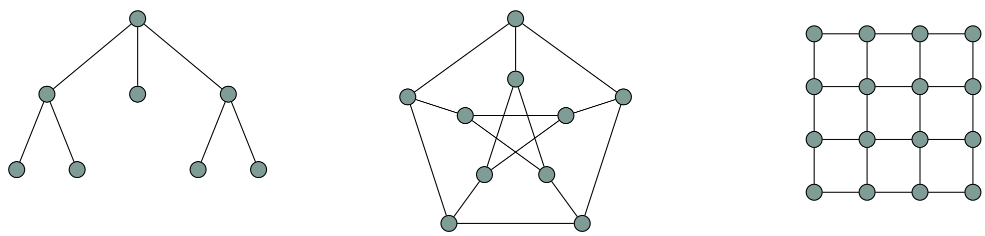
Math 107: Introduction to Graph Theory



Contents

Introduction

Welcome! These are the course notes for Math 107: Introduction to Graph Theory, taught by me (Tara Abrishami) at Stanford University in Winter 2026.

The purpose of these notes falls into three categories:

- To list and define the words and concepts you need to know for each class.
- To list the content we will cover in each class.
- To direct you toward additional resources you can consult.

The additional resources are drawn from the 6th edition of Reinhard Diestel's book *Graph Theory*, which is available for free download through the Stanford University library, and from the recorded video lectures of his class, which closely follow the book, available on Youtube. The definitions in these notes and the content of this course also follow the book, so together these should form a comprehensive and consistent set of resources.

Reading the course notes before each class is a requirement. All of the other materials are optional. If you enjoy reading textbooks, you can consult the book; if you learn well from watching videos, you can watch the video lectures (and Diestel is an engaging lecturer). If you learn best by doing, you can focus on thinking through the extra exercises given each week. While none of these is required, I expect that those who get the most out of this class will engage with some additional material in some way.

These notes will be continuously updated as we go along, so please make sure to refresh the page!

Last update: January 8, 2026.

1. Graphs, paths, cycles

1.1. **January 6th.**

- Graph, vertex, edge
- Adjacency, incidence, degree, neighborhood
- Minimum degree, maximum degree, average degree
- Subgraph, induced subgraph, spanning subgraph
- Graph isomorphism
- Complement of a graph

A (simple) **graph** $G = (V, E)$ consists of a set of **vertices** V and a set of **edges** E, where each edge is a pair of distinct vertices, called the **ends** of the edge. When G is a graph, we use $V(G)$ and $E(G)$ to refer to its vertex set and edge set, respectively. Two vertices v_1 and v_2 of a graph G are **adjacent** if there is an edge of G with ends v_1 and v_2. We denote this edge as v_1v_2. A vertex v of G is **incident** to an edge e of G if v is an end of e. Likewise, an edge e is **incident** to a vertex v of G if v is an end of e.

The **degree** of a vertex v of G, denoted $d(v)$, is the number of edges that v is incident to in G. The **neighborhood** of a vertex v, denoted $N(v)$, is the set of vertices u of G such that v is adjacent to u, i.e. uv is an edge of G. The size of the neighborhood of a vertex is equal to its degree. A vertex is **isolated** if it has degree zero.

The **minimum degree** of a graph G, denoted $\delta(G)$, is the minimum degree $d(v)$ of a vertex v of G. Similarly, the **maximum degree** of G, denoted $\Delta(G)$, is the maximum degree $d(v)$ of a vertex v of G. The **average degree** of G, denoted $d(G)$, is the average degree over all of its vertices; so $d(G) \coloneqq \frac{1}{|V(G)|} \sum_{v \in V(G)} d(v)$.

