EE 6383 – Nonlinear Control Systems

Topic 4: Nonlinear System Stability

Dr. James H. Taylor
Department of Electrical & Computer Engineering
telephone: 453-5101
e-mail: jtaylor@unb.ca
web site: www.ee.unb.ca/jtaylor/

February 18, 2003

Topic Outline

- **Lyapunov Stability**
  - Motivation
  - Definitions and Methods
  - Basic Theorem of Lyapunov (LaSalle)
  - Geometrical Interpretation
  - Comments
  - Examples
- **Absolute Stability**
  - Motivation
  - Methods and Definitions
  - Nyquist Revisited
  - Problem of Lur'e
  - Solution Due to Popov
  - The Circle Criterion (CC)
  - Significance of the Popov and Circle Criteria
  - Examples

References:

- LaSalle & Lefschetz, *Stability by Lyapunov's Direct Method with Applications*, Academic, 1961.
- Kalman & Bertram, "Control System Analysis and Design Via the 'Second Method' of Lyapunov", ASME *J. of Basic Engg.*, June 1960.
- Narendra & Taylor, *Frequency Domain Criteria for Absolute Stability*, Academic, 1973.

Motivation

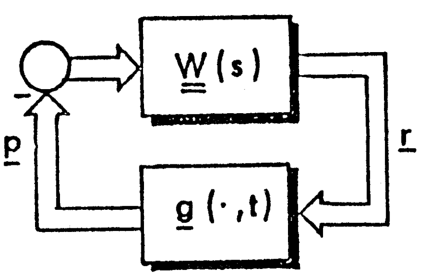
- Stability analyses/statements are a serious business
- No "loose" method of determining stability is fool-proof; carelessness is **very** dangerous
- The definition of stability itself is **extremely** subtle and so are the theorems
- There **are** rigorous methods of analysis (so there are **no** excuses!)
- ...and some rigorous methods are even easy to use!
- Lyapunov's Direct Method is also an important basis for:
  - Absolute stability theory
  - Adaptive control (model-reference adaptive control)
  - Sliding mode (variable structure) control
  - ...

SYSTEM MODELS

1. GENERAL CASE:

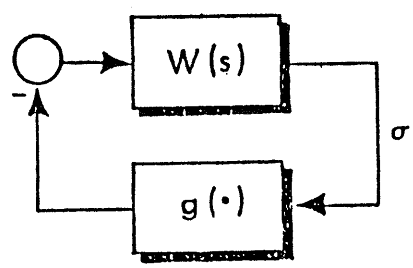
$$\underline{\dot{x}} = \underline{f}\,(\underline{x}, t)$$

2. USEFUL SPECIAL CASES:



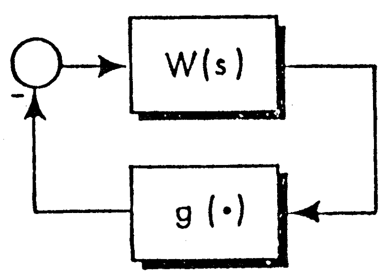
$$p_i = g_i(r_i, t)$$

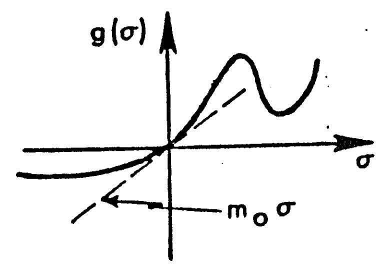
$$i = 1, 2, \ldots m$$



LOOSE METHOD 1:

SMALL-SIGNAL LINEARIZATION (TAYLOR SERIES)





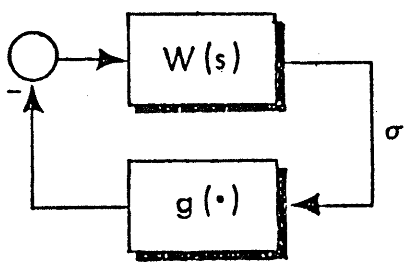
MISCONCEPTION: $-1/m_o \notin \mathcal{W}_R$ MEANINGFULLY GUARANTEES STABILITY

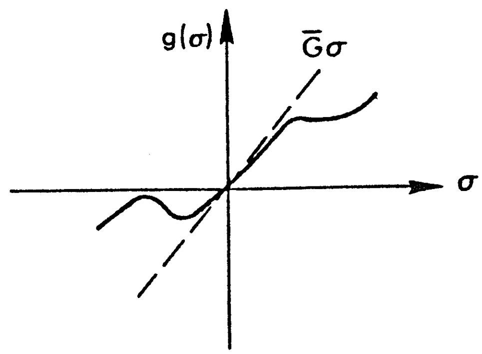
PROBLEM: THE ABOVE CONDITION GUARANTEES <u>LOCAL</u> STABILITY ONLY (ALSO CALLED <u>INFINITESIMAL</u> STABILITY)

LOOSE METHOD 2:

GLOBAL GAIN SECTOR LINEARIZATION

(THE AIZERMAN CONJECTURE ... 1949)

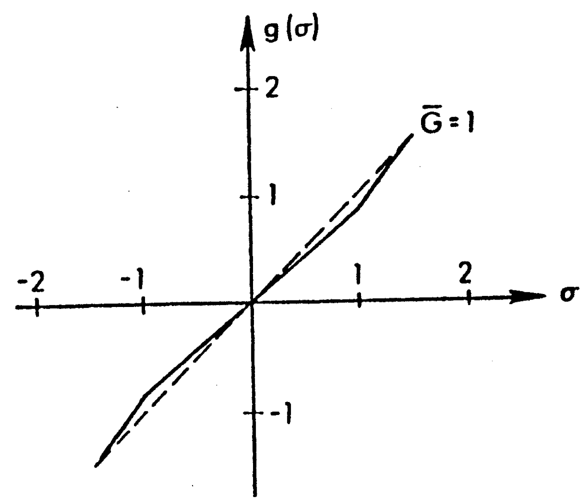




CONJECTURE: $-1/k \notin \mathcal{W}_R$ FOR $0 \leq k < \bar{G}$ GUARANTEES ASYMPTOTIC STABILITY.

