EE 6383 - Nonlinear Control Systems

Topic 6: Absolute Stability

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Lecture Outline

- Motivation
- Methods and Definitions
- Nyquist Revisited
- Problem of Lur'e
- Solution Due to Popov
- The Circle Criterion (CC)
- The Kalman-Yakubovich Lemma
- Informal Proof of the CC
- Significance of the Popov and Circle Criteria
- Hyperstability
- "Multiplier" Results
- Extensions for NLTV Systems
- Examples

References:

- Lefschetz, *Stability of Nonlinear Control Systems*, Academic, 1965.
- Aizerman & Gantmacher, *Absolute Stability of Regulator Systems*, Holden-Day, 1964.
- Narendra & Taylor, *Frequency Domain Criteria for Absolute Stability*, Academic, 1973.
- Narendra, ASME Books, Vol. 1, Chapter 2, 1978.
- Taylor, ASME Books, Vol. 2, Chapter 20, 1980.

Motivation (Recapitulation)

- Stability analysis is a serious business
- No loose method is fool-proof
  - Small-signal linearization
  - Gain sectors containing a nonlinearity (Aizerman conjecture)
  - Gain sectors based on max and min slope (Kalman conjecture)
  - Gain sectors based on a describing function
- There *are* rigorous methods – **No excuses!**
- $\star$ $\star$ Some of these are even easy to use!

Our Definition of Stability - UASIL

- **Given a system $\dot{x} = f(x, t)$ with equilibrium $x = 0$;**

- **The system is Uniformly Asymptotically Stable in the Large (UASIL) if:**

  1. For every $\epsilon > 0$ and t_0 there exists a $\delta(\epsilon) > 0$ such that $\lim_{\epsilon \to \infty} \delta(\epsilon) = \infty$, and $||x_0|| \leq \delta \Rightarrow ||x(t; x_0, t_0)|| \leq \epsilon$ for all $t \geq t_0$

  2. For some $\rho > 0$ and for every $\eta > 0$ there exists a $T(\eta, \rho)$ such that $||x(t; x_0, t_0)|| \leq \eta$ for all $||x_0|| \leq \rho$ and $t \geq t_0 + T$

This is a conservative (safe) definition for engineering usage.

Another definition (stated informally): any bounded input must result in a bounded output; the criteria given here are sufficient to guarantee either definition.

Nyquist Criterion Revisited

Given an open-loop transfer function $W(s) = \frac{(s+20)(s+30)}{(s+1)(s^2+2s+10)(s+200)}$:

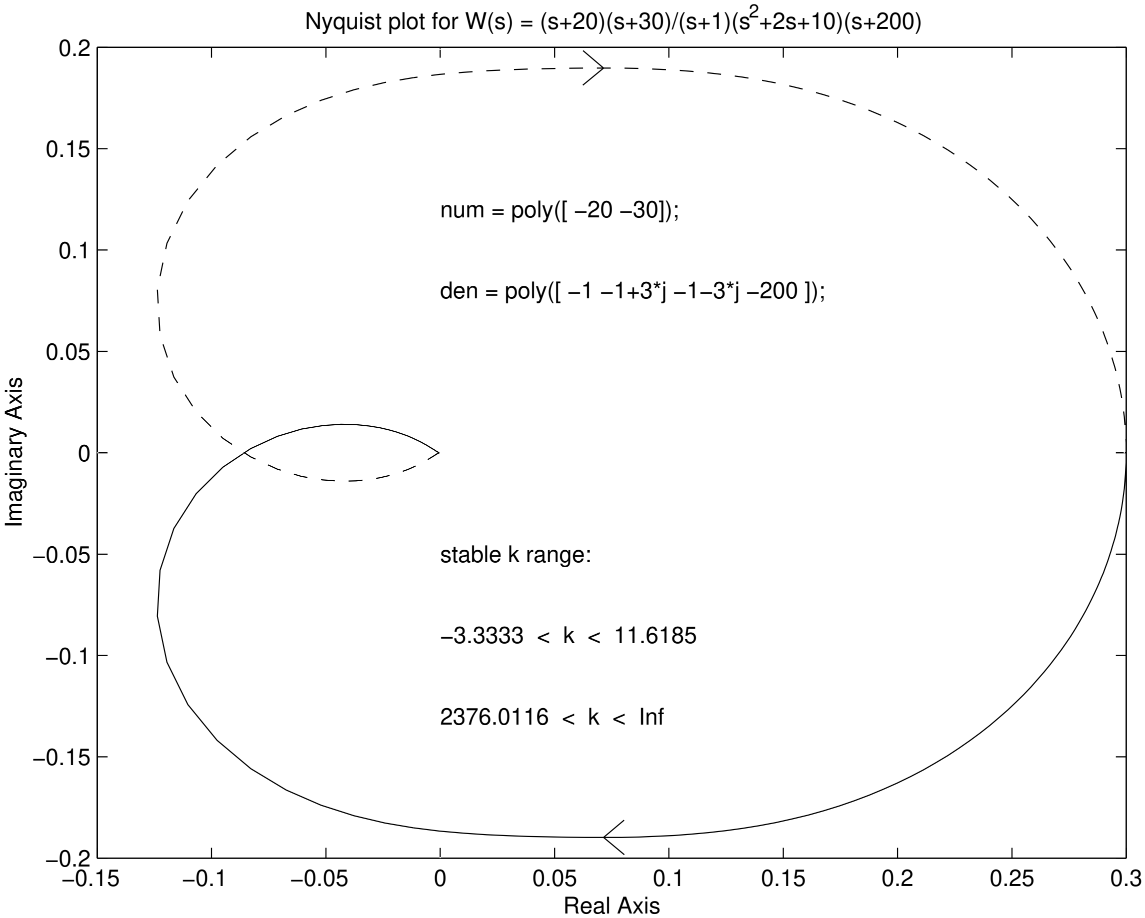


Figure 1: Condition for Asymptotic Stability: $-1/k \notin \mathcal{W}_{\mathcal{R}}$

The maximum useful stability range is -3.33 $< K <$ 11.62; if you want the "safety" of a gain margin of 5 (14 dB) then pick $K = 2.32$, etc.

Nyquist Criterion Revisited (continued)

Example: Consider the unstable plant: $W(s) = \frac{s+2}{s^2-4s-5}$

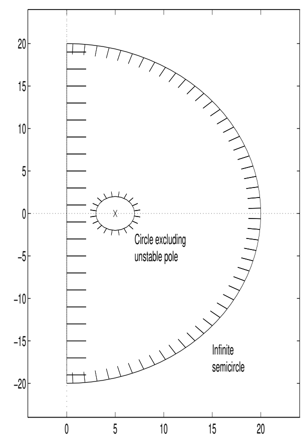


Figure 2: s-plane region mapped for Nyquist criterion

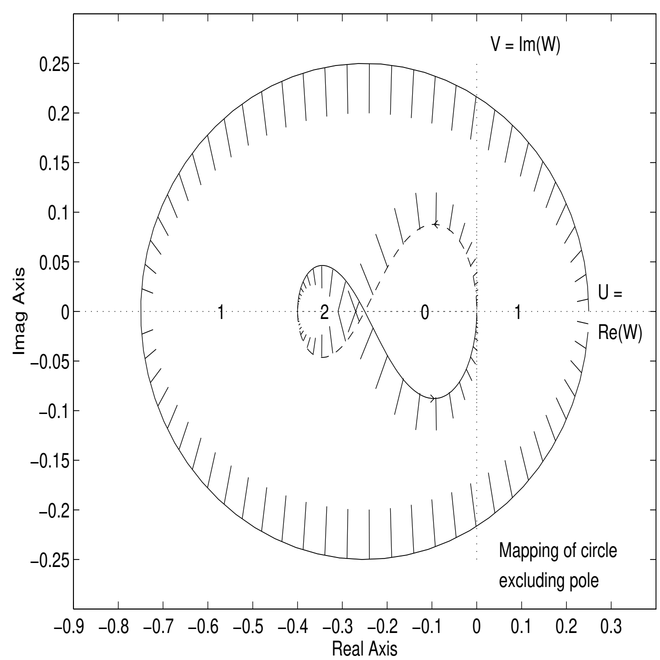


Figure 3: $W(s)$-map for the Nyquist criterion

Condition for asymptotic stability: $-1/k \notin \mathcal{W}_{\mathcal{R}}$

A New MATLAB Nyquist Tool

Another example: Consider a simple stable plant:

$$W(s) = \frac{s+1}{s^4 + 2s^3 + 25s^2 + 3s + 1} \tag{1}$$

$\rightarrow$

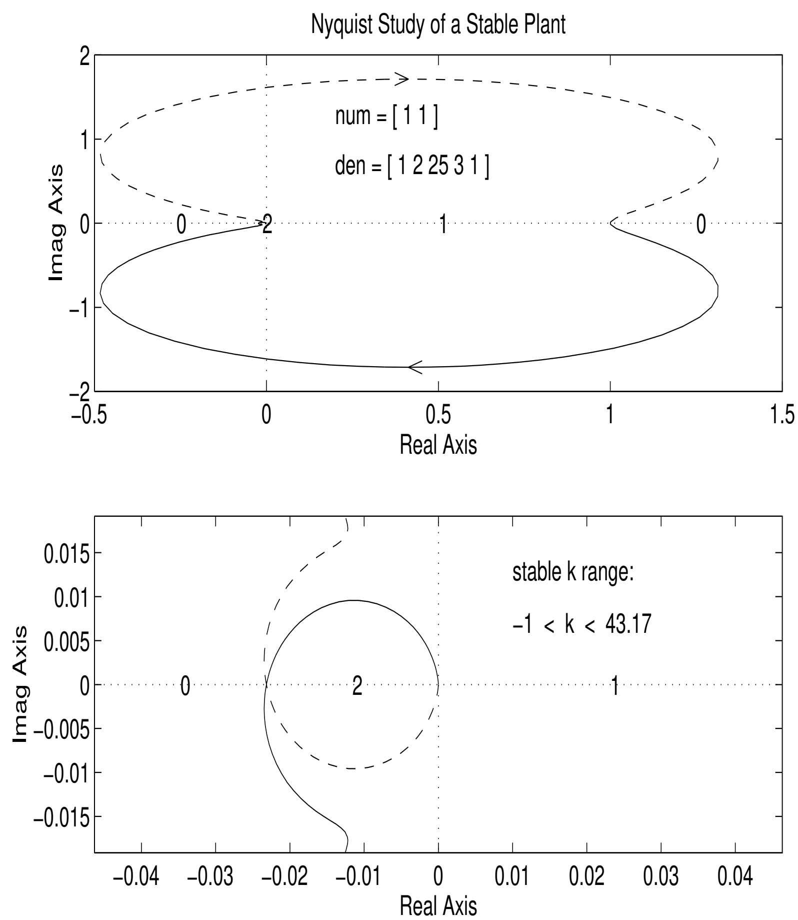


Figure 4: Nyquist Criterion Example (Stable Plant)

