EE 4323 – Nonlinear Control Systems

Module 10: Sinusoidal-Input Describing Function Methods – SIDF Analysis

James H. Taylor
Department of Electrical & Computer Engineering
Fredericton, New Brunswick, Canada E3B 5A3
E-mail: jtaylor@unb.ca Web site: www.ece.unb.ca/jtaylor

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Topic Outline

- Introduction
- Basic Concepts
- SIDF Calculations
- Qualitative Behavior of SIDFs; Heuristics
- Classical Harmonic Balance (Limit Cycle Conditions)
- Classical SIDF I/O Analysis ("Transfer Functions")
- Examples
- Systems with Multiple Nonlinearities
- Modern Algebraic SIDF Methods
- Examples

References:

- D. P. Atherton, *Nonlinear Control Engineering*, Van Nostrand, 1975 (Reprinted as Student Edition, 1982).
- A. Gelb & W. Vander Velde, *Multiple-Input DF's and Nonlinear System Design*, McGraw-Hill, 1968.
- J. H. Taylor, *Describing Functions*, an article in the *Electrical Engineering Encyclopedia*, John Wiley & Sons, Inc., New York, 1999.

Introduction

- Problem to be addressed: Analyzing Periodic Phenomena
  - **Nonlinear oscillations** (limit cycles)
  - **Response of a nonlinear system to sinusoidal inputs**
- Considerations:
  - Simulation is often too time-consuming and cumbersome, especially for parametric trade-off studies.
  - There are situations in which simulation is almost useless for studying periodic behavior (e.g., flight control systems)
  - It's the only method that handles **high-order systems** and/or **systems with multiple nonlinearities** with ease

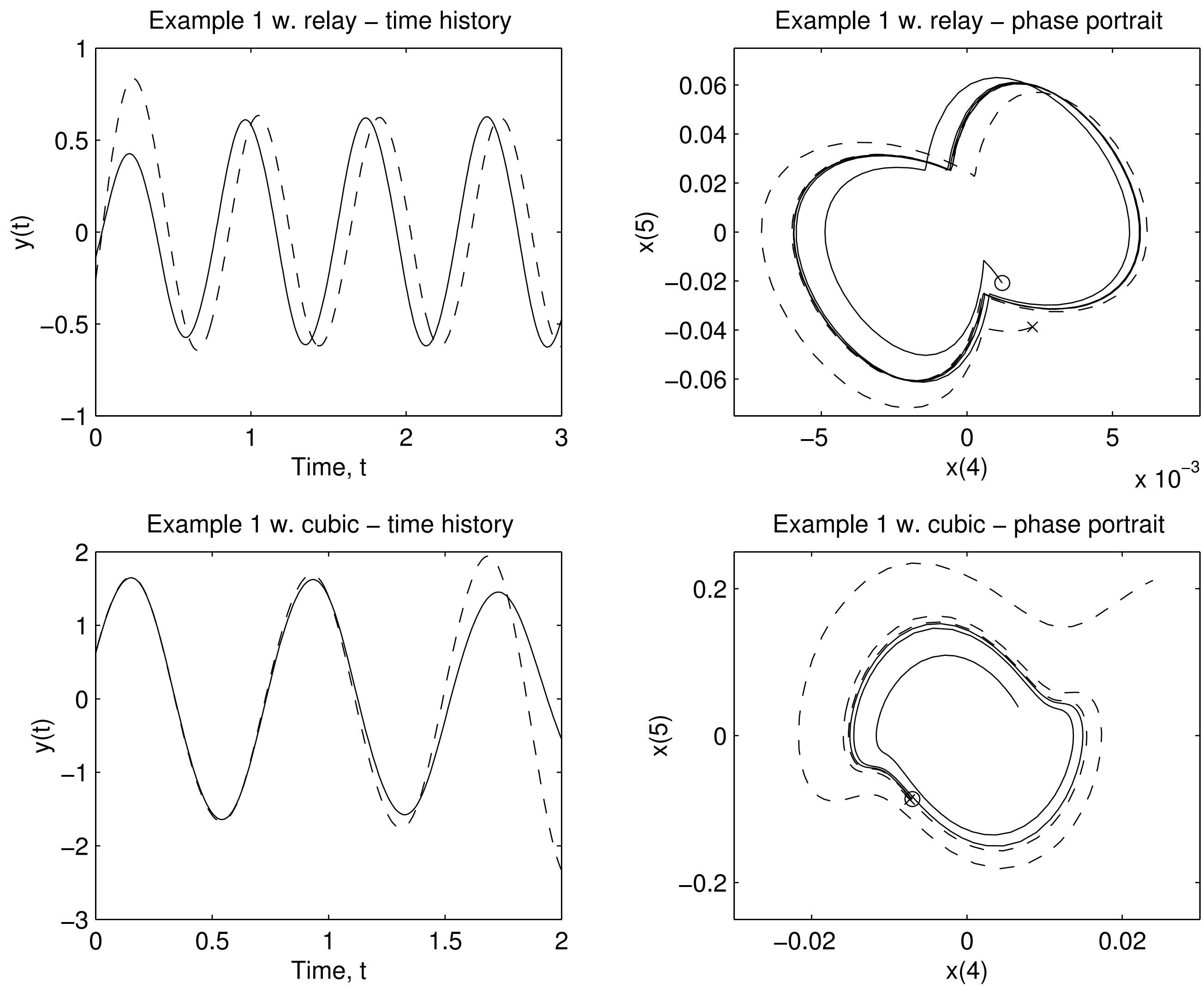
Importance of Periodic Effects

- A limit cycle may be **desired**, with a specified frequency and amplitude – can you design the system? This is the only practical way to build an oscillator . . .
- A limit cycle may be **unwanted but unavoidable** – is it small enough or slow enough to be acceptable?
- An unstable limit cycle is a **stability boundary** – is it large enough?
- A nonlinear system may be driven by **sinusoidal inputs** – how will it respond?
  - SIDF I/O models for different amplitudes → diagnosis
  - SIDF I/O models exhibit interesting phenomena, e.g., "jump resonance"
  - SIDF I/O models form an excellent basis for control system design – later

Definition of Limit Cycles

A simple limit cycle is a periodic trajectory in the state space, $x^*(t+T) = x^*(t)\,,\ \forall t$ where T is the period, such that all nearby trajectories:

- asymptotically approach $x^*(t)$ (a **stable** limit cycle) **or**
- diverge from $x^*(t)$ (an **unstable** limit cycle)



Note that **limit cycles** are not generally **ellipses**

Definition of Sinuoidal-Input Response

- Input: $u(t) = u_0 + a\cos(\omega t)$
- Output: **may be** periodic; to be safe, run a simulation:

$$y = \sum_{k=0}^{\infty} b_k \cos(k\omega t + \psi_k)$$

- "Transfer function" for the fundamental component:

$$G(j\omega; u_0, a) = \frac{b_1}{a}\exp(j\psi_1) \tag{1}$$

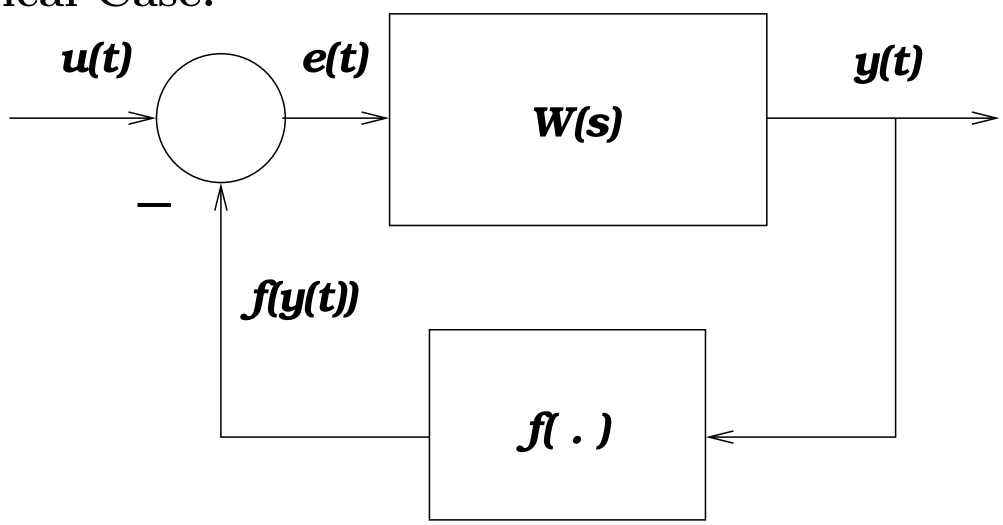
- Operating point ("DC level"): $b_0(u_0, a)$

Note that the "transfer function" and operating point are coupled

Hereafter we will call $G(j\omega; u_0, a)$ an **SIDF Input/Output Model** (SIDF I/O Model)

Basic System Models

Classical Case:



$$
\begin{aligned}
Y(s) \triangleq \mathcal{L}\,(y(t)) &= \frac{p(s)}{q(s)}\; E(s) \\
&\triangleq W(s)\; \mathcal{L}\,(e(t)) \qquad (2) \\
e(t) &= u(t) - f(y(t))
\end{aligned}
$$

Multivariable Case:

$$
\begin{aligned}
\dot{x} &= f(x, u(t)) \\
&\qquad\qquad (3) \\
y(t) &= h(x, u(t))
\end{aligned}
$$

For **limit cycle analysis:** $u(t) = u_0$

For **forced response:** $u(t) = u_0 + \mathrm{Re}\,[a \exp(j\omega t)]$

Basic System Model in MATLAB

- Given: $W(s) = 2/(s^2 + 3s + 7)$ and $f(y) = 4y^3$
- In ODE form: $\ddot{y} + 3\dot{y} + 7y + 8y^3 = 2u(t)$
- In state-space form (one realization): $x^T = [\, y \;\; \dot{y} \,]$

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -7x_1 - 3x_2 - 8x^3 + 2u(t)
\end{aligned}
$$

- In MATLAB:

```
function xdot = basic(t,x)
% Example in controllable canonical form:
% JH Taylor - 9 July 2002

num = 2; den = [ 1 3 7 ];
u = 3.5 * sin(10*t);
xdot(1) = x(2);
xdot(2) = - den(3)*x(1) - den(2)*x(2) + num*(u - 4*x(1)^3);
xdot = xdot(:);
```

- To run a simulation:

```
tspan = [ 0 6*pi/10 ]; % three cycles of sin(10*t)
x0 = [ 1.2 -3.4 ];     % arbitrary initial condition
[t,x] = ode45('basic',tspan,x0); % model is in basic.m
plot(t,x(:,1));        % plot first state only
```

Basic Idea of the Describing Function Method

- Knowledge of signal **form** and **amplitude** is essential in understanding the behavior of a nonlinear system

- Linear system approaches are the most powerful tools we have for analysis

- Replacing nonlinearities with **signal-dependent linear gains ("describing functions")** provides the best way to take advantage of linear system approaches to understand the behavior of a nonlinear system; this process is called "quasilinearization"

- You will see examples that use the *machinery* of Nyquist plots and root locus plots, ... but the **underlying theory** is entirely different. The Routh-Hurwitz technique can also be exploited, but it is difficult compared to the others

Classical Definition of a SIDF

- Given: a specific nonlinearity $f(v)$ and an input signal form, $v(t) = v_0 + \mathrm{Re}\,(a\exp(j\omega t))$

- Find: the quasilinear model

$$f(v) \stackrel{\sim}{=} f_0(v_0, a) + N(v_0, a) \cdot a\exp(j\omega t) \tag{4}$$

  such that **mean square approximation error** is minimized

- Method 1: $f_0(v_0, a)$ and $N(v_0, a)$ are determined by **Fourier analysis** (constant plus first harmonic terms)

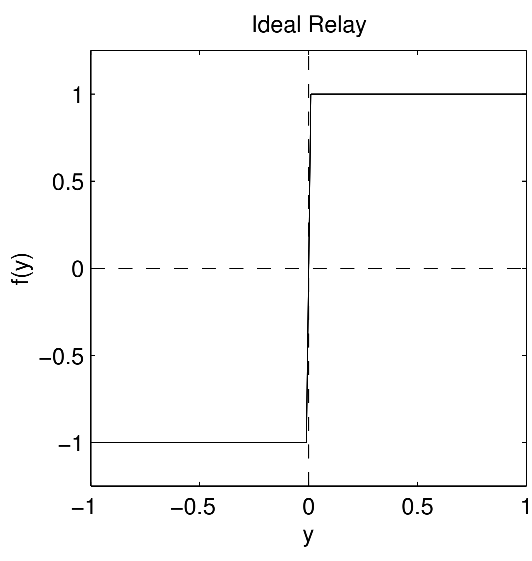
- Method 2: $f_0(v_0, a)$ and $N(v_0, a)$ are determined by using **trigonometric identities** (for power-law and product-type nonlinearities

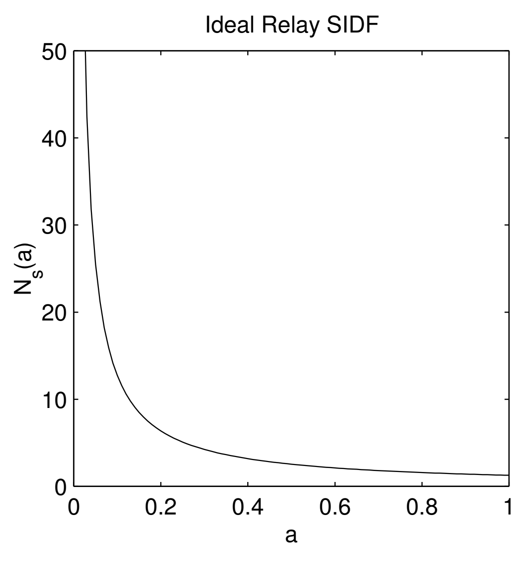
Calculating SIDFs – Piece-Wise-Linear Case