COUNTEREXAMPLE:

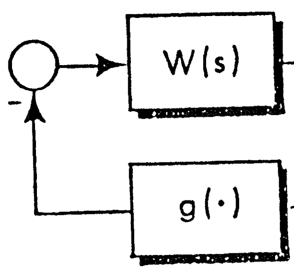
$$W(s) = \frac{-(s+1)}{(s^2+s+1)}$$

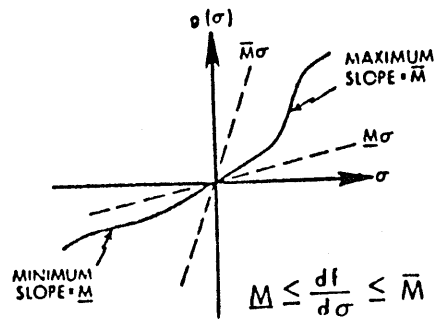


LOOSE METHOD 3:

GLOBAL INCREMENTAL LINEARIZATION

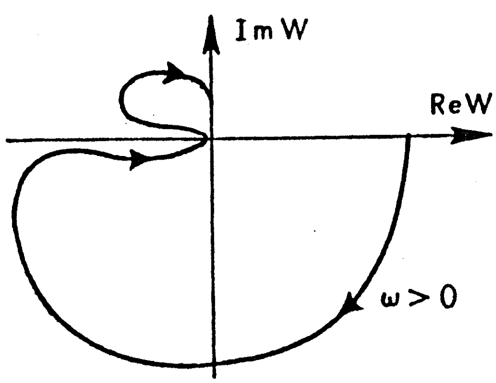
(THE KALMAN CONJECTURE ... 1957)

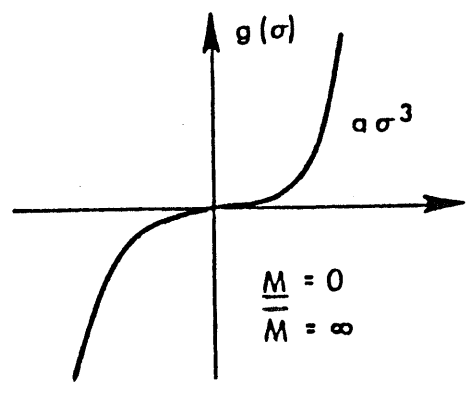




CONJECTURE: $-1/k \notin \mathcal{W}_R$ FOR $\underline{M} \le k \le \bar{M}$ GUARANTEES STABILITY.

COUNTEREXAMPLE:

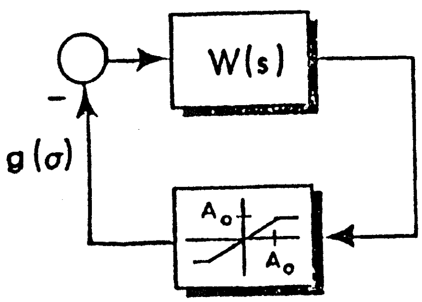




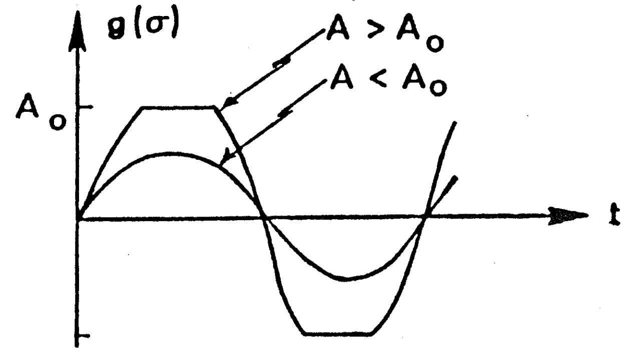
LOOSE METHOD 4:

THE DESCRIBING FUNCTION METHOD

("CONVERSE LIMIT CYCLE ANALYSIS")