The report that **newnyq** provides is:

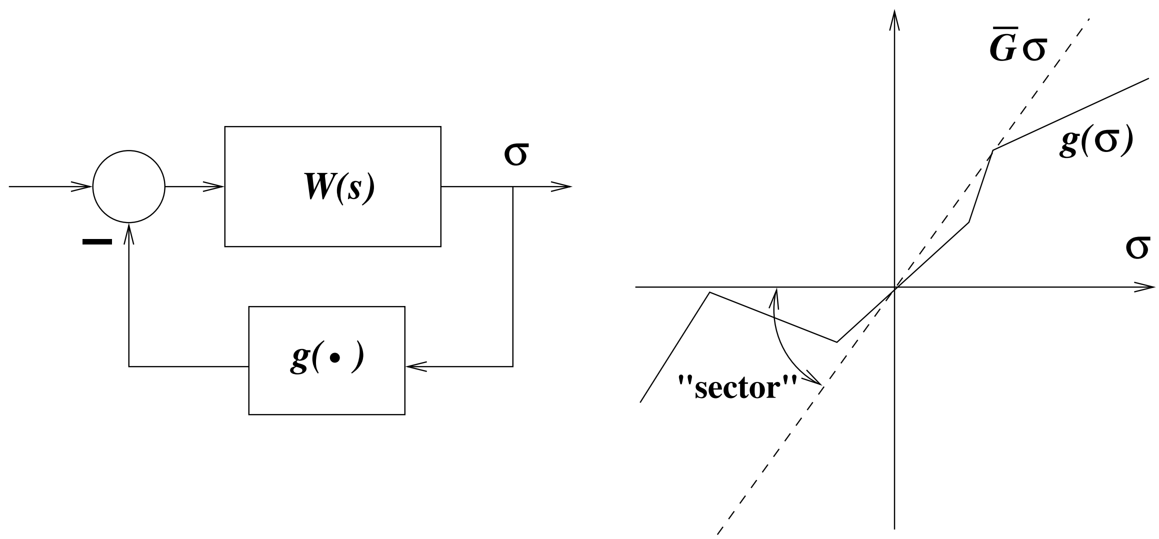
```
>> newnyq(num,den)


stable k range
-1  <  k  <  43.17
```

Nyquist Criterion Revisited (continued) - Why It Is So Important in Practice

- You do not need a **precise analytic model** – just $W(j\omega)$
- You have a direct graphical interpretation of the impact of **uncertainty**
- Empirical data is directly useful without a need to assume system order and curve fit

The Problem of Lur'e & Postnikov (1944)



- Question: What constraints must $W(s)$ satisfy for UASIL, given only that $0 < \frac{g(\sigma)}{\sigma} < \overline{G}$ (or "$g(\sigma)$ lies in the sector $(0, \overline{G})$")?
- This is called the Absolute Stability Problem; if $W(s)$ meets such constraints then the system is said to be Absolutely Stable.

Popov's Solution to the Lur'e-Postnikov Problem (1961)

- **The L-P system is absolutely stable if**:
  1. $W(s)$ is **stable**, and
  2. a **real** q **exists** such that

  $$T(s) = [W(s) + 1/\overline{G}] \cdot (1 + qs)^{\pm 1} \in \text{PR}$$

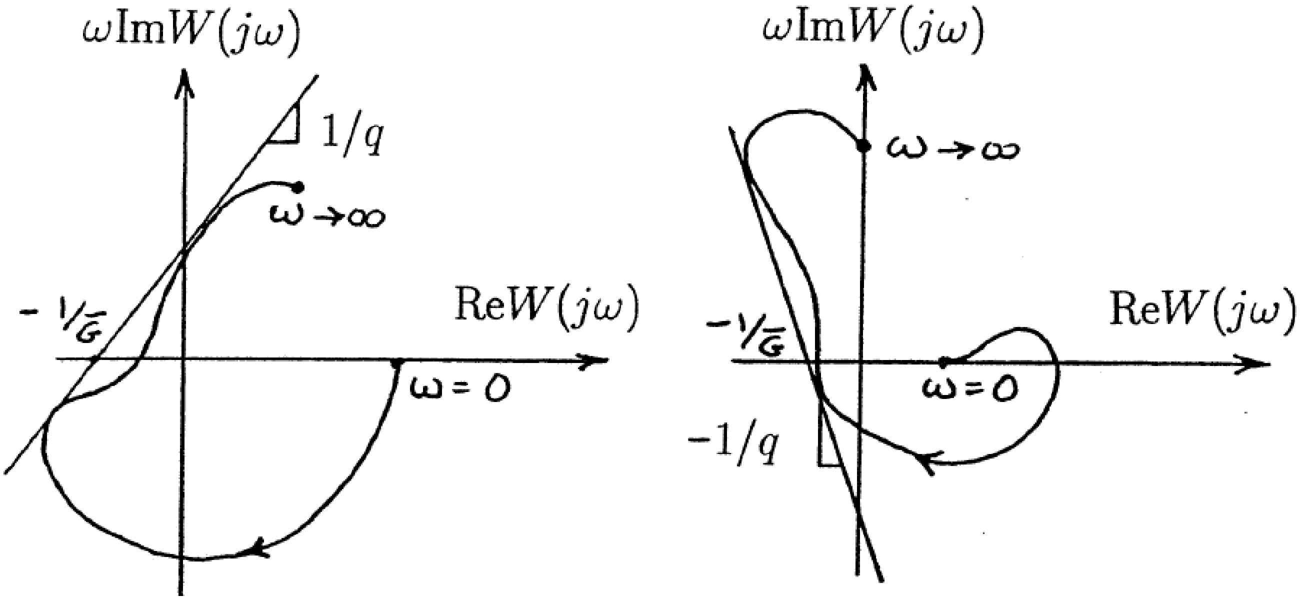
  where $(\ \cdot\ )^{\pm 1}$ denotes multiplication by $(1 + qs)$ or $1/(1+qs)$ and $\in$ PR signifies that $T(s)$ is positive real

- **Definition**: $T(s) \in \text{PR} \Leftrightarrow \operatorname{Re} T(s) \geq 0$ for all $\operatorname{Re} s \geq 0$ (for all $s \in \mathcal{R}$)

- $T(s) \in \text{PR} \Rightarrow T(s)$ has no poles or zeroes in the RHP; if so, you only need to consider $T(j\omega)$.

- **Important**: This condition is *sufficient* but not *necessary*, i.e., if the condition is not met that does not mean that the L-P system is unstable.

Geometrical Interpretation



These are <u>not</u> Nyquist plots

To show this, define $W(j\omega) = U + jV$; then

$$T(j\omega) = (U + \frac{1}{\overline{G}} + jV) \cdot (1 + qj\omega);$$

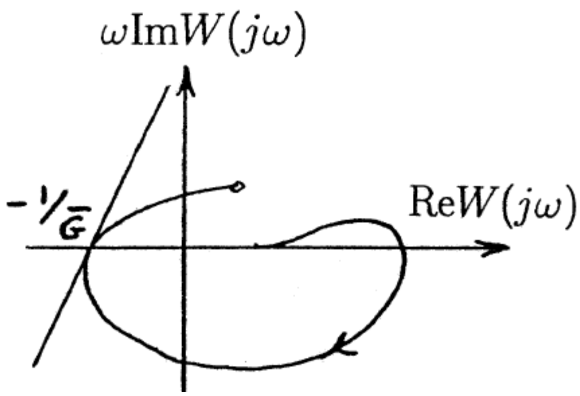
making the real part ≥ 0 requires

$$\omega V \leq (U + \frac{1}{\overline{G}})/q \tag{2}$$

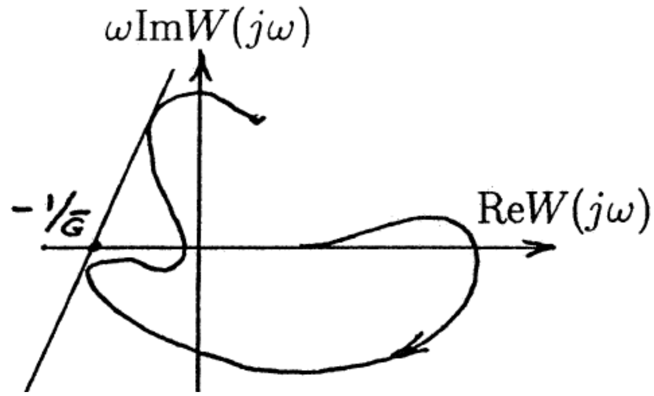
Relation of the Popov Criterion to the Aizerman Conjecture

The Aizerman conjecture is shown to be valid for any linear plant that satisfies the condition that the point $(-1/\overline{G}$, 0) lies both on the Popov <u>plot</u> and the Popov <u>line</u>:

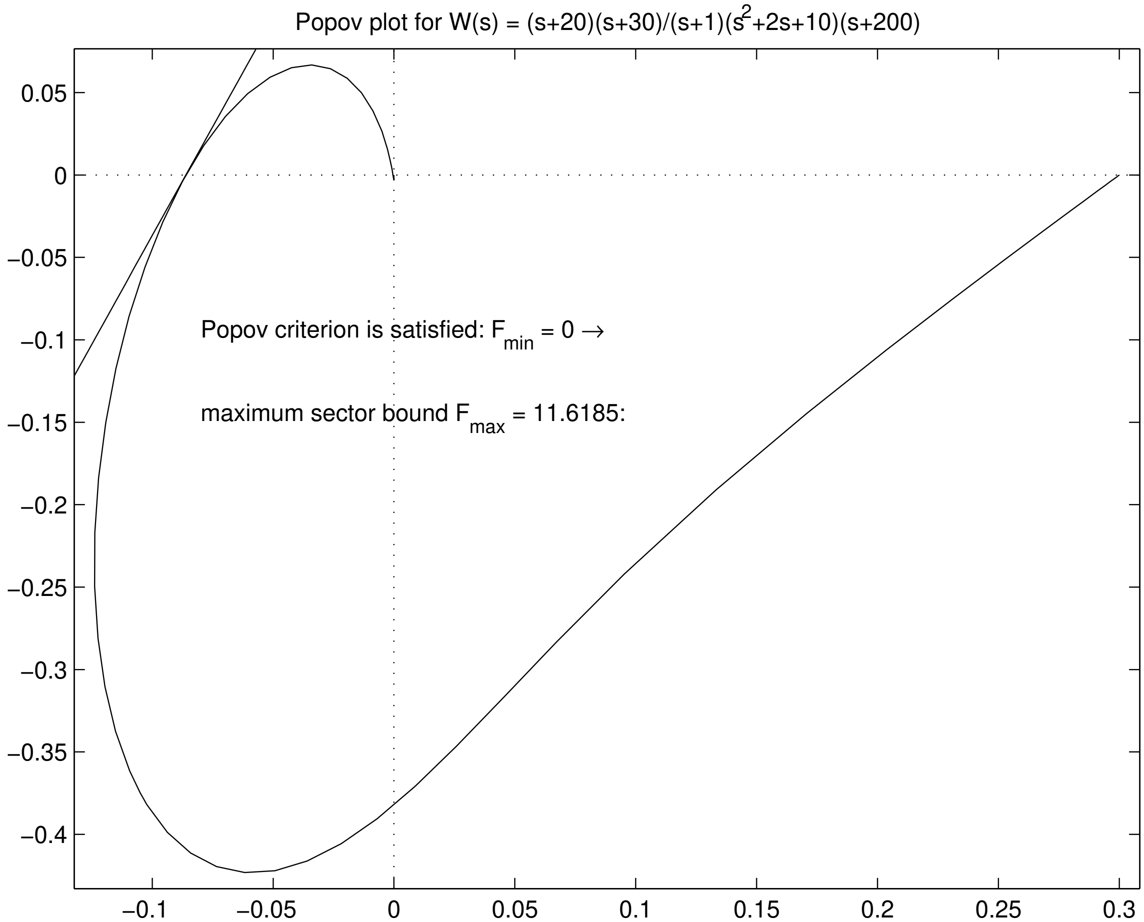
YES



NO



Popov Criterion Example



The closed-loop system with $W(s)$ and $g(\sigma)$ in the loop is *guaranteed* to be UASIL (absolutely stable) as long as $0 < \dfrac{g(\sigma)}{\sigma} < F_{\max} = 11.62$.

Note: in this case the nonlinearity is *time invariant*; also the upper bound $F_{\max}$ is equal to the Nyquist bound $K_{\max}$! As with Nyquist, if $F_{\max} = 2.32$ we have a gain margin of 5 (14 dB).

Popov Criterion Proof

Here's the model,

$$\dot{x} = Ax + b\tau, \qquad \sigma_0 = h^{\mathrm{T}}x + \rho\tau, \qquad \tau = -f(\sigma_0)$$

Here is the Lyapunov function candidate,

$$v(x) = \tfrac{1}{2}x^{\mathrm{T}}Px + \hat{\beta}_0\left\{\int_0^{\sigma_0} f(\zeta)\, d\zeta + \tfrac{1}{2}\rho\tau^2\right\}$$

...and *here* is $\dot{V}$!

$$\begin{aligned}\dot{v} = {} & \tfrac{1}{2}x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - f(\sigma_0)x^{\mathrm{T}}[Pb - \hat{\beta}_0A^{\mathrm{T}}h - \gamma_0 h] \\ & - [\hat{\beta}_0 h^{\mathrm{T}}b + \gamma_0(\rho + \bar{F}^{-1})]f^2(\sigma_0) - \gamma_0\sigma_0 f(\sigma_0)[1 - f(\sigma_0)/\bar{F}\sigma_0].\end{aligned}$$

The only additional algebraic manipulation performed on $\dot{v}$ is the inclusion of $\gamma_0[\sigma_0 f(\sigma_0)(f(\sigma_0)/\bar{F}\sigma_0) - f^2(\sigma_0)/\bar{F}]$, which is identically equal to zero.

We define some variables for use in the MKY lemma,

$$\begin{aligned}\tfrac{1}{2}\psi &\triangleq \beta_0 h^{\mathrm{T}}b + \gamma_0(\rho + \bar{F}^{-1}), \\ k &\triangleq \hat{\beta}_0 A^{\mathrm{T}}h + \gamma_0 h.\end{aligned} \tag{5-5}$$

The lemma then states that there is some matrix $P, P = P^{\mathrm{T}} > 0$; a matrix $M, M = M^{\mathrm{T}} \geqslant 0$; and a real vector q satisfying

(a) $A^{\mathrm{T}}P + PA = -qq^{\mathrm{T}} - M$,
(b) $Pb - k = \sqrt{\psi}q$,
(c) (q^{T}, A) is completely observable,

if and only if

(d) $H(s) = \frac{1}{2}\psi + k^{\mathrm{T}}(sI - A)^{-1}b \in \{PR\}$.

If $H(s)$ is indeed PR we have:

$$\begin{aligned}\dot{v} = {} & -\tfrac{1}{2}[x^{\mathrm{T}}q + \sqrt{\psi}\, f(\sigma_0)]^2 \\ & -\tfrac{1}{2}x^{\mathrm{T}}Mx - \gamma_0\sigma_0 f(\sigma_0)[1 - f(\sigma_0)/\bar{F}\sigma_0]\end{aligned}$$

So, finally,

absolute stability for the system determined by Eq. (5-3) are

(a) $\gamma_0 > 0$,
(b) $H(s) \triangleq [W(s) + \bar{F}^{-1}](\beta_0 s + \gamma_0)^{\pm 1} \in \{PR\}$ where $\beta_0 \geqslant 0, \gamma_0 > 0$

The Circle Criterion

The NLTV generalization of the Lur'e-Postnikov problem:

$$g(\sigma) \to g(\sigma, t)$$

- The NLTV system is UASIL if:
  1. $\underline{G} < \frac{g(\sigma,t)}{\sigma} < \overline{G}$
  2. $T(s) = \frac{1+\overline{G}W(s)}{1+\underline{G}W(s)} \in$ SPR

- **Definition**: $T(s) \in$ SPR ($T(s)$ is <u>strictly positive real</u>) $\Leftrightarrow$

$$\text{Re } T(s-\epsilon) \geq 0 \;\forall\; \text{Re } s \geq 0, \;\; \text{for some } \epsilon > 0$$

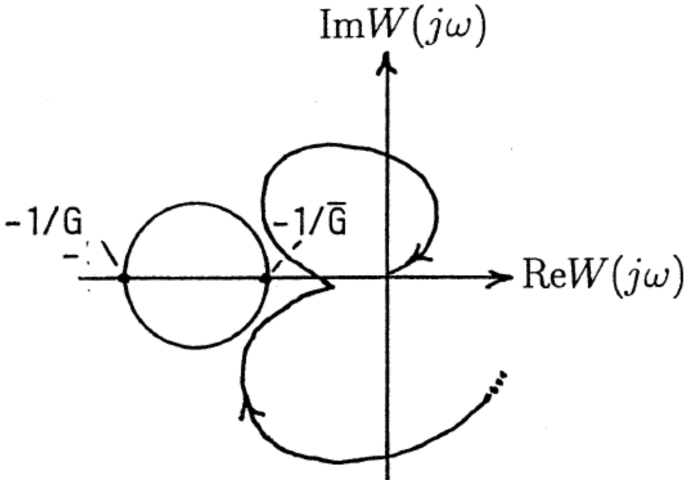
- This implies that there are <u>no poles or zeros in the RHP</u> and that $\text{Re}\, T(j\omega) > 0 \;\forall\omega$; this condition does <u>not</u> require that $\text{Re}\, T(j\omega) \geq \epsilon > 0 \;\forall\omega$

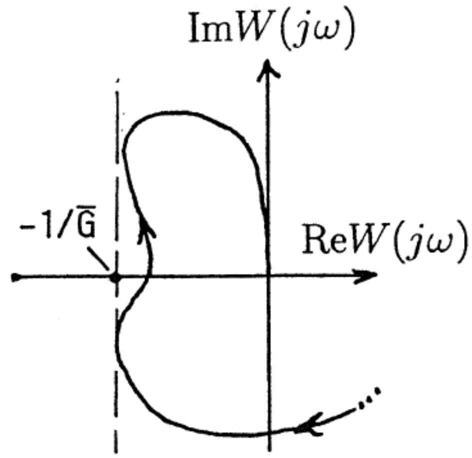
- This definition is strictly dictated by the T-K-Y Lemma; example: $\frac{1}{s+a} \in$ SPR by this definition

- **Important**: This condition is <u>*sufficient*</u> but not <u>*necessary*</u>, i.e., if the condition is not met that does not mean that the NLTV system is unstable.

