THE LOCAL COVER

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Abstract. This exposition explains the local cover of a graph, an object introduced by Diestel, Jacobs, Knappe, and Kurkofka in [1]. My aim is to help build intuition, motivate the definition and properties, and give a (more-or-less) self-contained description of the local cover that assumes only familiarity with graph theory as background.

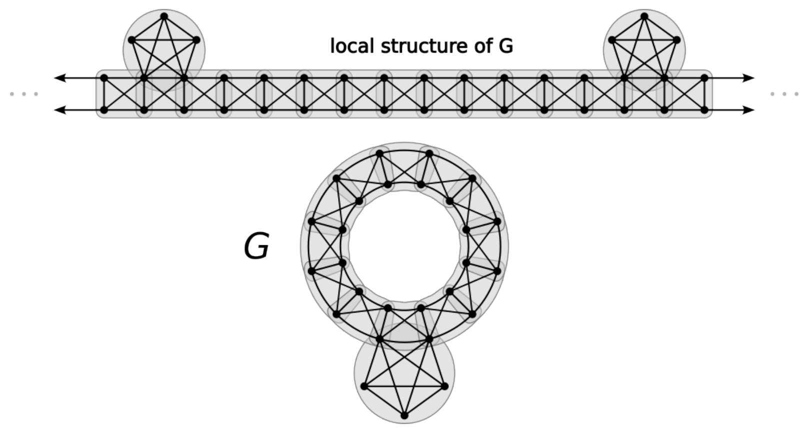

Figure 1. A graph G and its local structure. (Figure from Jan Kurkofka.)

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1. Introduction

In graph theory, knowledge of the so-called "local structure" of a graph often proves to be an insightful source of information. For example, several well-studied types of graphs, such as certain random graphs or expander graphs, are often described as "locally tree-like." In this context, a graph is typically considered *locally tree-like* if every small-radius ball of the graph looks like a tree.

Recently, Diestel, Jacobs, Knappe, and Kurkofka introduced a new way to define the "local structure" of a graph [1]. Instead of defining the local structure of a graph by looking separately into every small ball of a graph, they define a *single* graph, called the local cover, that simultaneously witnesses the local structure everywhere. In this framework, a graph is locally tree-like if and only if its local cover graph is a tree. An example illustrating a graph and its local structure is shown in Fig. 1.

Defining the local structure of a graph through the local cover has a number of advantages. One advantage is that the local cover has algebraic and topological properties in addition to combinatorial ones, expanding the available tools to study the local structure. Another advantage is that there is a natural way to use the local cover graph to also draw conclusions about the *global* structure of a graph. Being able to use knowledge of local structure to draw conclusions about global structure is a common goal in graph theory research, and the local cover is designed to help facilitate this kind of local-global analysis.

In this writeup, I explain how the local cover graph is constructed and describe several of its properties. My hope is that this document helps make the local cover more accessible and inspires those who are interested to learn more and work with the local cover themselves. Please feel welcome to email me if you have any thoughts or questions! **This is still a draft**, and it will likely be updated in the future.

2. Overview and preliminary definitions

The key idea of the local cover is that it is the simplest possible graph that represents the local structure of G everywhere. Because it's the "simplest possible" with respect to this property, it represents *only* the local structure of G: any global structure, i.e. structure not witnessed locally somehow, is ignored by the local cover.

To explain in detail how we construct the local cover, we first need to understand what we mean by "local structure." Intuitively, something is "local" in a graph if it appears in a "small part" of the graph. We can define a "small part" of a graph by using bounded-radius balls around vertices. Given a graph G, a vertex v of G, and a parameter r, the *(combinatorial) ball of radius $r/2$ around v*, denoted $B_G(v, r/2)$, is the subgraph of G with:

- vertex set the set of all vertices u of distance at most $r/2$ from v, and
- edge set the set of all edges uw such that $d(u, v) + d(w, v) < r$.

