Geometry Favorites

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updated July 4, 2024

(Note: here ∞_{XY}, $\infty_{\perp XY}$ refer to the points ∞ along in directions parallel and perpendicular to XY, respectively.)

-1 Credits + remarks

Inspired by chapter 11 of EGMO book, and **Geometry At Its Best** by Eric Shen. These are roughly in order of difficulty, but no promises!
Also thanks to collaborators…

Contents

0 Problems

Remark. Some attempt has been made to deviate from the aforementioned two famous geometry papers.

Problem 1 (SL 2009/G3). Let ABC be a triangle. The incircle of $\triangle ABC$ touches AB and AC at the points Z and Y, respectively. Let $G = \overline{BY} \cap \overline{CZ}$, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

Problem 2 (SL 2015/G4). Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

Problem 3 (USEMO 2023/4). Let ABC be an acute triangle with orthocenter H. Points A_1, B_1, C_1 are chosen in the interiors of sides BC, CA, AB, respectively, such that $\triangle A_1B_1C_1$ has orthocenter H. Define $A_2 = \overline{AH} \cap \overline{B_1C_1}$, $B_2 = \overline{BH} \cap \overline{C_1A_1}$, and $C_2 = \overline{CH} \cap \overline{A_1B_1}$.

Prove that triangle $A_2B_2C_2$ has orthocenter H.

Problem 4 (SL 2016/G7). Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.

Problem 5 (EGMO 2020/3). Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$, $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A, \angle C, \angle E$ are concurrent. Prove that the (interior) angle bisectors of $\angle B, \angle D, \angle F$ are also concurrent.

Problem 6 (IMO 2008/6). Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

Problem 7 (Iran TST 2018/1/4). Let ABC be a triangle ($\angle A \neq 90°$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N. Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that $\overline{AP}$ bisects $\overline{BC}$.

Problem 8 (Eric Shen). In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90°$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on $\overline{EF}$ such that $\overline{PQ} \parallel \overline{BC}$, prove that $\overline{AQ}$ bisects $\overline{BC}$.

Problem 9 (SL 2018/G5). Let ABC be a triangle with circumcircle ω and incenter I. A line ℓ meets the lines AI, BI, CI at points D, E, F respectively, all distinct from A, B, C, I. Prove that the circumcircle of the triangle determined by the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ is tangent to ω.

Problem 10 (SL 2009/G6). Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of triangles ABP and CDP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.

Problem 11 (MOP 2019 & USA TST 2019/6). Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90°$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

(a) (MOP 2019) Let E, F be the feet of the altitudes from B, C respectively. Prove that if $\overline{EF}$ touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.

(b) (USA TST 2019/6) Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to (DB_1C_1).

Problem 12 (ELMO SL 2024/G4). In quadrilateral $ABCD$ with incenter I, points W, X, Y, Z lie on sides AB, BC, CD, DA with $AZ = AW$, $BW = BX$, $CX = CY$, $DY = DZ$. Define $T = \overline{AC} \cap \overline{BD}$ and $L = \overline{WY} \cap \overline{XZ}$. Let points O_a, O_b, O_c, O_d be such that $\angle O_aZA = \angle O_aWA = 90^\circ$ (and cyclic variants), and $G = \overline{O_aO_c} \cap \overline{O_bO_d}$. Prove that $\overline{IL} \parallel \overline{TG}$.

Problem 13 (APMO 2014/5). Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.

Problem 14 (DeuX MO 2020/II/3). In triangle ABC with circumcenter O and orthocenter H, line OH meets $\overline{AB}, \overline{AC}$ at E, F respectively. Let ω be the circumcircle of triangle AEF with center S, meeting (ABC) again at $J \neq A$. Line OH also meets (JSO) again at $D \neq O$. Define $K = (JSO) \cap (ABC) \ (\neq J)$, $M = \overline{JK} \cap \overline{OH}$, and $G = \overline{DK} \cap (ABC) \ (\neq K)$. Prove that (GHM) and (ABC) are tangent to each other.

Problem 15 (IMO 2021/3). Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Problem 16 (USAMO 2021/6). Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.

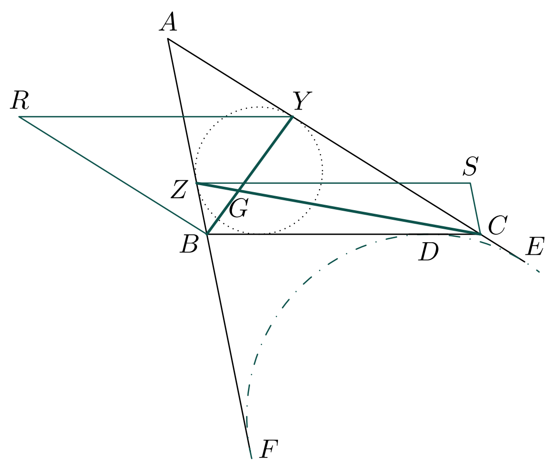
Problem 17 (SL 2021/G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

Problem 18 (USEMO 2020/3). Let ABC be an acute triangle with circumcenter O and orthocenter H. Let Γ denote the circumcircle of triangle ABC, and N the midpoint of OH. The tangents to Γ at B and C, and the line through H perpendicular to line AN, determine a triangle whose circumcircle we denote by ω_A. Define ω_B and ω_C similarly.
Prove that the common chords of ω_A,ω_B and ω_C are concurrent on line OH.

1 Solutions

1.1 SL 2009/G3, by Hossein Karke Abadi

Let ABC be a triangle. The incircle of $\triangle ABC$ touches AB and AC at the points Z and Y, respectively. Let $G = \overline{BY} \cap \overline{CZ}$, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.



This is a very "troll" problem. Let (R), (S), ω_a denote the point circles at R, S (radius $= 0$) and the A-excircle respectively. Let ω_a touch $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively. Also, for brevity, let $a = BC, b = CA, c = AB, s = (a + b + c)/2$.

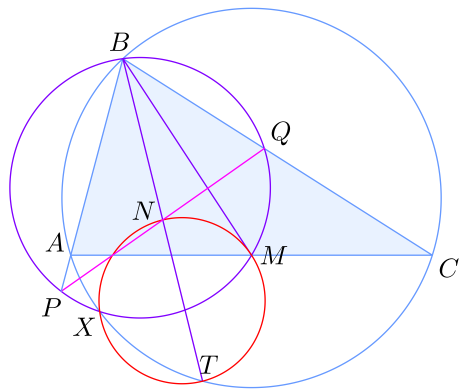
> **Claim –** $\overline{BY}$ is the radical axis of $(R), \omega_a$.

Proof. $BD = BR = s - c$, while $YE = YR = a$; because $\overline{BD}, \overline{YE}$ touch ω_a, B, Y have powers $(s-c)^2, a^2$ wrt each of $(R), \omega_a$ as promised. $\square$

By the claim, $G = \overline{BY} \cap \overline{CZ}$ must be the radical center of (R), (S), ω_a, implying the desired $GR = GS$.

1.2 SL 2015/G4

Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.



The answer is $\sqrt{2}$ only. Let $X = (ABC) \cap (BPMQ)$ $(\neq B)$, and let N be the midpoint of $\overline{BT}$.

> **Claim 1 –** $XNMT$ is cyclic, and $\overline{BM}$ is tangent to this circle.

Proof. Since N is also the midpoint of $\overline{PQ}$, there is a spiral similarity at X sending PNQ to AMC. Thus, we have

$$\measuredangle XMN = \measuredangle XAP = \measuredangle XTB,$$

proving the concyclicity. For the tangency, check that

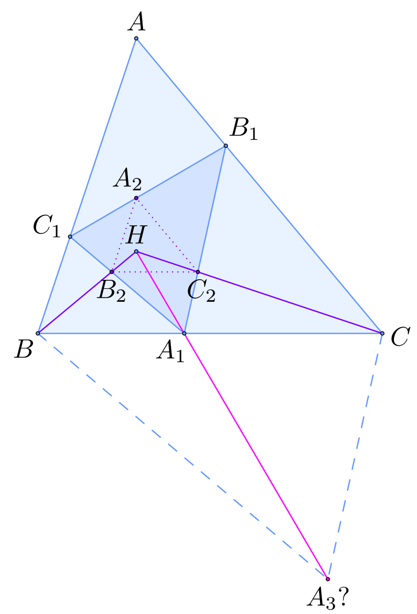
$$\measuredangle XNM = \measuredangle XPA = \measuredangle XPB = \measuredangle XMB. \qquad \square$$

By power of a point, $BM^2 = BN \cdot BT = \frac{BT^2}{2}$, so $\frac{BT}{BM} = \sqrt{2}$.

