls within the setting of Corollary 4.1.2. From it, we obtain

$$v(x, t) = 0, \text{ for } \|x\| > t + R, \quad t \geq 0, \tag{4.57}$$

where R corresponds to the radius of a ball containing the compact support of the initial conditions, see again Fig. 4.2.

2. The spherical average of $\Box v = v_t^2$ leads to the 1+1-dimensional equation $\Box(r\overline{v}) = r\overline{v_t^2}$, by (3.19). By Duhamel's principle (3.26) this equation has the following solution in 1+1 dimensions

$$\overline{v}(r,t) = \overline{v}^0(r,t) + \int\int_{T_{r,t}} \frac{\rho}{2r}\overline{v_t^2}(\rho,\tau)d\rho d\tau. \tag{4.58}$$

3. Fixing $(r,t) \in \Sigma$ from now on, where $\Sigma = \{(r,t)\,|\, r+R < t < 2r\}$, by symmetry considerations and the fact that $\overline{v}^0(r,t) = 0$ for $(r,t) \in \Sigma$, we can write the above equation as

$$\overline{v}(r,t) = \int\int_{T^*_{r,t}} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\rho d\tau, \;\; (r,t) \in \Sigma, \tag{4.59}$$

where $T^*_{r,t}$ is the trapezoid with vertices $\{(r,t),(0,t-r),(t-r,0),(t+r,0)\}$. For a sketch of the regions under consideration see Fig. (4.3).

4. Letting $c := t - r$, which satisfies $R < c < r$, we consider another region $S_{r,t} \subset T^*_{r,t}$ such that, by applying Cauchy-Schwarz's inequality to (4.59), we obtain

$$\overline{v}(r,t) \geq \int\int_{S_{r,t}} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\rho d\tau \geq \int_c^r\int_{\rho-R}^{\rho+c} \frac{\rho}{2r}\overline{v_\tau}^2(\rho,\tau)d\tau d\rho, \;\text{ for }\; R < c < r. \tag{4.60}$$

For a sketch of $S_{r,t}$ in this context see Fig. (4.4).

5. In the $S_{r,t}$ region, since $v \in C^2$ and $\overline{v}(\rho,\tau) = 0$ for $\tau \leq \rho - R$, we have $\overline{v}(\rho,\rho-R) = \overline{v}_\tau(\rho,\rho-R) = 0$. Using the fundamental theorem of calculus we have then

$$\overline{v}(\rho,\rho+c) = \int_{\rho-R}^{\rho+c} \partial_\tau\overline{v}(\rho,\tau)d\tau = \int_{\rho-R}^{\rho+c}\overline{v_\tau}(\rho,\tau)d\tau.$$

Using Cauchy-Schwarz's inequality in the above equation we obtain the following relation

$$\frac{1}{\Gamma(c)}\overline{v}^2(\rho,\rho+c) \leq \int_{\rho-R}^{\rho+c}\overline{v_\tau}^2(\rho,\tau)d\tau, \tag{4.61}$$

with $\Gamma(c) > 0$.

6. Using (4.60) and (4.61), along the characteristic $(r, c+r)$ we obtain

$$\overline{v}(r,c+r) \geq \frac{1}{2r\Gamma(c)}\int_c^r \rho\overline{v}^2(\rho,\rho+c)d\rho. \tag{4.62}$$

Defining

$$\alpha(r) := \overline{v}(r,c+r), \;\; \beta(r) := \int_c^r \rho\alpha^2(\rho)d\rho \geq 0, \tag{4.63}$$

one can obtain, from (4.62), the following ODE type inequality

$$\beta'(r) = r\alpha^2(r) \geq \frac{1}{4r\Gamma^2}\beta^2(r) \geq 0, \;\text{ for }\; r > c > R. \tag{4.64}$$

This is the crucial point, along the characteristic $(r, c+r)$, within Σ, one can study the solutions of the starting wave equation via this ODE type inequality.

7. From this last inequality, one can show by a contradiction argument that a global C^2 solution v of the IVP (4.56) must be zero along the characteristic $t = r + c$, for any $(r, t) \in \Sigma$ with $R < c < r$, concluding that $\overline{v}(r, t) = 0$ for $(r, t) \in \Sigma$.

8. This fact, together with (4.59), allows us to conclude that in the region $\|x\| + t > R$, with $t > 0$, we have $v_t(x, t) = 0$. This implies, via the starting wave-equation of the IVP, that $\Delta v(x, t) = 0$ in this same region.

9. Using $\Delta v(x, t) = 0$ and a fixed $t > R$, by the divergence theorem and the fact that $v(x, t)$ has compact support in x for a fixed t, we conclude that $v(x, t) = 0$ for $x \in \mathbb{R}^3$ and $t > R$.

10. The behaviour of the solution in the remaining region $T_{0,R}$, i.e. the triangle with vertices $\{(-R, 0), (0, R), (R, 0)\}$, can be obtained by time-reversing the wave equation from $t = R$ to $t = 0$ and using again the uniqueness result (4.1.2), showing that $v(x, t) = 0$ for $x \in \mathbb{R}^3$ and $t \geq 0$ if v is a global C^2 solution of $\Box v = v_t^2$.

To conclude, this result shows that a global C^2 solution v of the IVP (4.56) must vanish in $\mathbb{R}^3 \times \mathbb{R}_+$. This in turn implies, if one considers the IVP with non-trivial initial conditions, that blow-up must occur in finite time since there can be no non-vanishing C^2 solution defined for all $t \geq 0$. We prove this in detail in the following proposition.

Proposition 4.2.1. *Let $v \in C^3(\mathbb{R}^3 \times \mathbb{R}_+)$ be a solution of the IVP associated to the following equation*

$$\Box v = (v_t)^2, \tag{4.65}$$

with initial data $v|_{t=0}$ and $v_t|_{t=0}$ of compact support. Then $v(x, t) = 0$ in $\mathbb{R}^3 \times \mathbb{R}_+$.

Proof. Since $v|_{t=0}$ and $v_t|_{t=0}$ have compact support, there exists $R \in \mathbb{R}^+$ such that $v(x, 0) = v_t(x, 0) = 0$ for $\|x\| > R$. Furthermore, $\Box v = 0$ when $v_t = 0$, so we are in the conditions of the uniqueness theorem (Corollary 4.1.2) from which we conclude that

$$v(x, t) = 0, \text{ for } \|x\| > t + R, \;\; t \geq 0. \tag{4.66}$$

By the Euler-Poisson-Darboux equation in the quasilinear case (3.19), the spherical average of the wave equation in (4.65) is given by

$$\bar{v}_{tt} - \frac{2}{r}\bar{v}_r - \bar{v}_{rr} = \overline{v_t^2}, \tag{4.67}$$

in $\mathbb{R} \setminus \{0\} \times \mathbb{R}_+$ by even extension $\overline{v}(-r, t) = \overline{v}(r, t)$. Multiplying by r, one can rewrite (4.67) in the following form

$$\Box(r\overline{v}) = r\overline{v_t^2}. \tag{4.68}$$

Remark 4.2.1. *The symmetric extension is made in order for $r\overline{v}$ to be an odd function. This is the same idea used when obtaining d'Alembert's formula in the half-plane, Section 2.1.2, extending the problem to all of $\mathbb{R}$. This facilitates the analysis by avoiding issues with the boundary.*

This is a 1+1-dimensional non-homogeneous wave equation with known solution given by Duhamel's representation formula (3.26)

$$\overline{v}(r,t) = \overline{v}^0(r,t) + \int\int_{T_{r,t}} \frac{\rho}{2r}\overline{v_t^2}(\rho,\tau)d\rho d\tau, \tag{4.69}$$

where

$$\overline{v}^0(r,t) = \frac{1}{2r}\left[(r+t)\overline{v}(r+t,0) + (r-t)\overline{v}(r-t,0) + \frac{1}{2r}\int_{r-t}^{r+t} \rho\overline{v}_t(\rho,0)d\rho\right], \tag{4.70}$$

and where $T_{r,t}$ corresponds to the characteristic triangle with vertices $\{(r,t),(r-t,0),(r+t,0)\}$. Consider the wedge shaped region Σ given by

$$\Sigma = \{(r,t)\,|\,r+R<t<2r\}, \tag{4.71}$$

obtained by requiring that $t-r$, the point of intersection of the line of slope one passing through (r,t) with the t-axis, satisfies $R<t-r<r$, for integration considerations when we introduce the $S_{r,t}$ region ahead, and such that the compact support of the initial conditions is contained in $T_{r,t}$, i.e. we are not missing any information that could influence the solution at a fixed $(r,t)\in\Sigma$. Furthermore, for $(r,t)\in\Sigma$ we have

$$\overline{v}^0(r,t) = 0, \tag{4.72}$$

since $r+t>R$ and $r-t<-R$ implying $v(r+t,0)=v(r-t,0)=0$ and

$$\frac{1}{2r}\int_{r-t}^{r+t}\rho\overline{v}_t(\rho,0)d\rho = 0, \tag{4.73}$$

by symmetry considerations since the integrand $\rho\overline{v}_t(\rho,0)$ is odd in ρ and the region symmetric about $\rho=0$. Thus, for $(r,t)\in\Sigma$, equation (4.69) simply becomes

$$\overline{v}(r,t) = \int\int_{T_{r,t}} \frac{\rho}{2r}\overline{v_t^2}(\rho,\tau)d\rho d\tau, \quad (r,t)\in\Sigma. \tag{4.74}$$

The characteristic triangle $T_{r,t}$ can be written as the union $T_{r,t} = T_{0,t-r}\cup T^*_{r,t}$ where $T_{0,t-r}$ is the triangle with vertices $\{(r-t,0),(0,t-r),(t-r,0)\}$ and $T^*_{r,t}$ is the trapezoid with vertices

$$\{(r,t),(0,t-r),(t-r,0),(t+r,0)\}, \tag{4.75}$$

see Fig. 4.3. Thus, for $(r,t)\in\Sigma$, equation (4.74) becomes

$$\begin{aligned}\overline{v}(r,t) &= \int\int_{T_{r,t}} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\rho d\tau = \int\int_{T_{0,t-r}} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\rho d\tau + \int\int_{T^*_{r,t}} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\rho d\tau \\ &= \int\int_{T^*_{r,t}} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\rho d\tau,\end{aligned} \tag{4.76}$$

where again the integral over $T_{0,t-r}$ vanishes since it is symmetric with respect to ρ and $\rho\overline{v_t^2}$ is an odd function in ρ. Consider now the $S_{r,t}$ region defined by

$$S_{r,t} = \{(\rho,\tau)\,|\,t-r \leq \rho \leq r; \rho - R \leq \tau \leq \rho + t - r\}. \tag{4.77}$$

This region, which can be seen in Fig. 4.4, further justifies the introduction of Σ by having the upper edge of $S_{r,t}$ be along a characteristic where $r > t-r$. Clearly $S_{r,t} \subseteq T^*_{r,t}$ therefore

$$\overline{v}(r,t) \geq \int\int_{S_{r,t}} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\rho d\tau = \int_{t-r}^{r}\int_{\rho-R}^{\rho+t-r} \frac{\rho}{2r}\overline{v_\tau^2}(\rho,\tau)d\tau d\rho, \text{ for } (r,t)\in\Sigma. \tag{4.78}$$

Using Cauchy-Schwarz's inequality we obtain for the term $\overline{v_\tau^2}$

$$\begin{aligned}\overline{v_\tau^2} &= \frac{1}{4\pi}\int_{\|\xi\|=1} v_\tau^2 d\omega_\xi \\ &\geq \frac{1}{4\pi}\frac{1}{\int_{\|\xi\|=1} 1\, d\omega_\xi}\left(\int_{\|\xi\|=1} v_\tau d\omega_\xi\right)^2 \\ &= \left(\frac{1}{4\pi}\int_{\|\xi\|=1} v_\tau d\omega_\xi\right)^2 \\ &= \overline{v_\tau}^2,\end{aligned} \tag{4.79}$$

which, together with (4.78), results in

$$\overline{v}(r,t) \geq \int_{t-r}^{r}\int_{\rho-R}^{\rho+t-r} \frac{\rho}{2r}\overline{v_\tau}^2(\rho,\tau)d\tau d\rho, \text{ for } (r,t)\in\Sigma. \tag{4.80}$$

Since $\overline{v}(\rho,\tau) = 0$ for $\tau \leq \rho - R$ we have

$$\overline{v}(\rho,\rho-R) = \overline{v}_\tau(\rho,\rho-R) = 0. \tag{4.81}$$

A fixed $(r,t)\in\Sigma$ defines a characteristic defined by $c := t-r$ which satisfies $R < c < r$. Using the fundamental theorem of calculus one obtains

$$\overline{v}(\rho,\rho+c) = \int_{\rho-R}^{\rho+c} \partial_\tau\overline{v}(\rho,\tau)d\tau = \int_{\rho-R}^{\rho+c}\overline{v_\tau}(\rho,\tau)d\tau. \tag{4.82}$$

Squaring this expression and using Cauchy-Schwarz's inequality leads to

$$\overline{v}^2(\rho,\rho+c) \leq \left(\int_{\rho-R}^{\rho+c} 1 d\tau\right)\left(\int_{\rho-R}^{\rho+c}\overline{v_\tau}^2(\rho,\tau)d\tau\right). \tag{4.83}$$

Using (4.80) together with (4.83) we obtain, along the characteristic given by $c = t-r$, the following inequality for $\bar{v}$ along the characteristic

$$\overline{v}(r,c+r) \geq \frac{1}{2r\Gamma(c)}\int_c^r \rho\overline{v}^2(\rho,\rho+c)d\rho, \text{ for } r > c > R, \tag{4.84}$$

where we define

$$\Gamma(c) := \left(\int_{\rho-R}^{\rho+c} 1 d\tau\right) = c + R > 2R > 0. \tag{4.85}$$

The main point now is that, along the characteristic $c = t - r$ and within Σ, we obtain an ODE type inequality. Defining

$$\alpha(r) := \overline{v}(r, c + r), \;\; \beta(r) := \int_c^r \rho\alpha^2(\rho)d\rho \geq 0, \tag{4.86}$$

we have from (4.84)

$$\alpha(r) \geq \frac{1}{2r\Gamma(c)}\beta(r) \geq 0, \;\; r > c > R. \tag{4.87}$$

By the fundamental theorem of calculus $\beta'(r) = r\alpha^2(r)$. Therefore, taking the square of the inequality above, we get

$$\beta'(r) = r\alpha^2(r) \geq \frac{1}{4r\Gamma^2}\beta^2(r) \geq 0, \text{ for } r > c > R. \tag{4.88}$$

At this point, either $\beta(r) = 0$ for all $r > c > R$ or there exists an $r_0 > c$ such that $\beta(r) \geq \beta(r_0) > 0$ for all $r > r_0$, since $\beta'(r) > 0$. However, integrating (4.88) from r_0 to $r > r_0$ we obtain

$$\begin{aligned} \int_{r_0}^r \frac{\beta'(s)}{\beta^2(s)}ds &\geq \frac{1}{4\Gamma^2}\int_{r_0}^r \frac{1}{s}ds \\ \Leftrightarrow \quad -\frac{1}{\beta(s)}\bigg|_{s=r_0}^{s=r} &\geq \frac{1}{4\Gamma^2}\log\left(\frac{r}{r_0}\right), \end{aligned} \tag{4.89}$$

leading to

$$\frac{1}{\beta(r_0)} \geq \frac{1}{\beta(r_0)} - \frac{1}{\beta(r)} \geq \frac{1}{4\Gamma^2}\log\left(\frac{r}{r_0}\right), \text{ for } r > r_0. \tag{4.90}$$

Since by assumption $v \in C^2(\mathbb{R}^3 \times \mathbb{R}_+)$, we can take $r \to \infty$ which would make the $\log(r/r_0)$ term simultaneously unbounded and bounded by $1/\beta(r_0)$, a contradiction. Thus, there is no $r_0 > c$ for which $\beta(r_0) > 0$ and therefore we must have $\beta(r) = 0$ for all $r > c > R$.

By the definition of β this implies that $\alpha(r) = \overline{v}(r, c+r) = 0$ for $r > c > R$, i.e. along the characteristic defined by c. Since the argument is valid for any $(r, t) \in \Sigma$ we conclude

$$\overline{v}(r, t) = 0, \text{ for } (r, t) \in \Sigma. \tag{4.91}$$

This implies, from (4.76), that on the following region

$$\Lambda := \bigcup_{(r,t)\in\Sigma} T^*_{r,t} = \{(\rho, \tau) \,|\, \rho > 0; \tau > 0; \rho + \tau > R\} = \{(x, t) \,|\, \|x\| + t > R, \; t > 0\}, \tag{4.92}$$

we have $\overline{v_t^2}(r, t) = 0$. Thus, in the region $t > R - \|x\|$, $t > 0$, we have from the wave equation in (4.65) that

$$v_t^2(x, t) = 0 \implies v_t(x, t) = 0 \implies v_{tt}(x, t) = 0 \implies \Delta v(x, t) = 0. \tag{4.93}$$

Consider a fixed $t_0 > R$. For any $x \in \mathbb{R}^3$ such that $\|x\| > R + t_0$ we have $v(x, t_0) = 0$, by (4.66), so $v(x, t_0)$ has compact support in x within the ball $B_{R+t_0}(0) =: V$. Using the divergence theorem we obtain

$$\begin{aligned} \int_V v\Delta v dx &= \int_V \nabla \cdot (v\nabla v)dx - \int_V (\nabla v)^2 dx \\ &= \int_{\partial V} v\nabla v \cdot \hat{n} dS - \int_V (\nabla v)^2 dx \\ &= -\int_V (\nabla v)^2 dx = 0, \end{aligned} \tag{4.94}$$

where the first integral on the previous to last expression vanishes since v is zero on ∂V by compact support. Thus, $\nabla v(x, t_0) = 0$, for all $x \in \mathbb{R}^3$, leading to $v(x, t_0) = 0$ for all $x \in \mathbb{R}^3$ and $t_0 > R$. Since the argument is valid for all $t_0 > R$, by continuity of v we conclude that

$$v(x,t) = 0, \quad x \in \mathbb{R}^3, \quad t \geq R. \tag{4.95}$$

Remark 4.2.2. *Alternatively, equation (4.93) corresponds to Laplace's equation with trivial boundary conditions. Thus, by the strong maximum principle (Theorem 4, Section 2.2 of [Eva10]), we have $v(x, t_0) = 0$ for $x \in \mathbb{R}^3$ and $t_0 > R$.*

Finally, going backwards in time from $t = R$ to $t = 0$, if we let $\tilde{v}(x,t) = v(x, R-t)$ then $\tilde{v}$ is a solution of

$$\Box \tilde{v} = \tilde{v}_t^2, \text{ for } x \in \mathbb{R}^3,\, 0 \leq t \leq R, \tag{4.96}$$

with

$$\tilde{v}(x, 0) = \tilde{v}_t(x, 0) = 0, \tag{4.97}$$

since $\tilde{v}(x,0) = v(x,R) = 0$ and $\tilde{v}_t(x,0) = -v_t(x,R) = 0$. We are again in the conditions of the uniqueness theorem (Corollary 4.1.2) from which it follows that $\tilde{v}(x,t) = 0$ for $x \in \mathbb{R}^3$, $0 \leq t \leq R$ and thus also $v(x,t) = 0$ for $x \in \mathbb{R}^3$, $0 \leq t \leq R$. Together with (4.95) this concludes the proof. ■

In the following section we finally consider the proof of blow-up in 3+1 by Fritz John in its general setting.

