Weakly Compressible AB Diblock + Bare C-like Nanoparticles with Negative χ_{AC} and Smearing Derivation

1 Canonical Ensemble Derivation (with w_+, $w_{AB}^{(\pm)}$, $w_{AC}^{(\pm)}$, and $w_{AC}^{(\pm)}$ fields)

The system is composed of n_D AB diblock chains and n_P bare, A-like nanoparticles. Each diblock chain has $P_A + P_B = P$ segments. χ_{IJ} is the interaction strength between components I and J AB, AC, BC. Segment center densities are defined as

$$\hat{\rho}_{DA,c}(\mathbf{r}) = \sum_{i=1}^{n_D} \sum_{j=1}^{P_A} \delta(\mathbf{r} - \mathbf{r}_{i,j})$$

$$\hat{\rho}_{DB,c}(\mathbf{r}) = \sum_{i=1}^{n_D} \sum_{j=P_A+1}^{P} \delta(\mathbf{r} - \mathbf{r}_{i,j})$$

$$\hat{\rho}_{P,c}(\mathbf{r}) = \sum_{i=1}^{n_P} \delta(\mathbf{r} - \mathbf{r}_i)$$

For all of the polymer segments, the full (smeared) segment densities are given by:

$$\hat{\rho}_K(\mathbf{r}) = (h * \hat{\rho}_{K,c})(\mathbf{r})$$

where $K \in \{DA, DB\}$ and h is the segment density distribution function given by the Gaussian

$$h(\mathbf{r}) = \left(\frac{1}{2\pi a^2}\right)^{d/2} \exp\left(-\frac{|\mathbf{r}|^2}{2a^2}\right)$$

where a is the segment size and d is the number of dimensions. The full nanoparticle density distribution is given by

$$\hat{\rho}_P = (\Gamma * \hat{\rho}_{P,c})(\mathbf{r})$$

where

$$\Gamma(\mathbf{r}) = \frac{\rho_0}{2} \mathrm{erfc}\left(\frac{|\mathbf{r}| - R_P}{\xi}\right)$$

where R_P is the nanoparticle radius, ρ_0 is the bulk density, and ξ controls the nanoparticle interface width.
The harmonic bond potential between connected segments is given by

$$\beta U_0 = \sum_{i=1}^{n_D} \sum_{j=1}^{P-1} \frac{3\left|\mathbf{r}_{i,j+1} - \mathbf{r}_{i,j}\right|^2}{2b^2}$$

The nonbonded interaction potential is given by

$$\beta U_1 = \frac{\chi_{AB}}{\rho_0} \int d\mathbf{r} \hat{\rho}_{DA} \hat{\rho}_{DB} + \frac{-\chi_{AC}}{\rho_0} \int d\mathbf{r} \hat{\rho}_{DA} \hat{\rho}_P + \frac{\chi_{BC}}{\rho_0} \int d\mathbf{r} \hat{\rho}_{DB} \hat{\rho}_P$$

A Helfand incompressibility potential penalizes deviations away from ρ_0, and is given by

$$\beta U_2 = \frac{\kappa}{2\rho_0} \int d\mathbf{r} \left[\hat{\rho}_+(\mathbf{r}) - \rho_0\right]^2$$

where $\hat{\rho}_+ = \hat{\rho}_{DA} + \hat{\rho}_{DB} + \hat{\rho}_P$ is the local total density.

This gives us a canonical partition function of

$$Z_C = \frac{1}{n_D! n_P! \left(\lambda_T^3\right)^{n_D+n_P}} \int d\mathbf{r}^{n_D N_D} \int d\mathbf{r}^{n_P} \exp\left(-\beta U_0 - \beta U_1 - \beta U_2\right)$$

To prepare this for a particle-to-field transformation, let's define

$$\begin{aligned}
\hat{\rho}_{AB}^{(\pm)}(\mathbf{r}) =& \hat{\rho}_{DA}(\mathbf{r}) \pm \hat{\rho}_{DB}(\mathbf{r}) \\
\hat{\rho}_{AC}^{(\pm)}(\mathbf{r}) =& \hat{\rho}_{DA}(\mathbf{r}) \pm \hat{\rho}_{P}(\mathbf{r}) \\
\hat{\rho}_{BC}^{(\pm)}(\mathbf{r}) =& \hat{\rho}_{DB}(\mathbf{r}) \pm \hat{\rho}_{P}(\mathbf{r})
\end{aligned}$$

With these definitions, we can rewrite βU_1 as

$$\beta U_1 = \frac{\chi_{AB}}{4\rho_0} \int d\mathbf{r} \left(\hat{\rho}_{AB}^{(+)}(\mathbf{r})^2 - \hat{\rho}_{AB}^{(-)}(\mathbf{r})^2\right) + \frac{-\chi_{AC}}{4\rho_0} \int d\mathbf{r} \left(\hat{\rho}_{AC}^{(+)}(\mathbf{r})^2 - \hat{\rho}_{AC}^{(-)}(\mathbf{r})^2\right) + \frac{\chi_{BC}}{4\rho_0} \int d\mathbf{r} \left(\hat{\rho}_{BC}^{(+)}(\mathbf{r})^2 - \hat{\rho}_{BC}^{(-)}(\mathbf{r})^2\right)$$