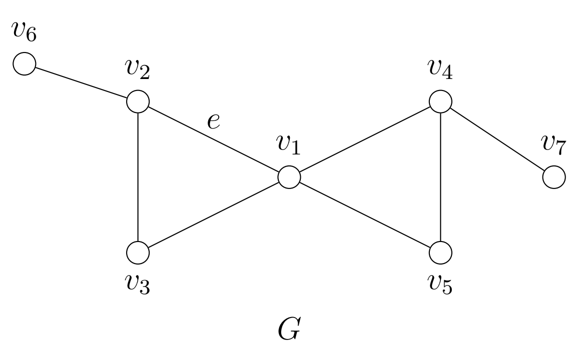


Figure 1. In this example graph G, vertex v_1 is adjacent to vertices v_2, v_3, v_4, and v_5. Vertices v_1 and v_2 are both incident with edge e, and v_1 and v_2 are also the ends of e. The neighborhood of v_1 is $N(v_1) = \{v_2, v_3, v_4, v_5\}$, and its degree is $d(v_1) = 4$.

A graph H is a **subgraph** of a graph G if H is formed from G by deleting vertices and edges. A graph H is an **induced subgraph** of a graph G if H is formed from G by deleting vertices. Observe that every induced subgraph is a subgraph, but not every subgraph is an induced subgraph. A subgraph H of a graph G is **spanning** if the vertex set of H equals the vertex set of G.

Subgraph Induced subgraph

FIGURE 2. This figure shows a subgraph of G on the left and an induced subgraph of G on the right. Neither of these subgraphs is spanning.

Let H and H' be graphs. An **isomorphism** from H to H' is a bijection $\varphi : H \to H'$ such that uv is an edge of H if and only if $\varphi(u)\varphi(v)$ is an edge of H'. Two graphs H and H' are **isomorphic** if there is an isomorphism from H to H'. We typically only care about graphs up to isomorphism. For example, since the graph H' in Fig. 3 is isomorphic to the graph H and H is a subgraph of G in Fig. 1, we would consider G to contain H' as a subgraph.

H H'

FIGURE 3. The graphs H and H' are isomorphic. Every isomorphism from H to H' sends v_2 to b. Any bijective mapping from $\{v_6, v_3, v_1\}$ to $\{a, c, d\}$ will create an isomorphism. For example, the mapping $\varphi : H \to H'$ such that $\varphi(v_6) = a, \varphi(v_3) = d, \varphi(v_1) = c, \varphi(v_2) = b$ is an isomorphism.

The **complement** of a graph $G = (V, E)$, denoted $\overline{G}$, is the graph with vertex set V and such that distinct vertices u and v in V are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.

1.1.1. **Class plan.** In class we will go over the following problems.

1. Write a formula for the average degree $d(G)$ of a graph $G = (V, E)$ in terms of $|V|$ and $|E|$.
2. For a graph $G = (V, E)$, write a formula for the number of edges of the complement $\overline{G}$ in terms of $|V|$ and $|E|$.
3. Is there a 3-regular graph with 9 vertices?
4. Prove that every graph with at least two vertices contains two distinct vertices with the same degree.

1.1.2. **References.** Optional references for today's class:

- Diestel Sections 1.1, 1.2
- Diestel Graph Theory:
  - Lecture 1: Introduction
  - Lecture 2: Invariants I up to minute 38:00.

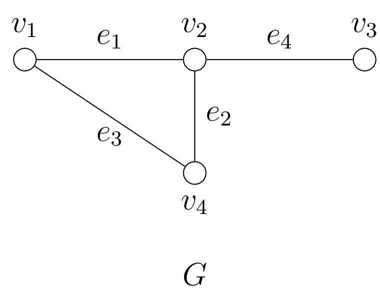
1.1.3. **Class log.**

- We discussed the problems above
- We also discussed the graph reconstruction problem. We observed that the number of vertices, number of edges, and degrees of each vertex can be reconstructed, as long as the graph has at least three vertices. We also observed that reconstruction fails for graphs with two vertices: the two graphs with two vertices are indistinguishable.