4.3 Blow-up in 3+1 dimensions - General Theorem (Theorems 1 and 2 of [Joh85a])

We divide the result in two parts. First, we prove a version of the theorem under a certain assumption ($K > 0$, where K is to be defined in the theorem's statement) using the ideas of Theorem 1 of [Joh85a] and the particular case of the equation ($\Box u = 2u_t u_{tt}$) treated in [Joh90]. Then, we extend the result to the $K = 0$ case following again the corresponding part in Theorem 1 of [Joh85a].

Theorem 4.3.1. *Let $n = 3$ and let $u \in C^2(\mathbb{R}^3 \times \mathbb{R}_+)$ be a solution of the IVP associated to the wave equation*

$$\Box u(x,t) = \phi(x,t,u,u',u''), \tag{4.98}$$

with initial data $u|_{t=0}$, $u_t|_{t=0}$ of compact support and where

$$\phi(x,t,u,u',u'') = A(x,t,u,u',u'') + \frac{\partial}{\partial t}\left(B(x,t,u,u')\right), \tag{4.99}$$

for functions $A \in C^2$, $B \in C^3$, satisfying the following properties

$$A(x,t,u,u',u'') \geq 0, \quad \forall x,t,u,u',u'', \tag{4.100}$$

$$A(x,t,0,0,u'') = 0, \;\; \forall x,t,u'', \tag{4.101}$$

$$\exists\, (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\} \; s.t. \; B(x,t,u,u') \geq (au + bu_t)^2, \;\; \forall x,t,u,u', \tag{4.102}$$

$$B(x,t,0,0) = 0, \;\; \forall x,t. \tag{4.103}$$

Assuming further that

$$K := \int_{\mathbb{R}^3} \big(u_t(x,0) - B(x,0,u(x,0),u'(x,0))\big)dx > 0, \tag{4.104}$$

then $u \equiv 0$.

Proof. Let u be a global C^2 solution of (4.98) and let ϕ, A, B satisfy the assumptions of Theorem 4.3.1. Since the initial data has compact support, there exists $R \in \mathbb{R}_+$ such that

$$u(x,0) = u_t(x,0) = 0, \;\text{ for } \|x\| \geq R. \tag{4.105}$$

By (4.102) and (4.103) B is non-negative and zero at $(x,t,u,u') = (x,t,0,0)$. This implies that $(x,t,0,0)$ is a minimum of B and therefore

$$(\partial_t B)(x,t,0,0) = (\partial_{x_i} B)(x,t,0,0) = (\partial_u B)(x,t,0,0) = 0, \;\; i = 1,2,3, \tag{4.106}$$

and

$$(\partial_{u'} B)(x,t,0,0) = 0, \tag{4.107}$$

leading to

$$\partial_t \left(B(x,t,u(x,t),u'(x,t))\right)(x,t,0,0) = 0. \tag{4.108}$$

Together with (4.101) this implies, from (4.99), that

$$\phi(x,t,0,0,u'') = 0. \tag{4.109}$$

Thus, we are in the conditions of Corollary 4.1.2 from which we conclude that

$$u(x,t) = 0, \;\; \text{for } \; \|x\| \geq t + R, \;\; t \geq 0. \tag{4.110}$$

Let

$$v(x,t) := \int_0^t u(x,s)ds. \tag{4.111}$$

By the fundamental theorem of calculus this implies

$$v_t(x,t) = u(x,t), \;\; \text{and} \;\; v(x,0) = 0. \tag{4.112}$$

From (4.110) v also satisfies

$$v(x,t) = 0, \;\text{ for } \|x\| \geq t + R, \;\; t \geq 0. \tag{4.113}$$

The wave equation in u (4.98) implies for v

$$\Box u = u_{tt} - \Delta u = v_{ttt} - \Delta v_t = \partial_t(\Box v) = \phi. \tag{4.114}$$

Thus, from (4.99), we have

$$\partial_t \left(\Box v(x,t) - (B(x,t,u(x,t),u'(x,t)))\right) = A(x,t,u(x,t),u'(x,t),u''(x,t)), \tag{4.115}$$

which, integrating from 0 to t,

$$\int_0^t \partial_s \left(\Box v - (B(x,s,u(x,s),u'(x,s)))\right) ds = \int_0^t A(x,s,u(x,s),u'(x,s),u''(x,s))\, ds, \tag{4.116}$$

and using again the fundamental theorem of calculus, leads to

$$\Box v(x,t) - B(x,t,u,u') - \Box v(x,0) + B(x,0,u(x,0),u'(x,0)) = \int_0^t A(x,s,u(x,s),u'(x,s),u''(x,s))\, ds. \tag{4.117}$$

From (4.111) we have

$$\Delta v(x,t) = \int_0^t \Delta u(x,s) ds, \tag{4.118}$$

therefore $\Delta v(x,0) = 0$ and

$$\Box v(x,0) = v_{tt}(x,0) - \Delta v(x,0) = u_t(x,0). \tag{4.119}$$

Defining

$$h(x) := u_t(x,0) - B(x,0,u(x,0),u'(x,0)), \tag{4.120}$$

(4.117) is equivalent to

$$\Box v(x,t) = B(x,t,u,u') + h(x) + \int_0^t A(x,s,u(x,s),u'(x,s),u''(x,s))\, ds. \tag{4.121}$$

From (4.100) the integral on the right satisfies

$$\int_0^t A(x,t,u(x,s),u'(x,s),u''(x,s))\, ds \geq 0, \tag{4.122}$$

which implies, together with (4.102), that

$$\Box v(x,t) \geq (au(x,t) + bu_t(x,t))^2 + h(x), \tag{4.123}$$

for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$. Note that, by (4.110) and (4.103),

$$h(x) = 0, \text{ for } \|x\| \geq R, \tag{4.124}$$

and by assumption (4.104),

$$K = \int_{\mathbb{R}^3} h(x) dx > 0. \tag{4.125}$$

From (4.121) let

$$w(x,t) := B(x,t,u,u') + h(x) + \int_0^t A(x,s,u(x,s),u'(x,s),u''(x,s))\, ds, \tag{4.126}$$

and define

$$p(x,t) := (au(x,t) + bu_t(x,t)) = (av_t(x,t) + bv_{tt}(x,t)), \tag{4.127}$$

and

$$q(x,t) := p^2(x,t). \tag{4.128}$$

It follows from (4.123) that

$$w \geq B + h \geq q + h. \tag{4.129}$$

We now introduce symmetrically extended spherical averages, centred at $x = 0$, i.e. given an arbitrary continuous function $f(x,t)$ defined in $(\mathbb{R}^3 \times \mathbb{R}_+)$ let

$$\overline{f}(r,t) := \frac{1}{4\pi} \int_{\|\xi\|=1} f(r\xi, t) d\omega_\xi, \tag{4.130}$$

for $t \geq 0$ and $r \in \mathbb{R}$ via the extension

$$\overline{f}(-r,t) = \overline{f}(r,t). \tag{4.131}$$

Taking the spherical average of (4.126) and using the Euler-Poisson-Darboux equation (3.19), with $n = 3$, we obtain

$$\overline{w}(r,t) = \overline{v}_{tt}(r,t) - \frac{2}{r}\overline{v}_r(r,t) - \overline{v}_{rr}(r,t). \tag{4.132}$$

Taking the spherical average of p we obtain

$$\overline{p}(r,t) = a\overline{v}_t(r,t) + b\overline{v}_{tt}(r,t), \tag{4.133}$$

and using Cauchy-Schwarz's inequality

$$\left(\int fg dx \right)^2 \leq \left(\int f^2 dx \right) \left(\int g^2 dx \right),$$

with $g = 1$ and $f = av_t + bv_{tt}$, we obtain

$$\begin{aligned} \overline{q} = \overline{p^2} &= \frac{1}{4\pi} \int_{\|\xi\|=1} (av_t + bv_{tt})^2 d\omega_\xi \\ &\geq \frac{1}{4\pi} \frac{1}{\int_{\|\xi\|=1} 1 \, d\omega_\xi} \left(\int_{\|\xi\|=1} (av_t + bv_{tt}) d\omega_\xi \right)^2 \\ &= \overline{p}^2, \end{aligned} \tag{4.134}$$

since

$$\int_{\|\xi\|=1} 1 \, d\omega_\xi = 4\pi. \tag{4.135}$$

Thus, we conclude that

$$\overline{q} \geq \overline{p}^2. \tag{4.136}$$

The order relation from (4.129) is naturally still maintained for spherical averages therefore

$$\overline{w} \geq \overline{B} + \overline{h} \geq \overline{q} + \overline{h}, \tag{4.137}$$

where all functions $\overline{v}, \overline{w}, \overline{p}, \overline{q}, \overline{h}$ and $\overline{B}$ are even in r, from the extension considered above. Furthermore, $\overline{v}, \overline{w}, \overline{p}, \overline{q}$ and $\overline{B}$ vanish for $|r| \geq t + R$ as a consequence of (4.110) and (4.103). Note that, by (4.124) the function

$$\overline{h}(r) = \frac{1}{4\pi} \int_{\|\xi\|=1} h(r\xi) d\omega_\xi \tag{4.138}$$

satisfies

$$\overline{h}(r) = 0, \text{ for } |r| \geq R, \tag{4.139}$$

and, by (4.125), we have

$$K = \int_0^\infty 4\pi r^2 \overline{h}(r) dr = 4\pi \int_0^R r^2 \overline{h}(r) dr > 0, \tag{4.140}$$

since $4\pi r^2 \overline{h}(r)$ is the value of the integral over the sphere of radius r.

As we have seen for $n = 3$, when obtaining (3.24), we can rewrite (4.132) as

$$\Box(r\overline{v}) = r\overline{w}, \tag{4.141}$$

which is a 1-dimensional non-homogeneous wave equation with solution given by Duhamel's representation formula (3.26)

$$\overline{v}(r,t) = \overline{v}^0(r,t) + \int\int_{T_{r,t}} \frac{\rho}{2r} \overline{w}(\rho,\tau) d\rho d\tau, \tag{4.142}$$

where

$$\overline{v}^0(r,t) = \frac{1}{2r}\left[(r+t)\overline{v}(r+t,0) + (r-t)\overline{v}(r-t,0) + \frac{1}{2r}\int_{r-t}^{r+t} \rho \overline{v}_t(\rho,0) d\rho\right], \tag{4.143}$$

and $T_{r,t}$ is the characteristic triangle with vertex (r,t) given by

$$T_{r,t} = \{(\rho,\tau) \,|\, \tau + \rho \leq t + r; \tau - \rho \leq t - r; \tau \geq 0\}, \tag{4.144}$$

or equivalently with the vertices given by $\{(r,t), (r-t,0), (r+t,0)\}$.

Remark 4.3.1. *Obtaining the wave equation (4.141) is the only point where $n = 3$ is required.*

We now introduce the region

$$\Sigma = \{(r,t) \,|\, r + R < t < 2r\}, \tag{4.145}$$

which was justified in the comments after (4.71). Notice that, fixing $(r,t) \in \Sigma$, due to (4.112) we have

$$\overline{v}(r+t,0) = \overline{v}(r-t,0) = 0. \tag{4.146}$$

Furthermore,

$$\frac{1}{2r}\int_{r-t}^{r+t} \rho \overline{v}_t(\rho,0) d\rho = \frac{1}{2r}\int_{-R}^{R} \rho \overline{v}_t(\rho,0) d\rho = 0, \tag{4.147}$$

since $\overline{v}(r,t) = 0$ for $|\rho| \geq R$, which follows from (4.113), and because $\rho \overline{v}_t(\rho,0)$ is an odd function in ρ due to the symmetrical extension of the spherical average defined above. Therefore

$$\overline{v}^0(r,t) = 0, \text{ for } (r,t) \in \Sigma, \tag{4.148}$$

and (4.142) becomes just

$$\overline{v}(r,t) = \int\int_{T_{r,t}} \frac{\rho}{2r}\overline{w}(\rho,\tau)d\rho d\tau, \text{ for } (r,t)\in\Sigma. \tag{4.149}$$

Note that $\overline{v}$ in (4.149) still depends on the initial data via $\overline{w}$. Figure 4.3 shows the geometric picture of the regions under consideration.

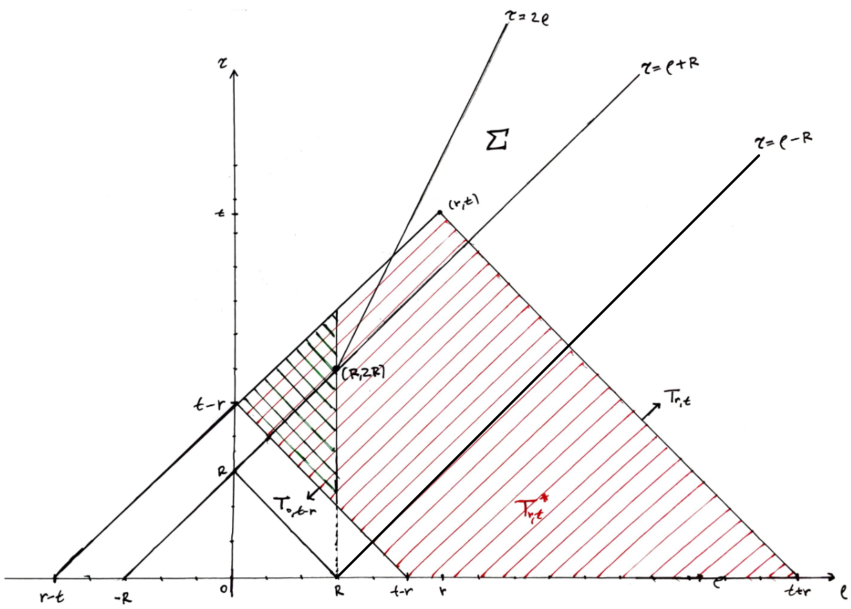

Figure 4.3: Wedge-shaped region Σ and $T^*_{r,t} = T_{r,t}\setminus T_{0,t-r}$.

Considering the characteristic triangle $T_{r,t}$ as the union

$$T_{r,t} = T_{0,t-r}\cup T^*_{r,t}, \tag{4.150}$$

where $T_{0,t-r}$ is the triangle with vertices $\{(r-t,0),(0,t-r),(t-r,0)\}$, and $T^*_{r,t}$ is the trapezoid with vertices $\{(r,t),(0,t-r),(t-r,0),(t+r,0)\}$, see again Figure 4.3, equation (4.149) becomes

$$\begin{aligned}\overline{v}(r,t) &= \int\int_{T_{r,t}} \frac{\rho}{2r}\overline{w}(\rho,\tau)d\rho d\tau \\ &= \int\int_{T_{0,t-r}} \frac{\rho}{2r}\overline{w}(\rho,\tau)d\rho d\tau + \int\int_{T^*_{r,t}} \frac{\rho}{2r}\overline{w}(\rho,\tau)d\rho d\tau \\ &= \int\int_{T^*_{r,t}} \frac{\rho}{2r}\overline{w}(\rho,\tau)d\rho d\tau, \quad (r,t)\in\Sigma,\end{aligned} \tag{4.151}$$

where the integral over $T_{0,t-r}$ vanishes since it is symmetric with respect to ρ and $\rho\bar{w}_t$ is an odd function in ρ, again due to the symmetrical extension of the spherical average defined above. Because $\rho>0$ in $T^*_{r,t}$ inequality (4.137) is preserved and we have

$$\overline{v}(r,t) \geq \int\int_{T^*_{r,t}} \frac{\rho}{2r}\bar{B}d\rho d\tau + \int\int_{T^*_{r,t}} \frac{\rho}{2r}\bar{h}(\rho)d\rho d\tau. \tag{4.152}$$

Due to (4.139) the integral on the right is simply (see Figure 4.3)

$$\int\int_{T^*_{r,t}} \frac{\rho}{2r}\bar{h}(\rho)d\rho d\tau = \int_0^R \left(\frac{\rho}{2r}\bar{h}(\rho)\int_{-\rho+(t-r)}^{\rho+(t-r)} d\tau\right) d\rho = 2\int_0^R \frac{\rho^2}{2r}\bar{h}(\rho)d\rho, \tag{4.153}$$

and from (4.140) we get

$$\int_0^R \frac{\rho^2}{r}\bar{h}(\rho)d\rho = \frac{K}{4\pi r} > 0. \tag{4.154}$$

and thus (4.152) becomes

$$\overline{v}(r,t) \geq \int\int_{T^*_{r,t}} \frac{\rho}{2r}\bar{B}\ d\rho d\tau + \frac{K}{4\pi r}. \tag{4.155}$$

Using inequalities (4.137) and (4.136) we further obtain

$$\overline{v}(r,t) \geq \int\int_{T^*_{r,t}} \frac{\rho}{2r}\overline{p}^2(\rho,\tau)\ d\rho d\tau + \frac{K}{4\pi r},\ \ (r,t)\in\Sigma. \tag{4.156}$$

Consider now the region $S_{r,t}$ defined as

$$S_{r,t} = \{(\rho,\tau)\,|\,t-r\leq\rho\leq r,\ \rho-R\leq\tau\leq\rho+t-r\}\,, \tag{4.157}$$

see Figure 4.4.

