Bayesian Q Estimation with Attribute Hierarchy

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Bayesian Models for Q

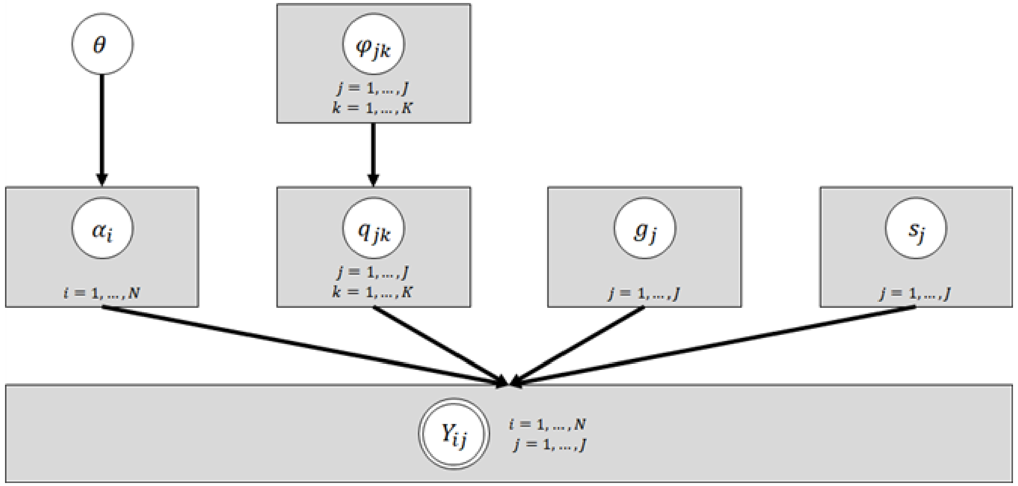
- Suggested by Chung(2014) & Chen(2018) → Same Gibbs Sampling method Essentially!

- Variables for Estimation

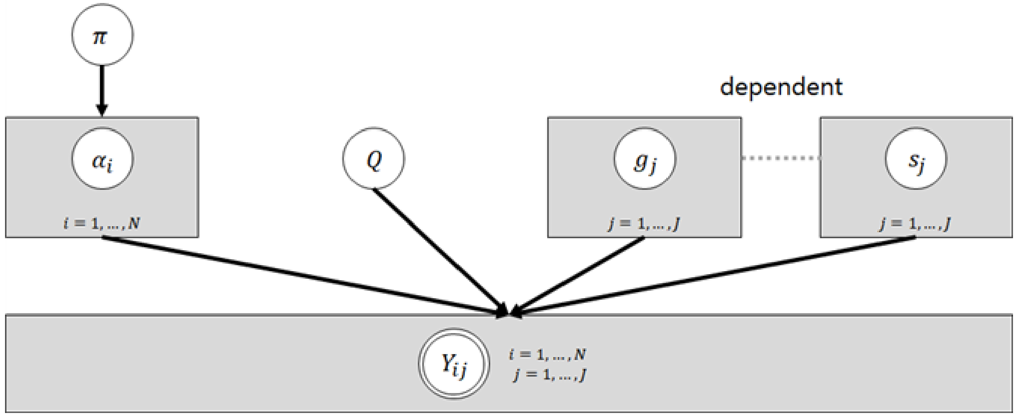
Variables	Notation	Prior
Attribute pattern	α_i, for $i=1,\dots,N$	For $c \in \{0,1,\dots,2^K-1\}$, $\Pr(\alpha_i = a_c) = \pi_c$
Class distribution	$\pi = (\pi_0,\dots,\pi_{2^K-1})$	$\pi \sim dirichlet(1,1,\dots,1)$
Item parameter	g_j and s_j, for $j=1,\dots,J$	$g_j \sim Beta(\alpha_g,\beta_g)$ and $s_j \sim Beta(\alpha_s,\beta_s)$, with constraint $0 \le g_j < 1-s_j \le 1$ $p(s_j, g_j) \propto s_j^{\alpha_s-1}(1-s_j)^{\beta_s-1} g_j^{\alpha_g-1}(1-g_j)^{\beta_g-1} I(0 \le g_j < 1-s_j \le 1)$
Q matrix	$Q=(q_{jk})_{J\times K}$	Chung) $\Pr(q_{jk}=1)=p_{jk}$, $p_{jk} \sim Beta(a, b)$
($\Pr(q_{jk}=1)$)	p_{jk}, for $j=1,\dots,J$, $k=1,\dots,K$	Chen) Uniform on $\mathbb{Q}$ (Identiablilty Q Set)

