NO TITLE

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Abstract

Acknowledgments

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2.2.1 General anelastic MHD

!!!! !!!! The anelastic model is an extention of MLT which includes stratification by including a first order pertubed field upon a static background. It is based on the assumption that convective flow is carried by the foreground field. This approach filters out sound waves

... DERIVATION LIKE SONDRE ON THE BLACKBOARD ...

The anelastic approximation is motivated by Mixing-Length Theory (MLT), which is a model where we look at how energy is transported over a characteristic mixing length before being dissapated. This model gives us the dominant balance for the lowest order equations and the relative sizes of pertubations to the first order 1999ApJS..121..247L. The MLT model has been verified against fully-compressible simulations which appear to support it's validity 1989ApJ...336.1022C, but needs modification around the bottom and top of a convective layer 1996ApJ...466..372C. This is where the anelastic approximation comes in, which is a an extention of MLT which includes stratification.

In the anelastic approximation we separate the thermodynamical quantities upon a background reference state and an overlaying pertubation. The background state can be set up with a non-trivial depth stratification without full compressibility, filtering out fast-moving sound waves 1999ApJS..121..247L. This has the effect of lowering the

required time-resolution due to not having to consider the acoustic time scale. Instead we only have to consider the much lower dynamic time scale, which is determined by the flow velocity and AlfvÃ©n speed.

The background state can be time dependent 1976ApJ...207..545T or time independent. The problem with a time dependent background state is that the entropy gradient can be strongly superadiabatic in an upper thermal boundary layer, resulting a high Mach flow number, making this layer supersonically unstable 1975JFM....68..695G. It is therefore prefered that the entropy background is constant in time.

(HERE: ADIABATIC PARAMETERS. PLASMA BETA. WHERE DOES THE ANELASTIC APPRXIMATION HOLD. EQUATION OF STATE (IDEAL GAS LAW WITH FIRST ORDER PERTUBATION?))

We will now show a simple example of how to use pertubation theory on the hydrodynamical equations. Assume that we are in one dimension with a constant flow and that the continuity equation has no sinks or sources. This gives us that the flux of the mass density ρ is ρu, where u is the fluid velocity, and we get that the continuity equation becomes !!!! !!!!

2.2.2 Ideal anelastic HD

We start by looking at the ideal case where the thermal diffusivity $K = 0$, the viscous stress $\mu = 0 \rightarrow \Pi = 0$, no magnetic field and no geometrical effects $\mathbf{g} = -g\hat{\mathbf{k}}$, where $\hat{k}$ is the unit vector in the upward direction (z-direction). This gives us the momentum equation

$$\rho_0 \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = -\nabla p_1 - \rho_1 g \hat{\mathbf{k}} \tag{2.1}$$

and the entropy equation

$$\rho_0 T_0 \left[\frac{\partial s_1}{\partial t} + (\mathbf{v} \cdot \nabla)(s_0 + s_1) \right] = 0 \tag{2.2}$$

in the ideal case. The other equations are unchanged. To solve this we start by writing the momentum equation w.r.t the time derivative

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0}\nabla p_1 - (\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{\rho_1}{\rho_0} g \hat{\mathbf{k}}, \tag{2.3}$$

Then we split this up into it's two components for the horizontal (x-direction) $\hat{\mathbf{i}}$ and the vertical direction $\hat{\mathbf{k}}$ giving us

$$\hat{\mathbf{i}}: \ \partial_t v_x = -\frac{1}{\rho_0}\partial_x p_1 - (v_x \partial_x + v_z \partial_z) v_x, \tag{2.4}$$

$$\hat{\mathbf{k}}: \ \partial_t v_z = -\frac{1}{\rho_0}\partial_z p_1 - (v_x \partial_x + v_z \partial_z) v_z - \frac{\rho_1}{\rho_0} g(z). \tag{2.5}$$

We then write the entropy equation w.r.t. the time derivative, giving us

$$\partial_t s_1 = -(v_x \partial_x + v_z \partial_z) s_1 - v_z \partial_z s_0. \tag{2.6}$$

2.2.3 Adding thermal diffusivity

2.2.4 Adding viscous stress

2.2.5 Adding the magnetic field

2.3 Numerical Methods

2.3.1 Discretization and derivatives

The simulation box is set up with N_z grid points in the z-direction and N_x in the x-direction. These have a resolution of Δz and Δx respectively. Then our grid points are at locations

$$\begin{aligned} z_i &= z_0 + i\Delta z,\ i \in [0, N_z], \\ x_j &= x_0 + j\Delta x,\ j \in [0, N_x]. \end{aligned}$$