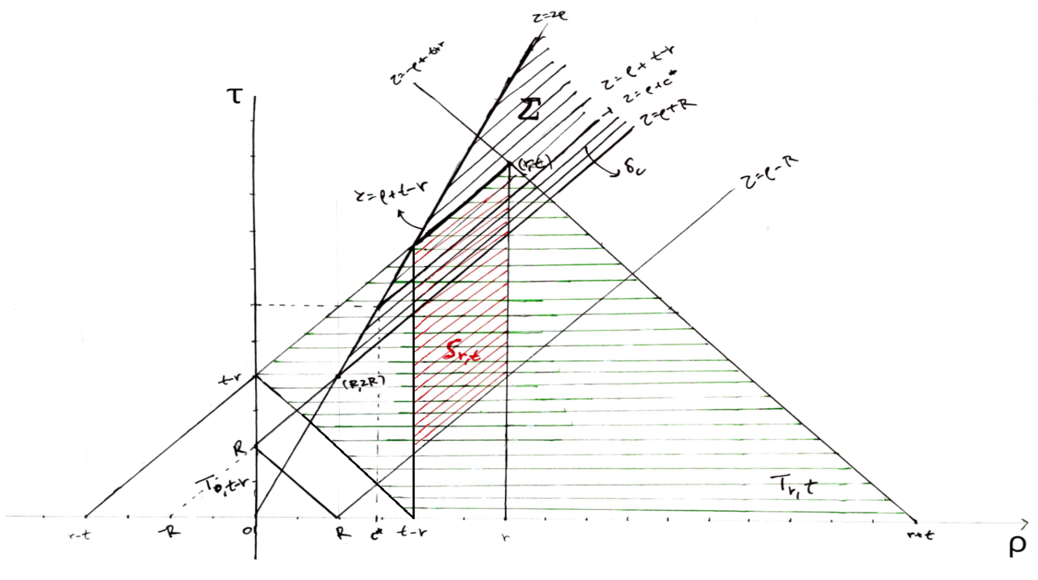

Figure 4.4: The regions $T^*_{r,t} = T_{r,t}\setminus T_{0,t-r}$, Σ and $S_{r,t}$.

Clearly $S_{r,t}\subseteq T^*_{r,t}$ therefore, from (4.156), we have

$$\begin{aligned}\overline{v}(r,t) &\geq \int\int_{S_{r,t}} \frac{\rho}{2r}\overline{p}^2(\rho,\tau)d\rho d\tau + \frac{K}{4\pi r}\\ &= \int_{t-r}^{r}\int_{\rho-R}^{\rho+t-r} \frac{\rho}{2r}\overline{p}^2(\rho,\tau)d\tau d\rho + \frac{K}{4\pi r},\ \ (r,t)\in\Sigma.\end{aligned} \tag{4.158}$$

Since, from (4.113), $\overline{v}(\rho,\tau)=0$ for $\tau \leq \rho - R$ we have

$$\overline{v}(\rho,\rho-R)=\overline{v}_\tau(\rho,\rho-R)=0, \tag{4.159}$$

therefore, using (4.133) with $a \neq 0$ and $b \neq 0$, we can write

$$\begin{aligned} & b\overline{v}_{\tau\tau}+a\overline{v}_\tau=\overline{p} \\ \Leftrightarrow \quad & e^{\frac{a}{b}\tau}(\overline{v}_{\tau\tau}+\frac{a}{b}\overline{v}_\tau)=e^{\frac{a}{b}\tau}\left(\frac{\overline{p}}{b}\right) \\ \Leftrightarrow \quad & \frac{\partial}{\partial_\tau}(\overline{v}_\tau e^{\frac{a}{b}\tau})=e^{\frac{a}{b}\tau}\left(\frac{\overline{p}}{b}\right) \end{aligned} \tag{4.160}$$

Integrating a first time up to some arbitrary τ satisfying $\rho - R \leq \tau \leq \rho + t - r$, we obtain

$$\begin{aligned} \int_{\rho-R}^{\tau}\partial_s(\overline{v}_s e^{\frac{a}{b}s})ds &= \int_{\rho-R}^{\tau} e^{\frac{a}{b}s}\left(\frac{\overline{p}}{b}\right)ds \\ \Leftrightarrow \quad \overline{v}_\tau(\rho,\tau)e^{\frac{a}{b}\tau} &= \int_{\rho-R}^{\tau} e^{\frac{a}{b}s}\left(\frac{\overline{p}}{b}\right)ds \\ \Leftrightarrow \quad \overline{v}_\tau(\rho,\tau) &= e^{-\frac{a}{b}\tau}\int_{\rho-R}^{\tau} e^{\frac{a}{b}s}\left(\frac{\overline{p}}{b}\right)ds, \end{aligned} \tag{4.161}$$

where we used (4.159). Integrating a second time in τ from $\tau = \rho - R$ up to $\tau = \rho + t - r$, i.e. up to the characteristic defined by the choice of $(r,t) \in \Sigma$, and using (4.159) once again, we obtain

$$\begin{aligned} \int_{\rho-R}^{\rho+t-r}\overline{v}_\tau(\rho,\tau)d\tau &= \int_{\rho-R}^{\rho+t-r} e^{-\frac{a}{b}\tau}\left(\int_{\rho-R}^{\tau} e^{\frac{a}{b}s}\left(\frac{\overline{p}}{b}\right)ds\right)d\tau \\ \Leftrightarrow \quad \overline{v}(\rho,\rho+t-r) &= \int_{\rho-R}^{\rho+t-r} e^{-\frac{a}{b}\tau}\left(\int_{\rho-R}^{\tau} e^{\frac{a}{b}s}\left(\frac{\overline{p}}{b}\right)ds\right)d\tau. \end{aligned} \tag{4.162}$$

Changing the order of integration we get

$$\begin{aligned} \overline{v}(\rho,\rho+t-r) &= \int_{\rho-R}^{\rho+t-r}\left(\int_{s}^{\rho+t-r} e^{-\frac{a}{b}\tau}e^{\frac{a}{b}s}\left(\frac{\overline{p}}{b}\right)d\tau\right)ds \\ &= \int_{\rho-R}^{\rho+t-r} -\left(\frac{b}{a}\right)e^{-\frac{a}{b}\tau}\Big|_{\tau=s}^{\tau=\rho+t-r} e^{\frac{a}{b}s}\left(\frac{\overline{p}}{b}\right)ds \\ &= \int_{\rho-R}^{\rho+t-r} -\left(\frac{b}{a}\right)\left(e^{-\frac{a}{b}(\rho+t-r)}-e^{-\frac{a}{b}s}\right)e^{\frac{a}{b}s}\left(\frac{\overline{p}}{b}\right)ds \\ &= \int_{\rho-R}^{\rho+t-r}\frac{1}{a}\left(1-e^{-\frac{a}{b}(\rho+t-r-s)}\right)\overline{p}(\rho,s)ds. \end{aligned} \tag{4.163}$$

If $b \neq 0$ and $a = 0$ from the first line of (4.163) we obtain

$$\int_{\rho-R}^{\rho+t-r}\frac{(\rho+t-r-s)}{b}\overline{p}(\rho,s)ds. \tag{4.164}$$

If $b = 0$ and $a \neq 0$ from the first line of (4.160), using the fundamental theorem of calculus we immediately get

$$\int_{\rho-R}^{\rho+t-r}\frac{1}{a}\overline{p}(\rho,s)ds. \tag{4.165}$$

Thus, renaming the variable s as τ, we finally obtain

$$\overline{v}(\rho, \rho+t-r) = \int_{\rho-R}^{\rho+t-r} k(\rho+t-r-\tau)\overline{p}(\rho,\tau)d\tau, \tag{4.166}$$

where

$$k(s) = \begin{cases} \frac{1}{a}(1-e^{-as/b}) & \text{for } a \neq 0, b \neq 0, \\ \frac{s}{b} & \text{for } a = 0, b \neq 0, \\ \frac{1}{a} & \text{for } a \neq 0, b = 0. \end{cases} \tag{4.167}$$

From (4.166), using Cauchy-Schwarz's inequality and the change of variable $s = \rho + t - r - \tau$ we obtain

$$\big(\overline{v}(\rho,\rho+t-r)\big)^2 \leq \bigg(\int_0^{R+t-r} k^2(s)ds\bigg)\bigg(\int_{\rho-R}^{\rho+t-r} (\overline{p}(\rho,\tau))^2 d\tau\bigg). \tag{4.168}$$

The region Σ can be represented as the union of half-lines

$$\Sigma = \bigcup_{c>R} \sigma_c, \tag{4.169}$$

where

$$\sigma_c = \left\{(\rho, c+\rho) \in \mathbb{R}^2, \text{ for } \rho > c > R\right\}, \tag{4.170}$$

with $c := t - r$ for a fixed $(r,t) \in \Sigma$, and defining a particular characteristic, see again Figure 4.4. Thus, fixing $(r,t) \in \Sigma$, and defining

$$\Gamma(c) := \int_0^{R+c} k^2(\rho)d\rho, \ \ \alpha(r) := \overline{v}(r, c+r), \ \text{ and } \ \beta(r) := \int_c^r \rho\alpha^2(\rho)d\rho, \tag{4.171}$$

from (4.168) we obtain

$$(\overline{v}(\rho,\rho+c))^2 \leq \int_{\rho-R}^{\rho+c} k^2(\rho+c-\tau)d\tau \int_{\rho-R}^{\rho+c} (\overline{p}(\rho,\tau))^2 d\tau = \Gamma(c)\int_{\rho-R}^{\rho+c} (\overline{p}(\rho,\tau))^2 d\tau. \tag{4.172}$$

Returning to (4.158), along the characteristic $(r, c+r) \in \Sigma$, we have

$$\alpha(r) \geq \frac{K}{4\pi r} + \frac{1}{2r}\int_c^r \rho\left(\int_{\rho-R}^{\rho+c} (\overline{p}(\rho,\tau))^2 d\tau\right)d\rho, \text{ for } r > c, \tag{4.173}$$

which, together with (4.172), and the fact that $\Gamma(c) > 0$ in all cases, leads to

$$\alpha(r) \geq \frac{K}{4\pi r} + \frac{1}{2r\Gamma(c)}\int_c^r \rho(\overline{v}(\rho,\rho+c))^2 d\rho = \frac{K}{4\pi r} + \frac{1}{2r\Gamma(c)}\beta(r) > 0, \text{ for } r > c, \tag{4.174}$$

or equivalently

$$r\alpha(r) \geq \frac{K}{4\pi} + \frac{1}{2\Gamma(c)}\int_c^r \rho\alpha^2(\rho)\, d\rho, \text{ for } r > c. \tag{4.175}$$

Suppose $W(r)$ is the solution of the equality in (4.175), i.e.

$$W(r) = \frac{K}{4\pi} + \frac{1}{2\Gamma(c)}\int_c^r \rho^{-1}W^2(\rho)d\rho, \ \text{ for } \ r > c. \tag{4.176}$$

It follows from Grönwall's lemma that $r\alpha(r) \geq W(r)$ for $r \geq c$. In detail, following the proof of Grönwall's lemma found in [BV], let

$$\Delta(r) := r\alpha(r) - W(r), \tag{4.177}$$

and

$$F(r) := \frac{1}{2\Gamma(c)} \int_c^r \frac{1}{\rho} \left(\rho\alpha(\rho) + w(\rho)\right) \Delta(\rho)\, d\rho. \tag{4.178}$$

From (4.175) and (4.176) we have

$$\begin{aligned} \Delta(r) &\geq \frac{1}{2\Gamma(c)} \int_c^r \frac{1}{\rho} \left((\rho\alpha)^2 - w^2\right) \, d\rho \\ &= \frac{1}{2\Gamma(c)} \int_c^r \frac{1}{\rho} \left(\rho\alpha(\rho) + w(\rho)\right) \Delta(\rho)\, d\rho \\ &= F(r), \end{aligned} \tag{4.179}$$

resulting in

$$F'(r) = \frac{\Delta(r)}{2\Gamma(c)} \left(\alpha(r) + \frac{w(r)}{r}\right) \geq \frac{1}{2\Gamma(c)} F(r) \left(\alpha(r) + \frac{w(r)}{r}\right). \tag{4.180}$$

Considering the integrating factor

$$\mu(r) = exp\left(-\frac{1}{2\Gamma(c)} \int_c^r \left(\alpha(\rho) + \frac{w(\rho)}{\rho}\right) \, d\rho\right), \tag{4.181}$$

it follows from (4.180) that

$$(\mu(r)F(r))' \geq 0. \tag{4.182}$$

Integrating from c to r and using the fact that $F(c) = 0$, we finally obtain

$$\mu(r)F(r) \geq 0, \tag{4.183}$$

which implies $F(r) \geq 0$ and, from (4.179), $\Delta(r) \geq 0$. Thus, we have

$$r\alpha(r) \geq W(r), \;\; \text{for} \;\; r \geq c. \tag{4.184}$$

In order to solve (4.176) for $W(r)$ we differentiate with respect to r and use the fundamental theorem of calculus to obtain

$$W'(r) = \frac{1}{2r\Gamma(c)} W^2(r). \tag{4.185}$$

This is a separable ODE which rearranging and integrating from c to r leads to

$$W(r) = \frac{K/(4\pi)}{1 - \frac{K}{8\pi\Gamma(c)} \log(r/c)}, \;\; \text{for} \;\; r \geq c, \tag{4.186}$$

where we crucially[1] use the fact that $W(c) = K/(4\pi) > 0$. This allows us to conclude that as $r \to r^*$, with

$$r^* = c \exp\left(\frac{8\pi\Gamma(c)}{K}\right), \tag{4.187}$$

we have $W(r^*) \to \infty$. This implies from (4.184) that r^* is an upper bound for the blow-up of $\alpha(r)$ and thus for $\overline{v}(r, c+r)$, which occurs along the characteristic defined by c satisfying $r > c > R$. Similarly, since $8\pi\Gamma(c)/K > 0$, blow-up happens at some $t \leq t^*$ for

$$t^* = r^* + c = c \exp\left(\frac{8\pi\Gamma(c)}{K}\right) + c \leq 2c \exp\left(\frac{8\pi\Gamma(c, R)}{K}\right), \tag{4.188}$$

[1] If $K = 0$ we no longer have $W(r) > 0$ in $r \geq c$, leading only to $W \equiv 0$ as a solution of (4.185).

recalling the dependence of Γ in R. Finally, blow-up of $\overline{v}$ implies blow-up of v since it cannot remain bounded when $\|x\| = r^*$ at $t = c + r^*$. ■

Remark 4.3.2. *Considering the characteristic defined by $c = R$, and recalling that R defines the size of the support of the initial conditions, notice that the upper bound on the time of blow-up (4.188) increases with R, for all possible conditions on a and b in (4.167). Thus, the smaller the support of the initial conditions the sooner we expect blow-up of the solution to happen.*

Example 4.3.1. *As mentioned at the beginning of the section, via the methods used in the proof above, Fritz John shows in [Joh90] that the following IVP*

$$\begin{cases} \square u = 2u_{tt}u_t, \\ u(x,0) = f(x) \in C_0^2(\mathbb{R}^3), \\ u_t(x,0) = g(x) \in C_0^2(\mathbb{R}^3), \end{cases} \tag{4.189}$$

with $f(x) = g(x) = 0$ for $\|x\| > R$ and satisfying

$$L = \frac{1}{4\pi}\int_{\mathbb{R}^3} \left(g(x) - g^2(x)\right) dx > 0, \tag{4.190}$$

has an upper bound for the time of blow-up given by

$$t^* = 2R\exp\left(\frac{16R^3}{3L}\right). \tag{4.191}$$

The above result is obtained by considering the characteristic $c = R$. Furthermore, since $L = K/4\pi$ as defined in [Joh90], and (4.102) is satisfied by taking $a = 0$ and $b = 1$, we have from the definition of Γ in (4.171)

$$\Gamma(c) = \int_0^{2R} s^2 ds = \frac{8R^3}{3}. \tag{4.192}$$

Thus, equation (4.188) becomes

$$t^* \leq 2c\exp\left(\frac{8\pi\Gamma}{K}\right) = 2R\exp\left(\frac{16R^3}{3L}\right), \tag{4.193}$$

verifying the result for this particular example.

We now consider the setting of Theorem 4.3.1 when $K \geq 0$. In this case, instead of obtaining an explicit blow-up result, one shows that a global C^2 solution must be the trivial solution.

Theorem 4.3.2. *If, in the conditions of Theorem 4.3.1, (4.104) is extended to the case $K \geq 0$, then $u \equiv 0$.*

Proof. As mentioned, for $K = 0$ we can not perform the steps leading to (4.186). Where before we used the fact that $W(c) = K/4\pi > 0$ in (4.176), it could now be the case that $W(r) = 0$ for $r > c$. Picking up from (4.174) we have

$$\overline{v}(r, c+r) = \alpha(r) \geq \frac{1}{2r\Gamma}\int_c^r \rho\alpha^2(\rho)d\rho = \frac{1}{2r\Gamma}\beta(r) \geq 0, \text{ for } r > c. \tag{4.194}$$

Since $\beta'(r) = r\alpha^2(r)$, taking the square of the inequality above we get

$$\beta'(r) = r\alpha^2(r) \geq \frac{1}{4r\Gamma^2}\beta^2(r), \text{ for } r > c. \tag{4.195}$$

Here, we either have $\beta(r) = 0$ for all $r > c$ or there must be an $r_0 > c$ such that $\beta(r) \geq \beta(r_0) > 0$ since $\beta'(r) > 0$ for all $r > r_0$. However, integrating (4.195) from r_0 to r we obtain

$$\begin{aligned} \int_{r_0}^{r} \frac{\beta'(s)}{\beta^2(s)} ds &\geq \frac{1}{4\Gamma^2} \int_{r_0}^{r} \frac{1}{s} ds \\ \Leftrightarrow \quad -\frac{1}{\beta(s)}\Bigg|_{s=r_0}^{s=r} &\geq \frac{1}{4\Gamma^2} \log\left(\frac{r}{r_0}\right), \end{aligned} \tag{4.196}$$

leading to

$$\frac{1}{\beta(r_0)} \geq \frac{1}{\beta(r_0)} - \frac{1}{\beta(r)} \geq \frac{1}{4\Gamma^2} \log\left(\frac{r}{r_0}\right), \text{ for } r > r_0. \tag{4.197}$$

Since by assumption v is globally defined, we can take $r \to \infty$ making the $\log(r/r_0)$ term simultaneously unbounded and bounded by $1/\beta(r_0)$, a contradiction. By the definition of β (4.171), this implies that $\alpha(r) = \overline{v}(r, c + r) = 0$ for $r > c > R$. Because the argument is valid for any $(r, t) \in \Sigma$ we conclude

$$\overline{v}(r, t) = 0, \text{ for } (r, t) \in \Sigma. \tag{4.198}$$

Since $\bar{B} \geq 0$, by assumption (4.102), (4.198) together with (4.152) and (4.154) (in the $K = 0$ case) implies that, on the following region

$$\Lambda := \bigcup_{(r,t)\in\Sigma} T^*_{r,t} = \{(\rho, \tau) \mid \rho > 0; \tau > 0; \rho + \tau > R\}, \tag{4.199}$$

$\overline{B}$ vanishes (see Figure 4.5).