This definition of $r/2$-ball is chosen to mimic the definition of the metric ball of radius $r/2$ (if G is viewed as a 1-complex). The $r/2$-balls also interact with cycles in an important way:

Remark 1. Consider a cycle C_k of length k. Intuitively speaking, we want C_k to be "r-local" for $k \leq r$ but not r-local for $k > r$. In other words, cycles of length at most r are "r-local cycles," whereas cycles of length greater than r are not "r-local." A definition using closed neighborhoods cannot capture the full spectrum of cycles in this way. For instance, both C_4 and C_5 are contained in $N^2[v]$ for each of their vertices v but not in $N[v]$ for any of their vertices v, so closed neighborhoods alone cannot say that a C_4 is "more local" than a C_5. By contrast, a C_4 is contained in $B(v, 4/2)$ for each of its vertices v, but a C_5 is not contained in $B(v, 4/2)$ for any of its vertices v; see Fig. 2. So the combinatorial ball of radius $4/2$ gives us the precision to say that C_4 is 4-local but C_5 is not 4-local.

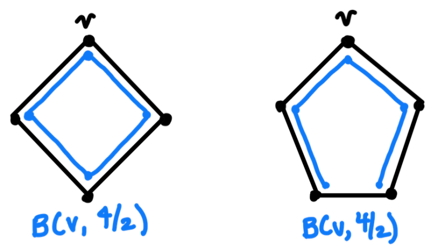

FIGURE 2. The 4/2-ball in a C_4 and in a C_5.

Now, given a graph G and parameter r, the *r-local structure of G* is the structure that appears in the balls of radius $r/2$ in G.

At the moment, this definition gives us a collection of subgraphs of G, the $r/2$-balls, that define its "local structure." But what we want is a *single* graph that captures the local structure everywhere. To do this, we use some tools from topology – specifically, the notion of a cover graph.

3. Coverings

In the remainder of this writeup, we assume all graphs are simple and connected. Given a graph G, a *covering of G* is a map $p : C \to G$ from a graph C to G such that:

- p is *surjective*: for every $v \in V(G)$ there is $c \in V(C)$ such that $p(c) = v$, and
- p is a *local homeomorphism*: p induces an isomorphism from $C[N[v]]$ to $G[N[p(v)]]$ for every $v \in V(C)$.

The map p is called a *covering map* of G, and the graph C is called a *cover* of G.

By the definition, every covering map of a graph G preserves neighborhoods everywhere. This is already a step in the direction of preserving the local structure. What if we ask for something stronger; namely, for the covering map to preserve $r/2$-balls everywhere? Then, every local piece of G would appear as a subgraph in the cover. Conversely, every local piece of the cover would correspond to a local piece of the original graph. In other words, the cover would represent the local structure everywhere – just what we're looking for.

Given graphs G and C, a covering $p : C \to G$ is *$r/2$-ball-preserving* if p induces an isomorphism from $B_C(v, r/2)$ to $B_G(p(v), r/2)$ for every vertex v of C. Now, we define the *r-local covering of G*, denoted $p_r : G_r \to G$, to be the *universal* $r/2$-ball-preserving covering of G. Here, *universal* means that for every $r/2$-ball-preserving covering $s : C \to G$ of G, there is an $r/2$-ball-preserving covering $q : G_r \to C$ and $p_r = s \circ q$. See Fig. 3 to visualize the universal property of p_r.

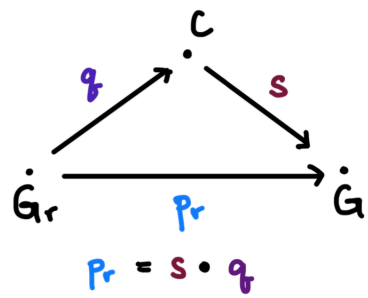

FIGURE 3. A diagram illustrating the universal property of G_r.

The property that p_r is universal is essential to defining something that captures "only" the local structure – otherwise, every graph automorphism gives an $r/2$-ball-preserving covering (for example). Universality also gives that the r-local covering map p_r is unique. It thus follows that the graph G_r, which we call the *r-local cover*, is also unique. Finally, the fact that p_r is universal is what guarantees that the graph G_r is the "simplest possible" graph that represents the local structure of G everywhere.