Ideal relay: $f(y) = D \cdot \operatorname{sgn}(y)$ where we assume no DC level, $y(t) = a\cos(\omega t)$

Set up and evaluate the integral for the first Fourier coefficient divided by a as follows:

$$\begin{aligned} N_s(a) &= \frac{1}{\pi\, a} \int_0^{2\pi} f(a\ \cos(x)) \cdot \cos(x)\, dx \\ &= \frac{4\, D}{\pi\, a} \int_0^{\pi/2} \cos(x)\, dx \quad \text{(by symmetry)} \\ &= \frac{4\, D}{\pi\, a} \end{aligned} \tag{5}$$





This makes good, intuitive sense; this SIDF was used to generate the **stable limit cycle** on slide 5

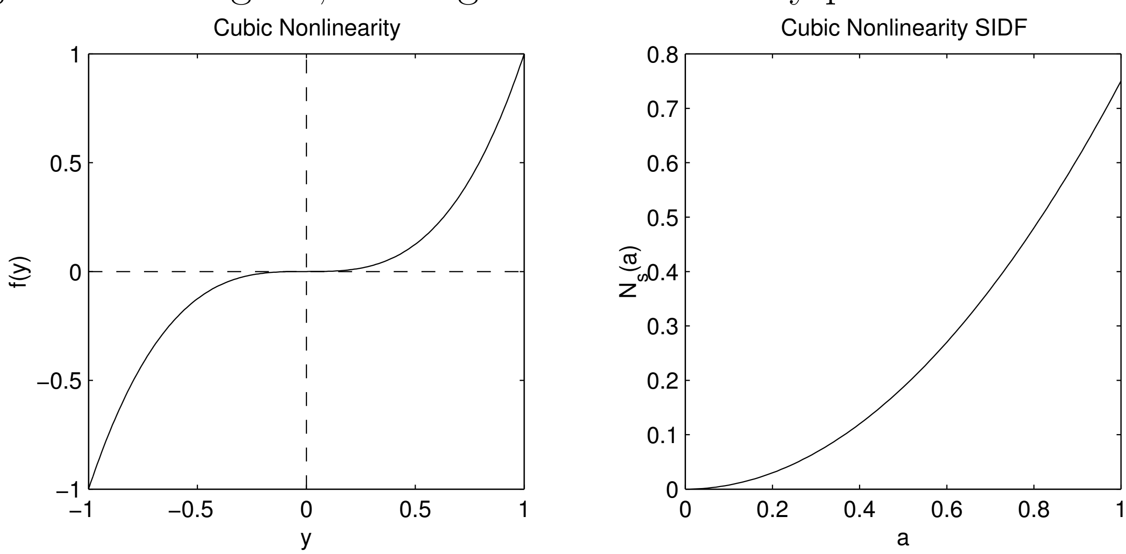
Calculating SIDFs – Trig. Identity Case

Cubic nonlinearity: $f(y) = K\, y^3(t)$; again, assume $y(t) = a\cos(\omega t)$

Directly write the Fourier expansion using trigonometric identities:

$$\begin{aligned} f(a\cos(\omega t)) &= K[\, a\cos(\omega t)\,]^3 \\ &= K\, a^3\, [\, \frac{3}{4}\cos(\omega t) + \frac{1}{4}\, \cos(3\omega t)\,] \\ &\cong \frac{3\, K\, a^2}{4} \cdot a\cos(\omega t) \end{aligned} \tag{6}$$

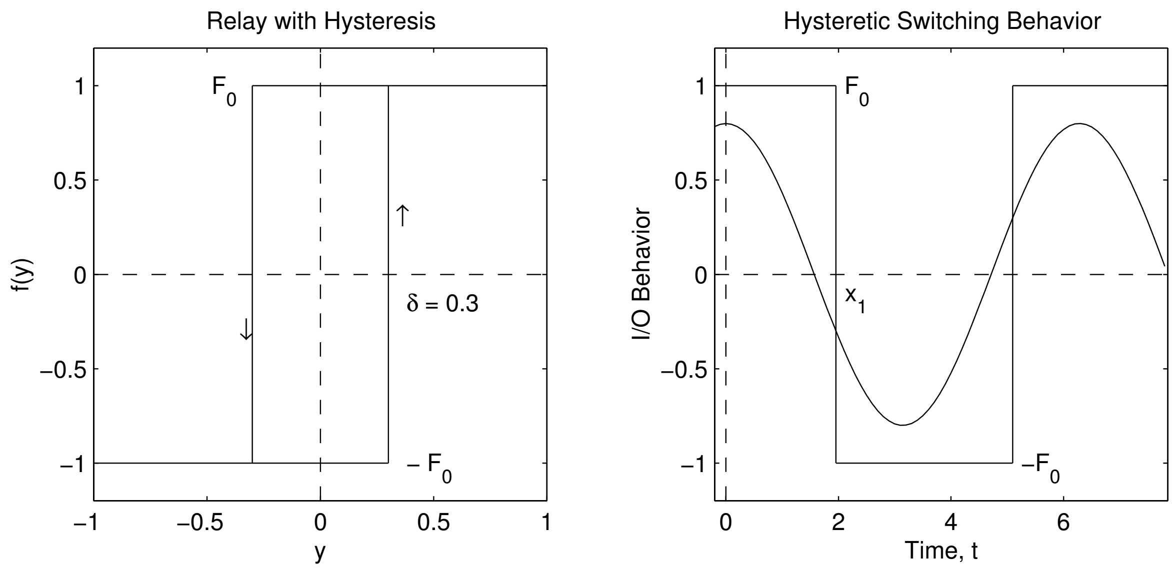
so $N_s(a) = 3\, K\, a^2/4$. Trigonometric identities are a shortcut to using Fourier integrals; use trig. identities for any power-law element.



This also makes good, intuitive sense; this SIDF was used to generate the **unstable limit cycle** on slide 5

Calculating SIDFs – Multi-valued Case

Setting up the Fourier integrals requires care:



$$
\begin{aligned}
N(a) &= \frac{1}{\pi\, a} \int_0^{2\pi} f(a\, \cos(x)) \cdot \exp(-jx)\, dx \\
&= \frac{2\, F_0}{\pi\, a} \left\{ \int_0^{x_1} \exp(-jx) dx - \int_{x_1}^{\pi} \exp(-jx) dx \right\} \quad \text{(by symmetry)}
\end{aligned}
$$

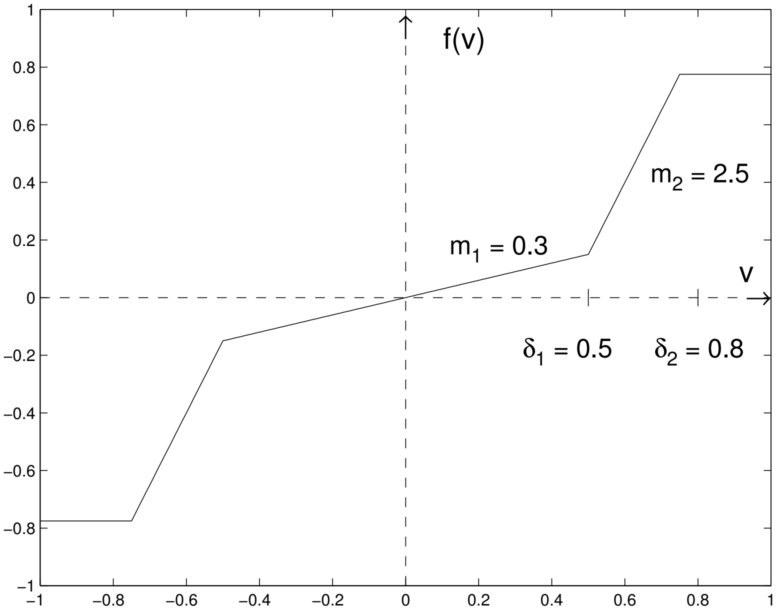
where $x_1 = \cos^{-1}(-\delta/a)$;

$$
= \begin{cases} \frac{4\, F_0}{\pi\, a} \left\{ \sqrt{1 - (\delta/a)^2} - j\, \delta/a \right\} & a > \delta \\ 0 & a \leq \delta \end{cases} \tag{7}
$$

Note that $N(a) \triangleq 0$ if $a \leq \delta$ …the relay **does not switch** $\Rightarrow$ the output is not periodic; more importantly, **this SIDF is complex-valued**, because it causes **lag**

Calculating SIDFs – Details

Given: a typical piecewise linear characteristic $f(v)$,



Set up the Fourier integral for this function for $a > \delta_2$:

- Integrate from 0 to $\pi/2$, by symmetry, and break the integral into three intervals, $[0, x_1]$, $[x_1, x_2]$, $[x_2, \pi/2]$ where $x_1 = cos^{-1}(\delta_2/a)$ and $x_2 = cos^{-1}(\delta_1/a)$

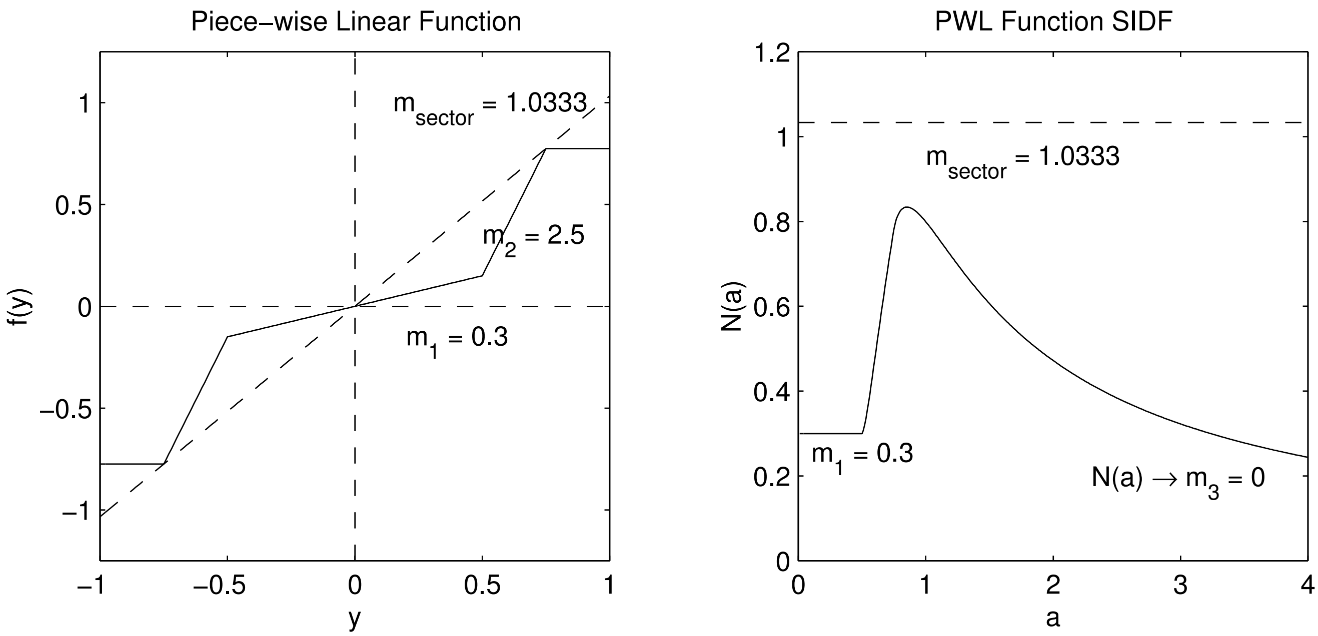
- Be sure to use the correct function for $f(v)$ in each interval, $f(v) = m_1 v$ in interval 3, $f(v) = m_2(v - \delta_1) + m_1\delta_1$ in interval 2, $f(v) = m_2\delta_2 - (m_2 - m_1)\delta_1 \triangleq D_0$ in interval 1

- Therefore the complete SIDF Fourier integral is:

$$N(a) = \frac{4}{\pi a}\left[\int_0^{x_1} D_0\, cos(x)dx + \int_{x_1}^{x_2}[m_2\, a\, cos(x) - (m_2 - m_1)\delta_1]cos(x)dx \right.$$
$$\left. + \int_{x_2}^{\pi/2} m_1\, a\, cos^2(x)dx\right]$$

Qualitative Behavior of SIDFs

Considering the same nonlinearity $f(v)$ –



- For small signals $N(a) \cong N(0) = [df/dv]_{v=0} = m_1$ (if the derivative exists)