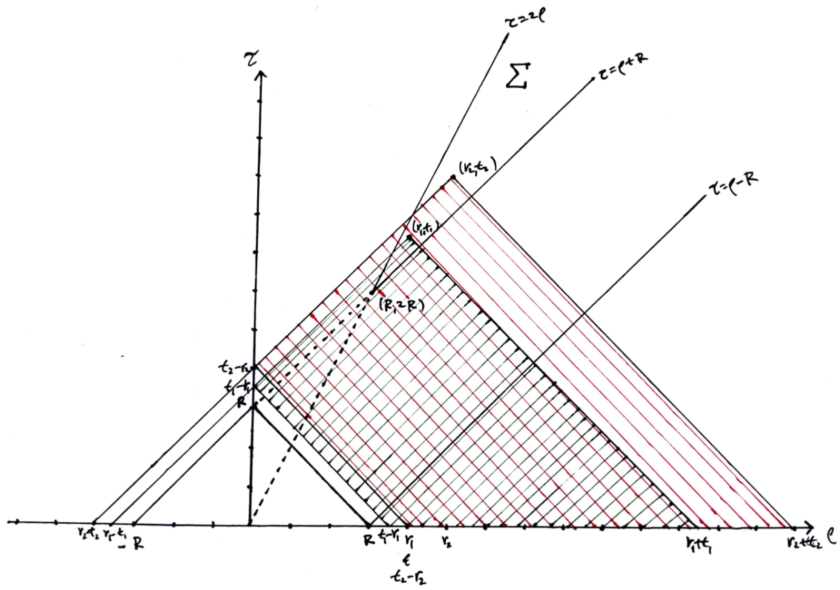

Figure 4.5: Points (r_i, t_i) with corresponding $T^*(r_i, t_i)$, $i = 1, 2$. The union for all (r, t) in Σ corresponds to the first quadrant minus $T_{0,R}$.

Again, by assumption (4.102), we conclude that

$$B(x,t,u(x,t),u'(x,t)) = 0, \text{ for } \|x\| + t > R, \, t > 0. \tag{4.200}$$

and thus, by (4.102), that

$$au(x,t) + bu_t(x,t) = 0, \text{ for } \|x\| + t > R, \, t > 0. \tag{4.201}$$

From (4.200), recalling also (4.98), (4.99) and (4.100), it follows that

$$\Box u = u_{tt} - \Delta u = A \geq 0, \text{ for } \|x\| + t > R, \, t > 0. \tag{4.202}$$

One can now show, from (4.201), (4.202) and (4.152), that

$$u(x,t) = 0, \text{ for } x \in \mathbb{R}^3, \, t > R. \tag{4.203}$$

If $b = 0$ and $a \neq 0$ this follows directly from (4.201) since

$$\left\{(x,t) \,|\, x \in \mathbb{R}^3, \, t > R\right\} \subset \{(x,t) \,|\, \|x\| + t > R, \, t > 0\}. \tag{4.204}$$

For $b \neq 0$ differentiating (4.201) with respect to t we have

$$au_t = -bu_{tt}, \tag{4.205}$$

and thus, using (4.201) again,

$$u_{tt} = \left(\frac{a}{b}\right)^2 u = m^2 u, \tag{4.206}$$

where we define $m := |a/b|$. By (4.202) we obtain the following PDE

$$m^2 u - \Delta u = A \geq 0, \text{ for } x \in \mathbb{R}^3, \, t > R, \tag{4.207}$$

using the set inclusion above (4.204).

Fix $t^* > R$ and $R^* > R + t^*$ and let $A(y) = A(y, t^*, u(y,t^*), u'(y,t^*), u''(y,t^*))$. From $u(y,t^*) = 0$ for $\|y\| > R + t^*$, one can show that (for the proof see Appendix (A.2.5))

$$u(x,t) = \frac{1}{(4\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{A(y)}{m} \underbrace{\left(\int_0^\infty \frac{e^{-s-\frac{m\|x-y\|^2}{4s}}}{s^{3/2}} ds\right)}_{(I)} dy \text{ for } \|x\| < R^*. \tag{4.208}$$

Remark 4.3.3. *For the solution of (4.207) Fritz John uses instead*

$$u(x;t^*) = \int_{\mathbb{R}^3} \frac{e^{-m\|x-y\|}}{4\pi\|x-y\|} A(y) dy, \text{ for } \|x\| < R^*, \tag{4.209}$$

which should be equivalent to (4.208). If $a = 0$, $b \neq 0$, the solution is given also by (4.209) with $m = 0$, corresponding to the solution of Poisson's equation.

Since $A(y) \geq 0$ for all y, from (4.100), and $u(x;t^*) = 0$ for $\|x\| > R + t^*$ from (4.110), since the integral in (4.208) is over all $\mathbb{R}^3$, and (I) in (4.208) is positive, it follows that A must vanish for all y. For a fixed $t^* > R$ we conclude then that

$$m^2 u(x,t^*) - \Delta u(x,t^*) = 0, \text{ for } x \in \mathbb{R}^3, \tag{4.210}$$

which, by (4.208), has solution $u(x,t^*) = 0$ for $\|x\| \leq t^* + R$.

Since, for each fixed $t > R$, $u(x,t)$ has compact support within $\|x\| \leq t + R$, we conclude that

$$u(x,t) = 0, \text{ for } x \in \mathbb{R}^3 \text{ and } t > R. \tag{4.211}$$

To finalize the argument one shows that $u(x,t) = 0$ for $x \in \mathbb{R}^3$ and $0 < t < R$ by going backwards in time. Letting $\tilde{u}(x,t) = u(-x, R-t)$ then $\tilde{u}_{tt} = u_{tt}$ and $\Delta\tilde{u} = \Delta u$, therefore $\tilde{u}$ is a solution of

$$\Box\tilde{u} = \phi(-x, R-t, \tilde{u}, -\tilde{u}', \tilde{u}''), \text{ for } x \in \mathbb{R}^3,\ 0 \leq t \leq R, \tag{4.212}$$

with

$$\tilde{u}(x,0) = \tilde{u}_t(x,0) = 0, \tag{4.213}$$

since $\tilde{u}(x,0) = u(-x,R) = 0$ and $\tilde{u}_t(x,0) = u_t(-x,R) = 0$ due to (4.211).

Remark 4.3.4. *One sets $x \to -x$ just so that on ϕ we can write the first derivatives of $\tilde{u}$ as $-\tilde{u}'$, otherwise $\tilde{u}_t$ and $\tilde{u}_{x_i}$ would have different signs.*

Since $\phi(-x, R-t, 0, 0, \tilde{u}'') = 0$, by the same reason as in (4.109), we are in the conditions of Corollary 4.1.2, from which it follows that $\tilde{u}(x,t) = 0$ for $x \in \mathbb{R}^3$, $0 \leq t \leq R$ and thus also $u(x,t) = 0$ for $x \in \mathbb{R}^3$, $0 \leq t \leq R$. Together with (4.211) we finally conclude that $u(x,t) = 0$ for $x \in \mathbb{R}^3$, $t \geq 0$. ■

The following theorem essentially "extends" this previous result for the time integral of the solution of (4.98) with the corresponding necessary conditions on the non-linearity.

Theorem 4.3.3. *Let $n = 3$ and consider $C = C(x,t,v,v',v'') \in C^3$, on all its arguments, where*

$$\exists\,(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\} \text{ s.t. } C(x,t,v,v',v'') \geq (av_t + bv_{tt})^2, \ \ \forall x,t,v,v',v'', \tag{4.214}$$

$$C(x,t,v,v',v'') = 0, \text{ for } v_t = v'_t = 0. \tag{4.215}$$

If $v \in C^3(\mathbb{R}^3 \times \mathbb{R}_+)$ is a solution of

$$\Box v = C(x,t,v,v',v''), \tag{4.216}$$

with $v(x,0), v_t(x,0)$ and $v_{tt}(x,0)$ of compact support, then $v \equiv 0$.

Proof. Let $u = v_t$, then

$$v(x,t) = v(x,0) + \int_0^t u(x,s)ds. \tag{4.217}$$

The time derivative of (4.216) leads to

$$\begin{aligned}\partial_t(\Box v) &= \partial_t\left(C(x,t,v,v',v'')\right)\\ \Leftrightarrow\ v_{ttt} - \Delta v_t &= \partial_t\left(C(x,t,v,v',v'')\right)\\ \Leftrightarrow\ \Box u &= \partial_t\left(C(x,t,v,v',v'')\right),\end{aligned} \tag{4.218}$$

where partial derivatives commute since $v \in C^3(\mathbb{R}^3 \times \mathbb{R}_+)$. Looking specifically at the dependence of C in u and u' we have

$$C(x,t,v,v',v'') = C(x,t,v,v_t,v_i,v_{tt},v_{ti},v_{ij}) = C(x,t,v,u,v_i,u_t,u_i,v_{ij}),\ \forall i,j = 1,2,3. \tag{4.219}$$

So, given $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$ and knowing v, we can define

$$B\left(x,t,u(x,t),u'(x,t)\right) := C\left(x,t,v(x,t),u(x,t),v_i(x,t),v_{tt}(x,t),u_i(x,t),v_{ij}(x,t)\right), \tag{4.220}$$

since a choice of (x,t) specifies the v, v_i and v_{ij}. In fact, we only care about B composed with the solution and its derivatives as a function of (x,t). From (4.216) we have

$$\Delta v = v_{tt} - C = u_t - C = u_t - B. \tag{4.221}$$

Thus, recalling the definition of K from Theorem 4.3.1, we have

$$\begin{aligned}K &= \int_{\mathbb{R}^3}\big(u_t(x,0) - B(x,0,v(x,0),u(x,0),v_i(x,0),v_{tt}(x,0),u_i(x,0),v_{ij}(x,0))\big)dx\\ &= \int_{\mathbb{R}^3}\Delta v(x,0)dx = \int_{\mathbb{R}^3}\nabla\cdot\nabla v(x,0)dx.\end{aligned} \tag{4.222}$$

Since v has compact support, at time $t=0$, using the divergence theorem and considering a compact region V containing the support of v, we obtain

$$K = \int_{\mathbb{R}^3}\nabla\cdot\nabla v(x,0)\,dx = \int_V \nabla\cdot\nabla v(x,0)\,dx = \int_{\partial V}\nabla v(x,0)\cdot\hat{n}\,dS = 0. \tag{4.223}$$

Thus, from assumptions (4.214, 4.215) together with the fact that $u(x,0) = v_t(x,0)$ and $u_t(x,0) = v_{tt}(x,0)$ have compact support, we are in the conditions of Theorem 4.3.2 for the wave equation (4.98), in the particular case where $A = 0$, and thus

$$\Box u = \partial_t\left(B(x,t,u(x,t),u'(x,t))\right). \tag{4.224}$$

By Theorem 4.3.2 then, we have $u(x,t) = v_t(x,t) = 0$ which implies $v_{tt}(x,t) = 0$ and $v_{ti}(x,t) = 0$ for $i = 1,2,3$, $t \geq 0$ and $x \in \mathbb{R}^3$. This, together with assumption (4.215), leads to

$$C(x,t,v,v',v'') = 0,\ \text{ for }\ t \geq 0,\ \ x \in \mathbb{R}^3. \tag{4.225}$$

and therefore, from (4.216),

$$\Box v = 0,\ \text{ for }\ t \geq 0,\ \ x \in \mathbb{R}^3. \tag{4.226}$$

Furthermore, because $v_{tt}(x,t) \equiv 0$, we have $\Delta v(x,t) = 0$ and in particular, at $t = 0$, we have

$$\Delta v(x,0) = 0. \tag{4.227}$$

Since $v(x,0)$ has compact support let V be a region such that $\operatorname{supp}(v(x,0)) \subset V$. By the divergence theorem we have

$$\int_{\mathbb{R}^3} \nabla \cdot \big(v(x,0)\nabla v(x,0)\big)dx = \int_V \nabla \cdot \big(v(x,0)\nabla v(x,0)\big)dx = \int_{\partial V} \big(v(x,0)\nabla v(x,0)\big) \cdot \hat{n} dS = 0, \tag{4.228}$$

which is also equal to

$$\int_V \nabla \cdot \big(v(x,0)\nabla v(x,0)\big)dx = \int_V \big(\nabla v(x,0)\big)^2 dx + \int_V v(x,0)\Delta v(x,0)dx = \int_V \big(\nabla v(x,0)\big)^2 dx, \tag{4.229}$$

since $\Delta v(x,0) = 0$ by (4.227), concluding that $\nabla v(x,0) = 0$. Thus, since v is continuous and zero outside the compact support, we conclude that $v(x,0) = 0$. Finally, since $v_t(x,t) = 0$, for $t \geq 0$ and $x \in \mathbb{R}^3$, we conclude that $v(x,t) = 0$ in $\mathbb{R}^3 \times \mathbb{R}_+$. ∎

Remark 4.3.5. *Note that the semi-linear wave-equation we considered in Section 4.2 can be obtained by taking*

$$C(x,t,v,v',v'') = (v_t)^2, \tag{4.230}$$

which satisfies all assumptions of the theorem with $b = 0$ and $a \geq 1$. For this particular example we have $K = 0$, as shown in equation (4.223) so we can not establish, via the proof of Theorem 4.3.1, an upper bound for the time of blow-up. However, Theorem 3 of [Joh85a], which we do not cover here, manages to obtain a lower and upper bounds for the time of blow-up for radially symmetric solutions of this particular equation.

In the following chapter we finally obtain (1.4) by considering the wave equation in (4.65) with a different geometrical background by considering a FLRW metric describing a decelerated expanding universe.

5

Semilinear Wave Equations on FLRW Spacetimes

The semilinear wave equation (4.65), considered in Section 4.2, can be seen as a special case of a more general parametrized wave equation depending on an underlying geometry specified by a choice of a Friedmann–Lemaître–Robertson–Walker (FLRW) metric. In particular, it can be seen as a special case of the following equation

$$\Box u = -\frac{2p}{(1-p)(t+1)} u_t + (u_t)^2, \tag{5.1}$$

in the $p = 0$ case, corresponding to the geometry of Minkowski spacetime, as shown in the following sections.

In this chapter we first compute the wave operator with an underlying geometry defined by a FLRW type metric for a general expansion term $a(t)$. Next, we set up the equivalent wave equation to (4.65), in this geometry, by performing a change of variables to conformal time. To finally obtain our equation of interest, we set a specific expansion term of the form $a(t) = t^p$, in order to obtain (5.1) which is the wave equation we want to understand in regards to blow-up phenomena for the particular case of decelerated expansions, i.e. for $p \in (0, 1)$. To perform this set-up we closely follow [CFO23] which, together with Proposition 4.2.1, are the main reason for studying (5.1) as we motivate later in Section 5.3.

5.1 Wave operator for a general FLRW metric

Consider a FLRW spacetime corresponding to the semi-Riemannian manifold $\mathcal{M} = (\mathbb{R}^n \times \mathbb{R}_+, \boldsymbol{g})$ with global coordinates $(t, x^1, ..., x^n)$ and metric given by

$$\boldsymbol{g} := -dt^2 + a^2(t)\sigma_{ij}(x)dx^i dx^j, \tag{5.2}$$

where σ_{ij} is a Riemannian metric in $\mathbb{R}^n$ and $a(t)$ corresponds to a scale factor of the universe defining in general whether it is static, expands or contracts. In our setting, $a(t)$ corresponds to an expansion, i.e.

$$\partial_t a > 0, \tag{5.3}$$

and we further assume that $a(t) > 0$ for all $t \geq 0$.

The covariant wave operator $\Box_{\boldsymbol{g}} u = \nabla^\mu \nabla_\mu u$, acting on a scalar function $u : M \to \mathbb{R}$, can be written in local coordinates as (see for example Proposition 2.46 of [Lee19])

$$\Box_{\boldsymbol{g}} u = \frac{1}{\sqrt{|\boldsymbol{g}|}} \partial_\alpha (\boldsymbol{g}^{\alpha\beta} \sqrt{|\boldsymbol{g}|} \partial_\beta u), \tag{5.4}$$

where $|\boldsymbol{g}| = -det(\boldsymbol{g}_{\alpha\beta}) > 0$ and $\boldsymbol{g}^{\alpha\beta}$ are the components of the inverse metric with matrix representations given by

$$(\boldsymbol{g}_{\alpha\beta}) = \left[\begin{array}{c|c} -1 & \mathbf{0}_{1\times n} \\ \hline \mathbf{0}_{n\times 1} & a^2 \sigma_{n\times n} \end{array}\right], \quad (\boldsymbol{g}^{\alpha\beta}) = \left[\begin{array}{c|c} -1 & \mathbf{0}_{1\times n} \\ \hline \mathbf{0}_{n\times 1} & a^{-2} \sigma^{-1}_{n\times n} \end{array}\right]. \tag{5.5}$$

For the metric in equation (5.2) we have

$$\begin{aligned} \Box_{\boldsymbol{g}} u &= \frac{1}{\sqrt{|\boldsymbol{g}|}} \left(\partial_\alpha (\boldsymbol{g}^{\alpha\beta}) \sqrt{|\boldsymbol{g}|} \partial_\beta u + \boldsymbol{g}^{\alpha\beta} \partial_\alpha (\sqrt{|\boldsymbol{g}|}) \partial_\beta u + \boldsymbol{g}^{\alpha\beta} \sqrt{|\boldsymbol{g}|} \partial_\alpha (\partial_\beta u) \right) \\ &= \underbrace{\partial_\alpha (\boldsymbol{g}^{\alpha\beta}) \partial_\beta u}_{(i)} + \underbrace{\frac{1}{\sqrt{|\boldsymbol{g}|}} \boldsymbol{g}^{\alpha\beta} \partial_\alpha (\sqrt{|\boldsymbol{g}|}) \partial_\beta u}_{(ii)} + \underbrace{\boldsymbol{g}^{\alpha\beta} \partial_\alpha (\partial_\beta u)}_{(iii)}. \end{aligned} \tag{5.6}$$

(i) When $\alpha = 0$, we have $\boldsymbol{g}^{\alpha\beta} \neq 0$ only if $\beta = 0$ which gives the term $\partial_t (\boldsymbol{g}^{00}) \partial_t u = 0$, since $\boldsymbol{g}^{00} = -1$. The remaining terms, for $\alpha \geq 1$, lead to $\partial_i (\boldsymbol{g}^{ij}) \partial_j u$, where $i, j = 1, ..., n$, and is equivalent to $a^{-2} \partial_i (\sigma^{ij}) \partial_j u$. Thus, term (i) becomes

$$\frac{1}{a^2} \partial_i (\sigma^{ij}) \partial_j u. \tag{5.7}$$

(ii) Note that $-det(\boldsymbol{g}_{\alpha\beta}) = det(a^2 \sigma_{ij}) = a^{2n} det(\sigma_{ij})$, thus, for this term we obtain

$$\begin{aligned} \frac{1}{a^n \sqrt{|\sigma|}} \boldsymbol{g}^{\alpha\beta} \partial_\alpha (a^n \sqrt{|\sigma|}) \partial_\beta u &= -\frac{1}{a^n \sqrt{|\sigma|}} (n a^{n-1} \dot{a} \sqrt{|\sigma|} + a^n \partial_t \sqrt{|\sigma|}) \partial_t u \\ &+ \frac{a^{-2} \sigma^{ij}}{a^n \sqrt{|\sigma|}} \partial_i (a^n \sqrt{|\sigma|}) \partial_j u, \end{aligned} \tag{5.8}$$

for $i, j = 1, ...n$, using $\boldsymbol{g}^{ij} = a^{-2} \sigma^{ij}$ and $|\sigma| = det(\sigma) > 0$. Since $|\sigma|$ does not depend on t and a does not depend on the x_i, $i = 1, ..., n$, term (ii) is finally given by

$$-\frac{n\dot{a}}{a} \partial_t u + \frac{\sigma^{ij}}{a^2 \sqrt{|\sigma|}} \partial_i (\sqrt{|\sigma|}) \partial_j u. \tag{5.9}$$

(iii) Again when $\alpha = 0$, we have $\boldsymbol{g}^{\alpha\beta} \neq 0$ only if $\beta = 0$ which gives the term $-\partial_t^2 u$, the remaining terms can be written as $\sigma^{ij} \partial_i \partial_j u$, where $i, j = 1, ..., n$. Thus, term (iii) becomes

$$-\partial_t^2 u + \frac{\sigma^{ij}}{a^2} \partial_i \partial_j u. \tag{5.10}$$

Finally, equation (5.6) becomes

$$\Box_{\boldsymbol{g}} u = \frac{1}{a^2} \partial_i (\sigma^{ij}) \partial_j u - \frac{n\dot{a}}{a} \partial_t u + \frac{\sigma^{ij}}{a^2 \sqrt{|\sigma|}} \partial_i (\sqrt{|\sigma|}) \partial_j u - \partial_t^2 u + \frac{\sigma^{ij}}{a^2} \partial_i \partial_j u, \tag{5.11}$$

which we can write as

$$\Box_{\boldsymbol{g}} u = -\partial_t^2 u - \frac{n\dot{a}}{a}\partial_t u + \frac{1}{a^2}\Delta_\sigma u, \tag{5.12}$$

where

$$\Delta_\sigma u = \frac{1}{\sqrt{|\sigma|}}\partial_i(\sigma^{ij}\sqrt{|\sigma|}\partial_j u). \tag{5.13}$$

Thus, equations (5.12) and (5.13) define the wave operator in local coordinates for general dimension n, scale factor $a(t)$ and Riemannian metric σ.

