EE 4323 - Industrial Control Systems
Module 8: Absolute Stability of Nonlinear Systems

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Lecture Outline

- Motivation
- Methods and Definitions
- Nyquist Revisited
- Problem of Lur'e
- Solution Due to Popov
- The Circle Criterion (CC)
- Significance of the Popov and Circle Criteria
- Examples

Acronyms: NL = nonlinear, NLTI = nonlinear time-invariant; NLTV = nonlinear time-varying; PR = positive real; SPR = strictly positive real; RHP = right half plane

References:

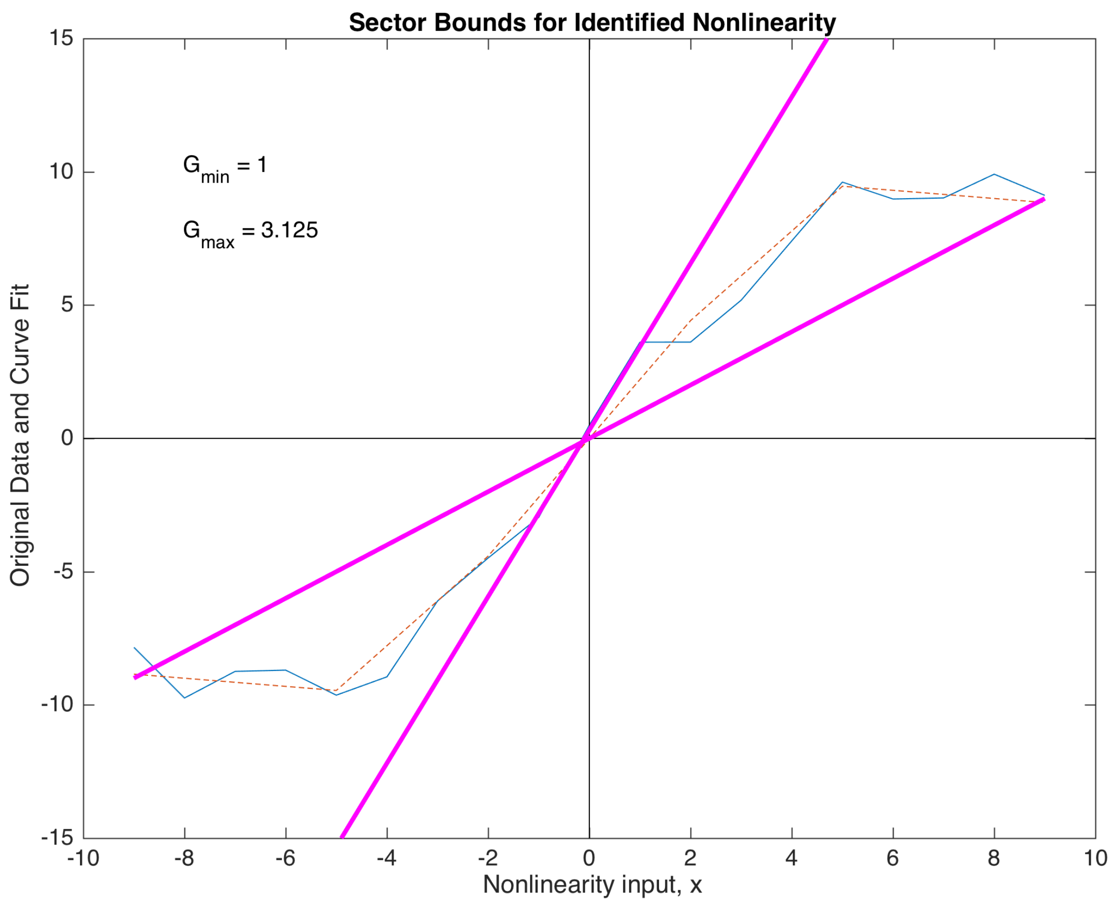
- Lefschetz, *Stability of Nonlinear Control Systems*, Academic, 1965.
- Aizerman & Gantmacher, *Absolute Stability of Regulator Systems*, Holden-Day, 1964.
- Narendra & Taylor, *Frequency Domain Criteria for Absolute Stability*, Academic, 1973.
- Narendra, ASME Books, Vol. 1, Chapter 2, 1978.
- Taylor, ASME Books, Vol. 2, Chapter 20, 1980.

Motivation

- Stability analysis is a serious business
- No loose method is fool-proof
  - Small-signal linearization
  - Gain sectors containing a nonlinearity (Aizerman conjecture)
  - Gain sectors based on max and min slope (Kalman conjecture)
  - Gain sectors based on a describing function
- There *are* rigorous methods
- $\star$ $\star$ Some of these are even easy to use!

Important Concept: Sector Bounds

The term **sector bounds** will be used throughout this module; here is an example (from static nonlinearity model ID):

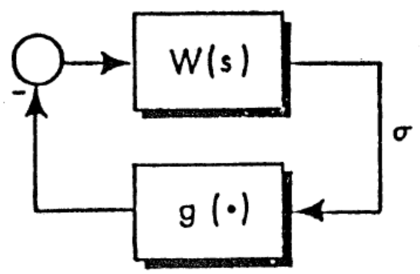


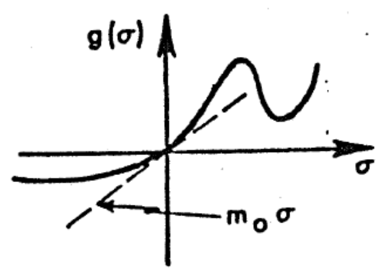
So, we say "this nonlinearity lies in the sector one to 3.125'. This, of course would be true **if we know or can insure that the input is constrained to** $-8 < x < 8$

Small-signal Linearization

LOOSE METHOD 1:

SMALL-SIGNAL LINEARIZATION (TAYLOR SERIES)





MISCONCEPTION: $-1/m_0 \notin \mathcal{W}_R$ MEANINGFULLY GUARANTEES STABILITY

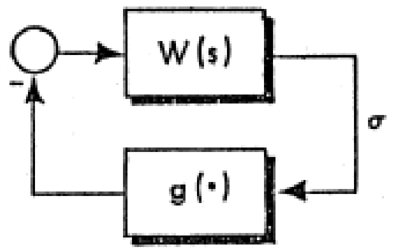
PROBLEM: THE ABOVE CONDITION GUARANTEES <u>LOCAL</u> STABILITY ONLY (ALSO CALLED <u>INFINITESIMAL</u> STABILITY)

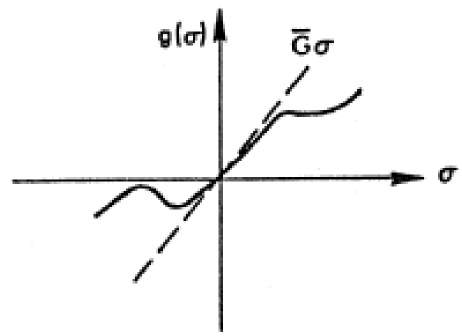
The Aizerman Conjecture

LOOSE METHOD 2:

GLOBAL GAIN SECTOR LINEARIZATION

(THE AIZERMAN CONJECTURE ... 1949)

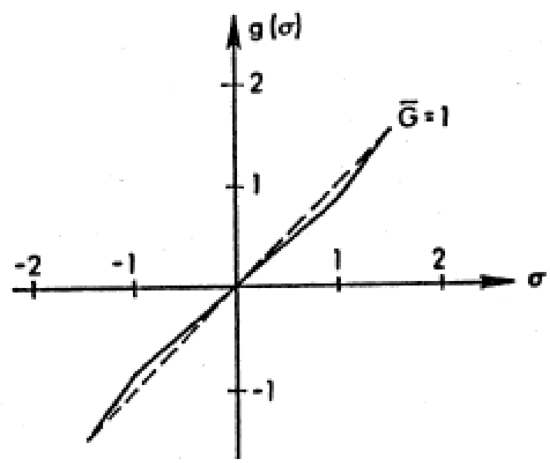




CONJECTURE : $-1/k \notin \mathcal{W}_R$ FOR $0 \le k < \bar{G}$ GUARANTEES ASYMPTOTIC STABILITY.