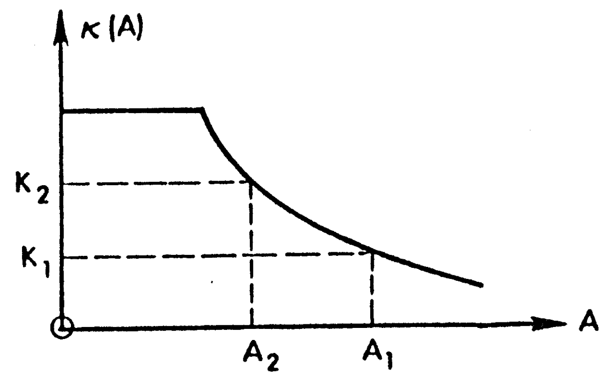


$\sigma = A \sin(\omega t)$

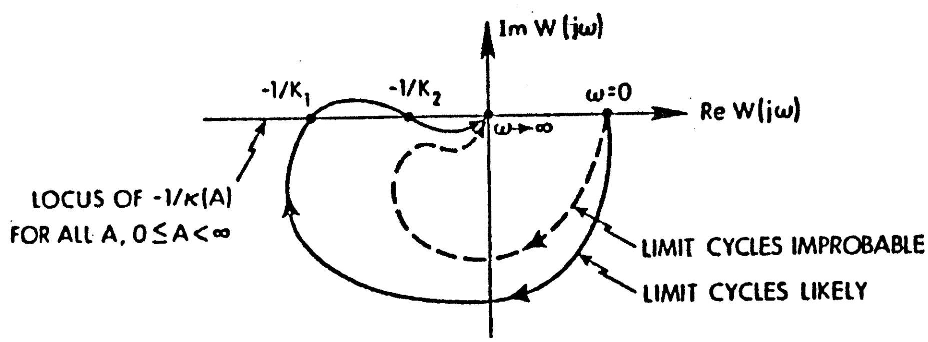


$g(\sigma) \approx \kappa(A)\,\sigma$

- W(s) IS "LOW PASS"
- g (·) IS "WELL - BEHAVED"



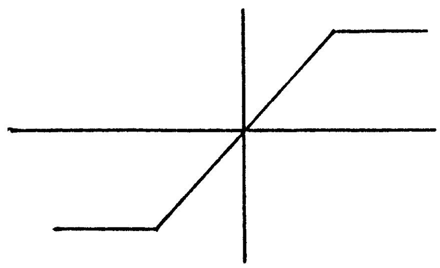
- APPROXIMATE CONDITIONS FOR LIMIT CYCLES:



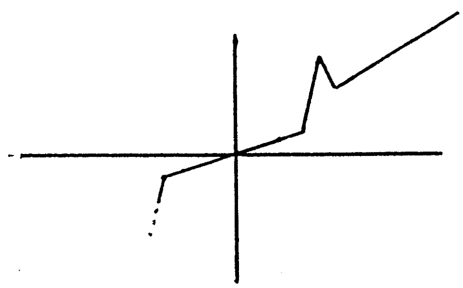
- RELATION TO STABILITY:
  - TEMPTATION: $-1/\kappa(A) \notin \mathcal{W}_R$ FOR ALL A GUARANTEES STABILITY
  - COUNTEREXAMPLES: BOTH CASES TREATED PREVIOUSLY

COMPARISON OF LOOSE METHODS

1. AIZERMAN, KALMAN AND THE DF METHOD MAY BE EQUIVALENT ...



2. ... OR THEY MAY BE VERY DIFFERENT



3. IT IS ALWAYS TRUE THAT: $\underline{M} \leq \underline{F} \leq DF \leq \overline{F} \leq \overline{M}$

BOTTOM LINE: ALL OF THESE ARE RISKY -

- ESPECIALLY IF THE SYSTEM IS HIGH ORDER
- ESPECIALLY IF THERE IS MORE THAN ONE IMPORTANT NONLINEARITY

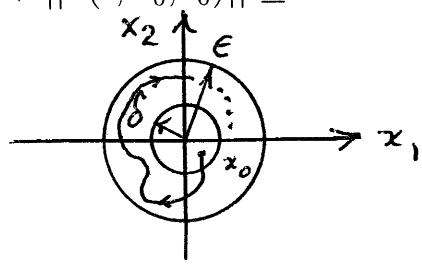
MORE SOPHISTICATED LOOSE METHODS ARE "SAFER", HOWEVER.

Definition of Stability (Lyapunov)

Given $\dot{x} = f(x)$ with equilibrium $x = 0$:

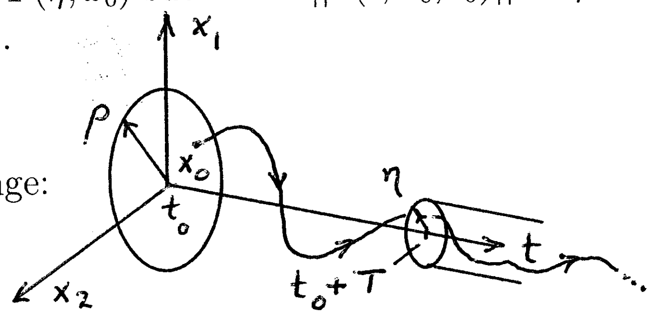
- Stability: The system is **stable** if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $||x_0|| \le \delta \Rightarrow ||x(t; x_0, t_0)|| \le \epsilon$ for all $t \ge t_0$.

  The "ϵ" challenge:

- Attractivity: The system is **attractive** if for some $\rho > 0$ and for any $\eta > 0$ there exists a $T(\eta, x_0)$ such that $||x(t; x_0, t_0)|| < \eta$ for all $||x_0|| \le \rho$ and $t > T + t_0$.

  The "little pipe" challenge:

- Asymptotic Stability: The system is **asymptotically stable** if it is both stable and attractive.

- Refinement # 1: if attractivity depends only on $||x_0||$, i.e., $T = T(\eta, \rho)$ the system is said to be **equiasymptotically stable**.