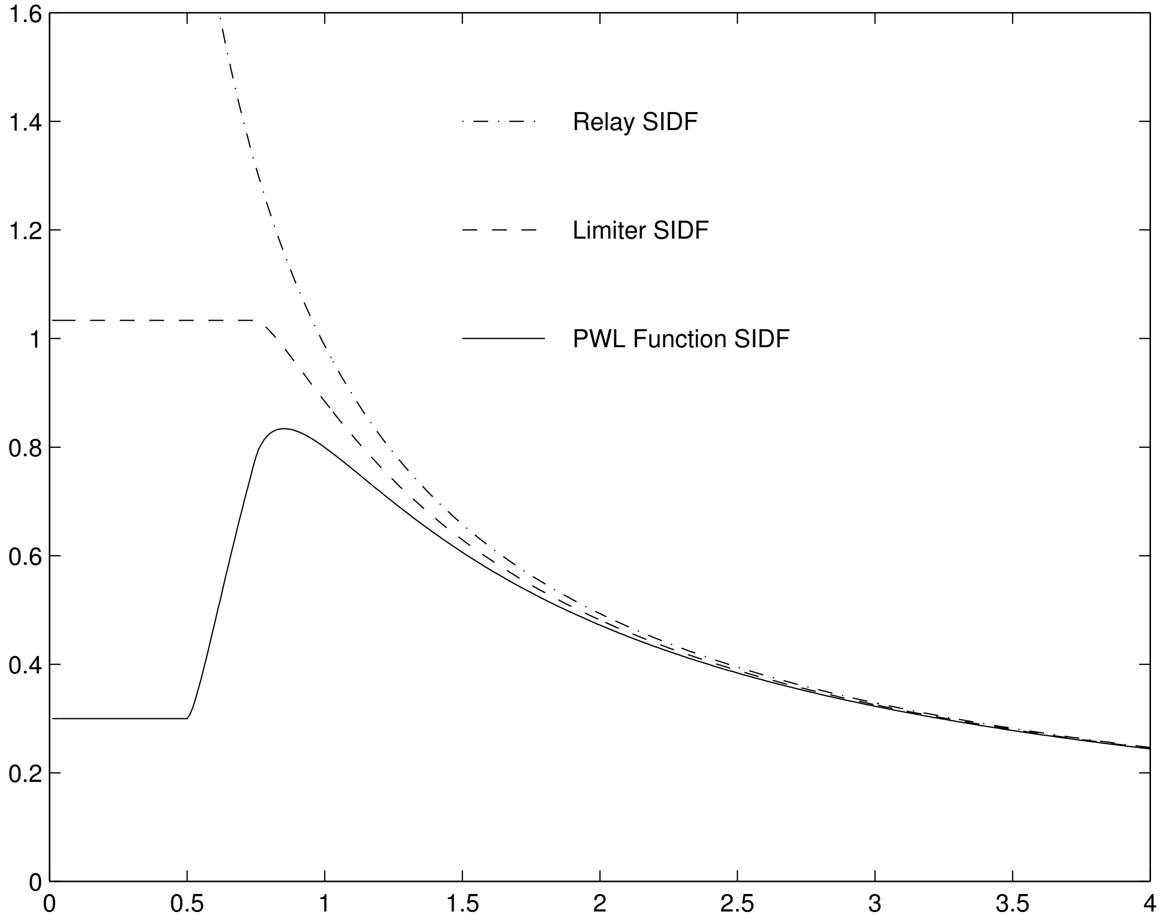
- The SIDF cannot lie outside the slopes of the enclosing sector

- The SIDF is always continuous, even though the nonlinearity derivative is discontinuous

- The SIDF always approaches the ultimate slope of the nonlinearity as $a \to \infty$ (zero for this example)

Qualitative Behavior of SIDFs (Cont'd)

- For large signals ($a > 2$) the "details" near the origin do not make much difference



Conclusion: In many cases we don't need to evaluate $N(a)$ exactly for a **qualitative analysis** ...

Calculating SIDFs in MATLAB

- First, define the basic “saturation function” used in calculating SIDFs for piece-wise-linear functions:

$$f_{\mathrm{sat}} = \begin{cases} \mathrm{sign}(x)\,, \quad |x| \geq 1 \\ 2\,[\,sin^{-1}(x) + x\sqrt{1-x^2}\,]/\pi\,, \quad |x| < 1 \end{cases} \tag{8}$$

- The SIDF for a general limiter is $N_{LIM}(a) = m f_{\mathrm{sat}}(\delta/a)$

- The SIDF for the piece-wise-linear example is $N_{PWL}(a) = m_1 f_{\mathrm{sat}}(\delta_1/a) + m_2\,[\,f_{\mathrm{sat}}(\delta_2/a) - f_{\mathrm{sat}}(\delta_1/a)\,]$

- Therefore the previous plots are obtained as follows:

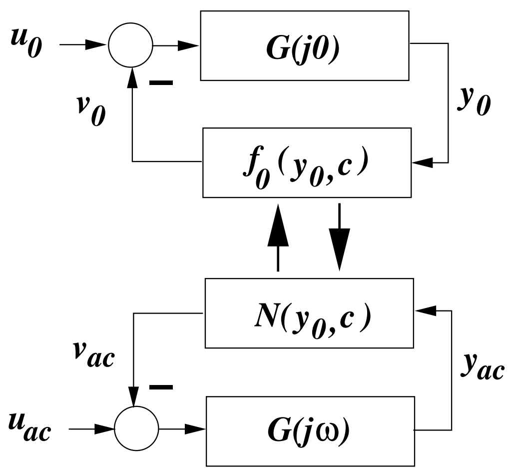
```
D = m1*d1 + m2*(d2 - d1); m_sect = D/d2;
av = 0.01:0.01:4.0;
for i = 1:length(av);
  DFqual(i) = m1*f_sat(d1/av(i)) + m2*(f_sat(d2/av(i))-f_sat(d1/av(i)));
  DFlim(i) = m_sect*f_sat(d2/av(i)); % limiter
  DFrel(i) = 4*D / (pi*av(i)); % relay
end
plot(av,DFqual,av,DFlim,'--',av,DFrel,'-.');
axis([0 4 0 1.6]);
```

where:

```
function f_sat = f_gvdv(x)
% saturation function "f" for calculating SIDFs for PWL functions
% Gelb & Vander Velde, Appendix B, p. 519
% JH Taylor - 18 June 2002
if abs(x) >= 1,
  fdf = sign(x);
else
  fdf = 2*(asin(x) + x*sqrt(1 - x*x))/pi;
end
```

Harmonic Balance – Limit Cycle Conditions

1. Classical Case:



$$
\begin{aligned}
u &= u_0 + a\cos(\omega t) \\
&\triangleq u_0 + u_{ac} \\[1em]
y &= y_0 + \mathrm{Re}\,[\, c \cdot \exp(j\omega t)\,] \\
&\triangleq y_0 + y_{ac} \\[1em]
v &= f_0(y_0, c) + \mathrm{Re}\,[\, N(y_0, c) \cdot c \cdot \exp(j\omega t)\,] \\
&\triangleq v_0 + v_{ac}
\end{aligned}
$$

DC Harmonic Balance: $y_0 = G(j0)[\, u_0 - f_0(y_0, c)\,]$ ...must be solved simultaneously with an AC harmonic balance relation (below) to obtain y_0 and c

AC Harmonic Balance:

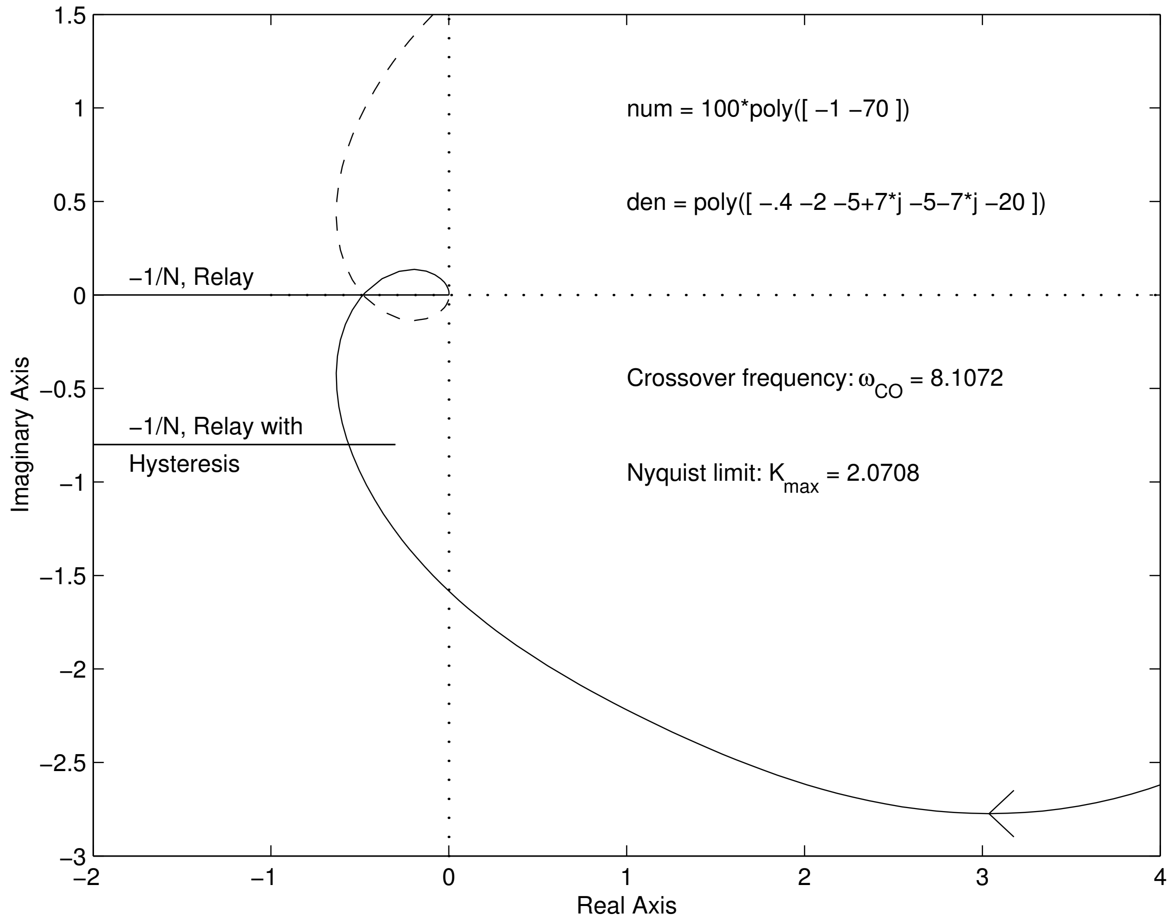
- Limit Cycles: $a = 0\,;\;\; G(j\omega) \cdot N(y_0, c) = -1$ must be satisfied for some $\{y_0,\, c,\, \omega\}$ for limit cycle prediction

- Forced Response: $a \neq 0\,;\;\; c = \frac{G(j\omega)}{1+N(y_0,c)\cdot G(j\omega)} \cdot a\,;$ solve for $c(j\omega;\, u_0,\, a)$ then $N(y_0, c)$ to obtain the overall "transfer function" $W(j\omega;\, u_0,\, a) = \frac{G(j\omega)}{1+N(y_0,c)\cdot G(j\omega)}$ (the frequency response from u_{ac} to y_{ac})

Classical Limit Cycle Analysis Using a Nyquist-plot Approach

Given $G(s) = \dfrac{s^2 + 71s + 70}{10s^5 + 32.4s^4 + 3468s^3 + 21616s^2 + 37712s + 11840}$:

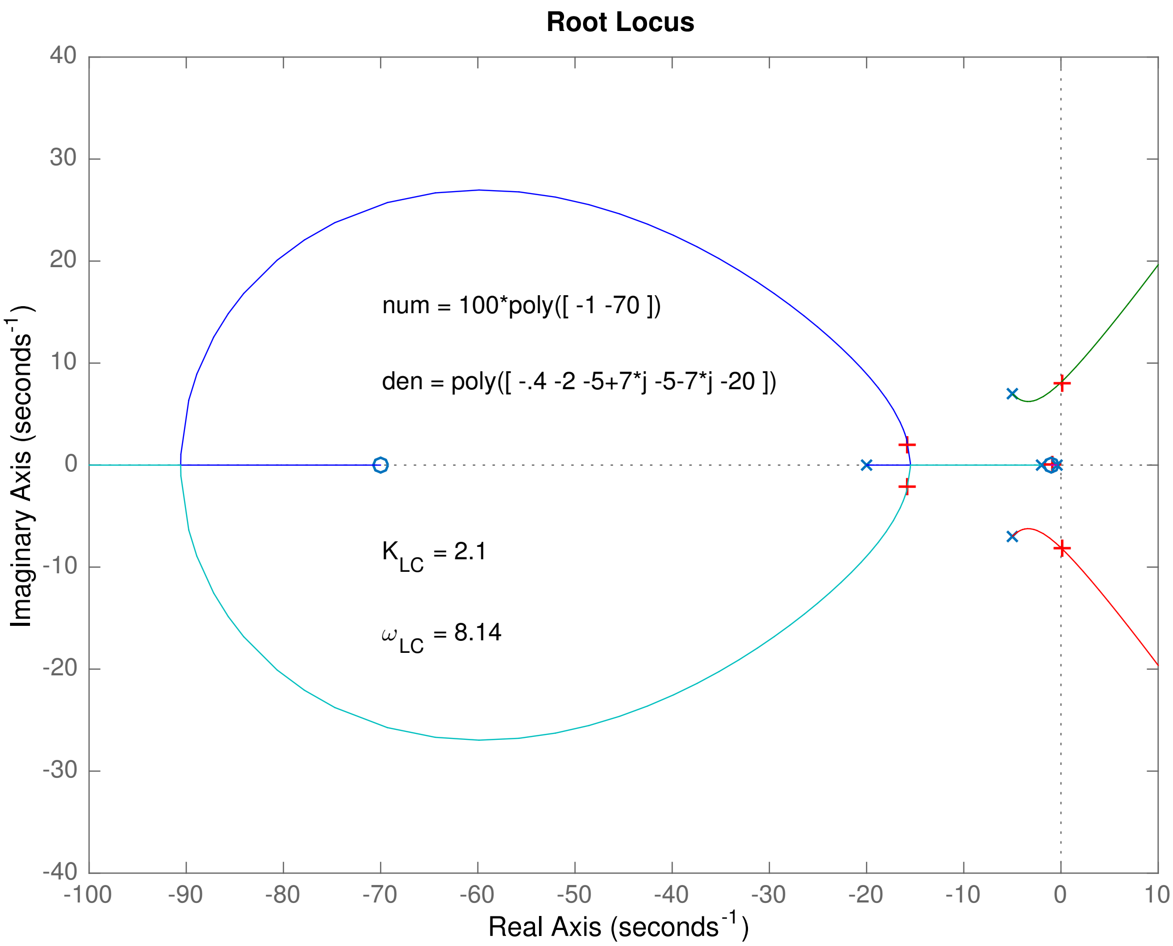
The condition $G(j\omega) \cdot N = -1$ (or $G(j\omega) = -1/N$) is easily investigated on a Nyquist plot:



Note: This is **not** the Nyquist test for stability!