From there, using the Gaussian functional integral, we get

$$\begin{aligned}
\exp(-\beta U_1) =& \frac{1}{\Omega_{AB}^{(+)}\Omega_{AB}^{(-)}\Omega_{AC}^{(+)}\Omega_{AC}^{(-)}\Omega_{BC}^{(+)}\Omega_{BC}^{(-)}} \int \mathcal{D}w_{AB}^{(+)} \int \mathcal{D}w_{AB}^{(-)} \int \mathcal{D}w_{AC}^{(+)} \int \mathcal{D}w_{AC}^{(-)} \int \mathcal{D}w_{BC}^{(+)} \int \mathcal{D}w_{BC}^{(-)} \\
&\times \exp\left(-\frac{\rho_0}{\chi_{AB}} \int d\mathbf{r} w_{AB}^{(+)}(\mathbf{r})^2 - i \int d\mathbf{r} \hat{\rho}_{AB}^{(+)}(\mathbf{r}) w_{AB}^{(+)}(\mathbf{r})\right) \exp\left(-\frac{\rho_0}{\chi_{AB}} \int d\mathbf{r} w_{AB}^{(-)}(\mathbf{r})^2 + \int d\mathbf{r} \hat{\rho}_{AB}^{(-)}(\mathbf{r}) w_{AB}^{(-)}(\mathbf{r})\right) \\
&\times \exp\left(-\frac{\rho_0}{\chi_{AC}} \int d\mathbf{r} w_{AC}^{(+)}(\mathbf{r})^2 + \int d\mathbf{r} \hat{\rho}_{AC}^{(+)}(\mathbf{r}) w_{AC}^{(+)}(\mathbf{r})\right) \exp\left(-\frac{\rho_0}{\chi_{AC}} \int d\mathbf{r} w_{AC}^{(-)}(\mathbf{r})^2 - i \int d\mathbf{r} \hat{\rho}_{AC}^{(-)}(\mathbf{r}) w_{AC}^{(-)}(\mathbf{r})\right) \\
&\times \exp\left(-\frac{\rho_0}{\chi_{BC}} \int d\mathbf{r} w_{BC}^{(+)}(\mathbf{r})^2 - i \int d\mathbf{r} \hat{\rho}_{BC}^{(+)}(\mathbf{r}) w_{BC}^{(+)}(\mathbf{r})\right) \exp\left(-\frac{\rho_0}{\chi_{BC}} \int d\mathbf{r} w_{BC}^{(-)}(\mathbf{r})^2 + \int d\mathbf{r} \hat{\rho}_{BC}^{(-)}(\mathbf{r}) w_{BC}^{(-)}(\mathbf{r})\right)
\end{aligned}$$

and

$$\exp(-\beta U_2) = \frac{1}{\Omega_+} \int \mathcal{D}w_+ \exp\left(-\frac{\rho_0}{2\kappa} \int d\mathbf{r} w_+(\mathbf{r})^2 + i \int d\mathbf{r} (\rho_0 - \hat{\rho}_+(\mathbf{r})) w_+(\mathbf{r})\right)$$

Notice that since $\chi_{AC} < 0$, the prefactors of $+1$ and $-i$ on the second terms in the exponentials with χ_{AC} are switched. (See Fredrickson eqns C.27 and C.28.) Before moving on, to make our lives easier, let's define

$$\Omega = \Omega_+ \Omega_{AB}^{(+)}\Omega_{AB}^{(-)}\Omega_{AC}^{(+)}\Omega_{AC}^{(-)}\Omega_{BC}^{(+)}\Omega_{BC}^{(-)}$$

Now the canonical partition function looks like

$$\begin{aligned}
Z_C =& \frac{1}{n_D! n_P! \left(\lambda_T^3\right)^{n_D+n_P}} \frac{1}{\Omega} \int \dots \int \mathcal{D}\{w\} \\
&\times \exp\left(-\frac{\rho_0}{2\kappa} \int d\mathbf{r} w_+(\mathbf{r})^2 + i\rho_0 \int d\mathbf{r} w_+ - \sum_{IJ \in \{AB,AC,BC\}} \frac{\rho_0}{\chi_{IJ}} \int d\mathbf{r} \left(w_{IJ}^{(+)}(\mathbf{r})^2 + w_{IJ}^{(-)}(\mathbf{r})^2\right)\right) \\
&\times \int d\mathbf{r}^{n_D N_D} \int d\mathbf{r}^{n_P} \exp\left(-\sum_{i=1}^{n_D} \sum_{j=1}^{P-1} \frac{3\left|\mathbf{r}_{i,j+1} - \mathbf{r}_{i,j}\right|^2}{2b^2} - i \int d\mathbf{r} w_+ \hat{\rho}_+\right) \\
&\times \exp\left(\int d\mathbf{r} \left(-i\hat{\rho}_{AB}^{(+)} w_{AB}^{(+)} + \hat{\rho}_{AB}^{(-)} w_{AB}^{(-)} + \hat{\rho}_{AC}^{(+)} w_{AC}^{(+)} - i\hat{\rho}_{AC}^{(-)} w_{AC}^{(-)} - i\hat{\rho}_{BC}^{(+)} w_{BC}^{(+)} + \hat{\rho}_{BC}^{(-)} w_{BC}^{(-)}\right)\right)
\end{aligned}$$

Using the definitions of $\hat{\rho}_{IJ}^{(\pm)}$, and defining

$$\begin{aligned}
w_A &= i\left(w_+ + w_{AB}^{(+)} + w_{AC}^{(-)}\right) - w_{AB}^{(-)} - w_{AC}^{(+)} \\
w_B &= i\left(w_+ + w_{AB}^{(+)} + w_{BC}^{(+)}\right) - w_{AB}^{(-)} - w_{AC}^{(-)} \\
w_C &= i\left(w_+ + w_{AC}^{(-)} + w_{BC}^{(+)}\right) - w_{AC}^{(+)} - w_{BC}^{(-)},
\end{aligned}$$

we can rewrite all the $\exp(\int d\mathbf{r} w\hat{\rho})$ type terms as

$$\prod_{j}^{n_D N_{DA}} \exp\left(-\omega_A(\mathbf{r}_j)\right) \prod_{k}^{n_D N_{DB}} \exp\left(-\omega_B(\mathbf{r}_k)\right). \prod_{l}^{n_P} \exp\left(-\omega_P(\mathbf{r}_l)\right).$$

where ω_A and ω_B are defined as

$$\omega_K(\mathbf{r}) = (h * w_K)(\mathbf{r})$$

and ω_P is defined as

$$\omega_P(\mathbf{r}) = (\Gamma * w_C)(\mathbf{r})$$

Additionally, defining the bond transition probability Φ as

$$\Phi(\mathbf{r}-\mathbf{r}') = \left(\frac{3}{2\pi b^2}\right)^{d/2} \exp\left(\frac{-3|\mathbf{r}-\mathbf{r}'|^2}{2b^2}\right),$$

