Formula to Calculate Weight for Low-Weight Weight 3 Inputs and Proof

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0.1 Equation and Proof

Proof. The polynomial representation of a weight-3 RTZ input is given by

$$Q(x) = x^{h\tau+t}(1 + x^{\beta\tau+1} + x^{\gamma\tau+2})$$

The impulse response of the RSC encoder is

$$(1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ \cdots)$$

and using the inpulse response, we can calculate the parity weight as well as Hamming weight of the turbo codeword.

Let
$\boldsymbol{\phi}_1 = (0\ 0\ 1),\ \boldsymbol{\phi}'_1 = (0\ 1\ 0),\ \boldsymbol{\phi}''_1 = (1\ 0\ 0)$,
$\boldsymbol{\phi}_2 = (0\ 1\ 1),\ \boldsymbol{\phi}'_2 = (1\ 1\ 0),\ \boldsymbol{\phi}''_2 = (1\ 0\ 1)$,
$\boldsymbol{\phi}_3 = (1\ 1\ 1)$.

Then, the weight-3 RTZ occurs since $\boldsymbol{\phi}_2 + \boldsymbol{\phi}'_2 + \boldsymbol{\phi}''_2 = \mathbf{0}_3$. Now, we consider the weight of the vector derived by the sumation of the followings vectors.

$$\begin{array}{c}(\mathbf{0}_{3i}\ \boldsymbol{\phi}_1\ \boldsymbol{\phi}'_2\ \cdots)\\(\mathbf{0}_{3j}\ \boldsymbol{\phi}_2\ \boldsymbol{\phi}''_2\ \cdots)\\(\mathbf{0}_{3k}\ \boldsymbol{\phi}_3\ \boldsymbol{\phi}_2\ \cdots)\end{array}$$

Where

$$k = h,\ j = \beta + h,\ i = \gamma + h$$

Without loss of generality, we can assume that all weight-3 RTZ inputs begin at the 0th position, ie $h = t = 0$. This is because the case where $h > 0$ or $t > 0$ is just a right-shifted version of the weight-3 RTZ. With this assumption, we have $k = 0,\ j = \beta,\ i = \gamma$ and we only need to consider cases where $k = 0,\ i \geq k$. To simplify calculation, we have included an addition table for all the vectors which is shown in Table 1

	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}'_1$	$\boldsymbol{\phi}''_1$	$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}'_2$	$\boldsymbol{\phi}''_2$	$\boldsymbol{\phi}_3$
$\boldsymbol{\phi}_1$	$\mathbf{0}_3$	–	–	–	–	–	–
$\boldsymbol{\phi}'_1$	$\boldsymbol{\phi}_2$	$\mathbf{0}_3$	–	–	–	–	–
$\boldsymbol{\phi}''_1$	$\boldsymbol{\phi}''_2$	$\boldsymbol{\phi}'_2$	$\mathbf{0}_3$	–	–	–	–
$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}'_1$	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_3$	$\mathbf{0}_3$	–	–	–
$\boldsymbol{\phi}'_2$	$\boldsymbol{\phi}_3$	$\boldsymbol{\phi}''_1$	$\boldsymbol{\phi}'_1$	$\boldsymbol{\phi}''_2$	$\mathbf{0}_3$	–	–
$\boldsymbol{\phi}''_2$	$\boldsymbol{\phi}''_1$	$\boldsymbol{\phi}_3$	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}'_2$	$\boldsymbol{\phi}_2$	$\mathbf{0}_3$	–
$\boldsymbol{\phi}_3$	$\boldsymbol{\phi}'_2$	$\boldsymbol{\phi}''_2$	$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}''_1$	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}'_1$	$\mathbf{0}_3$

Table 1: Truth Table

Furthermore, we consider 4 general cases for all possible values of i, j, k where $i \geq k$ These cases are $(=\ \ =)$, $(=\ \ <)$, $(<\ \ =)$ and $(<\ \ <)$

Case 0: $i = j = k$

For this case, the vectors to sum will be

$$\begin{array}{c} (\boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2' \ \cdots) \\ (\boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2'' \ \cdots) \\ (\boldsymbol{\phi}_3 \ \boldsymbol{\phi}_2 \ \cdots) \\ \hline (\boldsymbol{\phi}_2'' \ \mathbf{0}_3 \ \cdots) \end{array}$$

and the derived vector will be $(\boldsymbol{\phi}_2'' \ \mathbf{0}_3 \ \cdots)$ with a weight of $w_p = 2$

Case 1a: $i = k < j$

vectors to sum:

$$\begin{array}{r} (\boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2' \ \boldsymbol{\phi}_2' \ \boldsymbol{\phi}_2' \ \boldsymbol{\phi}_2' \ \cdots) \\ (\mathbf{0}_3 \ \cdots \ \mathbf{0}_3 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2'' \ \cdots) \\ +(\boldsymbol{\phi}_3 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \cdots) \\ \hline (\boldsymbol{\phi}_2' \ \boldsymbol{\phi}_2'' \ \boldsymbol{\phi}_2'' \ \boldsymbol{\phi}_2' \ \mathbf{0}_3 \ \cdots) \end{array}$$

derived vector : $(\boldsymbol{\phi}_2' \ (\boldsymbol{\phi}_2'')_{j-k-1} \ \boldsymbol{\phi}_2' \ \mathbf{0}_3 \ \cdots)$

Parity weight:

$$w_p = 2(j - i) + 2 = 2j + 2 \tag{0-1}$$

Case 1b: $j = k < i$

vectors to sum:

$$\begin{array}{r} (\mathbf{0}_3 \ \cdots \ \cdots \ \mathbf{0}_3 \ \boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2' \ \cdots) \\ (\boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2'' \ \cdots \ \boldsymbol{\phi}_2'' \ \boldsymbol{\phi}_2''\boldsymbol{\phi}_2'' \ \cdots) \\ +(\boldsymbol{\phi}_3 \ \boldsymbol{\phi}_2 \ \cdots \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \cdots) \\ \hline (\boldsymbol{\phi}_1'' \ \boldsymbol{\phi}_2' \ \cdots \ \boldsymbol{\phi}_2' \ \boldsymbol{\phi}_3 \ \mathbf{0}_3 \ \cdots) \end{array}$$

derived vector : $(\boldsymbol{\phi}_1'' \ (\boldsymbol{\phi}_2)_{i-k-1} \ \boldsymbol{\phi}_3 \ \mathbf{0}_3 \ \cdots)$

Parity weight:

$$w_p = 2(i - k) + 2 = 2i + 2 \tag{0-2}$$

Case 2a: $k < i = j$

vectors to sum:

$$\begin{array}{r}
(\mathbf{0}_3 \cdots \;\; \cdots \; \mathbf{0}_3 \; \phi_1 \; \phi_2' \;\; \cdots) \\
(\mathbf{0}_3 \cdots \;\; \cdots \; \mathbf{0}_3 \; \phi_2 \; \phi_2'' \;\; \cdots) \\
+(\phi_3 \; \phi_2 \;\; \cdots \; \phi_2 \; \phi_2 \; \phi_2 \;\; \cdots) \\
\hline
(\boldsymbol{\phi}_3 \; \boldsymbol{\phi}_2 \;\; \cdots \; \boldsymbol{\phi}_2 \; \boldsymbol{\phi}_1 \; \mathbf{0}_3 \;\; \cdots)
\end{array}$$

derived vector : $(\boldsymbol{\phi}_3 \; (\boldsymbol{\phi}_2)_{i-k-1} \; \boldsymbol{\phi}_1 \; \mathbf{0}_3 \;\; \cdots)$

