“ Progress So Far”

Kwame Ackah Bohulu

July 8, 2020

0.1 Notation

1. RTZ (Return-To-Zero) input :- A RTZ input is a binary input which causes a RSC encoder's final state to be return to zero after it has exited the zero state.

2. τ :- cycle length of the RSC encoder. For the 5/7 RSC encoder $\tau = 3$

3. N :- Interleaver length.

4. $\mathcal{N}$:- Integer set of $\{0, 1, \cdots, N-1\}$

5. $\mathbb{N}$: Indexed set of $\{0, 1, \cdots, N-1\}$ in the natural order.

6. We assume that $N/\tau = C$

7. $\mathcal{C}$ and $\mathbb{C}$ are definded in a similar manner.

8. $\mathcal{C}^t := \{c+t\}_{c \in \mathcal{C}}$ and $\mathbb{C}^t$ is the indexed set with the elements of $\mathcal{C}^t$ where $t = (0, 1, ..., \tau-1)$. Where it becomes necessary to distinguish between the elements of $\mathcal{C}^t$ and $\mathbb{C}^t$, we will write the elements of $\mathbb{C}^t$ as $c^{t'}_{x'}$ and the elements of $\mathcal{C}^t$ as c^t_x

9. Permutation matrix

$$\mathbf{\Pi} = \begin{bmatrix} \boldsymbol{\pi}^0 \\ \boldsymbol{\pi}^1 \\ \vdots \\ \boldsymbol{\pi}^{K-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_{\tau-1} \end{bmatrix} = \begin{bmatrix} \pi^{(i)}_t \end{bmatrix}^{K-1,\ \tau-1}_{i=0,\ t=0}$$

where $\pi^{(i)}_t \in \{0, 1, \tau-1\}$.

10. For the row vector $\boldsymbol{\pi}^{(i)}$, let $\mathscr{S}^e[\boldsymbol{\pi}^{(i)}]$ be the left-hand cycle shift of $\boldsymbol{\pi}^{(i)}$ and $\mathscr{S}^e[\boldsymbol{\pi}_t]$ be the up cycle shift of $\boldsymbol{\pi}_t$

11. We assume that the operation outputs the elements in $\mathbb{C}^t$ in order while t is appeared in $\boldsymbol{\pi}^k$. For example, $\boldsymbol{\pi}^0 = (0, 0, 1)$ outputs (c^0_0, c^0_1, c^1_0). From this example, we can see that the column index of i in $\pi^{(i)}$ represents the coset it belongs to before interleaving and the value $\pi^{(i)}_j$ specifies the coset after interleaving

12. Our goal is to find a prefer $\mathbf{\Pi}$ and $\mathbb{C}^t$, $t = 0, 1, \cdots, \tau-1$.

0.2 Cosets and RTZ inputs

1. a weight 2 input sequence
   - polynomial: $P(x) = x^{h\tau+t}(1 + x^{\alpha\tau}) = x^t(x^{h\tau} + x^{(h+\alpha)\tau})$
   - coset: the hth and $(h+\alpha)$th elements in $\mathbb{C}^t$

2. a weight 3 input sequence
   - polynomial: $Q(x) = x^{h\tau+t}(1 + x^{\beta\tau+1} + x^{\gamma\tau+2}) = x^{h\tau+t} + x^{(h+\beta)\tau+t+1} + x^{(h+\gamma)\tau+t+2}$. Notice that $h \leq \beta$ is not a necessary condition.
   - coset: the hth element in $\mathbb{C}^t$, $(h+\beta)$th element in $\mathbb{C}^{[t+1]_\tau}$, and $(h+\gamma)$th element in $\mathbb{C}^{[t+2]_\tau}$.

0.3 Representation of interleaver

If the mapping relationship between elements in $\boldsymbol{x}$ and $\boldsymbol{y}$ are read column wise as shown below

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 5 & 1 & 6 & 2 & 7 & 3 & 8 & 4 \end{bmatrix}$$

the interleaver is represented by $\mathbb{N} = \{0, 5, 1, 6, 2, 7, 3, 8, 4\}$.

Let $\mathbb{C}^0 = \{0, 6, 3\}$, $\mathbb{C}^1 = \{1, 7, 4\}$, and $\mathbb{C}^2 = \{5, 2, 8\}$. Then, the permutation matrix of $\mathbb{N}$ is $\mathbf{\Pi} = (0, 2, 1)$. Notice the row of $\mathbf{\Pi}$ takes cyclicly.