we can rewrite the canonical partition function equation as

$$\begin{aligned}
Z_C =& \frac{1}{n_D! n_P! \, (\lambda_T^3)^{n_D+n_P}} \frac{1}{\Omega} \int \cdots \int \mathcal{D}\{w\} \\
&\times \exp\left(-\frac{\rho_0}{2\kappa}\int d\mathbf{r} w_+(\mathbf{r})^2 + i\rho_0 \int d\mathbf{r} w_+ - \sum_{IJ} \frac{\rho_0}{\chi_{IJ}} \int d\mathbf{r} \left(w_{IJ}^{(+)}(\mathbf{r})^2 + w_{IJ}^{(-)}(\mathbf{r})^2\right)\right) \\
&\times \int d\mathbf{r}^{n_D N_D} \int d\mathbf{r}^{n_P} \prod_{j}^{n_D} \prod_{k}^{N_D-1} \Phi(\mathbf{r}_{j,k+1} - \mathbf{r}_{j,k}) \\
&\times \prod_{j}^{n_D N_{DA}} \exp\left(-\omega_A(\mathbf{r}_j)\right) \prod_{k}^{n_D N_{DB}} \exp\left(-\omega_B(\mathbf{r}_k)\right). \prod_{l}^{n_P} \exp\left(-\omega_P(\mathbf{r}_l)\right)
\end{aligned}$$

Then, we define Q_D as

$$Q_D = \frac{1}{V}\int d\mathbf{r} q_D(N_D, \mathbf{r})$$

where

$$q_D(j+1, \mathbf{r}) = \exp(-\omega_{X_{j+1}}(\mathbf{r})) \int d\mathbf{r}' \Phi(\mathbf{r}-\mathbf{r}') q(j, \mathbf{r})$$

where X_{j+1} is either A or B depending on type of segment $j+1$ and $q_D(1, \mathbf{r}) = \exp(-\omega_A(\mathbf{r}))$. We also define Q_P as

$$Q_P = \frac{1}{V}\int d\mathbf{r} \exp(-\omega_P)$$

With these definitions, we get

$$\begin{aligned}
Z_C =& \frac{V^{n_D+n_P}}{n_D! n_P! \, (\lambda_T^3)^{n_D+n_P}} \frac{1}{\Omega} \left(\frac{2\pi b^2}{3}\right)^{(d/2) n_D (N_D-1)} \int \cdots \int \mathcal{D}\{w\} \\
&\times \exp\left(-\frac{\rho_0}{2\kappa}\int d\mathbf{r} w_+(\mathbf{r})^2 + i\rho_0 \int d\mathbf{r} w_+ - \sum_{IJ} \frac{\rho_0}{\chi_{IJ}} \int d\mathbf{r} \left(w_{IJ}^{(+)}(\mathbf{r})^2 + w_{IJ}^{(-)}(\mathbf{r})^2\right)\right) \\
&\times Q_D^{n_D} Q_P^{n_P}
\end{aligned}$$

We can rewrite this as

$$
\begin{aligned}
Z_C =& \frac{V^{n_D+n_P}}{n_D! n_P! \left(\lambda_T^3\right)^{n_D+n_P}} \frac{1}{\Omega} \left(\frac{2\pi b^2}{3}\right)^{(d/2)n_D(N_D-1)} \\
& \int \dots \int \mathcal{D}\{w\} \exp\left(-\mathcal{H}[\{w\}]\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}[w_+, w_{AB}^{(\pm)}] =& \frac{\rho_0}{2\kappa} \int d\mathbf{r} w_+(\mathbf{r})^2 - i\rho_0 \int d\mathbf{r} w_+ + \sum_{IJ} \frac{\rho_0}{\chi_{IJ}} \int d\mathbf{r} \left(w_{IJ}^{(+)}(\mathbf{r})^2 + w_{IJ}^{(-)}(\mathbf{r})^2\right) \\
& - n_D \log Q_D - n_P \log Q_P
\end{aligned}
$$

2 Grand Canonical Derivation

The grand canonical partition function is then given by

$$
Z_G(\mu_D, \mu_P, V, T) = \sum_{n_D}^{\infty} \exp(\beta\mu_D n_D) \sum_{n_P}^{\infty} \exp(\beta\mu_P n_P) Z_C(n_D, n_P, V, T)
$$

Now let's define activities z_D and z_P as

$$
z_K = z_{K0} \exp\left(\beta\mu_K\right)
$$

where

$$
z_{D0} = \frac{1}{\lambda_T^3} \left(\frac{2\pi b^2}{3}\right)^{d/2(N_D-1)}
$$

and

$$
z_{P0} = \frac{1}{\lambda_T^3}
$$

Now we can rewrite equation Z_G as

$$
\begin{aligned}
Z_G(\mu_D, \mu_P, V, T) =& \frac{1}{\Omega} \int \dots \int \mathcal{D}\{w\} \\
& \times \exp\left(-\frac{\rho_0}{2\kappa} \int d\mathbf{r} w_+(\mathbf{r})^2 + i\rho_0 \int d\mathbf{r} w_+ - \frac{\rho_0}{\chi} \int d\mathbf{r} w_{AB}^{(+)}(\mathbf{r})^2 - \frac{\rho_0}{\chi} \int d\mathbf{r} w_{AB}^{(-)}(\mathbf{r})^2\right) \\
& \times \sum_{n_D}^{\infty} \frac{(z_D V Q_D)^{n_D}}{n_D!} \sum_{n_P}^{\infty} \frac{(z_P V Q_P)^{n_P}}{n_P!} \\
=& \frac{1}{\Omega} \int \dots \int \mathcal{D}\{w\} \\
& \times \exp\left(-\frac{\rho_0}{2\kappa} \int d\mathbf{r} w_+(\mathbf{r})^2 + i\rho_0 \int d\mathbf{r} w_+ - \sum_{IJ} \frac{\rho_0}{\chi_{IJ}} \int d\mathbf{r} \left(w_{IJ}^{(+)}(\mathbf{r})^2 + w_{IJ}^{(-)}(\mathbf{r})^2\right)\right) \\
& \times \exp(z_D V Q_D) \exp(z_P V Q_P)
\end{aligned}
$$