5.2 Obtaing the Semilinear Wave Equation (5.1)

We now set the equivalent wave equation to (4.65) using the wave operator $\Box_{\boldsymbol{g}}$ obtained in the previous section. Given the sign of the metric in (5.2), i.e. $(-,+,+,+)$, in order to recover equation (4.65) the nonlinearity ϕ we need to consider is given by

$$\phi(x,t,u,u',u'') = -(u_t)^2, \tag{5.14}$$

resulting in the following wave equation

$$\Box_{\boldsymbol{g}} u = -(u_t)^2. \tag{5.15}$$

From (5.12), using the assumption $a(t) > 0$, we can write

$$\frac{1}{a^2(t)}\left(a^2(t)\partial_t^2 u - \Delta_\sigma u\right) = -\frac{n\partial_t a \partial_t u}{a(t)} + (\partial_t u)^2. \tag{5.16}$$

Considering a change of variable $t \to \tau$ given by

$$\frac{d\tau}{dt} = \frac{1}{a(t)}, \quad \tau(t) = \tau(t_0) + \int_{t_0}^{t} \frac{1}{a(s)}ds, \quad t_0 > 0, \tag{5.17}$$

where τ is referred to as conformal time, we have

$$\begin{aligned}
\partial_t u &= (\partial_\tau u)\frac{d\tau}{dt} = \frac{\partial_\tau u}{a(t)},\\
\partial_t a &= \frac{\partial_\tau a}{a(t)},\\
\partial_t^2 u &= (\partial_\tau^2 u)\left(\frac{d\tau}{dt}\right)^2 + (\partial_\tau u)\frac{d^2\tau}{dt^2} = (\partial_\tau^2 u)\left(\frac{1}{a(t)}\right)^2 - \left(\frac{\partial_\tau u \partial_\tau a}{a^3(t)}\right),
\end{aligned} \tag{5.18}$$

and (5.16) becomes

$$\begin{aligned}
\frac{1}{a^2}\left(a^2\left(\frac{1}{a^2}\partial_\tau^2 u - \partial_\tau u\frac{\partial_\tau a}{a^3}\right) - \Delta_\sigma u\right) &= -\frac{n\partial_\tau a \partial_\tau u}{a^3} + (\partial_\tau u)^2\frac{1}{a^2}\\
\Leftrightarrow \qquad \partial_\tau^2 u - \Delta_\sigma u &= \frac{(1-n)\partial_\tau a \partial_\tau u}{a(\tau)} + (\partial_\tau u)^2,
\end{aligned} \tag{5.19}$$

leading to

$$\partial_\tau^2 u - \Delta_\sigma u = (1-n)\frac{\partial_\tau a(\tau)}{a(\tau)}\partial_\tau u + (\partial_\tau u)^2. \tag{5.20}$$

This sets a general version of our equation of interest for arbitrary dimension n, scale factor $a(t)$, satisfying $a(t) \geq 0$ for $t > 0$, and Riemannian metric σ.

To finally obtain (5.1) we consider (5.20) with $n = 3$, with matrix representation for σ given by $I_{3\times 3}$ and $a(t) = t^p$ for $p \in [0, 1)$. The metric in (5.2) becomes

$$\boldsymbol{g} := -dt^2 + t^{2p}\sum_{i=0}^{3}(dx^i)^2. \tag{5.21}$$

for $i = 1, 2, 3$. Using equation (5.17) we have

$$\tau(t) = \frac{t^{1-p}}{1-p} - \frac{t_0^{1-p}}{1-p} + \tau(t_0), \tag{5.22}$$

where we define

$$\theta_0(p) := \frac{t_0^{1-p}}{1-p} - \tau(t_0), \tag{5.23}$$

with $p \in [0, 1)$. Solving for $t(\tau)$ we obtain

$$t(\tau) = [(1-p)(\tau + \theta_0)]^{\frac{1}{1-p}}. \tag{5.24}$$

Thus

$$a(\tau) = t(\tau)^p = [(1-p)(\tau + \theta_0)]^{\frac{p}{1-p}}, \tag{5.25}$$

and

$$\partial_\tau a(\tau) = p\Big[(1-p)(\tau + \theta_0)\Big]^{\frac{2p-1}{1-p}}, \tag{5.26}$$

so that

$$\frac{\partial_\tau a(\tau)}{a(\tau)} = \frac{p}{(1-p)(\tau + \theta_0)}, \tag{5.27}$$

and finally from (5.20)

$$\Box u = \partial_\tau^2 u - \Delta u = -\frac{2p}{(1-p)(\tau + \theta_0)}\partial_\tau u + (\partial_\tau u)^2. \tag{5.28}$$

To recover (5.1) we simply set $\theta_0 = 1$, for all p, which we can obtain by fixing an appropriate $t_0 > 0$ and by considering $\tau(t_0) = 0$. In particular,

$$t_0 = (1-p)^{\frac{1}{1-p}}. \tag{5.29}$$

5.3 Motivation for Studying Equation (5.1)

As mentioned in the Introduction, for $n = 3$, the reasons for studying equation (5.1) are the following:

- for $p = 0$, which corresponds to Minkowski spacetime, i.e. $a(t) = 1$ and matrix representation for σ given by the identity $I_{3\times 3}$, we obtain

$$-\Box_{\boldsymbol{g}} u = \Box u = (\partial_t u)^2, \tag{5.30}$$

  which falls within Fritz John's corollary to Theorem 4.3.2, as shown by Proposition 4.2.1 and Remark 4.3.5. Thus, for non-trivial initial conditions of compact support, solutions of (5.30) blow up in finite time.

- for $p > 1$, corresponding to an accelerated expansion with $a(t) = t^p$ and again matrix representation for σ given by $I_{3\times 3}$, for (5.2), Theorem 2.1 and Remark 2.2 of [CFO23] show that $\Box_{\boldsymbol{g}} = -(\partial_t u)^2$ admits global non-trivial solutions for small initial data as a consequence of the accelerated character of the expansion.

Our goal is to understand what happens for $p \in (0, 1)$, in regards to global existence of solutions to the IVP associated to (5.28). In particular, the remaining sections of this chapter are devoted to (so far unsuccessful) attempts at proving blow-up of solutions for this IVP in different settings, first attempting a continuity argument for small p and a second attempt using the method by Klainerman and Sarnack when $p = 1/2$, following [NR23].

5.4 Solution Attempts

In this section we tackle the problem of understanding blow-up for solutions to the IVP in 3+1 dimensions associated to the following wave equation

$$\Box v(x,t) = -\frac{\gamma(p)}{t+1} v_t(x,t) + (v_t(x,t))^2, \quad \text{for } p \in (0,1), \tag{5.31}$$

with initial data $v|_{t=0}$ and $v_t|_{t=0}$ of compact support and where $\gamma(p) = 2p/(1-p)$. Let

$$\phi(v') := -\frac{\gamma(p)}{t+1} v_t + (v_t)^2. \tag{5.32}$$

Notice that this equation does not fall in the setting of previous theorems by Fritz John. First, it immediately fails to satisfy (4.55). Furthermore, assumption (4.214) no longer holds, we do not have non-negativity of ϕ, as in Theorem 4.3.3, see Figure 5.1.

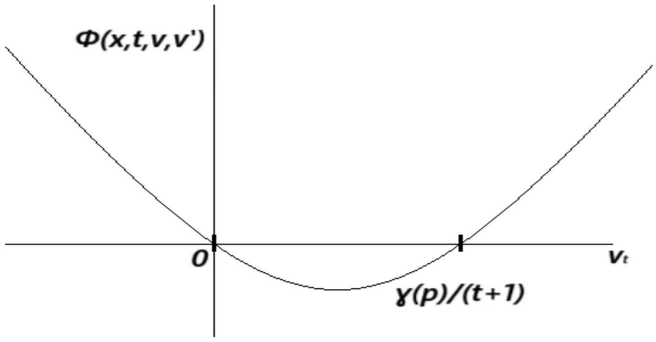

Figure 5.1: The right-hand side of (5.28) as a function of v_t.

In the following subsection we try to adapt the ideas used in the proof of Theorems 4.3.1 and 4.3.2 in an attempt to prove blow-up for small p. In particular, we comment on the obstructions that prevent us from continuing a similar argument.

5.4.1 Attempt for small $p > 0$

Let $v \in C^3(\mathbb{R}^3 \times \mathbb{R}_+)$ be a solution to the IVP associate to equation (5.31) for initial data of compact support. First we summarize the steps which we can still use in this case.

1. The nonlinearity $\phi(v')$ is C^2 in its arguments and satisfies $\phi(0) = 0$. Thus, we are in the setting of **Corollary** 4.1.2 (uniqueness theorem) and we have $v(x,t) = 0$, for $\|x\| > t + R$, $t \geq 0$, where $R > 0$ is such that both *supp*$(v(x,0))$, and *supp*$(v_t(x,0))$ are contained in $B(0,R)$.

2. Considering the **spherical average** of equation (5.31), by the Euler-Poisson-Darboux equation (3.19) we obtain

$$\Box(r\bar{v}) = -\frac{\gamma(p)}{t+1} r\bar{v}_t + r\overline{v_t^2}, \tag{5.33}$$

   where we are again considering the symmetric extension as in (4.131).

3. The solution to the 1+1-dimensional wave equation (5.33) is obtained by **Duhamel's principle** (3.26) and is given by

$$\bar{v}(r,t) = \bar{v}^0(r,t) + \frac{1}{2r}\int\int_{T_{r,t}} \left(-\frac{\gamma(p)\rho}{\tau+1}\bar{v}_\tau(\rho,\tau) + \rho\overline{v_\tau^2}(\rho,\tau)\right) d\tau d\rho, \tag{5.34}$$

   for $(r,t) \in \mathbb{R} \times \mathbb{R}_+$ and where $\bar{v}^0(r,t)$ is given by (4.143).

4. Considering the same **region** Σ as in (4.145) and due to parity of the integrand and symmetry as before, for $(r,t) \in \Sigma$ we have $\bar{v}^0(r,t) = 0$ and we can reduce the above to

$$\bar{v}(r,t) = \frac{1}{2r}\int\int_{T^*_{r,t}} \left(-\frac{\gamma(p)\rho}{\tau+1}\bar{v}_\tau(\rho,\tau) + \rho\overline{v_\tau^2}(\rho,\tau)\right) d\tau d\rho, \tag{5.35}$$

for $(r,t) \in \Sigma$. Using Cauchy-Schwarz, just as in (4.134), and commutation of time derivatives and the spherical average we can further write

$$\bar{v}(r,t) \geq \frac{1}{2r} \int\int_{T^*_{r,t}} \left(-\frac{\gamma(p)\rho}{\tau+1}\bar{v}_\tau(\rho,\tau) + \rho\bar{v}^2_\tau(\rho,\tau) \right) d\tau d\rho, \tag{5.36}$$

for $(r,t) \in \Sigma$.

5. From $(a-b)^2 \geq 0$ with $a = 1/(\tau+1)$ and $b = \overline{v}_\tau$ we get

$$-\frac{\overline{v}_\tau}{\tau+1} \geq -\frac{1}{2}\left(\overline{v}^2_\tau + \frac{1}{(\tau+1)^2}\right), \tag{5.37}$$

so we can write (5.36) as

$$\overline{v}(r,t) \geq \frac{1}{2r}\left(1 - \frac{\gamma(p)}{2}\right)\int\int_{T^*_{r,t}} \rho\overline{v}^2_\tau d\tau d\rho - \frac{\gamma(p)}{4r}\int\int_{T^*_{r,t}} \frac{\rho}{(\tau+1)^2} d\tau d\rho, \tag{5.38}$$

for $(r,t) \in \Sigma$ and where $c = t - r$ defines the characteristic.

Remark 5.4.1. *Estimate (5.37) can probably be improved upon.*

6. To guarantee non-negativity of the left term on the right we choose, from this point onward, $1 - \gamma(p)/2 \geq 0$, i.e. we restrict p such that $p \leq 1/2$. In this way the integral on the left can be restricted to $S_{r,t}$, as defined in (4.157). Thus, for $p \in [0, 1/2]$, we have

$$\overline{v}(r,t) \geq \frac{1}{2r}\left(1 - \frac{\gamma(p)}{2}\right)\int_c^r \int_{\rho-R}^{\rho+c} \rho\overline{v}^2_\tau d\tau d\rho - \frac{\gamma(p)}{4r}\int\int_{T^*_{r,t}} \frac{\rho}{(\tau+1)^2} d\tau d\rho, \tag{5.39}$$

for $(r,t) \in \Sigma$.

7. Using the fundamental theorem of calculus and the fact that $\overline{v}(\rho, \rho - R) = 0$ we have again

$$\int_{\rho-R}^{\rho+c} \overline{v}_\tau d\tau = \overline{v}(\rho, \rho+c), \tag{5.40}$$

which, using Cauchy-Schwarz's inequality as before, leads to

$$\overline{v}^2(\rho,\rho+c) = \left(\int_{\rho-R}^{\rho+c} \overline{v}_\tau d\tau\right)^2 \leq \left(\int_{\rho-R}^{\rho+c} 1 d\tau\right)\left(\int_{\rho-R}^{\rho+c} \overline{v}^2_\tau d\tau\right), \tag{5.41}$$

thus

$$\int_{\rho-R}^{\rho+c} \overline{v}^2_\tau d\tau \geq \frac{1}{\Gamma(c)}\left(\int_{\rho-R}^{\rho+c} \overline{v}_\tau d\tau\right)^2 = \frac{1}{\Gamma(c)}\overline{v}^2(\rho,\rho+c), \tag{5.42}$$

where $\Gamma(c) = c + R$.

8. So, for a fixed $(r,t) \in \Sigma$, along the characteristic defined by $c = t - r$ we can write

$$\overline{v}(r, c+r) \geq \frac{1}{2r\Gamma}\left(1 - \frac{\gamma(p)}{2}\right)\int_c^r \rho\overline{v}^2(\rho,\rho+c) d\rho - \frac{\gamma(p)}{4r}\int\int_{T^*_{r,c+r}} \frac{\rho}{(\tau+1)^2} d\tau d\rho. \tag{5.43}$$

All terms featuring $\gamma(p)$ are new compared to the equivalent expression (4.194) in the proof of Theorem 4.3.1. Rearranging terms to have non-negative quantities we get

$$\overline{v}(r,c+r)+\frac{\gamma(p)}{4r}\int\int_{T^*_{r,c+r}}\frac{\rho}{(\tau+1)^2}d\tau d\rho\geq\frac{1}{2r\Gamma}\left(1-\frac{\gamma(p)}{2}\right)\int_c^r\rho\overline{v}^2(\rho,\rho+c)d\rho\geq 0, \tag{5.44}$$

for $r>c>R$ and $p\in[0,1/2]$. Defining again $\alpha(r):=\overline{v}(r,c+r)$ and $\beta(r):=\int_c^r\rho\overline{v}^2(\rho,\rho+c)d\rho$ we have as before

$$\beta'(r)=r\alpha^2(r)\geq 0. \tag{5.45}$$

Let also

$$f(r):=\frac{1}{4}\int\int_{T^*_{r,c+r}}\frac{\rho}{(\tau+1)^2}d\tau d\rho, \tag{5.46}$$

then (5.44) becomes

$$\alpha(r)+\gamma(p)\frac{f(r)}{r}\geq\frac{1}{2r\Gamma}\left(1-\frac{\gamma(p)}{2}\right)\beta(r)\geq 0. \tag{5.47}$$

Squaring (5.47) and rearranging leads to

$$\alpha^2(r)\geq-\gamma^2(p)\frac{f^2(r)}{r^2}-2\frac{\gamma(p)}{r}\alpha(r)f(r)+\frac{1}{4r^2\Gamma^2}\left(1-\frac{\gamma(p)}{2}\right)^2\beta^2(r), \tag{5.48}$$

which, using $-2\alpha(r)f(r)\geq-\alpha^2(r)-f^2(r)$ and rearranging again, leads to

$$\alpha^2(r)\left(1+\frac{\gamma(p)}{r}\right)\geq-\frac{\gamma(p)}{r}f^2(r)\left(1+\frac{\gamma(p)}{r}\right)+\frac{1}{4r^2\Gamma^2}\left(1-\frac{\gamma(p)}{2}\right)^2\beta^2(r). \tag{5.49}$$

Multiplying by $r/(1+\gamma(p)/r)$ and using (5.45) we finally obtain

$$\beta'(r)=r\alpha^2(r)\geq-\gamma(p)f^2(r)+\frac{1}{16\Gamma^2}\frac{(2-\gamma(p))^2}{r+\gamma(p)}\beta^2(r). \tag{5.50}$$

Note that for $p=0$, and thus $\gamma(p)=0$, we recover exactly (4.197) as expected.