Bayesian Models for Q

Bayesian Model (Chung, 2014)



Bayesian Model (Chen, 2018)



Likelihood on DINA Model

$$P(Y_{ij}=1 \mid \alpha_i, q_j, s_j, g_j) = (1-s_j)^{\eta_{ij}} g_j^{1-\eta_{ij}} \quad , \text{ where } \quad \eta_{ij} = \prod_k \alpha_{ik}^{q_{ik}}$$

fully, $$P(Y \mid \alpha_i, q_j, s_j, g_j) = \prod_{i=1}^{N} \prod_{j=1}^{J} ((1-s_j)^{\eta_{ij}} g_j^{1-\eta_{ij}})^{y_{ij}} (s_j^{\eta_{ij}} (1-g_j)^{1-\eta_{ij}})^{1-y_{ij}}$$

Gibbs Sampling Process

[Step1] Update Attribute pattern

Prior(pmf) : $\Pr(\alpha_i = a_c \mid \pi) = \pi_c$ for each class $c = 0, 1, \ldots, 2^K - 1$

Likelihood : $P(\boldsymbol{Y_i} = \boldsymbol{y_i} \mid \alpha_i = a_c, Q, \boldsymbol{s}, \boldsymbol{g}) = \prod_{j=1}^{J} [(1-s_j)^{\eta_{c,j}} g_j^{1-\eta_{c,j}}]^{y_{ij}} [s_j^{\eta_{c,j}} (1-g_j)^{1-\eta_{c,j}}]^{1-y_{ij}}, \quad \eta_{c,j} = \prod_{k=1}^{K} a_{c(k)}^{q_{jk}}$

$\Rightarrow$ Posterior(pmf) : Prior x Likelihood

- Use the fact 'log(prior) + log(likelihood) = log(posterior)'
- Use appropriate matrices to update all attribute patterns at the same time

Gibbs Sampling Process

[Step2] Update Class Distribution

Prior : $\pi \sim dirichlet(1,1,\dots,1)$

Likelihood : $P(\alpha_1,\dots,\alpha_N \mid \pi) = \pi_0^{n_0}\pi_1^{n_1}\cdots\pi_{2^K-1}^{n_{2^K-1}}$ where n_c : # of attribute pattern a_c

$\Rightarrow$ Posterior : $\pi \mid \alpha \sim dirichlet(1+n_0,\ 1+n_1,\ \dots,1+n_{2^K-1})$

- Dirichlet distribution is the conjugate prior of Multinomial distribution!

Gibbs Sampling Process

[Step3] Update Item Parameter (Chen)

Prior : $p(s_j, g_j) \propto s_j^{\alpha_s - 1}(1-s_j)^{\beta_s - 1} g_j^{\alpha_g - 1}(1-g_j)^{\beta_g - 1} I(0 \le g_j < 1 - s_j \le 1)$

Likelihood : $P(y_{1j}, \ldots, y_{Nj} \mid \alpha, Q, s_j, g_j) = \prod_{i=1}^{N} [(1-s_j)^{\eta_{ij}} g_j^{1-\eta_{ij}}]^{y_{ij}} [s_j^{\eta_{ij}}(1-g_j)^{1-\eta_{ij}}]^{1-y_{ij}}$

$\Rightarrow$ Posterior : $p(s_j, g_j \mid y_{1j}, \ldots, y_{Nj}, \alpha, Q) \propto s_j^{\tilde{\alpha}_s - 1}(1-s_j)^{\tilde{\beta}_s - 1} g_j^{\tilde{\alpha}_g - 1}(1-g_j)^{\tilde{\beta}_g - 1} I(0 \le g_j < 1 - s_j \le 1)$

where $\tilde{\alpha}_s = \sum_{i=1}^{N} (1-y_{ij})\eta_{ij} + \alpha_s$, $\tilde{\beta}_s = \sum_{i=1}^{N} y_{ij}\eta_{ij} + \beta_s$, $\tilde{\alpha}_g = \sum_{i=1}^{N} y_{ij}(1-\eta_{ij}) + \alpha_g$, $\tilde{\beta}_g = \sum_{i=1}^{N} (1-y_{ij})(1-\eta_{ij}) + \beta_g$

- Chung(2014) used the fact that Beta distribution is the conjugate prior of Binomial distribution, ad conjugate prior and a quantile function for inverse transform to maintain $0 \le g_j < 1 - s_j \le 1$

  → Consequently, the same result!

