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Introduction to quantum information theory

I will first give an introduction to quantum information theory, which uses quantum mechanical concepts to perform information processing and transmission of information. In quantum mechanics, we can associate a Hilbert space $\mathbb{H}$ with every quantum system. Quantum states are operators $\rho : \mathbb{H} \to \mathbb{H}$, which we can represent as density matrices. In general, a complex $M \times M$ matrix is a density matrix if it is:

1. Hermitian, $\rho = \rho^\dagger$,
2. positive, $\rho \geq 0$,
3. normalized, $Tr\rho = 1$.

The set of density matrices is a convex set and its pure states obey $\rho^2 = \rho$. In quantum computing, we mostly deal with N 2-level systems called qubits, the composite space of which is $H = H_1 \otimes H_2 \otimes \ldots \otimes H_N$. In this space, there are states ρ which can not be expressed through a tensor product of states in the subsystems $\rho = \rho_1 \otimes \rho_2 \ldots \otimes \rho_N$. We call these states entangled states. States which can be expressed as such are called separable or product states. We can represent a qubit state as

$$\rho = \frac{1}{2}\left(\mathbb{I} + \sum_{i=1}^{3} \sigma_i \tau_i\right).$$

This is called the Bloch representation of the state, and is associated with the Bloch vector $\boldsymbol{\tau}$. The σ_i are generators of $SU(2)$, in our case these are the Pauli matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

For qubits, the positivity property is equivalent to $Tr\rho^2 \leq Tr\rho$, which implies $|\boldsymbol{\tau}| \leq 1$ and characterizes the Bloch-vector-space as a solid Ball with Radius 1, which is called the Bloch sphere.

Suppose $|0\rangle$ and $|1\rangle$ form an orthogonal basis for the 2-dimensional one qubit state space. An arbitrary vector in the space can then be written

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,.$$

where a and b are complex numbers. The normalization condition of quantum states is equivalent to $\langle\psi|\,\psi\rangle = 1$ and $|a|^2 + |b|^2 = 1$. The orthogonal basis vectors of the state space are called computational basis.

The basic model of transmitting quantum information has three steps: We send a state ρ through a quantum channel $\mathcal{N}$ and the reciever has to measure the outcome in order to extract information. Quantum channels can be understood either as geometrical transformations associated with the bloch representation, or as completely positive, trace preserving maps. A quantum channel has to be trace preserving i.e. $Tr(\mathcal{N}(\rho)) = Tr(\rho)$ in order for the outcome state to be normalized. It must be completely positive, i.e., the map $\mathbb{I} \otimes \mathcal{N}$ maps positive semidefinite hermitian matrices to positive semidefinite hermitian matrices for any identity matrix $\mathbb{I}$, in order for the outcome state to be positive. A completely positive map is trace preserving if and only if $\sum_i A_i^\dagger A_i = \mathbb{I}$.

By the Kraus representation Theorem [1] a linear map Ψ is completely positive if and only if there exist operators $\{A_i\}$ such that

$$\Psi\left(\rho\right) = \sum_i A_i \rho A_i^\dagger.$$

Maps that are both completely positive and trace preserving are called CPT maps. We discern between unital and non-unital maps. Unital maps map the identity to itself. Geometrically, we can interpret this as the image of the map having the same center as the bloch sphere. Unital maps can be expressed as convex combinations of the Pauli operators and the identity. Their action in the bloch sphere are different rotations with shrinking parameters, since the Pauli matrices are unitary.[2]

The most commonly used model for quantum computation is the quantum circuit model, which generalizes its classical analogue. To classify quantum algorithms we use, in parallel to classical complexity theory, quantum complexity classes. The two prominent complexity classes in classical computation are P and NP. P is the set of problems which can be solved by a deterministic Turing machine in polynomial time, while NP is the set of problems which can be solved by a nondeterministic Turing machine in polynomial time. The class QMA is the quantum analogue to of NP in a probabilistic setting, the class of all problems which can be solved by a quantum verifier probabilistically in polynomial time.[3] We call a problem complete, if any other problem in its class can be reduced to it. Reduction means that for predicates L_1 and L_2 there is a polynomial f, such that $L_1(x) = L_2(f(x))$. We say that f reduces L_1 to L_2 polynomially.[4]

