1 Elliptic solver

Second order elliptic linear partial differential equation. Elliptic because discriminant is -4.

Elliptic equation = equilibrium problem.

Inverse of matrix is inneficient in terms of FLOPs. A will be a symmetric square matrix. Gaussian elimination: Row reduction is called forward substitution in the litterature.

Vanilla Dirichlet boundary conditions = 0. General = any number.

Momentum equation

$$\rho_0\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]=-\nabla p_1+\rho_1\mathbf{g}. \tag{1}$$

We take the divergence of the momentum equation

$$\nabla\cdot\left[\rho_0\frac{\partial \mathbf{v}}{\partial t}+\rho_0(\mathbf{v}\cdot\nabla)\mathbf{v}\right]=-\nabla^2 p_1+\nabla\cdot\left(\rho_1\mathbf{g}\right). \tag{2}$$

$$\nabla\cdot\frac{\partial\left(\rho_0\mathbf{v}\right)}{\partial t}+\nabla\cdot\rho_0(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla^2 p_1-\partial_z\left(\rho_1 g\right). \tag{3}$$

Using Clairaut's theorem on the first term

$$\frac{\partial\left[\nabla\cdot\left(\rho_0\mathbf{v}\right)\right]}{\partial t}+\nabla\cdot\rho_0(\mathbf{v}\cdot\nabla)\mathbf{v}=-\nabla^2 p_1-\partial_z\left(\rho_1 g\right). \tag{4}$$

Then the first term is zero by the continuity equation, we solve for $\nabla^2 p_1$

$$\nabla^2 p_1=-\partial_z\left(\rho_1 g\right)-\nabla\cdot\rho_0(\mathbf{v}\cdot\nabla)\mathbf{v}. \tag{5}$$

Earlier we found that

$$(\mathbf{v}\cdot\nabla)\mathbf{v}=(v_x\partial_x+v_z\partial_z)v_x\hat{\mathbf{i}}+(v_x\partial_x+v_z\partial_z)v_z\hat{\mathbf{k}}. \tag{6}$$

Inserting this

$$\nabla^2 p_1=-\partial_z\left(\rho_1 g\right)-\nabla\cdot\rho_0\left[(v_x\partial_x+v_z\partial_z)v_x\hat{\mathbf{i}}+(v_x\partial_x+v_z\partial_z)v_z\hat{\mathbf{k}}\right]. \tag{7}$$

$$\nabla^2 p_1=-\partial_z\left(\rho_1 g\right)-\nabla\cdot\left[(\rho_0 v_x\partial_x+\rho_0 v_z\partial_z)v_x\hat{\mathbf{i}}+(\rho_0 v_x\partial_x+\rho_0 v_z\partial_z)v_z\hat{\mathbf{k}}\right]. \tag{8}$$

$$\nabla^2 p_1=-\partial_z\left(\rho_1 g\right)-\partial_x\left[\rho_0 v_x\partial_x v_x+\rho_0 v_z\partial_z v_x\right]-\partial_z\left[\rho_0 v_x\partial_x v_z+\rho_0 v_z\partial_z v_z\right]. \tag{9}$$

$$\begin{aligned}\nabla^2 p_1=&-\partial_z\left(\rho_1 g\right)\\&-\partial_x(v_x)\partial_x(\rho_0 v_x)-\rho_0 v_x\partial_x^2 v_x\\&-\partial_z(v_x)\partial_x(\rho_0 v_z)-\rho_0 v_z\partial_x\partial_z(v_x)\\&-\partial_x(v_z)\partial_z(\rho_0 v_x)-\rho_0 v_x\partial_z\partial_x(v_z)\\&-\partial_z(v_z)\partial_z(\rho_0 v_z)-\rho_0 v_z\partial_z^2 v_z.\end{aligned}$$

By the continuity equation we can remove some terms.

$$\begin{aligned}\nabla^2 p_1=&-\partial_z\left(\rho_1 g\right)\\&-\rho_0 v_x\partial_x^2 v_x\\&-\partial_z(v_x)\partial_x(\rho_0 v_z)-\rho_0 v_z\partial_x\partial_z(v_x)\\&-\partial_x(v_z)\partial_z(\rho_0 v_x)-\rho_0 v_x\partial_z\partial_x(v_z)\\&-\rho_0 v_z\partial_z^2 v_z.\end{aligned}$$

