Bit Error Probability for Turbo Codes

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Chapter 1

Detectors and Error Probability

1.1 Maximum Likelihood Detection (MLD)

Assume that there exists an input space $\mathbb{R}^M$ with $M = 2^k$ elements. From this input space,a k-bit binary information sequence $\boldsymbol{x}_m = (x_{m,0}, x_{m,1}, ...x_{m,k-1})$ is mapped to an element $\boldsymbol{c}_m = (c_{m,0}, c_{m,1}, ..., c_{m,n-1})$ in the output space $\mathbb{R}^N$ with $N = 2^n$ elements. The M elements in $\mathbb{R}^M$ form a code and $\mathcal{C} \subset \mathbb{R}^n$ The BPSK modulated codeword $\boldsymbol{y}_m = (y_{m,0}, y_{m,1}, ..., y_{m,n-1})$ is transmitted over the AWGN channel and is received at the receiver as $\boldsymbol{r} = (r_0, r_1, r_{n-1})$

The task of the receiver is to obtain an estimate of $\boldsymbol{x}_{,}$ $\hat{\boldsymbol{x}_m}$ from $\boldsymbol{r}$. The probability of a correct decision given $\boldsymbol{r}$ $P[\text{correct decision}|\boldsymbol{r}] = P[\hat{\boldsymbol{x}_m} \text{ sent}|\boldsymbol{r}]$ and the probability of a correct decision $P[\text{correct decision}] = \int P[\hat{\boldsymbol{x}_m} \text{ sent}|\boldsymbol{r}]p(\boldsymbol{r})d\boldsymbol{r}$

For optimal accuracy the receiver must decide in favor of the $\boldsymbol{x}_m$ that maximizes $P[\boldsymbol{x}_m|\boldsymbol{r}]$ upon observing $\boldsymbol{r}$

$$\begin{aligned} \hat{\boldsymbol{x}}_m &= \arg\max_{1\le m\le m} P[\boldsymbol{x}_m|\boldsymbol{r}] \\ &\arg\max_{1\le m\le M} P[\boldsymbol{c}_m|\boldsymbol{r}] \end{aligned} \tag{1-1}$$

This decision rule is known as *maximum a posteriori (MAP) probability rule* and it may be simplified to

$$\hat{\boldsymbol{x}}_m = \arg\max_{1\le m\le M} \frac{P_{x_m} p(\boldsymbol{r}|\boldsymbol{c}_m)}{p(\boldsymbol{r})} = \arg\max_{1\le m\le N} P_{x_m} p(\boldsymbol{r}|\boldsymbol{c}_m) \tag{1-2}$$

Since $p(\boldsymbol{r}_m)$ is independent of $\boldsymbol{x}_m$. In the case where $P_{x_m} = 1/M$

$$\hat{\boldsymbol{x}}_m = \arg\max_{1\le m\le M} p(\boldsymbol{r}|\boldsymbol{c}_m) \tag{1-3}$$

(1-3) is known as the *Maximum Likelihood (ML) rule.* It is worth noting that what the receiver is essentially doing is dividing an output space $\mathbb{R}^N$ into M decision spaces $D_1, D_2, ..., D_M$ and if $\boldsymbol{r} \in D_m,\ \hat{\boldsymbol{x}_m} = \boldsymbol{x}_m$

For the MAP detector $D_m = \{r \in \mathbb{R}^N : P[\boldsymbol{x}_m|\boldsymbol{r}] > P[\boldsymbol{x}'_m|\boldsymbol{r}],\ \forall\ 1 \le m \le M,\ m' \ne M\}$

1.2 Error Probability

From the above discussion, we realize that an error occurs if $r \notin D_m$ when $\boldsymbol{c}_m$ is sent. The symbol error probability of a receiver is given by

$$P_e = \sum_{m=1}^{M} P_{\boldsymbol{x}_m} P[\boldsymbol{r} \in D_m | \boldsymbol{c}_m \text{ sent}] = \sum_{m=1}^{M} P_{\boldsymbol{x}_m} P_{e|\boldsymbol{c}_m} \tag{1-4}$$

