“ Progress So Far”

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0.1 Notation

1. RTZ (Return-To-Zero) input :- A RTZ input is a binary input which causes a RSC encoder's final state to be return to zero after it has exited the zero state.
2. τ :- cycle length of the RSC encoder. For the 5/7 RSC encoder $\tau = 3$
3. N :- Interleaver length.
4. $\mathcal{N}$:- Integer set of $\{0, 1, \cdots, N-1\}$
5. $\mathbb{N}$: Indexed set of $\{0, 1, \cdots, N-1\}$ in the natural order.
6. We assume that $N/\tau = C$
7. $\mathcal{C}$ and $\mathbb{C}$ are definded in a similar manner.
8. $\mathcal{C}^t := \{c+t\}_{c\in\mathcal{C}}$ and $\mathbb{C}^t$ is the indexed set with the elements of $\mathcal{C}^t$ where $t = (0, 1, ..., \tau-1)$. Where it becomes necessary to distinguish between the elements of $\mathcal{C}^t$ and $\mathbb{C}^t$, we will write the elements of $\mathbb{C}^t$ as $c_{x'}^{t'}$ and the elements of $\mathcal{C}^t$ as c_x^t
9. Permutation matrix
$$\mathbf{\Pi} = \begin{bmatrix} \boldsymbol{\pi}^0 \\ \boldsymbol{\pi}^1 \\ \vdots \\ \boldsymbol{\pi}^{K-1} \end{bmatrix} = \left[\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_{\tau-1}\right] = \left[\pi_t^{(i)}\right]_{i=0,\ t=0}^{K-1,\ \tau-1}$$
where $\pi_t^{(i)} \in \{0, 1, \tau-1\}$.
10. For the row vector $\boldsymbol{\pi}^{(i)}$, let $\mathscr{S}^e[\boldsymbol{\pi}^{(i)}]$ be the left-hand cycle shift of $\boldsymbol{\pi}^{(i)}$ and $\mathscr{S}^e[\boldsymbol{\pi}_t]$ be the up cycle shift of $\boldsymbol{\pi}_t$
11. We assume that the operation outputs the elements in $\mathbb{C}^t$ in order while t is appeared in $\boldsymbol{\pi}^k$. For example, $\boldsymbol{\pi}^0 = (0, 0, 1)$ outputs (c_0^0, c_1^0, c_0^1). From this example, we can see that the column index of i in $\pi^{(i)}$ represents the coset it belongs to before interleaving and the value $\pi_j^{(i)}$ specifies the coset after interleaving
12. Our goal is to find a prefer $\mathbf{\Pi}$ and $\mathbb{C}^t$, $t = 0, 1, \cdots, \tau-1$.

0.2 Cosets and RTZ inputs

1. a weight 2 input sequence
   - polynomial: $P(x) = x^{h\tau+t}(1 + x^{\alpha\tau}) = x^t(x^{h\tau} + x^{(h+\alpha)\tau})$
   - coset: the hth and $(a+\alpha)$th elements in $\mathbb{C}^t$
2. a weight 3 input sequence
   - polynomial: $Q(x) = x^{h\tau+t}(1 + x^{\beta\tau+1} + x^{\gamma\tau+2}) = x^{h\tau+t} + x^{(h+\beta)\tau+t+1} + x^{(h+\gamma)\tau+t+2}$. Notice that $h \le \beta$ is not a necessary condition.
   - coset: the hth element in $\mathbb{C}^t$, $(h+\beta)$th element in $\mathbb{C}^{[t+1]_\tau}$, and $(h+\gamma)$th element in $\mathbb{C}^{[t+2]_\tau}$.

0.3 Representation of interleaver

If the mapping relationship between elements in $\boldsymbol{x}$ and $\boldsymbol{y}$ are read column wise as shown below

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 5 & 1 & 6 & 2 & 7 & 3 & 8 & 4 \end{bmatrix}$$

the interleaver is represented by $\mathbb{N} = \{0, 5, 1, 6, 2, 7, 3, 8, 4\}$.

Let $\mathbb{C}^0 = \{0, 6, 3\}$, $\mathbb{C}^1 = \{1, 7, 4\}$, and $\mathbb{C}^2 = \{5, 2, 8\}$. Then, the permutation matrix of $\mathbb{N}$ is $\mathbf{\Pi} = (0, 2, 1)$. Notice the row of $\mathbf{\Pi}$ takes cyclicly.

0.4 Coset Interleaver Design For Weight-2 RTZ inputs

From the definition of Weight-2 RTZ inputs in the previous section, we know that the index of the "1" bits are in the same coset $\mathcal{C}^t$ and represented by the elements c_a^t and $c_{a+\alpha}^t$. After interleaving, another weight-2 RTZ input occurs if the "1" bits are mapped to the elements $c_{a'}^{t'}$ and $c_{a'+\alpha'}^{t'}$ in $\mathbb{C}^{t'}$. Therefore, to convert a weigth-2 RTZ input into a non-RTZ input we need a permutation matrix $\mathbf{\Pi}$ which satisfies the condition below

$$\pi_j^{(i)} \neq \pi_j^{(i')}, \ |i - i'| \leq N_c \tag{0-1}$$

Since $\mathbf{\Pi}$ consisting of τ elements, the maximum length of column elements consisting of values different each other is τ. Thus, the cut-off interleaver length for which (**??**) is satisfied is $N_c = \tau = 3$.

1. One cycle permutation: Each row is permutation of the sequence $(0, 1, 2)$. Setting the element at the first row and first column to 0, there are exactly 4 permutation matrices that exist for cut-off length N_c. Let

$$\boldsymbol{\psi} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \ \boldsymbol{\psi}' = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

We then have

$$\begin{aligned} [\boldsymbol{\psi}, \mathscr{S}^1[\boldsymbol{\psi}], \mathscr{S}^2[\boldsymbol{\psi}]] &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} := \boldsymbol{\psi}(\boldsymbol{\psi}) \\ [\boldsymbol{\psi}', \mathscr{S}^1[\boldsymbol{\psi}'], \mathscr{S}^2[\boldsymbol{\psi}']] &= \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} := \boldsymbol{\psi}(\boldsymbol{\psi}') \\ [\boldsymbol{\psi}, \mathscr{S}^2[\boldsymbol{\psi}], \mathscr{S}^1[\boldsymbol{\psi}]] &= \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} := \boldsymbol{\psi}'(\boldsymbol{\psi}) \\ [\boldsymbol{\psi}', \mathscr{S}^2[\boldsymbol{\psi}'], \mathscr{S}^1[\boldsymbol{\psi}']] &= \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} := \boldsymbol{\psi}'(\boldsymbol{\psi}') \end{aligned} \tag{0-2}$$

2. Two cycle permutation: Two rows are permutation of the sequence $(0, 0, 1, 1, 2, 2)$.

   There are no permutation matrices that satisfying cut-off length N_c. This is because the sequence length is not divisible by N_c, there will always be 2 elements of the same value in each row of $\mathbf{\Pi}$

3. Three cycle permutation: Three rows are permutation of the sequence$(0, 0, 0, 1, 1, 1, 2, 2, 2)$.

   Example of the permutation matrices satisfying cut-off length $N_c = 9$ are shown in **??**

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \tag{0-3}$$

find all such matrices.

Table **??** shows all unique coset interleaving arrays of length N_c that convert weight-2 RTZ inputs to non-RTZ inputs. They are labeled from A to X. A coset interleaving array is unique if a shift of the elements in the array does not produce another another coset interleaving array.

A	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$	B	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$	C	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$	D	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$
E	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$	F	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$	G	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	H	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$
I	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$	J	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$	K	$\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$	L	$\begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$
M	$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$	N	$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$	O	$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$	P	$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
Q	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}$	R	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{bmatrix}$	S	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$	T	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$
U	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$	V	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$	W	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$	X	$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

Table 1: All unique coset interleaving arrays of length $N_c = 9$ for weight-2 RTZ inputs

The interleaver length used in turbo coding are way greater than N_c and it is not possible to transform weight-2 RTZ inputs into non-RTZ inputs for all values of i. All is not lost however, since not all weight-2 RTZ inputs produce low-weight codewords. The formula for calculating the Hamming weight (w_H) of the Turbo codeword produced by a weight-2 RTZ input occuring in both component codes is given by[SunTakeshita]

$$\begin{aligned} w_H =& 2 + (2 + \frac{\Delta_c}{\tau})w_0 + (2 + \frac{\Delta_{c'}}{\tau})w_0 \\ =& 6 + \Big(\frac{\Delta_c + \Delta_{c'}}{\tau}\Big)w_0, \;\; w_0 = 2 \end{aligned} \tag{0-4}$$

For all the Π in Table **??**, since $\Delta c = 9 = 3\tau$ and $\Delta c' := (c^{t'}_{a'+\alpha'} - c^{t'}_{a'})$ we have

$$w_H = 6 + \Big(3 + \frac{\Delta c'}{3}\Big) w_0,\ w_0 = 2 \tag{0-5}$$

0.5 Coset Interleaver Design For Weight-3 RTZ inputs

As mentioned earlier, a weight-3 RTZ input is formed when the indices of the "1" bits each occur in different cosets. It goes without saying that the simplest way to convert a weight-3 RTZ input into a non-RTZ input is to make sure that at least two of indices of the "1" bits occur within the same coset after interleaving.