The Hamiltonian of a system corresponds to its energy, the spectrum of the operator being the set of possible outcomes when measuring the total energy. A k-local-Hamiltonian is a hermitian matrix acting on N qubits, which can be written as a sum of Hamiltonians where each acts on at most k qubits. Physically, this corresponds to system, where the interaction energy between more than k qubits is neglible. Specifically, we look at 2-local-Hamiltonians on qubits of the form

$$H = H_1 + H_2.$$

where

$$H_1 = \sum_{j=1}^{3n} D_j P_j, \quad H_2 = \sum_{i,j=1}^{3n} C_{i,j} P_i P_j.$$

with the Pauli-operators

$$P_{3a-2} = X_a, \; P_{3a-1} = Y_a, \; P_{3a} = Z_a.$$

The minimal eigenvalue of such a systems corresponds to its ground state. Since the quantum state achieving this optimal value might be an entangled state which might not be computable in polynomial time, we are interested in finding the product state that achieves the best approximation. It is equivalent to finding the best approximation to the maximal eigenvalue, because $\lambda_{max}(-H) = \lambda_{min}(H)$. [5]
In the local Hamiltonian problem, we are to determine, whether the groundstate energy of a given k-local Hamiltonian is below one threshold or above another. It is equivalent to the maximum constraint satisfaction problem from classical computation. The 2-local Hamiltonian problem is QMA complete.[3] Specifically, we look at *traceless* 2-local Hamiltonians, as these are the quantum generalization of binary quadratic functions of the form

$$F(x) = x^T B x + v^T x, \quad x \in \{\pm 1\}^n, .$$

where $B \in \mathbb{R}^{n \times n}$ is a matrix with zero diagonal and $v \in \mathbb{R}^n$ a vector.

Chapter 3

Relevant classical methods

For productstate approximation algorithms, many techniques from classical computing are used and generalized. Finding the maximal eigenvalue of a traceless 2-local hamiltonian is the quantum analogue to maximizing a binary quadratic program (MaxQP): Given a matrix A with $a_{ii} = 0$ maximize

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \quad \text{s.t.} \quad x_i \in \{-1, 1\} \quad \forall\, i.$$

An important tool for solving such problems is the relaxation of a semidefinite program (SDP), which has been pioneered by Goemans and Williamson.
In semidefinite programing we try to maximize a linear function, such that an affine combination of symmetric matrices is positive semidefinite. An affine combination is a linear combination $\sum_{i=1}^{n} a_i x_i$ where x_i are elements of a vector space, such that $\sum_{i=1}^{n} a_i = 1$. Semidefinite programs are very useful, as they can be solved efficiently both in theory and in practice.[6] We can write a general SDP as:

$$\begin{aligned} \text{minimize} \quad & C \cdot X \\ \text{subject to} \quad & A_i \cdot X = b_i, \quad i = 1, \dots, m \\ & X \geq 0 \end{aligned}$$

where C and A_i are symmetric matrices and $b_i \in \mathbb{R}^m$ a vector. This is called the primal problem. The dual of a SDP is its reformulated version, such that instead of minimizing (maximizing) an objective function, we maximize (minimize) another:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{m} y_i b_i \\ \text{subject to} \quad & C - \sum_{i=1}^{m} y_i A_i \geq 0 \end{aligned}$$

If the optimal value of the primal and the dual problem are the same, we say that *strong dualitity* holds. While this is not the case in general, it usually holds for SDPs.