While the uniqueness is clear, it is not easy to see from this definition that the r-local covering of a graph G always exists. When local covering were first introduced in [1], they were defined differently. The definition in [1] is equivalent to the definition given above (by Lemmas 4.2 and 4.3 in [1]), but is based on ideas from algebraic topology. From that definition, one can show that the r-local covering indeed always exists. So the r-local covering is well-defined:

Theorem 1. *For every graph G and positive integer r, the r-local covering $p_r : G_r \to G$ exists and is unique.*

Next, we'll review the relevant concepts and explain how the local covering can be defined using algebraic topology.

4. Fundamental group of a graph

First, we need to understand the *fundamental group* of a graph. We follow the definitions from [1].

Let G be a connected graph and let x_0 be a fixed, arbitrary vertex of G called the *base vertex*. The elements of the fundamental group are the *closed walks* starting at x_0. We say that a walk W is *reduced* if it does not contain a subwalk of the form u-v-u. We can obtain a reduced walk W' from every walk W by a sequence of *reductions*: replacing a subwalk of the form u-v-u by the vertex u. Every walk corresponds to a unique reduced walk. We say that two walks are *equivalent* if they correspond to the same reduced walk, which defines an equivalence class on the set of closed walks starting at x_0. We typically consider the representative of the equivalence class to be its unique reduced walk.

We can now give the definition of the fundamental group of a graph. The *fundamental group of G with base x_0*, denoted $\pi_1(G, x_0)$, consists of the collection of closed walks in G starting from x_0, where the group operation is concatenation. Let's quickly check that $\pi_1(G, x_0)$ is a group:

Example 1. Let G be a graph and x_0 a fixed vertex of G. Then $\pi_1(G, x_0)$ is a group.

Proof. Since the concatenation of two closed walks starting from x_0 is again a closed walk starting from x_0, the group is closed. Concatenation of closed walks is associative. The unique identity is simply the closed walk $e = x_0$. Finally, we show that every element $W \in \pi_1(G, x_0)$ has an inverse W^-. Let $W = x_0\text{-}v_1 \ldots \text{-}v_m\text{-}x_0$ and let $W^- = x_0\text{-}v_m\text{-} \ldots \text{-}v_1\text{-}x_0$. In other words, W^- is the walk W traversed backwards. Now,

$$W \cdot W^- = x_0\text{-}v_1 \ldots \text{-}v_m\text{-}x_0\text{-}v_m \ldots \text{-}v_1\text{-}x_0.$$

By the reduction principle, the subwalk $v_m\text{-}v_0\text{-}v_m$ should be replaced by simply v_m, yielding:

$$\begin{aligned} W \cdot W^- &= x_0\text{-}v_1 \ldots \underbrace{v_m\text{-}v_0\text{-}v_m} \ldots \text{-}v_1\text{-}x_0 \\ &= x_0\text{-}v_1 \ldots \text{-}v_{m-1}\text{-}v_m\text{-}v_{m-1}\text{-} \ldots \text{-}v_1\text{-}x_0 \end{aligned}$$

Now, we can again replace the subwalk $v_{m-1}\text{-}v_m\text{-}v_{m-1}$ with v_{m-1}. Using this approach, we can iteratively reduce the walk until it becomes the identity x_0. □

4.1. **Cycles and the fundamental group.** In topology, the purpose of the fundamental group is to understand the loops of a topological space. In the graph setting, we want to use the fundamental group to understand the structure of cycles in a graph. Therefore, let's understand how cycles appear in the fundamental group.