Gibbs Sampling Process

[Step4] Update Q-matrix (Chung)

Prior : $\Pr(\boldsymbol{q}_j = a_c | p_{j1}, \ldots, p_{jk}) \propto \prod_{k=1}^{K} [p_{jk}^{a_{c(k)}} (1-p_{jk})^{1-\alpha_{c(k)}}]$ and $\Pr(\boldsymbol{q}_j = (0,0,\ldots,0)) = 0.$

Likelihood : $P(y_{1j}, \ldots, y_{Nj} \mid \boldsymbol{q}_j = a_c, \boldsymbol{\alpha}, s_j, g_j) = \prod_{i=1}^{N} [(1-s_j)^{\eta_{ij}^{(c)}} g_j^{1-\eta_{ij}^{(c)}}]^{y_{ij}} [s_j^{\eta_{ij}^{(c)}} (1-g_j)^{1-\eta_{ij}^{(c)}}]^{1-y_{ij}}$

Posterior(pmf) : Prior x Likelihood

- Update q-vector of Q-matrix one by one.
- Use the fact 'log(prior) + log(likelihood) = log(posterior)'

Gibbs Sampling Process

[Step5] Update Probability for q=1 (Chung)

Prior : $p_{jk} \sim Beta(a, a)$

Likelihood : $P(q_{jk} \mid p_{jk}) = \begin{cases} p_{jk} & \text{for } q_{jk} = 1 \\ 1 - p_{jk} & \text{for } q_{jk} = 0 \end{cases}$

$\Rightarrow$ Posterior : $p_{jk} \mid q_{jk} \sim Beta(a + q_{jk}, a + 1 - q_{jk})$

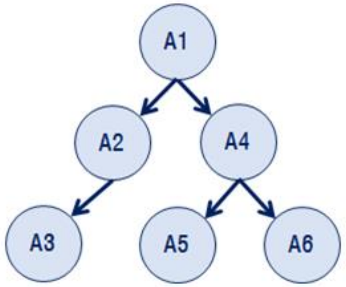
- Beta distribution is the conjugate prior of Binomial distribution!

Attribute Hierarchy

- Words : preceding attribute, following attribute, root attribute,

  adjacent matrix, access matrix, possible attribute pattern

- Expressions for the hierarchy information :

Direct Graph	Adjacent Matrix	Access matrix
	$A = \begin{bmatrix} 010100 \\ 001000 \\ 000000 \\ 000011 \\ 000000 \\ 000000 \end{bmatrix}$	$R = \begin{bmatrix} 111111 \\ 011000 \\ 001000 \\ 000111 \\ 000010 \\ 000001 \end{bmatrix}$

Develop a more efficient and accurate estimation method for Q-matrix with the hierarchy information

Bayesian Model - H

- Hierarchy structure -> fixed meaning of attributes
- A : adjacent matrix for hierarchy information.
- QI : Expert suggested matrix

  Each entries of QI means the probability of '(corresponding q-entry)=1

Examples of QI matrix

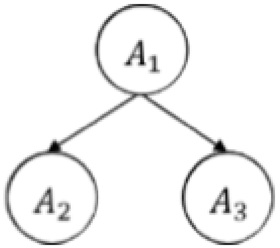
	A1	A2	A3
P1	1	0.3	0.3
P2	0.7	1	0.3
P3	0.7	0.3	0.7
P4	0.3	0.7	0.3
P5	0.3	0.7	0.7
:	:	:	:

	A1	A2	A3
P1	0.8	0.3	0.1
P2	0.9	0.7	0.2
P3	0.8	0.2	0.7
P4	0.1	0.9	0.4
P5	0.2	0.8	0.7
:	:	:	:

- λ : Prior Effect size (default $\lambda = 1$)

Bayesian Model - Strong H

[Possible Attribute Patterns]



$a_0=(0,0,0)$	$c_1=0$
$a_1=(0,0,1)$	X
$a_2=(0,1,0)$	X
$a_3=(0,1,1)$	X
$a_4=(1,0,0)$	$c_2=4$
$a_5=(1,0,1)$	$c_3=5$
$a_6=(1,1,0)$	$c_4=6$
$a_7=(1,1,1)$	$c_5=7$