1.2. **January 8th.**

- Walk, trail, path
- Connected, disconnected
- Distance, diameter
- Cycle, induced cycle, girth
- Acyclic, forest, tree, leaf
- Cut vertex, cut edge
- k-connected

Let G be a graph. A **walk** in G is a sequence of alternating vertices and edges $v_1e_1v_2 \ldots v_{k-1}e_{k-1}v_k$, such that each edge e_i has ends v_i and v_{i+1}. The vertices v_1 and v_k are the **ends** of the walk. A walk is **closed** if its two ends are the same. A **trail** of G is a walk where no edge is used more than once.



$$W = v_1e_1v_2e_2v_4e_3v_1e_1v_2e_4v_3$$

FIGURE 4. An example walk W in a graph G with ends v_1 and v_3.

A **path** is a graph $P = (V, E)$ with vertex and edge set

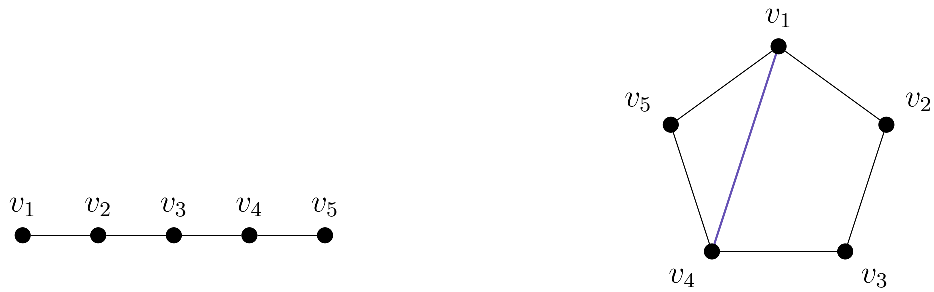
$$V = \{v_1, v_2, \ldots, v_k\}, \quad E = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{k-1}v_k\}.$$

The vertices v_1 and v_k are the **ends** of the path P. The **length** of a path P is the number of its edges. We will often denote a path with k vertices as P_k. (Observe that the path P_k has length $k-1$, not k.) Notice that every path defines a walk.

A path in a graph G is a subgraph of G isomorphic to a path. A walk in G defines a path if and only if no vertex is used more than once.

A graph G is **connected** if G contains a path with ends u and v for every pair of distinct vertices u and v in G. Intuitively, G is connected if you can move from any vertex to any other vertex by traversing the edges of G. A graph G is **disconnected** otherwise. A **connected component** of a graph G is a maximal connected subgraph of G.

If vertices u and v are the ends of a path P or a walk W, we say that P is a path from u to v and W is a walk from u to v. We may also say that P is a u-v path or W is



P_5 $\qquad$ C_5 with a chord

FIGURE 5. This figure shows the path of length 5, P_5, on the left, and the cycle of length 5, C_5, on the right. The purple edge is not part of the cycle, it is a chord.

u-v walk. The **distance** between two vertices u and v of a graph G, denoted $d(u, v)$, is the length of the shortest path from u to v in G. The **diameter** of a graph G, denoted $\text{diam}(G)$, is the maximum distance between any two vertices of G.

A **cycle** is a graph $C = (V, E)$ with vertex and edge set

$$V = \{v_1, v_2, \ldots, v_k\}, \quad E = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{k-1}v_k, v_kv_1\},$$

with $k \geq 3$. Deleting a single edge from a cycle yields a path. The **length** of a cycle is the number of its edges. We will often denote a cycle with k vertices as C_k. (Observe that the cycle C_k has length k.) Notice that every cycle defines a closed walk. The cycle C_3 of length three is also called a **triangle**.

A cycle in a graph G is a subgraph of G isomorphic to a cycle. The **girth** of a graph G, denoted $g(G)$, is the minimum length of a cycle in G. A **chord** of a cycle in a graph G is an edge of G between two vertices v_i and v_j such that v_iv_j is not an edge of the cycle. A cycle in a graph G that has no chords is an **induced cycle**.