And finally, we get

$$
Z_G(\mu_D, \mu_P, V, T) = \frac{1}{\Omega} \int \dots \int \mathcal{D}\{w\} \exp\left(-\mathcal{H}_G\left[\{w\}\right]\right)
$$

where

$$
\mathcal{H}_G[\{w\}] = \frac{\rho_0}{2\kappa} \int d\mathbf{r} w_+(\mathbf{r})^2 - i\rho_0 \int d\mathbf{r} w_+ + \sum_{IJ} \frac{\rho_0}{\chi_{IJ}} \int d\mathbf{r} \left(w_{IJ}^{(+)}(\mathbf{r})^2 + w_{IJ}^{(-)}(\mathbf{r})^2\right) \quad - z_D V Q_D - z_P V Q_P
$$

3 Canonical 1S Update Derivation

3.1 w_+ Field

First let's do the w_+ update derivation for the Canonical Ensemble.

$$w_+^{t+1} = w_+^t - \lambda\left[\frac{\delta \mathcal{H}}{\delta w_+^t} + \left(\frac{\delta \mathcal{H}}{\delta w_+^{t+1}}\right)_{lin} - \left(\frac{\delta \mathcal{H}}{\delta w_+^t}\right)_{lin}\right]$$

Taking the Fourier Transform,

$$\hat{w}_+^{t+1} = \hat{w}_+^t - \lambda\left[\frac{\delta \hat{\mathcal{H}}}{\delta w_+^t} + \left(\frac{\delta \hat{\mathcal{H}}}{\delta w_+^{t+1}}\right)_{lin} - \left(\frac{\delta \hat{\mathcal{H}}}{\delta w_+^t}\right)_{lin}\right]$$

Then, after plugging in the correct expressions, we can solve for $\hat{w}_+^{t+1}$ and take the inverse Fourier Transform to get w_+^{t+1}.

$$\begin{aligned}
\frac{\delta \mathcal{H}}{\delta w_+^t} &= \frac{\rho_0}{\kappa} w_+^t - i\rho_0 + i[(\rho_{DA,c} * h)(\mathbf{r}) + (\rho_{DB,c} * h)(\mathbf{r}) + (\rho_{P,c} * \Gamma)(\mathbf{r})] \\
\frac{\delta \hat{\mathcal{H}}}{\delta w_+^t} &= \frac{\rho_0}{\kappa} \hat{w}_+^t - i\rho_0 \delta(\mathbf{k}) + i[\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h} + \hat{\rho}_{P,c}\hat{\Gamma}] \\
\left(\frac{\delta \hat{\mathcal{H}}}{\delta w_+^t}\right)_{lin} &= \frac{\rho_0}{\kappa} \hat{w}_+^t - i\hat{h}^2 \phi_D \rho_0 N_D (\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB}) i \hat{w}_+^t - i\phi_P \rho_0 \hat{\Gamma}^2 i \hat{w}_+^t \\
&= \frac{\rho_0}{\kappa} \hat{w}_+^t + \hat{h}^2 \phi_D \rho_0 N_D (\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB}) \hat{w}_+^t + \phi_P \rho_0 \hat{\Gamma}^2 \hat{w}_+^t
\end{aligned}$$

Assembling the pieces, we get

$$\hat{w}_+^{t+1} = \hat{w}_+^t - \lambda\left[\frac{\rho_0}{\kappa} \hat{w}_+^t - i\rho_0 \delta(\mathbf{k}) + i(\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h} + \hat{\rho}_{P,c}\hat{\Gamma}) + A(\hat{w}_+^{t+1} - \hat{w}_+^t)\right]$$

where

$$\begin{aligned}
A &= \frac{1}{w_+^t}\left(\frac{\delta \hat{\mathcal{H}}}{\delta w_+^t}\right)_{lin} = \frac{1}{w_+^{t+1}}\left(\frac{\delta \hat{\mathcal{H}}}{\delta w_+^{t+1}}\right)_{lin} \\
&= \frac{\rho_0}{\kappa} + \hat{h}^2 \phi_D \rho_0 N_D (\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB}) + \phi_P \rho_0 \hat{\Gamma}^2
\end{aligned}$$

If we also let B and F equal

$$\begin{aligned}
B &= A - \frac{\rho_0}{\kappa} \\
&= \hat{h}^2 \phi_D \rho_0 N_D (\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB}) + \phi_P \rho_0 \hat{\Gamma}^2 \\
F &= -i\rho_0 \delta(\mathbf{k}) + i(\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h} + \hat{\rho}_{P,c}\hat{\Gamma})
\end{aligned}$$

Then

$$\begin{aligned}
\hat{w}_+^{t+1}(1 + \lambda A) &= \hat{w}_+^t - \lambda\left(F - B\hat{w}_+^t\right) \\
\hat{w}_+^{t+1} &= \frac{\hat{w}_+^t - \lambda\left(F - B\hat{w}_+^t\right)}{1 + \lambda A}
\end{aligned}$$

3.2 $w_{AB}^{(+)}$

For the $w_{AB}^{(+)}$ field, the relevant expressions are:

$$\begin{aligned}
\left.\frac{\delta\mathcal{H}}{\delta w_{AB}^{(+)}}\right|_t &= \frac{2\rho_0}{\chi_{AB}}\ w_{AB}^{(+)}\Big|_t + i[(\rho_{DA,c} * h)(\mathbf{r}) + (\rho_{DB,c} * h)(\mathbf{r})] \\
\left.\frac{\delta\hat{\mathcal{H}}}{\delta w_{AB}^{(+)}}\right|_t &= \frac{2\rho_0}{\chi_{AB}}\ \hat{w}_{AB}^{(+)}\Big|_t + i[\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h}] \\
\left.\frac{\delta\hat{\mathcal{H}}}{\delta w_{AB}^{(+)}}\right|_t^{lin} &= \frac{2\rho_0}{\chi_{AB}}\ \hat{w}_{AB}^{(+)}\Big|_t - i\hat{h}^2\phi_D\rho_0 N_D(\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB})i\ \hat{w}_{AB}^{(+)}\Big|_t \\
&= \frac{2\rho_0}{\chi_{AB}}\ \hat{w}_{AB}^{(+)}\Big|_t + \hat{h}^2\phi_D\rho_0 N_D(\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB})\ \hat{w}_{AB}^{(+)}\Big|_t
\end{aligned}$$

