CI Models from Graphs and Matroids

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The basis of the zero-margin part of the model from Hoşten-Sullivant

[6, Theorem 2.6] provides the dimension formula for the space of tables all of whose Δ margins are zero. (That is, for the kernel of the toric map parametrizing the model of Δ-independence.) Since the model is toric, the proof amounts to simply computing the vector space dimension of the kernel of the linear map induced by Δ by identifying the appropriate number of linearly independent vectors, which comprise the basis of the vectors space of tensors whose Δ-margins are all zero. Hoşten and Sullivant identify the basis as the exponents of adjacent minors as follows.

Let $S \notin \Delta$ be a non-face of the complex. An adjacent minor X supported on S is defined as:

$$X_{k_1 \dots k_n} = (-1)^{\sum_{j \in S} \epsilon_j},$$

where $k_j = i_j + \epsilon_j$, $\epsilon_j \in \{0,1\}$ and $j \in S$, and $k_j = 0$ otherwise.

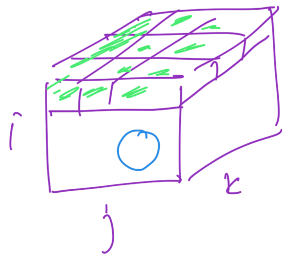
Let us see some small examples. For a $2 \times \cdots \times 2$ tensor, for example if $S = \{12\}$, this amounts to the following set of 2×2 minors located at the first slice along all indices in the complement of S:

$$\begin{aligned} X_{000\dots0} &= +1, & X_{010\dots0} &= -1, \\ X_{100\dots0} &= -1, & X_{110\dots0} &= +1. \end{aligned}$$

Similarly, an adjacent minor supported on $S = \{123\}$ then we get the $2 \times 2 \times 2$ minor:

$$\begin{aligned} X_{0000\dots0} &= +1, & X_{0010\dots0} &= -1, \\ X_{0100\dots0} &= -1, & X_{0110\dots0} &= +1, \\ X_{1000\dots0} &= -1, & X_{1010\dots0} &= +1, \\ X_{1100\dots0} &= +1, & X_{1110\dots0} &= -1. \end{aligned}$$

In [6, Example 2.4], where $n = 3$ and $S = \{2,3\}$, all nonzero entries (± 1) of the minors occur in the $i = 0$ slice, which are the green shaded cells in this picture:



That the adjacent minors vanish on tensors in the model variety is clear by construction, and their linear independence can be seen by recognizing that each has a unique last non-zero entry index.

the rest here is open still.

Generalizing this, we obtain:

Proposition 0.1. *Does the join variety V_M have the expected dimension? Meaning we just add the dim of hierarchical model from Hosten-Sullivant and $\prod d_i$? (minus 1.)*

Proof. tbd. □

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