A graph with no cycles is called **acyclic**; an acyclic graph is also called a **forest**. A graph is a **tree** if it is connected and acyclic. A vertex of degree one in a tree is called a **leaf**.

Let G be a graph. A **cut vertex** of G is a vertex v such that deleting v disconnects the connected component of G containing v. A **cut edge** of G is an edge $e = uv$ such that u and v are in different connected components of G_e, the graph formed by deleting e.

For an integer $k \geq 2$, a graph G is **k-connected** if $|V(G)| > k$ and for every set X of vertices of G of size less than k, the graph $G - X$ formed by deleting the vertices in X is connected. A graph is k-connected if you must delete at least k vertices to disconnect the graph. The **connectivity** of a graph G, denoted $\kappa(G)$, is the maximum integer k for which G is k-connected.

1.2.1. **Class plan.** In class we will go over the following problems.

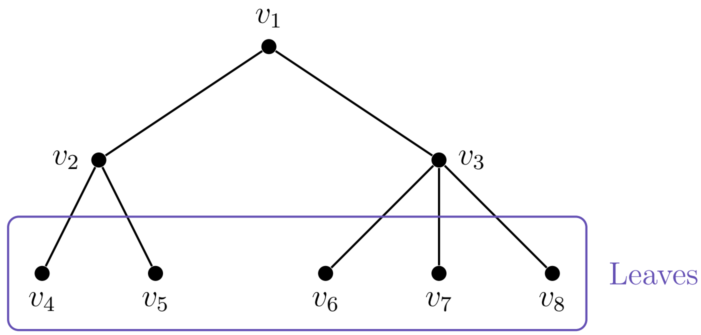


FIGURE 6. An example tree with eight vertices and five leaves.

1. Show that the components of a graph partition its vertex set.

2. Connectivity true or false:
   - Is every graph with minimum degree two 2-connected?
   - Does every 2-connected graph contain a cycle?
   - Is every graph that contains a cycle 2-connected?
   - Does every graph with minimum degree two contain a cycle?
   - Is $\kappa(G) \leq \delta(G)$ for every graph G?

3. Prove that every tree with at least two vertices has more leaves than vertices of degree at least three.

1.2.2. **References.** Optional references for today's class:

- Diestel Sections 1.3, 1.4, 1.8
- Diestel Graph Theory:
  - Lecture 2 from minute 38:00 to 1:04:00.
  - For more advanced material about connectivity: Lecture 2 from minute 1:04:00 to the end, and Lecture 3

1.2.3. **Class log.**

- We discussed the problems above
- We corrected the definition of k-connected. It is now correct in the notes above. To reiterate: a graph G is **k-connected** if $|V(G)| > k$ and if $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$.
- I've included two notes on problems from class below.
- I also included some notes on the graph reconstruction conjecture, since some people asked. **These are not necessary for class, they are just for fun if you are interested!**
- Also, here is a recent Numberphile video on graph reconstruction. This is also just for fun!

Notes from class.

First, let's redo the proof of the last connectivity true/false question using the correct definition of k-connected.

Claim 1.1. For every graph G,

$$\kappa(G) \leq \delta(G).$$

Proof. Let G be a graph and let v be a vertex of G of minimum degree. By the definition of k-connected, $\kappa(G) \leq |V(G)| - 1$. Let $X = N(v)$ be the set of neighbors of v, so $|X| = \delta(G)$. If $|V(G)| = \delta(G) + 1$, i.e. if the whole graph consists of v and its neighbors, then $\kappa(G) \leq |V(G)| - 1 = \delta(G)$, as desired.

Therefore, we may assume that $|V(G)| > \delta(G) + 1$, so G contains some vertex w that is non-adjacent to v. In particular, w is not in X. Now, the graph $G - X$ formed by deleting X yields a graph with at least two connected components, one consisting of the vertex v and one containing the vertex w. This proves that $\kappa(G) \leq \delta(G)$. □

I've next written down a proof of the last statement from class. I have added the assumption that we are working with trees with at least two vertices to better illustrate the base case.

