

“ Progress So Far”

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July 30, 2020

0.1 Notation

1. RTZ (Return-To-Zero) input :- A RTZ input is a binary input which causes a RSC encoder's final state to be return to zero after it has exited the zero state.
2. τ :- cycle length of the RSC encoder. For the 5/7 RSC encoder $\tau = 3$
3. N :- Interleaver length.
4. \mathcal{N} :- Integer set of $\{0, 1, \dots, N - 1\}$
5. \mathbb{N} : Indexed set of $\{0, 1, \dots, N - 1\}$ in the natural order.
6. We assume that $N/\tau = C$
7. \mathcal{C} and \mathbb{C} are defined in a similar manner.
8. $\mathcal{C}^t := \{c + t\}_{c \in \mathcal{C}}$ and \mathbb{C}^t is the indexed set with the elements of \mathcal{C}^t where $t = (0, 1, \dots, \tau - 1)$. Where it becomes necessary to distinguish between the elements of \mathcal{C}^t and \mathbb{C}^t , we will write the elements of \mathbb{C}^t as $c_{x'}^t$ and the elements of \mathcal{C}^t as c_x^t
9. Permutation matrix

$$\mathbf{\Pi} = \begin{bmatrix} \pi^0 \\ \pi^1 \\ \vdots \\ \pi^{K-1} \end{bmatrix} = [\pi_0, \pi_1, \dots, \pi_{\tau-1}] = \left[\pi_t^{(i)} \right]_{i=0, t=0}^{K-1, \tau-1}$$

where $\pi_t^{(i)} \in \{0, 1, \tau - 1\}$.

10. For the row vector $\boldsymbol{\pi}^{(i)}$, let $\mathcal{S}^e[\boldsymbol{\pi}^{(i)}]$ be the left-hand cycle shift of $\boldsymbol{\pi}^{(i)}$ and $\mathcal{S}^e[\boldsymbol{\pi}_t]$ be the up cycle shift of $\boldsymbol{\pi}_t$
11. We assume that the operation outputs the elements in \mathbb{C}^t in order while t is appeared in $\boldsymbol{\pi}^k$. For example, $\boldsymbol{\pi}^0 = (0, 0, 1)$ outputs (c_0^0, c_1^0, c_0^1) . From this example, we can see that the column index of i in $\boldsymbol{\pi}^{(i)}$ represents the coset it belongs to before interleaving and the value $\pi_j^{(i)}$ specifies the coset after interleaving
12. Our goal is to find a prefer $\mathbf{\Pi}$ and \mathbb{C}^t , $t = 0, 1, \dots, \tau - 1$.

0.2 Cosets and RTZ inputs

1. a weight 2 input sequence
 - polynomial: $P(x) = x^{h\tau+t}(1 + x^{\alpha\tau}) = x^t(x^{h\tau} + x^{(h+\alpha)\tau})$
 - coset: the h th and $(h + \alpha)$ th elements in \mathbb{C}^t
2. a weight 3 input sequence
 - polynomial: $Q(x) = x^{h\tau+t}(1 + x^{\beta\tau+1} + x^{\gamma\tau+2}) = x^{h\tau+t} + x^{(h+\beta)\tau+t+1} + x^{(h+\gamma)\tau+t+2}$. Notice that $h \leq \beta$ is not a necessary condition.
 - coset: the h th element in \mathbb{C}^t , $(h + \beta)$ th element in $\mathbb{C}^{[t+1]_\tau}$, and $(h + \gamma)$ th element in $\mathbb{C}^{[t+2]_\tau}$.

0.3 Representation of interleaver

If the mapping relationship between elements in \mathbf{x} and \mathbf{y} are read column wise as shown below

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 5 & 1 & 6 & 2 & 7 & 3 & 8 & 4 \end{bmatrix}$$

the interleaver is represented by $\mathbb{N} = \{0, 5, 1, 6, 2, 7, 3, 8, 4\}$.

Let $\mathbb{C}^0 = \{0, 6, 3\}$, $\mathbb{C}^1 = \{1, 7, 4\}$, and $\mathbb{C}^2 = \{5, 2, 8\}$. Then, the permutation matrix of \mathbb{N} is $\mathbf{\Pi} = (0, 2, 1)$. Notice the row of $\mathbf{\Pi}$ takes cyclicly.