1.3 USEMO 2023/4, by Ankan Bhattacharya

Let ABC be an acute triangle with orthocenter H. Points A_1, B_1, C_1 are chosen in the interiors of sides BC, CA, AB, respectively, such that $\triangle A_1B_1C_1$ has orthocenter H. Define $A_2 = \overline{AH} \cap \overline{B_1C_1}$, $B_2 = \overline{BH} \cap \overline{C_1A_1}$, and $C_2 = \overline{CH} \cap \overline{A_1B_1}$.

Prove that triangle $A_2B_2C_2$ has orthocenter H.



Working backwards: suffices to prove $\overline{AA_2H} \perp \overline{B_2C_2}$, or in other words, $\overline{BC} \parallel \overline{B_2C_2}$. $\iff$ $HB_2/HB = HC_2/HC$. $\iff$ There exists a point $A_3 \in \overline{HA_1}$ with $\overline{BA_3} \parallel \overline{A_1C_1}$ and $\overline{CA_3} \parallel \overline{A_1B_1}$. Indeed, this point would be chosen so that

$$\frac{HA_3}{HA} = \frac{HB_2}{HB} = \frac{HC_2}{HC},$$

lengths directed. In *still* other words, we now want $\overline{HA_1}$, $\overline{B\infty_{A_1C_1}}$, $\overline{C\infty_{A_1B_1}}$ concurrent.

For this we employ a massive cross-ratio chase:

$$\begin{aligned}
(\infty_{A_1C_1}\infty_{A_1B_1}; \infty_{\perp B_1C_1}\infty_{BC}) &\overset{\text{rotate } 90^\circ}{=} (\infty_{HB_1}\infty_{HC_1}; \infty_{B_1C_1}\infty_{HA}) \\
&\overset{H}{=} (B_1C_1; \infty_{B_1C_1}A_2) \\
&\overset{A}{=} (\overline{AC}, \overline{AB}; \overline{B_1C_1}, \overline{AH}) \\
&\overset{\text{rotate } 90^\circ}{=} (\overline{HB}, \overline{HC}; \overline{HA_1}, \overline{BC}) \\
&\overset{H}{=} (BC; A_1\infty_{BC})
\end{aligned}$$

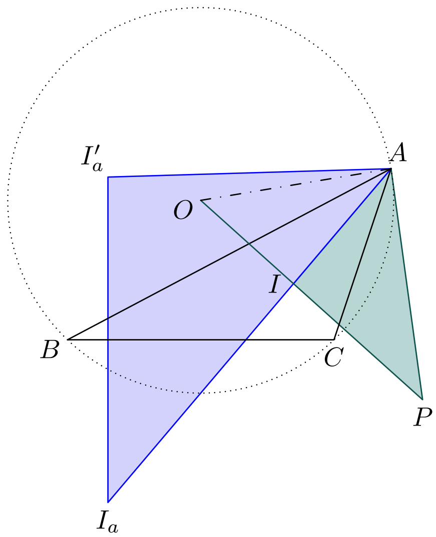
and the concurrence follows by prism lemma.

1.4 SL 2016/G7

Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120^\circ$.



Redefine P as the inverse of I wrt (ABC). For the first part we assert more strongly that:

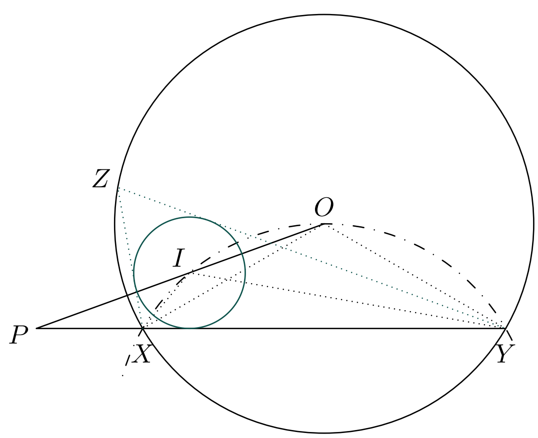
> **Claim –** $\triangle AI_aI'_a \overset{+}{\sim} \triangle API$.

Proof. By angle chasing, $\angle I_a = \angle P$ follows easily. We contend that $I_aI'_a/I_aA = IP/AP$; indeed, the first ratio equals $2\cos\angle BI_aC = 2\sin\frac{A}{2}$ because of similar triangles $I_aBC \overset{-}{\sim} \triangle I_aI_bI_c$, while

$$\frac{IP}{AP} = \frac{OP}{AP} - \frac{OI}{OA}\frac{OA}{AP} = \frac{OA}{AI} - \frac{OI^2}{OA\cdot AI} = \frac{R}{AI} - \frac{R^2-2rR}{R\cdot AI} = \frac{R-(R-2r)}{AI} = 2\sin\frac{A}{2},$$

so the ratios are equal. The similarity follows by SAS. □

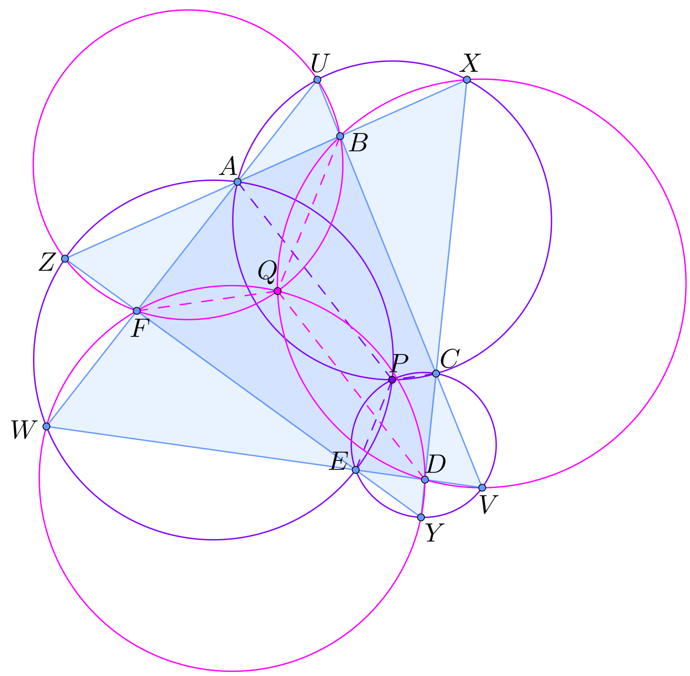
The claim clearly implies the isogonality.



For the second part, using Poncelet, let $Z \in (ABC)$ be the unique point so that $\triangle XYZ, ABC$ share a incircle and circumcircle. Inverting "P, X, Y collinear" wrt the circumcircle gives O, I, X, Y concyclic, or $\angle XOY = \angle XIY$.
As it's well-known that $\angle XOY = 2\angle Z$ and $\angle XIY = (\pi + \angle Z)/2$, we must have $\angle Z = \pi/3 \implies \angle XIY = 2\pi/3$ as needed.

1.5 EGMO 2020/3

Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$, $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A, \angle C, \angle E$ are concurrent. Prove that the (interior) angle bisectors of $\angle B, \angle D, \angle F$ are also concurrent.



Since $\angle A + \angle B = 240^\circ$ and cyclic variants, $\overline{AB}, \overline{CD}, \overline{EF}$ form an equilateral triangle, as do $\overline{BC}, \overline{DE}, \overline{FA}$. Label them UVW, XYZ as shown, and let the given concurrency point be P. By an angle chase, $P \in (ACXU), (CEYV), (EAZW)$, so it's the center of the spiral similarity s_1 mapping $U, V, W \to X, Y, Z$.

Claim – $\triangle UVW \cong \triangle XYZ$.

Proof. Recall that s_1 maps $\overline{UV} \to \overline{XY}$, but the fact that P lies on the bisector of $\angle C$ means that P is equidistant from these lines.

