CI Models from Graphs and Matroids

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The basis of the zero-margin part of the model from Hoşten-Sullivant

Let π_Δ be the monomial map parametrizing the hierarchical log-linear model of a simplicial complex Δ, and $K = \ker \pi_\Delta$ its kernel. The map is defined using facets of Δ, and thus K is precisely the set of tensors whose facet-marginals are identically zero. The dimension of K is provided in [6, Theorem 2.6] through a construction of an explicit basis of K as a vector space. This basis consists of *adjacent minors* supported on non-faces of Δ.

Hoşten and Sullivant identify the basis as *the exponents of adjacent minors* as follows.

Let $S \notin \Delta$ be a non-face of the complex. An adjacent minor X supported on S is defined as:

$$X_{k_1 \dots k_n} = (-1)^{\sum_{j \in S} \epsilon_j},$$

where $k_j = i_j + \epsilon_j$, $\epsilon_j \in \{0, 1\}$ and $j \in S$, and $k_j = 0$ otherwise.

Intermediate results (e.g., [6, Lemma 2.1]) imply that the union all adjacent minors supported on all non-empty subsets S is linearly independent and, in addition, that each table whose Δ-(facet-)margins are zero can be uniquely written as an integral combination of adjacent minors, providing a basis over $\mathbb{Z}$.

Internal notes

Disclaimer: This section may be repetitive but I'm extracting from it, moving above, and then merging with paper draft.

What they compute

[6, Theorem 2.6] provides the dimension formula for the space of tables all of whose Δ margins are zero. They do this within the context of hierarchical log-linear models defined by a simplicial complex Δ. These are toric models whose monomial parametrization is indexed by facets of Δ and whose sufficient statistics are table facet-marginals. Elements in the toric ideal of the model are then precisely tensors whose margins supported on the facets of Δ are all zero.

How they compute it

Since the model is toric, the proof amounts to simply computing the vector space dimension of the kernel of the linear map induced by Δ by identifying the appropriate number of linearly independent vectors, which then comprise the basis of the vectors space of tensors whose Δ-margins are all zero.

The basis and example

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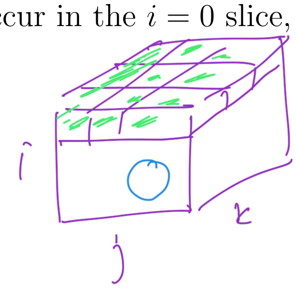
Let us see some small examples. For a $2 \times \cdots \times 2$ tensor, for example if $S = \{12\}$, this amounts to the following set of 2×2 minors located at the first slice along all indices in the complement of S:

$$\begin{aligned} X_{000\dots0} &= +1, & X_{010\dots0} &= -1, \\ X_{100\dots0} &= -1, & X_{110\dots0} &= +1. \end{aligned}$$

Similarly, an adjacent minor supported on $S = \{123\}$ then we get the $2 \times 2 \times 2$ minor:

$$\begin{aligned} X_{0000\dots0} &= +1, & X_{0010\dots0} &= -1, \\ X_{0100\dots0} &= -1, & X_{0110\dots0} &= +1, \\ X_{1000\dots0} &= -1, & X_{1010\dots0} &= +1, \\ X_{1100\dots0} &= +1, & X_{1110\dots0} &= -1. \end{aligned}$$

In [6, Example 2.4], where $n = 3$ and $S = \{2, 3\}$, all nonzero entries (± 1) of the minors occur in the $i = 0$ slice, which are the green shaded cells in this picture:



That the adjacent minors vanish on tensors in the model variety is clear by construction, and their linear independence can be seen by recognizing that each has a unique last non-zero entry index.

the rest here is open still.

Generalizing this, we obtain:

Proposition 0.1. *Does the join variety V_M have the expected dimension? Meaning we just add the dim of hierarchical model from Hosten-Sullivant and $\prod d_i$? (minus 1.)*

Proof. tbd. □

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