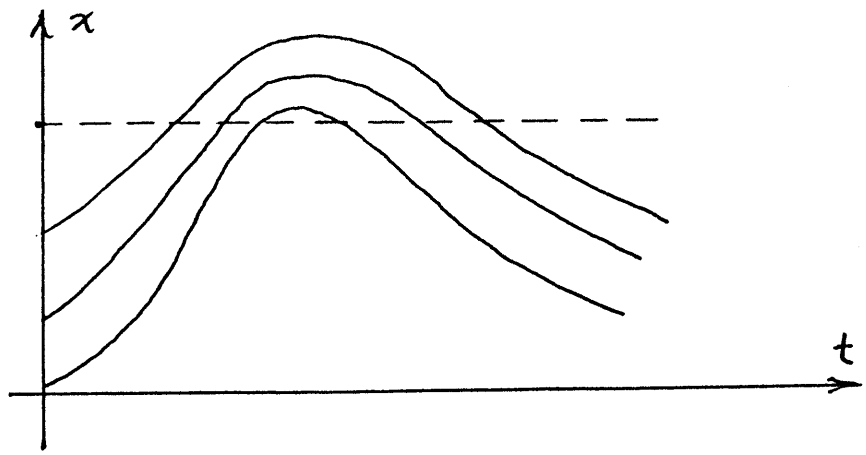
- Refinement # 2: if $\lim_{\epsilon \to \infty} \delta(\epsilon) = \infty$ the system is said to be **asymptotically stable in the large** (UASIL) or, alternatively, **asymptotically stable in the whole** (UASIW).

- **This is even more complicated for $\dot{x} = f(x, t)$!**

UASIL is usually what an engineer wants; however it may be necessary to settle for less ... e.g., it may only be possible to guarantee stability in some region; then the difficult task is determining that the stability region is **big enough**

DEFINITIONS OF STABILITY (CONT'D) -

1. WHY ATTRACTIVITY IS NOT ENOUGH



FOR SPECIFIC EXAMPLES, SEE HAHN. ANY UNSTABLE SYSTEM THAT "SELF DESTRUCTS" IS ATTRACTIVE.

DEFINITIONS OF STABILITY (CONT'D) -

4. WHY UNIFORMITY WITH RESPECT TO TIME IS IMPORTANT

GIVEN: $\dot{x} = 2u - x/t$

YIELDS:

a. initial condition response $= x_0 t_0 / t$ (stable)

b. step response $= t - t_0^2 / t$ (blows up)

THIS IS NOT DESIRABLE BEHAVIOR!

THE STABILITY THEOREM OF LYAPUNOV

The system $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ with equilibrium $\underline{x} = \underline{0}$ is **UASIW** if a scalar function $V(\underline{x}, t)$ with continuous partial derivatives in $\underline{x}$ and t can be found such that

a. $V(\underline{x}, t)$ is positive definite, i.e., a monotonically increasing function $\alpha(\rho)$ exists such that $\alpha(0) = 0$ and $0 < \alpha(\|\underline{x}\|) < V(\underline{x}, t)\ \forall t, x \neq 0$,

b. $V(\underline{x}, t)$ is radially unbounded, i.e., $\lim_{\rho \to \infty} \alpha(\rho) = \infty$,

c. $V(\underline{x}, t)$ is decrescent, i.e., a monotonic function $\beta(\rho)$ exists such that $\beta(0) = 0$ and $V(\underline{x}, t) < \beta(\|\underline{x}\|)$, $\forall\, t, x \neq 0$, and

d. $\dot{V}$ **along system trajectories** ($\dot{V} = \partial V/\partial t + \nabla V^T \dot{\underline{x}}$) is negative definite.

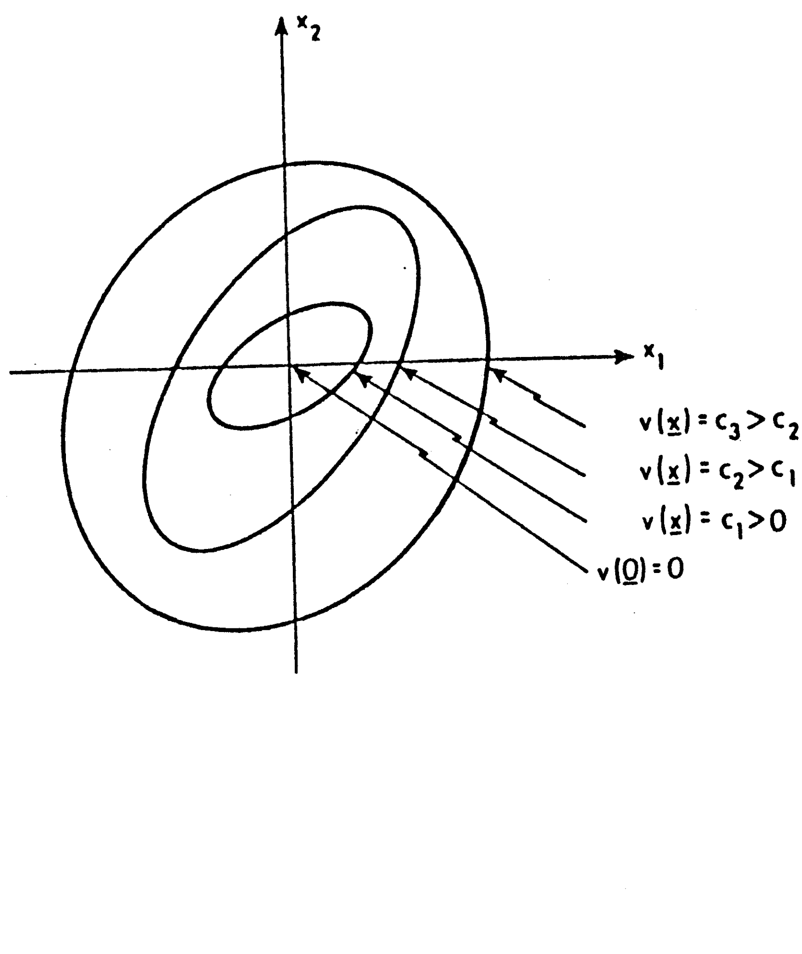
GEOMETRICAL INTERPRETATION

(A "GENERALIZED METRIC" VIEWPOINT)