Let G be a graph, x_0 a fixed vertex of G, and $\pi_1(G, x_0)$ the fundamental group of G. Consider a cycle C of G. Then, for any walk W from x_0 to C, the closed walk WCW^- is an element of $\pi_1(G, x_0)$. Observe that this representation is closed under conjugation: for any element Z of $\pi_1(G, x_0)$, the element $Z(WCW^-)Z^-$ is again of the form XCX^- for a walk $X = ZW$ from x_0 to C. We can likewise translate between any two elements of $\pi_1(G, x_0)$ of the form WCW^- and ZCZ^- by conjugation. First, observe that $X = ZW^-$ is in $\pi_1(G, x_0)$ as it is a closed walk starting at x_0, and then observe the following:

$$X(WCW^-)X^- = (ZW^-)WCW^-(WZ^-) = ZCZ^-.$$

We will therefore consider a cycle C to live in the fundamental group $\pi_1(G, x_0)$ in the conjugacy class $[WCW^-]$ where W is any element of $\pi_1(G, x_0)$.

4.2. **Generating the fundamental group.** The fundamental group of a graph can always be generated by a special subset of elements of the fundamental group. Let T be any spanning tree of G. A cycle C is a *fundamental cycle for T* if there is an edge e of $G - T$ such that C is the unique cycle of $T + e$. For every fundamental cycle C for T, there is a unique shortest path P_C from x_0 to C contained in T. Let L_T be the set of elements of $\pi_1(G, x_0)$ of the form $P_C C P_C^-$, where C is a fundamental cycle for T. Then, L_T is a generating set for $\pi_1(G, x_0)$. In particular:

Theorem 2 ([1], Lemma 4.1)**.** *Let G be a graph, x_0 a fixed vertex of G, and T a spanning tree of G. Then, every element of $\pi_1(G, x_0)$ is the product of elements of L_T.*

4.3. **The base vertex.** In the context of the local covering, we care about two aspects of the fundamental group: first, how cycles appear in the fundamental group, and second, how normal subgroups of the fundamental group behave. Given a group Γ, a subgroup S of Γ is *normal* if $\gamma s \gamma^{-1} \in S$ for every $\gamma \in \Gamma$ and $s \in S$. In other words, normal subgroups are those subgroups that are closed under conjugation by elements of the group.

Observe that in both contexts, the objects that we are interested in – cycles and normal subgroups – are invariant under conjugation. This means that the base vertex we choose for the fundamental group does not matter. Let x_0 and x_1 be two distinct vertices of G and consider the groups $\pi_1(G, x_0)$ and $\pi_1(G, x_1)$. Consider a cycle C and its representatives $[W_0 C W_0^-]$ and $[W_1 C W_1^-]$ in $\pi_1(G, x_0)$ and $\pi_1(G, x_1)$, respectively. Fix any walk Z from x_1 to x_0 in G. Now, conjugation by Z allows us to move from any representative of C in $\pi_1(G, x_0)$ to a representative of C in $\pi_1(G, x_1)$:

$$Z(W_0 C W_0^-)Z^- = (ZW_0)C(ZW_0)^-,$$

and ZW_0 is a walk from x_1 to C, so $(ZW_0)C(ZW_0)^-$ is in the class for C in $\pi_1(G, x_1)$. In an analogous way, we can conjugate by a walk from x_1 to x_0 in order to translate a normal subgroup N_0 of $\pi_1(G, x_0)$ into (the same) normal subgroup N_1 of $\pi_1(G, x_1)$.

Since we only care about objects invariant under conjugation and the base vertex thus does not matter, from now on we will refer to the fundamental group of G as $\pi_1(G)$ and implicitly assume an arbitrary fixed base point.

5. Covers and the fundamental group

Covers are related to the fundamental group in the following way. Let C and G be graphs and let $p : C \to G$ be a covering from C to G. Let $\pi_1(C)$ and $\pi_1(G)$ be the fundamental groups of C and G, respectively. Now, $p(\pi_1(C)) \subseteq \pi_1(G)$. In other words, the covering map p maps the fundamental group of C to the fundamental group of G. We call the subgroup $p(\pi_1(C))$ of $\pi_1(G)$ the *characteristic subgroup of p*. We will restrict our attention to coverings where $p(\pi_1(C))$ is a normal subgroup of $\pi_1(C)$; these are called *normal coverings*.