Classical Limit Cycle Analysis Using a Root Locus-plot Approach

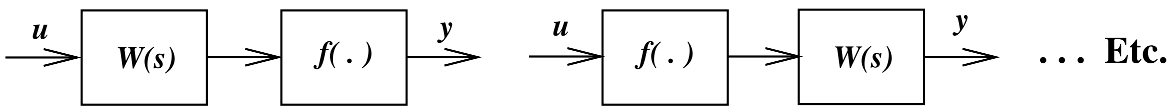
The condition that the root locus of $G(j\omega)$ has poles on the imaginary axis is easily investigated on a root locus plot:




```
clear; close all
den = poly([ -.4 -2 -5+7*j -5-7*j -20 ]);
num = 100*poly([ -1 -70 ]);
rlocus(num,den)
[K,poles] = rlocfind(num,den);
```

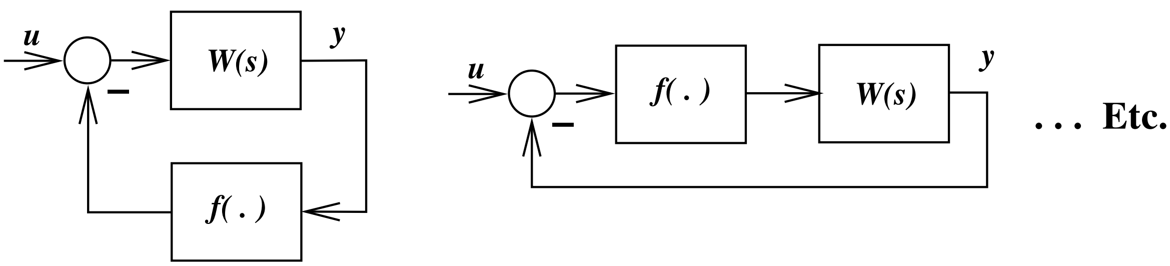
Limitations of SIDF Analysis

- Situations when SIDFs are exact:



The first harmonic of y is correct

- Situations when SIDFs are **not** exact:



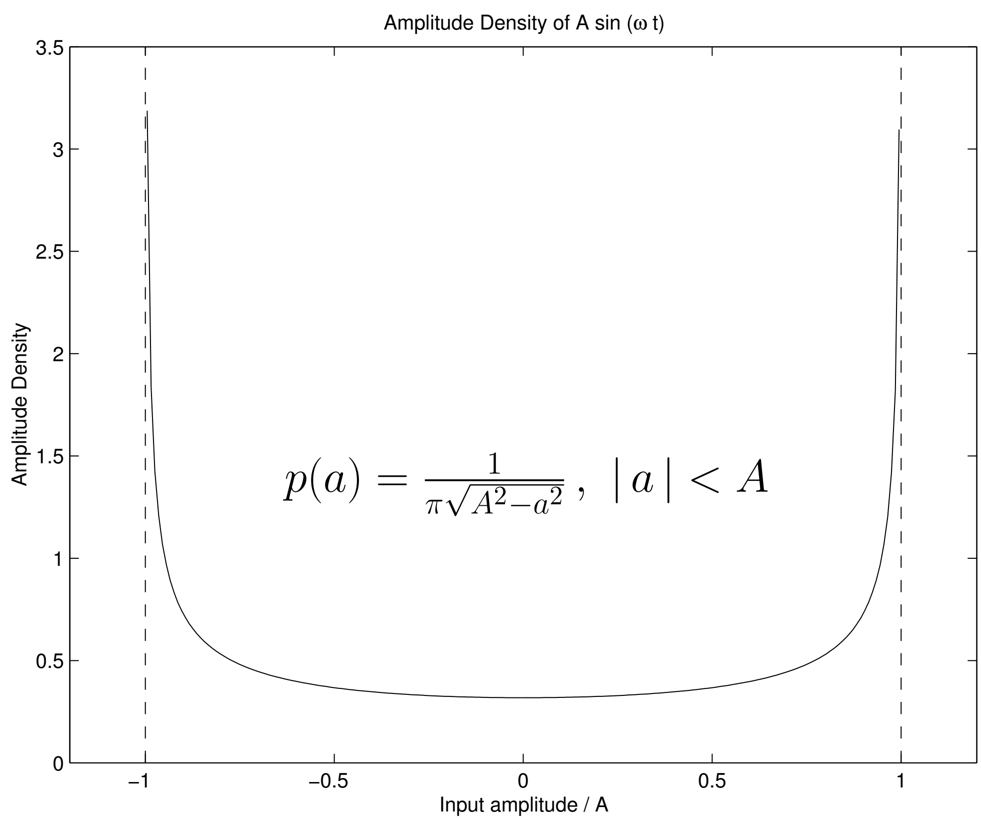
The first harmonic of y is approximate

- How to deal with inexact situations:
  - Consider the validity of the "low-pass filter hypothesis": The nonlinearity input is essentially sinusoidal due to the filtering of higher harmonics by $W(j\omega)$ – if true, SIDF results should be accurate.
  - Consider how well-behaved the system nonlinearity is
  - Look at simulation results, assess the importance of higher harmonics (distortion)

Limitations of SIDF Analysis (Cont'd)

Except for multi-valued nonlinearities (hysteresis, backlash etc.) the DF is not dependent on the assumption of periodicity – only the **amplitude distribution** matters

- For a triangular ("saw-tooth") wave the DF is the same as that for a uniformly distributed random variable
- In many control applications the sine-wave distribution is a good approximation:



Limitations of SIDF Analysis (Cont'd)

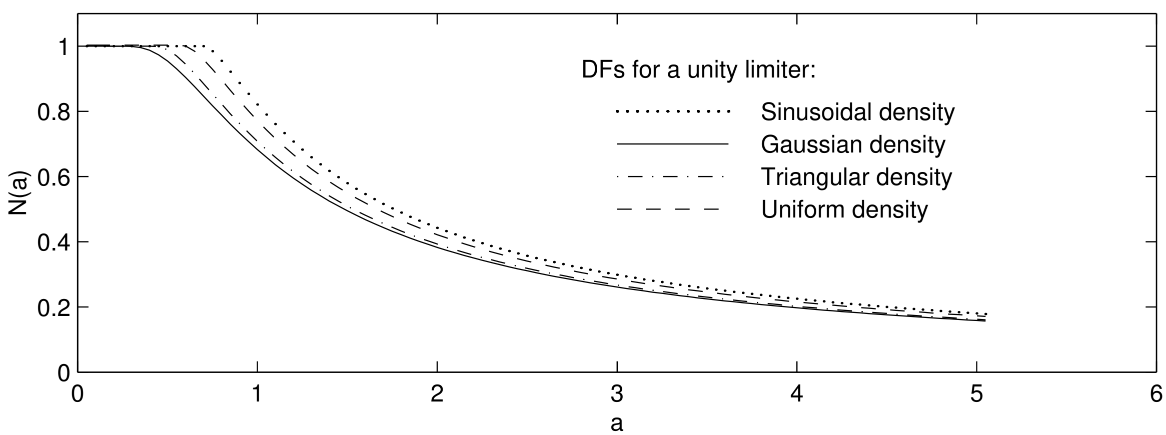
- The basic DF definition for $f(v)$ where v has a bias b is:

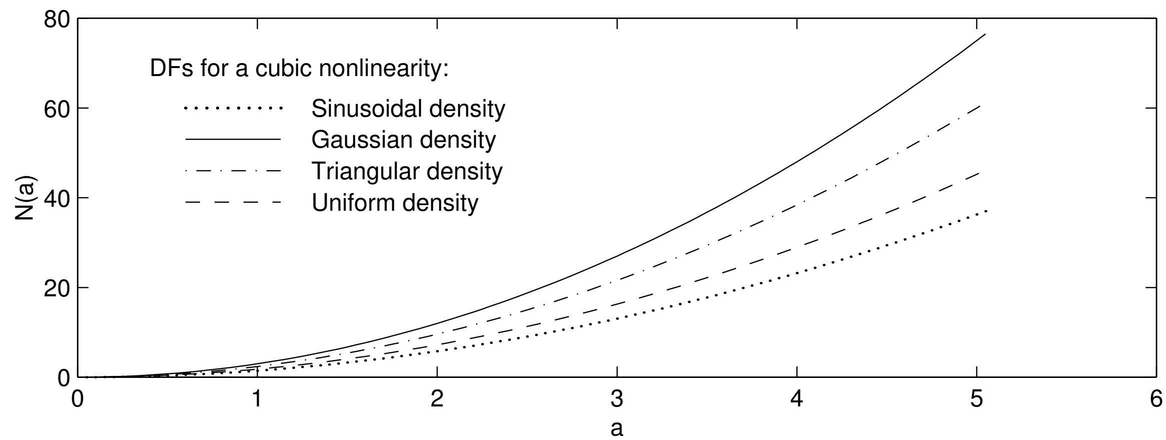
$$F_0(\sigma,\, b) \;=\; \int_{-\infty}^{\infty} f(b+z)\, p(z)\, dz \tag{9}$$

$$N_z(\sigma,\, b) \;=\; \frac{1}{\sigma^2}\int_{-\infty}^{\infty} z\, f(b+z)\, p(z)\, dz \tag{10}$$

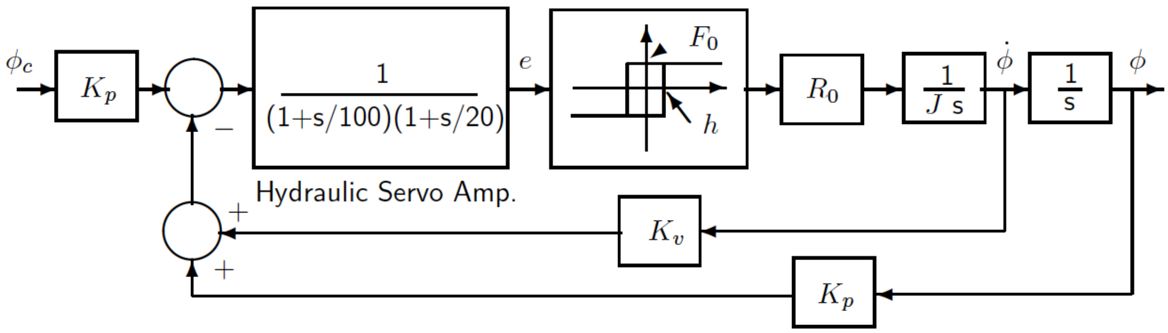
where σ is the generalized input signal amplitude, $\sigma = \sqrt{\mathrm{E}[(v-b)^2]}$ which for a sinusoidal signal is $\sigma_s = a/\sqrt{2}$

- For many nonlinearities the DF is not particularly sensitive to the amplitude distribution:

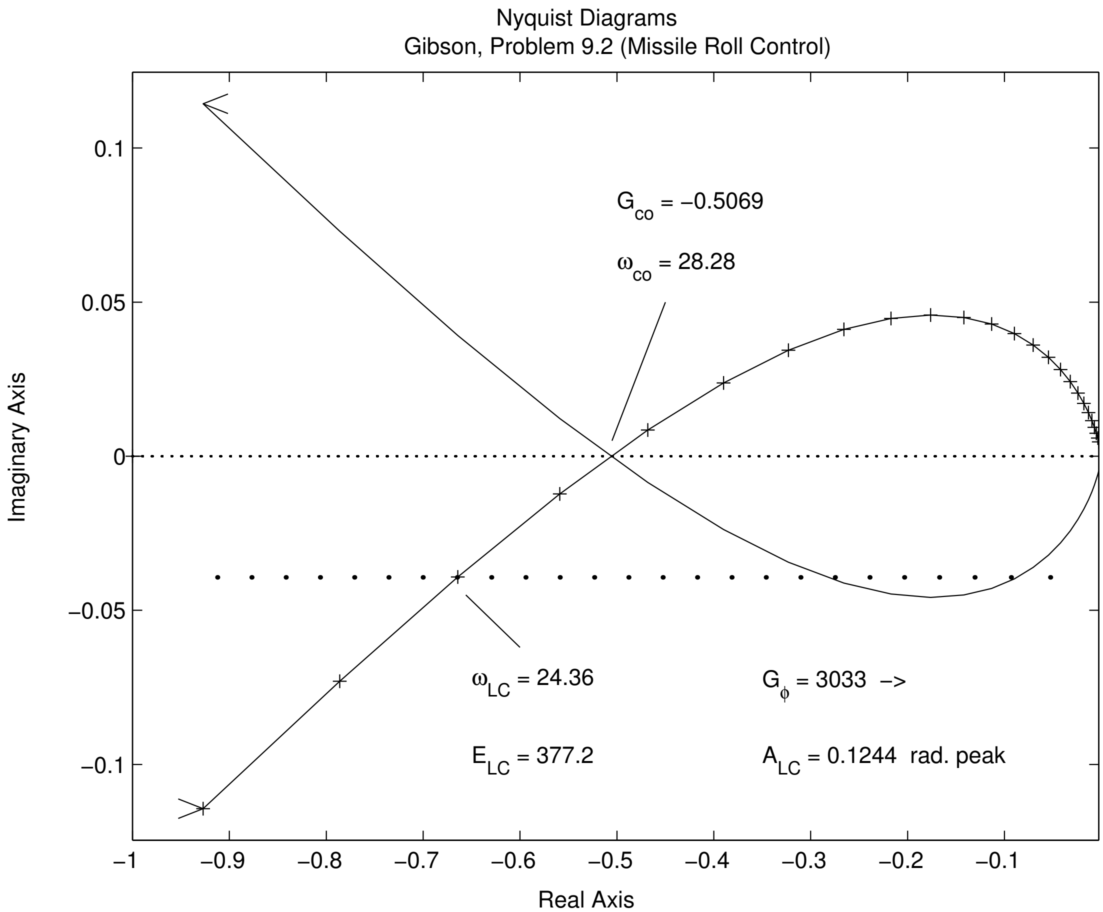




Example: Limit Cycle Analysis, Missile Roll-Control Loop



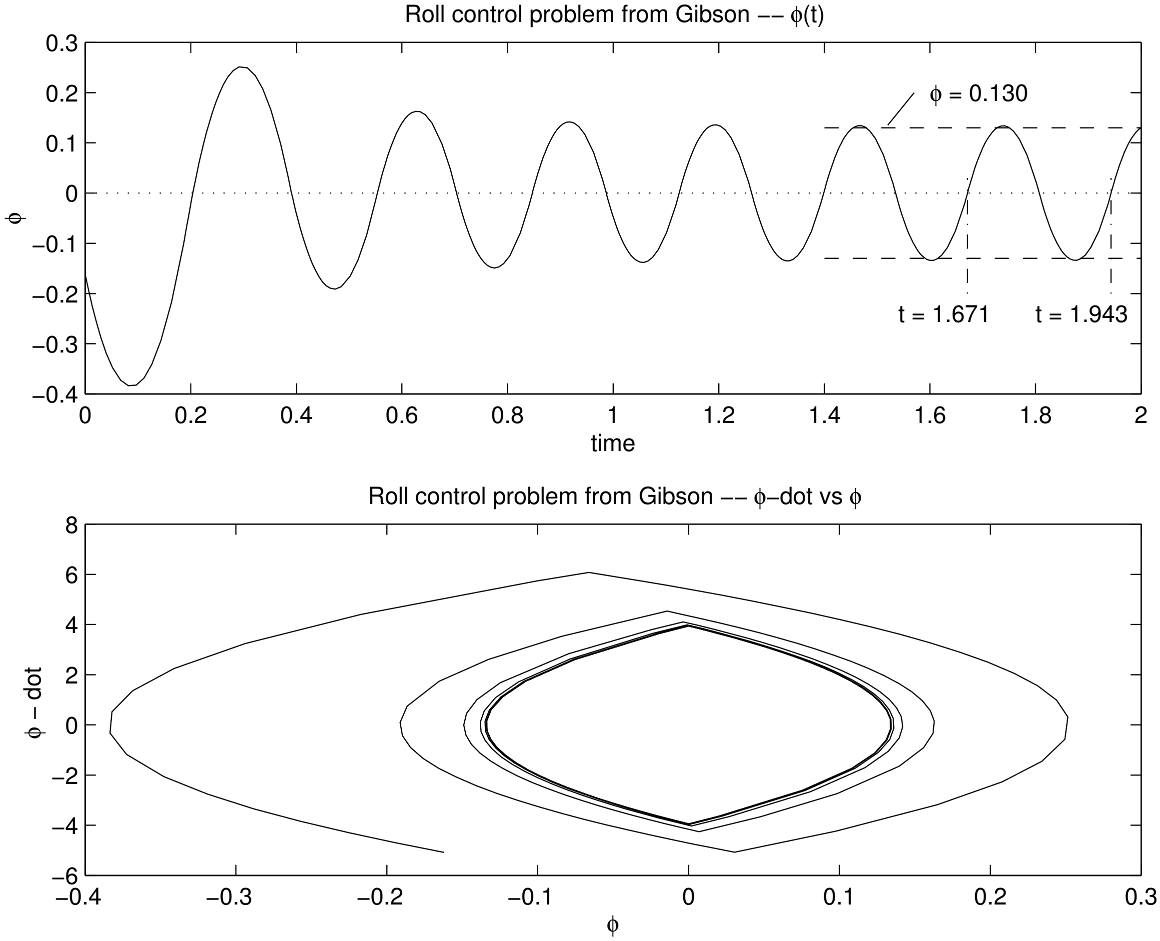
The relay-with-hysteresis implements "bang-bang" control, a simple robust method that is easy to implement



Note: G_ϕ = gain from e to ϕ

Limit Cycle Verification

Simulation provides a good verification:



$$T = 0.272 \text{ sec} \rightarrow \omega_{LC} = 23.1 \text{ rad/sec}$$

Compare with the SIDf result on the previus slide:

$$T = 0.258 \text{ sec} \rightarrow \omega_{LC} = 24.36 \text{ rad/sec}$$

The SIDF spproach is usually this accurate – excellent!

Harmonic Balance "Transfer Functions"

Two methods for generating the SIDF I/O model $G(j\omega; u_0, a)$:

1. Analytic approach: solve the AC Harmonic Balance equation for $c(j\omega; u_0, a)$, divide by a

   (a) Advantage: you can tell, for example, when solutions do not exist

   (b) Disadvantage: it's difficult to carry out if the nonlinear system is at all complicated

2. Simulation approach: develop a simulation model for the nonlinear dynamic system with a sinusoidal input, simulate to obtain the steady-state response, perform Fourier analysis of the result

   (a) Advantages: No need to assume that the input to each nonlinearity is sinusoidal, the number of system states and nonlinearities is relatively unimportant

   (b) Disadvantages: May be quite time consuming, may be difficult to interpret the results

Harmonic Balance "Transfer Function" – Classical Duffing's Equation

Duffing's Equation: $\ddot{x} + 2\zeta\dot{x} + x + x^3 = a\cos(\omega t)$

This represents, for example, a normalized mass-spring-damper system with a hardening spring; in the standard form $W(s) = 1/(s^2 + 2\zeta s + 1)$, $u(t) = a\cos(\omega t)$ and $f(\cdot) = x^3$

Let b be the amplitude of the fundamental component of x; then quasilinearize Duffing's equation to obtain:

$$b^2\left[(1 + \frac{3}{4}b^2 - \omega^2)^2 + (2\zeta\omega)^2\right] = a^2$$

or, if we let $B = b^2$,

$$B\left[(1 + \frac{3}{4}B - \omega^2)^2 + (2\zeta\omega)^2\right] = a^2$$

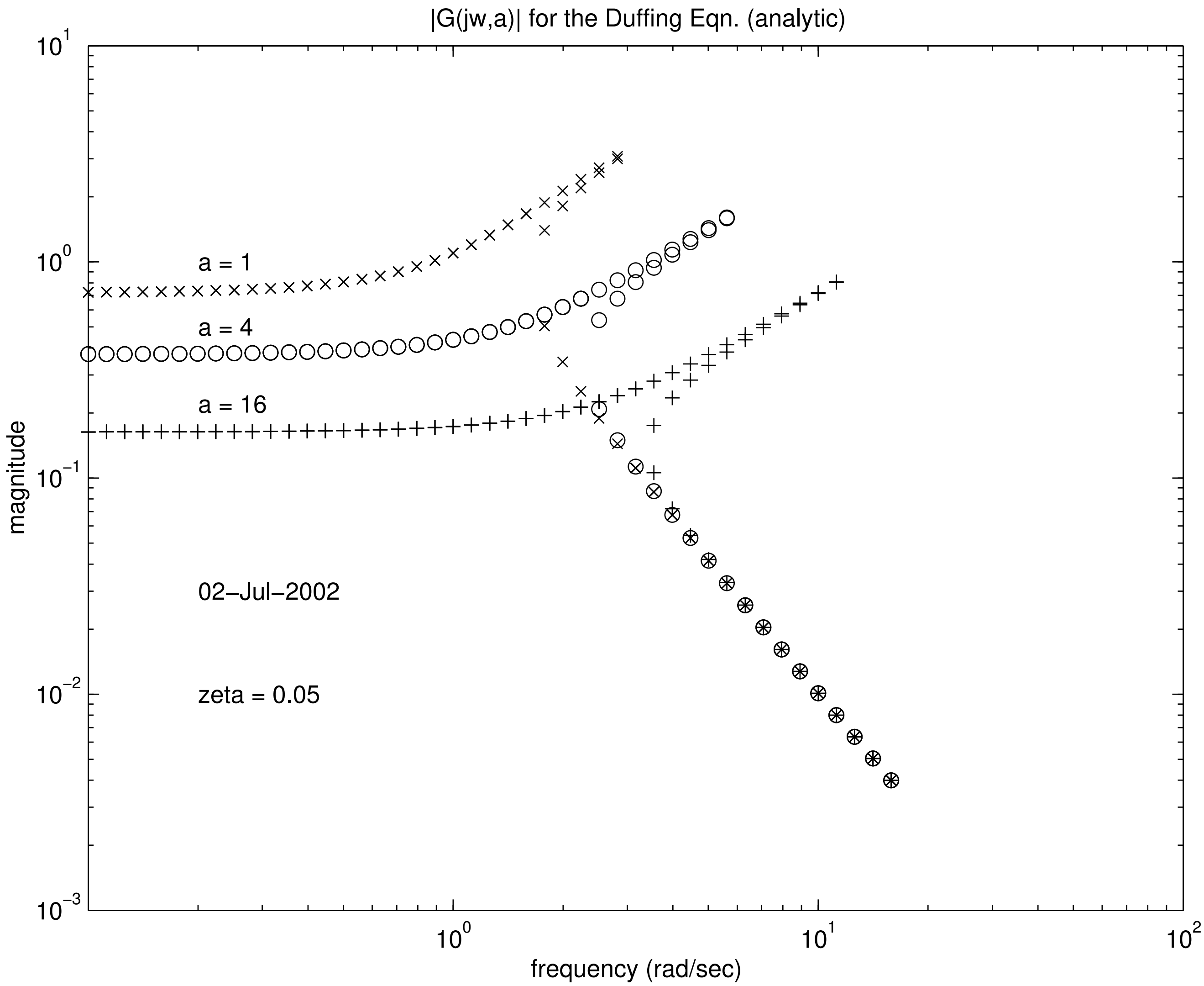
finally,

$$\frac{9}{16}B^3 + \frac{3}{2}(1-\omega^2)B^2 + \left[(1-\omega^2)^2 + (2\zeta\omega)^2\right]B - a^2 = 0 \quad (11)$$

The last simple polynomial equation may have 1 or 3 real roots, depending on a and ω:

Duffing's Equation "Transfer Function"

The results for several values of a are as follows:



Here we see a **jump resonance phenomenon**, which can be a real surprise to an experimentalist!

Solving the Duffing Problem in MATLAB

- First, define the polynomial in Eqn. 11 multiplied by 16/3:

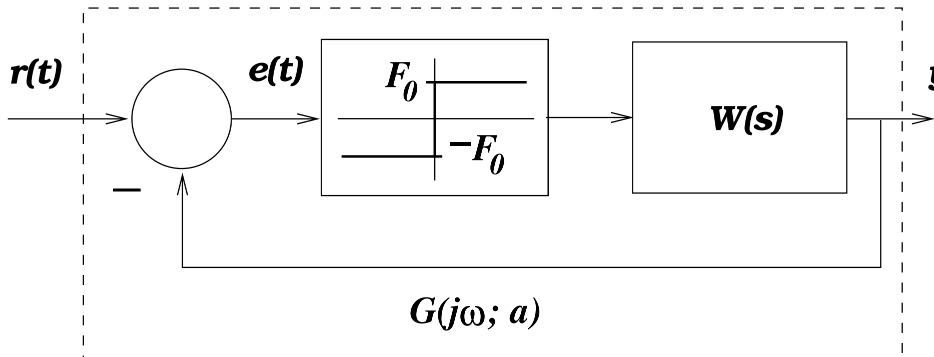
```
function soln = duff_poly(a,w,zeta)
% polynomial to be solved for Duffing's Equation
beta = 1 - w*w; gamma = 2*zeta*w; K = 16/3;
C(1) = 3; C(2) = 8*beta; C(3) = K*(beta^2 + gamma^2);
C(4) = - K*a*a;
soln = sqrt(roots(C)./(a*a));
```

- Now, set up loops for 3 amplitudes and 45 frequencies:

```
zeta = 0.050;
for jj=1:3 %% amplitude loop
  a = 4^(jj-1) %% a = 1, 4, 16
  av(jj) = a;
  for ii=1:45 %% frequency loop
    w = 10^((ii-21)/20)  %% w_min = 0.1, w_max = 10
    wv(ii) = w;
    G = duff_poly(a,w,zeta);
    % discard any complex conjugate
    if imag(G(1)) ~= 0 | imag(G(2)) ~= 0,
      for iii=1:3
        if imag(G(iii)) == 0, RG = G(iii); end
      end
      G(1) = RG; G(2) = RG; G(3) = RG; % only one real root exists
    end
    GM(ii,3*jj-2) = G(1); GM(ii,3*jj-1) = G(2); GM(ii,3*jj) = G(3);
  end % frequency loop
end % amplitude loop
%% plotting
loglog(wv,GM(:,1),'x',wv,GM(:,2),'x',wv,GM(:,3),'x', ...
      wv,GM(:,4),'o',wv,GM(:,5),'o',wv,GM(:,6),'o', ...
      wv,GM(:,7),'+',wv,GM(:,8),'+',wv,GM(:,9),'+');
title('|G(jw,a)| for the Duffing Eqn. (analytic)')
xlabel('frequency (rad/sec)');
ylabel('magnitude');
```