Geometrical Interpretation

General Case: Special Case, $\underline{G} = 0$:





These <u>are</u> Nyquist plots

To show this, define $W(j\omega) = U + jV$; then $\mathrm{Re}\, T(j\omega) > 0$ if:

$$\mathrm{Re}\,(U + \frac{1}{\overline{G}} + jV) \cdot (U + \frac{1}{\underline{G}} - jV) > 0$$

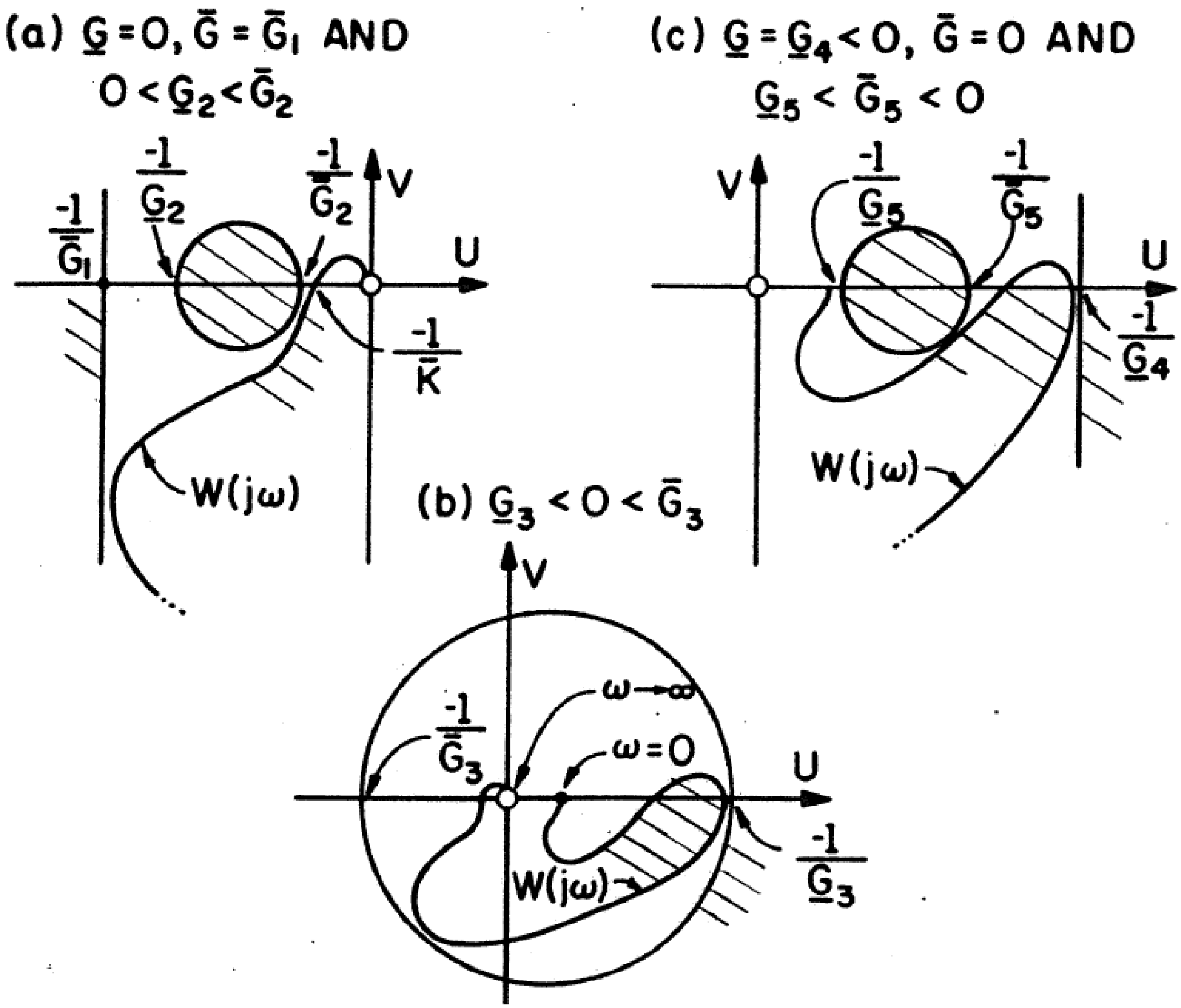
(assuming $0 < \underline{G} < \overline{G}$), therefore, constraining the real part of $T(j\omega)$ to be positive requires

$$(U + \frac{1}{\overline{G}}) \cdot (U + \frac{1}{\underline{G}}) + V^2 > 0 \tag{3}$$

which requires $U + jV$ to avoid the interior of a circle whose <u>diameter</u> is defined by the points $-1/k$ for $\underline{G} \leq k \leq \overline{G}$.

Geometrical Interpretation (Cont'd)

An **infinite number** of circles can be drawn, so one can (for example) trade off $\underline{G}$ against $\overline{G}$.



Be sure that the "interior" of the circle is not in $\mathcal{W}_{\mathcal{R}}$!
Be careful if $\underline{G} < 0 < \overline{G}$!

The Circle Criterion and "M-Circles"

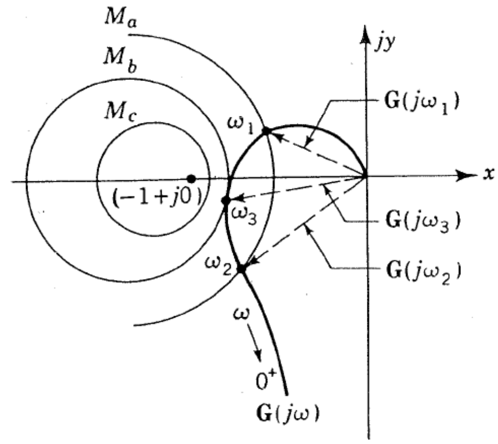


FIGURE 9-9
M contours and a **G**($j\omega$) plot.

Center: $x_0 = -\frac{M^2}{M^2-1}$

Radius: $r_0 = |\frac{M}{M^2-1}|$

Example 1: $M = 2 \rightarrow$
$x_0 = -4/3\,,\ r_0 = 2/3$

Relation to the CC:

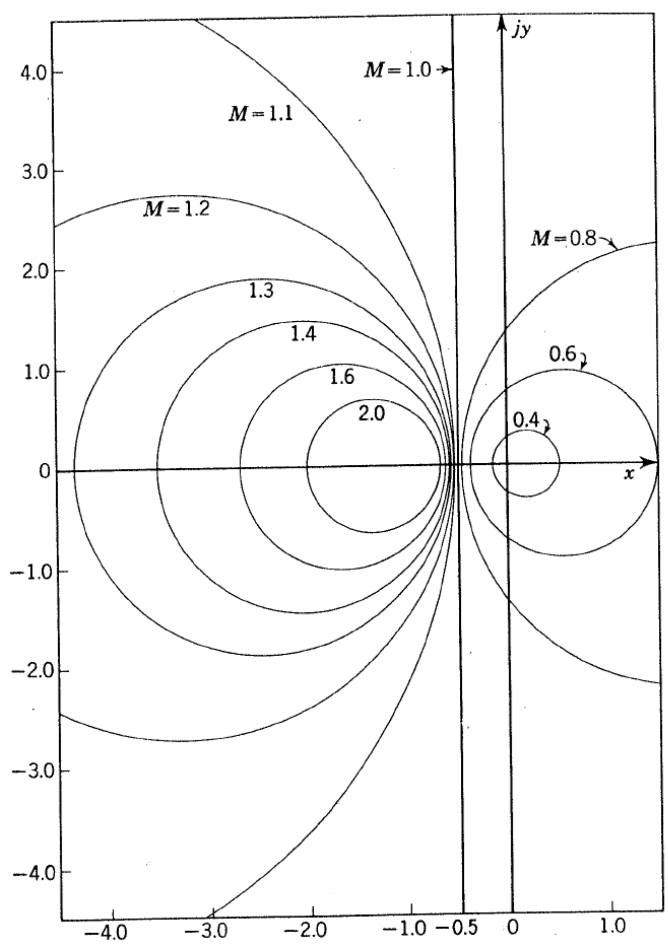
$\underline{G} = 0.5\,,\ \overline{G} = 1.5$

Example 2: $M = 1.2 \rightarrow$

$\underline{G} = 0.1667\,,\ \overline{G} = 1.833$

Example 3: $M = 1.0 \rightarrow$

$\underline{G} = 0\,,\ \overline{G} = 2$



Circle Criterion – A MATLAB Tool

Example: Consider the relatively simple stable plant:

$$W(s) = \frac{s+1}{s^4 + 2s^3 + 25s^2 + 3s + 1} \tag{4}$$

$\rightarrow$

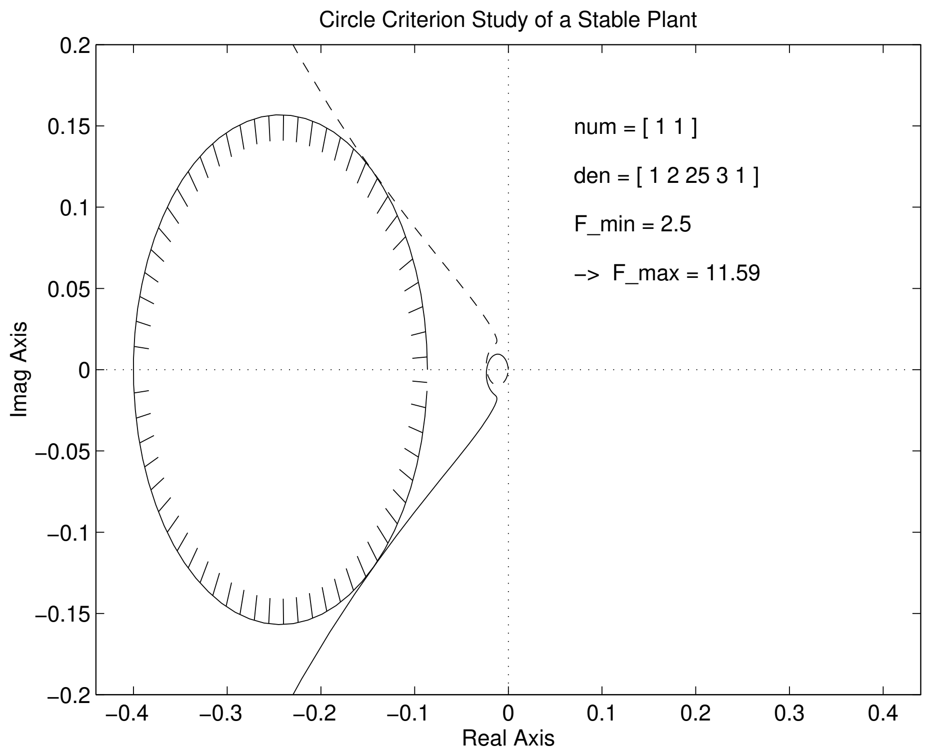


Figure 5: Circle Criterion Example (Stable Plant)