[Priors for Possible Attribute Pattern & Restricted Class Distribution]

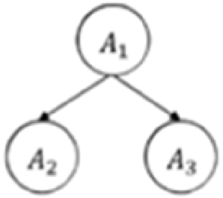
Variables	Notation	Prior
Attribute pattern	α_i, for $i=1,\dots,N$	For each classnum $c_l \in C_H$, $\Pr(\alpha_i=a_{c_l})=\pi_{c_l}$, and for $c\in\{0,1,\dots,2^K-1\}-C_H$, $\Pr(\alpha_i=a_c)=0$
Restricted Class distribution	$\pi=(\pi_{c_1},\dots,\pi_{c_L})$	$\pi\sim dirichlet(1,1,\dots,1)$

- In the situation of strong hierarchy, prior for attribute patterns other than possible attribute patterns set 0.

Bayesian Model - Strong H

[The equivalence relationship of q-vecor]

- In the situation of strong hierarchy, two different q-vectors can have the same likelihood
- Define the equivalence relationship if two different q-vectors have the same likelihood

Hierarchy	Att. Pattern	$q_j = (0,1,0)$	$q_j = (1,1,0)$
	$(0,0,0)$	$\eta_{ij} = 0$	$\eta_{ij} = 0$
	$(1,0,0)$	$\eta_{ij} = 0$	$\eta_{ij} = 0$
	$(1,0,1)$	$\eta_{ij} = 0$	$\eta_{ij} = 0$
	$(1,1,0)$	$\eta_{ij} = 1$	$\eta_{ij} = 1$
	$(1,1,1)$	$\eta_{ij} = 1$	$\eta_{ij} = 1$

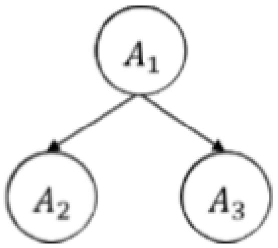
→ Equivalence Class : $\{(1,0,0)\},\ \{(0,1,0),\ (1,1,0)\},\ \{(0,0,1),\ (1,0,1)\}, \{(0,1,1),(1,1,1)\}$

Bayesian Model - Strong H

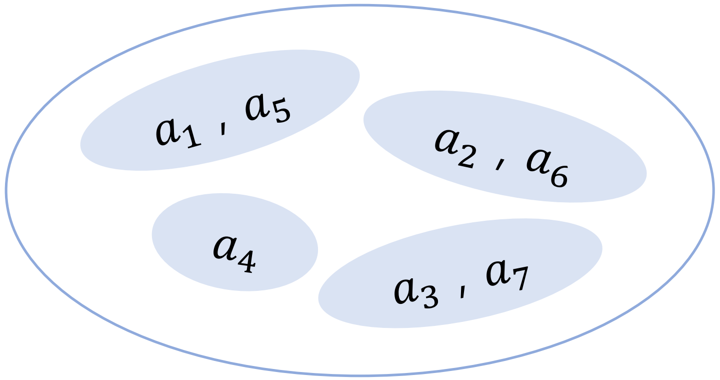
- (# of equivalence classes for q-vector) = (# of possible attribute patterns) - 1

  ∵ Each classes have 1-1 correspondence with a non-zero possible attribute pattern.

- Notation for the equivalence classes : $\widetilde{a}_{c_2}, \widetilde{a}_{c_3}, \cdots, \widetilde{a}_{c_l}$



$a_0 = (0,0,0)$	$c_1 = 0$
$a_1 = (0,0,1)$	X
$a_2 = (0,1,0)$	X
$a_3 = (0,1,1)$	X
$a_4 = (1,0,0)$	$c_2 = 4$
$a_5 = (1,0,1)$	$c_3 = 5$
$a_6 = (1,1,0)$	$c_4 = 6$
$a_7 = (1,1,1)$	$c_5 = 7$



For the efficiency, first, determine the equivalence classes for each q-vectors, then, decide q-vector by the closest q-vector form the experts' suggestion.