For SDPs we can use Slaters condition for strong duality, which states that if there is an x^* that is strictly feasible (i.e. all constraints are satisfied and inequalities hold), then the problem is strictly dual.[7]
The relaxation of a SDP was first proposed by Goemans and Williamson as part of an approximation algorithm for the max-cut problem, which is a special case of MaxQP. In max-cut, we are given a graph and are to find a partition of the vertices into two sets, such that the number of edges between the two sets is as large as possible. Goemans and Williamson start by formulating the problem into a semidefinite program. Given a vertex set $V = \{1, \dots n\}$ and non-negative weights $w_{i,j} = w_{j,i}$, maximize the objective function $\frac{1}{2}\sum_{i<j} w_{i,j}(1 - y_i y_j)$ such that $y_i \in S = \{-1, 1\} \quad \forall\, i \in V$. As this is in NP, we need to relax the constraints, which is accomplished by extending the objective function to a larger space, namely $S^n = \{-1, 1\}^n$. We then have to consider vectors v_i and look at the inner product $v_i \cdot v_j$
The algorithm proposed by Goemans and Williamson proceeds by partitioning the vertices of the graph based on randomized rounding. The rounding is based on a random hyperplane cut of the vectors. It has a approximation ratio of 0.878.
The first algorithm for approximating an optimal solution of MaxQP was proposed by Charikar and Wirth and has a $\Omega\left(\frac{1}{\log n}\right)$ approximation ratio. [8] It also uses relaxation of a SDP and randomized rounding, but instead of partitioning based on a random hyperplane cut through the origin, it takes into account the size of zhe projections of a random vector onto the solution vectors. For this, the relaxed semidefinite program is

$$\begin{aligned} \max \quad & \sum_{ij} a_{ij} v_i \cdot v_j \\ \text{s.t.} \quad & v_i \cdot v_i = 1 \quad \forall\, i \\ & v_i \in \mathbb{R}^n \end{aligned}$$

Using this, the algorithm, which can be solved in polynomial time is:

1. Obtain an optimal solution $\{v_i\}$ to the SDP
2. Create vector r in which the r_i are independently distributed over the normal distribution
3. Let $z_i = v_i \cdot r/T$, where $T = \sqrt{4 \log n}$
4. If $|z_i| > 1$ then $y_i = sgn(z_i)$, otherwise $y_i = z_i$
5. Round the y_i to ± 1

To proof that this fullfils the approximation ratio, the idea is to first proof that the y_i are a good approximation to the z_i, i.e. that $\Delta_{ij} = z_i z_j - y_i y_j$ is sufficiently small. The algorithm described in the following chapter parallelizes this procedure.

Chapter 4

Approximation algorithms for the groundstate energy of traceless 2-local-Hamiltonians

The aim is to find bounds for the maximum energy of traceless 2-local-Hamiltonians. Before looking at the algorithm itself, there is a prelimary lemma we have to look at. Hamiltonians of the kind $H = H_1 + H_2$ where $H_1 = \sum_{j=1}^{3n} D_j P_j$, $H_2 = \sum_{i,j=1}^{3n} C_{i,j} P_i P_j$ have terms, that are linear in Pauli operators. For the theorems presented in this paper, the following lemma will enable us to reduce this Hamiltonian to a purely quadratic one. We form a new $n+1$-qubit Hamiltonian:

$$H' = H_2 + Z_{n+1} H_1.$$

Lemma1 $\lambda_{max}(H') = \lambda_{max}(H)$. Moreover, given any $(n+1)$-qubit state ω we can efficiently compute an n-qubit state ϕ such that