Theorem 3 ([2], Theorem 1.38)**.** *Let G be a graph. Every normal covering of G corresponds to a unique normal characteristic subgroup, and every normal subgroup of $\pi_1(G)$ corresponds to a unique normal covering of G.*

The r-local covering $p_r : G_r \to G$ is a normal covering ([1], Section 4.2). Therefore, a consequence of Theorem 3 is that the local covering is determined completely by which (normal) subgroup of $\pi_1(G)$ is its characteristic subgroup.

The characteristic subgroup of a covering essentially identifies the elements of the fundamental group of G which appear isomorphically in the fundamental group of the cover. From this perspective, which subgroup should represent the r-local covering? Since the point of the fundamental group is to understand the behavior of cycles in a graph, we can think of this question as asking, which cycles in a graph are "r-local" cycles?

We know that the r-local covering $p_r : G_r \to G$ preserves $r/2$-balls everywhere. Since every cycle of length at most r appears in the $r/2$-ball around each of its vertices (recall Remark 1), the cycles of length at most r in G should appear isomorphically in the local cover G_r. However, short cycles are *not* the only cycles that appear in $r/2$-balls: a cycle of any length that appears in the neighborhood of a vertex is in an $r/2$-ball in G and thus should appear isomorphically in the cover.

Surprisingly, although short cycles are not the only r-local cycles, it turns out that they contain all of the necessary information about r-local cycles in the fundamental group. Let $\pi_1^r(G)$ be the subgroup of $\pi_1(G)$ generated by the short cycles of G. Specifically, $\pi_1^r(G)$ is the subgroup containing every element of $\pi_1(G)$ that can be written as the product of cycles $[WCW^-]$ where C has length at most r. Since cycles are represented by conjugacy classes in the fundamental group, $\pi_1^r(G)$ is a normal subgroup of G. We say that every cycle $[WCW^-]$ that appears in $\pi_1^r(G)$ is *generated by cycles of length at most r in the fundamental group*.

Now, we have the following cool equivalence:

Theorem 4 ([1], Section 4.2)**.** *Let G be a graph and r a positive integer. Then, the r-local covering $p_r : G_r \to G$ is the unique covering of G with characteristic subgroup $\pi_1^r(G)$.*

The fact that $p_r : G_r \to G$ is both the universal $r/2$-ball-preserving covering and the unique covering with characteristic subgroup $\pi_1^r(G)$ gives us a number of benefits. First, on an intuitive level, "short cycles" and "small-radius balls" are both simple and appealing definitions of local structure. Second, the fact that they are equivalent means that we have both combinatorial and topological tools at hand to study the local cover.

We also have the following two important properties.

Theorem 5. *Let G be a graph and $r \geq 3$ an integer. Then, $\pi_1(G_r) = \pi_1^r(G_r)$.*

Theorem 5 states that the fundamental group of every r-local cover G_r is equal to the r-local subgroup of the fundamental group. This is a way of saying that G_r is "determined by its local structure." Since G_r is meant to represent only the local structure of G, it makes sense that G_r is determined by its local structure.

The converse also holds: if a graph is "determined by its local structure" in this way, then it is equal to its local cover.

Theorem 6. *Let G be a graph and $r \geq 3$ an integer. Then, $G_r = G$ if and only if $\pi_1(G) = \pi_1^r(G)$.*

6. Deck transformations of the local cover

Much of the time, the r-local cover G_r of a graph G is an infinite graph, even when the original graph G is finite. For example, the r-local cover of a cycle C_k when $k > r$ is an infinite path; see Fig. 4.

At first, it might seem unwieldy that we end up with an infinite graph representing the "local" structure of a finite graph, but the r-local cover G_r has very strong symmetry properties. To explain what this means, let's start with some definitions.

Let G be a graph and let $p : C \to G$ be a covering of G. Let $v \in V(G)$. A vertex $u \in C$ is called a *lift of v (to C)* if $p(u) = v$. The set of all lifts of v (to C) is called the *fiber of v (in C)*.