For the temporal discretization we get a variable resolution Δt based on Von Neumann analysis, a start time t_0, usually set to zero, and an end time T. Then

$$t_n = t_0 + n\Delta t\ \in [t_0, T].$$

We name the foreground variables $f(z, x, t)$ and the background variables $h(z)$. These are then discretized as

$$\begin{aligned} f(z, x, t) \to f(z_i, x_i, t_n) &= f^n_{i,j}, \\ h(z) \to h(z_i) &= h_i. \end{aligned}$$

We then let the analytical derivatives go to a numerical derivative on the discretized variables, here for a derivative in the x-direction

$$\frac{\partial f(z, x, t)}{\partial x} \to \left[\frac{\partial f(z_i, x_j, t_n)}{\partial t}\right]^n_{i,j}.$$

These are handled using a number of known spacial and temporal schemes. For the temporal schemes we will use the Runge-Kutta methods and for the spacial methods we have to take care in which method used based on the form of the equation. As is shown in section 2.3.3 from Von Neumann analysis the advection equation is numerically unstable for a variety of spacial schemes based on the order of the Runge-Kutta method. Especially the central schemes are unstable and the advection terms should therefore be handled using upwind methods, or forward spacial (FS). Using the second order upwind method we have

$$\left[\frac{\partial f}{\partial x}\right]^n_{i,j} = \begin{cases} \frac{3f^n_{i,j} - 4f^n_{i,j-1} + f^n_{i,j-2}}{2\Delta x} & \text{if } v_x \geq 0 \\ \frac{-3f^n_{i,j} + 4f^n_{i,j+1} - f^n_{i,j+2}}{2\Delta x} & \text{if } v_x < 0 \end{cases} \tag{2.7}$$

In the following we will write out the momentum- and entropy-equations using the second order central scheme where possible and keep the brackets on derivatives including temporal methods or terms that needs an FS scheme.

$$\left[\frac{\partial v_x}{\partial t}\right]^n_{i,j} = -\frac{1}{\rho_{0,(i)}}\frac{p^n_{1,(i,j+1)} - p^n_{1,(i,j-1)}}{2\Delta x} - v^n_{x,(i,j)}\left[\frac{\partial v_x}{\partial x}\right]^n_{i,j} - v^n_{z,(i,j)}\left[\frac{\partial v_x}{\partial z}\right]^n_{i,j}, \tag{2.8}$$

$$\left[\frac{\partial v_z}{\partial t}\right]^n_{i,j} = -\frac{1}{\rho_{0,(i)}}\frac{p^n_{1,(i+1,j)} - p^n_{1,(i-1,j)}}{2\Delta z} - v^n_{x,(i,j)}\left[\frac{\partial v_z}{\partial x}\right]^n_{i,j} - v^n_{z,(i,j)}\left[\frac{\partial v_z}{\partial z}\right]^n_{i,j} - \frac{\rho^n_{1,(i,j)}}{\rho_{0,(i)}}g(z_i) \tag{2.9}$$

$$\left[\frac{\partial s_1}{\partial t}\right] = -v_{x,(i,j)}\left[\frac{\partial s_1}{\partial x}\right]^n_{i,j} - v_{z,(i,j)}\left[\frac{\partial s_1}{\partial z}\right]^n_{i,j} - v_{z,(i,j)}\frac{s_{0,(i+1,j)} - s_{0,(i-1,j)}}{2\Delta x}. \tag{2.10}$$

2.3.2 The Elliptic Equation

To find the pressure we take the divergence of the momentum equation **??**

$$\nabla \cdot \frac{\partial(\rho_0 \mathbf{v})}{\partial t} + \nabla \cdot \rho_0(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla^2 p_1 - \partial_z(\rho_1 g).$$

We use Clairaut's theorem on the first term

$$\frac{\partial[\nabla\cdot(\rho_0\mathbf{v})]}{\partial t} + \nabla \cdot \rho_0(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla^2 p_1 - \partial_z(\rho_1 g)$$

and get that the first term is zero due to the continuity equation. After simplification we end up with the expression

$$\nabla^2 p_1 = -\partial_z(\rho_1 g) - \rho_0\left[\partial_x\partial_z(v_x v_z) + v_z\partial_z^2 v_z + 2\partial_z(v_x)\partial_x(v_z)\right] - v_x\partial_x(v_z)\partial_z(\rho_0). \tag{2.11}$$