LYAPUNOV FUNCTION:

$V(\underline{x}, t)$ IS CHOSEN SUCH THAT:

- V BOUNDED $\Rightarrow$ $\underline{x}$ BOUNDED
- $V \rightarrow 0 \Rightarrow \underline{x} \rightarrow 0$



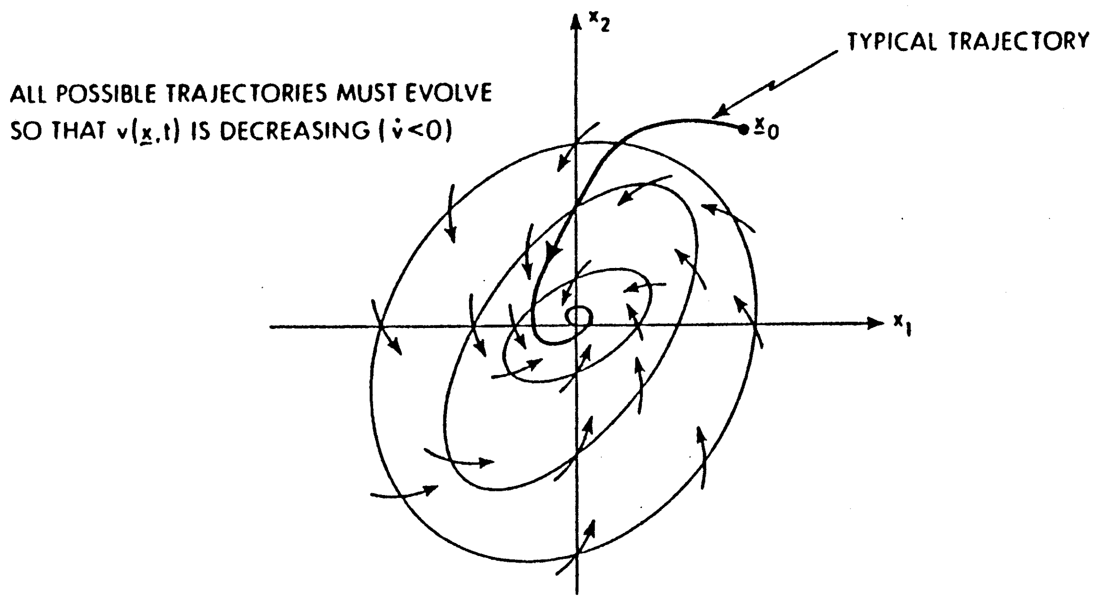
GEOMETRICAL INTERPRETATION (CONT'D)

- BEHAVIOR OF V IS DETERMINED BY ITS DIFFERENTIAL EQUATION:

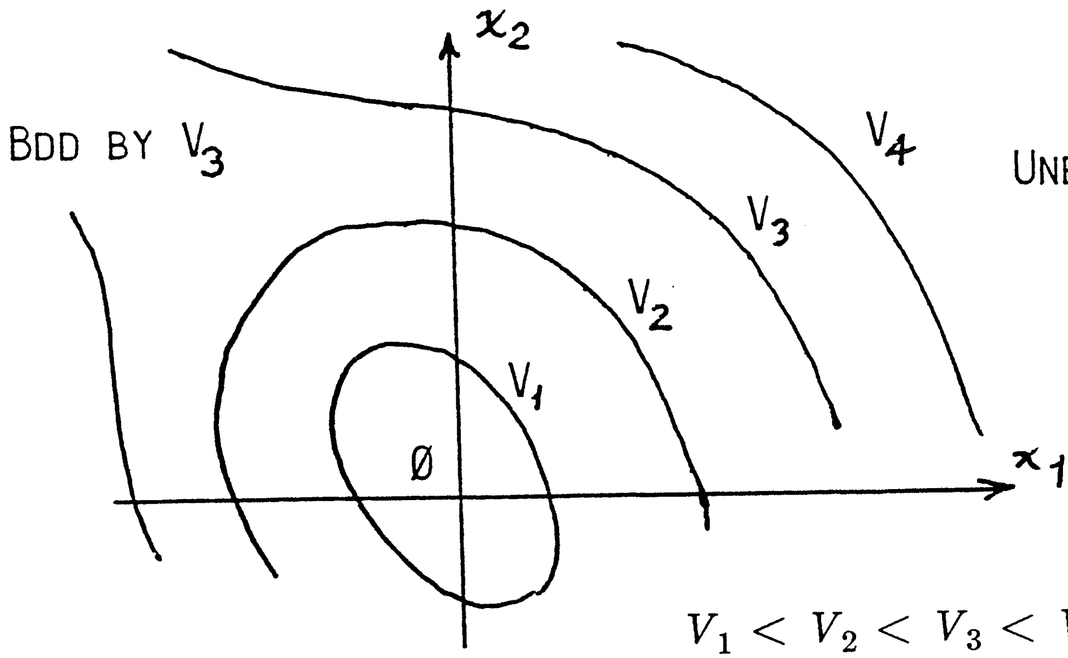
$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + (\underline{\nabla} V)^T \, \underline{f}(\underline{x}, t), \quad \underline{\nabla} V = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix}$$