Harmonic Balance "Transfer Functions" (Cont'd)

Closed-loop system with relay:



$r(t) = a\cos(\omega t)$ $\qquad\qquad$ $y(t) = \mathrm{Re}\,[\,c\exp(j\omega t)\,]$

Harmonic Balance Relation:

$$c = (a - c) \cdot \frac{4F_0}{\pi |\, a - c\,|}\, W(j\omega)$$

- Magnitude part:

$$\begin{aligned} M(j\omega) &\triangleq |\, W(j\omega)\,|; \\ |\, G(j\omega; a)\,| &\triangleq \frac{|\, c\,|}{a} = \frac{4F_0}{\pi\, a}\, M(j\omega) \end{aligned}$$

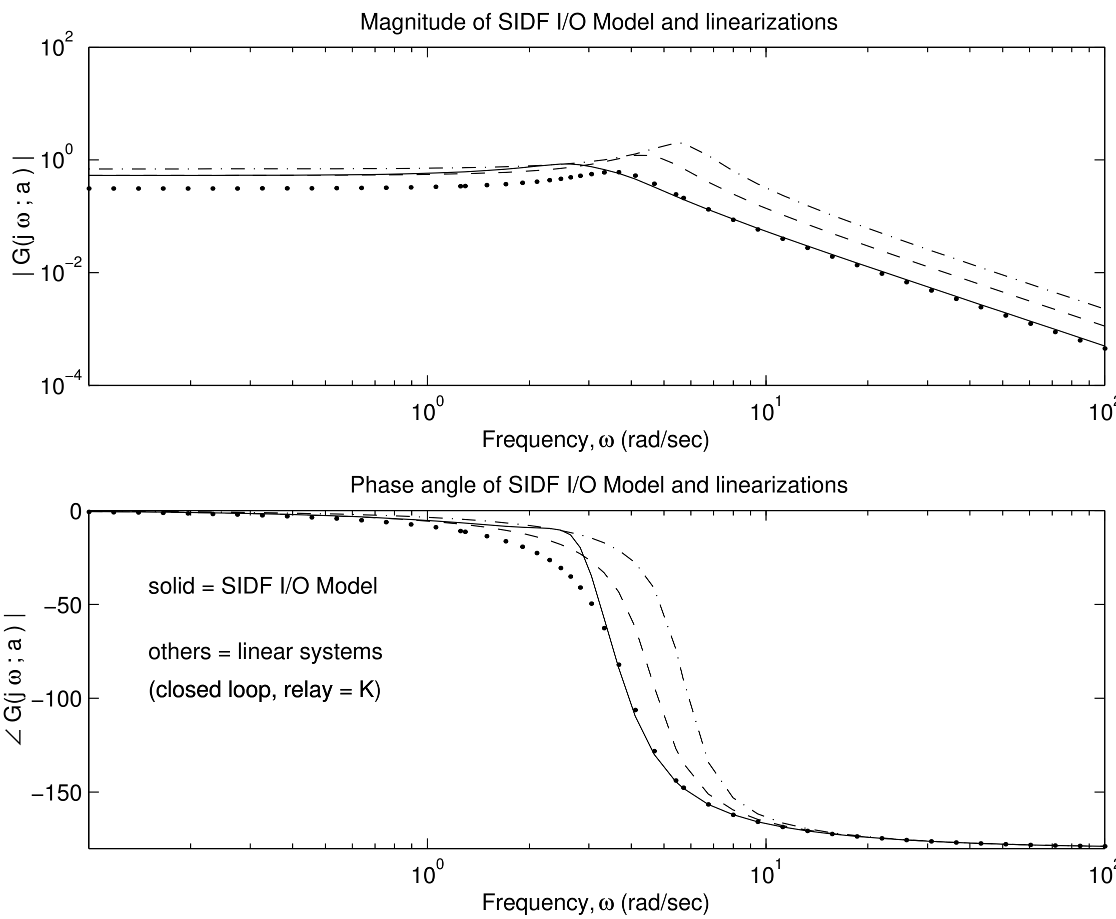
- Phase part:

$$\begin{aligned} \psi &\triangleq \angle W(j\omega)\ ; \\ \angle G(j\omega; a) &= \psi - \sin^{-1}\left(\frac{4F_0}{\pi a}\, M(j\omega)\, \sin(\psi)\right) \end{aligned}$$

Harmonic Balance "Transfer Functions" (Cont'd)

Closed-loop system with relay (cont'd)

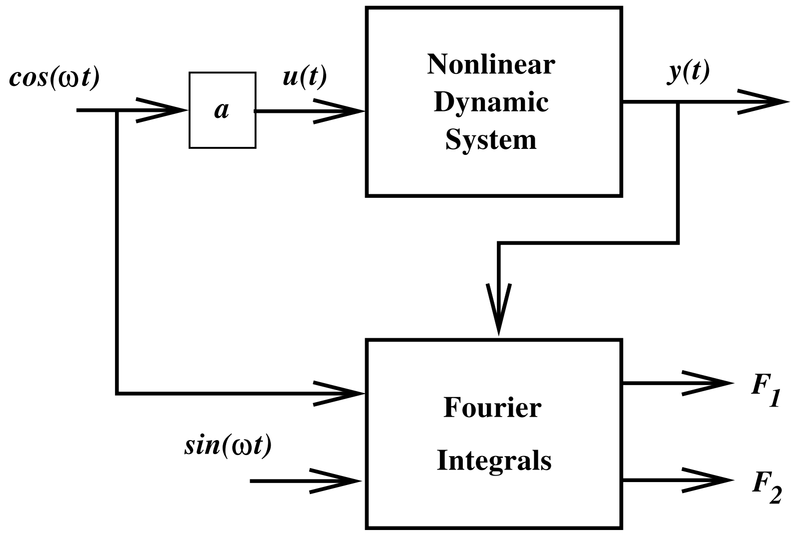
- The magnitude relation is quite straightforward (but it appears that the feedback disappears!)
- The phase relation can only be met if the input amplitude a is large enough that the argument of $\sin^{-1}$ is less than one at all frequencies
- Example: $W(s) = 45/(s^2 + 2s + 9)$, $F_0 = \pi/2$, $a = 18 \rightarrow$



Note that the SIDF transfer functions has charac- teristics that cannot be duplicated by any linear $G(j\omega)$

SIDF I/O Models by Simulation

The most efficient approach is to simulate and perform Fourier analysis simultaneously:



$$
\begin{aligned}
F_1^k &= \int_{(k-1)T}^{kT} y(t) \cdot \cos(\omega t) dt \\
F_2^k &= \int_{(k-1)T}^{kT} y(t) \cdot \sin(\omega t) dt
\end{aligned}
$$

from which we obtain:

$$
\begin{aligned}
\operatorname{Re} G(j\omega; u_0, a) &= \frac{\omega}{\pi a} F_1^k \\
\operatorname{Im} G(j\omega; u_0, a) &= -\frac{\omega}{\pi a} F_2^k
\end{aligned}
$$

Integrate for k cycles where k is sufficiently large that the magnitude and phase of $G(j\omega; u_0, a)$ have converged to your satisfaction.

SIDF I/O Model by Simulation in MATLAB

1. Add the Fourier integral states to your model:

```
function xdot = lim_filt2(t,x)
%  Second-order linear model with limiter; model is
%  augmented with Fourier integrals, to obtain G(jw,a)
%  JH Taylor, 10 July 2002
%
zeta = 0.15; global Ampl Freq
u = Ampl*sin(Freq*t);
xdot(1) = x(2);
xdot(2) = u - x(1) - 2*zeta*x(2);
%% define Y and set up the Fourier integrals:
if abs(x(1)) < 1
   y = x(1);
else
   y = sign(x(1));
end
xdot(3) = y*sin(Freq*t);
xdot(4) = y*cos(Freq*t);
xdot = xdot(:); %% end of model lim_filt2
```

2. Run a simulation to steady state and extract $G(j\omega)$:

```
function [mag,phase] = ggen(Model,MAGTOL,PHASETOL)
%% ggen(model,MAGTOL,PHASETOL) returns the magnitude
%% and phase of an ODE model defined in the file 'Model'.m
%% JH Taylor - University of New Brunswick - 7 July 2002

% Initialize:
global Ampl Freq Xdim;
k = 0; T = 2*pi/Freq; tspan = [ 0 T ]; x0 = zeros(Xdim,1);
[t,x] = ode45(Model,tspan,x0);
[nrows,ncols] = size(x);
xf = x(nrows,:);
mag0 = Freq/(pi*Ampl)*abs(xf(ncols-1)+j*xf(ncols));
```

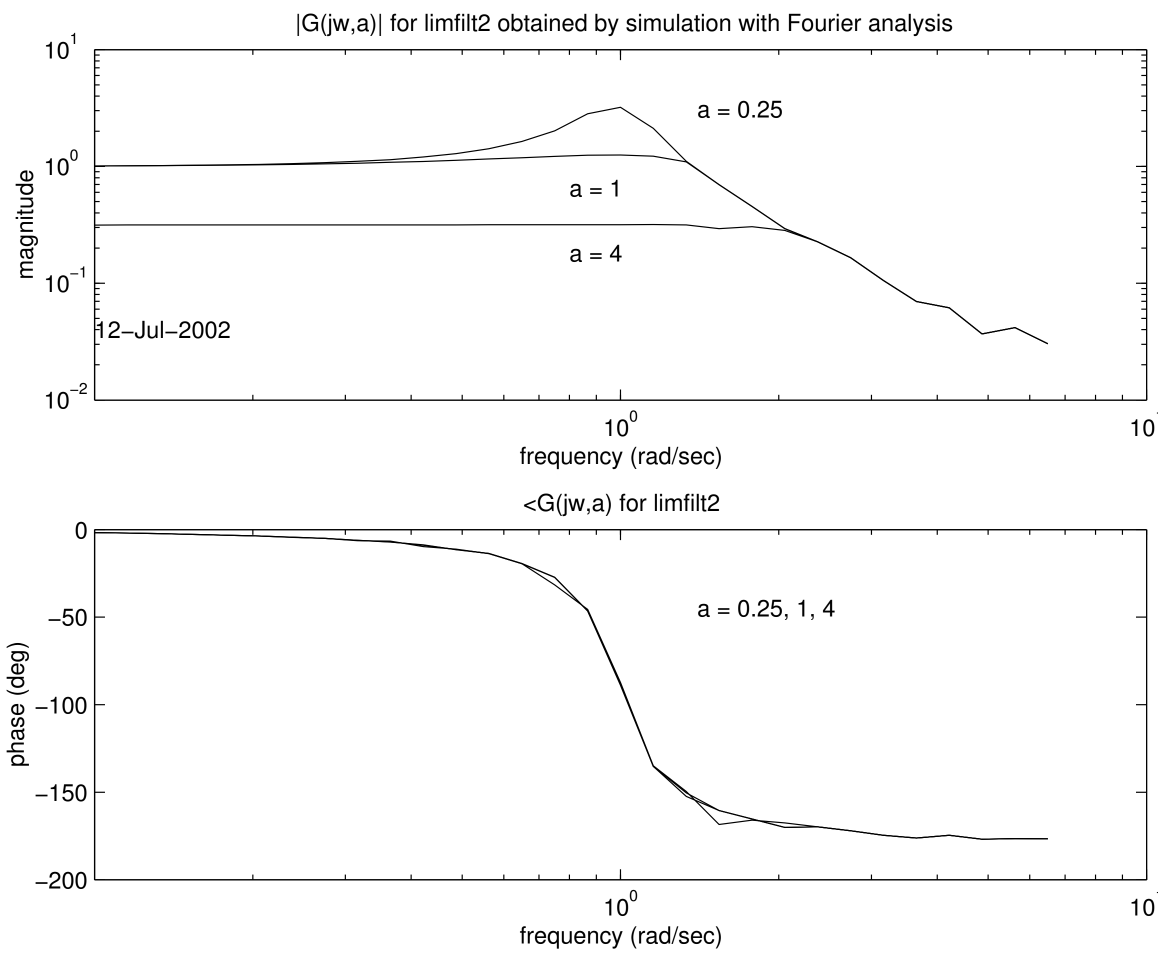
```
phase0 = atan2(xf(ncols),xf(ncols-1));
% Simulate cycle-by-cycle until convergence obtained:
while (k >= 0)
   k = k+1;
   x0 = xf; % initial condition from last cycle
   x0(ncols-1) = 0; % reset the Fourier states
   x0(ncols) = 0;
   [t,x] = ode45(Model,tspan,x0);
   [nrows,ncols] = size(x);
   xf = x(nrows,:);
   mag = Freq/(pi*Ampl)*abs(xf(ncols-1)+j*xf(ncols));
   phase = atan2(xf(ncols),xf(ncols-1));
   magdiff = abs(20*log10(mag/mag0));
   phasediff = (180/pi)*abs(phase-phase0);
   if ((magdiff >= MAGTOL) | (phasediff >= PHASETOL))
      mag0 = mag;
      phase0 = phase;
   else
      k = -1;
   end
end;
```