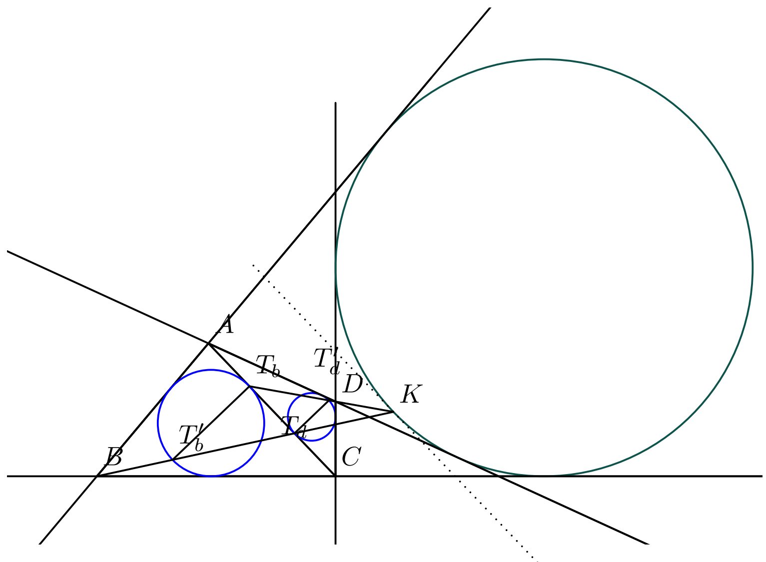
As this means that the spiral similarity above is in fact a rotation, we win. $\square$

To finish the problem, note that the center $Q = (BDVX) \cap (DFWY) \cap (FBUZ)$ of the rotation s_2 mapping $U, V, W \to Z, X, Y$ is equidistant from the pairs of sides $(\overline{UV}, \overline{XZ})$ and cyclic variants, so it lines on the bisectors of the angles $\angle B, \angle D, \angle F$ formed by those pairs of lines.

Remark. I wish I'd seen this problem before failing **USEMO 2020/5** in-contest...

1.6 IMO 2008/6, by Vladimir Shmarov

Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.



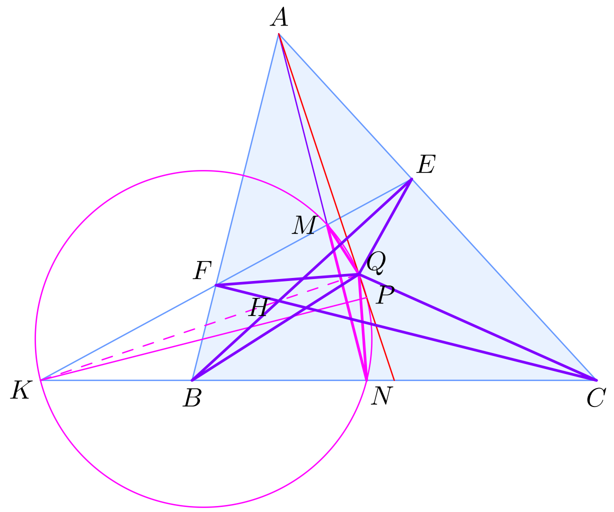
Rename ω_1, ω_2 to ω_b, ω_d; by Pitot-like reasoning we have $AB + AD = CB + CD$; let T_b, T_d be the intouch points on $\overline{AC}$; then T_b, T_d are isotomic by the obtained length condition.

If we let T'_b, T'_d be the antipodes of T_b, T_d on their respective circles, then an EGMO lemma (ch4) implies that B, T_d, T'_b and sym variant are collinear.

Construct the point K' on the "closer" side to the rest of the figure so that the tangent to ω at K is parallel to $\overline{AC}$. Then by homothety $K' \in \overline{BT_d}, \overline{DT_b}$, so this is the desired exsimilicenter.

1.7 Iran TST 2018/1/4, by Iman Maghsoudi & Hooman Fattahi

Let ABC be a triangle ($\angle A \neq 90°$), with altitudes $\overline{BE}, \overline{CF}$. The bisector of $\angle A$ intersects $\overline{EF}, \overline{BC}$ at M, N. Let P be a point such that $\overline{MP} \perp \overline{EF}$ and $\overline{NP} \perp \overline{BC}$. Prove that $\overline{AP}$ bisects $\overline{BC}$.



Construct $K = \overline{EF} \cap \overline{BC}$, Q as the A-Humpty point, H as the orthocenter of $\triangle ABC$, and $\omega = (KMN)$, so that the P given is the antipode of K on it. Let spiral similarity s at Q take $(E, F) \to (B, C)$. The main point of the problem is then:

Claim – $MKQN$ cyclic. In other words, $Q \in \omega$.

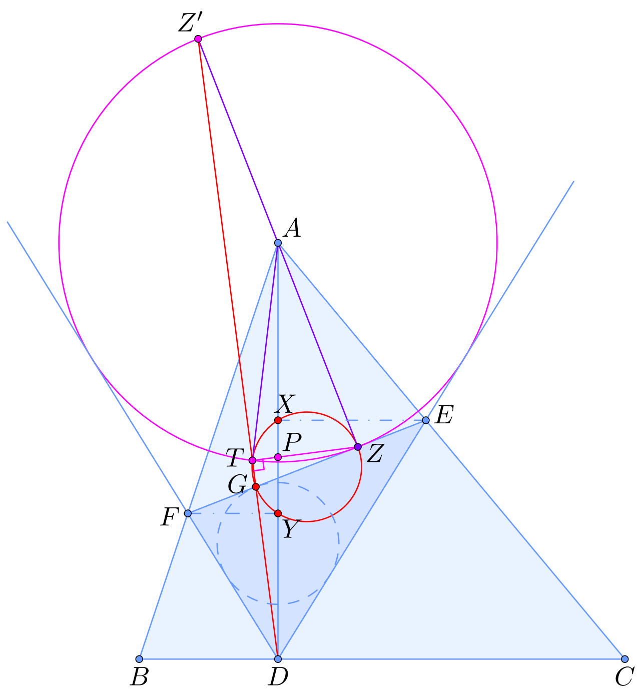
Proof. From angle bisector theorem,

$$\frac{EM}{MF} = \frac{EA}{AF} = \frac{BA}{AC} = \frac{BN}{NC} \Rightarrow (M \xrightarrow{s} N) \Rightarrow \measuredangle MQN = -\arg(s) = \measuredangle(\overline{EF}, \overline{BC}) = \measuredangle MKN. \qquad \square$$

Since P is the antipode of K on ω, $\angle KQP = 90° = \angle KQA$, implying that $P \in \overline{AQ}$, the A-median.

1.8 Mock AIME 2019/15', by Eric Shen & Raymond Feng

In $\triangle ABC$, let D, E, F be the feet of the altitudes from A, B, C respectively, and let O be the circumcenter. Let $Z = \overline{AO} \cap \overline{EF}$. There exists a point T such that $\angle DTZ = 90^\circ$ and $AZ = AT$. If $P = \overline{AD} \cap \overline{TZ}$, and Q lies on $\overline{EF}$ such that $\overline{PQ} \parallel \overline{BC}$, prove that $\overline{AQ}$ bisects $\overline{BC}$.



Construct points X, Y as the projections of E, F onto $\overline{AD}$ respectively. *

After drawing a diagram on Geogebra, we obtain:

Characterization of T

T is the harmonic conjugate of Z wrt XY – i.e. it lies on $\omega = (XYZ)$ so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of T lies on ω_a (defined as the circle at A thru Z) and (DZ),

Verification (inspired by USA TST 2015/1)

For $AZ = AT$, we use power of a point / length chase to get $AZ^2 = AX \cdot AY$ whence $\overline{AZ}$ touches ω. Hence, by harmonics $\overline{AT}$ is also tangent to ω, so this property follows.

$\angle DTZ = 90^\circ$ is much less straightforward. We define $Z' = 2A - Z$ and $G = E + F - Z$ as the antipodes of Z on the circle at A through Z. By a well-known lemma, D, Z', G collinear (along the cevian through the intouch point in $\triangle DEF$).

But also at the same time, T is on $\omega, \omega_a \implies \angle ZTG = \angle ZTZ' = \pi/2$ due to antipodes. Hence, $\angle DTZ = \pi/2$, completing the verification.

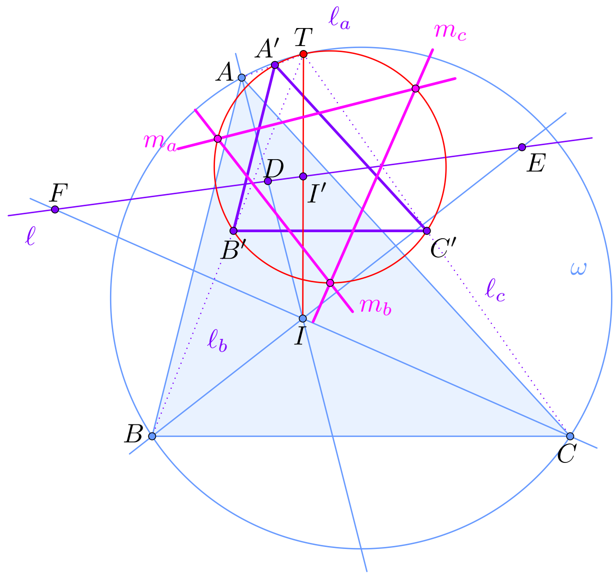
By this definition, we clearly have $(AP; XY) = -1$. From here (the chase is best discovered backwards), harmonic chasing suffices. Define $K = \overline{EF} \cap \overline{A\infty_{BC}}$. Then the bisection is established by

$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK; FE) \stackrel{\infty_{BC}}{=} (PA; YX) = -1.$$

*Eric Shen originally included these points in the problem statement (as seen in the 2019 version of "Geometry At Its Best"), but I guess the problem's made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

1.9 SL 2018/G5, by Denmark

Let ABC be a triangle with circumcircle ω and incenter I. A line ℓ meets the lines AI, BI, CI at points D, E, F respectively, all distinct from A, B, C, I. Prove that the circumcircle of the triangle determined by the perpendicular bisectors of $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ is tangent to ω.