We now have two directions for continuing the argument. As in the proof of Theorem (4.3.1), where $K>0$, we can solve for the equality in (5.50), a Riccati ordinary differential equation, and try to obtain blow-up directly or, as in the proof of Theorem (4.3.2), where $K=0$, argue that a global C^2 solution must be trivial.

Arguing for the trivial solution: Explicitly noting the underlying dependence in p from now on, we want to show $\beta(r,p)=0$, for $r>c$, for some $p\in(0,1/2]$. Assume it is false for all $p\in(0,1/2]$, i.e. there exists $r_0(p)>c$ such that $\beta(r_0(p);p)>0$ for all $p\in(0,1/2]$. Thus, from the fact that $\beta'(r;p)\geq 0$, equation (5.45), we have $\beta(r;p)\geq\beta(r_0(p);p)>0$ for $r>r_0(p)$. Dividing (5.50) by $\beta^2(r;p)$ and integrating from $r_0(p)$ to r we obtain

$$\begin{aligned}&\int_{r_0(p)}^r\frac{\beta'(s)}{\beta^2(s;p)}ds\geq\frac{(2-\gamma(p))^2}{16\Gamma^2}\int_{r_0(p)}^r\frac{1}{s+\gamma(p)}ds-\gamma(p)\int_{r_0(p)}^r\frac{f^2(s)}{\beta^2(s)}ds\\ \Leftrightarrow\quad&\frac{1}{\beta(r_0(p))}-\frac{1}{\beta(r)}\geq\frac{(2-\gamma(p))^2}{16\Gamma^2}\log\left(\frac{r+\gamma(p)}{r_0(p)+\gamma(p)}\right)-\gamma(p)\int_{r_0(p)}^r\frac{f^2(s)}{\beta^2(s)}ds.\end{aligned} \tag{5.51}$$

Which, rearranging terms, leads to

$$\beta(r;p) \geq \left(\frac{1}{\beta(r_0(p);p)} - \frac{(2-\gamma(p))^2}{16\Gamma^2} \log\left(\frac{r+\gamma(p)}{r_0(p)+\gamma(p)} \right) + \gamma(p) \int_{r_0(p)}^{r} \frac{f^2(s)}{\beta^2(s;p)} ds \right)^{-1}. \tag{5.52}$$

For a fixed $p \in (0, 1/2]$ we fix $\tilde{r} > r_0(p)$ such that

$$\frac{1}{\beta(r_0(p);p)} - \frac{(2-\gamma(p))^2}{16\Gamma^2} \log\left(\frac{\tilde{r}+\gamma(p)}{r_0(p)+\gamma(p)} \right) < 0, \tag{5.53}$$

which is clearly possible since the function is strictly decreasing and unbounded below. Over the integral term

$$\gamma(p) \int_{r_0(p)}^{\tilde{r}} \frac{f^2(s)}{\beta^2(s;p)} ds, \tag{5.54}$$

we have $\beta^2(s;p) \geq \beta^2(r_0(p);p) > 0$ for $s > r_0(p)$, it does not vanish. The term $f^2(s)$, recalling (5.46), increases with s, however we can control the integral by making p small enough via the $\gamma(p)$ factor. The goal would be to show that the denominator on the right-hand side of (5.52) starts positive at $r = r_0$, with value $1/\beta(r_0(p);p) > 0$, and eventually becomes negative for some $r \leq \tilde{r}$ by choosing p small enough.

The problem however is that changing p to a new (small) p^* leads to a new $\beta(r_0(p^*);p^*)$, possibly requiring a different (larger) $\tilde{r}$. Furthermore, $\beta^2(s;p^*)$ can become arbitrarily small. Thus, it seems we can not control the integral term (5.54) in this way.

In any case, even if $\beta(r;p) = 0$ for $r > c$ and $p \in (0, 1/2]$, which implies $\overline{v}(r,t) = 0$ for $(r,t) \in \Sigma$ via the definition of $\beta(r)$, from (5.38) and considering $r \neq 0$, we conclude

$$\frac{\gamma(p)}{2-\gamma(p)} \int\int_{\Lambda} \frac{\rho}{(\tau+1)^2} d\tau d\rho \geq \int\int_{\Lambda} \rho \overline{v}_{\tau}^2(\rho,\tau) d\tau d\rho \geq 0, \tag{5.55}$$

where Λ is given by (4.199). Or, from (5.35), that

$$\gamma(p) \int\int_{\Lambda} \frac{\rho \overline{v}_{\tau}}{(\tau+1)^2} d\tau d\rho = \int\int_{\Lambda} \rho \overline{v_{\tau}^2} d\tau d\rho. \tag{5.56}$$

This seems much weaker than the setting of Theorem 4.3.2. There, we have $\overline{B} \geq 0$, i.e. non-negativity on the right-hand side of (5.36), and also $B \geq 0$ allowing us to conclude that $B = 0$ in Λ and setting up (4.201).

Estimating the ODE: From (5.50) write

$$\beta'(r) \geq q_2(r)\beta^2(r) - q_1(r), \tag{5.57}$$

where

$$\begin{aligned} q_2(r) &:= \frac{1}{16\Gamma^2} \frac{(2-\gamma(p))^2}{r+\gamma(p)}, \\ q_1(r) &:= \gamma(p) f^2(r) \end{aligned} \tag{5.58}$$

We now have the following fact: the function $w(r)$ which satisfies equality in (5.57), i.e.

$$w'(r) = q_2(r)w^2(r) - q_1(r), \tag{5.59}$$

is such that we have

$$\beta(r) \geq w(r), \tag{5.60}$$

for $r \geq r_0 > c$, provided that $\beta(r_0) \geq w(r_0)$, the proof can be seen in Appendix (A.3.1). Note that (5.59), a Riccati equation, does not admit a trivial solution for $q_1(r) \neq 0$. We convert (5.59) to a second order linear ODE.

Let $v(r) = q_2(r)w(r)$, then

$$\begin{aligned} v'(r) &= q_2'(r)w(r) + q_2(r)w'(r) = q_2'(r)w(r) + q_2^2(r)w^2(r) - q_2(r)q_1(r) \\ &= \frac{q_2'(r)}{q_2(r)}v(r) + v^2(r) - q_2(r)q_1(r) \end{aligned} \tag{5.61}$$

recalling that $q_2(r) > 0$ for $r > c$ and $p \in (0, 1/2)$. We now consider solutions v of the form $v = -u'/u$ obtaining from (5.61) the second order linear homogeneous ODE with variable coefficients

$$u''(r) - \frac{q_2'}{q_2}(r)u'(r) - q_2(r)q_1(r)u(r) = 0. \tag{5.62}$$

In order to write this equation explicitly we first compute the $f(r)$ term defined in (5.46). The following figure shows the regions of integration.

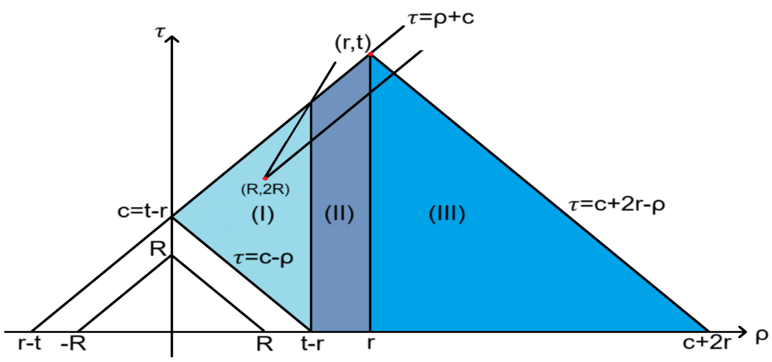

Figure 5.2: Regions of integration for determining $f(r)$.

Separating the integral in the three regions shown in Figure 5.2 and recalling that $c = t - r$ we have

$$\begin{aligned} f(r) &= \frac{1}{4}\left(\int\int_{(I)} \frac{\rho}{(\tau+1)^2}d\tau d\rho + \int\int_{(II)} \frac{\rho}{(\tau+1)^2}d\tau d\rho + \int\int_{(III)} \frac{\rho}{(\tau+1)^2}d\tau d\rho\right) \\ &= \frac{1}{4}\left(\underbrace{\int_0^c \int_{c-\rho}^{c+\rho} \frac{\rho}{(\tau+1)^2}d\tau d\rho}_{(I)} + \underbrace{\int_c^r \int_0^{c+\rho} \frac{\rho}{(\tau+1)^2}d\tau d\rho}_{(II)} + \underbrace{\int_r^{c+2r} \int_0^{c+2r-\rho} \frac{\rho}{(\tau+1)^2}d\tau d\rho}_{(III)}\right) \end{aligned} \tag{5.63}$$

1. **Region (I)**: Region I is compact so the integral takes some finite value depending on c. In particular we have

$$\begin{aligned}\int_0^c \int_{c-\rho}^{c+\rho} \frac{\rho}{(\tau+1)^2} d\tau d\rho &= \int_0^c \left(\frac{\rho}{c+1-\rho} - \frac{\rho}{c+1+\rho} \right) d\rho \\ &= ((c+1)\log(c+1) - c) - (-(c+1)\log((2c+1)/(c+1)) + c) \\ &= (c+1)\log(2c+1) - 2c.\end{aligned} \tag{5.64}$$

2. **Region (II)**: We have

$$\begin{aligned}\int_c^r \int_0^{c+\rho} \frac{\rho}{(\tau+1)^2} d\tau d\rho &= \int_c^r \left(\rho - \frac{\rho}{c+1+\rho} \right) d\rho \\ &= \frac{1}{2}(r^2 - c^2) - (r+c+1) + (c+1)\log(r+c+1) + (2c+1) - (c+1)\log(2c+1) \\ &= \frac{1}{2}(r^2 - c^2) - (r-c) + (c+1)\log\left(\frac{c+r+1}{2c+1} \right).\end{aligned} \tag{5.65}$$

3. **Region (III)**: We have

$$\begin{aligned}\int_r^{c+2r} \int_0^{c+2r-\rho} \frac{\rho}{(\tau+1)^2} d\tau d\rho &= \int_r^{c+2r} \left(\rho - \frac{\rho}{c+2r+1-\rho} \right) d\rho \\ &= \frac{1}{2}\left((c+2r)^2 - r^2\right) + (c+r) - (c+2r+1)\log(c+r+1).\end{aligned} \tag{5.66}$$

Thus, we conclude that

$$f(r) = \frac{r^2}{2} + \frac{cr}{2} - \frac{r}{2}\log(c+r+1). \tag{5.67}$$

Re-writing (5.62) we obtain

$$u'' + \left(\frac{1}{r+\gamma(p)} \right) u' - \frac{\gamma(p)}{16\Gamma^2} \frac{(2-\gamma(p))^2}{r+\gamma(p)} \left(\frac{r^2}{2} + \frac{cr}{2} - \frac{r}{2}\log(c+r+1) \right)^2 u = 0. \tag{5.68}$$

In particular, since $v(r) = -u'(r)/u(r)$, we want to determine zeroes of $u(r)$ occurring at some point $r > r_0 > c$.

ODE parameter continuity: Recall the definition of $\beta_0(r)$ obtained in the proof of proposition (4.2.1)

$$\beta_0(r) = \int_c^r \rho \overline{v}^2(\rho, \rho + c) d\rho, \tag{5.69}$$

where the included index reflects its correspondence to the $p = 0$ version of (5.31). At a certain point in the proof we either have $\beta_0(r) = 0$ in Σ, from which we conclude that $v = 0$ forcing trivial initial conditions or, for non-trivial initial conditions, there exists $r_0 > c$ such that $\beta_0(r_0) > 0$, over some characteristic $c = t - r$ fixed henceforth.

Considering equality in (4.88) we set the following IVP

$$W_0'(r) = \frac{1}{4r\Gamma^2} W_0^2(r), \;\; W_0(r_0) = \beta_0(r_0) > 0, \tag{5.70}$$

with solution given by

$$W_0(r) = \frac{1}{\frac{1}{W_0(r_0)} - \frac{1}{4\Gamma^2}\log\left(\frac{r}{r_0}\right)}, \tag{5.71}$$

which exists up to

$$\tilde{r} = r_0 \exp\left(\frac{4\Gamma^2}{W_0(r_0)}\right). \tag{5.72}$$

We consider the interval of existence of $W_0(r)$ within a compact set $[r_0, r_1]$ for some $r_0 < r_1 < \tilde{r}$. Recall that (5.70) corresponds to equality of

$$\beta'(r) = r\alpha^2(r) \geq -\gamma(p)f^2(r) + \frac{1}{16\Gamma^2}\frac{(2-\gamma(p))^2}{r+\gamma(p)}\beta^2(r), \tag{5.73}$$

obtained in (5.50), in the $p = 0$ case. Considering equality for the above equation we obtain the following ODE

$$W_p'(r) = -\gamma(p)f^2(r) + \frac{1}{16\Gamma^2}\frac{(2-\gamma(p))^2}{r+\gamma(p)}W_p^2(r). \tag{5.74}$$

We now make use of the following theorem, from [CL55], pertaining to continuous variation of solutions of ODEs with respect to parameters

Theorem 5.4.1. *Let D be a domain of (t,x) space, I_μ the domain $|\mu - \mu_0| < c$, $c > 0$, and D_μ the set of all (t,x,μ) satisfying $(t,x) \in D$, $\mu \in I_\mu$. Suppose f is a continuous function on D_μ bounded by a constant M there. For $\mu = \mu_0$ let*

$$x' = f(t,x,\mu) \quad x(\tau) = \xi \quad (4.1) \tag{5.75}$$

have a unique solution φ_0 on the interval $[a,b]$, where $\tau \in [a,b]$. Then there exists a $\delta > 0$ such that, for any fixed μ satisfying $|\mu - \mu_0| < \delta$, every solution φ_μ of (4.1) exists over $[a,b]$ and as $\mu \to \mu_0$

$$\varphi_\mu \to \varphi_0 \tag{5.76}$$

uniformly over $[a,b]$.

Considering both IVPs in $W_0(r)$ and $W_p(r)$ from (5.71) and (5.74) of the form

$$\begin{cases} W_0'(r) = g(r, W_0, p = 0), \\ W_p'(r) = g(r, W_p, p), \end{cases} \tag{5.77}$$

with initial condition $W_p(r) = W_0(r) = \beta_0(r_0) > 0$. From the above theorem, since g is continuous in its arguments and bounded for $|p| < 1/2$ and $r \in [r_0, r_1]$, given $\epsilon > 0$ there exists $\delta > 0$ such that if $|p| < \delta < 1/2$ we have

$$|W_p(r) - W_0(r)| < \epsilon. \tag{5.78}$$

If we let $\epsilon = W_0(r_0)/2$ there exists $\delta_0 > 0$ such that for $|p| < \delta_0 < 1/2$ we have in particular

$$0 < W_0(r_0)/2 < W_p(r_0) < (3/2)W_0(r_0). \tag{5.79}$$

We would now like to compare $W_p(r)$ with $\beta_p(r)$, for $r > r_0$, by using the result proved in (A.3.1) which requires an initial condition. However, since we only know $\beta_p(r)$ at c, where $\beta_p(c) = 0$ by definition, this would require the previous argument to be considered at c, instead of r_0, where $\beta_0(c) = 0$ forcing $W_0(r) = 0$ resulting only in $|W_p(r)| < \epsilon$ from the continuity argument, we lose the positivity result we wanted.

In any case we assume this result holds from now on (we have hope that the following may still be useful in a later argument). Suppose that, for $|p| < \delta_0$, $\beta_p(r) \geq W_p(r)$ for $r > r_0$, where $W_p(r_0) > W_0(r_0)/2$ and $\beta_p'(r) \geq 0$, from its definition. Recall from (5.52) that

$$\beta_p(r) \geq \left(\frac{1}{\beta_p(r_0)} - \frac{(2-\gamma(p))^2}{16\Gamma^2} \log\left(\frac{r+\gamma(p)}{r_0+\gamma(p)}\right) + \gamma(p) \int_{r_0}^{r} \frac{f^2(s)}{\beta_p^2(s)} ds \right)^{-1}, \tag{5.80}$$

for $|p| < 1/2$ and $r > c$. Let $r^* > r_0 > c$ be such that, for $|p| < \delta_0$, we have

$$\frac{2}{W_0(r_0)} - \frac{(2-\gamma(p))^2}{16\Gamma^2} \log\left(\frac{r^*+\gamma(p)}{r_0+\gamma(p)}\right) = A < 0. \tag{5.81}$$

Since $\beta_p(r_0) \geq W_0(r_0)/2$ for $|p| < \delta_0$ we have

$$\frac{1}{\beta_p(r_0)} - \frac{(2-\gamma(p))^2}{16\Gamma^2} \log\left(\frac{r^*+\gamma(p)}{r_0+\gamma(p)}\right) \leq A < 0. \tag{5.82}$$

Furthermore, we have

$$\int_{r_0}^{r^*} \frac{f^2(s)}{\beta_p^2(s)} ds \leq \frac{1}{\beta_p^2(r_0)} \int_{r_0}^{r^*} f^2(s) ds \leq M, \tag{5.83}$$

for some $M > 0$ since $\beta_p'(r) \geq 0$ and $f(s) \sim r^2$. Thus, there exists $0 < \delta_1 < \delta_0$ such that $\gamma(p)M < |A|/2$, for which we obtain

$$A + \gamma(p)M < 0, \tag{5.84}$$

provided $|p| < \delta_1$. This implies that the denominator on the right hand side of (5.80) goes from being positive at $r = r_0$ to negative at $r = r^*$, for values of $|p| < \delta_1$, and therefore blow-up of $\beta(r)$ must occur somewhere in between.

5.4.2 Klainerman and Sarnack method $(p = 1/2)$

In this section we consider the initial value problem (5.31) for $p = 1/2$. In this setting one case use the method of Klainerman and Sarnack [KS81] to map solutions u of (5.31) to a transformed solution $\hat{O}u$ of a simpler wave equation. We follow the procedure in [NR23], by Natário and Rossetti, where the method, among other results, is used to find explicit solutions for the homogeneous wave equation in a FLRW background.