0.4 Coset Interleaver Design For Weight-2 RTZ inputs

From the definition of Weight-2 RTZ inputs in the previous section, we know that the index of the “1” bits are in the same coset. Our aim is to make sure that the interleaver that we design is either able to break such weight-2 RTZ inputs or convert it into a large separation weight-2 RTZ. The condition to break weight-2 RTZs is given as

$$\pi_j^{(i)} \neq \pi_j^{(i')}, \quad |i - i'| \leq N_c \quad (0-1)$$

Since $\mathbf{\Pi}$ consisting of τ elements, the maximum length of column elements consisting of values different each other is τ . Thus, the cut-off interleaver length for which (0-1) is satisfied is $N_c = \tau = 3$. For this interleaver length, we investigate 3 different compositions of permutation matrices that can be used to achieve this condition in in 0-1

1. One cycle permutation: Each row is permutation of the sequence (0, 1, 2). Setting the element at the first row and first column to 0, there are exactly 4 permutation matrices that exist for cut-off length N_c . Let

$$\boldsymbol{\psi} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{\psi}' = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

We then have

$$\begin{aligned} [\boldsymbol{\psi}, \mathcal{S}^1[\boldsymbol{\psi}], \mathcal{S}^2[\boldsymbol{\psi}]] &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} := \boldsymbol{\psi}(\boldsymbol{\psi}) \\ [\boldsymbol{\psi}', \mathcal{S}^1[\boldsymbol{\psi}'], \mathcal{S}^2[\boldsymbol{\psi}']] &= \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} := \boldsymbol{\psi}(\boldsymbol{\psi}') \\ [\boldsymbol{\psi}, \mathcal{S}^2[\boldsymbol{\psi}], \mathcal{S}^1[\boldsymbol{\psi}]] &= \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} := \boldsymbol{\psi}'(\boldsymbol{\psi}) \\ [\boldsymbol{\psi}', \mathcal{S}^2[\boldsymbol{\psi}'], \mathcal{S}^1[\boldsymbol{\psi}']] &= \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} := \boldsymbol{\psi}'(\boldsymbol{\psi}') \end{aligned} \quad (0-2)$$

2. Two cycle permutation: Two rows are permutation of the sequence (0, 0, 1, 1, 2, 2).

There are no permutation matrices that satisfying cut-off length N_c . This is because the sequence length is not divisible by N_c , there will always be 2 elements of the same value in each row of $\mathbf{\Pi}$

3. Three cycle permutation: Three rows are permutation of the sequence(0, 0, 0, 1, 1, 1, 2, 2, 2).

Example of the permutation matrices satisfying cut-off length $N_c = 9$ are shown in 1

Table 1 shows all unique coset interleaving arrays of length N_c that convert weight-2 RTZ inputs to non-RTZ inputs. They are labeled from A to X . A coset interleaving array is unique if a shift of the elements in the array does not produce another another coset interleaving array.

A	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$	B	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$	C	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$	D	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$
E	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$	F	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$	G	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	H	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$
I	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$	J	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$	K	$\begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$	L	$\begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$
M	$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$	N	$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$	O	$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$	P	$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
Q	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}$	R	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{bmatrix}$	S	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}$	T	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix}$
U	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$	V	$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$	W	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$	X	$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

Table 1: All unique coset interleaving arrays of length $N_c = 9$ for weight-2 RTZ inputs

The interleaver length used in turbo coding are way greater than N_c and it is not possible to transform weight-2 RTZ inputs into non-RTZ inputs for all values of i . All is not lost however, since not all weight-2 RTZ inputs produce low-weight codewords. The formula for calculating the Hamming weight (w_H) of the Turbo codeword produced by a weight-2 RTZ input occurring in both component codes is given by[SunTakeshita]

$$\begin{aligned}
 w_H &= 2 + \left(2 + \frac{\Delta_c}{\tau}\right)w_0 + \left(2 + \frac{\Delta_{c'}}{\tau}\right)w_0 \\
 &= 6 + \left(\frac{\Delta_c + \Delta_{c'}}{\tau}\right)w_0, \quad w_0 = 2
 \end{aligned} \tag{0-3}$$