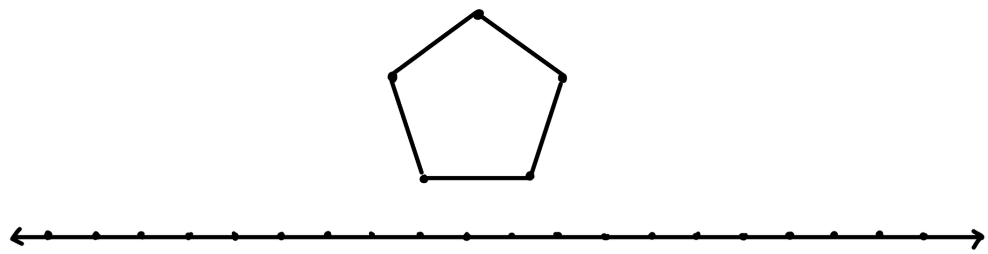

FIGURE 4. A cycle C_5 and its r-local cover for $r \leq 4$, the infinite path.

Observation 2. Let G be a graph and $p : C \to G$ a covering of G. For each vertex v of G, let $F(v)$ denote the fiber of v in C. Now, $\{F(v) \mid v \in V(G)\}$ is a partition of $V(C)$.

We will show that the symmetries of the graph C behave well with respect to the set of fibers defined in Observation 2. Let *Aut(G)* denote the set of automorphisms of the graph G, i.e. the set of isomorphisms $f : G \to G$.

Observation 3. For every graph G, the set $\mathrm{Aut}(G)$ forms a group with the function composition operation.

Given a covering $p : C \to G$, an automorphism f of C is called a *deck transformation of p* if f preserves the partition of C into fibers, as described in Observation 2. More specifically, an automorphism f of C is a deck transformation if for every vertex u of C, both u and $f(u)$ are in the same fiber $F(v)$ with $v \in V(G)$. Another way of saying this is the following: f is a deck transformation of C if $p \circ f = p$. Let $\mathcal{D}(p)$ denote the set of deck transformations of p.

Observation 4. Let G be a graph and $p : C \to G$ a covering of G. Then, the set of deck transformations $\mathcal{D}(p)$ is a subgroup of $\mathrm{Aut}(C)$.

The deck transformations $\mathcal{D}(p)$ of a covering $p : C \to G$ *act transitively on the fibers of p* if for every vertex v of G and for every two vertices u_1, u_2 of C such that $u_1, u_2 \in F(v)$, there is a deck transformation $f \in \mathcal{D}$ such that $f(u_1) = u_2$. It turns out that normal covers are precisely those whose deck transformations act transitively:

Theorem 7. *Let C and G be graphs and let $p : C \to G$ be a covering map. Then, the deck transformations of p act transitively on the fibers of p if and only if p is normal.*

Since local coverings are normal, we have the following:

Theorem 8. *Let G be a graph, let r be a positive integer, and let $p_r : G_r \to G$ be the r-local covering of G. Then, the deck transformations $\mathcal{D}(p_r)$ act transitively on the fibers of p_r.*

Theorem 8 is a way of formalizing the idea that G_r has "strong symmetry properties." One implication is the following. The *orbits* of the automorphism group $\mathrm{Aut}(G)$ of a graph G are sets $O_1, \ldots, O_k$ that form a partition of the vertex set $V(G)$ of G such that for every vertex v of G with $v \in O_i$, both of the following hold: (1) $f(v) \in O_i$ for every $f \in \mathrm{Aut}(G)$; and (2) for every $u \in O_i$ there exists $f \in \mathrm{Aut}(G)$ such that $f(v) = u$.

