Formula to Calculate Weight for Low-Weight Weight 3 Inputs and Proof

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0.1 Equation and Proof

Theorem 1. Let $Q(x) = x^{a\tau+t}(1 + x^{\beta\tau+1} + x^{\gamma\tau+2})$ be the polynomial representation of a weight 3 RTZ input. The Hamming weight, w_H of a turbo codeword generated by a weight-3 RTZ input is given by

$$7 + 2(l + l') \tag{0-1}$$

Proof. Since the impulse response is

$$(1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ \cdots)$$

Let $\boldsymbol{h}_1 = (1\ 1\ 1)$, $\boldsymbol{h}_2 = (0\ 1\ 1)$, $\boldsymbol{h}_3 = (0\ 0\ 1)$, $\boldsymbol{\phi}_1 = (0\ 1\ 1)$, $\boldsymbol{\phi}_2 = (1\ 0\ 1)$, and $\boldsymbol{\phi}_3 = (1\ 1\ 0)$. Then, the weight-3 RTZ occurs since $\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 + \boldsymbol{\phi}_3 = \mathbf{0}_3$.

Now, we consider the weight of the vector derived by the sumation of the followings vectors.

$$\begin{array}{c}(\mathbf{0}_{3i}\ \boldsymbol{h}_1\ \boldsymbol{\phi}_1\ \cdots)\\(\mathbf{0}_{3j}\ \boldsymbol{h}_2\ \boldsymbol{\phi}_2\ \cdots)\\(\mathbf{0}_{3k}\ \boldsymbol{h}_3\ \boldsymbol{\phi}_3\ \cdots)\end{array}$$

Case 1: $i = j = k$
For this case, the derived vector will be $(\mathbf{0}_{3i}\ \boldsymbol{h}_3\ \mathbf{0}_3\ \cdots)$ with a weight of 2

Case 2: $i < j,\ j = k$
For this case, the derived vector will be $(\mathbf{0}_{3i}\ \boldsymbol{h}_1\ (\boldsymbol{h}_2)_{j-i-1}\ \boldsymbol{h}_3\ \mathbf{0}_3\ \cdots)$ and the weight can be calculated as

$$w_p = 2(j - i) + 2$$

Case 3: $i < j < k$
For this case, the derived vector will be $(\mathbf{0}_{3i}\ \boldsymbol{h}_1\ (\boldsymbol{h}_2)_{j-i-1}\ \mathbf{0}_3\ (\boldsymbol{\phi}_3)_{k-j-1}\ \boldsymbol{h}_1\ \mathbf{0}_3\ \cdots)$. To calculate the weight of this derived vector, we get $2(j-1)+1$(weight for $(\mathbf{0}_{3i}\ \boldsymbol{h}_1\ (\boldsymbol{h}_2)_{j-i-1}\ \mathbf{0}_3)$ and $2(k-j)+1$ which is the weight for $((\boldsymbol{\phi}_3)_{k-j-1}\ \boldsymbol{h}_1\ \mathbf{0}_3\ \cdots)$. Therefore the hamming weight will be

$$w_p = 2(k - i) + 2$$

Case 4: $i < k < j$
For this case, the derived vector will be $(\mathbf{0}_{3i}\ \boldsymbol{h}_1\ (\boldsymbol{h}_2)_{k-i-1}\ (0\ 1\ 0)\ (\boldsymbol{\phi}_2)_{j-k-1}\ \boldsymbol{\phi}_3\ \mathbf{0}_3\ \cdots)$. The weight for this derived vector is the summation of $2(k-i)+2$ (the weight for $(\mathbf{0}_{3i}\ \boldsymbol{h}_1\ (\boldsymbol{h}_2)_{k-i-1}\ (0\,1\,0))$) and $2(j-k)$,(which is the weight for $((\boldsymbol{\phi}_2)_{j-k-1}\ \boldsymbol{\phi}_3\ \mathbf{0}_3\ \cdots)$). Therefore the hamming weight will be

$$w_p = 2(j - i) + 2$$

From the above cases, we conclude that the parity weight of a weight-3 RTZ input will be

$$w_p = 2(\max\{k, j\} - i) + 2$$

Let i', k', j', w'_p be similarly defind and correspond to a weight 3-RTZ input derived after interleaving. Then

$$w'_p = 2(\max\{k', j'\} - i') + 2$$

In this case, the hamming weight of the turbo codeword will be

$$\begin{aligned} w_H &= 3 + w_p + w_p' \\ &= 3 + 2(\max\{k, j\} - i) + 2 + 2(\max\{k', j'\} - i') + 2 \\ &= 7 + 2((\max\{k, j\} - i) + (\max\{k', j'\} - i')) \\ &= 7 + 2(l + l') \end{aligned} \tag{0-2}$$

where $l = (\max\{k, j\} - i)$ and $l' = (\max\{k', j'\} - i')$

$\square$