The report that `circle` provides is:

```
>> circle(num,den,2.5)

stable k range
-1  <  k  <  43.17
circle criterion is satisfied
maximum sector bound F_max = 11.59
```

A MATLAB Tool – More Examples

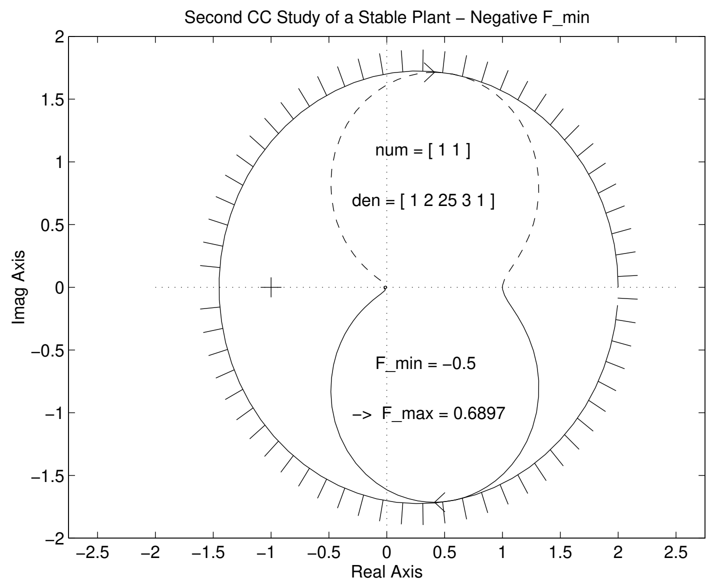


Figure 6: Circle Criterion Result for Negative F_min

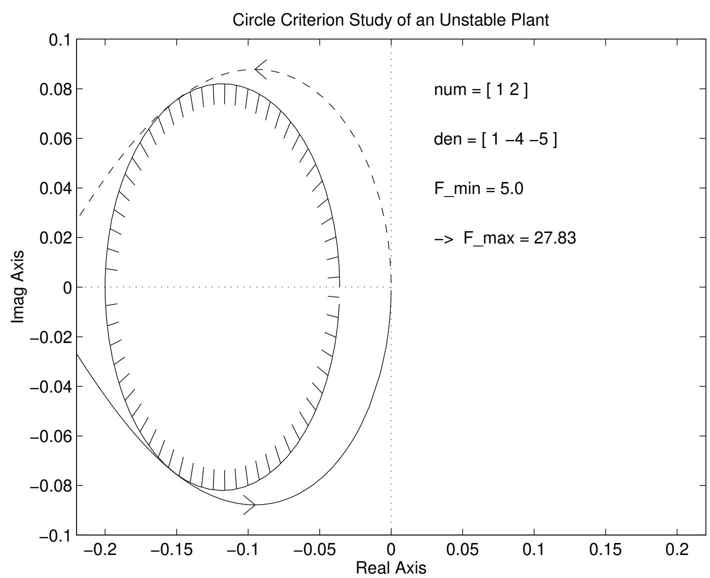
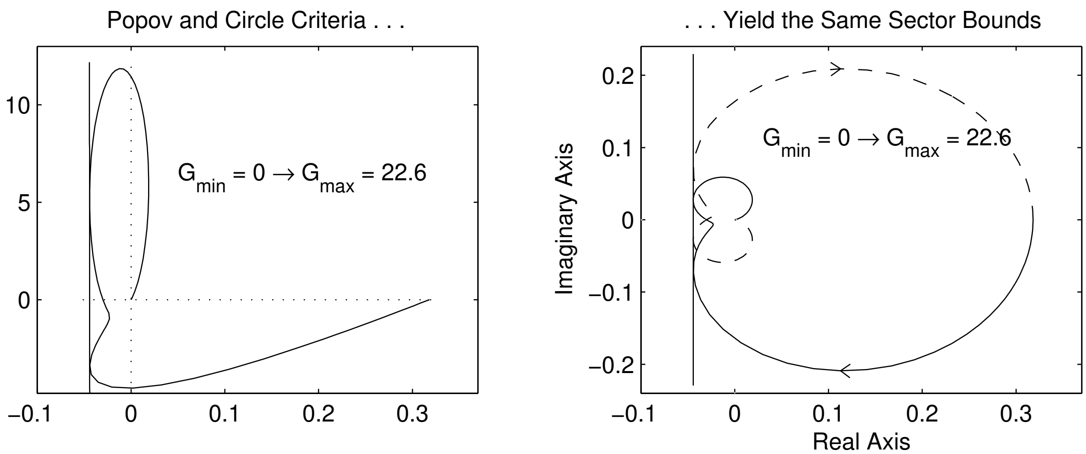


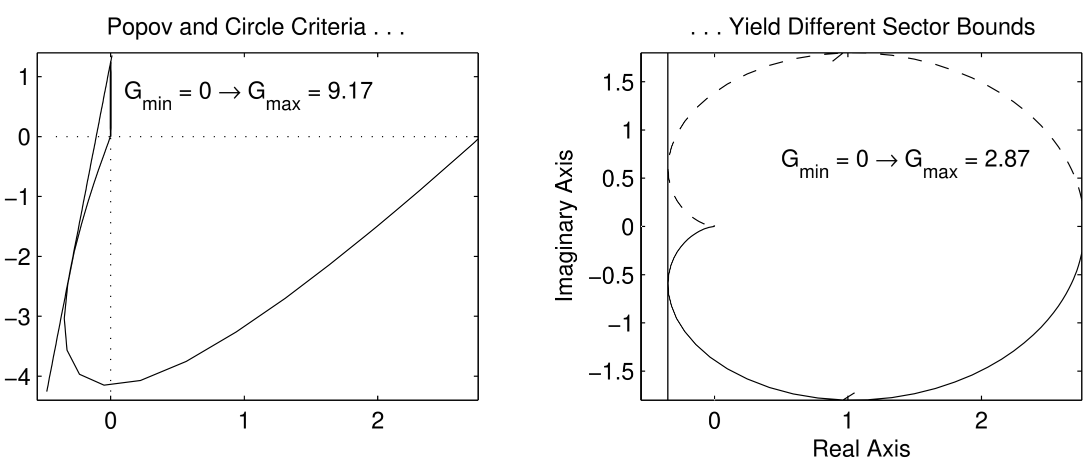
Figure 7: Circle Criterion Example (Unstable Plant)

Comparing the Popov and Circle Criteria

- The sector bounds may be the same …



- or $\overline{G}^{\text{Popov}}$ may be substantially larger than $\overline{G}^{\text{Circle}}$



- Since the negative real axis crossings and minimum values of $\mathrm{Re}(W(j\omega))$ are the same the Circle Criterion can **never** provide a larger $\overline{G}$

Significance of the Popov and Circle Criteria

- You do not need a precise analytic model for either the plant or the nonlinearity

- You have a direct graphical interpretation of the impact of uncertainty

- Empirical data is directly useable without a need to assume system order and curve fit

(These points look familiar, don't they?)

Taylor[1] Version of the Kalman-Yakubovich Lemma

Given: A asymptotically stable, (A, b) controllable, an arbitrary vector k and matrix $R = R^T > 0$

Then: The matrix $P = P^T > 0$ and vector q exist such that

$$\begin{aligned} A^T P + PA &= -qq^T - \epsilon R \\ Pb - k &= q \end{aligned}$$

if and only if ϵ is sufficiently small, and

$$T(s) = 1 + 2k^T (sI - A)^{-1} b \in \text{SPR}$$

Recall that if A is asymptotically stable then for any $Q = Q^T > 0$ one can solve $A^T P + PA = -Q$ for P and $P = P^T > 0$. This lemma is a very fundamental result in linear system theory.

[1] J. H. Taylor, "Strictly Positive Real Functions and the Lefschetz-Kalman-Yakubovich Lemma", IEEE *Trans. on Circuits and Systems, Vol. CAS-21,* No. 2, March 1974.

Informal Proof of the Circle Criterion

1. Given: $\dot{x} = A\,x - b\,g(\sigma, t)$ where $0 < g(\sigma)/\sigma < \overline{G}$

2. Choose the "Common Quadratic Lyapunov Function" $V = x^T P x$

3. Inspect the derivative of V along system trajectories:

$$\begin{aligned} \dot{V} &= x^T(A^T P + PA)x - 2x^T Pb\,g \\ &= x^T(A^T P + PA)x - gx^T(2Pb - \overline{G}c) - g^2 \\ &\quad -g(\overline{G}\sigma - g) \end{aligned}$$

   (where the second formulation is obtained by adding and subtracting terms that cancel)

