Dirichlet Potpourri

Youth EUCLID

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Series

Dirichlet's arith prog theorem

For any a, b with $a \perp b$, there are an infinite number of primes congruent to $a \pmod{b}$.

Recall the Riemann zeta:

$$\zeta(s) = \sum_{n \geq 1} 1/n^s.$$

$$\Rightarrow A(s) = \sum_{n \geq 1} a_n/n^s$$

Properties

Vanilla convolution: $a_n, b_n \to$

$$\sum_{k \in [0,n]} a_k b_{n-k}$$

Convolve $A, B \Rightarrow$

$$(A * B)(s) = \sum_{n \geq 1} n^{-s} \sum_{d|n} a_d b_{n/d}$$

Notable series

μ: $\sum_{d|n} \mu(d) = [n = 1]$

$$M(s)\zeta(s) = 1 \Rightarrow M = 1/\zeta.$$

$\sigma(n) = \sum_{d|n} d; \to$

$$\sum_{n\geq 1} \sigma(n)/n^s = \zeta(s) \sum_{n\geq 1} n/n^s = \zeta(s)\zeta(s-1).$$

$d(n) = \sum_{d|n} 1 \Rightarrow$

$$\sum_{n\geq 1} d(n)/n^s = \zeta(s)^2.$$

$\sum_{d|n} \phi(d) = n \Rightarrow$

$$\left(\sum_{n\geq 1} \phi(n)/n^s\right) \zeta(s) = \sum_{n\geq 1} n/n^s = \zeta(s-1).$$

$$\Rightarrow \sum_{n\geq 1} \phi(n)/n^s = \zeta(s-1)/\zeta(s).$$

More functions

Von Mangoldt lambda:

$$\Lambda(n) = \begin{cases} \ln p (n = \text{ prime power}); \\ 0 \text{ otherwise.} \end{cases}$$

$$\Lambda \to -\zeta'/\zeta.$$

Bounding zeta

Zeta bounds

$\forall s > 1$, ζ:

$$\zeta(s) = 1/(s-1) + O(1).$$

ζ':

$$\zeta'(s) = -1/(s-1)^2 + O(1).$$

ζ'/ζ:

$$(-\zeta'/\zeta)(s) = 1/(s-1) + O(1).$$

Proof.

As earlier set real $s > 1$. Bound term-by-term:

$$\int_{n-1}^{n} u^{-s} du \leq n^{-s} \leq \int_{n-1}^{n} u^{-s} du + 1.$$

Sum $\forall n \geq 1$:

$$\zeta(s) \geq \int_{1}^{\infty} du/u^s + smth \Rightarrow 1/(s-1) + O(1)$$

For ζ', do d/ds and observe that all sides monotonic (decreasing wrt x). □

Characters

A *Dirichlet character mod n* is an NT function χ such that

- χ totally multiplicative;
- χ determined by mod n; that is,

$$a \equiv b \pmod{n} \iff \chi(a) = \chi(b)$$

- Hence $\chi \in 0, \pm 1$.
- $a \not\perp n \iff \chi(a) = 0$

Basic char mod n: only 0 or 1, no negatives.
L-series:

$$L(s, \chi) = \sum_{\chi \geq 1} (n)/n^s.$$

Euler's factoring: for totally multiplicative f, we have

$$F(s) = \prod_p (1 + f(p)/p^s).$$

Mod 4 example

Lemma

There are $\phi(n)$ characters mod n.

Proof left as an exercise for the reader.

There are two chars mod 4, say the basic/main/principal one, χ_0, and the 'other' one, say χ_1.

* $(\chi_0 + \chi_1)/2 = [n \equiv 1 \pmod 4]$;
* $(\chi_0 - \chi_1)/2 = [n \equiv 3 \pmod 4]$;

Needed result:

$$\ln L(s, \chi) = \sum_{\geq 1} \Lambda(n) \ln n \chi(n)/n^s.$$

Proof cont'd

Claim

$L(s, \chi_0)$ diverges as $s \to 1$.

Proof.
By a known result, $\sum_p \chi(p)/p^s = \ln L(s, \chi) + O(1)$, which implies the result. $\square$

Claim

$L(s, \chi_1)$ is finite.

Proof.
Simply apply alternating series test or whatever. $\square$

Taking linear combos we get both cases of Dirichlet in one go for $\mod 4$.

Generalisation to general n

Lemma

For a nonprincipal char $\chi \pmod{n}$, $\sum_{k=1}^{n} \chi(k) = 0$.

Proof.
Simply observe that all such χ are totally multiplicative and there is at least one -1 val. This implies the result. □

Hence $L(s, \chi) = O(1)$ for all nonprinicpal χ and $s > 0$ (not 1). Finally, it takes a bit more work to show that

$$L(1, \chi) = \infty$$

for principal $\chi \pmod{n}$.

After taking linear combos and using the $O(1)$ bound from earlier, we finally get Dirichlet's arith prog theorem :0

The end?

Have a good sleep everyone :D