“ Progress So Far”

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0.1 Notation

1. RTZ (Return-To-Zero) input :- A RTZ input is a binary input which causes a RSC encoder's final state to be return to zero after it has exited the zero state.
2. τ :- cycle length of the RSC encoder. For the 5/7 RSC encoder $\tau = 3$
3. N :- Interleaver length.
4. $\mathcal{N}$:- Integer set of $\{0, 1, \cdots, N-1\}$
5. $\mathbb{N}$: Indexed set of $\{0, 1, \cdots, N-1\}$ in the natural order.
6. We assume that $N/\tau = C$
7. $\mathcal{C}$ and $\mathbb{C}$ are definded in a similar manner.
8. $\mathcal{C}^t := \{c+t\}_{c \in \mathcal{C}}$ and $\mathbb{C}^t$ is the indexed set with the elements of $\mathcal{C}^t$.
9. Operator $\mathbf{\Pi}\left(\mathbb{C}^0, \mathbb{C}^1, \cdots, \mathbb{C}^{\tau-1}\right)$ with a permutation matrix

$$\mathbf{\Pi} = \begin{bmatrix} \boldsymbol{\pi}_0 \\ \boldsymbol{\pi}_1 \\ \vdots \\ \boldsymbol{\pi}_{K-1} \end{bmatrix}$$

   where $\boldsymbol{\pi}_k$ is a length τ vector consisting of the elements in $(0, 1, \cdots, \tau - 1)$.

   We assume that the operation outputs the elements in $\mathbb{C}^t$ in order while t is appeared in $\boldsymbol{\pi}_k$. For example, $\boldsymbol{\pi}_0 = (0, 0, 1)$ outputs (c_0^0, c_1^0, c_0^1).

Our goal is to find a prefer $\mathbf{\Pi}$ and $\mathbb{C}^t$, $t = 0, 1, \cdots, \tau - 1$.

0.2 Cosets and RTZ inputs

1. a weight 2 input sequence
   - polynomial: $P(x) = x^{a\tau+t}(1 + x^{\alpha\tau})$
   - coset: the ath and $(a+\alpha)$th elements in $\mathbb{C}^t$
2. a weight 3 input sequence
   - polynomial: $Q(x) = x^{a\tau+t}(1 + x^{\beta\tau+1} + x^{\gamma\tau+2})$. Notice that $\alpha \leq \beta$ does not nessacery to be hold.
   - coset: the ath element in $\mathbb{C}^t$, $(a+\beta)$th element in $\mathbb{C}^{[t+1]_\tau}$, and $(a+\gamma)$th element in $\mathbb{C}^{[t+2]_\tau}$.

0.3 Representation of interleaver

If the mapping relationship between elements in $\boldsymbol{x}$ and $\boldsymbol{y}$ are read column wise as shown below

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 5 & 1 & 6 & 2 & 7 & 3 & 8 & 4 \end{bmatrix}$$

the interleaver is represented by $\mathbb{N} = \{0, 5, 1, 6, 2, 7, 3, 8, 4\}$.

Let $\mathbb{C}^0 = \{0, 6, 3\}$, $\mathbb{C}^1 = \{1, 7, 4\}$, and $\mathbb{C}^2 = \{5, 2, 8\}$. Then, the permutation matrix of $\mathbb{N}$ is $\mathbf{\Pi} = (0, 2, 1)$. Notice the row of $\mathbf{\Pi}$ takes cyclicly.

0.4 Coset Interleaver Design For Weight-2 RTZ inputs

From the definition of Weight-2 RTZ inputs in the previous section, we know that the index of the "1" bits are in the same coset. Let the indices of the "1" bits before interleaving be represented by i and $i + m\tau$, $m = 1, 2, ..., \lfloor \frac{N-1}{3} \rfloor$, $i = 0, 1, ..N - 1$. Let the indices of the "1" bits after interleaving be represented by $\Pi(i)$ and $\Pi(i + m\tau)$. Also let $t = (i - m\tau) - i = m\tau$ and $s = \Pi(i + m\tau) - \Pi(i)$ be the pre-interleaving separation and post interleaving separation respectively.