4. Use the T-K-Y lemma to complete the square: Let $2\,k = \overline{G}\,c$, then

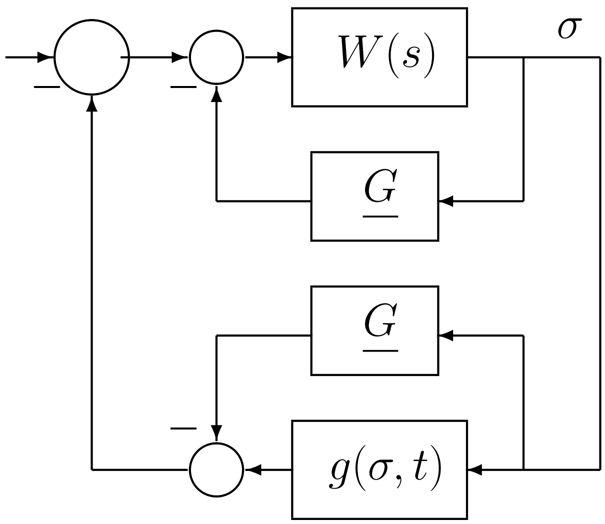
$$\dot{V} = -\epsilon x^T R x - [x^T q + g]^2 - g(\overline{G}\sigma - g)$$

   if and only if

$$1 + \overline{G}W(s) \in \text{SPR}$$

5. Note that $0 < g(\sigma)/\sigma < \overline{G}$ guarantees that $g(\overline{G}\sigma - g > 0$.

The General Finite Sector Transformation



$$\tilde{W} = \frac{W}{1+\underline{G}W}$$

$$\tilde{g} = g - \underline{G}\,\sigma$$

For the transformed system above,

$$\underline{G} < \frac{g(\sigma, t)}{\sigma} < \overline{G} \;\Rightarrow\; 0 < \frac{\tilde{g}(\sigma, t)}{\sigma} < \overline{G} - \underline{G}$$

Therefore,

- Using this transform on the $[\,0, \overline{G}\,]$ version of the Circle Criterion just proved, we must satisfy

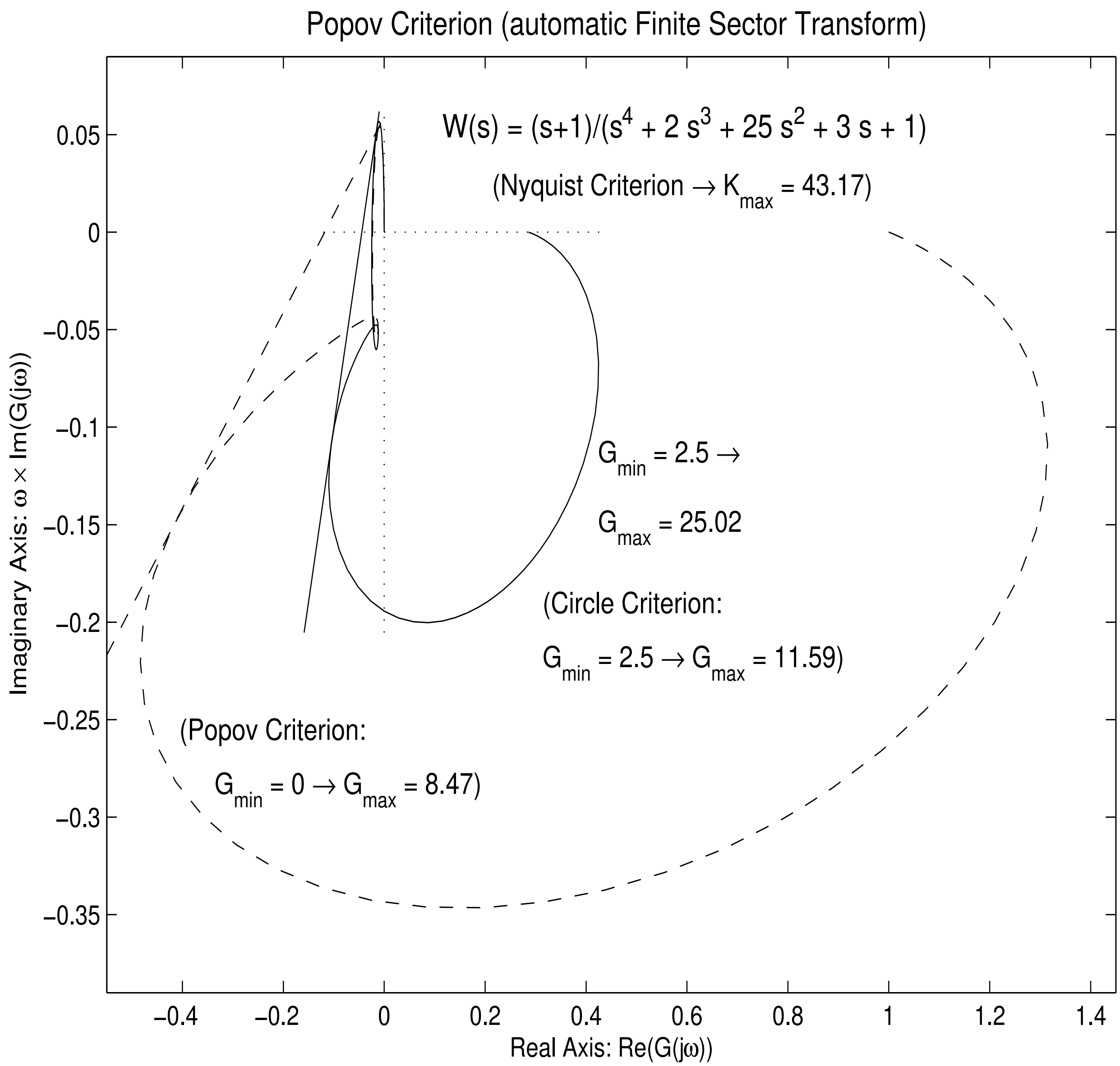
$$\tilde{T}(s) = 1 + \frac{(\overline{G} - \underline{G})W}{1 + \underline{G}W} = \frac{1 + \overline{G}W(s)}{1 + \underline{G}W(s)} \in \text{SPR}$$

- We can also obtain the best extension of the Popov Criterion in this way:

$$\tilde{T}(s) = \frac{1 + \overline{G}W}{1 + \underline{G}W} \cdot (1 + qs)^{\pm 1} \in \text{PR}$$

In other words, just draw the Popov plot for $\tilde{W}$, draw the Popov line to obtain $\tilde{\overline{G}} = (\overline{G} - \underline{G})$ and thus $\overline{G} = \tilde{\overline{G}} + \underline{G}$.

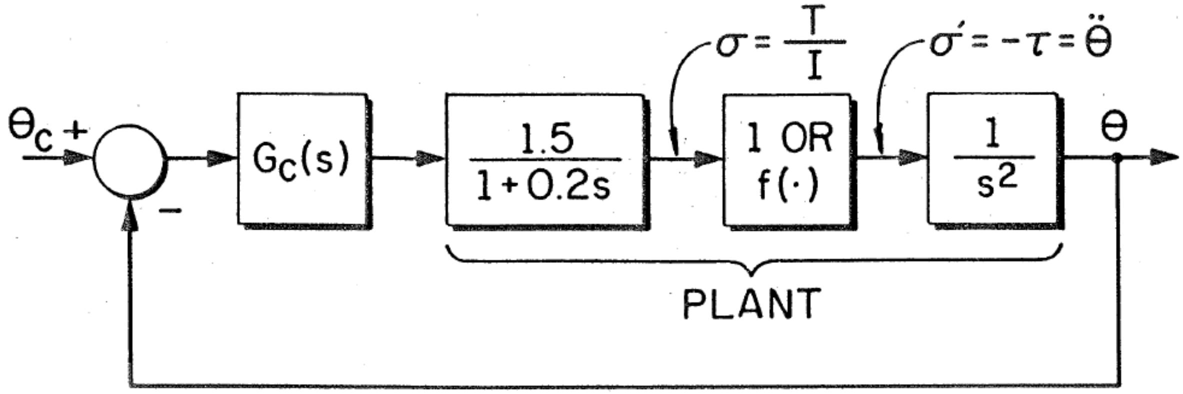
Applying the Finite Sector Transform (Popov Criterion)



The basic Popov result for $G_{min} = 0$ is shown by the dashed plot; the solid curve is the result of the `popov` routine using the finite sector transform to deal with $G_{min} = 2.5$ – so, the Parabola Criterion is no longer of any purpose.

Example: A "Semi-real World" Problem

Problem: Design a position control system, and assess the possible impact of nonlinearity in the plant (e.g., saturation)



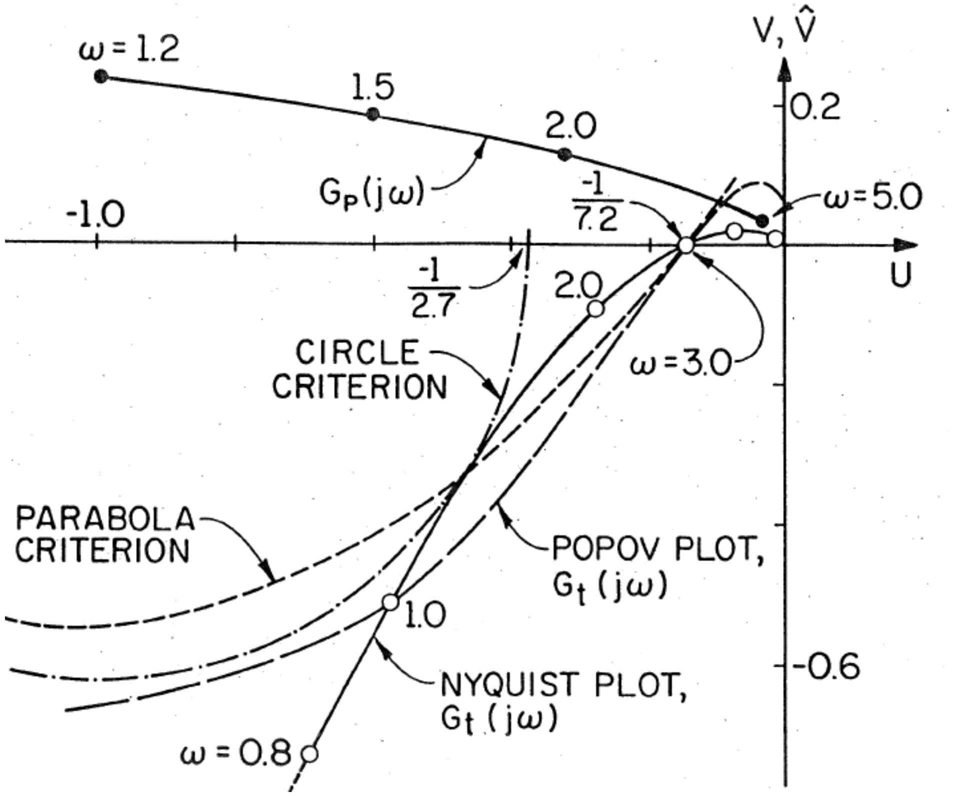
- *Plant*: Inertial load, $1\,/\,J\,s^2$, with additional motor lag; $f(\cdot)$ represents a possible motor nonlinearity
- *Design specs*: $M_M = 1.3$ (M-circle spec), Bandwidth = 1 rad/sec
- Standard linear compensator design methods (for $f(\sigma) = \sigma$) call for using a lead compensator[2], yielding:

$$G_c(s) = 1.132\;\frac{s+0.18}{s+2} \rightarrow G_t = 8.49\;\frac{s+0.18}{s^2(s+2)(s+5)}$$

[2]Note that the plant is unstable for any $K > 0$.