We call the right hand side $F(z,x,t)$ and write the equation on discretized form

$$\left[\nabla^2 p\right]^n_{i,j} = F^n_{i,j}. \tag{2.12}$$

We remove the sub- and super-script, keeping in mind that this is the pertubed pressure at timestep n. Using the second order central second derivative we get that

$$\frac{p_{i-1,j} - 2p_{i,j} + p_{i+1,j}}{\Delta z^2} + \frac{p_{i,j-1} - 2p_{i,j} + p_{i,j-1}}{\Delta x^2} = F_{i,j}. \tag{2.13}$$

We call $a = 1/\Delta z^2$, $b = 1/\Delta x^2$ and $c = -2(a+b)$. This gives us that

$$ap_{i-1,j} + ap_{i+1,j} + cp_{i,j} + bp_{i,j-1} + bp_{i,j-1} = F_{i,j} \tag{2.14}$$

TODO: SET UP AS MATRIX WITH FLAT F, p. TALK ABOUT THE STENCIL FOR THIS MATRIX AND HOW TO SOLVE USING JACOBI METHOD. THEN GAUSS-SEIDEL BUT NEED TO FIGURE OUT HOW TO HANDLE BOUNDARY.

2.3.3 Stability Analysis

We will use Von Neumann analysis of the advection equation to determine the required time-step sizes for our solver. The advection equation is

$$\frac{\partial u(t,x)}{\partial t} = -a\frac{\partial u(t,x)}{\partial x}, \tag{2.15}$$

where $u(t,x)$ is the exact solution and a is constant representing the velocity. This is discretized over a grid such that $y(t_n, x_j) = y_{n,j}$, where $t_n = n\Delta t$ and $x_j = j\Delta x$ for $n, j \in \mathbb{N}$, is our numerical solution. The advection equation then becomes

$$\left[\frac{\partial y}{\partial t}\right]_{n,j} = -a\left[\frac{\partial y}{\partial x}\right]_{n,j}. \tag{2.16}$$

The numerical solution is

$$y_{n,j} = u(t_n, x_j) + \epsilon_{n,j}, \tag{2.17}$$

where $\epsilon_{n,j}$ is the round-off error. The round-off error must also satisfy the discretized equation and this gives us that

$$\left[\frac{\partial \epsilon}{\partial t}\right]_{n,j} = -a\left[\frac{\partial \epsilon}{\partial x}\right]_{n,j}. \tag{2.18}$$

We expand the round-off error as a fourier series

$$\epsilon(t_n, x_j) = \sum_m E_m(t_n) e^{ik_m j\Delta x}, \tag{2.19}$$

where k_m is the wavenumber and $E_m(t_n)$ is the time-dependent amplitude of the error. When inserting this into our differential equation we get a linear difference equation, meaning that each of the terms behave like the entire series, so we can consider the growth of only one term

$$\epsilon_m(t_n, x_j) = E_m(t_n) e^{ik_m j\Delta x}. \tag{2.20}$$

We will show the calculations using the first-order upwind scheme with the second-order Runge-Kutta scheme. Since this should be true for any m we remove the subscipt, define $\beta \equiv k\Delta x$ and get that the spacial derivative is

$$\begin{aligned}\left[\frac{\partial \epsilon}{\partial x}\right]_{n,j} &= \frac{\epsilon_{n,j} - \epsilon_{n,j-1}}{\Delta x} \\ &= \frac{E(t_n)e^{i\beta j} - E(t_n)e^{i\beta(j-1)}}{\Delta x} \\ &= E(t_n)e^{i\beta j}\frac{1 - e^{-i\beta}}{\Delta x}.\end{aligned}$$

This gives us that the advection equation 2.18 becomes

$$\left[\frac{\partial \epsilon}{\partial t}\right]_{n,j} = e^{i\beta j}\left[\frac{\partial E(t_n)}{\partial t}\right]_{n,j} = -aE(t_n)e^{i\beta j}\frac{1-e^{-i\beta}}{\Delta x} \tag{2.21}$$

$$\left[\frac{\partial E(t_n)}{\partial t}\right]_{n,j} = -aE(t_n)\frac{1-e^{-i\beta}}{\Delta x}. \tag{2.22}$$

We define $\lambda = -\frac{a}{\Delta x}\left(1-e^{-i\beta}\right)$ which gives us

$$\mu = \Delta t\lambda = -C\left(1-e^{-i\beta}\right), \tag{2.23}$$

where $C \equiv a\Delta t/\Delta x$ is the Courant number. This means that the differential equation for the time-dependent error is

$$\left[\frac{\partial E(t_n)}{\partial t}\right]_{n,j} = \lambda E_n. \tag{2.24}$$