To convert a weigth-2 RTZ input into a non-RTZ input the following condition should be met.

$$\begin{aligned} \Pi(i) \bmod 3 \neq \Pi(i + m\tau) \bmod 3,\ \forall\ i, m \\ \text{OR} \\ s \bmod 3 \neq 0 \end{aligned} \tag{0-1}$$

In the coset representation, this condition means the same elements does not appear in the same columns.

Next, we consider the following question.

1. Is there a cut-off interleaver length (N_c) beyond which the condition in (0-1) cannot be met.

   Since $\mathbf{\Pi}$ consisting of τ elements, the maximum length of column elements consisting of different each other is τ. Thus, the cut-off interleaver length is $N_c = 9 = \tau^2$.

2. What is the best course of action to take when the condition in (0-1) cannot be met

1. One cycle permutation: Each row is permutation of the sequence $(0, 1, 2)$. Setting the element at the first row and first column to 0, there are exactly 2 permutation matrices that exist for cut-off length $N_c = 9$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} \tag{0-2}$$

2. Two cycle permutation: Two rows are permutation of the sequence $(0, 0, 1, 1, 2, 2)$.

   There are no permutation matrices that satisfying cut-off length $N_c = 9$. This is because the sequence length is not divisible by N_c, there will always be 2 elements of the same value in each row of $\mathbf{\Pi}$

3. Three cycle permutation: Three rows are permutation of the sequence$(0,0,0,1,1,1,2,2,2)$. Example of the permutation matrices satisfying cut-off length $N_c = 9$ are shown in 1

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \tag{0-3}$$

find all such matrices.

Table 1 shows all unique coset interleaving arrays of length N_c that convert weight-2 RTZ inputs to non-RTZ inputs. They are labeled from A to X. A coset interleaving array is unique if a shift of the elements in the array does not produce another another coset interleaving array.

A	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$	B	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$	C	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$	D	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$
E	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$	F	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$	G	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	H	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$
I	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$	J	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$	K	$\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$	L	$\begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$
M	$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$	N	$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$	O	$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$	P	$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
Q	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}$	R	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{bmatrix}$	S	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$	T	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$
U	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$	V	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$	W	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$	X	$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

Table 1: All unique coset interleaving arrays of length $N_c = 9$ for weight-2 RTZ inputs

The interleaver length used in turbo coding are way greater than N_c and it is not possible to transform weight-2 RTZ inputs into non-RTZ inputs for all values of i. All is not lost however, since not all weight-2 RTZ inputs produce low-weight codewords. The formula for calculating the Hamming weight (w_H) of the Turbo codeword produced by a weight-$2'$ RTZ input occuring in both component codes is given by[SunTakeshita]

$$\begin{aligned} w_H =& 2 + (2 + \frac{t}{3})w_0 + (2 + \frac{s}{3})w_0 \\ =& 6 + \Big(\frac{t+s}{3}\Big)w_0, \ w_0 = 2 \end{aligned} \tag{0-4}$$

For (0-3), since $t = 9$ and $s = \Delta c := c^t_{a+\tau} - c^t_a$ (where t in Δc denotes the index of coset, is not the same with t in (0-4)), we have

$$w_H = 6 + \Big(3 + \frac{\Delta c}{3}\Big)w_0, \ w_0 = 2 \tag{0-5}$$

0.5 Coset Interleaver Design For Weight-3 RTZ inputs

As mentioned earlier, a weight-3 RTZ input is formed when the indices of the "1" bits each occur in different cosets. It goes without saying that the simplest way to convert a weight-3 RTZ input into a non-RTZ input is to make sure that at least two of indices of the "1" bits occur within the same coset after interleaving.