Example: Position Control System (Cont'd)

Now, assess the possible impact of nonlinearity:

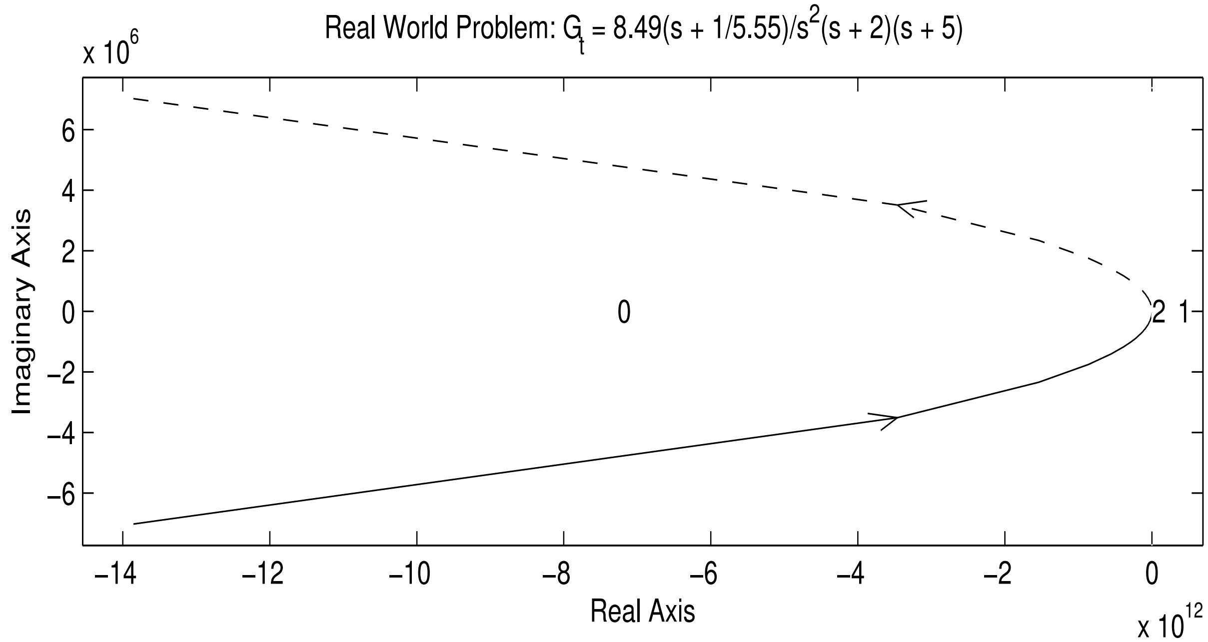


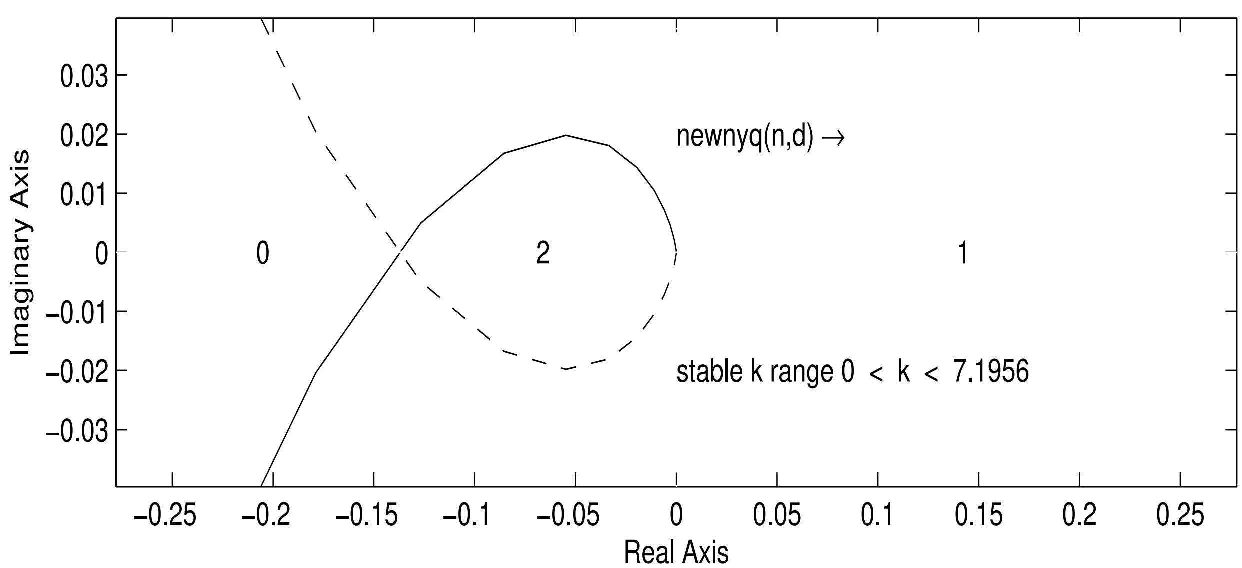
- Nyquist: $0 < k < 7.2$
- Parabola Criterion: $0.54 < \frac{g(\sigma)}{\sigma} < 7.19$ (NLTI!)
- Circle Criterion: $0.61 < \frac{g(\sigma,t)}{\sigma} < 2.7$ (NLTV!!)

(from a pre-MATLAB "by hand" analysis; the Parabola Criterion is an obsolete extension of the Popov Criterion (derived before the finite sector transform trick was conceived))

Example: Position Control System (Cont'd)

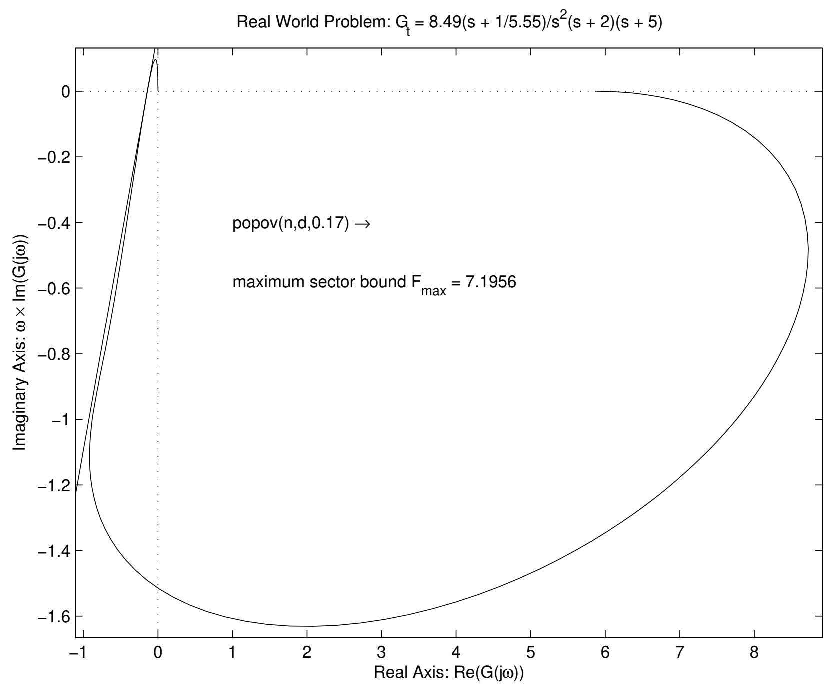
Finally, assess the possible impact of nonlinearity using MATLAB tools; first, the Nyquist gain margin:

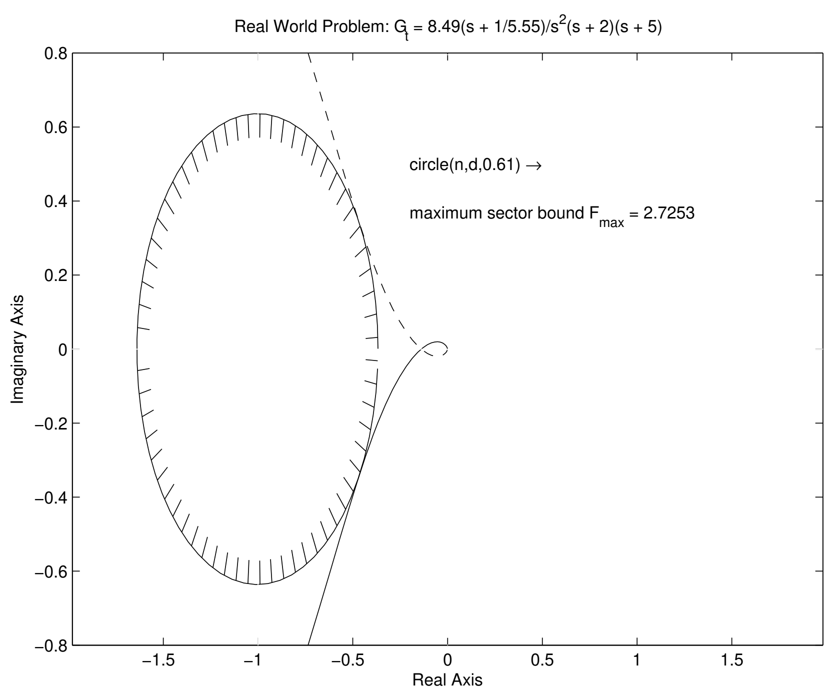




Example 1: Position Control System (Cont'd)