Recalling our starting point, just as in (5.15), we are interested in understanding blow-up phenomena for the solution of

$$\Box_{\boldsymbol{g}} u = -(u_t)^2, \tag{5.85}$$

with a (3+1)-dimensional FLRW metric given by

$$\boldsymbol{g} := -dt^2 + t^{2p} \sum_{i=1}^{3} (dx^i)^2. \tag{5.86}$$

As seen in the previous sections, by changing to conformal time τ, given by equation (5.17) (which we relabel as t onwards) we obtain (5.28). For $p = 1/2$ this leads to

$$\Box u = -\frac{2}{(t+1)} u_t + (u_t)^2, \tag{5.87}$$

Briefly, the goal of the method of Klainerman and Sarnack is to find an invertible operator $\hat{O}$ which maps solutions u of the wave-equation in a given FLRW spacetime to solutions $\hat{O}u$ of a wave equation in a simpler geometry, recovering u by applying the inverse of this operator.

Following [NR23], in our present case of flat spacial slices, we assume there exists an operator $\hat{O}$ which commutes with Δ and satisfies

$$\hat{O}\left(\partial_t^2 u + \frac{2}{t+1}\partial_t u\right) = \partial_t^2(\hat{O}u). \tag{5.88}$$

Then, from equation (5.87), $\hat{O}u$ satisfies the following wave equation

$$\partial_t^2(\hat{O}u) - \Delta(\hat{O}u) = \hat{O}\left((u_t)^2\right). \tag{5.89}$$

We admit the existence of an operator $\hat{O}$ of the form

$$\hat{O}u = f(t)u_t + g(t)u, \tag{5.90}$$

where f and g are functions to be determined. Substituting in (5.88) we obtain

$$\left(\frac{2}{t+1}f(t) - 2f'(t)\right)u_{tt} + \left(-\frac{2}{(t+1)^2}f(t) - f''(t) - 2g'(t) + \frac{2}{t+1}g(t)\right)u_t - g''(t)u = 0. \tag{5.91}$$

This equation holds, independently of u, if the following constraints are satisfied

$$\begin{cases} \frac{2}{t+1}f(t) - 2f'(t) = 0, \\ -\frac{2}{(t+1)^2}f(t) - f''(t) - 2g'(t) + \frac{2}{t+1}g(t) = 0, \\ g''(t) = 0, \end{cases} \tag{5.92}$$

which leads to

$$\begin{cases} f(t) = k(t+1), \\ -\frac{2}{(t+1)^2}f(t) - f''(t) - 2g'(t) + \frac{2}{t+1}g(t) = 0, \\ g(t) = at + b, \end{cases} \tag{5.93}$$

with $a, b, k \in \mathbb{R}$, $(a, b, k) \neq (0, 0, 0)$, and provided that $b - k - a = 0$, as obtained from the middle condition in (5.93). Letting $k = 0$, we have $b = a$, and, setting $b = a = 1$, we have

$$\hat{O}u = (t+1)u, \tag{5.94}$$

eliminating the u_t term in (5.90). Note that there is no combination of $(a,b,k) \neq (0,0,0)$ eliminating only u. Thus, equation (5.89) becomes

$$\partial_t^2\left((t+1)u\right) - \Delta\left((t+1)u\right) = (t+1)(u_t)^2. \tag{5.95}$$

Setting $v = (t+1)u$, equation (5.95) becomes

$$\Box v = \frac{(v_t)^2}{t+1} - \frac{2vv_t}{(t+1)^2} + \frac{v^2}{(t+1)^3}, \tag{5.96}$$

which we can also write as

$$\Box v = \frac{1}{(t+1)^3}\left((t+1)v_t - v\right)^2. \tag{5.97}$$

This equation is not in the setting of Theorem 4.3.3, since the right-hand side does not necessarily vanish for $v_t = 0$. It is also not obvious whether it can be put in a form that matches the setting of Theorem 4.3.2 since, if v or v_t are non-zero, we must have $B(x,t,v,v') > 0$, condition (4.102), and thus a $\partial_t B$ term must necessarily appear on the right-hand side of the wave equation, i.e. we can not just absorb everything in the $A(x,t,v,v',v'')$ term.

In any case, we now proceed similarly to the proof of Theorem 4.2.1. Taking the spherical average of equation (5.97) and using the Euler-Poisson-Darboux equation (3.19) we obtain

$$\partial_t^2(\overline{v}) - \frac{2}{r}(\overline{v})_r - (\overline{v})_{rr} = \frac{1}{(t+1)^3}\overline{\left((t+1)v_t - v\right)^2} \tag{5.98}$$

which, multiplying by r, leads to

$$\Box(r\overline{v}) = \frac{r}{(t+1)^3}\overline{\left((t+1)v_t - v\right)^2}. \tag{5.99}$$

Defining $\psi(r,t) := r\overline{v}(r,t)$, by Duhamel's representation formula (3.26) the solution to the above equation is given by

$$\begin{aligned}\psi(r,t) = \frac{\psi(r-t,0)+\psi(r+t,0)}{2} &+ \frac{1}{2}\int_{r-t}^{r+t}(\partial_t\psi)(\rho,0)d\rho \\ &+ \frac{1}{2}\int\int_{T_{r,t}} \frac{\rho}{(\tau+1)^3}\overline{\left((\tau+1)v_\tau - v\right)^2}d\rho d\tau,\end{aligned} \tag{5.100}$$

for $t > 0$. Leading to, for $r > 0$,

$$\overline{v}(r,t) = \overline{v}^0(r,t) + \int\int_{T_{r,t}} \frac{\rho}{(\tau+1)^3}\overline{\left((\tau+1)v_\tau - v\right)^2}d\rho d\tau, \tag{5.101}$$

for $t > 0$, $r > 0$, where

$$\overline{v}^0(r,t) = \frac{(r+t)v(r+t,0)+(r-t)v(r-t,0)}{2r} + \frac{1}{2r}\int_{r-t}^{r+t}\rho\overline{v}_t(\rho,0)d\rho. \tag{5.102}$$

Consider again the wedge shaped region Σ given by

$$\Sigma = \{(r,t)\,|\, r+R<t<2r\}, \tag{5.103}$$

by choosing $(r,t)\in\Sigma$ we have, by the same reasons as in (4.146) and (4.147),

$$\overline{v}^0(r,t) = 0, \tag{5.104}$$

and thus

$$\overline{v}(r,t) = \frac{1}{2r}\int\int_{T_{r,t}} \frac{\rho}{(\tau+1)^3}\overline{((\tau+1)v_\tau - v)^2}d\rho d\tau, \quad \text{for } \ (r,t)\in\Sigma. \tag{5.105}$$

For the same region $S_{r,t}$ and the same reasoning as in Theorem 4.2.1, for fixed $(r,t)\in\Sigma$, and defining $c := t-r$, we can write

$$\overline{v}(r,t) \geq \frac{1}{2r}\int\int_{S_{r,t}} \frac{\rho}{(\tau+1)^3}\overline{((\tau+1)v_\tau - v)^2}d\rho d\tau = \frac{1}{2r}\int_c^r\int_{\rho-R}^{\rho+c} \frac{\rho}{(\tau+1)^3}\overline{((\tau+1)v_\tau - v)^2}d\tau d\rho. \tag{5.106}$$

Using (4.79) we obtain

$$\overline{v}(r,t) \geq \frac{1}{2r}\int_c^r\int_{\rho-R}^{\rho+c} \frac{\rho}{(\tau+1)^3}\left((\tau+1)\overline{v}_\tau - \overline{v}\right)^2 d\tau d\rho. \tag{5.107}$$

which, so far, has not shown itself very amenable to obtaining an ODE type inequality.

Bibliography

[Ali99] Serge Alinhac, *Blowup of small data solutions for a quasilinear wave equation in two space dimensions*, 1999, (https://arxiv.org/abs/math/9901146).

[BV] L. Barreira and C. Valls, *Equações Diferenciais: Teoria Qualitativa*, IST Press.

[CFO23] João L. Costa, Anne T. Franzen, and Jesús Oliver, *Semilinear Wave Equations on Accelerated Expanding FLRW Spacetimes*, Annales Henri Poincaré **24** (2023), no. 9, 3185–3207, (https://arxiv.org/abs/2201.05210).

[CL55] Earl A. Coddington and Norman Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.

[Con] Peter Constantin, *Blow up for quasilinear wave equations*, (https://web.math.princeton.edu/~const/blowup.pdf).

[Eva10] Lawrence C. Evans, *Partial differential equations*, American Mathematical Society, Providence, R.I., 2010.

[FMO+24] David Fajman, Maciej Maliborski, Maximilian Ofner, Todd Oliynyk, and Zoe Wyatt, *Phase transition between shock formation and stability in cosmological fluids*, (https://arxiv.org/abs/2405.03431).

[Joh85a] Fritz John, *Blow-Up for Quasi-Linear Wave Equations in Three Space Dimensions*, pp. 566–588, Birkhäuser Boston, Boston, MA, 1985.

[Joh85b] ———, *Formation of Singularities in One-Dimensional Nonlinear Wave Propagation*, pp. 469–497, Birkhäuser Boston, Boston, MA, 1985.

[Joh90] ———, *Nonlinear Wave Equations, Formation of Singularities*, University Lecture Series, vol. 2, American Mathematical Society, 1990.

[Kla80] S. Klainerman, *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math. **33** (1980), no. 1, 43–101.

[KS81] S. Klainerman and P. Sarnak, *Explicit solutions of $\Box u = 0$ on the Friedmann-Robertson-Walker space-times*, Ann. Inst. H. Poincaré Sect. A (N.S.) **35** (253-257) (1981).

[Lee19] J.M. Lee, *Introduction to Riemannian Manifolds*, Graduate Texts in Mathematics, Springer International Publishing, 2019.

[Luk] Jonathan Luk, *Introduction to Nonlinear Wave Equations*, (https://web.stanford.edu/~jluk/NWnotes.pdf).

[NR23] José Natário and Flavio Rossetti, *Explicit formulas and decay rates for the solution of the wave equation in cosmological spacetimes*, Journal of Mathematical Physics **64** (2023), no. 3, (https://arxiv.org/abs/2112.00771).

[Rin] Hans Ringström, *Non-linear wave equations*, (https://people.kth.se/~hansr/nlw.pdf).

[Sog95] C.D. Sogge, *Lectures on Nonlinear Wave Equations*, Monographs in analysis, International Press, 1995.

A

Appendix

A.1 Appendix to Chapter 2

A.1.1 d'Alembert's Formula via change of variables

Consider new variables ξ and η given by

$$\begin{cases} \xi = x + t, \\ \eta = x - t, \end{cases} \quad \Longrightarrow \begin{cases} x = \frac{1}{2}(\xi + \eta), \\ t = \frac{1}{2}(\xi - \eta). \end{cases} \tag{A.1}$$

The wave equation in (2.3), over the variables ξ and η, becomes

$$u_{\xi\eta} = 0 \;\Leftrightarrow\; \partial_\eta(\partial_\xi u) = 0 \;\Longrightarrow\; \partial_\xi u = f(\xi), \tag{A.2}$$

for some unknown function f. Integrating in ξ we obtain

$$\begin{aligned} \int_a^\xi \partial_s u(s,\eta)\, ds &= \int_a^\xi f(s)\, ds, \\ \Leftrightarrow\; u(\xi,\eta) &= \underbrace{u(a,\eta)}_{=:G(\eta)} + \underbrace{\int_a^\xi f(s)\, ds}_{=:F(\xi)} = G(\eta) + F(\xi), \end{aligned} \tag{A.3}$$

for some $a \in \mathbb{R}$, with $a < \xi$, leading to

$$u(x,t) = F(x+t) + G(x-t). \tag{A.4}$$

Using the initial conditions of (2.3), we obtain

$$u(x,0) = F(x) + G(x) = g(x), \tag{A.5}$$

and

$$u_t(x,0) = F'(x) - G'(x) = h(x). \tag{A.6}$$

Differentiating (A.5) together with (A.6) leads to

$$\begin{cases} F'(x) + G'(x) = g'(x), \\ F'(x) - G'(x) = h(x), \end{cases} \tag{A.7}$$

which we can solve for $F'(x)$ and $G'(x)$ obtaining

$$\begin{cases} F'(x) = \frac{1}{2}\big(g'(x) + h(x)\big), \\ G'(x) = \frac{1}{2}\big(g'(x) - h(x)\big). \end{cases} \tag{A.8}$$

From (A.8), integrating the first equation from 0 to $x+t$, the second from 0 to $x-t$, and using the fundamental theorem of calculus leads to

$$\begin{cases} F(x+t) = F(0) + \frac{1}{2}\big(g(x+t) - g(0)\big) + \frac{1}{2}\int_0^{x+t} h(s)ds, \\ G(x-t) = G(0) + \frac{1}{2}\big(g(x-t) - g(0) + \frac{1}{2}\int_{x-t}^{0} h(s)ds. \end{cases} \tag{A.9}$$

Thus, using (A.4), and the fact that $g(0) = F(0) + G(0)$ from (A.5), leads again to *d'Alembert's formula*

$$u(x,t) = \frac{1}{2}\big[g(x+t) + g(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds, \;\; x \in \mathbb{R}, \;\; t \geq 0. \tag{A.10}$$

The solution is of the form

$$u(x,t) = F(x+t) + G(x-t), \tag{A.11}$$

for appropriate functions F and G which depend on the initial conditions. Physically, the solution represents a disturbance propagating to the left and another one propagating to the right respectively.

A.1.2 Energy Methods

Definition A.1.1. ***(Energy)***
Let $\mathcal{U} \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary $\partial\mathcal{U}$, $\mathcal{U}_T := \mathcal{U} \times (0,T]$, $\Gamma_T = \overline{\mathcal{U}}_T \setminus \mathcal{U}_T$, $T > 0$, and u a $C^2(\overline{\mathcal{U}}_T)$ solution of the following initial/boundary-value problem

$$\begin{cases} u_{tt} - \Delta u = f(x,t), \;\; (x,t) \in \mathcal{U} \times (0,T], \\ u(x,0) = g(x), \;\; x \in \Gamma_T \cup (\mathcal{U} \times \{t=0\}), \\ u_t(x,0) = h(x), \;\; x \in \mathcal{U}. \end{cases} \tag{A.12}$$

We define the energy $E : [0,T] \to \mathbb{R}$, associated to u, as

$$E(t) := \frac{1}{2}\int_{\mathcal{U}} u_t^2(x,t) + \|\nabla u(x,t)\|^2 dx, \;\; t \in [0,T]. \tag{A.13}$$

The following theorem establishes uniqueness of solutions for the linear wave equation.

Theorem A.1.1. ***(Uniqueness)***
There exists at most one function $u \in C^2(\overline{\mathcal{U}}_T)$ which is solution of the initial value problem (A.12).

Proof. Suppose there is another solution $\tilde{u}$ of (A.12). Then, if we let $w := u - \tilde{u}$, w solves

$$\begin{cases} w_{tt} - \Delta w = 0, \;\; (x,t) \in \mathcal{U} \times (0,T], \\ w(x,0) = 0, \;\; x \in \Gamma_T \\ w_t(x,0) = 0, \;\; x \in \mathcal{U}. \end{cases} \tag{A.14}$$

Differentiating the energy associated to w, as defined above, and integrating by parts we obtain

$$\begin{aligned}\dot{E}(t) := \frac{dE}{dt}(t) = \frac{1}{2}\frac{d}{dt}\Big(\int_{\mathcal{U}} w_t^2 + \|\nabla w\|^2 dx\Big) &= \int_{\mathcal{U}} w_t w_{tt} + \nabla w \cdot \nabla w_t \, dx \\ &= \int_{\partial\mathcal{U}} w_t \nabla w \cdot \hat{n} \, dS + \int_{\mathcal{U}} w_t w_{tt} - w_t \Delta w \, dx \\ &= \int_{\mathcal{U}} w_t (w_{tt} - \Delta w) \, dx = 0,\end{aligned} \tag{A.15}$$

where the boundary term disappears since $w(x,t) = 0$ on $\partial\mathcal{U}$ and thus also $w_t(x,t) = 0$ in $\partial\mathcal{U}$. We conclude that $\dot{E}(t) = 0$ which implies that $E(t) = E(0)$.

From the initial conditions of (A.14) we have $E(0) = 0$ therefore, since $E(t) = E(0) = 0$, we have $w_t(x,t) = 0$ and $\nabla w(x,t) = 0$ in $\mathcal{U}_T$. Thus, w is constant in $\mathcal{U}_T$ and since $w(x,0) = 0$ in $\overline{\mathcal{U}}$ we conclude that $w(x,t) = 0$ in $\overline{\mathcal{U}}_T$ and therefore $u = \tilde{u}$ in $\overline{\mathcal{U}}_T$. ■

One can also understand the **domain of dependence** of the solution of the wave-equation via energy methods. Defining the backwards wave cone $K(x_0,t_0)$, with vertex (x_0,t_0), by

$$K(x_0,t_0) := \{(x,t) \,|\, 0 \le t \le t_0,\ \|x - x_0\| \le t_0 - t\}, \tag{A.16}$$

we have the following result.

Theorem A.1.2. ***(Finite speed of propagation)***
Let $u \in C^2$ be a solution of

$$u_{tt} - \Delta u = 0, \tag{A.17}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, and let (x_0,t_0) be such that $B(x_0,t_0) \subseteq \mathbb{R}^n$. If the initial conditions vanish in $B(x_0,t_0) \times \{t = 0\}$ then $u(x,t) = 0$ for $(x,t) \in K(x_0,t_0)$.