Using this with the second order Runge-Kutta scheme, the slopes for the time-dependent error is

$$k_1 = \lambda E_n,$$
$$k_2 = \lambda\left(E_n + \frac{\Delta t}{2}k_1\right) = E_n\left(\lambda + \Delta t\lambda^2\right).$$

And the next time-step for the error is

$$\begin{aligned} E_{n+1} &= E_n + \Delta t\left(\frac{k_1}{2} + \frac{k_2}{2}\right) \\ &= E_n\left(1 + \Delta t\lambda + \frac{1}{2}\left(\Delta t\lambda\right)^2\right) \\ &= E_n\left(1 + \mu + \frac{1}{2}\mu^2\right). \end{aligned}$$

This gives us the amplification factor

$$g = \frac{E_{n+1}}{E_n} = \left(1 + \mu + \frac{1}{2}\mu^2\right). \tag{2.25}$$

We require $|g| \leq 1$, meaning that the time-dependent error does not grow in time. If this is any bigger than 1 the error will grow exponentially, giving an unstable numerical solution. Following the same steps for some other schemes we get the following equations for spacial schemes:

$$\text{First order upwind} : \mu = -C\left(1-e^{-i\beta}\right),$$
$$\text{Second order upwind} : \mu = -\frac{C}{2}\left(3-4e^{-i\beta}+e^{-2i\beta}\right),$$
$$\text{Second order central} : \mu = -\frac{C}{2}\left(e^{i\beta}-e^{-i\beta}\right),$$
$$\text{Fourth order central} : \mu = -\frac{C}{12}\left(-e^{2i\beta}+8e^{i\beta}-8e^{-i\beta}+e^{-2i\beta}\right).$$

And for the temporal schemes:

$$\begin{aligned}
\text{First order RK} &: g = 1 + \mu, \\
\text{Second order RK} &: g = 1 + \mu + \frac{1}{2}\mu^2, \\
\text{Third order RK} &: g = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3, \\
\text{Fourth order RK} &: g = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3 + \frac{1}{24}\mu^4.
\end{aligned}$$

In figures 2.1, 2.2, 2.3 and 2.4 we see the amplification factor for different C and β. Using the periodicity of β we can pick Δx and a corresponding Δt given the magnitude of a.

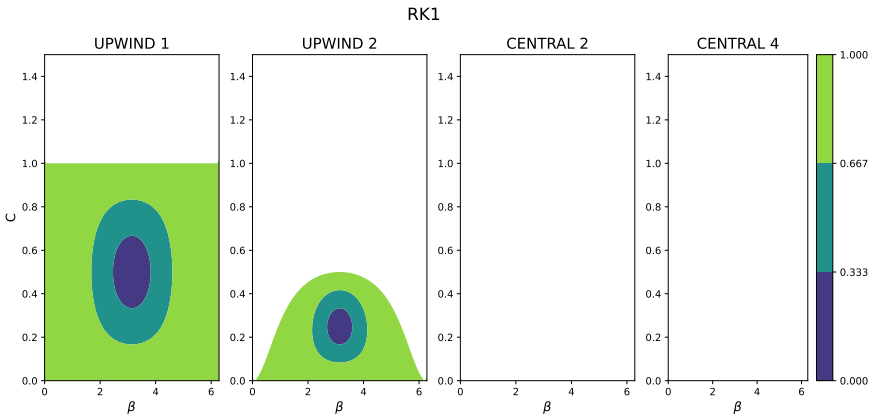

Figure 2.1: Amplification factor magnitude for the first-order Runge Kutta scheme.

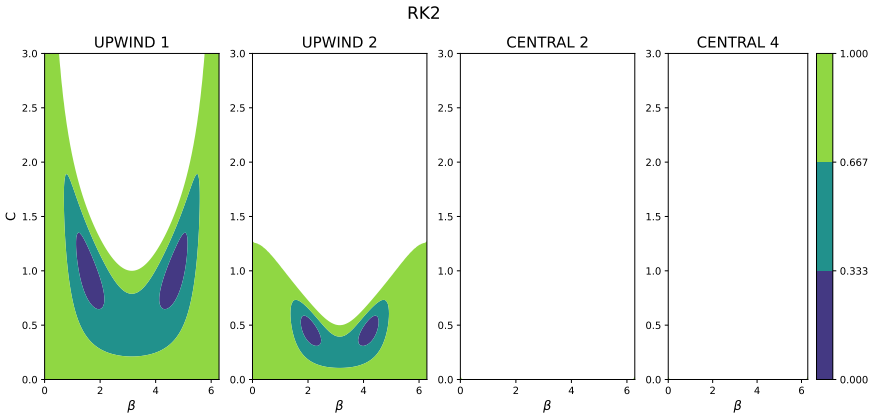

Figure 2.2: Amplification factor magnitude for the second-order Runge Kutta scheme.

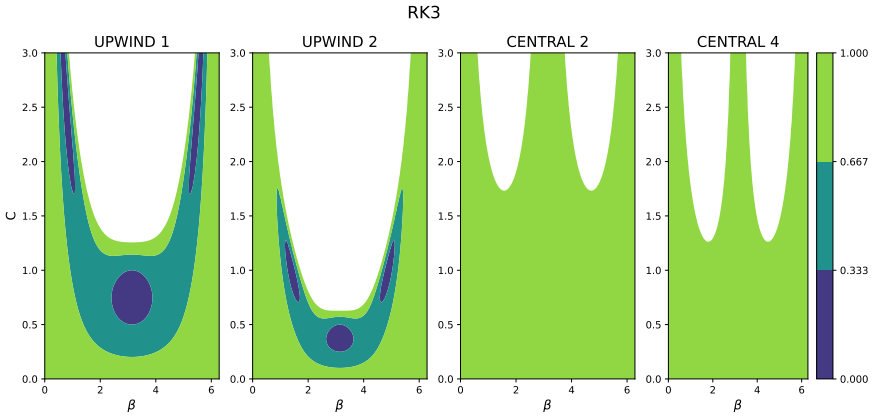

Figure 2.3: Amplification factor magnitude for the third-order Runge Kutta scheme.

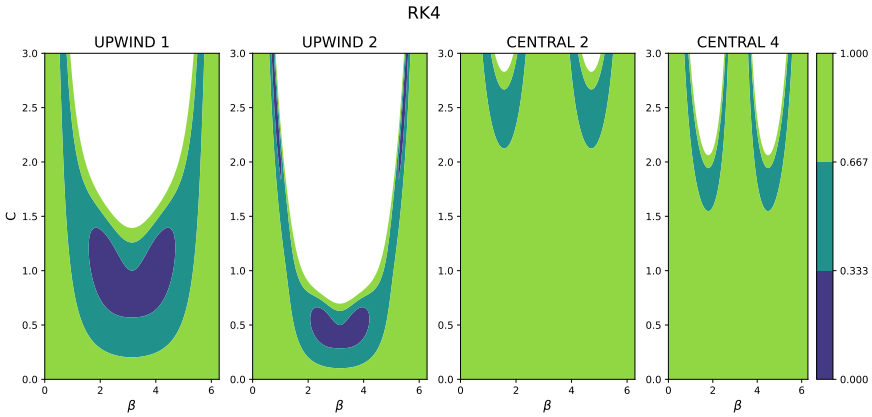

Figure 2.4: Amplification factor magnitude for the fourth-order Runge Kutta scheme.

2.3.4 Background Initialization

We base our background on reference values from the standard solar model, Solar S, 1996Sci...272.1286C. This data is downloaded from https://users-phys.au.dk/ jcd/solar$_m$*odels/cptrho.l5bi.d* temperature $T_0(r_r)$ and pressure $p_0(r_r)$ are picked at a radius $r_r = 0.71R_\odot$, which is the bottom of the convection zone 1991ApJ...378..413C. The mass at this point is calculated by

$$m(r_r) = \int_0^{r_r} dm = 4\pi \int_0^{r_r} \rho(r') r'^2 dr', \tag{2.26}$$

using cumulative trapezoidal integration. Having this we can use the following equations to integrate up and down from the reference point.

We are using an ideal gas model and therefore can not use this data as our entire background. This is because the temperature gradient of the background would not be large enough to produce convection numerically. Instead we will integrate out from our reference values and create a hydrostatic background. We will force convective instability by setting the superadiabaticity parameter

$$\Delta\nabla = \left(\frac{\partial \ln T}{\partial \ln p}\right)_* - \left(\frac{\partial \ln T}{\partial \ln p}\right)_{ad} = \nabla_* - \nabla_{ad} > 0,$$

where ∇_* is the adiabatic temperature gradient of the star and $\nabla_{ad} = 0.4$ is the adiabatic temperature gradient for an ideal gas. We therefore set

$$\nabla_* - \nabla_{ad} = \begin{cases} k & \text{if } r \geq 0.7R_\odot \\ 0 & \text{if } r < 0.7R_\odot \end{cases}$$

For an ideal gas we have $\nabla_{ad} = 0.4$ and we can then integrate our variables using the following equations.