Claim 1.2. Every tree with at least two vertices has more leaves than vertices of degree at least three.

Proof. We prove the claim by induction on the number of vertices. The base case is when the tree has two vertices. There is a single two-vertex tree, the graph consisting of two vertices with an edge between them. This tree has two leaves and zero vertices of degree at least three, so the statement holds for the base case.

Let us assume that the statement holds for all trees with n vertices, for $n \geq 2$. We will show using this assumption that the statement holds for trees with $n+1$ vertices.

Consider a tree T with $n+1$ vertices. Let v be a leaf of T, and let T' be the tree formed by deleting v from T. Now, T' is a tree with n vertices, so, by the induction hypothesis, T' has more leaves than vertices of degree three.

What happens to the number of leaves and the number of vertices of degree at least three when we move from T' to T? Let u be the (unique) neighbor of v in T. Every vertex of T' except u has the same degree in T as in T'. Therefore, T has one more leaf than T', unless u is a leaf of T', in which case T has the same number of leaves as T'. Also, T has the same number of vertices of degree at least three as T', unless u has degree two in T', in which case T has one more vertex of degree at least three as T'. In particular, if T has one more vertex of degree at least three than T', then T also has one more leaf than T'. Therefore, T has more leaves than vertices of degree at least three. This completes the proof. □

If you want to see the ending in more detail, you can also do out the cases. For example, we could replace the last paragraph above with the following:

Cases for u. Let u be the unique neighbor of v in T. There are three possibilities: u is a leaf of T', u has degree two in T', or u has degree at least three in T'.

First, suppose that u is a leaf in T'. Then, T has the same number of leaves as T': the vertex v is a leaf of T but not of T', and the vertex u is a leaf of T' but not of T. The number of vertices of degree at least three is also the same in T and in T'. Therefore, the statement holds for T in this case.

Next, suppose that u has degree two in T'. Then, T has one more leaf than T': every leaf of T' is also a leaf of T, and v is a leaf of T and not of T'. Also, T has one more vertex of degree at least three than T': every vertex of degree at least three in T' has degree at least three in T, and u has degree three in T but not in T'. Therefore, the statement holds for T in this case.

Finally, suppose that u has degree at least three in T'. Then, T has the same number of vertices of degree at least three as T'. Also, T has one more leaf than T': every leaf of T' is a leaf of T, and v is a leaf of T but not of T'. Therefore, the statement holds for T in this case.

Since the statement holds in all cases, this proves that the statement holds for T. This completes the proof by induction. $\square$

Graph Reconstruction notes.

This section is just for fun! You do not need to know this for class.

First let's review the set-up of graph reconstruction. Let G be a graph. The set of vertex-deleted subgraphs of G is called the **deck** of G. Specifically, the deck of G is the set of graphs $\{G - v \mid v \in V(G)\}$, where $G - v$ is the graph formed by deleting v. A graph G is **reconstructible** if G is uniquely determined (up to isomorphism) by its deck.

Here are some types of graphs which are known to be reconstructible.

- Graphs where every vertex has the same degree (these are called **regular** graphs)
- Graphs that contain a vertex adjacent to every other vertex
- Disconnected graphs
- Trees

The first two have simple proofs that you can try to find on your own. The second two are harder. Come to office hours if you have questions about any of this or want to see the proofs!

2. Trees and other basic notions

2.1. January 13th.

- Spanning tree
- Bipartite graph, r-partite graph
- Separator, separation

A **spanning tree** of a graph G is a spanning subgraph T of G such that T is a tree.