Where

$$P_{e|m} = \sum_{1 \leq m' \leq M,\ m' \neq m} \int_{D_{m'}} p(\boldsymbol{r}|\boldsymbol{c}_m) d\boldsymbol{r}$$

is the error probability when the message $\boldsymbol{x}_m$ is sent and

$$P_e = \sum_{m=1}^{M} P_{\boldsymbol{x}_m} \sum_{1 \leq m' \leq M,\ m' \neq m} \int_{D_{m'}} p(\boldsymbol{r}|\boldsymbol{c}_m) d\boldsymbol{r} \tag{1-5}$$

(1-5) gives the *symbol error probability*

We define $D_{mm'} = \{p(\boldsymbol{r}|\boldsymbol{c}'_m) > p(\boldsymbol{r}|\boldsymbol{c}_m)\}$ and we see that $D_{m'} \leq D_{mm'}$

Again, we define the *pairwise error probability* $P_{m \to m'}$ as

$$P_{m \to m'} = \int_{D_{mm'}} p(\boldsymbol{r}|\boldsymbol{c}_m) d\boldsymbol{r} \tag{1-6}$$

fixing (1-6) into (1-5) we get

$$\begin{aligned} P_e &\leq \sum_{m=1}^{M} P_{\boldsymbol{x}_m} \sum_{1 \leq m' \leq M,\ m' \neq m} \int_{D_{mm'}} p(\boldsymbol{r}|\boldsymbol{c}_m) d\boldsymbol{r} \\ &\leq \sum_{m=1}^{M} P_{\boldsymbol{x}_m} \sum_{1 \leq m' \leq M,\ m' \neq m} P_{m \to m'} \\ &\leq \frac{1}{M} \sum_{m=1}^{M} \sum_{1 \leq m' \leq M,\ m' \neq m} P_{m \to m'} \end{aligned} \tag{1-7}$$

where $P_{x_m} = 1/M$ for the case where the messages are equiprobable

1.3 Symbol Error Probability for Linear Block Codes

Without loss of generality, we assume that the all zero codeword $\mathbf{0}$ From the above discussion, we realize that an error occurs if the receiver decides upon $\boldsymbol{c}_m \neq \mathbf{0}$ as the codeword that was transmitted and this event is defined by the *pairwise error probability* $P_{\mathbf{0} \to c_m}$. The symbol error probability of the linear block code is then

$$P_e = \sum_{c_m \in \mathcal{C},\ c_m \neq \mathbf{0}} P_{\mathbf{0} \to c_m} \tag{1-8}$$

Since codewords with the same weight have the same $P_{\mathbf{0}\to c_m}$ we have

$$P_e = \sum_{i=d_{\min}}^{n} A_i P_2(i) \tag{1-9}$$

Where A_i is the number of codewords if a given weight i and $P_2(i)$ is the PEP of codewords with weight i

1.3.1 Upper bound on Pairwise Error Probability (PEP) for AWGN channel

We attempt to find the PEP for the case of the AWGN channel

$$P_{m\to m'} \tag{1-10}$$

1.4 Bit Error Probability for Linear Block codes

Let N be the block length of a code with 2^N codewords. From the previous section, we saw that for AWGN channels

$$P_{\mathbf{0}\to c_m} = Q(\sqrt{\frac{d_E^2(\boldsymbol{c}_m)}{2N_o}}) \tag{1-11}$$

For BPSK modulation $d_E^2(c_m) = 4E_bR_c\mathrm{w}(c_m)$ and we have

$$P_{\mathbf{0}\to c_m} = Q\left(\sqrt{\frac{2E_bR_c\mathrm{w}(\boldsymbol{c}_m)}{N_o}}\right) \tag{1-12}$$

the corresponding bit error probability when $\boldsymbol{c}_m$ is transmitted is given by

$$P_b(\mathbf{0}\to c_m) = \frac{w(\boldsymbol{x}_m)}{N} Q\left(\sqrt{\frac{2E_bR_c\mathrm{w}(\boldsymbol{c}_m)}{N_o}}\right) \tag{1-13}$$

Finally using the union bound, the average bit error probability is bounded by

$$P_b = \frac{1}{N}\sum_{m=1}^{2^N-1} w(\boldsymbol{x}_m) Q\left(\sqrt{\frac{2E_bR_c\mathrm{w}(\boldsymbol{c}_m)}{N_o}}\right) \tag{1-14}$$