For all the $\mathbf{\Pi}$ in Table 1, since $\Delta_c = 9 = 3\tau$ and $\Delta_{c'} := (c_{(h'+\alpha')}^t - c_{(h')}^t)$ we have

$$w_H = 6 + \left(3 + \frac{\Delta_{c'}}{3}\right)w_0, \quad w_0 = 2 \tag{0-4}$$

0.5 Coset Interleaver Design For Weight-3 RTZ inputs

As mentioned earlier, a weight-3 RTZ input is formed when the indices of the “1” bits each occur in different cosets. It goes without saying that the simplest way to convert a weight-3 RTZ input into a non-RTZ input is to make sure that at least two of indices of the “1” bits occur within the same coset after interleaving.

$$w_H = 7 + 2(l + l') \quad (0-5)$$

Unique permutation matrices which meet this criteria are shown in Table 2 and they are labeled from A to L

A	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$	B	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$
C	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}$	D	$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$
E	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	F	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$
G	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$	H	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$
I	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$	J	$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$
K	$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$	L	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$

Table 2: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs

Depending on which permutation matrix is chosen from Table 2, Equation 0-5 can be simplified.

In general w_H for turbo codewords as a result of weight-3 RTZ inputs can be written as

$$w_H = 3 + w_p + w'_p$$

, where w_p, w'_p refer to the pre-interleaving parity weight and the post-interleaving parity weight respectively. The value of w_p for the pre-interleaving weight-3 is dependent on the elements in \mathcal{C}^t

Let $(c_{(h)}^t, c_{(h+\beta)}^{t+1}, c_{(h+\gamma)}^{t+2})$ be the vector representing a weight-3 RTZ input Without loss of generality, we can assume that $h = t = 0$. We then have

$$l = \max(\beta, \gamma) \quad (0-6)$$

And

$$w_p = 2(\max(\beta, \gamma)) + 2 \quad (0-7)$$

By deciding on the Π we can easily calculate all values of l and w_p w'_p, β', γ' and l' are similarly defined and are dependent on the elements in \mathcal{C}^t , $t = 0, 1, \dots, \tau - 1$

As an example, Table 3 shows all the weight-3 RTZ inputs and the corresponding equations for calculating w_H

RTZ index	l	w_p	w_H
(0 4 8)	2	6	$11 + 2(\max(\beta', \gamma'))$
(0 5 7)	2	6	$11 + 2(\max(\beta', \gamma'))$
(1 3 8)	2	6	$11 + 2(\max(\beta', \gamma'))$
(1 5 6)	1	4	$9 + 2(\max(\beta', \gamma'))$
(2 3 7)	1	4	$9 + 2(\max(\beta', \gamma'))$
(2 4 6)	1	4	$9 + 2(\max(\beta', \gamma'))$
(0 8 13)	4	10	$15 + 2(\max(\beta', \gamma'))$
(0 4 17)	5	12	$17 + 2(\max(\beta', \gamma'))$
(0 13 17)	5	12	$17 + 2(\max(\beta', \gamma'))$
(0 7 14)	4	6	$15 + 2(\max(\beta', \gamma'))$
(0 5 16)	5	6	$17 + 2(\max(\beta', \gamma'))$
(0 14 16)	5	6	$17 + 2(\max(\beta', \gamma'))$
(1 8 12)	3	8	$13 + 2(\max(\beta', \gamma'))$
(1 3 17)	5	12	$17 + 2(\max(\beta', \gamma'))$
(1 12 17)	5	12	$17 + 2(\max(\beta', \gamma'))$
(1 6 14)	4	10	$15 + 2(\max(\beta', \gamma'))$
(1 5 15)	4	10	$15 + 2(\max(\beta', \gamma'))$
(1 14 15)	4	10	$15 + 2(\max(\beta', \gamma'))$
(2 7 12)	3	8	$13 + 2(\max(\beta', \gamma'))$
(2 3 16)	4	10	$15 + 2(\max(\beta', \gamma'))$
(2 12 16)	4	10	$15 + 2(\max(\beta', \gamma'))$
(2 6 13)	3	8	$13 + 2(\max(\beta', \gamma'))$
(2 4 15)	4	10	$15 + 2(\max(\beta', \gamma'))$
(2 13 15)	4	10	$15 + 2(\max(\beta', \gamma'))$

Table 3: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs

0.6 Coset Design

Once the permutation matrix is decided upon, we have the necessary constraints which will help us design \mathbb{C}^t with respect to weight 2 and weight 3 RTZ inputs. We settle on

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

which we will refer to as $\Pi^{(0)}$. It was decided upon because in the design process, we only need to focus on only one of the cosets, say \mathbb{C}^0 and replicate the results for the remaining cosets. We will make use of the Almost Linear Interleaver(ALI) is the design of \mathbb{C}^0 . The interleaving equation for the ALI(L,D) Interleaver is given by

$$\pi(h) = D \cdot h + \left\lfloor \frac{h}{A} \right\rfloor \bmod L$$

where $A = L/C$, $L = N/3$ and $C = \gcd(D, L)$ Also, D is the period of the interleaver and $h=0,1,\dots,L-1$.

