Approximation algorithms for ground state energies of multi-qutrit systems

Luca Göcke

Outline

- Approximation algorithms for quantum many-body systems
- Implementation of specific models
- Generalization to qutrits

Approximation algorithms for quantum many-body systems

Find product state approximaiton to maximal (minimal) eigenvalue of traceless 2-local Hamiltonians $H = H_1 + H_2$ where

$$H_1 = \sum_{j=1}^{3n} D_j P_j, \qquad H_2 = \sum_{i,j=1}^{3n} C_{i,j} P_i P_j \tag{1}$$

with the Pauli-operators $P_{3a-2} = X_a,\ P_{3a-1} = Y_a,\ P_{3a} = Z_a$

Approximation algorithms for quantum many-body systems

Theorem

There is an efficient classical algorithm which, given H of the form (1), outputs a product state $|\phi\rangle = |\phi_1\rangle \otimes \ldots \otimes |\phi_n\rangle$ such that with probability at least $\frac{2}{3}$

$$\langle\phi| H |\phi\rangle \geq \frac{\lambda_{max}(H)}{O(\log n)}.$$

Moreover, each single-qubit state ϕ_i in an eigenstate of one of the Pauli operators X, Y or Z.

Approximation algorithms for quantum many-body systems

- $H' = H_2 + Z_{n+1} H_1$

Lemma

$\lambda_{max}(H') = \lambda_{max}(H)$. Moreover, given any $(n+1)$-qubit state ω we can efficiently compute an n-qubit state ϕ such that

$$\langle\phi| H |\phi\rangle \geq \langle\omega| H' |\omega\rangle .$$

If ω is a tensor product of single qubit stabilizer states then so is ϕ.

Approximation algorithms for quantum many-body systems

- $H' = H_2 + Z_{n+1} H_1$
- Since H_1 and Z_{n+1} commute, they share a set of common eigenvectors:

$$Z_{n+1} H_1 \left|\psi\right\rangle = \lambda(Z_{n+1})\lambda(H_1)\left|\psi\right\rangle = \pm\lambda(H_1)\left|\psi\right\rangle = \lambda(Z_{n+1}H_1)\left|\psi\right\rangle$$

- Operations that conserve the spectrum:

$$\left(Y^{\otimes n}\left(H_2 + H_1\right) Y^{\otimes n}\right)^T = H_2 - H_1$$

The semidefinite program

For M hermitian:

$$\begin{aligned} \max \quad & Tr\,(CM) \\ \text{s.t.} \quad & M_{i,i} = 1 \\ & M \geq 0 \end{aligned}$$

- Relaxation method pioneered by Goemans and Williamson
- $Tr(CM) = \sum_{i,j} C_{ij} M_{ij}$
- Assume M is real, symmetric
- $M_{i,j} = \langle v^i, v^j \rangle$ for some unit vectors $v^1, v^2, \ldots, v^{3n+1}$.

The algorithm

1. Solve the relaxed semidefinite program, obtaining an optimal set of vectors v_i
2. Let $|r\rangle$ be a vector of $3n$ independently and identically distributed $N(0,1)$ random variables
3. Let $z_i = \langle r, v^i \rangle / T$ with $T = c\sqrt{\log n}$ and $c = O(\log n)$
4. If $|z_i| > \frac{1}{\sqrt{3}}$: $y_i = \frac{sgn(z_i)}{\sqrt{3}}$, otherwise $y_i = z_i$

Output: $\rho_a = \frac{1}{2}\left(\mathbb{1} + y_{3a-2}P_{3a-2} + y_{3a-1}P_{3a-1} + y_{3a}P_{3a}\right)$

Proof ideas

- Show that $\mathbb{E}_r |\Delta_{i,j}|$, with $\Delta_{ij} = z_i z_j - y_i y_j$ is suffiently small
- Show $T = c\sqrt{\log n}$ and $c = O(\log n)$ is suffient
- Use theorem due to Lieb to show $\langle\phi| H |\phi\rangle \geq \frac{\lambda_{max}(H)}{O(\log n)}$ with probability at least $\frac{2}{3}$

Implementation

- PICOS, interfacing CVXOPT, for the SDP
- Families of n-qubit Hamiltonians
- Plot the average of o iterations of the algorithm over a range of n qubits
- 4 to 5200 qubits, 25 steps, 20 iterarions per step

Implemented models: Transverse field Ising model

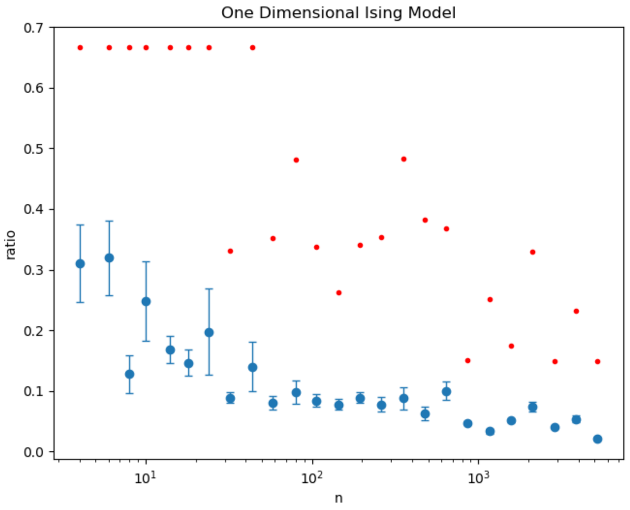
$$H = \alpha \sum_i Z_i + \beta \sum_i X_i X_{i+1}$$