COUNTEREXAMPLE:

$$W(s) = \frac{-(s+1)}{(s^2+s+1)}$$

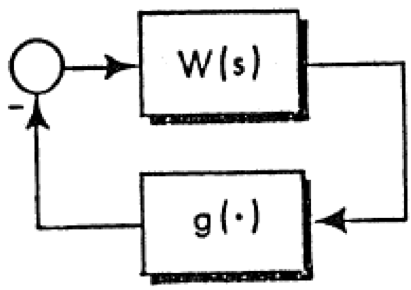


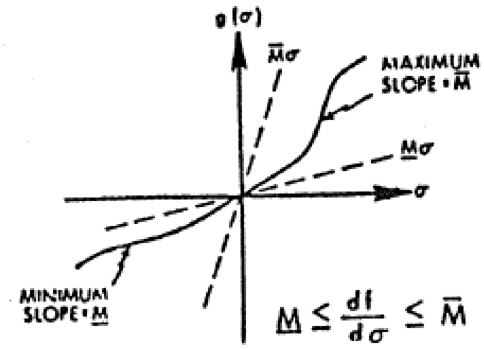
The Kalman Conjecture

LOOSE METHOD 3:

GLOBAL INCREMENTAL LINEARIZATION

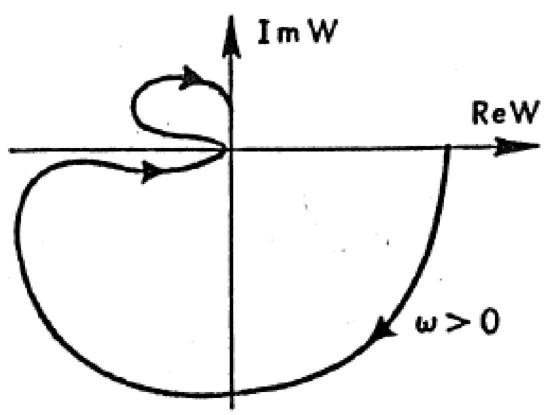
(THE KALMAN CONJECTURE ... 1957)

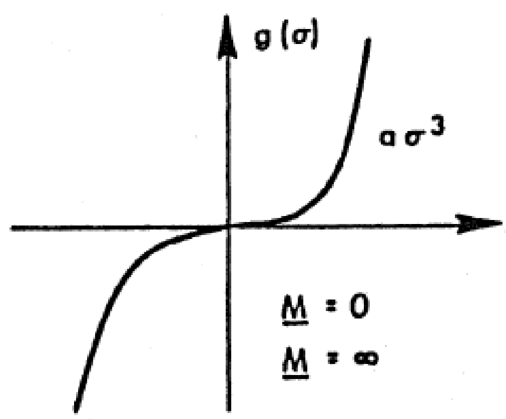




CONJECTURE : $-1/k \notin \mathcal{W}_R$ FOR $\underline{M} \le k \le \bar{M}$ GUARANTEES STABILITY.

COUNTEREXAMPLE :





The Bottom Line: All Loose Methods are Risky . . .

- especially if the system is of high order, and
- especially if the system has more than one important nonlinearity

More sophisticated "loose methods" (e.g., the Kalman Conjecture) are "safer", however – but why gamble?

Our Definition of Stability - UASIL

- **Given a NLTV system** $\dot{x} = f(x,t)$ **with equilibrium** $x = 0$**;**

- **The system is Uniformly Asymptotically Stable in the Large (UASIL) if:**

  1. For every $\epsilon > 0$ and t_0 there exists a $\delta(\epsilon) > 0$ such that $\lim_{\epsilon \to \infty} \delta(\epsilon) = \infty$, and $||x_0|| \leq \delta \Rightarrow ||x(t; x_0, t_0)|| \leq \epsilon$ for all $t \geq t_0$

  2. For some $\rho > 0$ and for every $\eta > 0$ there exists a $T(\eta, \rho)$ such that $||x(t; x_0, t_0)|| \leq \eta$ for all $||x_0|| \leq \rho$ and $t \geq t_0 + T$

This is a conservative (safe) definition for engineering purposes.

Another definition (stated informally): any bounded input must result in a bounded output; the criteria given here are sufficient to guarantee either definition.

Note: there are **many** definitions of stability for NLTV systems that vary in subtle but important ways.

Nyquist Criterion Revisited

Given an open-loop transfer function $W(s) = \frac{(s+20)(s+30)}{(s+1)(s^2+2s+10)(s+200)}$:

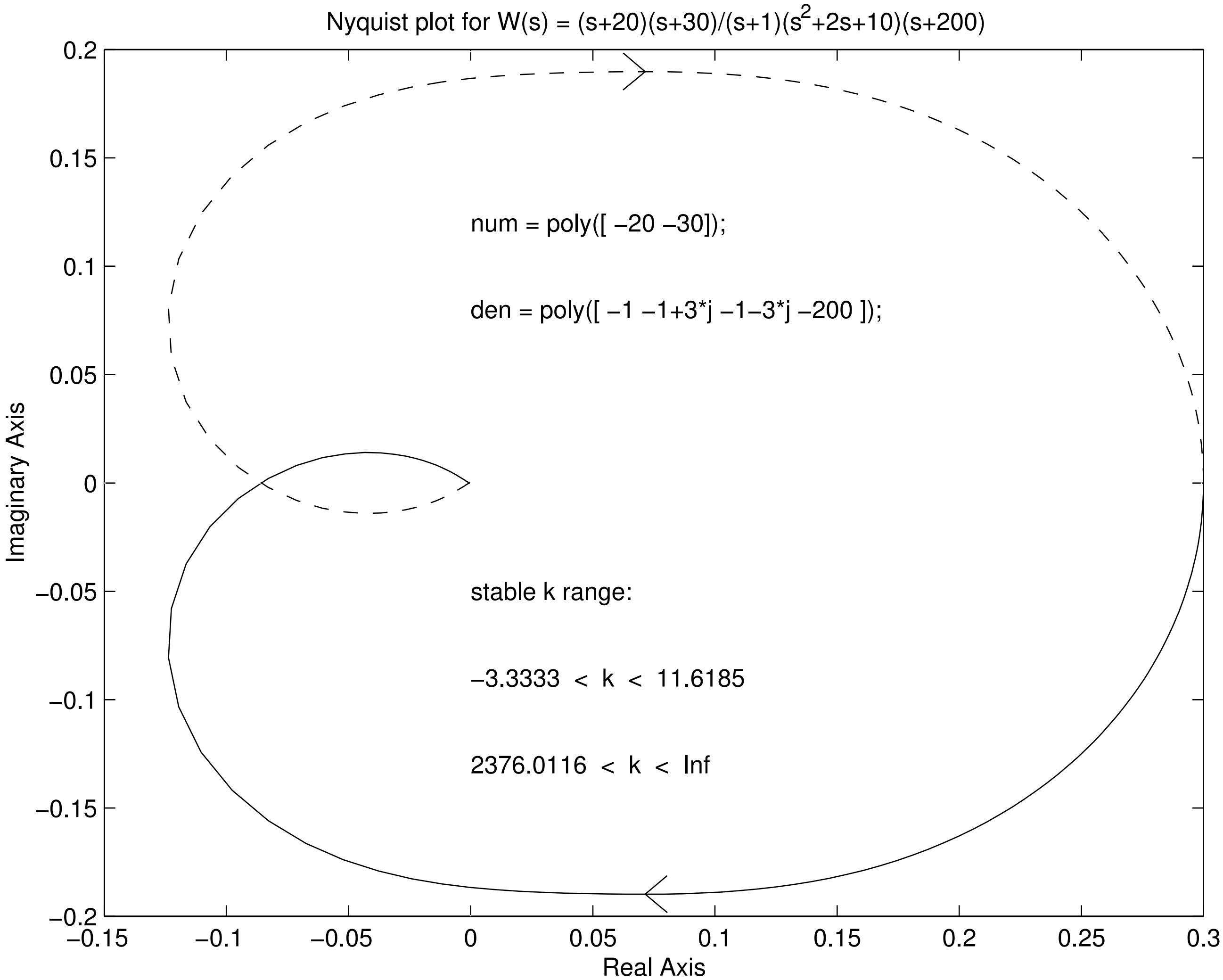


Figure 1: Condition for Asymptotic Stability: $-1/k \notin \mathcal{W}_{\mathcal{R}}$

The maximum useful stability range is -3.33 $< K <$ 11.62; if you want the "safety" of a gain margin of 5 (14 dB) then pick $K = 2.32$, etc.

Nyquist Criterion Revisited (continued)

Example: Consider the unstable plant: $W(s) = \frac{s+2}{s^2-4s-5}$

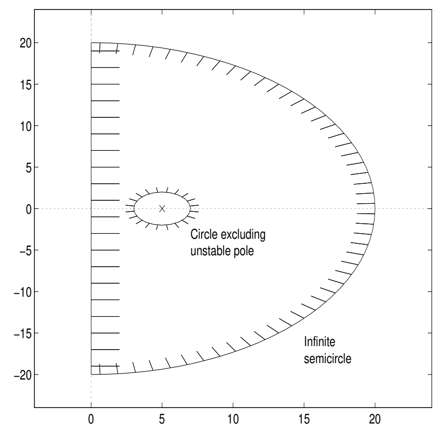


Figure 2: s-plane region $\mathcal{R}$ mapped for Nyquist criterion

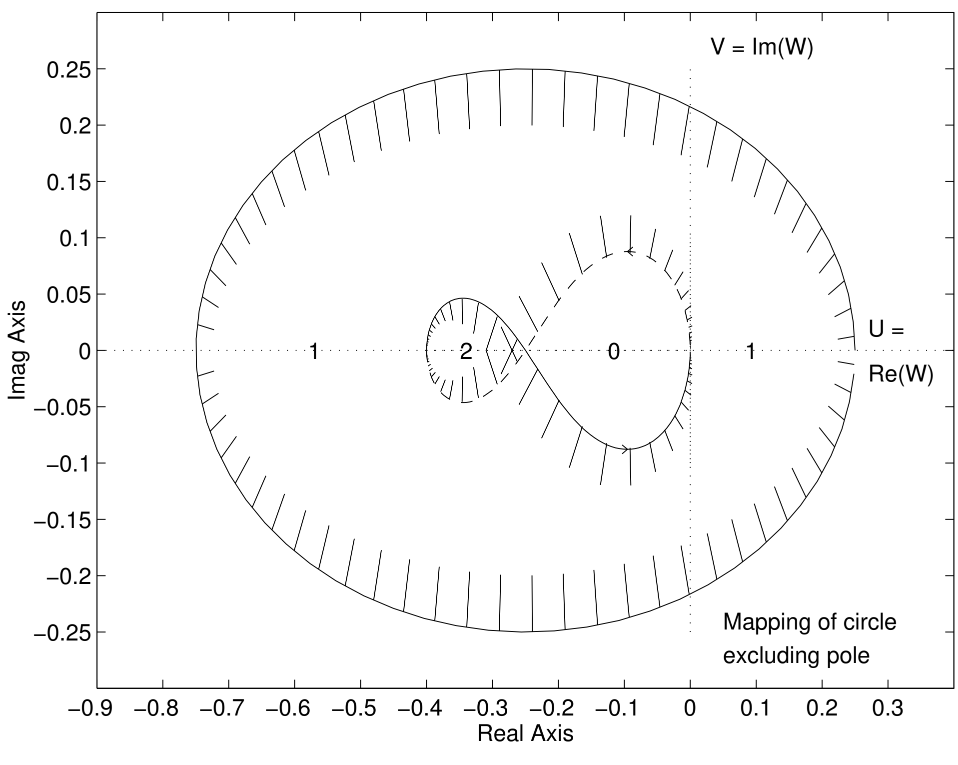


Figure 3: $W(s)$-map $\mathcal{W}_{\mathcal{R}}$ for the Nyquist criterion

Condition for asymptotic stability: $-1/k \notin \mathcal{W}_{\mathcal{R}}$

A New MATLAB Nyquist Tool

Another example: Consider a simple stable plant:

$$W(s) = \frac{s+1}{s^4 + 2s^3 + 25s^2 + 3s + 1} \qquad (1)$$

yields

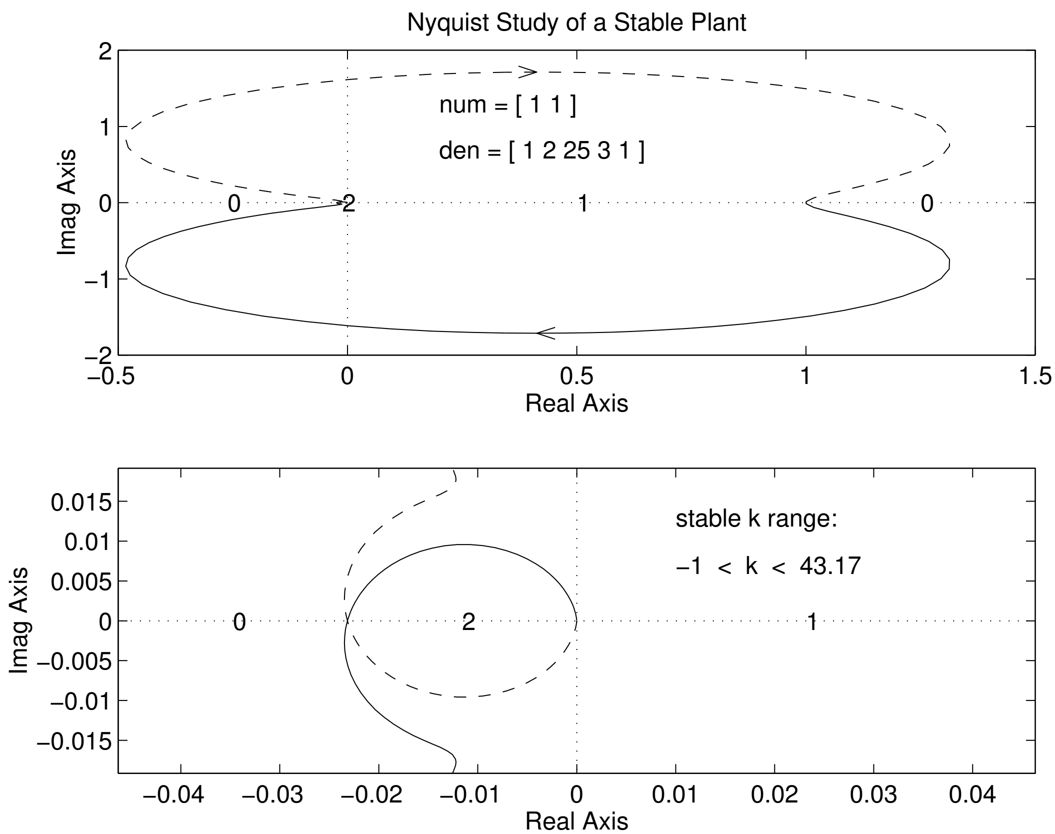


Figure 4: Nyquist Criterion Example (Stable Plant)

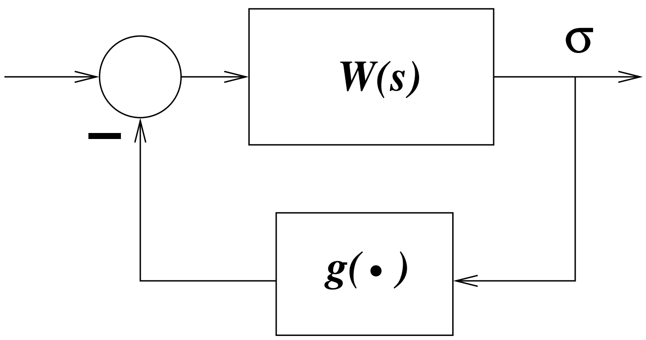
The report that **`newnyq`** provides is:

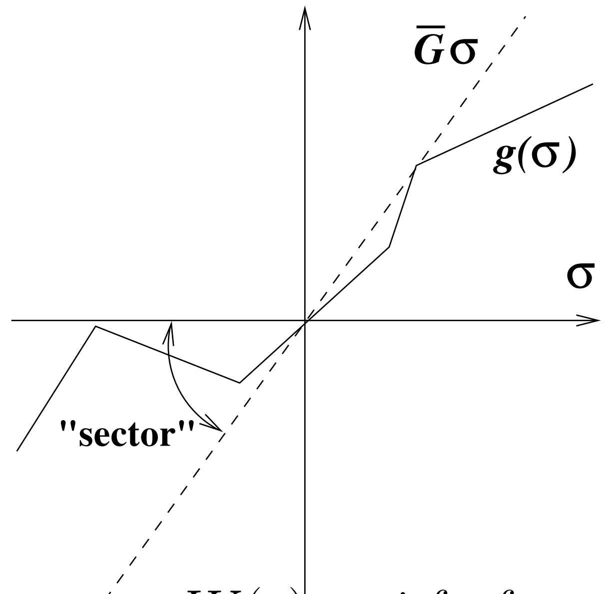
```
>> newnyq(num,den)

stable k range
-1  <  k  <  43.17
```

Nyquist Criterion Revisited (continued) - Why it is So Important in Practice

- You do not need a precise analytic model – just $W(j\omega)$
- You have a direct graphical interpretation of the impact of uncertainty
- Experimental frequency-response data is directly useful without a need to assume system order and curve fit

The Problem of Lur'e & Postnikov (1944)





- Question: What constraint must $W(s)$ satisfy for UASIL, given only that $0 < \frac{g(\sigma)}{\sigma} < \overline{G}$ (or "$g(\sigma)$ lies in the sector $(0, \overline{G})$")?
- This is called the **Absolute Stability Problem**; if $W(s)$ meets a certain constraint the system is said to be **Absolutely Stable**.
- Aizerman conjectured that satisfying the Nyquist criterion for $k \in [0, \overline{G}]$ guarantees stability; this has been disproved by counterexamples.
- Note: the nonlinearity is *time invariant*

Popov's Solution to the Lur'e-Postnikov Problem (1961)

- **The L-P system is <u>absolutely stable</u> if**:
  1. $W(s)$ is **stable**, and
  2. a **real** q **exists** such that

  $$T(s) = [W(s) + 1/\overline{G}] \cdot (1 + qs)^{\pm 1} \in \mathrm{PR}$$

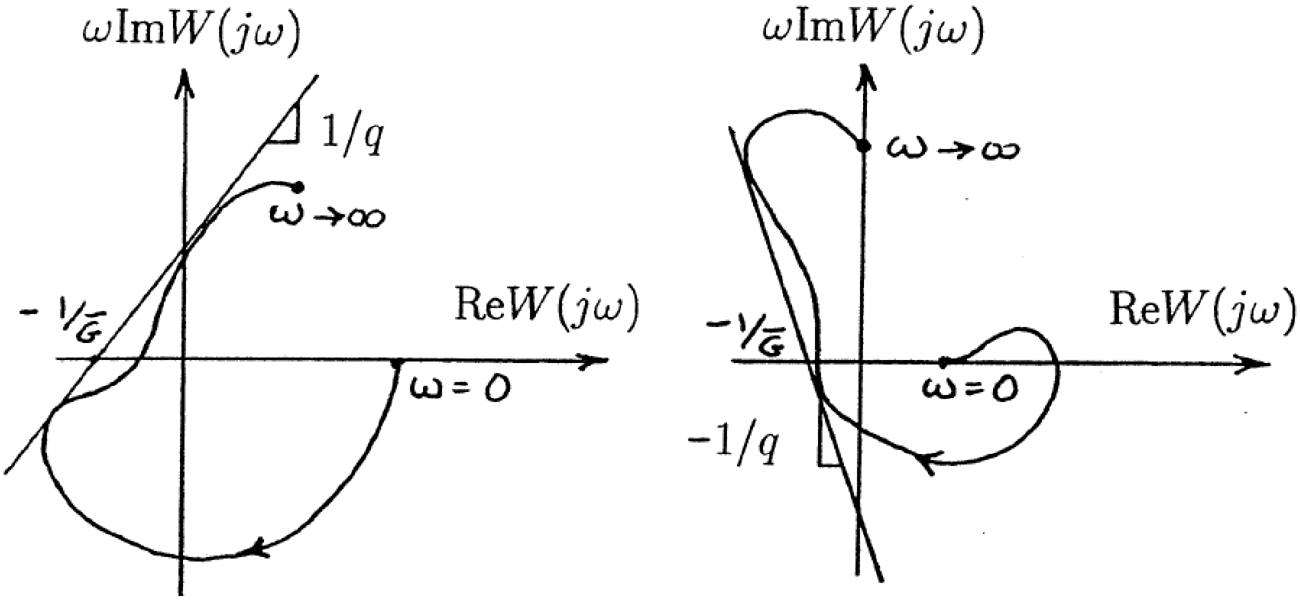
  where $(\ \cdot\ )^{\pm 1}$ denotes multiplication by $(1 + qs)$ or $1/(1 + qs)$ and $\in$ PR signifies that $T(s)$ is **<u>positive real</u>**