3. Here is the main executive:

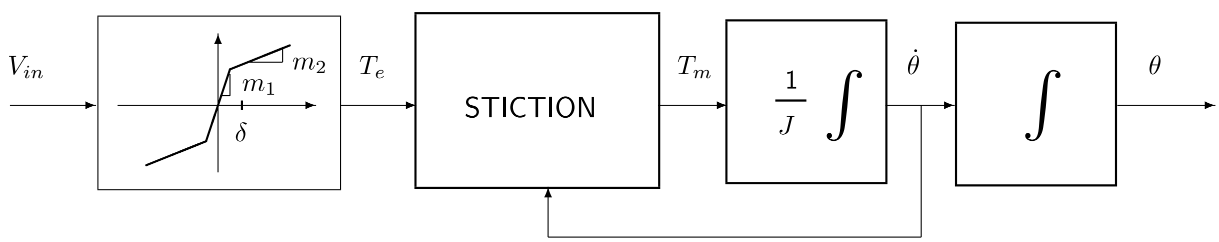
```
%% script for generating a set of G(jw,a) for model "mdl"
%%  JH Taylor  5 July 2002

global Ampl Freq Xdim; dpr = 180/pi; % degrees/radian
mtol = 1;  % magnitude tolerance (dB)
ptol = 5;  % phase tolerance (deg)
mdl = 'lim_filt2' % model = lim_filt2.m (2nd order filter + limiter)
Xdim = 4;  % # states, **including Fourier integrals**
%
% amplitude loop
for jj=1:3
   Ampl = 4^(jj-2) %% Ampl = .25, 1, 4
   av(jj) = Ampl;
```

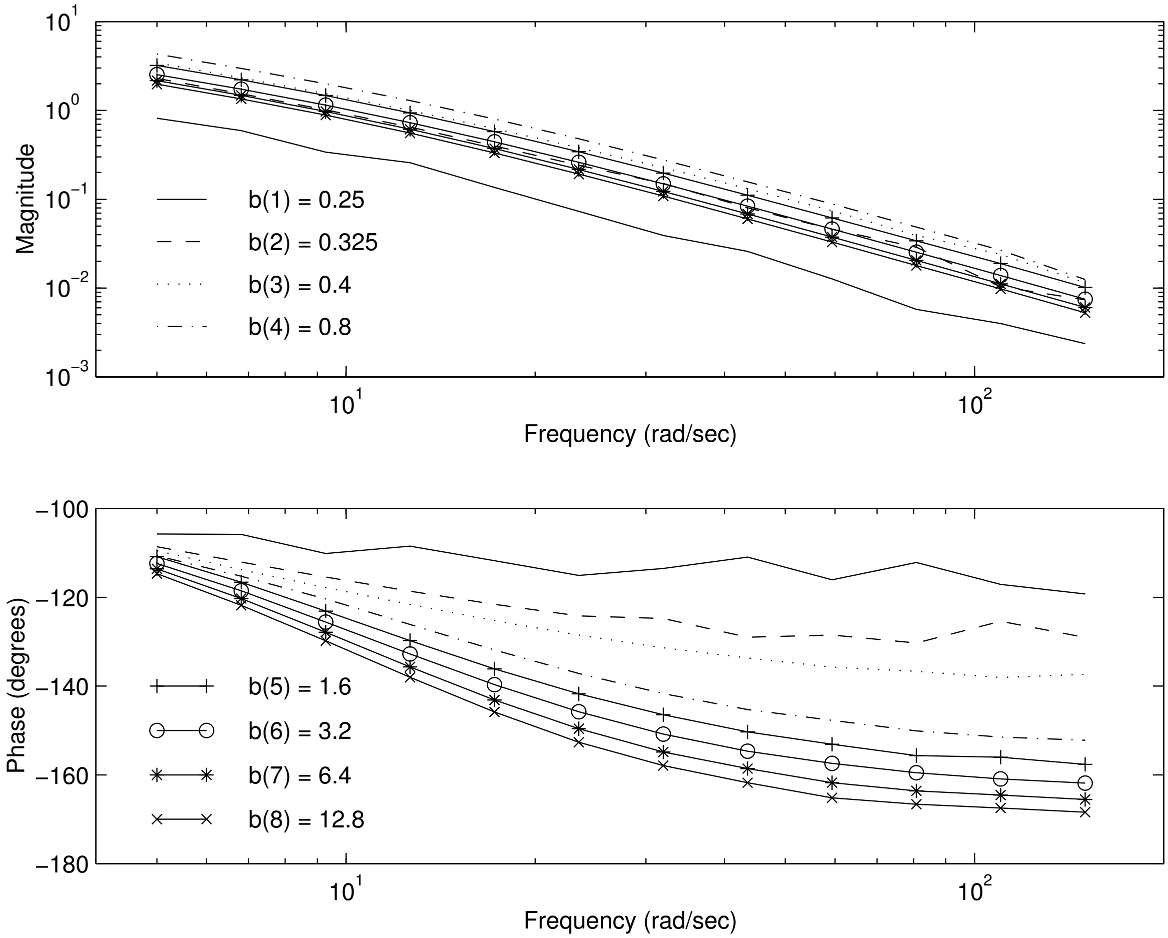
```
   % frequency loop
   for ii=1:30
      Freq = 10^((ii-17)/16)  %% w_min = 0.1, w_max ~= 6.5
      wv(ii) = Freq;
      [mag(ii,jj),phase(ii,jj)] = ggen(mdl,mtol,ptol);
   end % frequency loop
end % amplitude loop
phase = phase .*dpr;  %% change radians to  degrees
%% routine plotting commands for "Bode plots" omitted
```

4. Finally, here is the main result:



Example: SIDF I/O Model, Electromechanical System, by Simulation





Power of Classical SIDF Approach

When will SIDF limit cycle predictions be "good"?

- When $-1/N(a)$ definitely cuts $G(j\omega)$ (not a near miss or near hit)
- When only one limit cycle is predicted (no "nesting")
- When $G(3j\omega_{LC})$ is far from $-1/N(a)$ where ω_{LC} is the predicted limit cycle frequency

When will SIDF I/O models be "good"?

- When the nonlinear system is not highly resonant
- When higher harmonics are not dominant predicted limit cycle frequency

Modern SIDF Analysis

- Given: $\dot{x} = f(x,u)$ with $u(t) = u_0 + \mathrm{Re}\,[\,a\exp(j\omega t)\,]$
- Assume: $x(t) \stackrel{\sim}{=} x_c + \mathrm{Re}\,[\,b\exp(j\omega t)\,]$
- Quasilinearize the entire state-space system:

$$\begin{aligned} f(x,u) = \; & f_B(u_0, a, x_c, b) \\ & + \; \mathrm{Re}\,[\,A_{DF}(u_0, a, x_c, b) \cdot b\exp(j\omega t)\,] \\ & + \; \mathrm{Re}\,[\,B_{DF}(u_0, a, x_c, b) \cdot a\exp(j\omega t)\,] \end{aligned} \tag{12}$$

- Therefore DC harmonic balance is given by $0 = f_B(u_0, a, x_c, b)$
- ... and AC harmonic balance is given by:
  - Nonlinear Oscillations: $a = 0$, find $b \neq 0$ such that $[\,j\omega_{LC}I - A_{DF}\,]^{-1}b = 0$ (“A_{DF} has pure imaginary eigenvalues and b is the corresponding eigenvector”), i.e., limit cycles are predicted if solutions $b\,,\ \omega_{LC}$ exist
  - Forced Response: $b = [\,j\omega I - A_{DF}\,]^{-1}B_{DF} \cdot a$

SIDFs for Multivariable Functions

- Single-input nonlinearities $f(v)$ are quasilinearized as before
- Multi-variable nonlinearities $f(v_1, v_2, \ldots)$ are more complicated; products and powers of states are easiest to do:

$$
\begin{aligned}
\text{Given: } f(x) &= x_1 x_2^2 \\
&= (x_{10} + \mathrm{Re}[\, a_1 \exp(j\omega t)\,])(x_{20} + \mathrm{Re}[\, a_2 \exp(j\omega t)\,])^2 \\
&= \ldots \\
&\stackrel{\sim}{=} [\, x_{10}\, x_{20}^2 + \frac{1}{2} x_{10} \,|\, a_2 \,|^2 + x_{20}\, a_1 \bullet a_2 \\
&\quad + [\, x_{20}^2 + \frac{1}{4} |\, a_2 \,|^2 \,] \cdot x_{1,AC} \\
&\quad + [\, 2x_{10}\, x_{20} + \frac{1}{2} a_1 \bullet a_2 \,] \cdot x_{2,AC}
\end{aligned} \tag{13}
$$

(via trigonometric identities and eliminating higher harmonic terms), where $\bullet$ denotes dot product, $a_1 \bullet a_2 = \mathrm{Re}\, a_1 \cdot \mathrm{Re}\, a_2 + \mathrm{Im}\, a_1 \cdot \mathrm{Im}\, a_2$

Handling multivariable functions represents a **significant generalization** over the classical approach

Multivariable Limit Cycle Example

The following second-order differential equation has been derived to describe the local behavior of solutions to a two-mode panel flutter model:

$$\ddot{\chi} + (\alpha + \chi^2)\,\dot{\chi} + (\beta + \chi^2)\,\chi = 0 \tag{14}$$

Heuristically, it is reasonable to predict that limit cycles may occur for negative α (so the second term provides damping that is negative for small values of χ but positive for large values). Observe also that there are three singularities if β is negative: $\chi_0 = 0, \pm\sqrt{-\beta}$. Making the usual choice of state vector, $x = [\chi \;\; \dot{\chi}]^T$, the corresponding state vector differential equation is

$$\dot{x} = \begin{bmatrix} \dot{\chi} \\ \ddot{\chi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} x - \begin{bmatrix} 0 \\ x_1^2\,(x_1 + x_2) \end{bmatrix} \tag{15}$$

The SIDF assumption for this system of equations is that

$$\begin{aligned} x_1 = \chi &\;\cong\; \chi_c + a_1 \cos(\omega t) \\ x_2 = \dot{\chi} &\;\cong\; -a_1\,\omega \sin(\omega t) \end{aligned}$$

($x_2 = \dot{\chi}$ so x_2 has a center value of 0 and $a_2 = -j\,\omega\,a_1$)

Therefore, the combined nonlinearity in Eqn. 15 may be quasilinearized to obtain

$$\begin{aligned} x_1^2\,(x_1 + x_2) &= (\chi_c + a_1 \cos(\omega t))^2 (\chi_c + a_1 \cos(\omega t) - a_1\,\omega \sin(\omega t)\,) \\ &\cong (\chi_c^3 + \frac{3}{2}\chi_c\,a_1^2) + (3\,\chi_c^2 + \frac{3}{4}a_1^2)\,a_1 \cos(\omega t) \\ &\qquad + (\chi_c^2 + \frac{1}{4}a_1^2)\,(-a_1\,\omega \sin(\omega t)) \\ &\triangleq f_B + N_1 \cdot a_1 \cos(\omega t) + N_2 \cdot (-a_1\,\omega \sin(\omega t)) \end{aligned} \tag{16}$$

This result is obtained by expanding the first expression using trigonometric identities and discarding all terms except the fundamental ones ($k = 0, 1$).

Therefore, we require that

$$F_{DF} = \begin{bmatrix} 0 \\ -\chi_c\,(\beta + \chi_c^2 + \frac{3}{2}a_1^2) \end{bmatrix} = 0 \tag{17}$$

$$A_{DF} = \begin{bmatrix} 0 & 1 \\ -(\beta + 3\chi_c^2 + \frac{3}{4}a_1^2) & -(\alpha + \chi_c^2 + \frac{1}{4}a_1^2) \end{bmatrix} \triangleq \begin{bmatrix} 0 & 1 \\ -\omega_{LC}^2 & 0 \end{bmatrix} \tag{18}$$

(so A_{DF} has imaginary eigenvalues $\pm j\,\omega_{LC}$; again, the canonical form of A_{DF} ensures harmonic balance, not “pure imaginary eigenvalues”).

Multivariable Limit Cycle Example (Cont'd)

Relation 17 shows two possibilities:

- **Case 1:** $\chi_c = 0$, in which case Eqn. 18 yields

$$a_1 = 2\sqrt{-\alpha}\,, \quad \omega_{LC} = \sqrt{\beta - 3\,\alpha} \tag{19}$$

The amplitude a_1 and frequency ω_{LC} must be real for limit cycles to exist. Thus, as conjectured, $\alpha < 0$ is required. for a LC to exist centered about the origin, and β must satisfy $\beta > 3\,\alpha$, so β can take on any positive value but cannot be more negative than $3\,\alpha$.

- **Case 2:** $\chi_c = \pm\sqrt{(\beta - 6\,\alpha)/5} \triangleq \pm\,\chi_{c0}$, yielding

$$a_1 = 2\sqrt{(\alpha - \beta)/5}\,, \quad \omega_{LC} = \sqrt{\beta - 3\,\alpha} \tag{20}$$

For the two limit cycles in Case 2 to exist, centered at $\pm\chi_{c0}$, it is necessary that $3\,\alpha < \beta < \alpha$, so again limit cycles cannot exist unless $\alpha < 0$. One additional constraint must be imposed: $|\,\chi_c\,| > a_1$ must hold, or the two limit cycles will "overlap"; this condition reduces the permitted range of β to $2\,\alpha < \beta < \alpha$.