The value of a coset element at position h will be $3\pi(h)$. The resulting coset interleaver will be referred to by the notation CI(N,D)

For weight-2 RTZ inputs, the weight can be calculated as $w_H = 6 + \left(3 + \frac{\Delta_{c'}}{3}\right)w_0$, $w_0 = 2$ where $\Delta_{c'} := (c_{(h+\alpha')}^t - c_{(h)}^t)$, $\alpha' = 3$. With respect to the ALI, $c_{(h+3)}^t = 3(\pi(h+3))$ and $c_{(h)}^t = 3(\pi(h))$. Without loss of generality, we can set $h = 0$ and $\Delta_{c'} = 3(\pi(3))$. therefore have

$$\begin{aligned} w_H &= \min \left\{ 6 + 2 \left(3 + \frac{3(\pi(3))}{3} \right), 6 + 2 \left(3 + \frac{K - 3(\pi(3))}{3} \right) \right\} \\ w_H &= \min \left\{ 6 + 2 \left(3 + \pi(3) \right), 6 + 2 \left(3 + (K - \pi(3)) \right) \right\} \end{aligned} \quad (0-8)$$

For weight 3-RTZ inputs, the various equations are given in Table 3. The positions where a weight-3 inputs occur are know, but they are given with respect to the complete interleaver and need to be scaled down to a single coset. Let h , $h + \beta$, $h + \gamma$ be the inputs representing where the weight-3 RTZ inputs occur due to $\Pi^{(0)}$. Then the scaled down versions will be $h^{(0)}$, $(h + \beta)^{(0)}$, $(h + \gamma)^{(0)}$ and are calculated using the equation

$$f(x) = x \bmod 3 + 3 \left(\left\lfloor \frac{x}{9} \right\rfloor \right) \quad (0-9)$$

We feed $h^{(0)}$, $(h + \beta)^{(0)}$, $(h + \gamma)^{(0)}$ into the ALI and we get

$$\mathbf{s} = (\pi(h^{(0)}), \pi((h + \beta)^{(0)}), \pi((h + \gamma)^{(0)})) - \min(\pi(h^{(0)}), \pi((h + \beta)^{(0)}), \pi((h + \gamma)^{(0)})) \quad (0-10)$$

and

$$l' = \max(\beta, \gamma) = \mathbf{s}_{\max} - \mathbf{s}_{\min} = \mathbf{s}_{\max} \quad (0-11)$$

Below is a summary of the steps involved in determining the Hamming weight for all weight-3 RTZ associated with $\Pi^{(0)}$

1. convert h , $h + \beta$, $h + \gamma$ into $h^{(0)}$, $(h + \beta)^{(0)}$, $(h + \gamma)^{(0)}$ using (0-9)

2. input the indices into the ALI equation to obtain $susing(0 - 10)$
3. find the value of l' and Hamming weight using (0-11) and corresponding equation in Table 3
4. repeat above steps for all RTZ-inputs in Table ?? and their shifted versions, where each shift is a multiple of τ^2
5. Finally, the least Hamming weight value associated with weight-3 RTZ inputs is selected.

If we assume that the start position for all the cosets is the same, then the above calculations with respect to the weight-3 RTZ inputs are sufficient, however if we desire to adjust the start position for each coset a slight change in notation will be needed.

First off we adjust the notation for a Coset interleaver from $CI(N, D)$ to $CI(N, D, s_1, s_2)$ where s_1, s_2 indicate the start position for $\mathbb{C}^1, \mathbb{C}^2$ respectively. It is worth noting that the value of D used in the ALI(L,D) is the same for all the cosets, just that $\mathbb{C}^1, \mathbb{C}^2$ are shifted by s_1, s_2 respective positions to the right.

This means that values of $h^{(0)}, (h + \beta)^{(0)}, (h + \gamma)^{(0)}$ need to be adjusted by 0, s_1, s_2 respectively. This statement is made assuming that $h^{(0)}, (h + \beta)^{(0)}, (h + \gamma)^{(0)}$ are in $\mathbb{C}^0, \mathbb{C}^1, \mathbb{C}^2$ respectively.

Below is a summary of the steps involved in determining the Hamming weight for all weight-3 RTZ associated with $\Pi^{(0)}$ when $s_1, s_2 > 0$

1. convert $h, h + \beta, h + \gamma$ into $h^{(0)}, (h + \beta)^{(0)}, (h + \gamma)^{(0)}$ using (0-9)
2. input the indices $h^{(0)}, (h + \beta)^{(0)} + s_1, (h + \gamma)^{(0)} + s_2$ into the ALI equation to obtain $susing(0 - 10)$
3. find the value of l' and Hamming weight using (0-11) and corresponding equation in Table 3
4. repeat above steps for all RTZ-inputs in Table ?? and their shifted versions, where each shift is a multiple of τ^2
5. Finally, the least Hamming weight value associated with weight-3 RTZ inputs is selected.

0.7 Simulation Results and Discussion

Simulations for the coset interleaver $CI(N, D, s_1, s_2)$ interleaver are done for $s_1, s_2 = 0$. The values for $D = \{28, 29, \dots, 35\}$, $N = 261$ and

$$\Pi = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

The minimum weight for each interleaver with respect to weight-2 and weight 3 RTZs are shown in Table(??) and the simulation results are shown in Figure() As expected, the interleaver designed with $CI(261, 29)$ performs the worst, but the performance of the other interleavers does not perform as expected of the data from Table 4. Further examination of the simulation results reveals that even though the other interleavers have high Hamming weight related to weight-2 and weight-3 RTZ inputs, the minimum distance of the code seem to be bound at by RTZ

D	$w_H^{(2)}$	$w_H^{(3)}$
28	18	115
29	14	123
30	18	121
31	24	103
32	30	89
33	36	75
34	42	61
35	48	49

Table 4: Minimum Hamming weight for weight-2 and weight-3 RTZ using $\text{CI}(N, D, s_1, s_2)$, where $s_1, s_2 = 0$

inputs with higher weights, specifically those of weight-4. Specifically weight-4 input of the form $(1 + x^\tau) + x^{(\tau^2)}(1 + x^\tau)$ is transformed into weight-4 input of the form $(1 + x) + x^\tau(1 + x)$ after interleaving. In the end, the Hamming weight as a result of these combinations is

$$w_H = w_m + w_p + w_{p'} = 4 + 8 + 2 = 14$$

where w_m is the weight of the RTZ-input.

Specifically weight-4 inputs of the form $(1 + x^\tau) + x^{\tau^2}(1 + x^\tau)$ are transformed into weight-4 inputs of the form $(1 + x) + x^3(1 + x)$ after interleaving because the start index for picking the elements for all the cosets are the same and these causes somewhat well separated pre-interleaving inputs to be bunched together post-interleaving inputs. This is shown in the graph of the input output relation of any of the interleavers.

This can be remedied by making sure that the start position for the other cosets are different and this can be accomplished by shifting the elements of the cosets other than \mathbb{C}^0 a certain value to the left. Let a, b be the factor by which the \mathbb{C}^1 and \mathbb{C}^2 are shifted. We proceed to re-design $\text{CI}(23, 0, 0)$ by introducing $s_1 = 30$ and $s_2 = 7$. The simulation results for $\text{CI}(261, 23, 0, 0)$ and $\text{CI}(261, 23, 30, 7)$ are shown in Figure() as can be seen adjusting the start position of the other cosets, the error-correcting performance is greatly improved. A further investigation of the simulation results reveals that for $\text{CI}(261, 23, 30, 7)$, the Hamming distance for the turbo code is again bounded by weight-4 RTZ inputs. A specific example is that the weight-4 input of the form $(1 + x^\tau) + (x^{155})(1 + x^\tau)$ is transformed into another weight-4 RTZ input of the form $(1 + x^\tau) + (x^{244})(1 + x^\tau)$. Then

$$w_H = w_m + w_p + w_{p'} = 4 + 8 + 8 = 20$$

. The multiplicity of such weight-4 RTZ inputs is extremely high.

Since we know that a weight-4RTZ input is just a combination of 2 weight-2 RTZ inputs