0.4 Coset Interleaver Design For Weight-2 RTZ inputs

From the definition of Weight-2 RTZ inputs in the previous section, we know that the index of the "1" bits are in the same coset $\mathcal{C}^t$ and represented by the elements $\pi_t^{(h)}$ and $\pi_t^{(h+\alpha)}$. After interleaving, another weight-2 RTZ input occurs if the "1" bits are mapped to the elements $\pi_t^{(h')}$ and $\pi_t^{(h'+\alpha')}$ in $\mathbb{C}^t$. Therefore, to convert a weigth-2 RTZ input into a non-RTZ input we need a permutation matrix $\mathbf{\Pi}$ which satisfies the condition below

$$\pi_j^{(i)} \neq \pi_j^{(i')}, \ |i - i'| \leq N_c \tag{0-1}$$

Since $\mathbf{\Pi}$ consisting of τ elements, the maximum length of column elements consisting of values different each other is τ. Thus, the cut-off interleaver length for which (0-1) is satisfied is $N_c = \tau = 3$.

1. One cycle permutation: Each row is permutation of the sequence $(0, 1, 2)$. Setting the element at the first row and first column to 0, there are exactly 4 permutation matrices that exist for cut-off length N_c. Let

$$\boldsymbol{\psi} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \ \boldsymbol{\psi}' = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

   We then have

$$\begin{aligned}
[\boldsymbol{\psi}, \mathscr{S}^1[\boldsymbol{\psi}], \mathscr{S}^2[\boldsymbol{\psi}]] &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} := \boldsymbol{\psi}(\boldsymbol{\psi}) \\
[\boldsymbol{\psi}', \mathscr{S}^1[\boldsymbol{\psi}'], \mathscr{S}^2[\boldsymbol{\psi}']] &= \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} := \boldsymbol{\psi}(\boldsymbol{\psi}') \\
[\boldsymbol{\psi}, \mathscr{S}^2[\boldsymbol{\psi}], \mathscr{S}^1[\boldsymbol{\psi}]] &= \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} := \boldsymbol{\psi}'(\boldsymbol{\psi}) \\
[\boldsymbol{\psi}', \mathscr{S}^2[\boldsymbol{\psi}'], \mathscr{S}^1[\boldsymbol{\psi}']] &= \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} := \boldsymbol{\psi}'(\boldsymbol{\psi}')
\end{aligned} \tag{0-2}$$

2. Two cycle permutation: Two rows are permutation of the sequence $(0,0,1,1,2,2)$.

   There are no permutation matrices that satisfying cut-off length N_c. This is because the sequence length is not divisible by N_c, there will always be 2 elements of the same value in each row of $\mathbf{\Pi}$

3. Three cycle permutation: Three rows are permutation of the sequence$(0,0,0,1,1,1,2,2,2)$.

   Example of the permutation matrices satisfying cut-off length $N_c = 9$ are shown in 1

Table 1 shows all unique coset interleaving arrays of length N_c that convert weight-2 RTZ inputs to non-RTZ inputs. They are labeled from A to X. A coset interleaving array is unique if a shift of the elements in the array does not produce another another coset interleaving array.