- **Definition**: $T(s) \in \mathrm{PR} \Leftrightarrow \mathrm{Re}\, T(s) \geq 0$ for all $\mathrm{Re}\, s \geq 0$ (for all $s \in \mathcal{R}$)

- $T(s) \in \mathrm{PR} \Rightarrow T(s)$ has no poles or zeroes in the RHP; if so, you only need to consider $T(j\omega)$.

- **Important**: This condition is <u>*sufficient*</u> but not <u>*necessary*</u>, i.e., if the condition is not met that does not mean that the L-P system is unstable (contrary to the Aizerman conjecture).

Geometrical Interpretation



These are <u>not</u> Nyquist plots!

To show this, define $W(j\omega) = U + jV$; then

$$T(j\omega) = (U + \frac{1}{G} + jV) \cdot (1 + qj\omega);$$

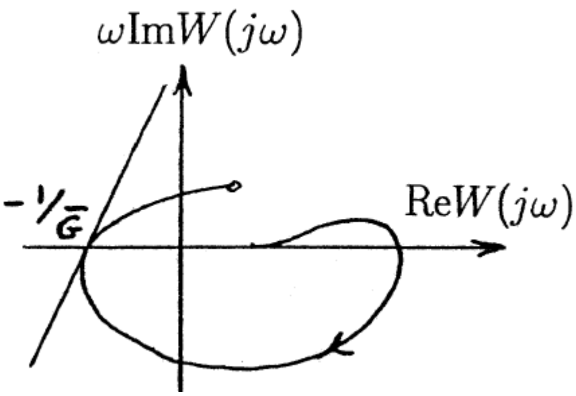
making the real part ≥ 0 requires

$$\omega V \leq (U + \frac{1}{G})/q \tag{2}$$

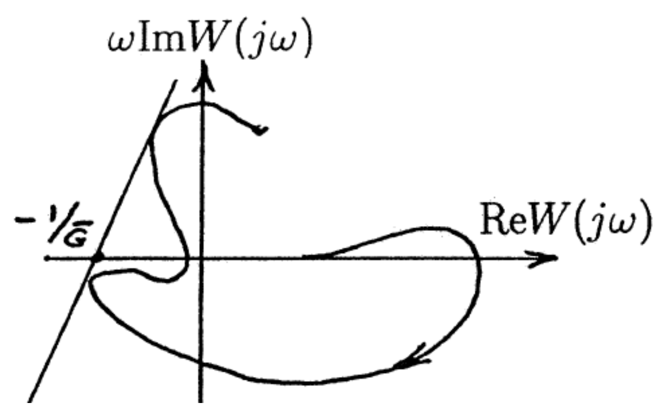
Relation of the Popov Criterion to the Aizerman Conjecture

The Aizerman conjecture is shown to be valid for any linear plant that satisfies the condition that the point $(-1/\overline{G},\ 0)$ lies both on the Popov plot and the Popov line:

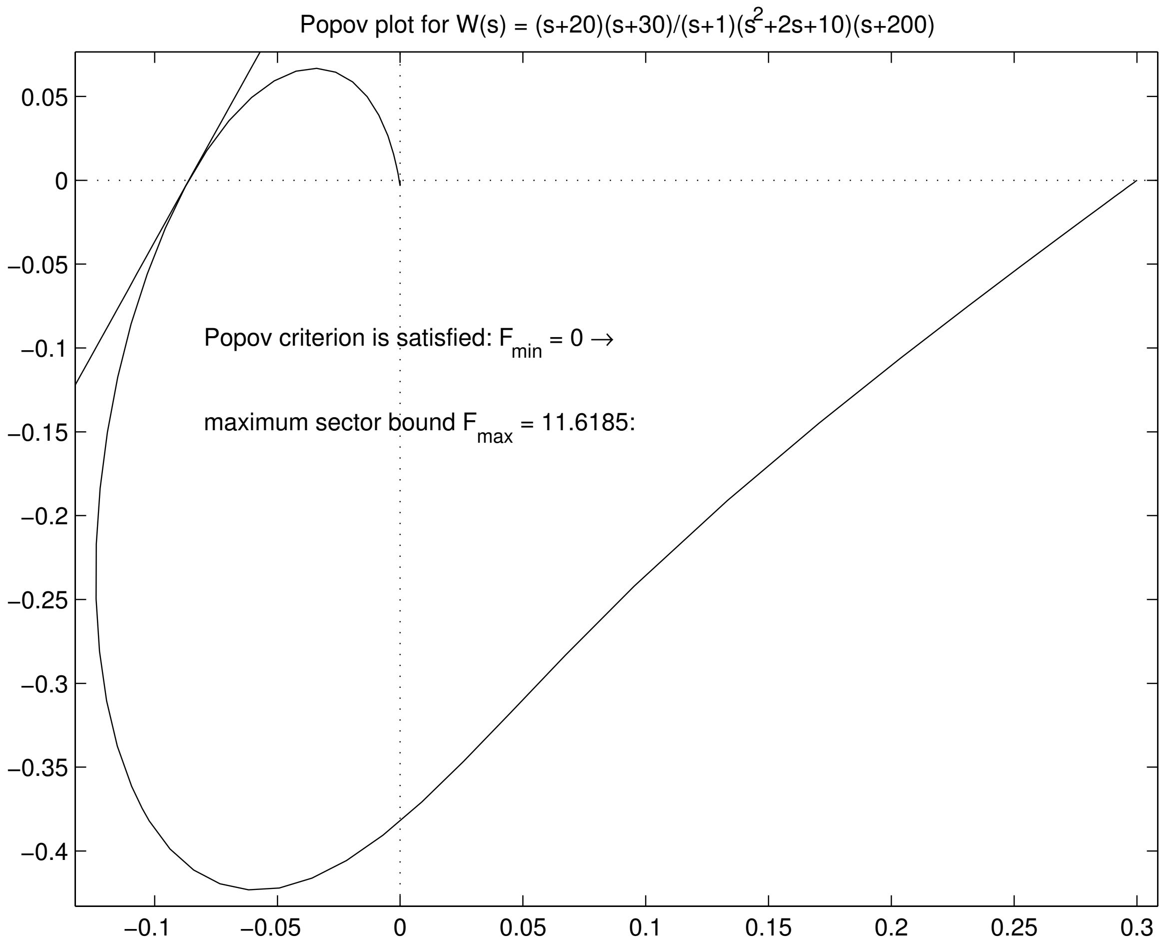
YES



NO



Popov Criterion Example – Another New matlab Tool



The closed-loop system with $W(s)$ and $g(\sigma)$ in the loop is *guaranteed* to be UASIL (absolutely stable) as long as $0 < \dfrac{g(\sigma)}{\sigma} < F_{\max} = 11.62$.

The upper bound $F_{\max}$ is equal to the Nyquist bound $K_{\max}$ *for this case*! As with Nyquist, if $F_{\max} = 2.32$ we have a *gain margin* of 5 (14 dB).

The Circle Criterion

The NLTV generalization of the Lur'e-Postnikov problem:

$$g(\sigma) \to g(\sigma, t)$$

- The NLTV system is UASIL if:
  1. $\underline{G} < \frac{g(\sigma,t)}{\sigma} < \overline{G}$
  2. $T(s) = \frac{1+\overline{G}W(s)}{1+\underline{G}W(s)} \in \text{SPR}$

- **Definition**: $T(s) \in \text{SPR}$ ($T(s)$ is <u>strictly positive real</u>) $\Leftrightarrow$

  $$\text{Re } T(s-\epsilon) \geq 0 \ \forall \ \text{Re } s \geq 0, \ \text{ for an arbitrarily small } \epsilon > 0$$

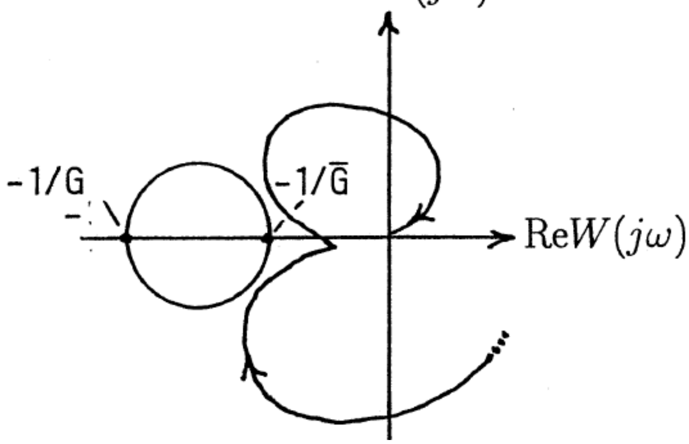
- This implies that there are <u>no poles or zeros in the RHP</u>and that Re $T(j\omega) > 0 \ \forall \omega$; this condition does <u>not</u> require that Re $T(j\omega) \geq \epsilon > 0 \ \forall \omega$

- This definition is dictated by the T-K-Y Lemma; example: $\frac{1}{s+a} \in \text{SPR}$ by this definition

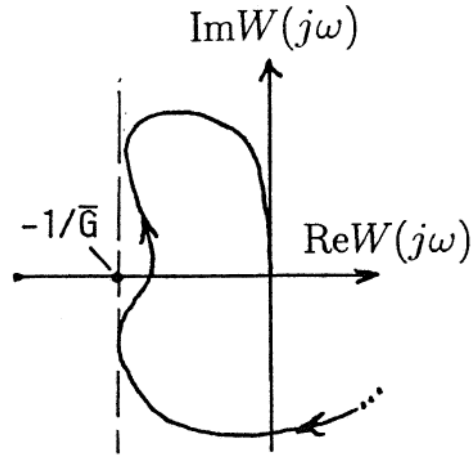
- **Important**: This condition is <u>*sufficient*</u> but not <u>*necessary*</u>, i.e., if the condition is not met that does not mean that the NLTV system is unstable.

Geometrical Interpretation

General Case:



Special Case, $\underline{G} = 0$:



These <u>are</u> Nyquist plots!

To show this, define $W(j\omega) = U + jV$; then $\mathrm{Re}\, T(j\omega) > 0$ if:

$$\mathrm{Re}\,(U + \frac{1}{\overline{G}} + jV) \cdot (U + \frac{1}{\underline{G}} - jV) > 0$$

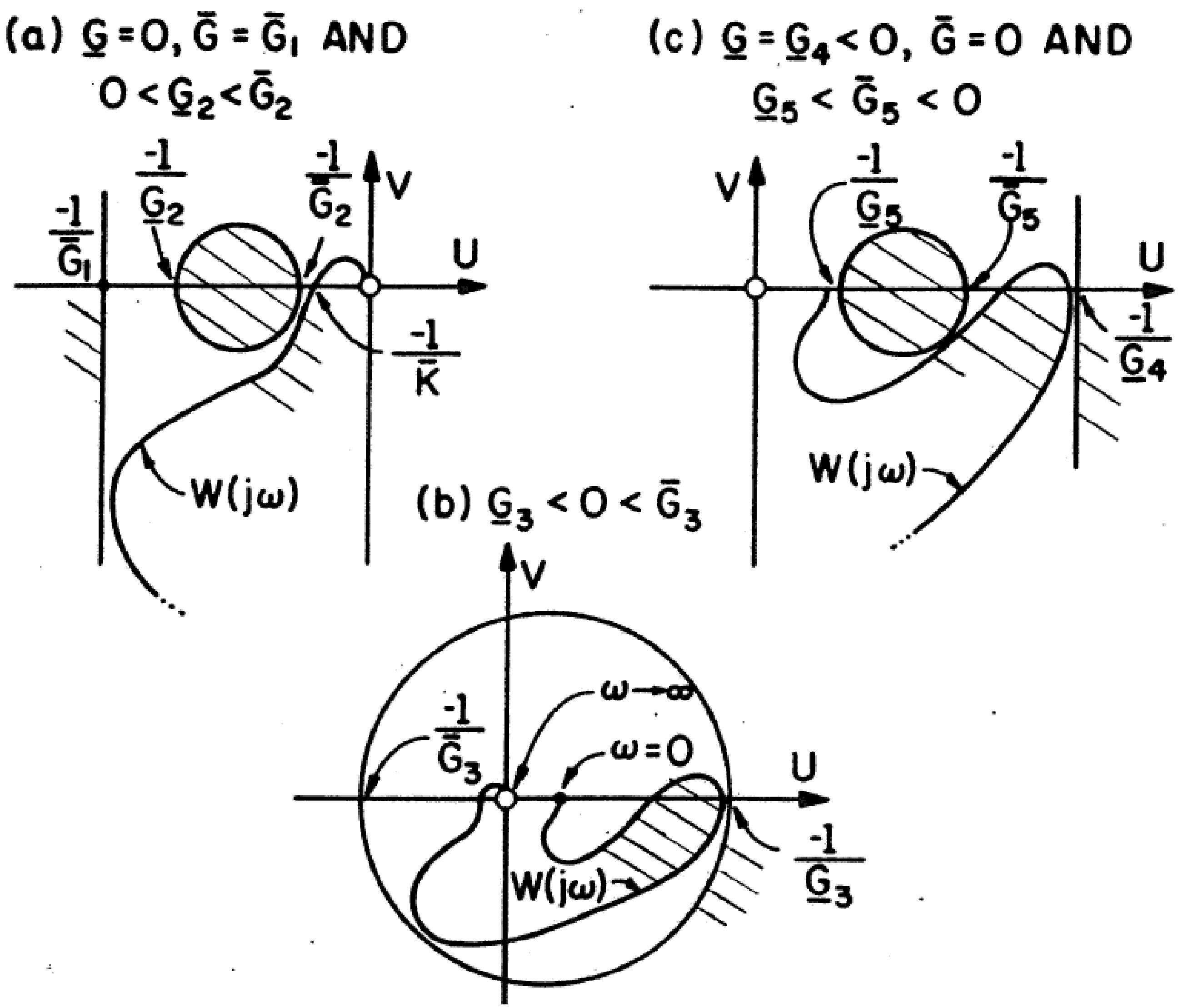
(assuming $0 < \underline{G} < \overline{G}$), therefore, constraining the real part of $T(j\omega)$ to be positive requires

$$(U + \frac{1}{\overline{G}}) \cdot (U + \frac{1}{\underline{G}}) + V^2 > 0 \tag{3}$$

which requires $U+jV$ to avoid the interior of a circle whose <u>diameter</u> is defined by the points $-1/k$ for $\underline{G} \leq k \leq \overline{G}$.

Geometrical Interpretation (Cont'd)

An **infinite number** of circles can be drawn, so one can (for example) trade off $\underline{G}$ against $\overline{G}$.



Be sure that the "interior" of the circle is not in $\mathcal{W}_{\mathcal{R}}$!
Be careful if $\underline{G} < 0 < \overline{G}$!

The Circle Criterion and "M-Circles"

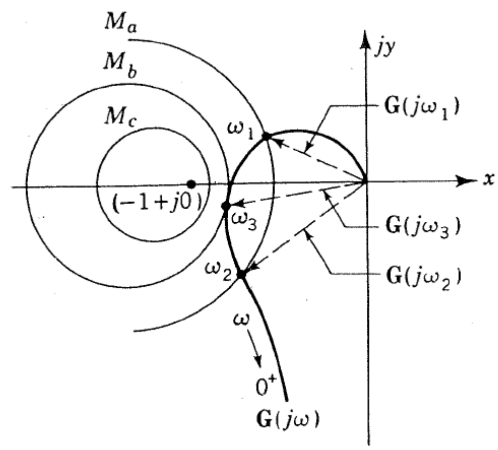


FIGURE 9-9
M contours and a $\mathbf{G}(j\omega)$ plot.

Center: $x_0 = -\frac{M^2}{M^2-1}$

Radius: $r_0 = |\frac{M}{M^2-1}|$

Example 1: $M = 2 \rightarrow$
$x_0 = -4/3\,,\ r_0 = 2/3$

Relation to the CC:

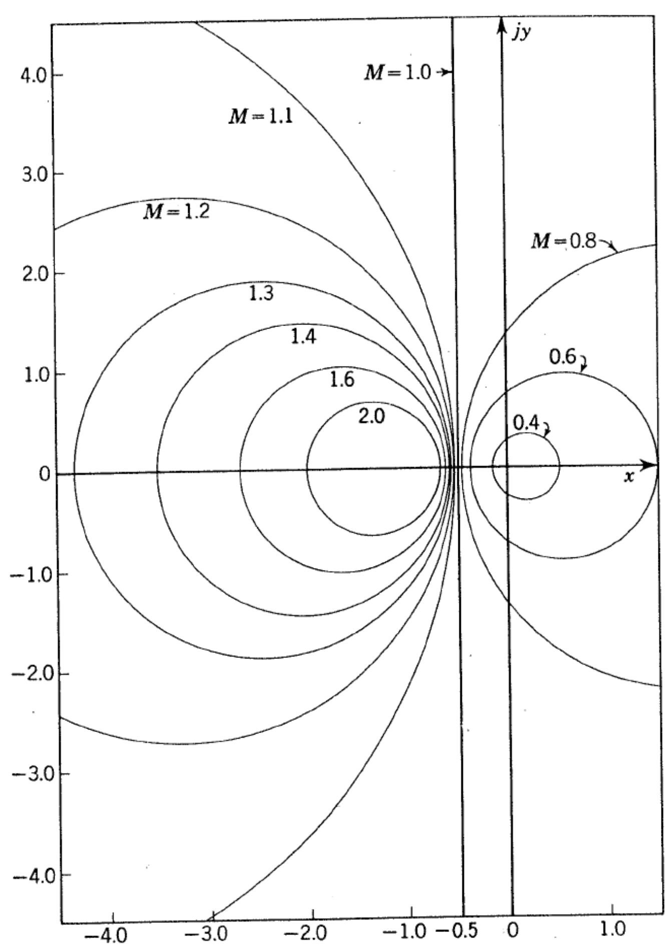
$\underline{G} = 0.5\,,\ \overline{G} = 1.5$

Example 2: $M = 1.2 \rightarrow$

$\underline{G} = 0.1667\,,\ \overline{G} = 1.833$

Example 3: $M = 1.0 \rightarrow$

$\underline{G} = 0\,,\ \overline{G} = 2$

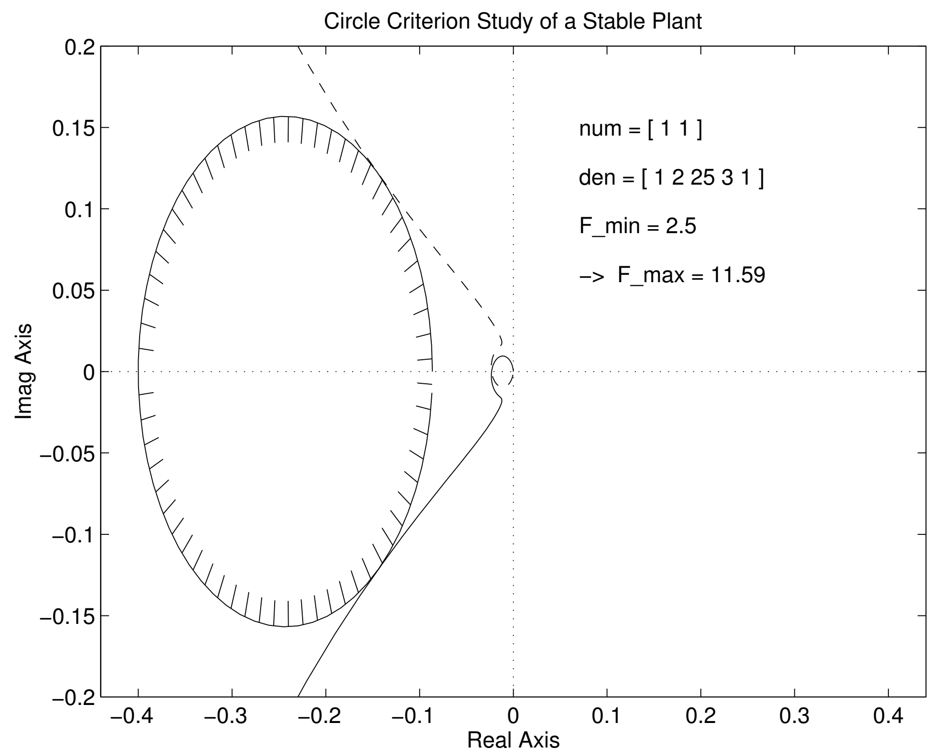


Circle Criterion – A MATLAB Tool

Example: Consider the relatively simple stable plant:

$$W(s) = \frac{s+1}{s^4 + 2s^3 + 25s^2 + 3s + 1} \tag{4}$$

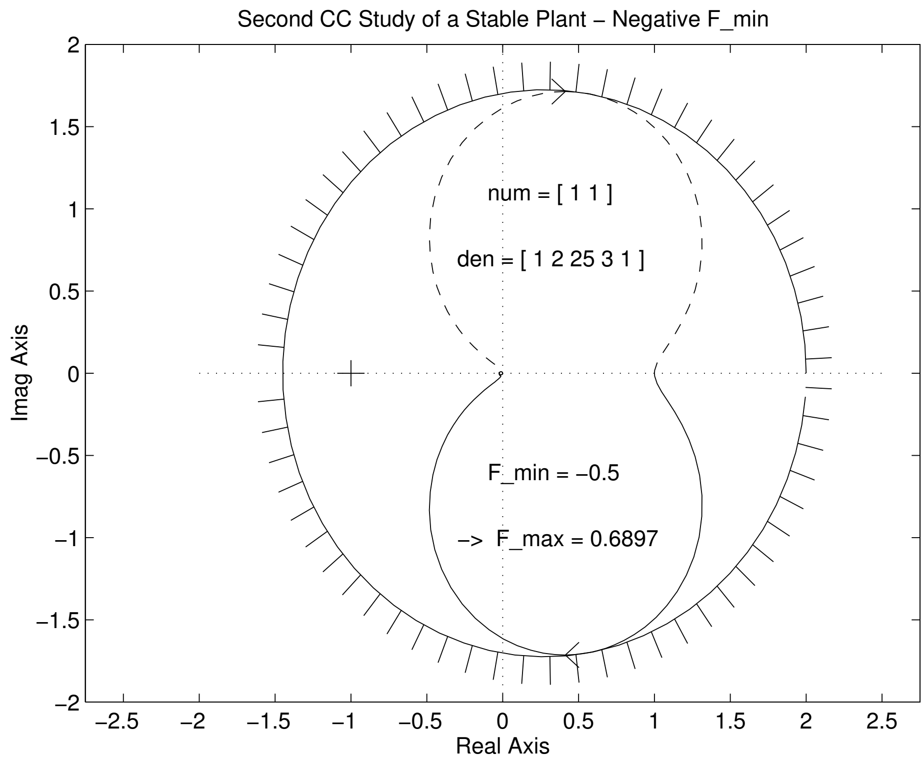
yields

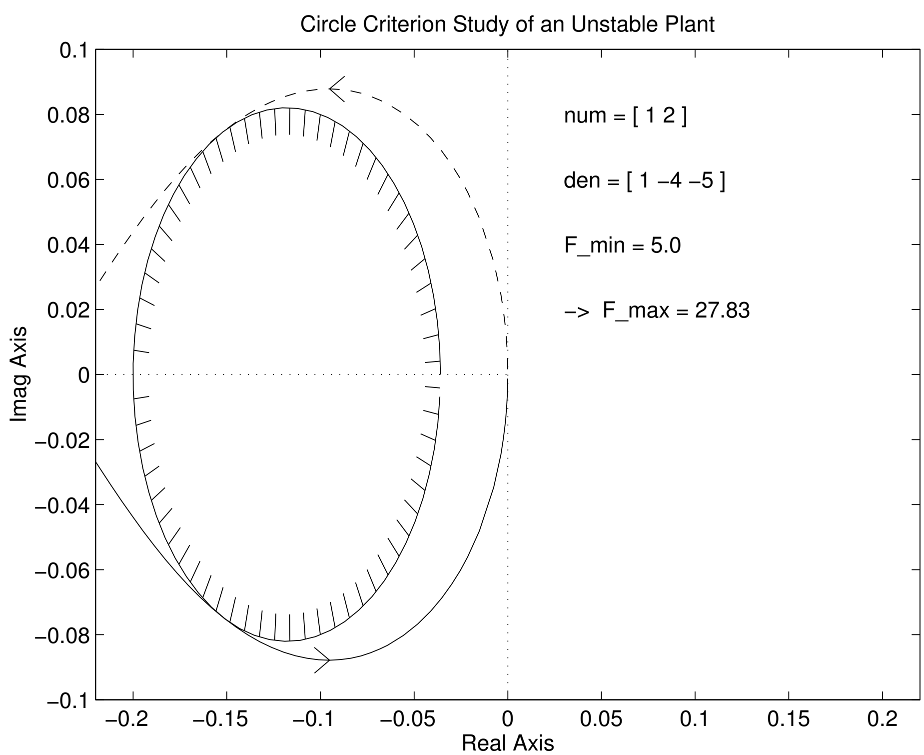


The report that `circle` provides is:

```
>> circle(num,den,2.5)

stable k range
-1  <  k  <  43.17
circle criterion is satisfied
maximum sector bound F_max = 11.59
```

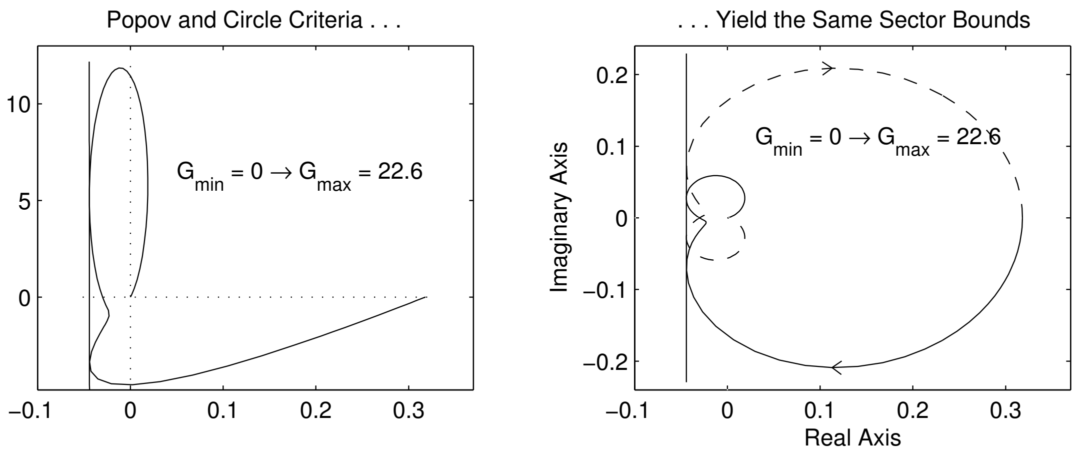
A MATLAB Tool – More Examples



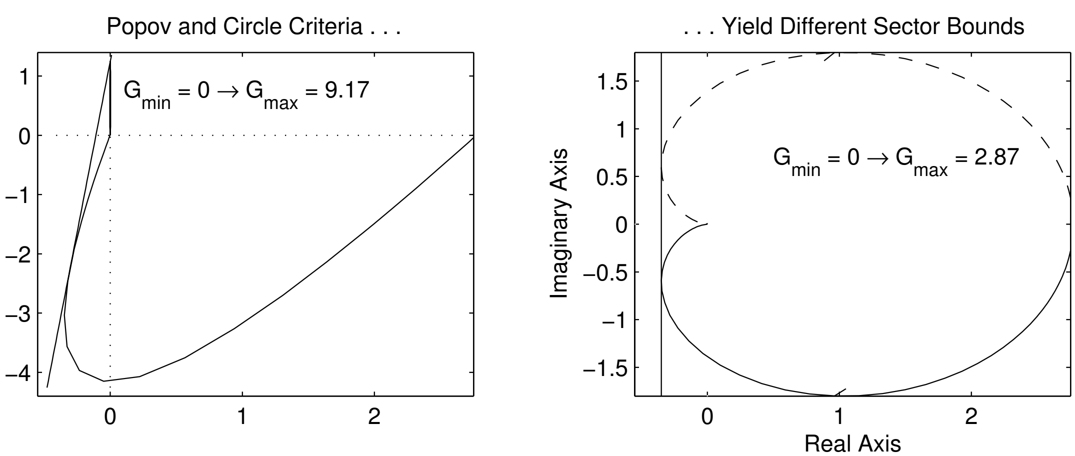


Comparing the Popov and Circle Criteria

- The sector bounds may be the same ...



- or $\overline{G}^{\text{Popov}}$ may be substantially larger than $\overline{G}^{\text{Circle}}$



- Since the negative real axis crossings and minimum values of $\text{Re}(W(j\omega))$ are the same the Circle Criterion can **never** provide a $\overline{G}$ larger than the Popov Criterion

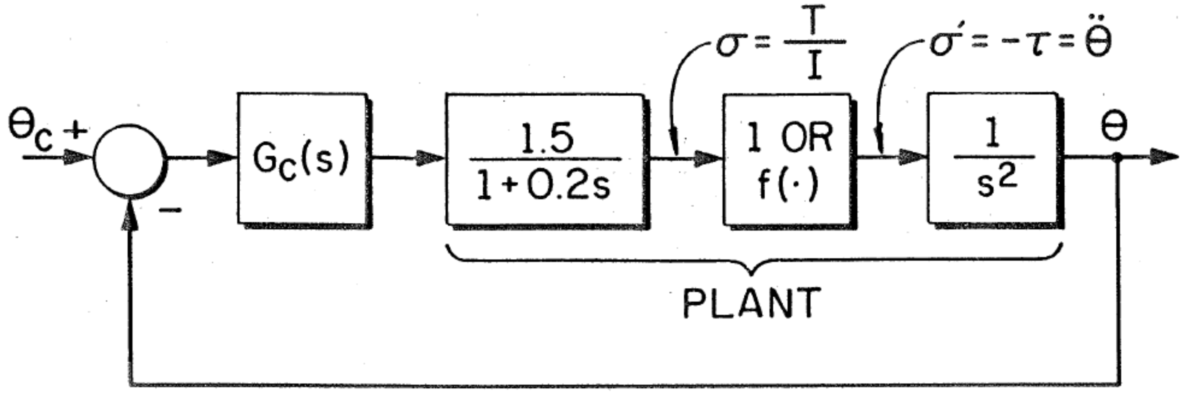
Significance of the Popov and Circle Criteria

- You do not need a precise analytic model for either the plant or the nonlinearity
- You have a direct graphical interpretation of the impact of uncertainty
- Experimental frequency-response data is directly useable without a need to assume system order and curve fit

(These points look familiar, don't they?)

Example 1: A Simple "Real World" Problem

Problem: Design a bf position control system, and assess the possible impact of nonlinearity in the plant (e.g., saturation):

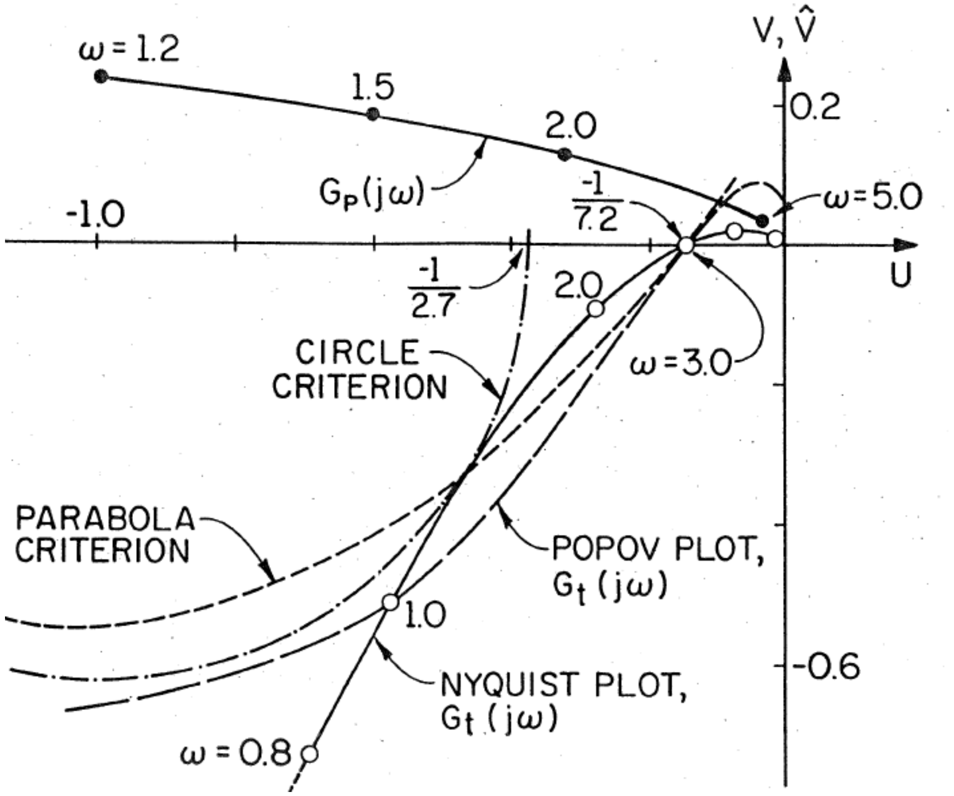


- *Plant*: Inertial load, $1\,/\,J\,s^2$, with additional motor lag
- *Design specs*: $M_M = 1.3$, Bandwidth = 1 rad/sec
- Standard linear compensator design methods ($f(\sigma) = \sigma$) call for using a lead compensator; note that the plant is unstable for any $K \leq\ 0$. This yields:

$$G_c(s) = 1.132\ \frac{s + 1/5.55}{s + 2} \rightarrow G_t = 8.49\ \frac{s + 1/5.55}{s^2(s+2)(s+5)}$$

Example 1 (Position Control System - Cont'd)

Now, assess the possible impact of nonlinearity:



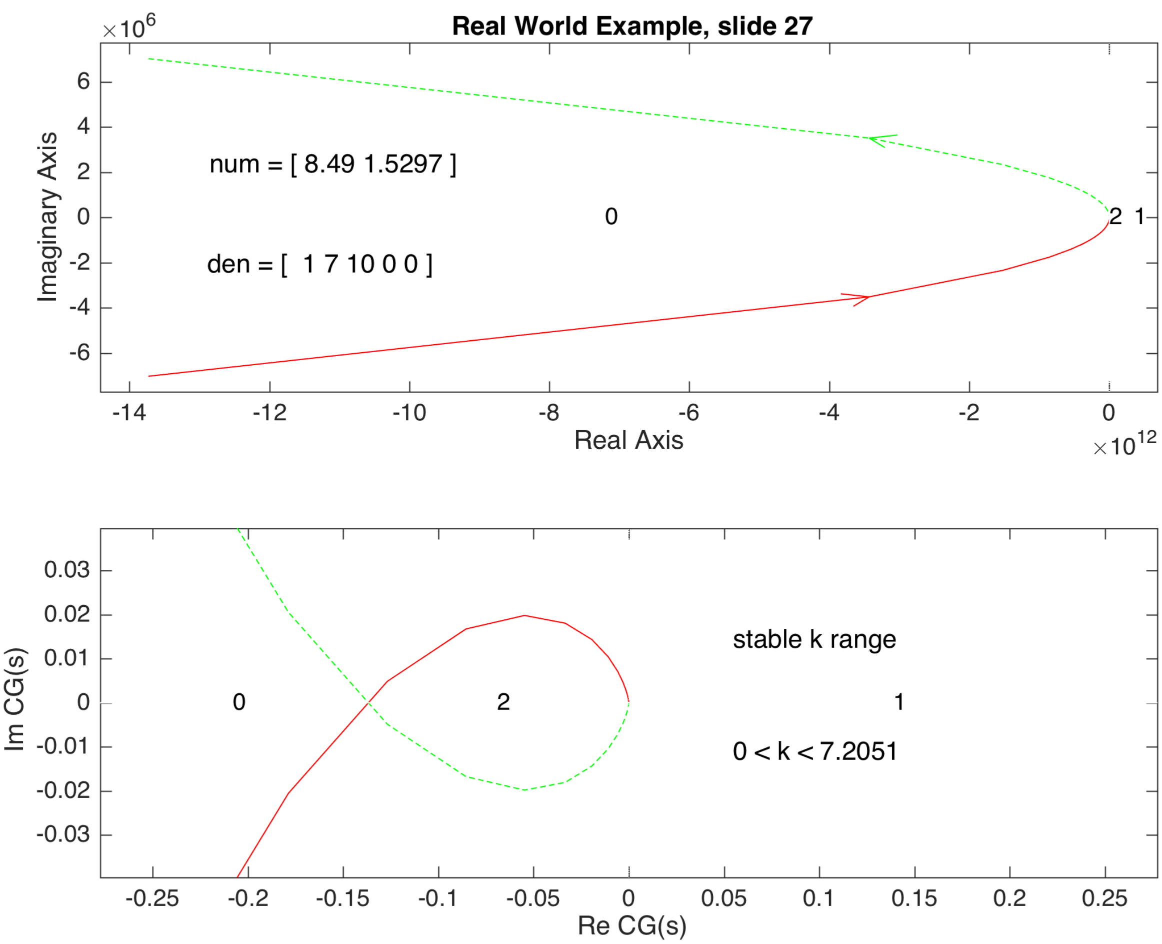
- Nyquist: $0 < k < 7.2$
- Parabola Criterion: $0.54 < \frac{g(\sigma)}{\sigma} < 7.19$ (NLTI!)
- Circle Criterion: $0.61 < \frac{g(\sigma,t)}{\sigma} < 2.7$ (NLTV!!)