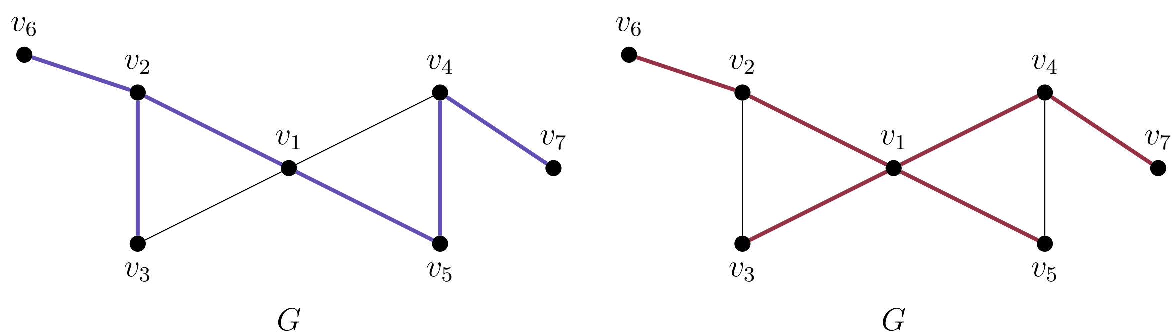


Figure 7. Two different spanning trees, one in purple on the left and one in red on the right, of a graph G.

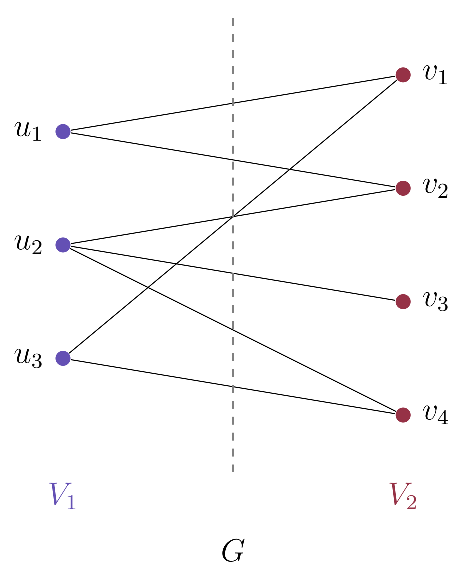


Figure 8. An example of a bipartite graph G with bipartition V_1, V_2.

A graph $G = (V, E)$ is **bipartite** if the vertex set of G can be partitioned into two parts V_1, V_2, called a **bipartition** of V, such that every edge of G has one end in V_1 and one end in V_2. Given an integer $r \geq 2$, a graph $G = (V, E)$ is **r-partite** if the vertex set of G

can be partitioned into r parts $V_1, \ldots, V_r$ such that every edge of G has ends in distinct parts.

2.1.1. **Class plan.** In class we will go over the following problems.

1. Prove that every connected graph has a spanning tree.

2. Prove that a graph G is bipartite if and only if every cycle of G is even.

3. Prove the following characterizations of trees.

   Theorem 2.1. The following are equivalent for every graph T.
   (a) T is a tree.
   (b) For every two vertices u, v of T, there is a unique path of T from u to v.
   (c) T is connected but for every edge e of T the graph $T - e$ formed by deleting edge e is disconnected.
   (d) T has no cycle but for every pair of non-adjacent vertices u, v of T, the graph $T + xy$ formed by adding edge xy to T contains a cycle.
   (e) T is connected and $|E(T)| = |V(T)| - 1$.
   (f) T is acyclic and $|E(T)| = |V(T)| - 1$.

4. A **cycle decomposition** of a graph G is a set of subgraphs $\{C_1, \ldots, C_k\}$ of G such that each subgraph C_i is a cycle and each edge of G appears in exactly one subgraph C_i. Prove that a graph G admits a cycle decomposition if and only if the degree of every vertex of G is even.

5. A **Eulerian tour** of a graph G is a closed walk of G that uses each edge of G exactly once. A graph is **Eulerian** if it has an Eulerian tour. Prove that a graph G is Eulerian if and only if G is connected and the degree of every vertex of G is even.