$$\langle\phi| H |\phi\rangle \geq \langle\omega| H' |\omega\rangle ..$$

If ω is a tensor product of single qubit stabilizer states then so is ϕ. The idea is now, that for any $n - qubit$ Hamiltonian with linear terms, there is a purely quadratic $(n+1)$-qubit Hamiltonian that has the same maximal eigenvalue and has an at best equally good product state approximation. Therefore, the bounds that we proof for quadratic Hamiltonians are valid also for Hamiltonians with linear terms. This enables us to set $H_1 = 0$ The proof idea is that all eigenvalues of H' are either eigenvalues of $H_2 - H_1$ or $H_1 + H_2$, and that $H_2 - H_1$ can be obtained from $H_1 + H_2$ by operations that conserve the spectrum. We can then choose the product state $|\phi\rangle$ according to $|\omega\rangle$, such that its eigenvalues will always be at least equal.
The last statement in the lemma references an elegant concept that is very useful to quantum error correction. We say an operator A stabilizes a state $|\psi\rangle$ if $A|\psi\rangle = |\psi\rangle$. Conversely, a state is called a stabilizer state of an operator, if it is in its $+1$-eigenspace. For practicality, we look at operators from the n-qubit Pauli group. This is favorable because they are unitary and their eigenvalues (± 1) differ significantly

from another, such that we can easily perform phase estimations to find out the eigenvalue. If we are given a set of operators $S = \{A, B, C \ldots\}$, we know that any errors (which are also from the Pauli group) either commute or anticommute with elements in S. One can correct any error E that anticommutes with S, and if the error lies in S it is correctable if they commute with S.[9] We say an operator commutes with a group, or is in the normalizer of the group, if for some $A, B \in S$: $EA = BE$ with possibly $A \neq B$. This criterion is very easy to check and gives us a useful mathematical toolbox.
We will now look at the algorithm presented, and then understand the ideas as to why it is indeed accurate. It is largely in parallel to the classical max-cut approximation algorithm by Goemans and Williamson. In our case, the semidefinite program is:

$$\begin{aligned} max\ tr\left(CM\right) \\ s.t. M_{i,i} = 1 \\ M \geq 0 \end{aligned}$$

where M is a real symmetric matrix. The ideal solution M is connected to our state in the following way:[5]

$$M_{i,j} = tr\left(\rho P_i P_j\right) \ \ i, j = 1 \ldots 3n.$$

Setting M as a real symmetric matrix is without loss of generality, because if the Pauli operators act on different matrices they commute, and therefore the matrix entry is real in this case. If they act on the same qubit, the matrix entry is purely imaginary because the operators anticommute. We can eliminate these terms using $M' = \frac{M+M*}{2}$ because they represent linear terms. This does not change the outcome of the objective function and is therefore fully without loss of generality. From this perspective the constraints can be understood in the following way:

$$M_{i,i} = tr\left(\rho P_i P_i\right) = tr\left(\rho\right) = 1$$

since $P_i P_i = 1$

$$M \geq 0 \leftrightarrow x^T M x = tr\left(\rho\left(\sum_i^{3n} x_i P_i\right)\left(\sum_j^{3n} x_j P_j\right)\right) = tr\left(\rho X^2\right) \geq 0$$

where $X = \sum_i^{3n} x_i P_i$ and since $X^2, \rho \geq 0$
Since M is real and symmetric, we can express any matrix element as $M_{i,j} = \left\langle v^i \middle| v^j \right\rangle$ for some unit vectors $v^1, v^2, \ldots, v^{3n+1}$. The vectors have unit norm, since $M_{i,i} = 1$. What we have done until now substitutes for the first step in the classical version. In the quantum version, we can interpret the geometry in a different, more instructive

way. Since we have to preserve the mathematical structure that states have to fulfill. Geometrically, this means that our blochvectors are restricted to a sphere of radius one. Since we will still project our optimal vectors obtained from the semidefinite program to a random hyperplane through the origin, we have to make sure trough rounding, that the projections correspond to valid bloch vector components. We ask the following question: What is the maximal value that a component of a valid bloch vector can have, if all three components have the same value? In other words, if we pick every value without knowing the others, what is the maximal value that we can assign to it so that it is part of the unit sphere?