This gives us

$$\hat{w}_{AB}^{(+)}\Big|_{t+1} = \hat{w}_{AB}^{(+)}\Big|_t - \lambda\left[\frac{2\rho_0}{\chi_{AB}}\ \hat{w}_{AB}^{(+)}\Big|_t + i(\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h}) + A(\ \hat{w}_{AB}^{(+)}\Big|_{t+1} - \hat{w}_{AB}^{(+)}\Big|_t)\right]$$

where

$$A = \frac{2\rho_0}{\chi_{AB}} + \hat{h}^2\phi_D\rho_0 N_D(\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB})$$

If we let B and F equal

$$\begin{aligned}
B &= A - \frac{2\rho_0}{\chi_{AB}} \\
&= \hat{h}^2\phi_D\rho_0 N_D(\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB}) \\
F &= +i(\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h})
\end{aligned}$$

Then

$$\begin{aligned}
\hat{w}_{AB}^{(+)}\Big|_{t+1}(1 + \lambda A) &= \hat{w}_{AB}^{(+)}\Big|_t - \lambda\left(F - B\ \hat{w}_{AB}^{(+)}\Big|_t\right) \\
\hat{w}_{AB}^{(+)}\Big|_{t+1} &= \frac{\hat{w}_{AB}^{(+)}\Big|_t - \lambda\left(F - B\ \hat{w}_{AB}^{(+)}\Big|_t\right)}{1 + \lambda A}
\end{aligned}$$

3.3 $w_{AB}^{(-)}$ Field

For the $w_{AB}^{(-)}$ field, the relevant expressions are:

$$\begin{aligned}
\left.\frac{\delta\mathcal{H}}{\delta w_{AB}^{(-)}}\right|_t &= \frac{2\rho_0}{\chi_{AB}}\ w_{AB}^{(-)}\Big|_t - (\rho_{DA,c} * h)(\mathbf{r}) + (\rho_{DB,c} * h)(\mathbf{r}) \\
\left.\frac{\delta\hat{\mathcal{H}}}{\delta w_{AB}^{(-)}}\right|_t &= \frac{2\rho_0}{\chi_{AB}}\ \hat{w}_{AB}^{(-)}\Big|_t - \hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h} \\
\left.\frac{\delta\hat{\mathcal{H}}}{\delta w_{AB}^{(-)}}\right|_t^{lin} &= \frac{2\rho_0}{\chi_{AB}}\ \hat{w}_{AB}^{(-)}\Big|_t
\end{aligned}$$

Note that we don't do the weak inhomogeneity expansion here because the $w_{AB}^{(-)}$ field tends to be much less stiff than the w_+ fields and so doesn't need the extra approximation. Now we get

$$\begin{aligned}
\hat{w}_{AB}^{(-)}\Big|_{t+1} &= \hat{w}_{AB}^{(-)}\Big|_{t} - \lambda\left[\frac{2\rho_0}{\chi_{AB}}\ \hat{w}_{AB}^{(-)}\Big|_{t} - \hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h} + \frac{2\rho_0}{\chi_{AB}}(\ \hat{w}_{AB}^{(-)}\Big|_{t+1} - \ \hat{w}_{AB}^{(-)}\Big|_{t})\right] \\
\hat{w}_{AB}^{(-)}\Big|_{t+1}(1+\lambda\frac{2\rho_0}{\chi_{AB}}) &= \hat{w}_{AB}^{(-)}\Big|_{t} - \lambda\left(-\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h}\right) \\
\hat{w}_{AB}^{(-)}\Big|_{t+1} &= \frac{\hat{w}_{AB}^{(-)}\Big|_{t} - \lambda\left(-\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{DB,c}\hat{h}\right)}{\left(1+\lambda\frac{2\rho_0}{\chi_{AB}}\right)}
\end{aligned}$$

3.4 $w_{AC}^{(-)}$

For the $w_{AC}^{(-)}$ field, the relevant expressions are:

$$\begin{aligned}
\frac{\delta\mathcal{H}}{\delta w_{AC}^{(-)}}\Bigg|_{t} &= \frac{2\rho_0}{\chi_{AC}}\ w_{AC}^{(-)}\Big|_{t} + i[(\rho_{DA,c} * h)(\mathbf{r}) + (\rho_{P,c} * \Gamma)(\mathbf{r})] \\
\frac{\delta\hat{\mathcal{H}}}{\delta w_{AC}^{(-)}}\Bigg|_{t} &= \frac{2\rho_0}{\chi_{AC}}\ \hat{w}_{AC}^{(-)}\Big|_{t} + i[\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{P,c}\hat{\Gamma}] \\
\frac{\delta\hat{\mathcal{H}}}{\delta w_{AC}^{(-)}}\Bigg|_{t}^{lin} &= \frac{2\rho_0}{\chi_{AC}}\ \hat{w}_{AC}^{(-)}\Big|_{t} - i\hat{h}^2 f_A\phi_D\rho_0 N_D(\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB})i\ \hat{w}_{AC}^{(-)}\Big|_{t} - i\phi_P\rho_0\hat{\Gamma}^2 i\ \hat{w}_{AC}^{(-)}\Big|_{t} \\
&= \frac{2\rho_0}{\chi_{AC}}\ \hat{w}_{AB}^{(-)}\Big|_{t} + \hat{h}^2 f_A\phi_D\rho_0 N_D(\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB})\ \hat{w}_{AC}^{(-)}\Big|_{t} + \phi_P\rho_0\hat{\Gamma}^2\ \hat{w}_{AC}^{(-)}\Big|_{t}
\end{aligned}$$

This gives us

$$\hat{w}_{AC}^{(-)}\Big|_{t+1} = \hat{w}_{AC}^{(-)}\Big|_{t} - \lambda\left[\frac{2\rho_0}{\chi_{AC}}\ \hat{w}_{AC}^{(-)}\Big|_{t} + i(\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{P,c}\hat{\Gamma}) + A(\ \hat{w}_{AC}^{(-)}\Big|_{t+1} - \ \hat{w}_{AC}^{(-)}\Big|_{t})\right]$$

where

$$A = \frac{2\rho_0}{\chi_{AC}} + \hat{h}^2 f_A\phi_D\rho_0 N_D(\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB}) + \phi_P\rho_0\hat{\Gamma}^2$$