A	$\begin{bmatrix}0&0&0\\1&1&1\\2&2&2\end{bmatrix}$	B	$\begin{bmatrix}0&0&0\\1&1&2\\2&2&1\end{bmatrix}$	C	$\begin{bmatrix}0&0&0\\1&2&1\\2&1&2\end{bmatrix}$	D	$\begin{bmatrix}0&0&0\\1&2&2\\2&1&1\end{bmatrix}$
E	$\begin{bmatrix}0&0&0\\2&1&1\\1&2&2\end{bmatrix}$	F	$\begin{bmatrix}0&0&0\\2&1&2\\1&2&1\end{bmatrix}$	G	$\begin{bmatrix}0&0&0\\2&2&1\\1&1&2\end{bmatrix}$	H	$\begin{bmatrix}0&0&0\\2&2&2\\1&1&1\end{bmatrix}$
I	$\begin{bmatrix}0&0&1\\1&1&0\\2&2&2\end{bmatrix}$	J	$\begin{bmatrix}0&0&1\\1&2&0\\2&1&2\end{bmatrix}$	K	$\begin{bmatrix}0&0&1\\2&1&0\\1&2&2\end{bmatrix}$	L	$\begin{bmatrix}0&0&1\\2&2&0\\1&1&2\end{bmatrix}$
M	$\begin{bmatrix}0&0&2\\1&1&0\\2&2&1\end{bmatrix}$	N	$\begin{bmatrix}0&0&2\\1&2&0\\2&1&1\end{bmatrix}$	O	$\begin{bmatrix}0&0&2\\2&1&0\\1&2&1\end{bmatrix}$	P	$\begin{bmatrix}0&0&2\\2&2&0\\1&1&1\end{bmatrix}$
Q	$\begin{bmatrix}0&1&0\\1&0&1\\2&2&2\end{bmatrix}$	R	$\begin{bmatrix}0&1&0\\1&0&2\\2&2&1\end{bmatrix}$	S	$\begin{bmatrix}0&1&0\\1&2&1\\2&0&2\end{bmatrix}$	T	$\begin{bmatrix}0&1&0\\2&0&1\\1&2&2\end{bmatrix}$
U	$\begin{bmatrix}0&1&0\\2&0&2\\1&2&1\end{bmatrix}$	V	$\begin{bmatrix}0&1&0\\2&2&1\\1&0&2\end{bmatrix}$	W	$\begin{bmatrix}0&1&1\\1&2&0\\2&0&2\end{bmatrix}$	X	$\begin{bmatrix}0&2&0\\2&0&2\\1&1&1\end{bmatrix}$

Table 1: All unique coset interleaving arrays of length $N_c = 9$ for weight-2 RTZ inputs

The interleaver length used in turbo coding are way greater than N_c and it is not possible to transform weight-2 RTZ inputs into non-RTZ inputs for all values of i. All is not lost however, since not all weight-2 RTZ inputs produce low-weight codewords. The formula for calculating the Hamming weight (w_H) of the Turbo codeword produced by a weight-2 RTZ input occuring in both component codes is given by[SunTakeshita]

$$\begin{aligned} w_H =& 2 + (2 + \frac{\Delta_c}{\tau})w_0 + (2 + \frac{\Delta_{c'}}{\tau})w_0 \\ =& 6 + \Big(\frac{\Delta_c + \Delta_{c'}}{\tau}\Big)w_0,\ w_0 = 2 \end{aligned} \tag{0-3}$$

For all the Π in Table 1, since $\Delta c = 9 = 3\tau$ and $\Delta c' := (c^{t'}_{a'+\alpha'} - c^{t'}_{a'})$ we have

$$w_H = 6 + \Big(3 + \frac{\Delta c'}{3}\Big)w_0,\ w_0 = 2 \tag{0-4}$$

0.5 Coset Interleaver Design For Weight-3 RTZ inputs

As mentioned earlier, a weight-3 RTZ input is formed when the indices of the "1" bits each occur in different cosets. It goes without saying that the simplest way to convert a weight-3 RTZ input into a non-RTZ input is to make sure that at least two of indices of the "1" bits occur within the same coset after interleaving.

$$w_H = 7 + 2(l + l') \tag{0-5}$$

Unique permutation matrices which meet this criteria are shown in Table 2 and they are labeled from A to L

A	$\begin{bmatrix}0&0&0\\1&1&1\\2&2&2\end{bmatrix}$	B	$\begin{bmatrix}0&0&0\\1&1&2\\1&2&2\end{bmatrix}$
C	$\begin{bmatrix}0&0&0\\1&1&2\\2&1&2\end{bmatrix}$	D	$\begin{bmatrix}0&0&0\\1&1&2\\2&2&1\end{bmatrix}$
E	$\begin{bmatrix}0&0&0\\2&2&1\\1&1&2\end{bmatrix}$	F	$\begin{bmatrix}0&0&0\\2&2&1\\1&2&1\end{bmatrix}$
G	$\begin{bmatrix}0&0&0\\2&2&1\\2&1&1\end{bmatrix}$	H	$\begin{bmatrix}0&0&1\\0&1&1\\2&2&2\end{bmatrix}$
I	$\begin{bmatrix}0&0&1\\1&1&2\\2&0&2\end{bmatrix}$	J	$\begin{bmatrix}0&0&2\\0&2&2\\1&1&1\end{bmatrix}$
K	$\begin{bmatrix}0&0&2\\2&2&1\\1&0&1\end{bmatrix}$	L	$\begin{bmatrix}0&1&0\\1&1&2\\2&0&2\end{bmatrix}$

Table 2: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs

Depending on which permutation matrix is chosen from Table 2, Equation 0-5 can be simplified.

