“ Progress So Far”

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0.1 Notation

1. RTZ (Return-To-Zero) input :- A RTZ input is a binary input which causes a RSC encoder's final state to be return to zero after it has exited the zero state.
2. τ :- cycle length of the RSC encoder. For the 5/7 RSC encoder $\tau = 3$
3. N :- Interleaver length.
4. $\mathcal{N}$:- Integer set of $\{0, 1, \cdots, N-1\}$
5. $\mathbb{N}$: Indexed set of $\{0, 1, \cdots, N-1\}$ in the natural order.
6. We assume that $N/\tau = C$
7. $\mathcal{C}$ and $\mathbb{C}$ are definded in a similar manner.
8. $\mathcal{C}^t := \{c+t\}_{c \in \mathcal{C}}$ and $\mathbb{C}^t$ is the indexed set with the elements of $\mathcal{C}^t$ where $t = (0, 1, ..., \tau-1)$. Where it becomes necessary to distinguish between the elements of $\mathcal{C}^t$ and $\mathbb{C}^t$, we will write the elements of $\mathbb{C}^t$ as $c^{t'}_{x'}$ and the elements of $\mathcal{C}^t$ as c^t_x
9. Permutation matrix

$$\mathbf{\Pi} = \begin{bmatrix} \boldsymbol{\pi}^0 \\ \boldsymbol{\pi}^1 \\ \vdots \\ \boldsymbol{\pi}^{K-1} \end{bmatrix} = \left[\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_{\tau-1}\right] = \left[\pi_t^{(i)}\right]_{i=0,\ t=0}^{K-1,\ \tau-1}$$

   where $\pi_t^{(i)} \in \{0, 1, \tau-1\}$.

10. For the row vector $\boldsymbol{\pi}^{(i)}$, let $\mathscr{S}^e[\boldsymbol{\pi}^{(i)}]$ be the left-hand cycle shift of $\boldsymbol{\pi}^{(i)}$ and $\mathscr{S}^e[\boldsymbol{\pi}_t]$ be the up cycle shift of $\boldsymbol{\pi}_t$
11. We assume that the operation outputs the elements in $\mathbb{C}^t$ in order while t is appeared in $\boldsymbol{\pi}^k$. For example, $\boldsymbol{\pi}^0 = (0, 0, 1)$ outputs (c_0^0, c_1^0, c_0^1). From this example, we can see that the column index of i in $\pi^{(i)}$ represents the coset it belongs to before interleaving and the value $\pi_j^{(i)}$ specifies the coset after interleaving
12. Our goal is to find a prefer $\mathbf{\Pi}$ and $\mathbb{C}^t$, $t = 0, 1, \cdots, \tau-1$.

0.2 Cosets and RTZ inputs

1. a weight 2 input sequence
   - polynomial: $P(x) = x^{h\tau+t}(1 + x^{\alpha\tau}) = x^t(x^{h\tau} + x^{(h+\alpha)\tau})$
   - coset: the hth and $(h+\alpha)$th elements in $\mathbb{C}^t$
2. a weight 3 input sequence
   - polynomial: $Q(x) = x^{h\tau+t}(1 + x^{\beta\tau+1} + x^{\gamma\tau+2}) = x^{h\tau+t} + x^{(h+\beta)\tau+t+1} + x^{(h+\gamma)\tau+t+2}$. Notice that $h \le \beta$ is not a necessary condition.
   - coset: the hth element in $\mathbb{C}^t$, $(h+\beta)$th element in $\mathbb{C}^{[t+1]_\tau}$, and $(h+\gamma)$th element in $\mathbb{C}^{[t+2]_\tau}$.

0.3 Representation of interleaver

If the mapping relationship between elements in $\boldsymbol{x}$ and $\boldsymbol{y}$ are read column wise as shown below

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 5 & 1 & 6 & 2 & 7 & 3 & 8 & 4 \end{bmatrix}$$

the interleaver is represented by $\mathbb{N} = \{0, 5, 1, 6, 2, 7, 3, 8, 4\}$.

Let $\mathbb{C}^0 = \{0, 6, 3\}$, $\mathbb{C}^1 = \{1, 7, 4\}$, and $\mathbb{C}^2 = \{5, 2, 8\}$. Then, the permutation matrix of $\mathbb{N}$ is $\boldsymbol{\Pi} = (0, 2, 1)$. Notice the row of $\boldsymbol{\Pi}$ takes cyclicly.

0.4 Coset Interleaver Design For Weight-2 RTZ inputs

From the definition of Weight-2 RTZ inputs in the previous section, we know that the index of the "1" bits are in the same coset. Our aim is to make sure that the interleaver that we design is either able to break such weight-2 RTZ inputs or convert it into a large separation weight-2 RTZ. The condition to break weight-2 RTZs is given as

$$\pi_j^{(i)} \neq \pi_j^{(i')},\ |i - i'| \leq N_c \tag{0-1}$$

Since $\boldsymbol{\Pi}$ consisting of τ elements, the maximum length of column elements consisting of values different each other is τ. Thus, the cut-off interleaver length for which (0-1) is satisfied is $N_c = \tau = 3$. For this interleaver length, we investigate 3 different compositions of permutation matrices that can be used to achieve this condition in in 0-1

1. One cycle permutation: Each row is permutation of the sequence $(0, 1, 2)$. Setting the element at the first row and first column to 0, there are exactly 4 permutation matrices that exist for cut-off length N_c. Let

$$\boldsymbol{\psi} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},\ \boldsymbol{\psi}' = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

We then have

$$\begin{aligned}
[\boldsymbol{\psi}, \mathscr{S}^1[\boldsymbol{\psi}], \mathscr{S}^2[\boldsymbol{\psi}]] &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} := \boldsymbol{\psi}(\boldsymbol{\psi}) \\
[\boldsymbol{\psi}', \mathscr{S}^1[\boldsymbol{\psi}'], \mathscr{S}^2[\boldsymbol{\psi}']] &= \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} := \boldsymbol{\psi}(\boldsymbol{\psi}') \\
[\boldsymbol{\psi}, \mathscr{S}^2[\boldsymbol{\psi}], \mathscr{S}^1[\boldsymbol{\psi}]] &= \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} := \boldsymbol{\psi}'(\boldsymbol{\psi}) \\
[\boldsymbol{\psi}', \mathscr{S}^2[\boldsymbol{\psi}'], \mathscr{S}^1[\boldsymbol{\psi}']] &= \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} := \boldsymbol{\psi}'(\boldsymbol{\psi}')
\end{aligned} \tag{0-2}$$

2. Two cycle permutation: Two rows are permutation of the sequence $(0, 0, 1, 1, 2, 2)$.

There are no permutation matrices that satisfying cut-off length N_c. This is because the sequence length is not divisible by N_c, there will always be 2 elements of the same value in each row of $\mathbf{\Pi}$

3. Three cycle permutation: Three rows are permutation of the sequence$(0, 0, 0, 1, 1, 1, 2, 2, 2)$.

   Example of the permutation matrices satisfying cut-off length $N_c = 9$ are shown in 1

Table 1 shows all unique coset interleaving arrays of length N_c that convert weight-2 RTZ inputs to non-RTZ inputs. They are labeled from A to X. A coset interleaving array is unique if a shift of the elements in the array does not produce another another coset interleaving array.

A	$\begin{bmatrix}0&0&0\\1&1&1\\2&2&2\end{bmatrix}$	B	$\begin{bmatrix}0&0&0\\1&1&2\\2&2&1\end{bmatrix}$	C	$\begin{bmatrix}0&0&0\\1&2&1\\2&1&2\end{bmatrix}$	D	$\begin{bmatrix}0&0&0\\1&2&2\\2&1&1\end{bmatrix}$
E	$\begin{bmatrix}0&0&0\\2&1&1\\1&2&2\end{bmatrix}$	F	$\begin{bmatrix}0&0&0\\2&1&2\\1&2&1\end{bmatrix}$	G	$\begin{bmatrix}0&0&0\\2&2&1\\1&1&2\end{bmatrix}$	H	$\begin{bmatrix}0&0&0\\2&2&2\\1&1&1\end{bmatrix}$
I	$\begin{bmatrix}0&0&1\\1&1&0\\2&2&2\end{bmatrix}$	J	$\begin{bmatrix}0&0&1\\1&2&0\\2&1&2\end{bmatrix}$	K	$\begin{bmatrix}0&0&1\\2&1&0\\1&2&2\end{bmatrix}$	L	$\begin{bmatrix}0&0&1\\2&2&0\\1&1&2\end{bmatrix}$
M	$\begin{bmatrix}0&0&2\\1&1&0\\2&2&1\end{bmatrix}$	N	$\begin{bmatrix}0&0&2\\1&2&0\\2&1&1\end{bmatrix}$	O	$\begin{bmatrix}0&0&2\\2&1&0\\1&2&1\end{bmatrix}$	P	$\begin{bmatrix}0&0&2\\2&2&0\\1&1&1\end{bmatrix}$
Q	$\begin{bmatrix}0&1&0\\1&0&1\\2&2&2\end{bmatrix}$	R	$\begin{bmatrix}0&1&0\\1&0&2\\2&2&1\end{bmatrix}$	S	$\begin{bmatrix}0&1&0\\1&2&1\\2&0&2\end{bmatrix}$	T	$\begin{bmatrix}0&1&0\\2&0&1\\1&2&2\end{bmatrix}$
U	$\begin{bmatrix}0&1&0\\2&0&2\\1&2&1\end{bmatrix}$	V	$\begin{bmatrix}0&1&0\\2&2&1\\1&0&2\end{bmatrix}$	W	$\begin{bmatrix}0&1&1\\1&2&0\\2&0&2\end{bmatrix}$	X	$\begin{bmatrix}0&2&0\\2&0&2\\1&1&1\end{bmatrix}$

Table 1: All unique coset interleaving arrays of length $N_c = 9$ for weight-2 RTZ inputs

The interleaver length used in turbo coding are way greater than N_c and it is not possible to transform weight-2 RTZ inputs into non-RTZ inputs for all values of i. All is not lost however, since not all weight-2 RTZ inputs produce low-weight codewords. The formula for calculating the Hamming weight (w_H) of the Turbo codeword produced by a weight-2 RTZ input occuring in both component codes is given by[SunTakeshita]

$$\begin{aligned} w_H =& 2 + (2 + \frac{\Delta_\pi}{\tau})w_0 + (2 + \frac{\Delta_{\pi'}}{\tau})w_0 \\ =& 6 + \Big(\frac{\Delta_\pi + \Delta_{\pi'}}{\tau}\Big)w_0, \;\; w_0 = 2 \end{aligned} \tag{0-3}$$

For all the Π in Table 1, since $\Delta\pi = 9 = 3\tau$ and $\Delta\pi' := (\pi_t^{(h'+\alpha')} - \pi_t^{(h')})$ we have

$$w_H = 6 + \Big(3 + \frac{\Delta\pi'}{3}\Big)w_0, \;\; w_0 = 2 \tag{0-4}$$

0.5 Coset Interleaver Design For Weight-3 RTZ inputs

As mentioned earlier, a weight-3 RTZ input is formed when the indices of the "1" bits each occur in different cosets. It goes without saying that the simplest way to convert a weight-3 RTZ input into a non-RTZ input is to make sure that at least two of indices of the "1" bits occur within the same coset after interleaving.

$$w_H = 7 + 2(l + l') \tag{0-5}$$

Unique permutation matrices which meet this criteria are shown in Table 2 and they are labeled from A to L

A	$\begin{bmatrix}0&0&0\\1&1&1\\2&2&2\end{bmatrix}$	B	$\begin{bmatrix}0&0&0\\1&1&2\\1&2&2\end{bmatrix}$
C	$\begin{bmatrix}0&0&0\\1&1&2\\2&1&2\end{bmatrix}$	D	$\begin{bmatrix}0&0&0\\1&1&2\\2&2&1\end{bmatrix}$
E	$\begin{bmatrix}0&0&0\\2&2&1\\1&1&2\end{bmatrix}$	F	$\begin{bmatrix}0&0&0\\2&2&1\\1&2&1\end{bmatrix}$
G	$\begin{bmatrix}0&0&0\\2&2&1\\2&1&1\end{bmatrix}$	H	$\begin{bmatrix}0&0&1\\0&1&1\\2&2&2\end{bmatrix}$
I	$\begin{bmatrix}0&0&1\\1&1&2\\2&0&2\end{bmatrix}$	J	$\begin{bmatrix}0&0&2\\0&2&2\\1&1&1\end{bmatrix}$
K	$\begin{bmatrix}0&0&2\\2&2&1\\1&0&1\end{bmatrix}$	L	$\begin{bmatrix}0&1&0\\1&1&2\\2&0&2\end{bmatrix}$

Table 2: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs

Depending on which permutation matrix is chosen from Table 2, Equation 0-5 can be simplified.