If we let B and F equal

$$\begin{aligned}
B &= A - \frac{2\rho_0}{\chi_{AC}} \\
&= \hat{h}^2 f_A\phi_D\rho_0 N_D(\hat{g}_{AA} + 2\hat{g}_{AB} + \hat{g}_{BB}) + \phi_P\rho_0\hat{\Gamma}^2 \\
F &= +i(\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{P,c}\hat{\Gamma})
\end{aligned}$$

Then

$$\begin{aligned}
\hat{w}_{AC}^{(-)}\Big|_{t+1}(1+\lambda A) &= \hat{w}_{AC}^{(-)}\Big|_{t} - \lambda\left(F - B\ \hat{w}_{AC}^{(-)}\Big|_{t}\right) \\
\hat{w}_{AC}^{(-)}\Big|_{t+1} &= \frac{\hat{w}_{AC}^{(-)}\Big|_{t} - \lambda\left(F - B\ \hat{w}_{AC}^{(-)}\Big|_{t}\right)}{1+\lambda A}
\end{aligned}$$

3.5 $w_{AC}^{(+)}$ Field

For the $w_{AC}^{(+)}$ field, the relevant expressions are:

$$\left.\frac{\delta\mathcal{H}}{\delta w_{AC}^{(+)}}\right|_t = \frac{2\rho_0}{\chi_{AC}}\left. w_{AC}^{(+)}\right|_t - (\rho_{DA,c} * h)(\mathbf{r}) - (\rho_{P,c} * \Gamma)(\mathbf{r})$$
$$\left.\frac{\delta\hat{\mathcal{H}}}{\delta w_{AC}^{(+)}}\right|_t = \frac{2\rho_0}{\chi_{AC}}\left. \hat{w}_{AC}^{(+)}\right|_t - \hat{\rho}_{DA,c}\hat{h} - \hat{\rho}_{P,c}\hat{\Gamma}$$
$$\left.\frac{\delta\hat{\mathcal{H}}}{\delta w_{AC}^{(+)}}\right|_t^{lin} = \frac{2\rho_0}{\chi_{AC}}\left. \hat{w}_{AC}^{(+)}\right|_t$$

Note that we don't do the weak inhomogeneity expansion here because the $w_{AC}^{(+)}$ field tends to be much less stiff than the w_+ fields and so doesn't need the extra approximation. Now we get

$$\left.\hat{w}_{AC}^{(+)}\right|_{t+1} = \left.\hat{w}_{AC}^{(+)}\right|_t - \lambda\left[\frac{2\rho_0}{\chi_{AC}}\left.\hat{w}_{AC}^{(+)}\right|_t - \hat{\rho}_{DA,c}\hat{h} - \hat{\rho}_{P,c}\hat{\Gamma} + \frac{2\rho_0}{\chi_{AC}}(\left.\hat{w}_{AC}^{(+)}\right|_{t+1} - \left.\hat{w}_{AC}^{(+)}\right|_t)\right]$$
$$\left.\hat{w}_{AC}^{(+)}\right|_{t+1}(1+\lambda\frac{2\rho_0}{\chi_{AC}}) = \left.\hat{w}_{AC}^{(+)}\right|_t - \lambda\left(-\hat{\rho}_{DA,c}\hat{h} - \hat{\rho}_{P,c}\hat{\Gamma}\right)$$
$$\left.\hat{w}_{AC}^{(+)}\right|_{t+1} = \frac{\left.\hat{w}_{AC}^{(+)}\right|_t - \lambda\left(-\hat{\rho}_{DA,c}\hat{h} - \hat{\rho}_{P,c}\hat{\Gamma}\right)}{\left(1+\lambda\frac{2\rho_0}{\chi_{AC}}\right)}$$

3.6 $w_{BC}^{(+)}$

For the $w_{BC}^{(+)}$ field, the relevant expressions are:

$$\left.\frac{\delta\mathcal{H}}{\delta w_{AC}^{(-)}}\right|_t = \frac{2\rho_0}{\chi_{AC}}\left. w_{AC}^{(-)}\right|_t + i[(\rho_{DA,c} * h)(\mathbf{r}) + (\rho_{P,c} * \Gamma)(\mathbf{r})]$$
$$\left.\frac{\delta\hat{\mathcal{H}}}{\delta w_{AC}^{(-)}}\right|_t = \frac{2\rho_0}{\chi_{AC}}\left. \hat{w}_{AC}^{(-)}\right|_t + i[\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{P,c}\hat{\Gamma}]$$
$$\begin{aligned}\left.\frac{\delta\hat{\mathcal{H}}}{\delta w_{AC}^{(-)}}\right|_t^{lin} &= \frac{2\rho_0}{\chi_{AC}}\left. \hat{w}_{AC}^{(-)}\right|_t - i\hat{h}^2 f_A\phi_D\rho_0 N_D(\hat{g}_{AA}+2\hat{g}_{AB}+\hat{g}_{BB})i\left.\hat{w}_{AC}^{(-)}\right|_t - i\phi_P\rho_0\hat{\Gamma}^2 i\left.\hat{w}_{AC}^{(-)}\right|_t \\ &= \frac{2\rho_0}{\chi_{AC}}\left. \hat{w}_{AB}^{(-)}\right|_t + \hat{h}^2 f_A\phi_D\rho_0 N_D(\hat{g}_{AA}+2\hat{g}_{AB}+\hat{g}_{BB})\left.\hat{w}_{AC}^{(-)}\right|_t + \phi_P\rho_0\hat{\Gamma}^2\left.\hat{w}_{AC}^{(-)}\right|_t\end{aligned}$$

This gives us

$$\left.\hat{w}_{AC}^{(-)}\right|_{t+1} = \left.\hat{w}_{AC}^{(-)}\right|_t - \lambda\left[\frac{2\rho_0}{\chi_{AC}}\left.\hat{w}_{AC}^{(-)}\right|_t + i(\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{P,c}\hat{\Gamma}) + A(\left.\hat{w}_{AC}^{(-)}\right|_{t+1} - \left.\hat{w}_{AC}^{(-)}\right|_t)\right]$$

where

$$A = \frac{2\rho_0}{\chi_{AC}} + \hat{h}^2 f_A\phi_D\rho_0 N_D(\hat{g}_{AA}+2\hat{g}_{AB}+\hat{g}_{BB}) + \phi_P\rho_0\hat{\Gamma}^2$$