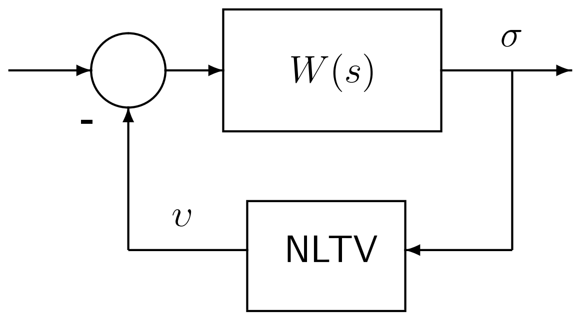
Assess the possible impact of nonlinearity using MATLAB tools, now Popov and Circle criteria sector limits:





Hyperstability

V. M. Popov's "Standard Feedback Configuration"



- **The standard configuration is said to be asymptotically hyperstable in the large whenever it can be said to be UASIL for all feedback blocks satisfying the Popov integral condition:**

$$\int_{t_0}^{t} \sigma\, \upsilon\, dt > -\psi^2 \ \ \forall\, t > t_0$$

- **The condition for asymptotic hyperstability in the large is $W(s) \in$ SPR**
- **This problem is a generalization of absolute stability; the Popov integral condition subsumes $g(\sigma, t)\ /\ \sigma > 0$, and the NLTV block can even be dynamic**
- **Note that the original $W(s)$ and NLTV would rarely satisfy these conditions; one must find $Z(s)$ such that $W(s) \cdot Z(s)$ is SPR and NLTV$\cdot Z^{-1}(s)$ satisfies the Popov integral condition.**
- **These conditions guarantee that the linear plant is *strictly passive* and the feedback block is *passive***

Absolute Stability "Multiplier Results"

Absolute stability "multiplier results" can be obtained in several ways; hyperstability/embedding (described informally with the $Z(s)$ / $Z^{-1}(s)$ trick above) may be the easiest to grasp:

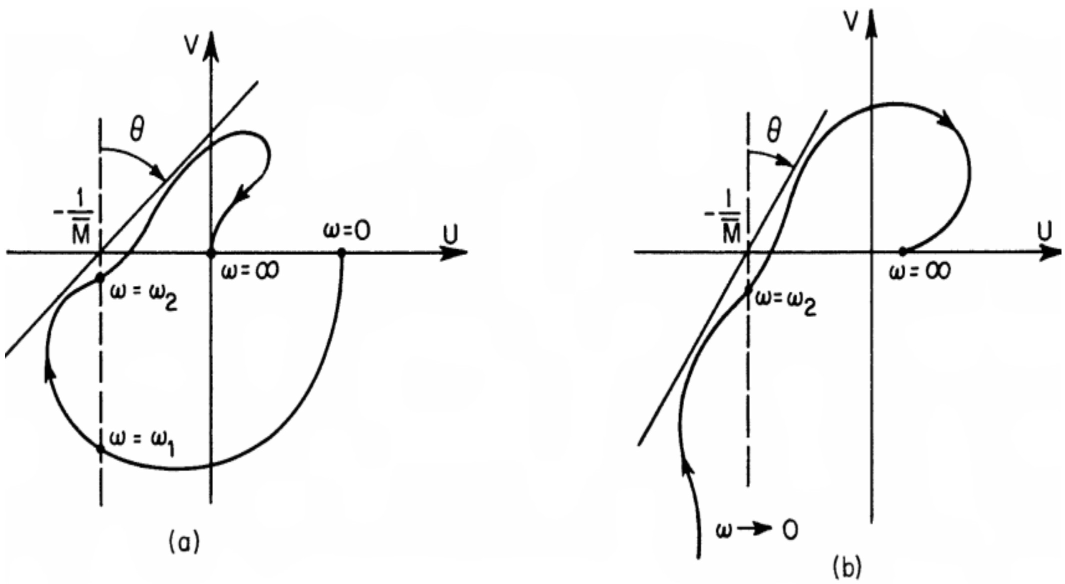
1. **Define a useful class of nonlinearities, e.g., slope-bounded, $\underline{M} < dg/d\sigma < \overline{M}$ (recall the Kalman Conjecture!)**
2. **Prove that $Z(s) \in \{Z_{RL}\}$ or $Z(s) \in \{Z_{RC}\}$ followed by a slope-bounded nonlinearity satisfies the Popov integral condition (these classes represent driving-point impedances that can be realized with resistances and inductances or resistances and capacitances respectively)**
3. **Then the condition for the UASIL of the standard feedback system with slope-bounded nonlinearities is (after the usual finite sector transforms etc.):**

$$\frac{1+\overline{M}W(s)}{1+\underline{M}W(s)} \cdot Z(s) \in \text{SPR}$$

   where $Z(s) \in \{Z_{RL}\}$ or $\{Z_{RC}\}$

Monotonic Nonlinearities and RL/RC Multipliers - the Off-Axis Circle Criterion

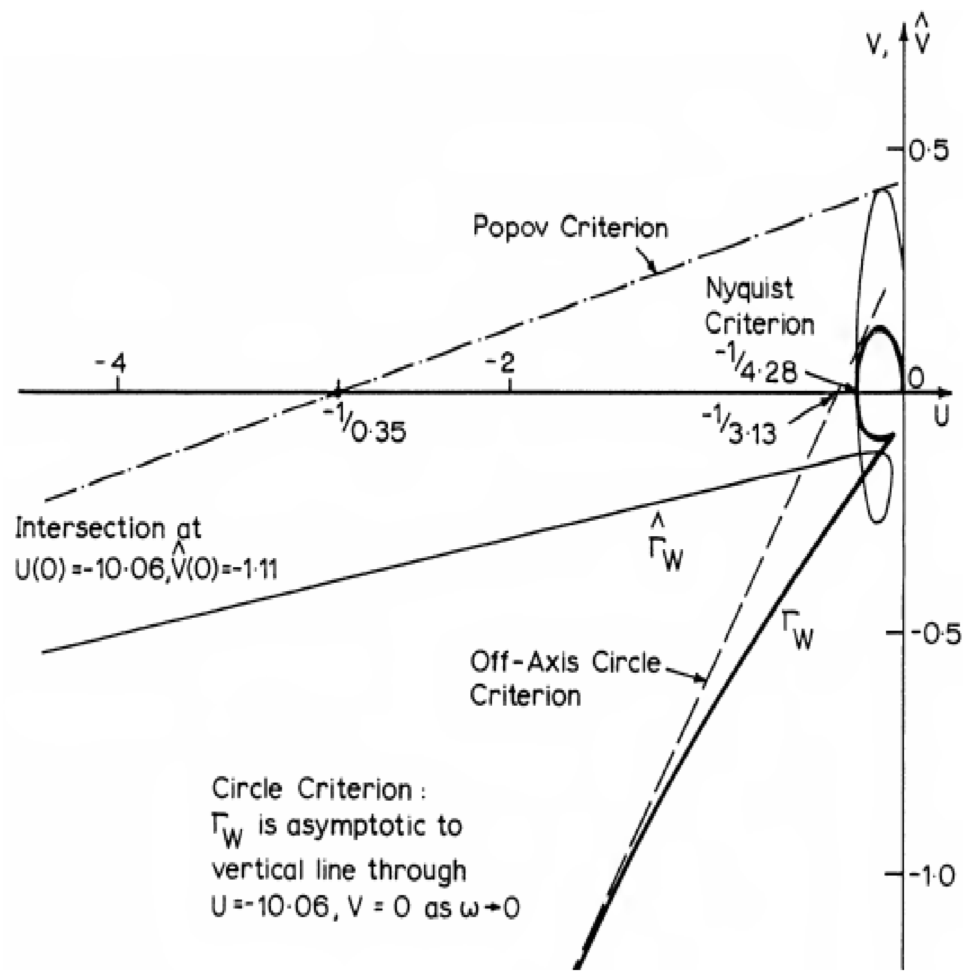
The existence of $Z(s) \in \{Z_{RL}\}$ or $\{Z_{RC}\}$ satisfying $\frac{1+\overline{M}W(s)}{1+\underline{M}W(s)} \cdot Z(s) \in \text{SPR}$ is guaranteed by a simple geometric condition imposed on the Nyquist plot:



- **Note that one <u>must not</u> draw the Nyquist locus for negative frequency (or you lose the benefit of this condition)**
- **The Off-Axis Circle Criterion is effective for "gentle" slope-bounded nonlinearities (no zig-zags); it reveals when the KalmanConjecture might be correct.**
- **Derived by Cho & Narendra, 1968, by showing at an RL or RC multiplier can always be found such that $(W(s)+1/\overline{M}) \cdot Z(s)$ is SPR; Cho was a fellow grad student**

Example 2: Comparison of Criteria

$$W(s) = \frac{3(s+1)}{s^2(s^2+s+25)}$$



CC:	$\mathbf{1} < \frac{g(\sigma,t)}{\sigma} < \mathbf{2.22}$	**PC:**	$\mathbf{1} < \frac{g(\sigma)}{\sigma} < \mathbf{2.70}$
OACC:	$\mathbf{1} < \frac{dg(\sigma}{d\sigma} < \mathbf{7.25}$	**NYQ:**	$\mathbf{0} < k < \mathbf{8.0}$

On the other hand, using the Popov Criterion with the finite sector transform $\rightarrow \mathbf{1} < \frac{g(\sigma)}{\sigma} < \mathbf{3.7}$

Summary and Conclusions

- **Absolute stability criteria <u>are</u> rigorous**
- **The exact (analytic) form of $g(\sigma, t)$ is not required - only "features" are important (e.g., its sector $(\underline{G}, \overline{G})$)**
- **There is no need for a state-space model of the system – only the frequency response is needed**
- **There are many more absolute stability criteria – some are not too useful (too complicated), however**
- **The multi-nonlinearity case has been dealt with in a parallel fashion – but most results lack a simple graphical interpretation**
- **Less restrictive results take a lot more work to obtain!**