- $\dot{V} < 0$ GUARANTEES ASYMPTOTIC STABILITY

$$(\dot{V} < 0 \Rightarrow V \rightarrow 0 \Rightarrow \underline{x} \rightarrow \underline{0})$$



WHY RADIAL UNBOUNDEDNESS IS IMPORTANT



$V_1 < V_2 < V_3 < V_4 < \ldots$

TRAJECTORIES CAN GO TO ∞ EVEN THOUGH $\dot{V} < 0$

WHY DECRESCENCE IS IMPORTANT

GIVEN: $\dot{x} = -\,x/t$

$V = (1 + a\,t)\,x^2$

YIELDS: $\dot{V} = -(a + 2/t)x^2$

IF YOU NEGLECT DECRESCENCE, YOU WOULD INCORRECTLY DECLARE THIS SYSTEM TO BE UASIL. ($a > 0$ IS NEEDED TO MAKE $\dot{V}$ NEGATIVE DEFINITE.) THEN YOU MAY MAKE SOME WRONG ASSUMPTIONS ABOUT THE EFFECT OF INPUTS (SEE PREVIOUS EXAMPLE).

LaSalle's Theorem

A special case for nonlinear *time-invariant* systems provides a notable weakening of the condition on $\dot{V}$ for asymptotic stability:

The system $\dot{x} = f(x)$ with equilibrium $x = 0$ is UASIW if a scalar function $V(x)$ exists such that

- $V(0) = 0$ and the elements of $\nabla V = [\partial V / \partial x_i]$ are continuous with respect to x,
- $V(x)$ is positive definite,
- $V(x)$ is radially unbounded, and
- $\dot{V} = \nabla V^T \dot{x} \leq 0$ for all x and $\dot{V}(x(t; x_0, t_0)) \equiv 0$ cannot occur along any trajectory of $\dot{x} = f(x)$ other than $x(t; x_0, t_0) \equiv 0$

Some observations:

- The function $V(x)$ is both positive definite and radially unbounded if there is some $\epsilon > 0$ such that $V(x) > \epsilon x^T x = \epsilon ||x||^2$ for all x
- The condition on $\dot{V}$ is more simply (but weakly) stated $\dot{V}$ is negative definite (there is some $\delta > 0$ such that $\dot{V} < -\delta x^T x$)

QUADRATIC LYAPUNOV FUNCTIONS
(AND LINEAR SYSTEMS)

1. QUADRATIC LYAPUNOV FUNCTIONS: $V = \underline{x}^T P \underline{x}$; P MUST BE A SYMMETRIC, POSITIVE DEFINITE MATRIX; $P = P^T > 0$:

$$p_{11} > 0, \quad \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} > 0, \quad |P| > 0$$

2. A QUADRATIC LYAPUNOV FUNCTION $\underline{x}^T P(t)\, \underline{x}$ **ALWAYS** EXISTS FOR A STABLE LINEAR SYSTEM, $\dot{\underline{x}} = A(t)\, \underline{x}$

3. YOU CAN READILY **SOLVE** FOR P (ONE P OR ALL P's) IN THE LINEAR TIME-INVARIANT CASE: GIVEN $\dot{\underline{x}} = A\, \underline{x}$,

$$V = \underline{x}^T P \underline{x} \rightarrow \dot{V} = \underline{x}^T (A^T P + P A) \underline{x} = -\, \underline{x}^T Q\, \underline{x}\,;$$

CHOOSE ANY $Q = Q^T > 0$ AND SOLVE FOR P; $P = P^T > 0$ IFF THE SYSTEM IS STABLE.

QUADRATIC LYAPUNOV FUNCTIONS (CONT'D)

GIVEN:

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \underline{x}, \quad V = \underline{x}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & 1 \end{bmatrix} \underline{x};$$