6. (Time permitting) Let $k \geq 1$ be an integer. Prove that every graph with average degree at least $4k$ has a k-connected subgraph. In fact, prove that every such graph G has a $(k+1)$-connected subgraph H with average degree at least $2k$.

2.1.2. **References.** Optional references for today's class:

- Diestel Sections 1.5, 1.6
- Diestel Graph Theory:
  - Lecture 4
  - Lecture 5 up to minute 30:00

2.2. **January 15th.**

- Minor
- Subdivision, contraction
- Topological minor
- Independent set, clique

Let G and X be graphs. We say that **G is an IX** if there is a partition $\{V_x \mid x \in V(X)\}$ of $V(G)$ satisfing the following:

- V_x is a connected subset of G for every $x \in V(X)$;
- for every pair $x, y \in V(X)$, the edge xy is in X if and only if G contains an edge with one end in V_x and one end in V_y.

The sets V_x are called the **branch sets** of the IX. If G is an IX, then we say that X is a **contraction minor** of G, and that X is formed from G by contracting each branch set V_x.

If a graph G contains an IX as a subgraph, then X is a **minor** of G. The IX is the **model** of X in G.

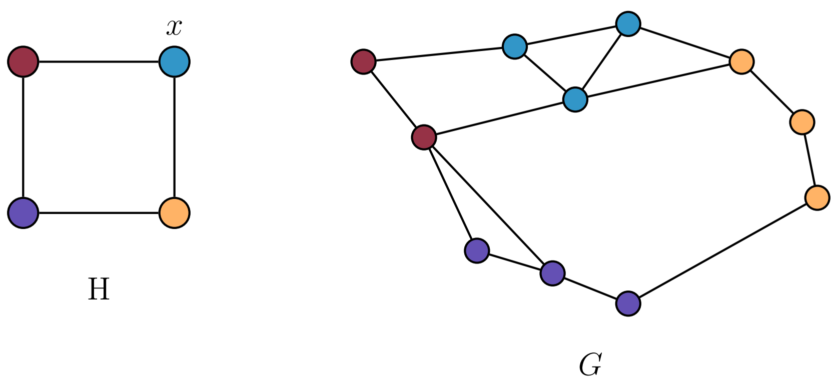


FIGURE 9. In this figure, the graph G on the right is an IH, where H is the graph on the left. The vertex-partition of G giving the branch sets is indicated by the vertex colors. For example, V_x for the labeled vertex x on the left consists of the set of the blue vertices of G.

Let X be a graph. A **subdivision** of X is a graph G formed from X by replacing edges of X with paths of length at least one. If G is a subdivision of X, then we say that **G is a TX**. If a graph G contains a TX as a subgraph, then X is a **topological minor** of G.

Let G be a graph. A vertex-subset I of G is **independent** if there is no edge of G with both endpoints in I. A vertex-subset K of G is a **clique** if for every pair of vertices x, y in K, the edge xy is in G. The clique consisting of n vertices is denoted K_n.

2.2.1. **Class plan.** In class we will go over the following problems.

1. What is the relationship between subgraphs, induced subgraphs, minors, subdivisions, and topological minors?
2. Prove that a graph G contains a graph X as a minor if and only if X can be formed from G by deleting vertices, deleting edges, and contracting edges.
3. Describe in a word the class of graphs consisting of all graphs that do not have a triangle minor.
4. Prove that if a graph G contains a graph X as a minor and $\Delta(X) \leq 3$, then G contains X as a topological minor.
5. Prove that if a graph G contains a graph H as a minor, and H contains a graph J as a minor, then G contains J as a minor. What if we replace some instances of the word "minor" with subgraph? induced subgraph? subdivision?

2.2.2. **References.** Optional references for today's class:

- Diestel Section 1.8
- Diestel Graph Theory Lecture 5 from minute 30:00 to minute 1:17:00.