In general w_H for turbo codewords as a result of weight-3 RTZ inputs can be written as

$$w_H = 3 + w_p + w_p'$$

, where w_p, w_p' refer to the pre-interleaving parity weight and the post-interleaving parity weight respectively. The value of w_p for the pre-interleaving weight-3 is dependent on the elements in $\mathcal{C}^t$

Let $(c_\alpha^t,\ c_{\alpha+\beta}^{t+1},\ c_{\alpha+\gamma}^{t+2})$ be the vector representing a weight-3 RTZ input, where

$$\begin{aligned} c_\alpha^t &= \alpha\tau + t \\ c_{\alpha+\beta}^{t+1} &= (\alpha+\beta)\tau + t + 1 \\ c_{\alpha+\gamma}^{t+2} &= (\alpha+\gamma)\tau + t + 2 \end{aligned} \tag{0-6}$$

if c_α^t is the smallest value in the vector, then

$$l = \begin{cases} \beta, & \beta > \gamma \\ \gamma & \beta < \gamma \end{cases} \tag{0-7}$$

And

$$w_p = \begin{cases} 2\beta + 2, & i \geq k, \ \beta > \gamma \\ 2\beta & i < k, \ \beta > \gamma \\ 2\gamma + 2, & i \geq k, \ \beta < \gamma \\ 2\gamma & i < k, \ \beta < \gamma \end{cases} \tag{0-8}$$

where $i = \lfloor \frac{c_*^2}{3} \rfloor$, $k = \lfloor \frac{c_*^0}{3} \rfloor$ and $*$ is the element position in $\mathcal{C}^t$
By deciding on the Π we can easily calculate all values of l and w_p

w_p' on the other hand is dependent on the elements in $\mathbb{C}^t, \ t = 0, 1, .., \tau-1$ Let $(c_{\alpha'}^{t'}, \ c_{\alpha'+\beta'}^{t'+1}, \ c_{\alpha'+\gamma'}^{t'+2})$ be the vector representing a weight-3 RTZ input, where

$$\begin{aligned} c_{\alpha'}^{t'} &= \alpha\tau + t' \\ c_{\alpha'+\beta'}^{t'+1} &= (\alpha + \beta)\tau + t' + 1 \\ c_{\alpha'+\gamma'}^{t'+2} &= (\alpha + \gamma)\tau + t' + 2 \end{aligned} \tag{0-9}$$

if $c_{\alpha'}^{t'}$ is the smallest value in the vector, then

$$l' = \begin{cases} \beta, & \beta > \gamma \\ \gamma & \beta < \gamma \end{cases} \tag{0-10}$$

And

$$w_p' = \begin{cases} 2\beta + 2, & i' \geq k', \ \beta > \gamma \\ 2\beta & i' < k', \ \beta > \gamma \\ 2\gamma + 2, & i' \geq k', \ \beta < \gamma \\ 2\gamma & i' < k', \ \beta < \gamma \end{cases} \tag{0-11}$$

where $i' = \lfloor \frac{c_*^2}{3} \rfloor$,$k' = \lfloor \frac{c_*^0}{3} \rfloor$ and $*$ represents its position in $\mathbb{C}^t$
As an example, Table 3 shows all the weight-3 RTZ inputs and the corresponding equations for calculating w_H

To Do

1. add elements for inter-block Weight-3 RTZ inputs

RTZ index	l	w_p	w_H
(0 4 8)	2	6	$\begin{cases} 11+2\beta, & \beta > \gamma \\ 11+2\gamma, & \beta < \gamma \end{cases}$
(0 5 7)	2	6	$\begin{cases} 11+2\beta, & \beta > \gamma \\ 11+2\gamma, & \beta < \gamma \end{cases}$
(1 3 8)	2	6	$\begin{cases} 11+2\beta, & \beta > \gamma \\ 11+2\gamma, & \beta < \gamma \end{cases}$
(1 5 6)	2	4	$\begin{cases} 9+2\beta, & \beta > \gamma \\ 9+2\gamma, & \beta < \gamma \end{cases}$
(2 3 7)	2	4	$\begin{cases} 9+2\beta, & \beta > \gamma \\ 9+2\gamma, & \beta < \gamma \end{cases}$
(2 4 6)	2	4	$\begin{cases} 9+2\beta, & \beta > \gamma \\ 9+2\gamma, & \beta < \gamma \end{cases}$

Table 3: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs