Formula to Calculate Weight for Low-Weight Weight 3 Inputs and Proof

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0.1 Equation and Proof

Proof. The polynomial representation of a weight-3 RTZ input is given by

$$Q(x) = x^{h\tau+t}(1 + x^{\beta\tau+1} + x^{\gamma\tau+2})$$

The impulse response of the RSC encoder is

$$(1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ \cdots)$$

and using the inpulse response, we can calculate the parity weight as well as Hamming weight of the turbo codeword.

Let
$\boldsymbol{\phi}_1 = (0\ 0\ 1),\ \boldsymbol{\phi}_1' = (0\ 1\ 0),\ \boldsymbol{\phi}_1'' = (1\ 0\ 0),$
$\boldsymbol{\phi}_2 = (0\ 1\ 1),\ \boldsymbol{\phi}_2' = (1\ 1\ 0),\ \boldsymbol{\phi}_2'' = (1\ 0\ 1),$
$\boldsymbol{\phi}_3 = (1\ 1\ 1).$

Then, the weight-3 RTZ occurs since $\boldsymbol{\phi}_2 + \boldsymbol{\phi}_2' + \boldsymbol{\phi}_2'' = \mathbf{0}_\tau = \mathbf{0}_3$. Now, we consider the weight of the vector derived by the sumation of the followings vectors.

$$\begin{array}{lll} (\mathbf{0}_{3(\gamma+h)} & \boldsymbol{\phi}_1\ \boldsymbol{\phi}_2' & \cdots) \\ (\mathbf{0}_{3(\beta+h)} & \boldsymbol{\phi}_2\ \boldsymbol{\phi}_2'' & \cdots) \\ (\mathbf{0}_{3h} & \boldsymbol{\phi}_3\ \boldsymbol{\phi}_2 & \cdots) \end{array}$$

Without loss of generality, we can assume that all weight-3 RTZ inputs begin at the 0th position, ie $h = t = 0$. This is because the case where $h > 0$ or $t > 0$ is just a right-shifted version of the weight-3 RTZ. With this assumption, we we only need to consider cases where $h = 0,\ \gamma \geq h$. To simplify calculation, we have included an addition table for all the vectors which is shown in Table **??**

	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_1'$	$\boldsymbol{\phi}_1''$	$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}_2'$	$\boldsymbol{\phi}_2''$	$\boldsymbol{\phi}_3$
$\boldsymbol{\phi}_1$	$\mathbf{0}_3$	$-$	$-$	$-$	$-$	$-$	$-$
$\boldsymbol{\phi}_1'$	$\boldsymbol{\phi}_2$	$\mathbf{0}_3$	$-$	$-$	$-$	$-$	$-$
$\boldsymbol{\phi}_1''$	$\boldsymbol{\phi}_2''$	$\boldsymbol{\phi}_2'$	$\mathbf{0}_3$	$-$	$-$	$-$	$-$
$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}_1'$	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_3$	$\mathbf{0}_3$	$-$	$-$	$-$
$\boldsymbol{\phi}_2'$	$\boldsymbol{\phi}_3$	$\boldsymbol{\phi}_1''$	$\boldsymbol{\phi}_1'$	$\boldsymbol{\phi}_2''$	$\mathbf{0}_3$	$-$	$-$
$\boldsymbol{\phi}_2''$	$\boldsymbol{\phi}_1''$	$\boldsymbol{\phi}_3$	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_2'$	$\boldsymbol{\phi}_2$	$\mathbf{0}_3$	$-$
$\boldsymbol{\phi}_3$	$\boldsymbol{\phi}_2'$	$\boldsymbol{\phi}_2''$	$\boldsymbol{\phi}_2$	$\boldsymbol{\phi}_1''$	$\boldsymbol{\phi}_1$	$\boldsymbol{\phi}_1'$	$\mathbf{0}_3$

Table 1: Truth Table

Furthermore, we consider 4 general cases for all possible values of i, j, k where $i \geq k$ These cases are $(=\ \ =)$, $(=\ \ <)$, $(<\ \ =)$ and $(<\ \ <)$

Case 0: $\gamma = \beta = h$
For this case, the vectors to sum will be

$$\begin{array}{c} (\boldsymbol{\phi}_1\ \boldsymbol{\phi}_2'\ \cdots) \\ (\boldsymbol{\phi}_2\ \boldsymbol{\phi}_2''\ \cdots) \\ (\boldsymbol{\phi}_3\ \boldsymbol{\phi}_2\ \cdots) \\ \hline (\boldsymbol{\phi}_2''\ \mathbf{0}_3\ \cdots) \end{array}$$

and the derived vector will be $(\boldsymbol{\phi}_2''\ \mathbf{0}_3\ \cdots)$ with a weight of $w_p = 2$

Case 1a: $\gamma = h < \beta$
vectors to sum:

$$\begin{array}{r}
(\boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2' \ \boldsymbol{\phi}_2' \ \boldsymbol{\phi}_2' \ \boldsymbol{\phi}_2' \ \cdots) \\
(\mathbf{0}_3 \ \cdots \ \mathbf{0}_3 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2'' \ \cdots) \\
+(\boldsymbol{\phi}_3 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \cdots) \\
\hline
(\boldsymbol{\phi}_2' \ \boldsymbol{\phi}_2'' \ \boldsymbol{\phi}_2'' \ \boldsymbol{\phi}_2' \ \mathbf{0}_3 \ \cdots)
\end{array}$$

derived vector : $(\boldsymbol{\phi}_2' \ (\boldsymbol{\phi}_2'')_{\beta-h-1} \ \boldsymbol{\phi}_2' \ \mathbf{0}_3 \ \cdots)$
Parity weight:

$$w_p = 2(\beta - h) + 2 = 2\beta + 2 \tag{0-1}$$

Case 1b: $\beta = h < \gamma$
vectors to sum:

$$\begin{array}{r}
(\mathbf{0}_3 \ \cdots \ \cdots \ \mathbf{0}_3 \ \boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2' \ \cdots) \\
(\boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2'' \ \cdots \ \boldsymbol{\phi}_2'' \ \boldsymbol{\phi}_2''\boldsymbol{\phi}_2'' \ \cdots) \\
+(\boldsymbol{\phi}_3 \ \boldsymbol{\phi}_2 \ \cdots \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_2 \ \cdots) \\
\hline
(\boldsymbol{\phi}_1'' \ \boldsymbol{\phi}_2' \ \cdots \ \boldsymbol{\phi}_2' \ \boldsymbol{\phi}_3 \ \mathbf{0}_3 \ \cdots)
\end{array}$$

derived vector : $(\boldsymbol{\phi}_1'' \ (\boldsymbol{\phi}_2)_{\gamma-h-1} \ \boldsymbol{\phi}_3 \ \mathbf{0}_3 \ \cdots)$
Parity weight:

$$w_p = 2(\gamma - h) + 2 = 2\gamma + 2 \tag{0-2}$$

Case 2a: $h < \gamma = \beta$
vectors to sum:

$$\begin{array}{r} (\mathbf{0}_3 \cdots \ \cdots \ \mathbf{0}_3 \ \phi_1 \ \phi_2' \ \cdots) \\ (\mathbf{0}_3 \cdots \ \cdots \ \mathbf{0}_3 \ \phi_2 \ \phi_2'' \ \cdots) \\ +(\phi_3 \ \phi_2 \ \cdots \ \phi_2 \ \phi_2 \ \phi_2 \ \cdots) \\ \hline (\boldsymbol{\phi}_3 \ \boldsymbol{\phi}_2 \ \cdots \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_1 \ \mathbf{0}_3 \ \cdots) \end{array}$$

derived vector : $(\boldsymbol{\phi}_3 \ (\boldsymbol{\phi}_2)_{\gamma-h-1} \ \boldsymbol{\phi}_1 \ \mathbf{0}_3 \ \cdots)$
Parity weight:

$$w_p = 2(\gamma - h) + 2 = 2\gamma + 2 \tag{0-3}$$

Case 3a: $h < \gamma < \beta$
vectors to sum:

$$\begin{array}{r} (\mathbf{0}_3 \cdots\cdots \ \mathbf{0}_3 \ \phi_1 \ \phi_2' \ \cdots \ \phi_2' \ \phi_2' \ \phi_2' \cdots) \\ (\mathbf{0}_3 \cdots \ \cdots \ \cdots \ \cdots \ \cdots \ \mathbf{0}_3 \ \phi_2 \ \phi_2'' \cdots) \\ +(\phi_3 \ \phi_2 \ \cdots \ \phi_2 \ \phi_2 \ \phi_2 \ \cdots \ \phi_2 \ \phi_2 \ \phi_2 \cdots) \\ \hline (\boldsymbol{\phi}_3 \ \boldsymbol{\phi}_2 \cdots \ \boldsymbol{\phi}_2 \ \boldsymbol{\phi}_1' \ \boldsymbol{\phi}_2'' \cdots \ \boldsymbol{\phi}_2'' \ \boldsymbol{\phi}_2' \ \mathbf{0}_3 \cdots) \end{array}$$

derived vector : $(\boldsymbol{\phi}_3 \ (\boldsymbol{\phi}_2)_{\gamma-h-1} \ \boldsymbol{\phi}_1' \ (\boldsymbol{\phi}_2'')_{\beta-\gamma-1} \ \boldsymbol{\phi}_2' \ \mathbf{0}_3 \ \cdots)$
Parity weight:

$$\begin{aligned} w_p &= 2(\gamma - h) + 2 + 2(\beta - i) \\ &= 2(\beta - h) + 2 \\ &= 2\beta + 2 \end{aligned} \tag{0-4}$$

Case 3b: $h < \beta < \gamma$

$$\begin{array}{r} (\mathbf{0}_3 \ \cdots \ \cdots \ \cdots \ \cdots \ \cdots \ \mathbf{0}_3 \ \phi_1 \ \phi_2' \cdots) \\ (\mathbf{0}_3 \ \cdots\cdots \ \mathbf{0}_3 \ \phi_2 \ \phi_2'' \ \cdots \ \phi_2'' \ \phi_2'' \ \phi_2'' \cdots) \\ +(\phi_3 \ \phi_2 \ \cdots \ \phi_2 \ \phi_2 \ \phi_2 \ \cdots \ \phi_2 \ \phi_2 \ \phi_2 \cdots) \\ \hline (\boldsymbol{\phi}_3 \ \boldsymbol{\phi}_2 \cdots \ \boldsymbol{\phi}_2 \ \mathbf{0}_3 \ \boldsymbol{\phi}_2' \cdots \ \boldsymbol{\phi}_2' \ \boldsymbol{\phi}_3 \ \mathbf{0}_3 \cdots) \end{array}$$

derived vector : $(\boldsymbol{\phi}_3 \ (\boldsymbol{\phi}_2)_{j-k-1} \ \mathbf{0}_3 \ (\boldsymbol{\phi}_2')_{i-j-1} \ \boldsymbol{\phi}_3 \ \mathbf{0}_3 \ \cdots)$
Parity weight:

$$\begin{aligned} w_p &= 2(\beta - h) + 1 + 2(\gamma - \beta) + 1 \\ &= 2(\gamma - h) + 2 \\ &= 2\gamma + 2 \end{aligned} \tag{0-5}$$

From all the above cases we can conclude that the parity weight for a weight-3 RTZ sequence may be calculated as

$$w_p = 2l + 2 \tag{0-6}$$

where $l = \max\{\gamma, \beta\} - k = \max\{\gamma, \beta\}$ since $k = 0$

Assuming that after interleaving, another weight-3 RTZ input is produced. Let γ', β', h', l' and w'_p be similarly defined. Then the Hamming weight w_H of the turbo codeword produced can be calculated as

$$w_H = 7 + 2(l + l') \tag{0-7}$$

□