Following the pattern in the previous section for weight-2 RTZ inputs, we need a permutation matrix $\mathbf{\Pi}$ for weight-3 RTZ events. The best permutation matrix would be one that totally gets rid of all weight-3 RTZ interleavers of length $N_c = 9$. This is not possible for reasons that will be made clear soon. It is however possible to control the kind weight-3 RTZ that occurs. To further explain this, we introduce the concept of layers and layer distances. We take $\mathcal{N}$ and assuming that $\tau | N$, we feed the elements of $\mathcal{N}$ into a $N/\tau \times \tau$ matrix $\boldsymbol{N}$. Furthermore we label colums 0 to τ k, j, i respectively. The layer is defined as the row where an element of $\mathcal{N}$ exists. Furthermore given two element in $\boldsymbol{N}$, the layer distance is the difference between the row indices, with the index of the first row set to 0. It is clear that each column corresponds to a coset and therefore a weight-3 RTZ input occurs when the indices of the "1" bits each occur in different columns. When $N = N_c = 9$, the number of elements in each coset is always equal to the number of cosets and therefore it is impossible to completely get rid of weight-3 RTZ inputs when $N = N_c$.

Let l, l' be the pre-interleaving layer distance and the post-interleaving layer distance. We know that the codeword weight of a turbo code due to weight-3 RTZ inputs is given by

$$w_H = \begin{cases} 3 + 2(l + l'), & i < k,\ i' < k' \\ 7 + 2(l + l') & i \geq k,\ i' \geq k' \\ 5 + 2(l + l') & i \geq k,\ i' < k' \text{ or } i < k\ i' \geq k' \end{cases} \tag{0-6}$$

where k', i' is similarly defined but with respect to Π. To increase the value of w_H , we need to make l' as large as possible for N_c when $i' < k'$ or $i \geq k$

Unique permutation matrices which meet this criteria are shown in Table **??** and they are labeled from A to L

Depending on which permutation matrix is chosen from Table **??**, Equation **??** can be simplified.

In general w_H for turbo codewords as a result of weight-3 RTZ inputs can be written as

$$w_H = 3 + w_p + w'_p$$

, where w_p, w'_p refer to the pre-interleaving parity weight and the post-interleaving parity weight respectively. The value of w_p for the pre-interleaving weight-3 is dependent on the elements in $\mathcal{C}^t$

Let $(c^t_\alpha,\ c^{t+1}_{\alpha+\beta},\ c^{t+2}_{\alpha+\gamma})$ be the vector representing a weight-3 RTZ input, where

$$\begin{aligned} c^t_\alpha &= \alpha\tau + t \\ c^{t+1}_{\alpha+\beta} &= (\alpha + \beta)\tau + t + 1 \\ c^{t+2}_{\alpha+\gamma} &= (\alpha + \gamma)\tau + t + 2 \end{aligned} \tag{0-7}$$

A	$\begin{bmatrix}0&0&0\\1&1&1\\2&2&2\end{bmatrix}$	B	$\begin{bmatrix}0&0&0\\1&1&2\\1&2&2\end{bmatrix}$
C	$\begin{bmatrix}0&0&0\\1&1&2\\2&1&2\end{bmatrix}$	D	$\begin{bmatrix}0&0&0\\1&1&2\\2&2&1\end{bmatrix}$
E	$\begin{bmatrix}0&0&0\\2&2&1\\1&1&2\end{bmatrix}$	F	$\begin{bmatrix}0&0&0\\2&2&1\\1&2&1\end{bmatrix}$
G	$\begin{bmatrix}0&0&0\\2&2&1\\2&1&1\end{bmatrix}$	H	$\begin{bmatrix}0&0&1\\0&1&1\\2&2&2\end{bmatrix}$
I	$\begin{bmatrix}0&0&1\\1&1&2\\2&0&2\end{bmatrix}$	J	$\begin{bmatrix}0&0&2\\0&2&2\\1&1&1\end{bmatrix}$
K	$\begin{bmatrix}0&0&2\\2&2&1\\1&0&1\end{bmatrix}$	L	$\begin{bmatrix}0&1&0\\1&1&2\\2&0&2\end{bmatrix}$

Table 2: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs

if c_α^t is the smallest value in the vector, then

$$l = \begin{cases} \beta, & \beta > \gamma \\ \gamma & \beta < \gamma \end{cases} \tag{0-8}$$