If we let B and F equal

$$\begin{aligned}B &= A - \frac{2\rho_0}{\chi_{AC}} \\ &= \hat{h}^2 f_A\phi_D\rho_0 N_D(\hat{g}_{AA}+2\hat{g}_{AB}+\hat{g}_{BB}) + \phi_P\rho_0\hat{\Gamma}^2 \\ F &= +i(\hat{\rho}_{DA,c}\hat{h} + \hat{\rho}_{P,c}\hat{\Gamma})\end{aligned}$$

Then

$$\hat{w}_{AC}^{(-)}\Big|_{t+1}(1+\lambda A) = \hat{w}_{AC}^{(-)}\Big|_{t} - \lambda\left(F - B\,\hat{w}_{AC}^{(-)}\Big|_{t}\right)$$

$$\hat{w}_{AC}^{(-)}\Big|_{t+1} = \frac{\hat{w}_{AC}^{(-)}\Big|_{t} - \lambda\left(F - B\,\hat{w}_{AC}^{(-)}\Big|_{t}\right)}{1+\lambda A}$$

3.7 $w_{BC}^{(-)}$ Field

For the $w_{BC}^{(-)}$ field, the relevant expressions are:

$$\frac{\delta \mathcal{H}}{\delta w_{AC}^{(+)}}\Bigg|_{t} = \frac{2\rho_0}{\chi_{AC}}\,w_{AC}^{(+)}\Big|_{t} - (\rho_{DA,c} * h)(\mathbf{r}) - (\rho_{P,c} * \Gamma)(\mathbf{r})$$

$$\frac{\delta \hat{\mathcal{H}}}{\delta w_{AC}^{(+)}}\Bigg|_{t} = \frac{2\rho_0}{\chi_{AC}}\,\hat{w}_{AC}^{(+)}\Big|_{t} - \hat{\rho}_{DA,c}\hat{h} - \hat{\rho}_{P,c}\hat{\Gamma}$$

$$\frac{\delta \hat{\mathcal{H}}}{\delta w_{AC}^{(+)}}\Bigg|_{t}^{lin} = \frac{2\rho_0}{\chi_{AC}}\,\hat{w}_{AC}^{(+)}\Big|_{t}$$

Note that we don't do the weak inhomogeneity expansion here because the $w_{AC}^{(+)}$ field tends to be much less stiff than the w_+ fields and so doesn't need the extra approximation. Now we get

$$\hat{w}_{AC}^{(+)}\Big|_{t+1} = \hat{w}_{AC}^{(+)}\Big|_{t} - \lambda\left[\frac{2\rho_0}{\chi_{AC}}\,\hat{w}_{AC}^{(+)}\Big|_{t} - \hat{\rho}_{DA,c}\hat{h} - \hat{\rho}_{P,c}\hat{\Gamma} + \frac{2\rho_0}{\chi_{AC}}(\,\hat{w}_{AC}^{(+)}\Big|_{t+1} - \hat{w}_{AC}^{(+)}\Big|_{t})\right]$$

$$\hat{w}_{AC}^{(+)}\Big|_{t+1}(1+\lambda\frac{2\rho_0}{\chi_{AC}}) = \hat{w}_{AC}^{(+)}\Big|_{t} - \lambda\left(-\hat{\rho}_{DA,c}\hat{h} - \hat{\rho}_{P,c}\hat{\Gamma}\right)$$

$$\hat{w}_{AC}^{(+)}\Big|_{t+1} = \frac{\hat{w}_{AC}^{(+)}\Big|_{t} - \lambda\left(-\hat{\rho}_{DA,c}\hat{h} - \hat{\rho}_{P,c}\hat{\Gamma}\right)}{\left(1+\lambda\frac{2\rho_0}{\chi_{AC}}\right)}$$

4 Calculating Densities

4.1 Canonical Ensemble

In the Canonical Ensemble, the polymer densities are given by

$$\rho_{DA,c} = -n_D\frac{\delta \log Q_D}{\delta\omega_A(\mathbf{r})} = \frac{n_D}{VQ_D}\sum_{j=1}^{P_A} q_D(j,\mathbf{r})e^{\omega_A(\mathbf{r})}q_D^{\dagger}(P-j,\mathbf{r})$$

$$\rho_{DB,c} = -n_D\frac{\delta \log Q_D}{\delta\omega_B(\mathbf{r})} = \frac{n_D}{VQ_D}\sum_{j=P_A+1}^{P} q_D(j,\mathbf{r})e^{\omega_B(\mathbf{r})}q_D^{\dagger}(P-j,\mathbf{r})$$

and the particle density is given by

$$\rho_P(\mathbf{r}) = -n_P\frac{\delta \log Q_P}{\delta\omega_P(\mathbf{r})} = \frac{n_P}{VQ_P}e^{-\omega_P(\mathbf{r})}$$

4.2 Grand Canonical Ensemble

In the Grand Canonical Ensemble, the polymer densities are given by

$$\rho_{DA,c} = -z_D V \frac{\delta Q_D}{\delta \omega_A(\mathbf{r})} = z_D \sum_{j=1}^{P_A} q_D(j,\mathbf{r}) e^{\omega_A(\mathbf{r})} q_D^\dagger(P-j,\mathbf{r})$$

$$\rho_{DB,c} = -z_D \frac{\delta Q_D}{\delta \omega_B(\mathbf{r})} = z_D \sum_{j=P_A+1}^{P} q_D(j,\mathbf{r}) e^{\omega_B(\mathbf{r})} q_D^\dagger(P-j,\mathbf{r})$$

and the particle density is given by

$$\rho_P(\mathbf{r}) = -z_P \frac{\delta Q_P}{\delta \omega_P(\mathbf{r})} = z_P e^{-\omega_P(\mathbf{r})}$$

5 Comparing z_D values with 2 Field Model

From the 2-field model derivation,

$$z_{D0,2} = \frac{1}{\lambda_T^3} \exp\left(-\frac{N_D \chi}{4}\right) \left(\frac{2\pi b^2}{3}\right)^{d/2(N_D-1)}$$