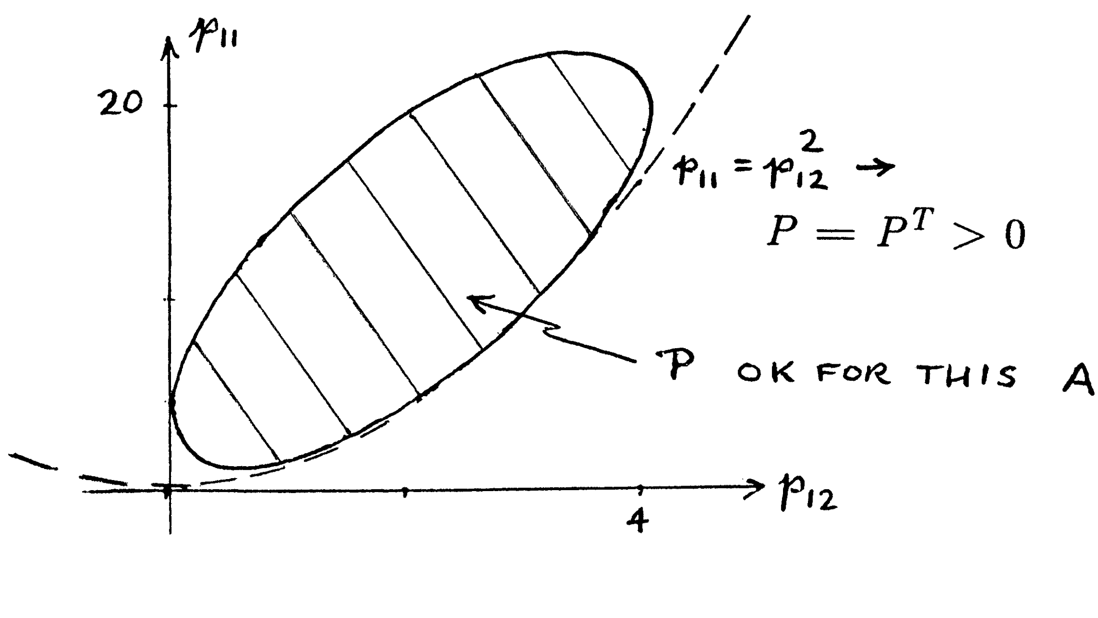
YIELDS:

$$Q = \begin{bmatrix} 8p_{12} & (4 + 4p_{12} - p_{11}) \\ (4 + 4p_{12} - p_{11}) & 8 - 2p_{12} \end{bmatrix}$$

THEN: $p_{11} > p_{12}^2 \rightarrow V$ IS POSITIVE DEFINITE;

$$p_{11}^2 - 8p_{11}(1 + p_{12}) + 16(2p_{12}^2 - 2p_{12} + 1) < 0$$

$\rightarrow \dot{V}$ IS NEGATIVE DEFINITE.



INSTABILITY

THE INSTABILITY THEOREM OF LYAPUNOV:

The system $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ with equilibrium $\underline{x} = \underline{0}$ is **unstable** if a scalar function $V(\underline{x}, t)$ with continuous partial derivatives in $\underline{x}$ and t can be found such that:

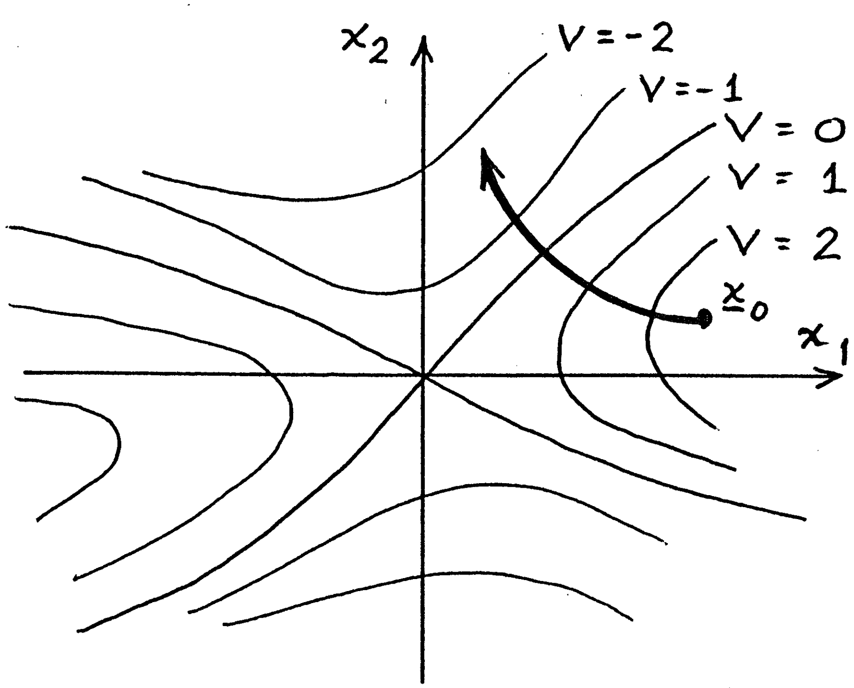
a. $V(\underline{0}, t) = 0$,

b. $V(\underline{x}, t)$ is **sign-indefinite** in some region about $\underline{x} = \underline{0}$,

c. $V(\underline{x}, t)$ is decrescent,

d. $\dot{V}$ along system trajectories $(\dot{V} = \partial V/\partial t + \nabla V^T \dot{\underline{x}})$ is (locally) negative definite.