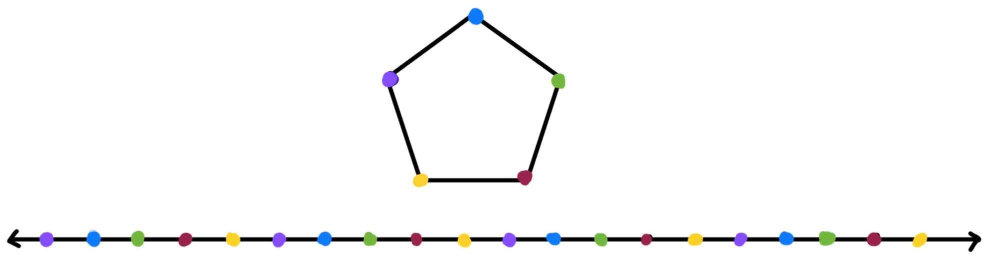

FIGURE 5. The cycle C_5 and its r-local cover for $r \leq 4$. The fibers are drawn with colors corresponding to their vertices.

An infinite graph is called *quasi-transitive* if its automorphism group has finitely many orbits. A consequence of Theorem 8 is that every fiber $F(v)$ is contained in a single orbit of $\mathrm{Aut}(G_r)$. Therefore, the number of fibers is an upper bound for the number of orbits, and so G_r is quasi-transitive. In particular, $\mathrm{Aut}(G_r)$ has at most $|V(G)|$ orbits, since there are at most $|V(G)|$ fibers.

7. The binary cycle space

Let G be a graph, r a positive integer, and $p_r : G_r \to G$ the r-local cover of G. We discussed in Section 5 that the r-local cover G_r is generated by short cycles in the fundamental group (meaning that its fundamental group is equal to the r-local subgroup of its fundamental group). In this section, we will show that there is also another, simpler setting in which the local cover is "generated by short cycles."

Let $|E|$ be the number of edges of G. Fix an ordering of the edges of G. Now, we can assign every cycle C of G to a vector E_C in $\{0,1\}^{|E|}$, where E_C is one in exactly the indices corresponding to the edges of C.

The span of the vectors E_C for every cycle C of G, with binary addition, forms a space called the *binary cycle space* of G, denoted $\mathcal{Z}(G)$. Now, we can again look at what happens if we only want to consider elements generated by short cycles. To that end, the subspace of $\mathcal{Z}(G)$ spanned by vectors corresponding to cycles of length at most r is denoted $\mathcal{Z}^r(G)$.

Now we have:

Theorem 9. *Let G be a graph, $r \geq 3$ a positive integer, and x_0 a fixed vertex of G. Suppose that $\pi_1(G, x_0) = \pi_1^r(G, x_0)$. Then, $\mathcal{Z}(G) = \mathcal{Z}^r(G)$.*

The fact that $\mathcal{Z}(G) = \mathcal{Z}^r(G)$ can be stated informally as "the graph G is generated by short cycles in the binary cycle space." Theorem 9 states that if a graph is generated by short cycles in its fundamental group, it is also generated by short cycles in its binary cycle space. This in particular implies the following:

Theorem 10 ([1], Lemma 4.6)**.** *Let G be a graph, $r \geq 3$ a positive integer, and G_r the r-local cover of G. Then, $\mathcal{Z}(G_r) = \mathcal{Z}^r(G_r)$.*

Being generated by short cycles in the binary cycle space is quite intuitive: a cycle C is generated by cycles of length at most r in the binary cycle space if there exists a set

$\{C_1, \ldots, C_k\}$ of cycles, each of length at most r, such that C is the symmetric difference $\triangle\{C_1, \ldots, C_k\}$. (An edge e is in the symmetric difference $\triangle\{C_1, \ldots, C_k\}$ if and only if it appears in an odd number of cycles of $\{C_1, \ldots, C_k\}$.)

By Theorem 9, being generated by short cycles in the fundamental group implies being generated by short cycles in the binary cycle space. Unfortunately, the converse does not hold. Nevertheless, if a graph G contains a cycle that is *not* generated in the binary cycle space, this witnesses that the cycle is also not generated in the fundamental group.