In general w_H for turbo codewords as a result of weight-3 RTZ inputs can be written as

$$w_H = 3 + w_p + w'_p$$

, where w_p, w'_p refer to the pre-interleaving parity weight and the post-interleaving parity weight respectively. The value of w_p for the pre-interleaving weight-3 is dependent on the elements in $\mathcal{C}^t$

Let $(\pi_t^{(h)},\ c_{t+1}^{(h+\beta)},\ c_{t+2}^{(h+\gamma)})$ be the vector representing a weight-3 RTZ input Without loss of generality, we can assume that $h = t = 0$. We then have

$$l = \max(\beta, \gamma) \tag{0-6}$$

And

$$w_p = 2(\max(\beta, \gamma)) + 2 \tag{0-7}$$

By deciding on the Π we can easily calculate all values of l and w_p w'_p, β', γ' and l' are similarly defined and are dependent on the elements in $\mathbb{C}^t,\ t = 0, 1, .., \tau - 1$

As an example, Table 3 shows all the weight-3 RTZ inputs and the corresponding equations for calculating w_H

RTZ index	l	w_p	w_H
(0 4 8)	2	6	$11 + 2(\max(\beta', \gamma'))$
(0 5 7)	2	6	$11 + 2(\max(\beta', \gamma'))$
(1 3 8) $\equiv$ (0 2 7)	2	6	$11 + 2(\max(\beta', \gamma'))$
(1 5 6) $\equiv$ (0 4 5)	1	4	$9 + 2(\max(\beta', \gamma'))$
(2 3 7) $\equiv$ (0 1 5)	1	4	$9 + 2(\max(\beta', \gamma'))$
(2 4 6) $\equiv$ (0 2 4)	1	4	$9 + 2(\max(\beta', \gamma'))$
(0 8 13)	4	10	$15 + 2(\max(\beta', \gamma'))$
(0 4 17)	5	12	$17 + 2(\max(\beta', \gamma'))$
(0 13 17)	5	12	$17 + 2(\max(\beta', \gamma'))$
(0 7 14)	4	6	$15 + 2(\max(\beta', \gamma'))$
(0 5 16)	5	6	$17 + 2(\max(\beta', \gamma'))$
(0 14 16)	5	6	$17 + 2(\max(\beta', \gamma'))$
(1 8 12) $\equiv$ (0 7 11)	3	8	$13 + 2(\max(\beta', \gamma'))$
(1 3 17) $\equiv$ (0 2 16)	5	12	$17 + 2(\max(\beta', \gamma'))$
(1 12 17) $\equiv$ (0 11 16)	5	12	$17 + 2(\max(\beta', \gamma'))$
(1 6 14) $\equiv$ (0 5 13)	4	10	$15 + 2(\max(\beta', \gamma'))$
(1 5 15) $\equiv$ (0 4 14)	4	10	$15 + 2(\max(\beta', \gamma'))$
(1 14 15) $\equiv$ (0 13 14)	4	10	$15 + 2(\max(\beta', \gamma'))$
(2 7 12) $\equiv$ (0 5 10)	3	8	$13 + 2(\max(\beta', \gamma'))$
(2 3 16) $\equiv$ (0 1 14)	4	10	$15 + 2(\max(\beta', \gamma'))$
(2 12 16) $\equiv$ (0 10 14)	4	10	$15 + 2(\max(\beta', \gamma'))$
(2 6 13) $\equiv$ (0 4 11)	3	8	$13 + 2(\max(\beta', \gamma'))$
(2 4 15) $\equiv$ (0 2 13)	4	10	$15 + 2(\max(\beta', \gamma'))$
(2 13 15) $\equiv$ (0 11 13)	4	10	$15 + 2(\max(\beta', \gamma'))$

Table 3: All unique permutation matrices of length $N_c = 9$ for weight-3 RTZ inputs

To Do

1. add elements for inter-block Weight-3 RTZ inputs