Following the pattern in the previous section for weight-2 RTZ inputs, we need a permutation matrix $\mathbf{\Pi}$ for weight-3 RTZ events. The best permutation matrix would be one that totally gets rid of all weight-3 RTZ interleavers of length $N_c = 9$. This is not possible for reasons that will be made clear soon. It is however possible to control the kind weight-3 RTZ that occurs. To further explain this, we introduce the concept of layers and layer distances. We take $\mathcal{N}$ and assuming that $\tau | N$, we feed the elements of $\mathcal{N}$ into a $N/\tau \times \tau$ matrix $\boldsymbol{N}$. Furthermore we label colums 0 to τ k, j, i respectively. The layer is defined as the row where an element of $\mathcal{N}$ exists. Furthermore given two element in $\boldsymbol{N}$, the layer distance is the difference between the row indices, with the index of the first row set to 0. It is clear that each column corresponds to a coset and therefore a weight-3 RTZ input occurs when the indices of the "1" bits each occur in different columns. When $N = N_c = 9$, the number of elements in each coset is always equal to the number of cosets and therefore it is impossible to completely get rid of weight-3 RTZ inputs when $N = N_c$.

Let l, l' be the pre-interleaving layer distance and the post-interleaving layer distance. We know that the codeword weight of a turbo code due to weight-3 RTZ inputs is given by

$$w_H = \begin{cases} 3 + 2(l + l'), & i < k, \ i' < k' \\ 7 + 2(l + l') & i \geq k, \ i' \geq k' \\ 5 + 2(l + l') & i \geq k, \ i' < k' \text{ or } i < k \ i' \geq k' \end{cases} \tag{0-6}$$

where k', i' is similarly defined but with respect to Π. To increase the value of w_H , we need to make l' as large as possible for N_c when $i' < k'$ or $i \geq k$

Unique permutation matrices which meet this criteria are shown in Table 2 and they are labeled from A to L

Depending on which permutation matrix is chosen from Table 2, Equation 0-6 can be simplified. As an example, Table 3 shows all the weight-3 RTZ inputs and the corresponding equation for calculating w_H

A	$\begin{bmatrix}0&0&0\\1&1&1\\2&2&2\end{bmatrix}$	B	$\begin{bmatrix}0&0&0\\1&1&2\\1&2&2\end{bmatrix}$
C	$\begin{bmatrix}0&0&0\\1&1&2\\2&1&2\end{bmatrix}$	D	$\begin{bmatrix}0&0&0\\1&1&2\\2&2&1\end{bmatrix}$
E	$\begin{bmatrix}0&0&0\\2&2&1\\1&1&2\end{bmatrix}$	F	$\begin{bmatrix}0&0&0\\2&2&1\\1&2&1\end{bmatrix}$
G	$\begin{bmatrix}0&0&0\\2&2&1\\2&1&1\end{bmatrix}$	H	$\begin{bmatrix}0&0&1\\0&1&1\\2&2&2\end{bmatrix}$
I	$\begin{bmatrix}0&0&1\\1&1&2\\2&0&2\end{bmatrix}$	J	$\begin{bmatrix}0&0&2\\0&2&2\\1&1&1\end{bmatrix}$
K	$\begin{bmatrix}0&0&2\\2&2&1\\1&0&1\end{bmatrix}$	L	$\begin{bmatrix}0&1&0\\1&1&2\\2&0&2\end{bmatrix}$

Table 2: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs

RTZ index	i, k condition	l	w_p	w_H
(0 4 8)	$i > k$	2	6	$(3+6)+(2l'+2) = 11+2l'$
(0 5 7)	$i > k$	2	6	$(3+6)+(2l'+2) = 11+2l'$
(1 3 8)	$i > k$	2	6	$(3+6)+(2l'+2) = 11+2l'$
(1 5 6)	$i < k$	2	4	$(3+4)+(2l') = 7+2l'$
(2 3 7)	$i < k$	2	4	$(3+4)+(2l') = 7+2l'$
(2 4 6)	$i < k$	2	4	$(3+4)+(2l') = 7+2l'$

Table 3: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs