1 Von Neumann analysis

We will analyse the advection equation to determine the required step-sizes for our solver. The advection equation is

$$\frac{\partial u(t,x)}{\partial t} = -a\frac{\partial u(t,x)}{\partial x}, \tag{1}$$

where $u(t,x)$ is the exact solution and a is constant. This is discretized over a grid such that $y(t_n,x_j) = y_{n,j}$, where $t_n = n\Delta t$ and $x_j = j\Delta x$ for $n, j \in \mathbb{N}$, is our numerical solution. Then the advection equation becomes

$$\left[\frac{\partial y}{\partial t}\right]_{n,j} = -a\left[\frac{\partial y}{\partial x}\right]_{n,j}. \tag{2}$$

The numerical solution

$$y_{n,j} = u(t_n, x_j) + \epsilon_{n,j}, \tag{3}$$

where $\epsilon_{n,j}$ is the round-off error. The round-off error must also satisfy the discretized equation and this gives us that

$$\left[\frac{\partial \epsilon}{\partial t}\right]_{n,j} = -a\left[\frac{\partial \epsilon}{\partial x}\right]_{n,j}. \tag{4}$$

We expand the round-off error as a fourier series

$$\epsilon(t_n, x_j) = \sum_m E_m(t_n)e^{ik_m j\Delta x}, \tag{5}$$

where k_m is the wavenumber and $E_(t_n)$ is the time-dependent amplitude of the error. When inserting this into our differential equation we get a linear difference equation, meaning that each of the terms behave like the entire series so we can consider the growth of only one term

$$\epsilon_m(t_n, x_j) = E_m(t_n)e^{ik_m j\Delta x}. \tag{6}$$

We will show the calculations using the first-order upwind scheme with the second-order Runge-Kutta scheme. Since this should be true for any m we remove the subscipt, define $\beta \equiv k\Delta x$ and get that the spacial derivative is

$$\begin{aligned}
\left[\frac{\partial \epsilon}{\partial x}\right]_{n,j} &= \frac{\epsilon_{n,j} - \epsilon_{n,j-1}}{\Delta x} \\
&= \frac{E(t_n)e^{i\beta j} - E(t_n)e^{i\beta(j-1)}}{\Delta x} \\
&= E(t_n)e^{i\beta j}\frac{1 - e^{-i\beta}}{\Delta x}.
\end{aligned}$$

This gives us that the advection equation 4 becomes

$$\left[\frac{\partial \epsilon}{\partial t}\right]_{n,j} = e^{i\beta j}\left[\frac{\partial E(t_n)}{\partial t}\right]_{n,j} = -aE(t_n)e^{i\beta j}\frac{1 - e^{-i\beta}}{\Delta x} \tag{7}$$

$$\left[\frac{\partial E(t_n)}{\partial t}\right]_{n,j} = -aE(t_n)\frac{1 - e^{-i\beta}}{\Delta x}. \tag{8}$$

We define $\lambda = -\frac{a}{\Delta x}\left(1 - e^{-i\beta}\right)$ which gives us

$$\mu = \Delta t\lambda = -C\left(1 - e^{-i\beta}\right), \tag{9}$$

where $C \equiv a\Delta t/\Delta x$ is the Courant number. This means that the differential equation for the time-dependent error is

$$\left[\frac{\partial E(t_n)}{\partial t}\right]_{n,j} = \lambda E(t_n). \tag{10}$$

Using this with the second order Runge-Kutta scheme, the slopes for the time-dependent error is

$$\begin{aligned}
k_1 &= \lambda E_n, \\
k_2 &= \lambda\left(E_n + \frac{\Delta t}{2}k_1\right) = E_n\left(\lambda + \frac{\Delta t \lambda^2}{2}\right).
\end{aligned}$$

And the next time-step for the error is

$$\begin{aligned}
E_{n+1} &= E_n + \Delta t\left(\frac{k_1}{2} + \frac{k_2}{2}\right) \\
&= E_n\left(1 + \Delta t\lambda + \frac{1}{2}\left(\Delta t\lambda\right)^2\right) \\
&= E_n\left(1 + \mu + \frac{1}{2}\mu^2\right).
\end{aligned}$$

This gives us the amplification factor

$$g = \frac{E_{n+1}}{E_n} = \left(1 + \mu + \frac{1}{2}\mu^2\right). \tag{11}$$

We require $|g| \leq 1$, meaning that the time-dependent error does not grow in time. If this is any bigger than 1 the error will grow exponentially, giving an unstable numerical solution. Following the same steps for some other schemes we get the following equations for spacial schemes:

$$\begin{aligned}
\text{First order upwind} &: \mu = -C\left(1 - e^{-i\beta}\right), \\
\text{Second order upwind} &: \mu = -\frac{C}{2}\left(3 - 4e^{-i\beta} + e^{-2i\beta}\right), \\
\text{Second order central} &: \mu = -\frac{C}{2}\left(e^{i\beta} - e^{-i\beta}\right), \\
\text{Fourth order central} &: \mu = -\frac{C}{12}\left(-e^{2i\beta} + 8e^{i\beta} - 8e^{-i\beta} + e^{-2i\beta}\right).
\end{aligned}$$

And for the temporal schemes:

$$\begin{aligned}
\text{First order RK} &: g = 1 + \mu, \\
\text{Second order RK} &: g = 1 + \mu + \frac{1}{2}\mu^2, \\
\text{Third order RK} &: g = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3, \\
\text{Fourth order RK} &: g = 1 + \mu + \frac{1}{2}\mu^2 + \frac{1}{6}\mu^3 + \frac{1}{24}\mu^4.
\end{aligned}$$

In figures 1, 2, 3 and 4 we see the amplification factor for different C and β. Using the periodicity of the error we can set $k = 1$ and pick Δx and Δt depending on the magnitude of a.

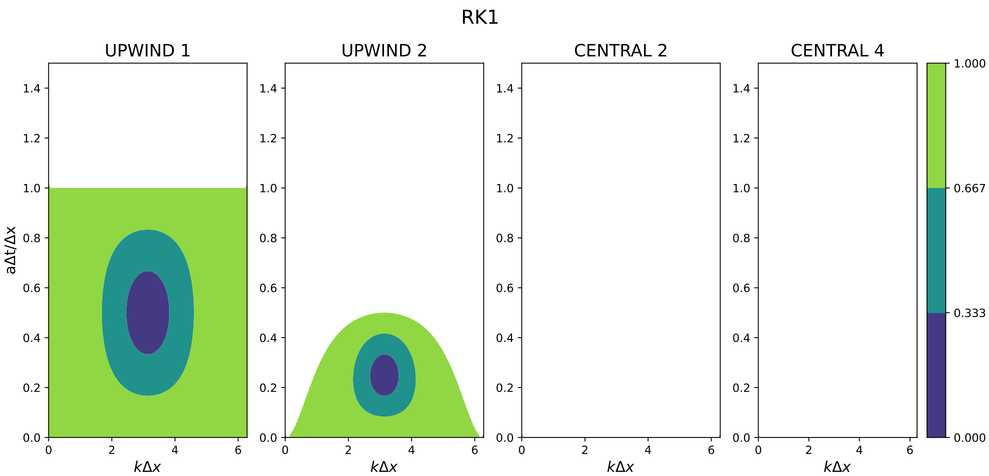

Figure 1: Amplification factor magnitude for the first-order Runge Kutta scheme.

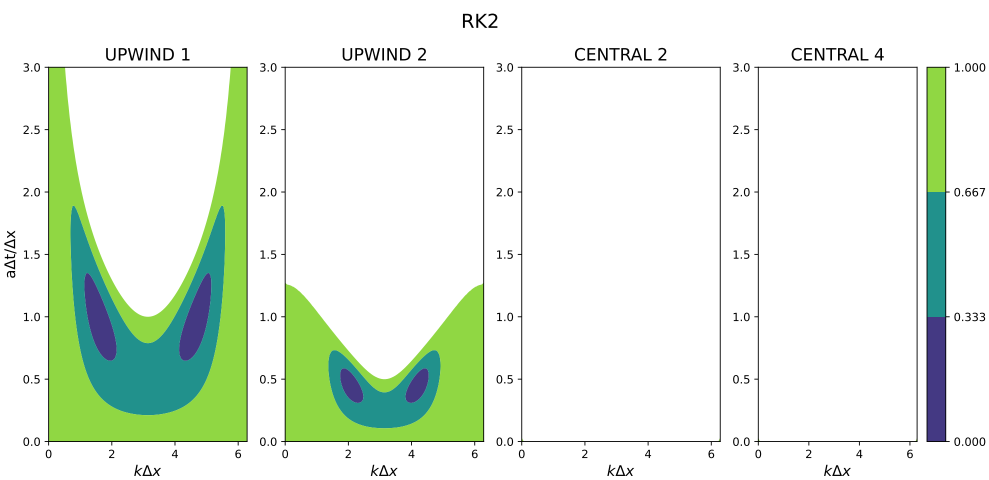

Figure 2: Amplification factor magnitude for the second-order Runge Kutta scheme.

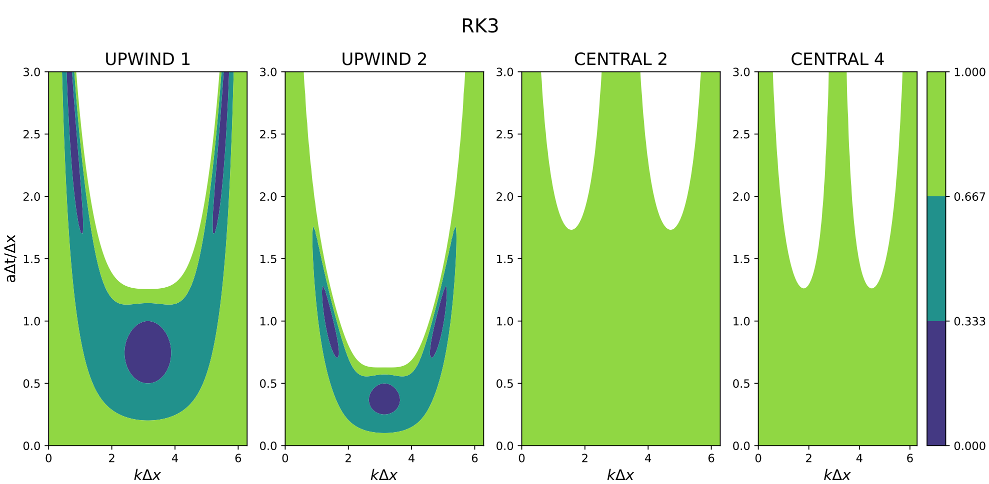

Figure 3: Amplification factor magnitude for the third-order Runge Kutta scheme.

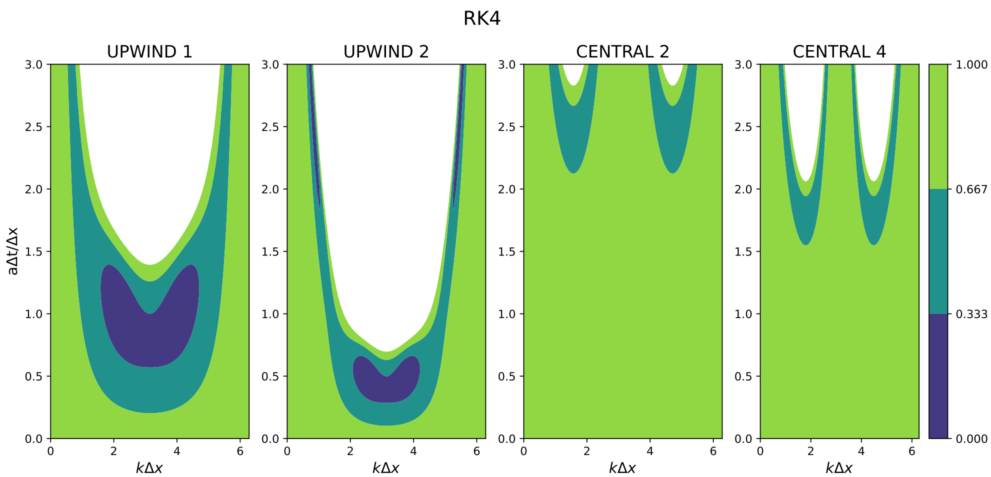

Figure 4: Amplification factor magnitude for the fourth-order Runge Kutta scheme.