Solution by **TheUltimate123**.

Let ℓ_a and cyclic variants be the reflections of ℓ in the perpendicular bisectors x_a of $\overline{AD}$, etc.

> **Claim –** $\ell_a, \ell_b, \ell_c, \omega$ concur at a point T.

Proof. Because

$$\measuredangle(\ell_b, \ell_c) = 2\measuredangle(x_b, x_c) = 2\measuredangle BIC = \measuredangle BAC,$$

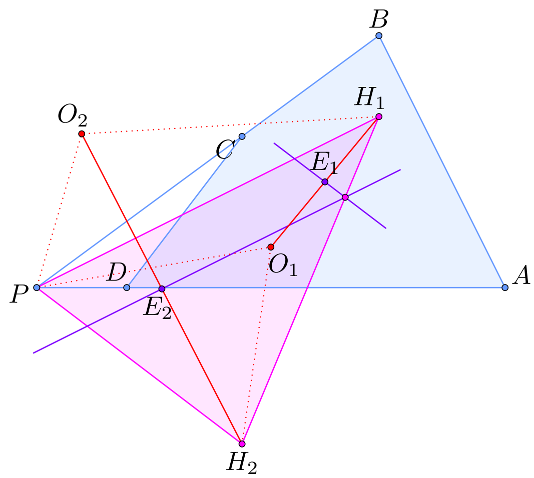
$\ell_b \cap \ell_c \in \omega$; the result follows by symmetry. □

Let $I' = \overline{TI} \cap \ell$, and consider the homothety h at T mapping $I \longrightarrow I'$. Let P' denote the image of point P under h, so I' is the incenter of $\triangle A'B'C'$. Since $\overline{A'I'} \parallel \overline{ADI}$ while $A' \in \ell_a$ and $I' \in \ell$, m_a is also the perpendicular bisector of $\overline{AI}$.

From here it follows that the pairwise intersections of m_a, m_b, m_c are just the arc midpoints in $(A'B'C')$. By h, $(A'B'C')$, (ABC) tangent at T, hence done.

1.10 SL 2009/G6, by Eugene Bilopitov (Ukraine)

Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of triangles ABP and CDP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.



Trying not to bash excessively…consider the problem wrt $\triangle PH_1H_2$. Observe that by isogonals, $\angle O_2PH_1 = \angle H_1PO_2$, so they've equal sines and

$$\frac{PH_1}{PO_1} = 2\cos P = \frac{PH_2}{PO_2} \Rightarrow [PO_2H_1] = [PO_1H_2] \Rightarrow h_1(O_1) = -h_2(O_2) \overset{\text{linearity}}{\Rightarrow} \boxed{h_1(E_1) + h_2(E_2) = 1}$$

in barycentrics wrt $\triangle PH_1H_2$, where $p(X)$ denotes the P-coordinate of X, and similarly for the H_k. This means that the three desired lines (which can be defined as those through E_1, E_2 parallel to $\overline{PH_2}$, $\overline{PH_1}$ respectively) concur at

$$\boxed{0P + h_1(E_1)\cdot H_1 + h_2(E_2)\cdot H_2} \in \overline{H_1H_2}$$

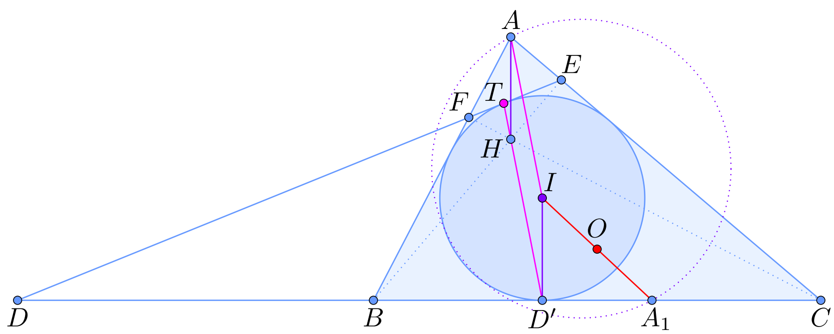
which is a valid barycentric point because of the first boxed equation.

1.11 MOP + USA TST, by Ankan Bhattacharya

Let ABC be a triangle with incenter I, and let D be a point on line BC satisfying $\angle AID = 90^\circ$. Let the excircle of triangle ABC opposite the vertex A be tangent to $\overline{BC}$ at A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and C, respectively.

1.11.1 MOP 2019/(?)

Let E, F be the feet of the altitudes from B, C respectively. Prove that if $\overline{EF}$ touches the incircle, then quadrilateral $AB_1A_1C_1$ is cyclic.



Call the incircle ω.

> **Claim 1 –** D, E, F are collinear.

Proof. We will prove that the tangent line from D is antiparallel to $\overline{BC}$ wrt $\angle A$. Indeed, this line is found by reflecting $\overline{DBC}$ over $\overline{DI}$, a line perpendicular to $\overline{AI}$, so we win. □

Let ω touch $\overline{DEF}$ at a point T, and let D' denote the A-intouch point.

> **Claim 2 –** $\overline{AI} \parallel \overline{HD'}$; hence $AID'H$ is a parallelogram and $AH = r$, the inradius of $\triangle ABC$.

Proof. Because $BCEF$ is tangential, it follows by degenerate Brianchon that lines BE, CF, DT' concur, i.e. $H \in \overline{TD'}$. Observe that $DT = DD'$; then $\overline{THD'} \perp \overline{DI}$ by symmetry, while $\overline{AI} \perp \overline{DI}$ is given; the lines are thus parallel as claimed. □

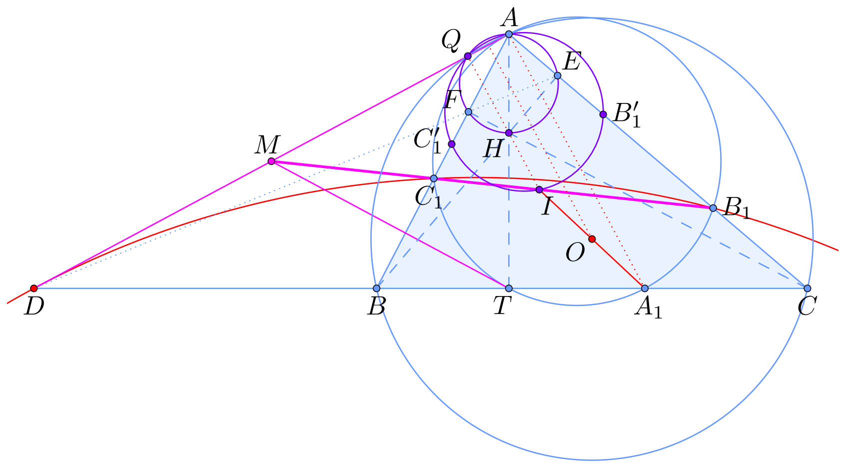
Now, let ω_a, etc denote (AB_1C_1), etc, respectively. We observe that because the perpendicular from A_1 to $\overline{BC}$ and its cyclic variants all concur at the point $2O - I$, it follows that all three circles must concur at this point by Miquel spam.

But because $r/2 = AH/2$ is the distance from O to $\overline{BC}$, we actually have $2O - I = A_1$ (also because of their feet onto $\overline{BC}$). Hence $A_1 \in \omega_a$ as desired.

> *Remark.* I know for a fact that this was a problem during the Red 2019 tests- Eric Shen seems to have thoroughly enjoyed it in contest.

1.11.2 USA TST 2019/6

Prove that if quadrilateral $AB_1A_1C_1$ is cyclic, then $\overline{AD}$ is tangent to the circumcircle of $\triangle DB_1C_1$.



