Formula to Calculate Weight for Low-Weight Weight 3 Inputs and Proof

Kwame Ackah Bohulu

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1 Equation and Proof

We begin by defining the following terms.

$$\begin{aligned}
&\alpha_0 = (011),\ \alpha_1 = (101),\ \alpha_2 = (110),\\
&\alpha_0^{'} = (111),\ \alpha_1^{'} = (011),\ \alpha_2^{'} = (001),\\
&u = c_1 \mod 3,\ v = c_2 \mod 3,\\
&d_i = \left\lfloor \frac{c_i - c_{i-1} - 2}{3} \right\rfloor,\ i = \{1,2\}
\end{aligned} \tag{1}$$

Theorem 1.1. Given a low-weight weight 3 input $B(x) = x^{c_0} + x^{c_1} + x^{c_2}$, the parity sequence weight for the convolutional codeword can be calculated as

$$\begin{aligned}
&w_H(\alpha_0^{'}) + w_H(\alpha_0 + \alpha_u^{'}) + w_H(\alpha_0 + \alpha_u + \alpha_v^{'})\\
&+ d_1\left(w_H(\alpha_0)\right) + d_2\left(w_H(\alpha_0 + \alpha_u)\right)
\end{aligned} \tag{2}$$

where $w_H(\cdot)$ is the Hamming weight of the sequence. All binary sequence additions are done in GF(2)

Proof. Given $B(x)$ it is possible to write it in a tabular form as shown below.We assume that $B(x)$ is written in its simplest form, which makes $c_0 = 0$ and it is always located at row zero and column zero. We can see that u, v is just used to determine which column c_1, c_2 occur in. Using the knowledge the impulse response, it is possible to find the parity weight of the codeword.

c_0		
	c_1	
		c_2

c_0		
		c_1
	c_2	

The table below shows a tabular representation of the impulse response with respect to the position of c_0, c_1 and c_2 for the table on the left.

1	1	1
0	1	1
0	1	1
0	1	1
0	1	1
0	1	1
0	1	1

0	0	0
0	0	0
0	0	0
0	1	1
1	0	1
1	0	1
1	0	1

0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	0
0	0	1

To calculate the parity weight, we just need to find the Hamming weight of each row , sum the Hamming weight of each row and then find the cummulative sum. Using the previously defined term, we may rewrite the below in the form below.

$\alpha_0^{'}$	$\mathbf{0}$	$\mathbf{0}$
α_0	$\mathbf{0}$	$\mathbf{0}$
α_0	$\mathbf{0}$	$\mathbf{0}$
α_0	$\alpha_1^{'}$	$\mathbf{0}$
α_0	α_1	$\mathbf{0}$
α_0	α_1	$\mathbf{0}$
α_0	α_1	$\alpha_2^{'}$

where $\mathbf{0} = (000)$. We note that depending on the value of u, v, the above table will be slightly different.

Finding the Hamming weight of for each corrsponding row and adding them up results in the equation below

$$\begin{aligned} &w_H(\alpha_0^{'}) + w_H(\alpha_0 + \alpha_1^{'}) + w_H(\alpha_0 + \alpha_1 + \alpha_2^{'}) \\ &+ d_1\left(w_H(\alpha_0)\right) + d_2\left(w_H(\alpha_0 + \alpha_1)\right) \end{aligned} \tag{3}$$

where $d_i,\ i = \{1, 2\}$ is the number of non-overlapping rows between c_0, c_1 and c_1, c_2 respectively and is given by

$$d_i = \left\lfloor \frac{c_i - c_{i-1} - 2}{3} \right\rfloor,\ i = \{1, 2\}$$

Taking into account the possibility of c_1, c_2 appearing in different columns we rewrite the above equation as

$$\begin{aligned} &w_H(\alpha_0^{'}) + w_H(\alpha_0 + \alpha_u^{'}) + w_H(\alpha_0 + \alpha_u + \alpha_v^{'}) \\ &+ d_1\left(w_H(\alpha_0)\right) + d_2\left(w_H(\alpha_0 + \alpha_u)\right) \end{aligned} \tag{4}$$

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