$$\|z\| = \sqrt{z_1^2 + z_2^2 + z_3^2} = \sqrt{3z_1^2} = 1.$$

Therefore the maximal value, above which we should round down is $\frac{1}{\sqrt{3}}$. If we round such that no bloch vector component can be above this value, we will always have a valid state. Visually, this is the same as fitting a cube inside the bloch sphere and reducing our state space to inside the cube. The edges of the cube which touch the sphere are the pure states that are possible if all components are ± 1. The projections are $z_i = \frac{\langle r|v^i\rangle}{c\sqrt{\log n}}$ with $c = O(\log n)$. The algorithm looks like this:

1. Solve the relaxed semidefinite program, obtaining an optimal set of vectors v_i
2. Let $|r\rangle$ be a vector of $3n$ indepently and identically distributed $N(0,1)$ random variables
3. If $|z_i| > \frac{1}{\sqrt{3}}$, we round down: $y_i = \frac{sgn(z_i)}{\sqrt{3}}$. Otherwise $y_i = z_i$.

As output we take $\rho = \rho_1 \otimes \ldots \otimes \rho_n$ where:

$$\rho_a = \frac{1}{2}\left(\mathbb{I} + y_{3a-2}P_{3a-2} + y_{3a-1}P_{3a-1} + y_{3a}P_{3a}\right)..$$

The energy of this system is then

$$Tr\left(H\rho\right) = y^T C y.$$

One now has to think about why this works and how good the approximation is. In the introduction, we have found the best product approximation that can be found for the EPR-state. We call the highest eigenvalue achievable by a product state:

$$\lambda_{sep}(H) = \max_{\phi_1,\ldots,\phi_n} \langle\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_n| H |\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_n\rangle .$$

We know the following:
Suppose H is traceless 2-local Hamiltonian. then

$$\lambda_{sep}(H) \geq \frac{1}{9}\lambda_{max}(H).$$

We can proof this using entanglement-breaking depolarizing channels. Depolarizing channels are a simple model for noise in quantum information theory. [10] They are implemented by a map Δ_λ, which maps a state ρ into a linear combination of itself and the identity matrix:[11]

$$\Delta_\lambda(\rho) = \lambda\rho + \frac{1-\lambda}{d}\mathbb{I}, .$$

where d is the dimension of state. The parameter λ must satisfy

$$-\frac{1}{d^2-1} \leq \lambda \leq_1 .$$

This channel maps pure states to mixed states and all output states have eigenvalues $\lambda + \frac{1-\lambda}{d}$ (multiplicity 1) and $\frac{1-\lambda}{d}$ (multiplicity $d-1$). For $\lambda = 0$ we get the maximally noisy channel, for $\lambda = 1$ the identity. In our case we look at entanglement breaking channels. These are channels for which the output state is always seperable, i.e. if any entangled density matrix is mapped to a seperable one.[**horodecki08**] Here, the relevant map $\mathcal{E}_\delta$ is defined by its action on the Pauli group:

$$\mathcal{E}_\delta(I) = I \quad \mathcal{E}_\delta(P) = \delta P \quad P \in \{X, Y, Z\}.$$

Therefore, the action on a qubit state in bloch representation is:

$$\mathcal{E}_\delta(\rho) = \frac{1}{2}\mathbb{I} + \delta \sum_{i=1}^{3n} \tau_i P_i.$$

Geometrically, this reduces the length of any bloch vector by a factor δ. Generally, a CPT map Φ can be written as $\Phi(\rho) = \frac{1}{2}\left(\mathbb{I} + (\boldsymbol{t} + T\tau)P_i\right)$ where $\boldsymbol{t}$ is a vector and T a matrix.[12] We can write this as $\boldsymbol{T} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{t} & T \end{pmatrix}$, where we can assume without loss of generality that T is diagonal, which follows directly from the Kraus representation Theorem. Then $\boldsymbol{T}$ has the canonical form

$$\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}.$$

If $\boldsymbol{t} = 0$, this channel is unital. Unital qubit channels are entanglement breaking if and only if $\sum_j |\lambda_j| \leq 1$ (after T was diagonalized)[12] This implies that qubit channels of the form we look at in the paper are entanglement breaking for $\delta \leq \frac{1}{3}$ as we have $\lambda_1 = \lambda_2 = \lambda_3 = \delta$. Using our definition of $\mathcal{E}$, we see

$$Tr\left(\sigma P_{j_1} P_{j_2} \dots P_{j_L}\right) = \frac{1}{3^L} Tr\left(\rho P_{j_1} P_{j_2} \dots P_{j_L}\right).$$