From MOP 2019, we make the following observations:

- By its converse, D, E, F collinear; then, if T is the foot from A to $\overline{BC}$, we have $(TD; BC) = -1$.
- As A_1 is the Bevan point $2O - I$, its projections onto $\overline{AC}, \overline{AB}$ are B_1, C_1 respectively. It follows that A, A_1 are antipodes on ω_a.
- Since $BCEF$ is bicentric, if the incircle touches $\overline{AC}, \overline{AB}$ at B_1', C_1', then $BC_1'/FC_1' = CB_1'/EB_1'$, so the A- incenter and orthocenter Miquel points coincide, say at $Q \in (ABC)$.

From the last item, $\angle AQI = \angle AQH = 90^\circ$.

Claim – $\overline{AD}$ touches ω_a.

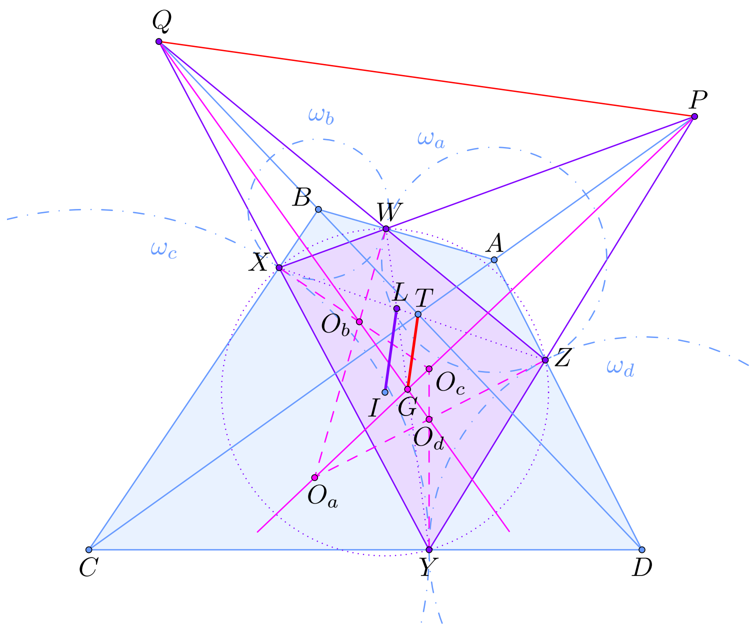
Proof. Since $(ABC) \cap (AH) = \{A, Q\}$, the projection of O onto $\overline{AQD}$ is $\frac{A+Q}{2}$. At the same time, the above implies Q is the projection of I onto $\overline{AQD}$. By linearity the projection of $A_1 = 2O - I$ onto $\overline{AD}$ is $2\frac{A+Q}{2} - Q = A$– in other words, $\angle A_1AD = 90^\circ$. This proves the tangency as $\overline{AA_1}$ is a diameter of ω_a. $\square$

Let $M = \frac{A+D}{2}$, so $\overline{MT}$ touches ω_a as well by symmetry in the perpendicular bisector $M\infty_{BC}$ of $\overline{AT}$. Now, $(AT; B_1C_1) \overset{A}{=} (DT; CB) = -1$ means $M \in \overline{B_1C_1}$.

Finish by power of a point converse: $MD^2 = MA^2 = MB_1 \cdot MC_1$ gives the needed tangency.

1.12 ELMO SL 2024/G4 (Nyan)

In quadrilateral $ABCD$ with incenter I, points W, X, Y, Z lie on sides AB, BC, CD, DA with $AZ = AW$, $BW = BX$, $CX = CY$, $DY = DZ$. Define $T = \overline{AC} \cap \overline{BD}$ and $L = \overline{WY} \cap \overline{XZ}$. Let points O_a, O_b, O_c, O_d be such that $\angle O_aZA = \angle O_aWA = 90°$ (and cyclic variants), and $G = \overline{O_aO_c} \cap \overline{O_bO_d}$. Prove that $\overline{IL} \parallel \overline{TG}$.



Draw the circle at A through W, Z and its cyclic variants, which we respectively call $\omega_a, \dots, \omega_d$. Then $IW = IX = IY = IZ$ follows by symmetry about $\overline{AI}$ and its cyclic variants.

Claim 1 – $\overline{O_aO_c}$ is the radical axis of ω_b, ω_d.

Proof. O_a has power $O_aW^2 = O_aZ^2$ wrt $\omega_d, \omega_a, \omega_b$ so it's their radical center. □

The crux of the problem is:

Claim 2 – Let P, Q be the exsimilicenters of (ω_b, ω_d), (ω_a, ω_c). Then $P \in \overline{WX}, \overline{YZ}, \overline{O_aO_c}, \overline{BD}$ and similarly $Q \in \overline{WZ}, \overline{XY}, \overline{O_bO_d}, \overline{AC}$.

Proof. We show that the first four lines pass through the exsimilicenter of ω_b and ω_d.

- $P = \overline{WX} \cap \overline{YZ}$ and $Q = \overline{WZ} \cap \overline{XY}$ follow by Monge on all sets of 3 circles;
- $P \in \overline{AC}$ by design;
- $P \in \overline{O_aO_c}$ are obtained from radical axis theorem on $(WXYZ)$, ω_b, ω_d in conjunction with claim 1;

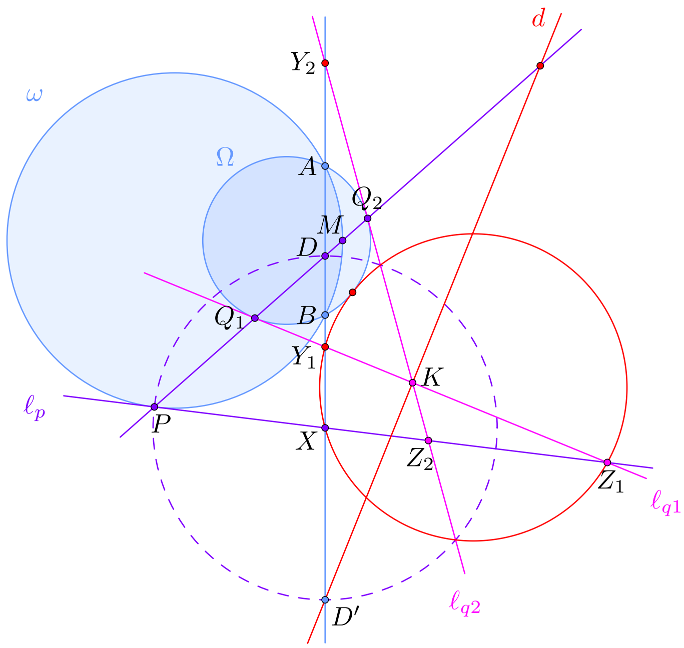
□

To finish, note that we have the orthocentric systems:

- $PQIL$ via Brocard on $WXYZ$;
- $PQTG$ because $\overline{O_aO_c} \perp \overline{BD}$ and $\overline{O_bO_d} \perp \overline{AC}$.

1.13 APMO 2014/5, by Ilya Bogdanov & Medeubek Kungozhin

Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω). A chord MP of circle ω intersects Ω at Q (Q lies inside ω). Let ℓ_P be the tangent line to ω at P, and let ℓ_Q be the tangent line to Ω at Q. Prove that the circumcircle of the triangle formed by the lines ℓ_P, ℓ_Q and AB is tangent to Ω.



We'll consider both Q's at once, the one inside and outside. Call them Q_1, Q_2 in any order. Define (here $k = 1, 2$):

- $X = \ell_p \cap \overline{AB}$, $Y_k = \ell_{qk} \cap \overline{AB}$, $Z_k = \ell_{qk} \cap \ell_p$;
- D and $D' = 2X - D$ as the intersections of the internal and external bisectors of $\angle APB$ with $\overline{AB}$, respectively, so that $XP = XD = XD'$;
- $K = \ell_{q1} \cap \ell_{q2}$ as the pole of $\overline{Q_1Q_2}$ wrt Ω, so that $KQ_1 = KQ_2$.

Claim 1 – $Y_1Y_2Z_1Z_2$ is cyclic.

Proof. Note that triangles PXD, KQ_1Q_2 are both isosceles. Then

$$\measuredangle(\ell_p, \ell_{q1}) = \measuredangle XPD + \measuredangle PQ_1K \overset{\text{isosceles}}{=} -\measuredangle XDP - \measuredangle PQ_2K = -\measuredangle(\overline{AB}, \ell_{q2}),$$

whence the quadrilateral formed by ℓ_p, ℓ_{q1}, $\overline{AB}$, ℓ_{q2} (in order) is cyclic. □

Let i denote inversion at X with power $XP^2 = XD^2 = XA \cdot XB$ (last equality by midpoints of harmonic bundles lemma).