And

$$w_p = \begin{cases} 2\beta + 2, & i \geq k,\ \beta > \gamma \\ 2\beta & i < k,\ \beta > \gamma \\ 2\gamma + 2, & i \geq k,\ \beta < \gamma \\ 2\gamma & i < k,\ \beta < \gamma \end{cases} \tag{0-9}$$

where $i = \lfloor \frac{c_*^2}{3} \rfloor$, $k = \lfloor \frac{c_*^0}{3} \rfloor$ and $*$ is the element position in $\mathcal{C}^t$
By deciding on the Π we can easily calculate all values of l and w_p

w'_p on the other hand is dependent on the elements in $\mathbb{C}^t,\ t = 0, 1, .., \tau - 1$ Let $(c_{\alpha'}^{t'},\ c_{\alpha'+\beta'}^{t'+1},\ c_{\alpha'+\gamma'}^{t'+2})$ be the vector representing a weight-3 RTZ input, where

$$\begin{aligned} c_{\alpha'}^{t'} &= \alpha\tau + t' \\ c_{\alpha'+\beta'}^{t'+1} &= (\alpha + \beta)\tau + t' + 1 \\ c_{\alpha'+\gamma'}^{t'+2} &= (\alpha + \gamma)\tau + t' + 2 \end{aligned} \tag{0-10}$$

if $c_{\alpha'}^{t'}$ is the smallest value in the vector, then

$$l' = \begin{cases} \beta, & \beta > \gamma \\ \gamma & \beta < \gamma \end{cases} \tag{0-11}$$

RTZ index	i,k condition	l	w_p	w_H
(0 4 8)	$i > k$	2	6	$\begin{cases} 11+2\beta, & i' \geq k',\ \beta > \gamma \\ 11+2\gamma, & i' \geq k',\ \beta < \gamma \\ 9+2\beta & i' < k',\ \beta > \gamma \\ 9+2\gamma & i' < k',\ \beta < \gamma \end{cases}$
(0 5 7)	$i > k$	2	6	$\begin{cases} 11+2\beta, & i' \geq k',\ \beta > \gamma \\ 11+2\gamma, & i' \geq k',\ \beta < \gamma \\ 9+2\beta & i' < k',\ \beta > \gamma \\ 9+2\gamma & i' < k',\ \beta < \gamma \end{cases}$
(1 3 8)	$i > k$	2	6	$\begin{cases} 11+2\beta, & i' \geq k',\ \beta > \gamma \\ 11+2\gamma, & i' \geq k',\ \beta < \gamma \\ 9+2\beta & i' < k',\ \beta > \gamma \\ 9+2\gamma & i' < k',\ \beta < \gamma \end{cases}$
(1 5 6)	$i < k$	2	4	$\begin{cases} 9+2\beta, & i' \geq k',\ \beta > \gamma \\ 9+2\gamma, & i' \geq k',\ \beta < \gamma \\ 7+2\beta & i' < k',\ \beta > \gamma \\ 7+2\gamma & i' < k',\ \beta < \gamma \end{cases}$
(2 3 7)	$i < k$	2	4	$\begin{cases} 9+2\beta, & i' \geq k',\ \beta > \gamma \\ 9+2\gamma, & i' \geq k',\ \beta < \gamma \\ 7+2\beta & i' < k',\ \beta > \gamma \\ 7+2\gamma & i' < k',\ \beta < \gamma \end{cases}$
(2 4 6)	$i < k$	2	4	$\begin{cases} 9+2\beta, & i' \geq k',\ \beta > \gamma \\ 9+2\gamma, & i' \geq k',\ \beta < \gamma \\ 7+2\beta & i' < k',\ \beta > \gamma \\ 7+2\gamma & i' < k',\ \beta < \gamma \end{cases}$

Table 3: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs

And

$$w'_p = \begin{cases} 2\beta+2, & i' \geq k',\ \beta > \gamma \\ 2\beta & i' < k',\ \beta > \gamma \\ 2\gamma+2, & i' \geq k',\ \beta < \gamma \\ 2\gamma & i' < k',\ \beta < \gamma \end{cases} \tag{0-12}$$

where $i' = \lfloor \frac{c_*^2}{3} \rfloor$,$k' = \lfloor \frac{c_*^0}{3} \rfloor$ and * represents its position in $\mathbb{C}^t$
As an example, Table **??** shows all the weight-3 RTZ inputs and the corresponding equations for calculating w_H

To Do

1. add elements for inter-block Weight-3 RTZ inputs