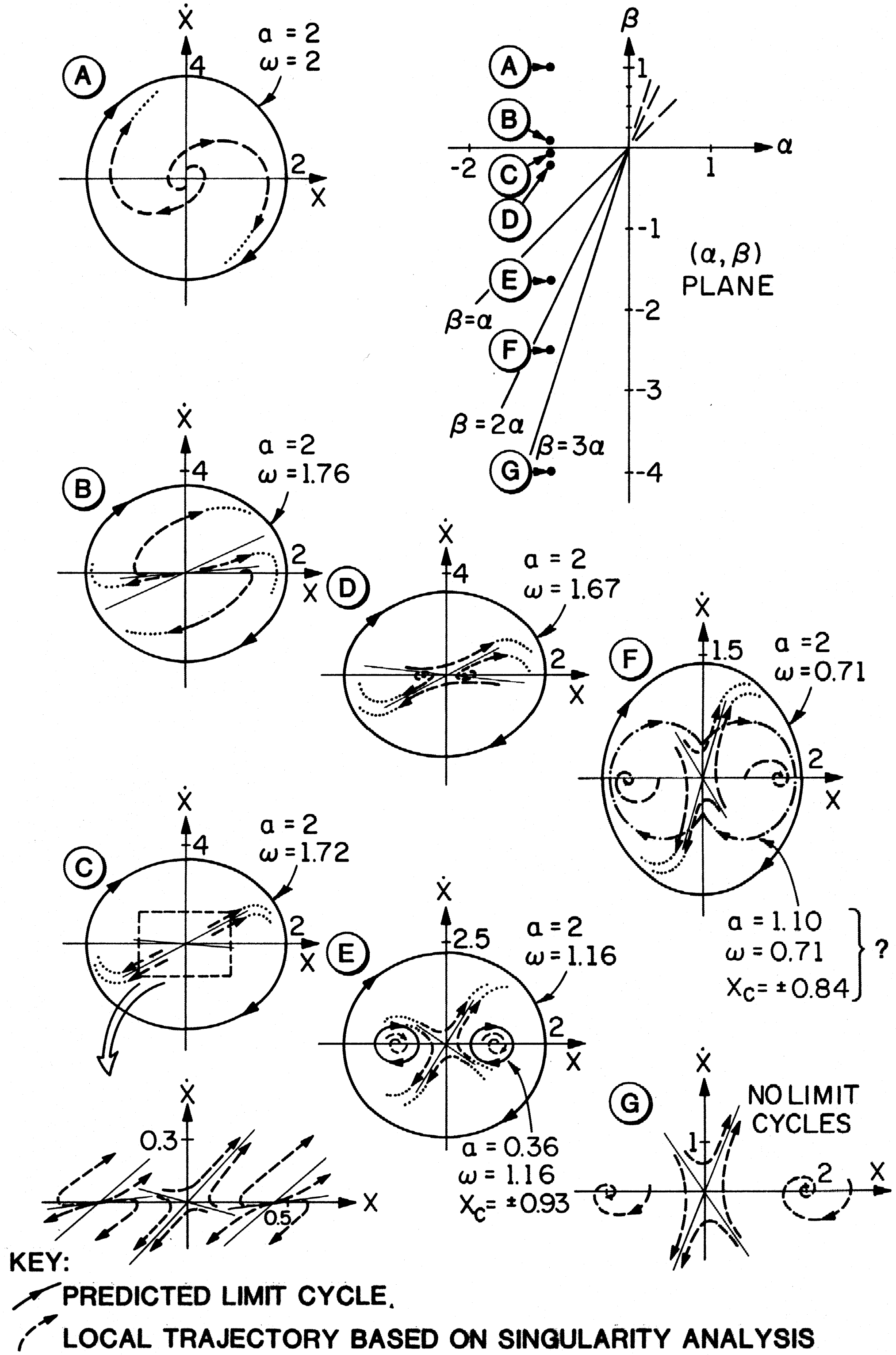
The stability of the Case 1 LC can be determined as follows: Take any $\epsilon > 0$ and perturb the LC amplitude to a slightly larger value, e.g., $a_1^2 = -4\,\alpha + \epsilon$. Substituting into Eqn. 18 yields

$$A_{DF} = \begin{bmatrix} 0 & 1 \\ -(\beta - 3\alpha + \frac{3}{4}\epsilon) & -\frac{1}{4}\epsilon \end{bmatrix} \tag{21}$$

which for $\epsilon > 0$ has "slightly stable eigenvalues". Thus a trajectory perturbed just outside the LC will decay, indicating that the Case 1 LC is stable. A similar analysis of the Case 2 LC is more complicated (since a perturbation in a_1 produces a shift in χ_c that must be considered), and thus is omitted.

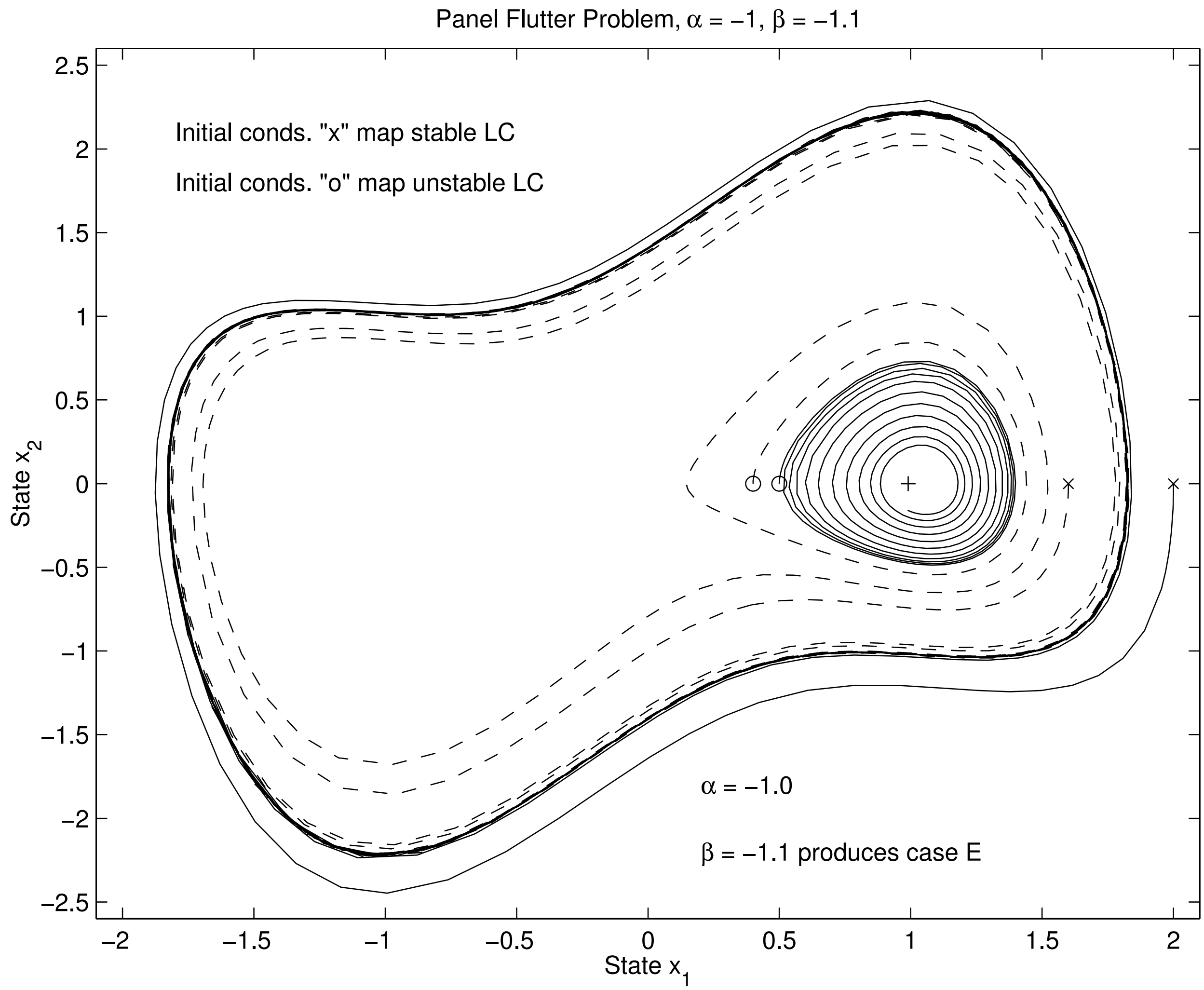
Based on the SIDF-based LC analysis outlined above, the behavior of the original system Eqn. 15 is portrayed for $\alpha = -1$ and seven values of β below:

Multivariable Limit Cycle Example (Cont'd)



Multivariable Limit Cycle Example (Cont'd)

Here is one simulation confirmation of these SIDF results, corresponding to Case E:



The outer limit cycle is clearly stable, while the inner limit cycle is unstable; the equilibrium inside the inner limit cycle is stable. There is another unstable limit cycle centered (approximately) at $x_{10} \cong -1$, $x_{20} = 0$

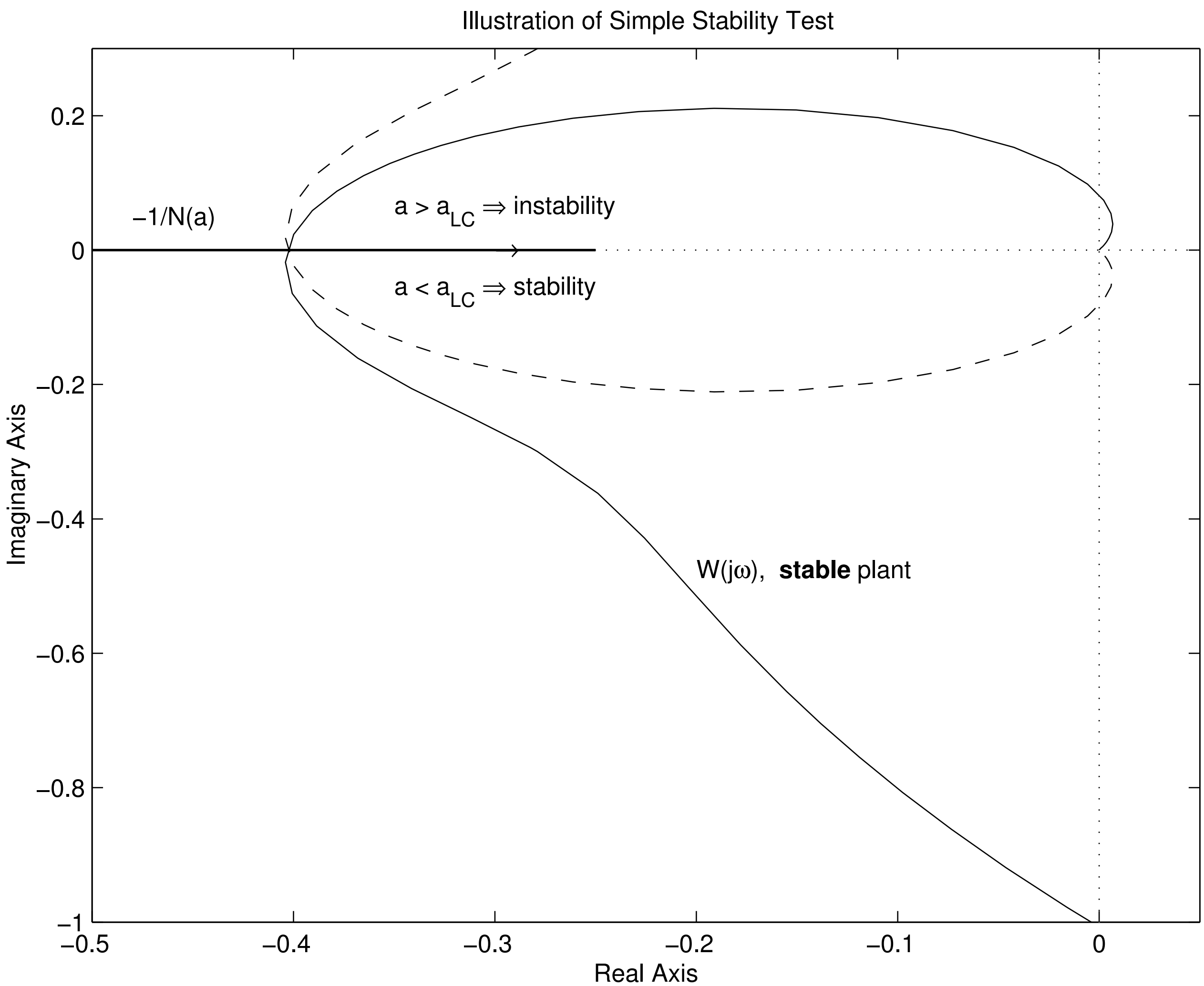
Limit Cycle Stability

The SIDF approach can also yield information on **limit cycle stability** – assume a limit cycle is predicted with amplitude a_{LC}, then:

- The limit cycle is **stable** if $a > a_{LC}$ moves the pure imaginary eigenvalues into the *left* half plane and $a < a_{LC}$ moves the pure imaginary eigenvalues into the *right* half plane
- The limit cycle is **unstable** if the converse is true
- Otherwise the limit cycle is **structurally unstable** (this is an uncommon "borderline" case)
- These conditions are easy to check in cases where there is no bias (DC level), otherwise the coupling between the center value and amplitude (y_c, a) must be taken into account

Limit Cycle Stability (Cont'd)

Here is a limit cycle stability test in the no bias case:



In other words: if $-1/N(a)$ for $a > a_{LC}$ moves inside the RHP map of $W(s)$ the limit cycle is unstable, and conversely.

Another test works if there is no bias and there is only one limit cycle predicted: The limit cycle is stable if the enclosed equilibrium is unstable, and conversely.

SIDF Methods: Conclusions

- SIDF techniques are very powerful for studying periodic behavior (nonlinear oscillations, forced response), even in high order and highly nonlinear dynamic system models, even where discontinuous and multi-valued functions exist
- One of the key uses of this approach is exploration:
  - Finding areas in parameter space where limit cycles exist and boundaries where bifurcations occur
  - Determining how a nonlinear system's response to sinusoidal inputs changes as model parameters change
- SIDF analysis and simulation are highly complementary; both have important roles to play