And from the 3-field model derivation, (this document)

$$z_{D0,3} = \frac{1}{\lambda_T^3} \left(\frac{2\pi b^2}{3}\right)^{d/2(N_D-1)}$$

Therefore, to compare 2-field and 3-field simulations, we need to take this into account to make sure μ_D matches between them. Thus, given a 3-field model using $z_{D,3}$, the corresponsing value of $z_{D,2}$ necessary to match a 2-field model is

$$z_{D,2} = \exp\left(-\frac{N_D \chi}{4}\right) z_{D,3}$$

6 Simplification for Homogeneous System

In a homogeneous system, the fields w_+, $w_{AB}^{(+)}$, and $w_{AB}^{(-)}$ are all constants. To simplify solving the equations, it is convenient to use only real numbers by solving for iw_+, $iw_{AB}^{(+)}$, and $w_{AB}^{(-)}$. With this in mind, we can rewrite the Grand Canonical Hamiltonian as

$$H_G = -\frac{\rho_0 V}{2\kappa}(iw_+)^2 - \rho_0 V(iw_+) - \frac{\rho_0 V}{\chi}(iw_{AB}^{(+)})^2 + \frac{\rho_0 V}{\chi}(w_{AB}^{(-)})^2 - z_D V Q_D - z_P V Q_P$$

If we solve with simple Euler equations, we get

$$
\begin{aligned}
iw_{+}^{t+1} &= iw_{+}^{t} - i\lambda_{+}\frac{\partial H_G/V}{\partial w_{+}^{t}} \\
&= iw_{+}^{t} + \lambda_{+}\frac{\partial H_G/V}{\partial iw_{+}^{t}} \\
iw_{AB}^{(+)}\Big|_{t+1} &= iw_{AB}^{(+)}\Big|_{t} - i\lambda_{+}\left.\frac{\partial H_G/V}{\partial w_{AB}^{(+)}}\right|_{t} \\
&= iw_{AB}^{(+)}\Big|_{t} + \lambda_{+}\left.\frac{\partial H_G/V}{\partial iw_{AB}^{(+)}}\right|_{t} \\
w_{AB}^{(-)}\Big|_{t+1} &= w_{AB}^{(-)}\Big|_{t} - \lambda_{-}\left.\frac{\partial H_G/V}{\partial w_{AB}^{(-)}}\right|_{t}
\end{aligned}
$$

With this in mind, we can rewrite the partition functions as

$$
\begin{aligned}
Q_D &= \exp(-P_A w_A - P_B w_B) \\
Q_P &= \exp(-\rho_0 V_P w_A)
\end{aligned}
$$

Let's note that

$$
\begin{aligned}
\rho_D &= \rho_{DA} + \rho_{DB} \\
&= -z_D\frac{\partial Q_D}{\partial \omega_A} - z_D\frac{\partial Q_D}{\partial \omega_B} \\
&= -z_D\frac{\partial Q_D}{\partial w_A} - z_D\frac{\partial Q_D}{\partial w_B} \\
&= z_D P \exp(-P_A w_A - P_B w_B) \\
&= z_D P Q_D
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_{P,c} &= -z_P\frac{\partial Q_P}{\partial \omega_P} \\
&= z_P \exp(-\rho_0 V_P w_A) \\
\rho_{P,c} &= z_P Q_P \\
\rho_P &= \rho_0 V_P z_P Q_P
\end{aligned}
$$

with the following derivatives:

$$
\frac{\partial H_G/V}{\partial iw_{+}} = -\frac{\rho_0}{\kappa}(iw_{+}) - \rho_0 - \frac{2\rho_0}{\chi}(iw_{AB}^{(+)}) + \frac{2\rho_0}{\chi}(w_{AB}^{(-)}) - z_D\frac{\partial Q_D}{\partial iw_{+}} - z_P\frac{\partial Q_P}{\partial iw_{+}}
$$

With this in mind, we can rewrite the partition functions as

$$
\begin{aligned}
Q_D &= \exp(-fPw_A - (1-f)Pw_B) \\
&= \exp(-fP(iw_{+} + iw_{AB}^{(+)} - w_{AB}^{(-)}) - (1-f)P(iw_{+} + iw_{AB}^{(+)} + w_{AB}^{(-)})) \\
&= \exp(-P(iw_{+} + iw_{AB}^{(+)}) + (2f-1)Pw_{AB}^{(-)})
\end{aligned}
$$

and

$$\begin{aligned} Q_P &= \exp(-\rho_0 V_P w_A) \\ &= \exp(-\rho_0 V_P (iw_+ + iw_{AB}^{(+)} - w_{AB}^{(-)})) \end{aligned}$$

Then we can rewrite the Grand Canonical Hamiltonian as

$$\begin{aligned} H_G = &-\frac{\rho_0 V}{2\kappa}(iw_+)^2 - \rho_0 V(iw_+) - \frac{\rho_0 V}{\chi}(iw_{AB}^{(+)})^2 + \frac{\rho_0 V}{\chi}(w_{AB}^{(-)})^2 \\ &-z_D V \exp(-P(iw_+ + iw_{AB}^{(+)}) + (2f-1)Pw_{AB}^{(-)}) \\ &-z_P V \exp(-\rho_0 V_P (iw_+ + iw_{AB}^{(+)} - w_{AB}^{(-)})) \end{aligned}$$

If we solve with simple Euler equations, we get

$$\begin{aligned} w_+^{t+1} &= w_+^t - \lambda_+ \frac{\partial H_G/V}{\partial w_+^t} \\ w_{AB}^{(+)}\Big|_{t+1} &= w_{AB}^{(+)}\Big|_t - \lambda_+ \left.\frac{\partial H_G/V}{\partial w_{AB}^{(+)}}\right|_t \\ w_{AB}^{(-)}\Big|_{t+1} &= w_{AB}^{(-)}\Big|_t - \lambda_- \left.\frac{\partial H_G/V}{\partial w_{AB}^{(-)}}\right|_t \end{aligned}$$

with the following derivatives:

$$\frac{\partial H}{\partial w_+^t} = -\frac{\rho_0}{\kappa}(iw_+) - \rho_0 - \frac{2\rho_0}{\chi}(iw_{AB}^{(+)}) + \frac{2\rho_0}{\chi}(w_{AB}^{(-)})$$