(From a pre-MATLAB "by hand" analysis; the Parabola Criterion is an **obsolete** extension of the Popov Criterion)

Example 1: Position Control System - Cont'd

Finally, assess the possible impact of nonlinearity using MATLAB tools:

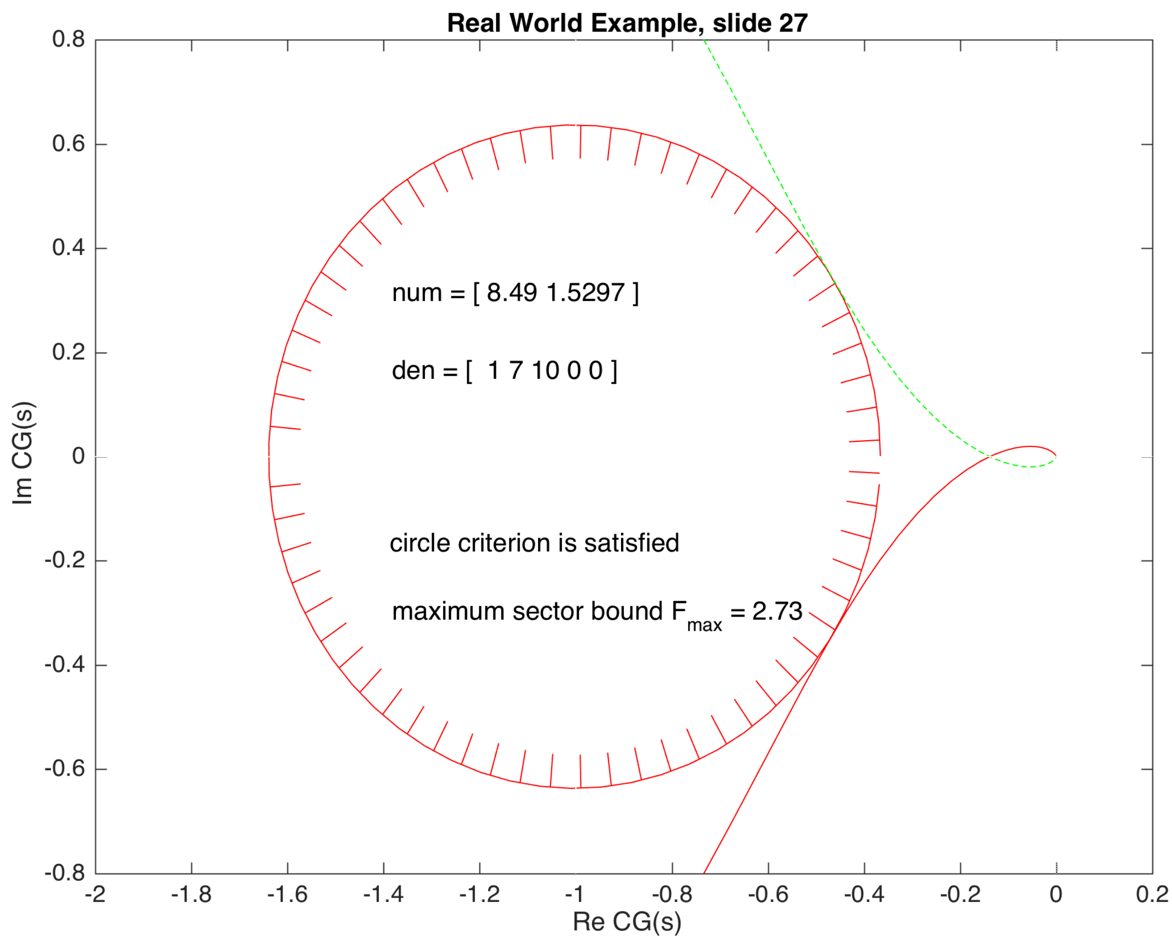
- First, check the stability range using `newnyq`:



So far, so good – the same previous upperbound is given by `newnyq`

Example 1: Position Control System - Cont'd

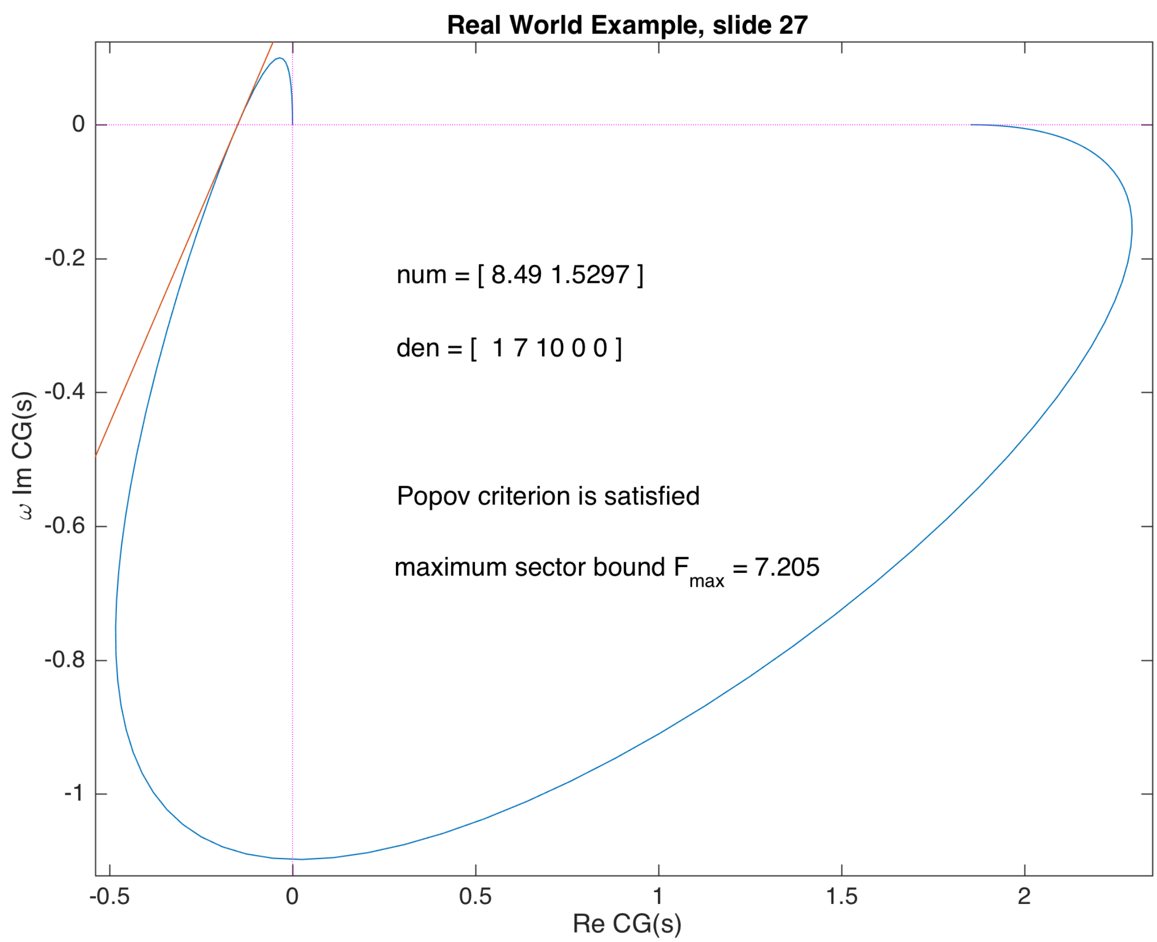
- Now, check the `circle` criterion result:



This also comfirms the "by hand" result

Example 1: Position Control System - Cont'd

- Lastly, check the **popov** criterion result:



Even the "by hand" parabola criterion was very close!