Since the binary cycle space is much easier to work with than the fundamental group, it might be useful to understand whether there are natural cases where we can instead work in the binary cycle space:

Question 1. *Are there any natural conditions under which generation in the binary cycle space implies generation in the fundamental group?*

8. Examples

We have now discussed several definitions and properties of local covers. In this section, we explain how to check if a graph C is the r-local cover of a graph G, and go through several examples. To address the former point, we use the following characterization:

Theorem 11 (Paul Knappe)**.** *Let G and C be graphs and r a positive integer. Then, C is the r-local cover of G if and only if there is a covering map $p : C \to G$ such that all of the following hold:*

(1) the deck transformations $\mathcal{D}_p$ of p act transitively on the fibers of p;
(2) for every cycle Q of length at most r in G, there is a cycle O in C such that p induces an isomorphism from O to Q; and
(3) there exists a spanning tree T of C such that every fundamental cycle for T is generated by cycles of length at most r in the fundamental group of C.

Let's understand why Theorem 11 holds. By Theorem 4, given a candidate covering map $p : C \to G$, we need to check that the characteristic subgroup of p is precisely $\pi_1^r(G, x_0)$. Point *(1)* guarantees that the covering map p is normal. Therefore, to check that the characteristic subgroup of p contains $\pi_1^r(G, x_0)$, it suffices to check that the characteristic subgroup of p contains the r-local lollipops, which is done by Point *(2)*.

Finally, we need to check that the characteristic subgroup of p contains only $\pi_1^r(G, x_0)$. Therefore, we need to show that every element of the fundamental group $\pi_1(C, c_0)$ maps to an element of $\pi_1^r(G, x_0)$. Since we know that the fundamental cycles of any spanning tree of C generate $\pi_1(C, c_0)$ by Theorem 2, it suffices to check this property for the fundamental cycles of some spanning tree, which is done in Point *(3)*.

Theorem 11 is the best way we currently know to verify whether a graph C is the r-local cover of a graph G in general. However, in many cases, the graph G has enough structure that we can directly determine the local cover. In the rest of this section, we review a few basic examples of local covers.

Example 2. Let G be a graph with $n := |V(G)|$. The 2-local cover G_2 of G is a tree. The n-local cover G_n of G is G itself. As r grows from 2 to n, the r-local cover reflects a growing amount of the structure of G.

Example 3. Let T be a tree. The r-local cover of T is T itself for every positive integer r.

Example 4. Let G be a cycle of length k. The r-local cover of G is an infinite path if $r < k$ (see Fig. 4). The r-local cover of G is G itself if $r \geq k$.

Example 5. Let G be a graph with girth g. The r-local cover of G is a tree if $r < g$.

Example 6. Let G be the $n \times n$ grid. The r-local cover of G is a tree if $r \leq 3$. The r-local cover of G is G itself if $r \geq 4$.

Example 7. Let G be the toroidal grid, i.e. the grid embedded in a torus. Let k be the minimum length of a cycle that "goes around the torus." The r-local cover of G is the infinite planar grid if $4 \leq r < k$.

9. Problems

There are many interesting open problems related to the local cover. Any problem that aims to formalize the idea that the local cover G_r represents "only" the local structure would be very interesting. I will likely update this section to include some problems of that flavor in the near future. For now, I want to mention two problems related to the definition of the local covering:

Question 2. *Is there a proof that the r-local covering $p_r : G_r \to G$ exists, using only the definition that p_r is the universal $r/2$-ball-preserving covering?*

Question 3. *Is there a proof that the r-local covering $p_r : G_r \to G$ is a normal covering, using only the definition that p_r is the universal $r/2$-ball-preserving covering?*

It is known that the r-local covering exists and is normal, but the known proofs both use the definition of $p_r : G_r \to G$ as the unique covering with characteristic subgroup $\pi_1^r(G)$. Since the universal $r/2$-ball-preserving covering is a very natural definition, it would be quite interesting to know whether these key properties can be proved using this definition.

References

[1] R. Diestel, R.W. Jacobs, P. Knappe, and J. Kurkofka. Canonical graph decompositions via coverings. *arXiv:2207.04855v6*, 2022.
[2] A. Hatcher. *Algebraic Topology.* 2001.

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