Bayesian Model - Strong H

- qnum: the equivalence class number for a q-vector (For q-vectors in $\widetilde{a_{c_l}}$, qnum = l)
- Qnum: the collection of qnum

[Priors for Qnum & qnum distribution]

Variables	Notation	Prior
Qnum	$Qn=(qn_1,\ldots,qn_J)^T$	For $c_l \in C_H - \{c_1\}$, $\Pr(qn_j = l \mid \phi) = \Pr(\boldsymbol{q}_j \in \widetilde{a_{c_l}}) = \phi_{c_l}$
qnum distribution	$\phi_j = (\phi_{c_2}^{(j)}, \ldots, \phi_{c_L}^{(j)})$, for $j=1,\ldots,J$	$\phi_j \sim dirichlet(\lambda p_{c_2}^{(j)},\ldots,\lambda p_{c_L}^{(j)})$,for $p_{c_l}^{(j)} = \sum_{a_c \in \widetilde{a_{c_l}}} P(\boldsymbol{q}_j = a_c)/(1-p_0^{(j)})$ Each $P(\boldsymbol{q}_j = a_c)$ and $p_0^{(j)} = P(\boldsymbol{q}_j = (0,0,\ldots,0))$ can be computed by QI matrix. (Defaul : $\lambda = 1$)

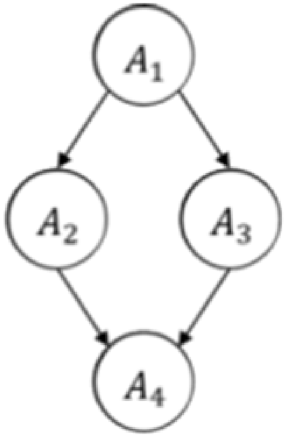
Data Simulation

- N = 1000 (same with Chung(2014))
- Hierarchy structure & true Q matrices (used by Chen el.(2016))

Structure1 ($K=3$, $J=10$)

	A_1	A_2	A_3
I1	1	0	0
I2	0	1	0
I3	0	0	1
I4	1	0	0
I5	0	1	0
I6	0	0	1
I7	1	1	0
I8	1	0	1
I9	0	1	1
I10	1	1	1

Structure2 ($K=4$, $J=18$)



	A_1	A_2	A_3	A_4
I1	1	0	0	0
I2	0	1	0	0
I3	0	0	1	0
I4	0	0	0	1
I5	1	0	0	0
I6	0	1	0	0
I7	0	0	1	0
I8	0	0	0	1
I9	1	1	0	0
I10	1	0	1	0
I11	1	0	0	1
I12	0	1	1	0
I13	0	1	0	1
I14	0	0	1	1
I15	1	1	1	0
I16	1	1	0	1
I17	1	0	1	1
I18	0	1	1	1

Data Simulation

- Master probability for root attributes & conditional master probablity

	Structure1	Structure2
	$P(A_1=1)=r_1,\ P(A_1=0)=1-r_1,$ $P(A_2=1\mid A_1=1)=r_{2,1},$ $P(A_2=1\mid A_1=0)=r_{2,0}$ $P(A_3=1\mid A_2=1)=r_{3,1},$ $P(A_3=1\mid A_2=0)=r_{3,0}$	$P(A_1=1)=r_1,\ P(A_1=0)=1-r_1,$ $P(A_2=1\mid A_1=1)=r_{2,1},\ P(A_2=1\mid A_1=0)=r_{2,0}$ $P(A_3=1\mid A_1=1)=r_{3,1},\ P(A_3=1\mid A_1=0)=r_{3,0}$ $P(A_4=1\mid A_2A_3=1)=r_{4,1},\ P(A_4=1\mid A_2A_3=0)=r_{4,0}$
Default	$r_1=0.7,\ r_{2,1}=0.8,\ r_{3,1}=0.8$ Strong H : $r_{2,0}=r_{3,0}=0$ Weak H : $r_{2,0},\ r_{3,0}\sim Unif[0,0.1]$	$r_1=0.7,\ r_{2,1}=0.8,\ r_{3,1}=0.8,\ r_{4,1}=0.8$ Strong H : $r_{2,0}=r_{3,0}=r_{4,0}=0$ Weak H : $r_{2,0},\ r_{3,0},\ r_{4,0}\sim Unif[0,0.1]$

- Default for item parameters : $s_j=g_j=0.2$ (same with Chung(2014))

Future Work

- Accuracy of Bayesian Estimation - SH
- Suggest appropriate λ
- Develop Bayesian Model - Weak Hierarchy
- Compare previous methods, in terms of accuracy and efficiency
- Studies about the effects by various factors
- …