Mass of shell with thickness dr

$$\frac{dm}{dr} = 4\pi r^2 \rho_0(r). \tag{2.27}$$

Hydrostatic equilibrium condition

$$\frac{p_0}{r} = -\frac{Gm(r)}{r^2}\rho_0(r), \tag{2.28}$$

where G is Newtons gravitational constant.

????

$$\frac{dT_0}{dr} = \nabla_* \frac{T_0}{p_0}\frac{dp_0}{dr}. \tag{2.29}$$

We also have the **entropy gradient**

$$\frac{ds}{dr} = -\frac{c_p}{H}\Delta\nabla,$$

where the pressure scale height

$$H = -\frac{dr}{d\ln p_0} = -p_0 \frac{dr}{dp_0},$$

the spesific heat capacity at constant pressure 1999ApJS..121..247L

$$c_p = \frac{r_*}{1 - 1/\gamma},$$

and r_* is the spesific gas constant and γ is the adiabatic parameter. The gas constant is taken from the ideal gas law on the from

$$p = \frac{k_B}{\mu m_u} T,$$

where k_B is Boltzmanns constant m_u is the atomic mass unit and μ is the mean molecular mass. Using the chemical abundaces of Hydrogen Ab(H) = 12, Helium Ab(He) = 10.93 and metals Ab(Metals) = 0.012 2007SSRv..130..105G we can calculate their number densities by

$$n_E = 10^{\mathrm{Ab(E)}},$$

the mass densities

$$\rho_E = n_E \cdot m_E,$$

where m_e is the mass of the element, and the fractional abundance

$$X_E = \frac{\rho_E}{\rho} = \frac{n_E m_E}{\sum_i n_i m_i},$$

where ρ is the total density. This gives us that $n_{\mathrm{H}} = 10^{12}$, $n_{\mathrm{He}} = 10^{10.93}$ and $n_{\mathrm{metals}} = 10^{0.012} \approx 1$. For simplification we firstly approximate $n_{\mathrm{metals}} = 0$, since $n_{\mathrm{metals}} \approx 1 \ll n_{\mathrm{H}}, n_{\mathrm{He}}$, secondly $m_{\mathrm{He}} \approx 4m_{\mathrm{H}}$ and thirdly $m_{\mathrm{H}} \approx m_u$. Then the the fractional abundance for Hydrogen, Helium and metals are

$$\begin{aligned} X_{\mathrm{H}} &= \frac{n_{\mathrm{H}}}{n_{\mathrm{H}} + 4n_{\mathrm{He}}} \approx 0.746, \\ X_{\mathrm{He}} &= \frac{n_{\mathrm{He}}}{n_{\mathrm{H}} + 4n_{\mathrm{He}}} \approx 0.063, \\ X_{\mathrm{metals}} &= 1 - X_{\mathrm{H}} - X_{\mathrm{He}} \approx 0.190 \end{aligned}$$

We will assume full ionization due to the high temperatures and densities of the convection zone as can be seen in figures 2.5 and 2.6 from the standard solar model. Then the total number of particles due to Hydrogen is

$$\frac{2X_{\mathrm{H}}\rho}{m_u},$$

by one proton and electron. The total number of particles due to Helium is

$$\frac{3X_{\mathrm{He}}\rho}{4m_u},$$

by one alpha particle and two electrons. The metals will have one nucleus and a number of electrons equal to the number of protons in the nucleus per particle. This means that the ratio of free particles to the number of particles in the nucleus is always $(Z+1)/A$, where Z is the number of protons and A is the atomic weight. We can approximate this to $Z/A \approx 1/2m_u$ and get that the total number of particles due to the metals is

$$\frac{X_{\text{metals}}\rho}{2m_u}.$$

Then the total number of particles

$$n_{total} = \frac{\rho}{m_u}\left(2X_{\text{H}} + \frac{3X_{\text{He}}}{4} + \frac{X_{\text{metals}}}{2}\right).$$

This gives us the mean molecular weight per particle

$$\mu = \frac{1}{2X_{\text{H}} + 3X_{\text{He}}/4 + X_{\text{metals}}/2} \approx 0.61. \tag{2.30}$$

We can then calculate the spesific gas constant

$$r_* = \frac{k_B}{m_u\mu},$$

and the **entropy gradient**

$$\frac{ds}{dr} = p_0\frac{k_B}{\mu m_u}\frac{\Delta\nabla}{1-1/\gamma}\frac{dp_0}{dr}. \tag{2.31}$$

We integrate this up to a radius r_e and down to a radius r_b, limiting dr for numerical stability. The limitation on dr is implemented by picking a $p < 1$ such that

$$dr = \frac{pV}{f},$$

for a variable V with

$$f = \frac{dV}{dr}.$$

This is done for all the variables and the smalles dr is picked. For updating the density in each grid point we use the ideal gas **equation of state**

$$\rho = \frac{p}{T}\frac{m_u\mu}{k_B}. \tag{2.32}$$

When we have the background set up we can interpolate this to a grid. We have done this using linear interpolation and the comparrison between our background model and the Solar S model from radius $r = 0.6R_\odot$ to $0.97R_\odot$ can be seen in figures 2.5 for temperature, 2.6 for pressure, 2.7 for density and 2.8 for gravitational acceleration.

2.3.5 Code testing, maybe not in theory

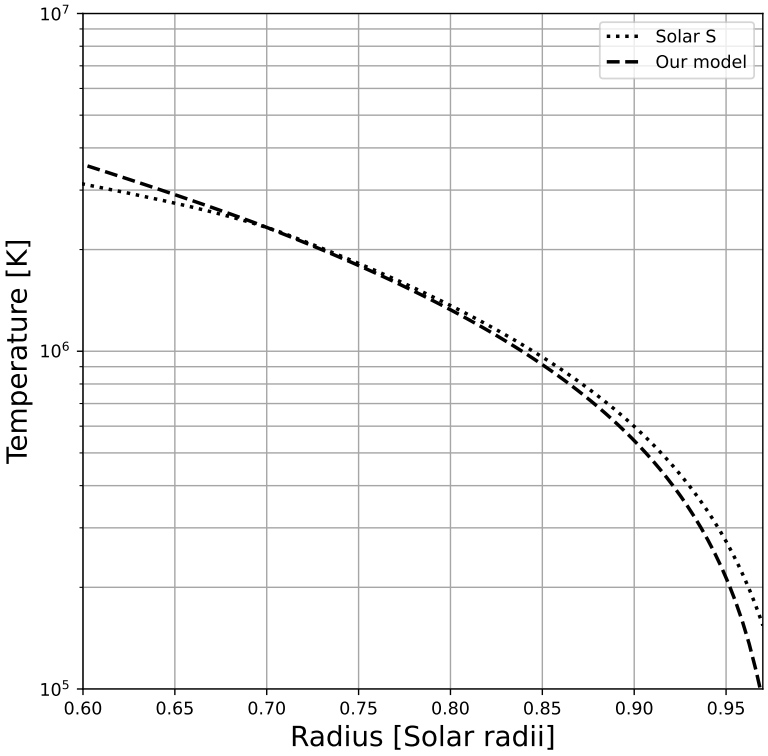

Figure 2.5: Temperature as a function of radius for the integrated hydrostatic model in dashed lines and the Solar S model in dotted line.

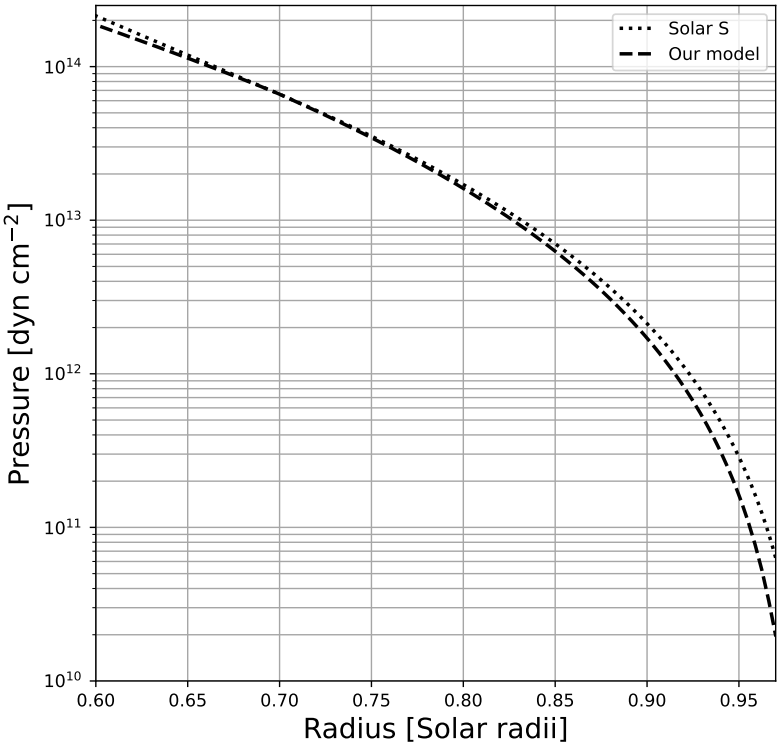

Figure 2.6: Pressure as a function of radius for the integrated hydrostatic model in dashed lines and the Solar S model in dotted line.

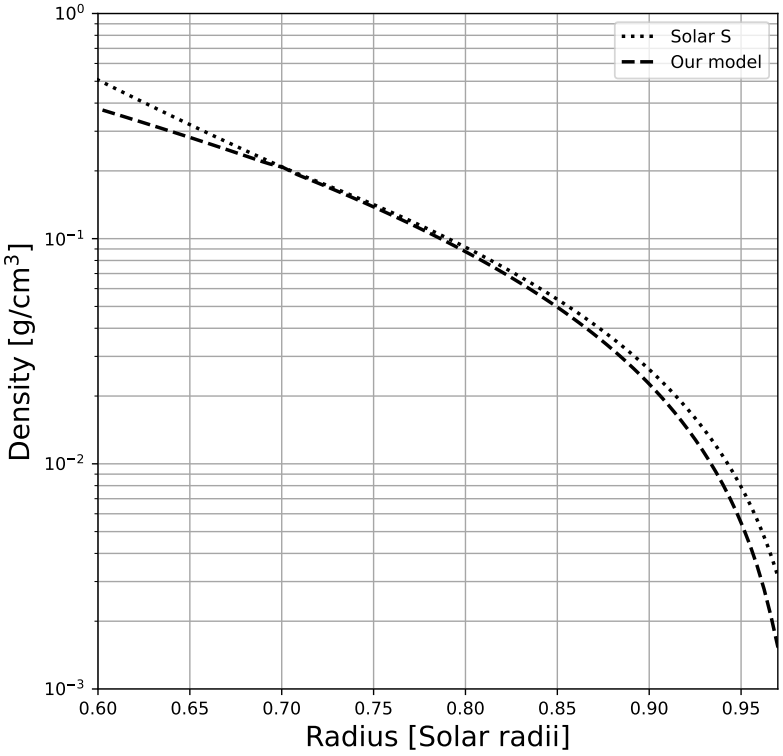

Figure 2.7: Density as a function of radius for the integrated hydrostatic model in dashed lines and the Solar S model in dotted line.

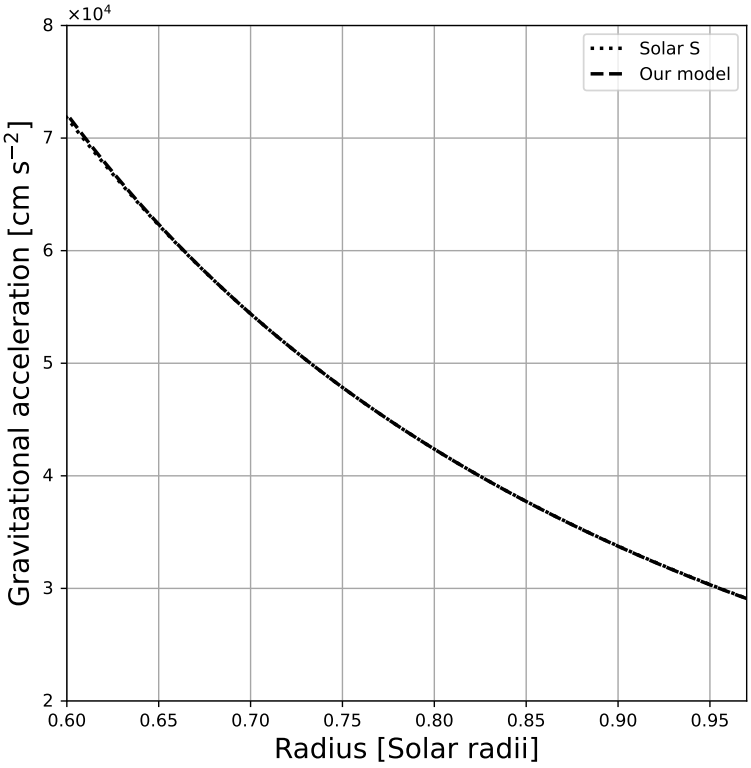

Figure 2.8: Gravitational acceleration as a function of radius for the integrated hydrostatic model in dashed lines and the Solar S model in dotted line.

Appendices

Bibliography