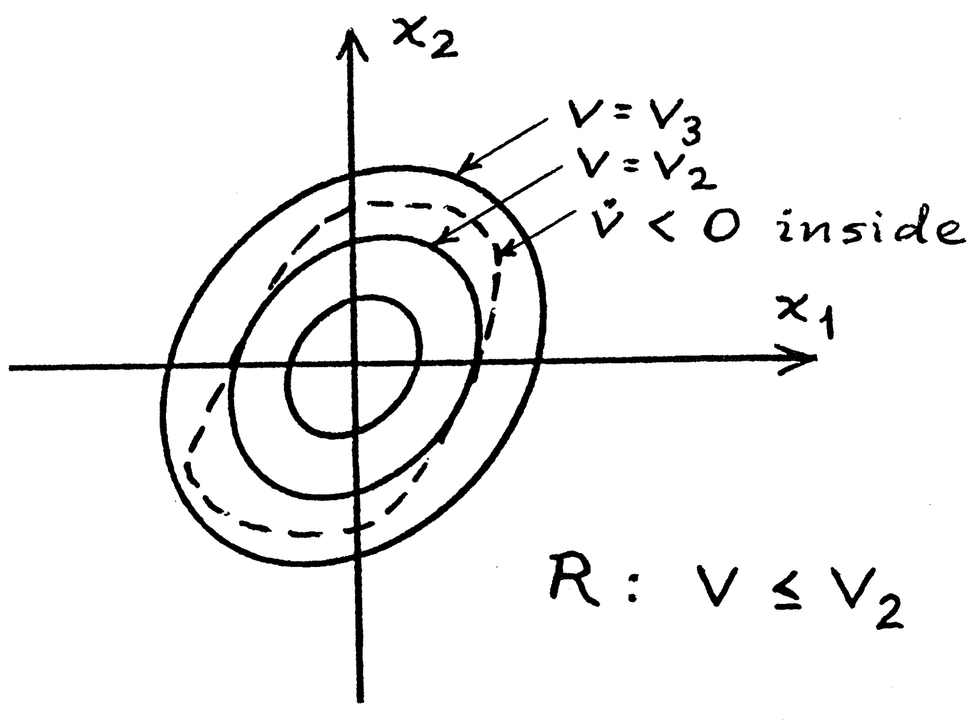
GEOMETRICAL INTERPRETATION:



STABILITY REGIONS

Informally, a region **R** is in the stability region of the system $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ with equilibrium $\underline{x} = \underline{0}$ if **R** is defined by $V(\underline{x}, t) \leq V_0$ where V_0 is the largest positive constant such that $\dot{V}$ is negative definite in **R**.

GEOMETRICAL INTERPRETATION:



→ Zubov's method

EXAMPLE 1 - "LEGITIMIZING" SSL

GIVEN: $\dot{\underline{x}} = \underline{f}(\underline{x}, t) = A(t)\underline{x} +$ higher-order terms (HOT)

THE NONLINEAR SYSTEM IS UAS **IN SOME NEIGHBORHOOD OF THE ORIGIN** IFF THE LINEARIZED SYSTEM IS UAS.

PROOF:

Choose $V = \underline{x}^T P(t)\,\underline{x}$ that proves the UAS of $\dot{\underline{x}} = A(t)\underline{x}$ (Such a Lyapunov function is guaranteed to exist.) Then, for this choice of V,

$$\dot{V} = \underline{x}^T[A(t)^T P + PA(t) + \dot{P}]\underline{x} + 2\underline{x}^T P * HOT$$

By this selection of V, the $\underline{x}^T[\ldots]\,\underline{x}$ term is negative definite, and for "sufficiently small" $||\underline{x}\,||$, this quadratic term will dominate the remainder. QED.

EXAMPLE 2 - BARBASHIN'S RESULT

GIVEN:

$$D^3x + a_3 D^2 x + g(\dot{x}) + f(x) = 0$$

THIS SYSTEM IS UASIL IF:

a. $a_3 > 0$

b. $\dfrac{f(x)}{x} \geq \epsilon_1 > 0$

c. $a_3 \cdot \dfrac{g(\dot{x})}{\dot{x}} - \dfrac{df}{dx} \geq \epsilon_2 > 0$

COMPARE WITH $D^3x + a_3 D^2 x + a_2 \dot{x} + a_1 x = 0 \rightarrow a_3 > 0$ AND $a_3 a_2 > a_1 > 0$ FOR ASYMPTOTIC STABILITY; THIS RESULT IS A LOOSE VALIDATION OF THE KALMAN CONJECTURE (cf KALMAN & BERTRAM).

SUMMARY AND CONCLUSIONS

1. WHEN **STABILITY** IS AT ISSUE, YOU SHOULD BE VERY CAUTIOUS.
2. THE LDM IS A VERY POWERFUL TOOL FOR NON-LINEAR SYSTEMS STABILITY ANALYSIS.
3. IT IS NOT EASY TO APPLY TO MOST SPECIFIC PROBLEMS.
4. SOME VERY POWERFUL GENERIC RESULTS HAVE BEEN OBTAINED USING THE LDM.

COMMENTS

1. THE LYAPUNOV DIRECT METHOD (1892!) AND ITS EXTENSIONS REMAIN THE MOST POWERFUL TOOL AVAILABLE FOR THE STABILITY ANALYSIS OF NON-LINEAR SYSTEMS.

2. YOU OFTEN DO NOT EVEN NEED TO KNOW THE NONLINEARITY(IES) OR THE DYNAMICS EXACTLY IN ORDER TO USE THE LDM RIGOROUSLY.

3. IT IS NOT EASY TO APPLY, IN GENERAL.

4. HOWEVER, GEOMETRICAL THINKING IS VERY USE-FUL, AND THE RELATIONS BETWEEN CONDITIONS AND STABILITY DEFINITIONS IS QUITE TRANSPARENT.

5. THE LDM HAS BEEN USED TO PROVE SOME VERY POWERFUL AND EASILY-APPLIED RESULTS (**ABSOLUTE STABILITY**).

6. THERE ARE SOME INTIMATE RELATIONSHIPS BETWEEN THE LDM AND OPTIMIZATION.

7. MODEL-REFERENCE ADAPTIVE CONTROL MAKES MUCH USE OF THE LDM.

8. SLIDING-MODE CONTROL MAKES MUCH USE OF THE LDM.