Parity weight:

$$w_p = 2(i-k) + 2 = 2i + 2 \tag{0-3}$$

Case 3a: $k < i < j$

vectors to sum:

$$\begin{array}{r}
(\mathbf{0}_3 \cdots\cdots \; \mathbf{0}_3 \; \phi_1 \; \phi_2' \;\; \cdots \; \phi_2' \; \phi_2' \; \phi_2' \cdots) \\
(\mathbf{0}_3 \cdots \;\; \cdots \;\; \cdots \;\; \cdots \;\; \cdots \; \mathbf{0}_3 \; \phi_2 \; \phi_2'' \cdots) \\
+(\phi_3 \; \phi_2 \;\; \cdots \; \phi_2 \; \phi_2 \; \phi_2 \;\; \cdots \; \phi_2 \; \phi_2 \; \phi_2 \cdots) \\
\hline
(\boldsymbol{\phi}_3 \; \boldsymbol{\phi}_2 \cdots \; \boldsymbol{\phi}_2 \; \boldsymbol{\phi}_1' \; \boldsymbol{\phi}_2'' \cdots \; \boldsymbol{\phi}_2'' \; \boldsymbol{\phi}_2' \; \mathbf{0}_3 \cdots)
\end{array}$$

derived vector : $(\boldsymbol{\phi}_3 \; (\boldsymbol{\phi}_2)_{i-k-1} \; \boldsymbol{\phi}_1' \; (\boldsymbol{\phi}_2'')_{j-i-1} \; \boldsymbol{\phi}_2' \; \mathbf{0}_3 \;\; \cdots)$

Parity weight:

$$\begin{aligned}
w_p &= 2(i-k) + 2 + 2(j-i) \\
&= 2(j-k) + 2 \\
&= 2j + 2
\end{aligned} \tag{0-4}$$

Case 3b: $k < j < i$

$$\begin{array}{r}
(\mathbf{0}_3 \;\; \cdots \;\; \cdots \;\; \cdots \;\; \cdots \;\; \cdots \; \mathbf{0}_3 \; \phi_1 \; \phi_2' \cdots) \\
(\mathbf{0}_3 \;\; \cdots\cdots \; \mathbf{0}_3 \; \phi_2 \; \phi_2'' \;\; \cdots \; \phi_2'' \; \phi_2'' \; \phi_2'' \cdots) \\
+(\phi_3 \; \phi_2 \;\; \cdots \; \phi_2 \; \phi_2 \; \phi_2 \;\; \cdots \; \phi_2 \; \phi_2 \; \phi_2 \cdots) \\
\hline
(\boldsymbol{\phi}_3 \; \boldsymbol{\phi}_2 \cdots \; \boldsymbol{\phi}_2 \; \mathbf{0}_3 \; \boldsymbol{\phi}_2' \cdots \; \boldsymbol{\phi}_2' \; \boldsymbol{\phi}_3 \; \mathbf{0}_3 \cdots)
\end{array}$$

derived vector : $(\boldsymbol{\phi}_3 \; (\boldsymbol{\phi}_2)_{j-k-1} \; \mathbf{0}_3 \; (\boldsymbol{\phi}_2')_{i-j-1} \; \boldsymbol{\phi}_3 \; \mathbf{0}_3 \;\; \cdots)$

Parity weight:

$$\begin{aligned}
w_p &= 2(j-k) + 1 + 2(i-j) + 1 \\
&= 2(i-k) + 2 \\
&= 2i + 2
\end{aligned} \tag{0-5}$$

From all the above cases we can conclude that the parity weight for a weight-3 RTZ sequence may be calculated as

$$w_p = 2l + 2 \tag{0-6}$$

where $l = \max\{i, j\} - k = \max\{\gamma, \beta\}$ since $k = 0$

Assuming that after interleaving, another weight-3 RTZ input is produced. Let i', j', k', l' and w'_p be similarly defined. Then the Hamming weight w_H of the turbo codeword produced can be calculated as

$$w_H = 7 + 2(l + l') \tag{0-7}$$

$\square$