By ρ_0 being constant in the x-direction we get

$$\begin{aligned}\nabla^2 p_1 =& -\partial_z\left(\rho_1 g\right)\\ &- v_x \partial_x^2(\rho_0 v_x)\\ &- \partial_z(v_x)\partial_x(\rho_0 v_z) - \rho_0 v_z \partial_x \partial_z(v_x)\\ &- \partial_x(v_z)\partial_z(\rho_0 v_x) - \rho_0 v_x \partial_z \partial_x(v_z)\\ &- \rho_0 v_z \partial_z^2 v_z.\end{aligned}$$

And we again use the continuity equation

$$\begin{aligned}\nabla^2 p_1 =& -\partial_z\left(\rho_1 g\right)\\ &- \partial_z(v_x)\partial_x(\rho_0 v_z) - \rho_0 v_z \partial_x \partial_z(v_x)\\ &- \partial_x(v_z)\partial_z(\rho_0 v_x) - \rho_0 v_x \partial_z \partial_x(v_z)\\ &- \rho_0 v_z \partial_z^2 v_z.\end{aligned}$$

$$\nabla^2 p_1 = -\partial_z\left(\rho_1 g\right) - \rho_0\left[v_z \partial_x \partial_z(v_x) + v_x \partial_z \partial_x(v_z) + v_z \partial_z^2 v_z\right] - \partial_x(v_z)\partial_z(\rho_0 v_x) - \partial_z(v_x)\partial_x(\rho_0 v_z) \tag{10}$$

We again use that ρ_0 is constant in the x-direction

$$\nabla^2 p_1 = -\partial_z\left(\rho_1 g\right) - \rho_0\left[v_z \partial_x \partial_z(v_x) + v_x \partial_z \partial_x(v_z) + v_z \partial_z^2 v_z + \partial_z(v_x)\partial_x(v_z)\right] - \partial_x(v_z)\partial_z(\rho_0 v_x) \tag{11}$$

This can be simplified to

$$\nabla^2 p_1 = -\partial_z\left(\rho_1 g\right) - \rho_0\left[v_z \partial_x \partial_z(v_x) + v_x \partial_z \partial_x(v_z) + v_z \partial_z^2 v_z + \partial_z(v_x)\partial_x(v_z)\right] - \partial_x(v_z)\partial_z(\rho_0 v_x) \tag{12}$$

Simplifying the bracket term

$$\nabla^2 p_1 = -\partial_z\left(\rho_1 g\right) - \rho_0\left[\partial_x \partial_z(v_x v_z) + v_z \partial_z^2 v_z + \partial_z(v_x)\partial_x(v_z)\right] - \partial_x(v_z)\partial_z(\rho_0 v_x) \tag{13}$$

Expanding the last term

$$\nabla^2 p_1 = -\partial_z\left(\rho_1 g\right) - \rho_0\left[\partial_x \partial_z(v_x v_z) + v_z \partial_z^2 v_z + \partial_z(v_x)\partial_x(v_z)\right] - \rho_0 \partial_x(v_z)\partial_z(v_x) - v_x \partial_x(v_z)\partial_z(\rho_0) \tag{14}$$

$$\nabla^2 p_1 = -\partial_z\left(\rho_1 g\right) - \rho_0\left[\partial_x \partial_z(v_x v_z) + v_z \partial_z^2 v_z + \partial_z(v_x)\partial_x(v_z) + \partial_x(v_z)\partial_z(v_x)\right] - v_x \partial_x(v_z)\partial_z(\rho_0) \tag{15}$$

$$\nabla^2 p_1 = -\partial_z\left(\rho_1 g\right) - \rho_0\left[\partial_x \partial_z(v_x v_z) + v_z \partial_z^2 v_z + 2\partial_z(v_x)\partial_x(v_z)\right] - v_x \partial_x(v_z)\partial_z(\rho_0) \tag{16}$$

THERE MUST BE SOME WAY TO SIMPLIFY THIS BUT I CAN'T SEE IT!