Proof. Fix $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ and define the local energy

$$e(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + \|\nabla u\|^2 dx, \quad t \in [0,t_0], \tag{A.18}$$

which corresponds to the energy restricted to $B(x_0,t_0-t)$. Differentiating the local energy and setting $B := B(x_0,t_0-t)$ in intermediate steps to ease notation, we have

$$\begin{aligned}\dot{e}(t) &= \frac{1}{2}\frac{d}{dt}\Big(\int_0^{t_0-t}\int_{\partial B(x_0,r)} (u_t^2 + \|\nabla u\|^2) dS(x) dr\Big) = \\ &= -\frac{1}{2}\int_{\partial B} (u_t^2 + \|\nabla u\|^2)\, dS(x) + \int_B (u_t u_{tt} + \nabla u \cdot \nabla u_t)\, dx \\ &= -\frac{1}{2}\int_{\partial B} (u_t^2 + \|\nabla u\|^2)\, dS(x) + \int_{\partial B} u_t \nabla u \cdot \hat{n}\, dS(x) + \int_B u_t(u_{tt} - \Delta u)\, dx \\ &= \int_{\partial B(x_0,t_0-t)} \big(u_t \nabla u \cdot \hat{n} - \frac{1}{2}u_t^2 - \frac{1}{2}\|\nabla u\|^2\big)\, dS(x),\end{aligned} \tag{A.19}$$

where we used Leibniz's integral rule, integration by parts as before and the fact that u satisfies (A.17). Using Cauchy-Schwartz's inequality and the fact that $(a-b)^2 \geq 0$ with $a = u_t$ and $b = \|\nabla u\|$, we obtain

$$\left|u_t \nabla u \cdot \hat{n}\right| \leq |u_t| \|\nabla u\| \leq \frac{1}{2} u_t^2 + \frac{1}{2} \|\nabla u\|^2. \tag{A.20}$$

As such, we conclude that $\dot{e}(t) \leq 0$ which implies $e(t) \leq e(0) = 0$ and therefore $e(t) = 0$ for all $t \in [0, t_0]$. Thus, $u_t = \nabla u = 0$ inside $K(x_0, t_0)$ and, since $u(x, 0) = u_t(x, 0) = 0$ in $B(x_0, t_0)$, this implies that $u(x, t) = 0$ in $K(x_0, t_0)$. ∎

Remark A.1.1. *The theorem states that if the initial conditions are zero in the base of the cone, then u is zero in the cone. This shows that the speed of propagation must satisfy $v \leq 1$, since initial conditions outside the base of the cone have no influence on the solution inside the cone. For $n = 3$ (and odd $n \geq 3$ in general) one can conclude that $v = 1$ via the representation formula.*

Remark A.1.2. *This theorem also shows uniqueness within the cone. Given arbitrary initial conditions in the base of the cone, then any two solutions u and v must coincide within the cone. Consider arbitrary initial conditions in $B(x_0, t_0) \times \{t = 0\}$ and suppose the initial value problem has solutions u and v within the cone $K(x_0, t_0)$. We can consider an initial-value problem for the difference $w = u - v$, which has trivial initial conditions in $B(x_0, t_0) \times \{t = 0\}$. By linearity, w satisfies the initial-value problem above and we conclude that $u = v$ in the cone $K(x_0, t_0)$.*

A.2 Appendix to Chapter 4

A.2.1 Proof of property (4.9)

Proof. Recalling the definition of $\psi(x, \lambda)$, equation (4.7), where x is such that $\|x - x_0\| < a$ and $0 \leq \lambda < a$, we have

$$\max |\psi(x, \lambda)| \leq \frac{\max_x \left(|2\lambda(a^2 - \|x - x_0\|^2)|\right)}{\min_x \left(|a^2 + \lambda^2 + \sqrt{(a^2 - \lambda^2)^2 + 4\lambda^2 \|x - x_0\|^2}|\right)} = \frac{2\lambda a^2}{2a^2} = \lambda, \tag{A.21}$$

which occurs at $x = x_0$. ∎

A.2.2 Proof of property (4.10)

Proof. We define $\rho := \|x - x_0\|$ and compute the partial derivative

$$\psi_\lambda = \frac{2(a^2 - \rho^2)}{a^2 + \lambda^2 + \sqrt{(a^2 - \lambda^2)^2 + 4\lambda^2\rho^2}} - 4\lambda^2 (a^2 - \rho^2) \left(\frac{1 + \frac{\lambda^2 - a^2 + 2\rho^2}{\sqrt{(a^2 - \lambda^2)^2 + 4\lambda^2 \rho^2}}}{\left(a^2 + \lambda^2 + \sqrt{(a^2 - \lambda^2)^2 + 4\lambda^2\rho^2}\right)^2} \right). \tag{A.22}$$

The left term is at most $(2a^2)/(2a^2) = 1$, when $\rho = 0$. The numerator of the second term we can write as

$$-4\lambda^2 (a^2 - \rho^2) \left(1 + \frac{\lambda^2 - a^2}{\sqrt{(a^2 - \lambda^2)^2 + 4\lambda^2\rho^2}} + \frac{2\rho^2}{\sqrt{(a^2 - \lambda^2)^2 + 4\lambda^2\rho^2}}\right) \tag{A.23}$$

Sending the right most term to zero, since it is negative and we are looking for an upper bound, and reducing the remaining terms to the same denominator we get

$$-4\lambda^2(a^2-\rho^2)\left(\frac{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}+\lambda^2-a^2}{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}}\right). \tag{A.24}$$

Since the term involving the square root is negative, and will be largest for $x = x_0$, we get

$$-4\lambda^2(a^2-\rho^2)\left(\frac{a^2-\lambda^2+\lambda^2-a^2}{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}}\right) = 0, \tag{A.25}$$

concluding that $\psi_\lambda \leq 1$. We now show that $\psi_\lambda \geq 0$ in order to conclude that $|\psi_\lambda| \leq 1$. We can write (A.22) as

$$\psi_\lambda = 2(a^2-\rho^2)\left(\frac{a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}}{\left(a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}\right)^2} - \frac{\lambda\left(2\lambda+\frac{2\lambda(\lambda^2-a^2)+4\lambda\rho^2}{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}}\right)}{\left(a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}\right)^2}\right). \tag{A.26}$$

Since the denominator is always positive and $(a^2-\rho^2) > 0$ we have

$$\operatorname{sgn}(\psi_\lambda) = \operatorname{sgn}\left(a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2} - \lambda\left(2\lambda+\frac{2\lambda(\lambda^2-a^2)+4\lambda\rho^2}{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}}\right)\right). \tag{A.27}$$

For $\lambda \neq 0$ this is equivalent to

$$\operatorname{sgn}\left(2\lambda^2\left(\frac{a^2}{2\lambda^2}+\frac{1}{2}+\frac{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}}{2\lambda^2}\right) - 2\lambda^2\left(1+\frac{\lambda^2-a^2+2\rho^2}{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}}\right)\right), \tag{A.28}$$

and, because $a > \lambda$, the term $a^2/(2\lambda^2)$ is larger than $1/2$ which, together with the other $1/2$, takes care of the one on the right. Thus, we can simply look at

$$\operatorname{sgn}\left(\frac{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}}{2\lambda^2} - \frac{\lambda^2-a^2+2\rho^2}{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}}\right). \tag{A.29}$$

Reducing to the same denominator and again ignoring positive terms we obtain

$$(a^2-\lambda^2)^2+4\lambda^2\rho^2-2\lambda^2(\lambda^2-a^2+2\rho^2) = (a^2-\lambda^2)^2 - 2\lambda^2\underbrace{(\lambda^2-a^2)}_{<0} > 0, \tag{A.30}$$

since $0 \leq \lambda < a$. If $\lambda = 0$ expression (A.27) is just $2a^2 > 0$ so we conclude that $|\psi_\lambda| \leq 1$. ■

A.2.3 Proof of property (4.11)

Proof. We first compute the partial derivative

$$\begin{aligned}
\partial_i\psi &= \frac{-4\lambda(x_i-x_{0,i})}{a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\|x-x_0\|^2}} \\
&\qquad - \frac{4\lambda^2(x_i-x_{0,i})(2\lambda(a^2-\|x-x_0\|^2))}{\sqrt{(a^2-\lambda^2)^2-4\lambda^2\|x-x_0\|^2}\left(a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\|x-x_0\|^2}\right)^2} \\
&= \left(\frac{-4\lambda(x_i-x_{0,i})}{a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\|x-x_0\|^2}}\right)\times \\
&\qquad \left(1+\frac{2\lambda^2(a^2-\|x-x_0\|^2)}{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\|x-x_0\|^2}(a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\|x-x_0\|^2})}\right).
\end{aligned} \tag{A.31}$$

Thus

$$\sum_i (\partial_i \psi)^2 = \frac{16\lambda^2 \|x-x_0\|^2}{(a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\|x-x_0\|^2})^2} \times \left(1+\frac{2\lambda^2(a^2-\|x-x_0\|^2)}{\sqrt{(a^2-\lambda^2)^2+4\lambda^2\|x-x_0\|^2}\big(a^2+\lambda^2+\sqrt{(a^2-\lambda^2)^2+4\lambda^2\|x-x_0\|^2}\big)}\right)^2.$$

Letting $\rho := \|x-x_0\|^2$, $\Omega := \sqrt{(a^2-\lambda^2)^2+4\lambda^2\rho^2}$ and $k := a^2+\lambda^2+\Omega$ we have

$$\begin{aligned}\sum_i (\partial_i \psi)^2 &= \frac{16\lambda^2\rho^2}{k^2}\left(\frac{k\Omega+2\lambda^2(a^2-\rho^2)}{\Omega k}\right)^2 \\ &= 16\lambda^2 \underbrace{\frac{\rho^2}{\Omega^2}}_{=:F_1(\rho)} \underbrace{\left(\frac{k\Omega+2\lambda^2(a^2-\rho^2)}{k^2}\right)^2}_{=:F_2(\rho)}.\end{aligned} \tag{A.32}$$

Note that

$$\begin{aligned} k^2 &= (a^2+\lambda^2)^2 + 2(a^2+\lambda^2)\Omega + (a^2-\lambda^2)^2 + 4\lambda^2\rho \\ &= 2\big((a^2+\lambda^2)\Omega + (a^2-\lambda^2)^2 + 4\lambda^2\rho + 2\lambda^2(a^2-\rho)\big) \\ &= 2\big(k\Omega + 2\lambda^2(a^2-\rho)\big), \end{aligned} \tag{A.33}$$

therefore $F_2(\rho) = 1/2$ and

$$\sum_i (\partial_i \psi)^2 = 4\lambda^2 \frac{\rho^2}{\Omega^2}. \tag{A.34}$$

We now compute

$$\frac{dF_1}{d\rho} = \frac{2\rho\Omega^2 - 2\Omega\Omega'\rho^2}{\Omega^4} = \frac{2\rho}{\Omega^4}(\Omega^2 - 4\lambda^2\rho^2) = \frac{2\rho}{\Omega^4} = (a^2-\lambda^2)^2 > 0, \tag{A.35}$$

where $\Omega' = 4\lambda^2\rho/\Omega$. Thus, recalling (4.11), we have

$$\theta = \sum_i (\partial_i \psi)^2 = 4\lambda^2 \frac{\rho^2}{\Omega^2}, \tag{A.36}$$

furthermore, since $0 \le \rho < a$, we have $\partial F_1/\partial\rho > 0$, and the maximum value of θ occurs at $\rho = a$, we conclude

$$\theta = \sum_i (\partial_i \psi)^2 \le \sum_i (\partial_i \psi)^2\Big|_{\rho=a} = \frac{4\lambda^2 a^2}{(a^2-\lambda^2)^2+4\lambda^2 a^2} = \left(\frac{2\lambda a}{a^2+\lambda^2}\right)^2 < 1. \tag{A.37}$$

■

A.2.4 Proof of property (4.26)

Proof. Applying Taylor's theorem to ϕ at $p := (x,t,0,0,u'')$, we have

$$\begin{aligned} \phi(x,t,u,u',u'')\big|_p &= \phi(x,t,0,0,u'') + u\frac{\partial\phi}{\partial u}\Big|_p + u'\cdot\frac{\partial\phi}{\partial u'}\Big|_p + R(u,u') \\ &= u\frac{\partial\phi}{\partial u}\Big|_p + u'\cdot\frac{\partial\phi}{\partial u'}\Big|_p + R(u,u'), \end{aligned} \tag{A.38}$$

where $\phi(x,t,0,0,u'') = 0$ from assumption (4.2) and $R(u,u')$ is a remainder term of second order terms in u and u'. Using the triangle inequality and Cauchy-Schwarz's inequality we obtain

$$\begin{aligned} |\phi(x,t,u,u',u'')| &= \left| u \frac{\partial \phi}{\partial u}\bigg|_{(x,t,0,0,u'')} + u' \cdot \frac{\partial \phi}{\partial u'}\bigg|_{(x,t,0,0,u'')} + R(u,u') \right| \\ &\leq |R(u,u')| + \left(\left| \frac{\partial \phi}{\partial u}\bigg|_p \right| + \left\| \frac{\partial \phi}{\partial u'}\bigg|_p \right\| \right) (|u| + \|u'\|). \end{aligned} \tag{A.39}$$

Using $|u| \leq \sqrt{u^2 + a\|u'\|^2}$, $\|u'\| \leq \sqrt{1/a}\sqrt{u^2 + a\|u'\|^2}$ and assuming for now $\sqrt{u^2 + a\|u'\|^2} \neq 0$, we have

$$|\phi(x,t,u,u',u'')| \leq \frac{|R(u,u')|}{\sqrt{u^2 + a\|u'\|^2}} \sqrt{u^2 + a\|u'\|^2} + 2 \left(\left| \frac{\partial \phi}{\partial u}\bigg|_p \right| + \frac{1}{\sqrt{a}} \left\| \frac{\partial \phi}{\partial u'}\bigg|_p \right\| \right) \sqrt{u^2 + a\|u'\|^2}. \tag{A.40}$$

Since x, t, u, u', u'' are bounded in $K(x_0, a)$, then ϕ, ϕ_u and $\phi_{u'}$ are bounded, because ϕ is C^2, and so is $|R(u,u')|$. Defining then

$$M' := \max \left\{ \frac{|R(u,u')|}{\sqrt{u^2 + a\|u'\|^2}}, \left| \frac{\partial \phi}{\partial u}\bigg|_p \right|, \frac{1}{\sqrt{a}} \left\| \frac{\partial \phi}{\partial u'}\bigg|_p \right\| \right\}, \tag{A.41}$$

we obtain

$$|\phi(x,t,u,u',u'')| \leq 5M' \sqrt{u^2 + a\|u'\|^2}. \tag{A.42}$$

If however $\sqrt{u^2 + a\|u'\|^2} = 0$, then $u = 0$ and $u' = 0$. From (4.2), equation (A.42) still holds therefore, letting $M := 5M'$, we conclude

$$|\phi(x,t,u,u',u'')| \leq M \sqrt{u^2 + a\|u'\|^2}. \tag{A.43}$$

■

A.2.5 Proof of equation (4.208)

For the properties of the Fourier transform used in the following proof see Theorem 2 of Section 4.3 in [Eva10].

Proof. Consider first $a \neq 0$, $b \neq 0$, and thus $m = |a/b| \neq 0$. Let

$$\tilde{u}(x,t^*) := u(x/m, t^*), \tag{A.44}$$

and

$$\tilde{A}(x,t^*,\tilde{u}(x,t),\tilde{u}'(x,t),\tilde{u}''(x,t)) := A(x/m,t^*,u(x/m,t^*),u'(x/m,t^*),u''(x/m,t^*)). \tag{A.45}$$

Then, since $\Delta\tilde{u}(x,t) = 1/m^2 \Delta u(x/m,t)$, equation (4.207) becomes

$$\tilde{u}(x,t^*) - \Delta\tilde{u}(x,t^*) = \frac{1}{m^2} \tilde{A}\left(x,t^*,\tilde{u}(x,t),\tilde{u}'(x,t),\tilde{u}''(x,t)\right) =: \frac{1}{m^2} \tilde{A}(x) \geq 0, \tag{A.46}$$

Since we are under the assumption that $u(x;t^*) \in C^2(\mathbb{R}^3)$ we consider the domain to be $x \in \mathbb{R}^3$, i.e. we do not impose explicit boundary conditions. To solve this, we consider the Fourier transform of the above equation in the space domain, following Example 1 of Section 4.3.1 of [Eva10] which provides a method for finding the solution of $-\Delta u + u = f$ in $\mathbb{R}^n$. The Fourier transform requires all functions in (A.46) to be in $L^2(\mathbb{R}^3)$ which is the case since u has compact support for each fixed t^*. This leads to

$$\begin{aligned} & \mathcal{F}(\tilde{u} - \Delta\tilde{u}) = \mathcal{F}(\tilde{A}/m^2) \\ \Leftrightarrow \quad & \mathcal{F}(\tilde{u})(z,t^*) = \frac{\mathcal{F}(\tilde{A}/m^2)}{1+\|z\|^2} \\ \Leftrightarrow \quad & \tilde{u}(x,t^*) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(\tilde{A}/m^2)}{1+\|z\|^2}\right) = \frac{\tilde{A} * B}{m^2(2\pi)^{3/2}}, \end{aligned} \tag{A.47}$$

where $*$ is the convolution operator and

$$B(x) := \mathcal{F}^{-1}\left(\frac{1}{1+\|z\|^2}\right). \tag{A.48}$$

Again, as seen in Example 1 of Section 4.3.1 of [Eva10], one can compute the inverse Fourier transform to obtain

$$B(x) = \frac{1}{2^{3/2}} \int_0^\infty \frac{e^{-s-\frac{\|x\|^2}{4s}}}{s^{3/2}} ds, \quad \text{for } \; x \neq 0. \tag{A.49}$$

And thus

$$\tilde{u}(x,t) = \frac{1}{(4\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{\tilde{A}(z)}{m^2} \left(\int_0^\infty \frac{e^{-s-\frac{\|x-z\|^2}{4s}}}{s^{3/2}} ds \right) dz. \tag{A.50}$$

Performing the change of variables $z \to my$ and evaluating at mx we obtain

$$u(x,t) = \tilde{u}(mx,t) = \frac{1}{(4\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{A(y)}{m} \left(\int_0^\infty \frac{e^{-s-\frac{m\|x-y\|^2}{4s}}}{s^{3/2}} ds \right) dy. \tag{A.51}$$

If $a = 0$, $b \neq 0$, equation (4.207) corresponds to Poisson's equation which has a known solution (see Theorem 1 of Section 2.2 of [Eva10]) given by

$$u(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{A(y)}{\|x-y\|} dy. \tag{A.52}$$

∎

A.3 Appendix to Chapter 5

A.3.1 Proof of equation (5.60)

Proof. Let $\Delta(r) := \beta(r) - w(r)$ and let $r_0 > c$ be such that $\beta(r_0) \geq w(r_0)$. We want to prove that, for $r \geq r_0$, we have $\Delta(r) \geq 0$. Since

$$\begin{aligned} \Delta'(r) &\geq q_2(r)(\beta^2(r) - w^2(r)) \\ &= q_2(r)(\beta(r) - w(r))(\beta(r) + w(r)) \\ &= \Delta(r) q_2(r)(\beta(r) + w(r)) \\ &= \Delta(r) g(r), \end{aligned} \tag{A.53}$$

where $g(r) := q_2(r)(\beta(r) + w(r))$. We obtain then

$$\Delta'(r) \geq \Delta(r)g(r). \tag{A.54}$$

Since $g(r)$ is continuous by assumption we consider the integrating factor

$$\mu(r) = e^{-\int_{r_0}^{r} g(t)dt}, \tag{A.55}$$

and we can write

$$\mu(r)\Delta'(r) - g(r)\mu(r)\Delta(r) \geq 0, \tag{A.56}$$

which is equivalent to

$$(\mu(r)\Delta(r))' \geq 0. \tag{A.57}$$

Since $\mu(r_0)\Delta(r_0) \geq 0$ and $(\mu(r)\Delta(r))' \geq 0$, for all $r > r_0 > c$, we have $\mu(r)\Delta(r) \geq 0$ for $r \geq 0$ and thus $\Delta(r) \geq 0$ for $r \geq r_0$, concluding the proof. ■