Claim 2 – i swaps Y_1, Y_2 as well.

Proof. Consider the polar $\overline{KD'}$ of D wrt Ω, which we call d. Then

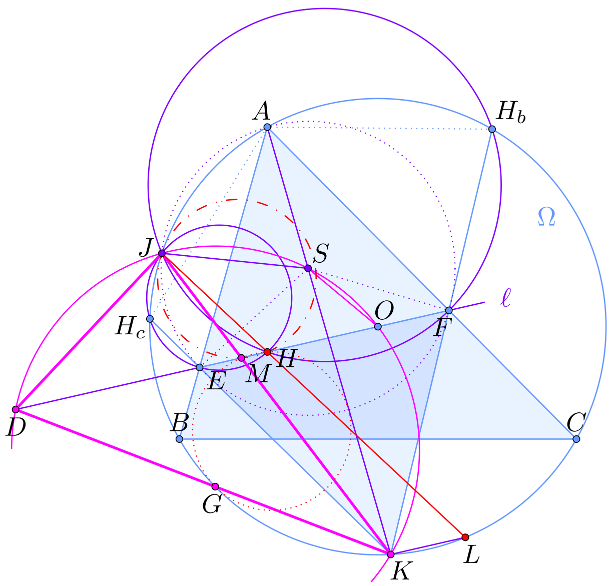
$$(Y_1Y_2; DD') \overset{K}{=} (Q_1, Q_2; D, d \cap \overline{Q_1DQ_2}) = -1,$$

the last harmonic bundle holding by definition of polar. The claim follows by another application of midpoints of harmonics bundles lemma. □

By the previous two claims and power of a point at X, i also swaps (Z_1, Z_2). Applying i to the given "$\overline{Y_2Z_2}$ touches Ω" yields (XY_1Z_1) also tangent to Ω, concluding the proof.

1.14 DeuX MO 2020/II/3, by Hao Minyan (China)

In triangle ABC with circumcenter O and orthocenter H, line OH meets $\overline{AB}, \overline{AC}$ at E, F respectively. Let ω be the circumcircle of triangle AEF with center S, meeting (ABC) again at $J \neq A$. Line OH also meets (JSO) again at $D \neq O$. Define $K = (JSO) \cap (ABC) \ (\neq J)$, $M = \overline{JK} \cap \overline{OH}$, and $G = \overline{DK} \cap (ABC) \ (\neq K)$. Prove that (GHM) and (ABC) are tangent to each other.



Solution by **crazyeyemoody907**, **v4913**.

Let $\Omega = (ABC)$, H_b, H_c be the respective reflections of H in $\overline{AC}, \overline{AB}$, and $\ell = \overline{EFOH}$. Redefine $K = \overline{H_cE} \cap \overline{H_bF}$ (we'll see this is an equivalent definition). As $\overline{EA}, \overline{FA}$ are external angle bisectors wrt $\triangle KEF$, we have $\angle EKF = \pi - 2A$.

> **Claim 1 –** $J \in (HEH_c), (HFH_b)$.

Proof. Let $J' = (HEH_c) \cap (HFH_b) \ (\neq H)$. Then:

$$\measuredangle H_cJ'H_b = \measuredangle H_cJ'H + \measuredangle HJ'H_b = \measuredangle H_cEH + \measuredangle HFH_b = \measuredangle(\overline{H_cE}, \overline{H_bF}) = \measuredangle H_bKH_c = \measuredangle H_bAH_c \Rightarrow J' \in \Omega.$$

The construction of J' implies that $\overline{J'E}, \overline{J'F}$ respectively bisect $\angle H_cJ'H$, $\angle H_bJ'H$, and thus

$$\angle EJ'F = \frac{1}{2}\angle H_bJ'H_c = \angle BAC = \angle EAF \Rightarrow J' \in (AEF),$$

finishing the claim. □

Let $L = \overline{JH} \cap \Omega \ (\neq J)$; then, as JH_cKL, JH_cEH cyclic, $\ell \parallel \overline{KL}$ by Reim. By homothety, (JHM) touches $(JKL) = \Omega$.

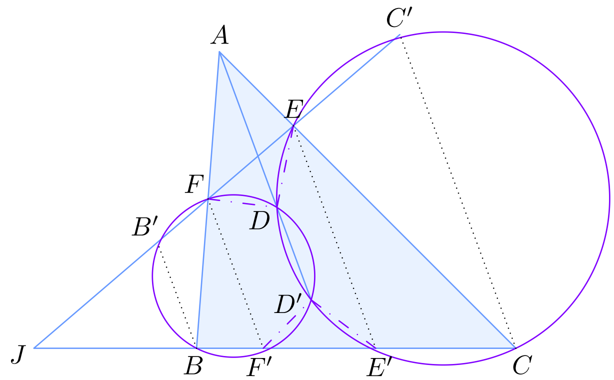
> **Claim 2 –** For the K defined in solution, $K \in \overline{AS}, (JSO)$.

Proof. Since $\measuredangle ESF = 2\measuredangle BAC = \measuredangle EKF$, we have $KESF$ cyclic; as $SE = SF, AH_b = AH_c, A, S$ both lie on bisector of $\angle EKF$. Next, we prove that O is the midpoint of $\widehat{JSK}$ on (JSK). Because $\overline{OS}$ is the perpendicular bisector of $\overline{AJ}$ by symmetry, it externally bisects $\angle JSK$ as $K \in \overline{AS}$. At the same time, $OJ = OK$ means O is on the perpendicular bisector of $\overline{JK}$. These two properties imply that O is the claimed arc midpoint. □

From here, as $DJKO$ cyclic and $OJ = OK$, $\overline{DO}$ bisects $\angle JDK$, and $G = \overline{DK} \cap \Omega$ is the reflection of J in ℓ by symmetry. Reflecting "(JHM) touches Ω" over ℓ completes the proof.

1.15 IMO 2021/3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.



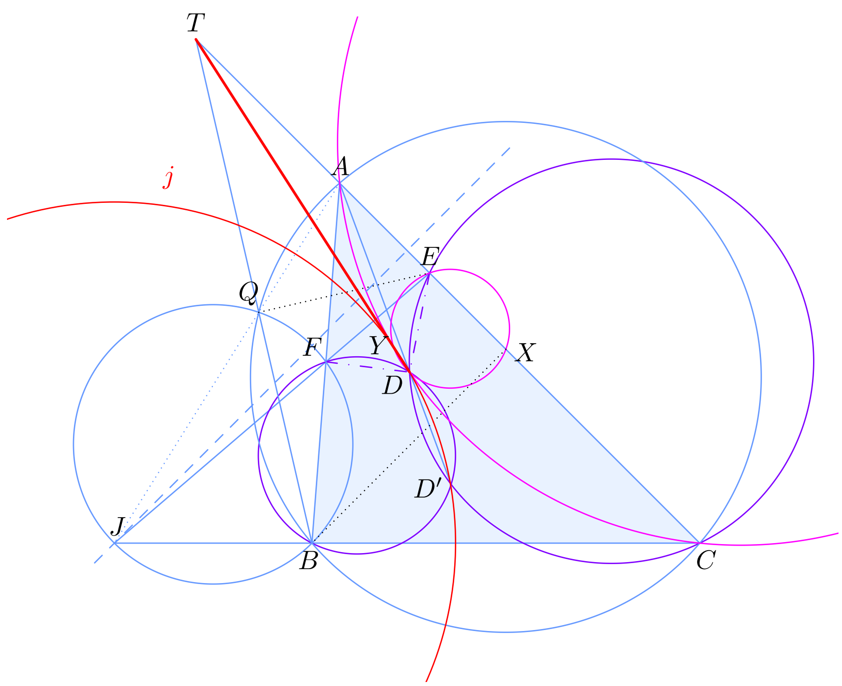
Solution by **v4913**.

Let $J = \overline{EF} \cap \overline{BC}$, and $D' \in \overline{AD}$ be the isogonal conjugate of D wrt $\triangle ABC$. The given angle conditions imply that $BDD'F$, $CDD'E$ are cyclic, while power of a point at A implies $BCEF$ cyclic as well.

Claim 1 – J is the exsimilicenter of (EDC), (FDB); hence, $JD = JD'$ by symmetry.