. With this, we can proof the Theorem due to Lieb.
We consider the n-qubit state ψ satisfying $\langle\psi| H |\psi\rangle = \lambda_{max}(H)$ With the identity shown above, the depolarized state

$$\sigma = \mathcal{E}_{\frac{1}{3}}^{\otimes n} \left(|\psi\rangle \langle\psi|\right).$$

is seperable and

$$\lambda_{sep}(H) \geq Tr(\sigma H) = \frac{1}{9} \langle\psi| H |\psi\rangle .$$

which is the wanted statement.
This gives a clear bound on how good a seperable state can be in the worst case.

Chapter 5

Testing specific models

As an elementary example, let us look at a two qubit Hamiltonian:

$$H = X_1X_2 + Z_1Z_2.$$

The state achieving the maximal eigenvalue $\lambda_{max} = 2$ is the EPR-state $|EPR\rangle = \frac{\langle 00|+\langle 11|}{\sqrt{2}}$. This is a maximally entangled state. To find out the product state which approximates this the best, look at a general product state and maximize the overlap.

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = (a_1 |0\rangle + b_1 |1\rangle) \otimes (a_2 |0\rangle + b_2 |1\rangle) .$$

with $a_1^2 + b_1^2 = a_2 + b_2^2 = 1$

$$\max_{\psi_1,\psi_2} \left(|\langle \mathrm{EPR}|\, \psi\rangle|^2\right) = \max \left(\left|\frac{1}{\sqrt{2}} (a_1a_2 + b_1b_2)\right|^2\right) = \frac{1}{2}.$$

With either $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$ or $b_1 = b_2 = 1$ and $a_1 = a_2 = 0$. Therefore, the product states with the maximal overlap are $|00\rangle$ and $|11\rangle$ with maximal eigenvalue $\lambda_{sep} = 1$, the approximation ratio being $\frac{\lambda_{sep}}{\lambda_{max}} = 0.5$

Chapter 6

Generalization to qutrits

We can write a state ρ as

$$\rho = \frac{1}{M}\mathbb{I} + \sum_{i=1}^{M^2-1} \tau_i \sigma_i$$

where σ_i are generators of $SU(M)$ obeying

$$\sigma_i \sigma_j = \frac{2}{M}\delta_{ij} + d_{ijk}\sigma_k + i f_{ijk}\sigma_k.$$

f_{ijk} is totally antisymmetric and equals the Levi-Civita-Symbol for $M = 2$, d_{ijk} is totally symmetric and vanishing for $M = 2$. This is the Bloch representation of quantum states. We can construct the generators as follows:[13]

$$\{\sigma_i\}_{i=1}^{M^2-1} = \{u_{jk}, v_{jk}, w_l\}.$$

where

$$u_{jk} = |k\rangle\langle k| + |k\rangle\langle j|\,,\ v_{jk} = -i(|j\rangle\langle k| - |k\rangle\langle j|),$$

$$w_l = \sqrt{\frac{2}{l(l+1}} \sum_{j=1}^{l} \left(|j\rangle\langle j| - l\,|l+1\rangle\langle l+1|\right),$$

$$1 \leq j \leq k \leq M, 1 \leq l \leq M-1$$

The τ_i are the components of the $M^2 - 1$ dimensional bloch vector and are the expectation values of the σ_i:

$$\tau_i = Tr(\rho\sigma_i)$$

For $M \geq 3$ there are bloch vectors which do not correspond to a positive semi-definite matrix. The space spanned by the bloch-vectors is therefore n[illegible] solid ball with radius 1. The generators of $SU(3)$ are the Gell-Mann-matrices.

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