- Transform the Hamiltonian into a quadratic form of Fermi operators via Jordan-Wigner transformation:

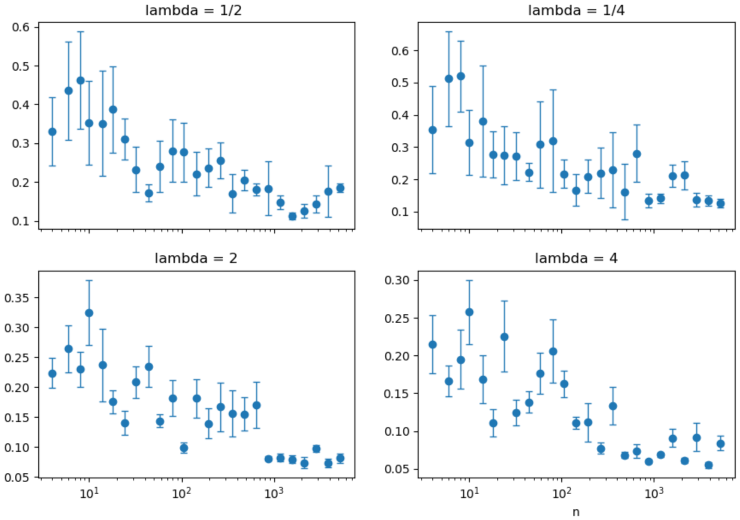
$$X_i = 1 - c_i^\dagger c_i, \quad Z_i = -\prod_{j<i} (1 - c_i^\dagger c_i)(c_i + c_i^\dagger)$$

- Diagonalize via discrete Fourier transform and a unitary transformation to a set of operators whose fermionic number is conserved (Bogoliubov transformation)

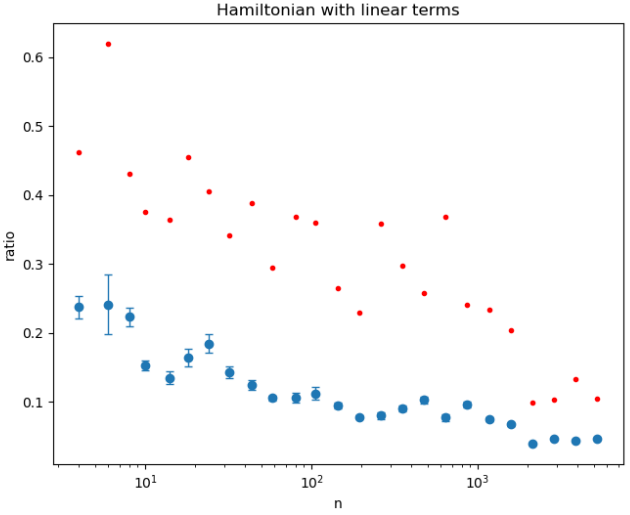
$\alpha = 0, \quad \beta = 1$



$\lambda = \alpha/\beta$



$$H = X_1X_2 + Z_1Z_2 + X_3 + X_4 + X_5X_6 + Z_5Z_6 + X_7 + X_8 \ldots$$



Next steps

- Exactly solve the semidefinite program
- Find optimal values for the constant c

The qutrit Bloch-space

We represent a state ρ with the help of a $d^2 - 1$-dimensional Bloch vector $\boldsymbol{\tau}$.

$$\rho = \frac{1}{d}\mathbb{1} + \sum_{i=1}^{d^2-1} \tau_i \sigma_i$$

For $d \geq 3$, there exist Bloch vectors with $|\tau| \leq 1$ which do not correspond to a positive semi-definite matrix.

Generalizing the Pauli matrices

The Gell-Mann matrices:

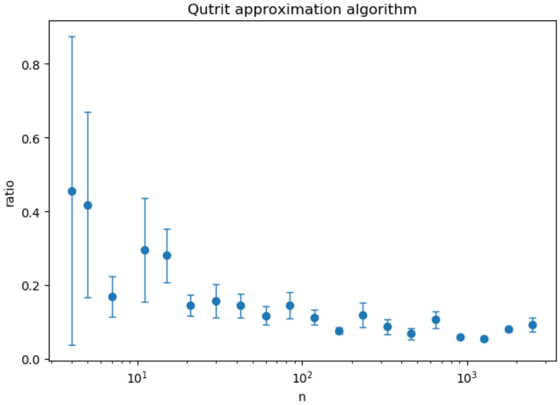
$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Generalizing the algorithm to qutrit systems

- The Bloch space for qutrits has a solid sphere of radius $\frac{1}{2}$, i.e. all states corresponding to $|\boldsymbol{\tau}| \leq \frac{1}{2}$ are valid states
- Adapt the algorithm to this smaller sphere the cut-off then being $\frac{1}{2\sqrt{8}}$
- $H = \sum_i \lambda_1^i \lambda_1^{i+1}$ with $\lambda_1^{n+1} = \lambda_1^1$



Qutrit approximation algorithm
ratio
0.0
0.2
0.4
0.6
0.8
n
10^1
10^2
10^3

Ideas for future work

- Analytically investigate the effiency of this algorithm
- Find more efficient rounding schemes that take into account the geometry of the Bloch space