Proof. Construct $E_1 = (CDD'E) \cap \overline{BC}$ $(\neq C)$, $F_1 = (BDD'F) \cap \overline{BC}$ $(\neq B)$. By isogonality, $DF = D'F'$ and $DE = D'E'$ whence $DD'E'E$, $DD'F'F$ are both cyclic isosceles trapezoids. $\overline{DD'}, \overline{EE'}, \overline{FF'}$ share a perpendicular bisector b, and in fact, this is the bisector of $\angle J$, i.e. $JE = JE'$, $JF = JF'$.

Reflect B, C over b to obtain B', C'; then, because $JB/JF' = JB/JF = JE/JC = JE'/JC$, there is a homothety at J mapping $(B, B', F, F') \to (E', E, C', C)$ and thus their circumcircles $(BB'DD') \to (CC'DD')$ as well. $\square$



Let $Y = (ADC) \cap (EXD) \ (\neq D)$, Q be the Miquel point of $ABCJEF$, and j the circle at J through D, D'. Observing that $\overline{O_1O_2}$ is the perpendicular bisector of $\overline{DY}$, it remains to prove $Y \in j$.

> **Claim 2 –** $XQEB$ is cyclic.

Proof. This is a simple angle chase: using cyclic quadrilaterals $(ABCQ)$, $(JFBQ)$, $(ECJQ)$, and $(AEFQ)$, we obtain

$$\measuredangle EQB = \measuredangle EQA + \measuredangle AQB = \measuredangle ACB + \measuredangle EFA = 2\measuredangle ACB = \measuredangle EXB$$ □

Next, we characterize the radical axis of j, (JBF)– it's perpendicular to the line of centers and through A:

> **Claim 3 –** The line through B and the center of (JBF) is perpendicular to $\overline{AC}$.

Proof. This is equivalent to "t_b, the tangent to (JBF) at J, is parallel to $\overline{AC}$". Because $\measuredangle(\overline{BJC}, t_b) = \measuredangle BFJ = \measuredangle JCE$, the result follows. □

Because $\text{Pow}(A, j) = AD \cdot AD' = AQ \cdot AJ = \text{Pow}(A, (JBQF))$, A is on the radical axis of j, (JBF). By the previous claim, it follows that $\overline{AC}$ is the radical axis of j, (JBF).

To finish, define $T = \overline{DY} \cap \overline{AC} \cap \overline{BQ}$ as the radical center of (JBF), (ABC), (EXD), (ADC), and the phantom point $Y' = \overline{TD} \cap j \ (\neq D)$. Because T is on $\overline{AC}$, the radical axis of j, (JBF), we have (lengths directed)

$$TY' \cdot TD = \text{Pow}(T, j) = \text{Pow}(T, (JBF)) = \text{Pow}(T, (ABCQ)) = TA \cdot TC = TY \cdot TD \Rightarrow Y = Y',$$

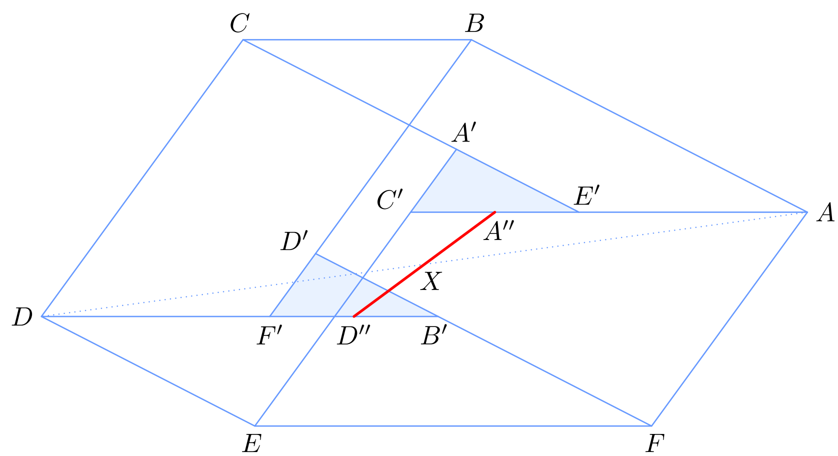
the end!

1.16 USAMO 2021/6, by Ankan Bhattacharya

Let $ABCDEF$ be a convex hexagon satisfying $\overline{AB} \parallel \overline{DE}$, $\overline{BC} \parallel \overline{EF}$, $\overline{CD} \parallel \overline{FA}$, and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let X, Y, and Z be the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. Prove that the circumcenter of $\triangle ACE$, the circumcenter of $\triangle BDF$, and the orthocenter of $\triangle XYZ$ are collinear.



Construct parallelogram $CDEA'$ and cyclic variants: $A' = C + E - D$, etc. We may compute using vectors that $\triangle B'D'F'$ is a translation of $\triangle A'C'E'$ by the vector $(B + D + F) - (A + C + E)$. In particular, they're congruent.

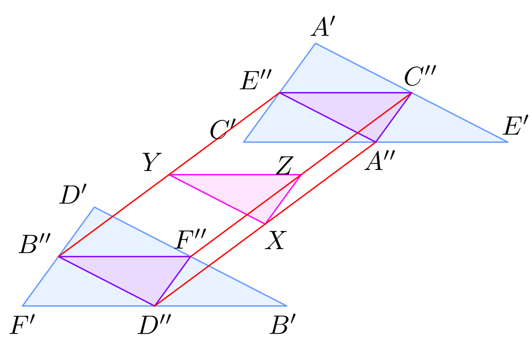
> **Claim 1 –** A, C, E have same power wrt $(A'C'E')$; in other words, $\triangle ACE, A'C'E'$ share a circumcenter.

Proof. Observing that $\operatorname{Pow}(A, (A'C'E')) = AC' \cdot AE' = BC \cdot EF$ by parallelograms, this claim follows by the given length condition. □

Next, construct $A'' = \frac{C'+E'}{2}$ and cyclic variants. The circumcenter of $\triangle A'C'E'$ is then the orthocenter of $\triangle A''C''E''$.

> **Claim 2 –** $X = \frac{A''+D''}{2}$.

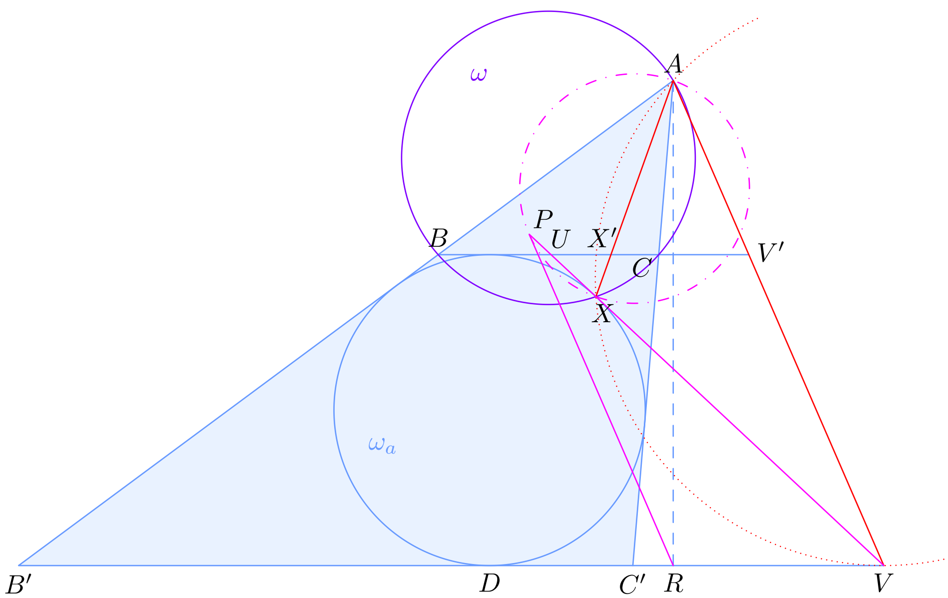
Proof. Using vectors, $B' + C' = E' + F' = A + D \Rightarrow \frac{A+D}{2} = \frac{B'+C'+E'+F'}{4} = \frac{A''+D''}{2}$. □



By claim 2 + symmetry, $\triangle XYZ$ is the vector average of (congruent) triangles $A''C''E''$, $B''D''F''$, so their orthocenters are collinear.

1.17 SL 2021/G8

Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.



Solution by **crazyeyemoody907**.

Let the antipode of the A-extouch point be D, and the tangent to ω_a at D intersect $\overline{AB}, \overline{AC}$ at B', C' respectively. Also, construct the tangent line to ω_a at X, meeting $\overline{BC}, \overline{B'C'}$ at U, V respectively. Finally, let $X' = \overline{AX} \cap \overline{BC}$, $V' = \overline{AV} \cap \overline{BC}$.

Claim 1 – $AXUV'$ cyclic.

Proof. Apply DDIT to A, $UXV\infty_{BC}$ (inconic ω_a), and project onto $\overline{BC}$, to obtain an involutive pairing $(BC; UV'; \infty_{BC}X')$– or equivalently, $X'B \cdot X'C = X'U \cdot X'V'$. By power of a point, $X'B \cdot X'C = X'A \cdot X'X$, so the claim follows from $X'U \cdot X'V = X'A \cdot X'X$. □

Claim 2 – $\overline{DV}$ is tangent to (AXV).

Proof. Angle chase using previous claim, and the fact that $\overline{BC} \parallel \overline{B'C'}$:

$$\measuredangle XAV \overset{\text{claim 1}}{=} \measuredangle XUV' = \measuredangle XVD.$$ □

Redefine R as the foot from A to $\overline{B'C'}$. It remains to show,

Claim 3 – $\overline{PR}$ touches (APX').

Proof. Since $\angle VPA = \angle VRA = 90°$, $APRV$ cyclic, so we may anglechase as follows:

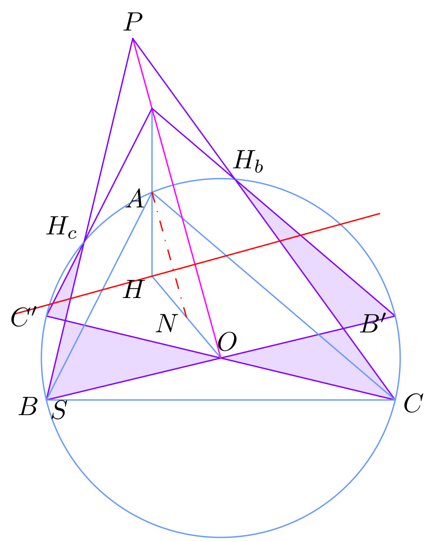
$$\measuredangle APR = \measuredangle AVR \overset{\text{claim 2}}{=} \measuredangle AXV = \measuredangle AXP.$$ □

1.18 USEMO 2020/3, by Anant Mudgal

Let ABC be an acute triangle with circumcenter O and orthocenter H. Let Γ denote the circumcircle of triangle ABC, and N the midpoint of $\overline{OH}$. The tangents to Γ at B and C, and the line through H perpendicular to line AN, determine a triangle whose circumcircle we denote by ω_A. Define ω_B and ω_C similarly.

Prove that the common chords of ω_A, ω_B, and ω_C are concurrent on line OH.

Let H_a, A' denote the respective reflections of H in $\overline{BC}$, A in O, and their symmetric variants.



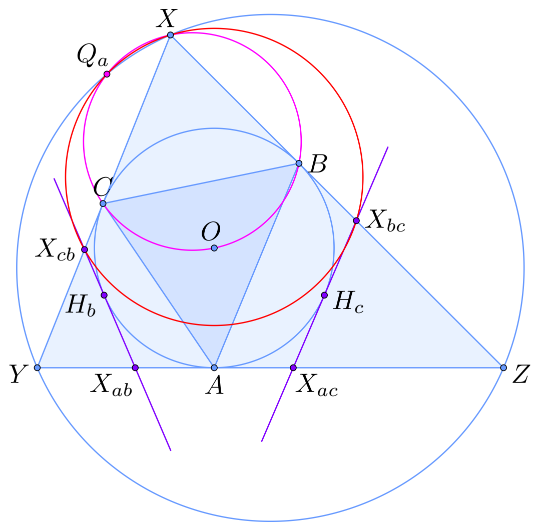
Claim 1 – The polar ℓ_a of $\overline{BH_c} \cap \overline{CH_b}$ passes through H and is perpendicular to $\overline{AN}$.

Proof. Let $P = \overline{BH_c} \cap \overline{CH_b}$ and $S = 2A - H$. $H \in \ell_a$ is just Brokard, so it suffices to prove $\overline{AN} \parallel \overline{OP}$. By Pascal on $BB'H_bCC'H_c$, we have P, O, S collinear. Taking a homothety at H with scale factor $\frac{1}{2}$ maps the latter two points to N, A, which implies the required parallel lines. □

In $\triangle ABC$, let X_bc be the pole of $\overline{BH_c}$ wrt Γ (and 5 other variants), X, Y, Z be the poles of the sides, D, E, F be the feet of the altitudes. Clearly, $\ell_a = \overline{X_{bc}X_{cb}}$.

> *Note.* Here, the condition $\triangle ABC$ acute comes in: Γ is the incircle, not excircle, of $\triangle XYZ$.

We'll show that $\overline{XD}$ is the radical axis of ω_b, ω_c. (By a somewhat-known configuration (say, **Brazil 2013/6**), $\overline{XD} \cap \overline{YE} \cap \overline{ZF}$ lies on the Euler line.) Also let Q_a, Q_b, Q_c be the SD points of $\triangle XYZ$.



> **Claim 2 –** Q_a lies on ω_a.

Proof. By spiral similarity, it suffices to prove $YX_{bc}/YC = ZX_{cb}/ZB$. By antiparallel lines, $\triangle XYZ \stackrel{-}{\sim} \triangle X_{ab}YX_{cb}, X_{ac}X_{bc}Z$. But since Γ is the Y-excircle of $\triangle X_{ab}YX_{cb}$, we have $YX_{cb}/YC = a/s$. Similarly $ZX_{bc}/ZB = a/s$ as well.
(In some awful notation, $a = YZ$, $b = ZX$, $c = XY$ and $s = \frac{a+b+c}{2}$.) □

Let $L = \overline{YQ_b} \cap \overline{ZQ_c}$.

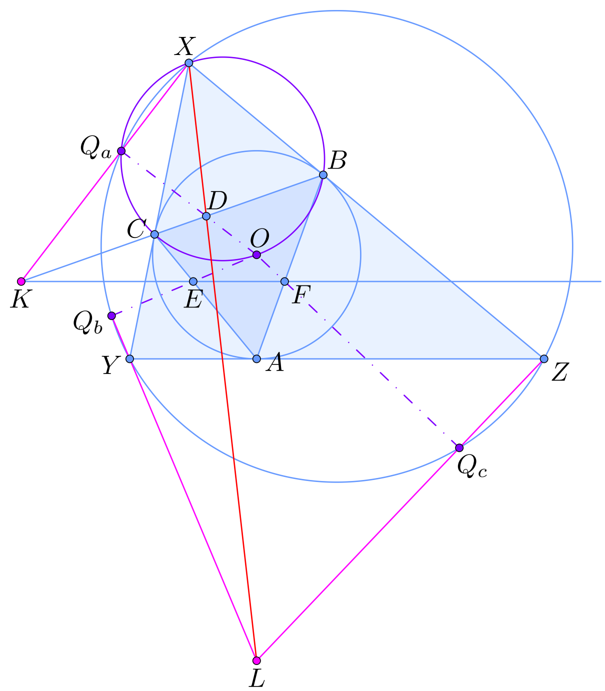
> **Claim 3 –** $\overline{XL}$ is the radical axis of ω_b, ω_c.

Proof. By antiparallel lines again, $YZX_{ba}X_{ca}$ cyclic, so that

$$\operatorname{Pow}(X, \omega_b) = XX_{ca} \cdot XY = XX_{ba} \cdot XZ = \operatorname{Pow}(X, \omega_c), \text{ while}$$

$$\operatorname{Pow}(L, \omega_b) = LY \cdot LQ_b = LZ \cdot LQ_c = \operatorname{Pow}(L, \omega_c).$$ □

It remains to prove X, D, L collinear.



Claim 4 – L is the pole of $\overline{EF}$ wrt Γ.

Proof. Since Q_a is the inverse of D wrt Γ and $\angle OQ_aX = 90°$, $\overline{XQ_a}$ is the polar of D wrt Γ. Similarly, $\overline{YQ_b}$, $\overline{ZQ_c}$ are the respective polars of E, F wrt Γ. The claim is then established by la Hire. $\square$

Claim 5 – $\overline{BC}, \overline{EF}, \overline{XQ_a}$ concurrent.

Proof. Let $K = \overline{EF} \cap \overline{BC}$ so that $(KD; BC) = -1$. Because $\overline{Q_aO}$ bisects $\angle BQ_aC$, $\angle KQ_aO = 90° = \angle AQ_aO \Rightarrow X, Q_a, K$ collinear. $\square$

Taking poles wrt Γ in the last claim gives the desired collinearity.

Remark. The problem can be bary'd wrt $\triangle XYZ$ after the first claim, but it's monstrous from my experience a long time ago, oops