Ridiculous geometry compilation

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2022-4

I wanted to make a record of these and they don't exactly belong in a list of favorites, so here we are. ~~Also this definitely doesn't show off the power of Geogebra.~~

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0 Problems

Problem 1 (USA TST 2021/2). Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

Problem 2 (DIMO 2022/6). In triangle $\triangle ABC$, M is the midpoint of arc BAC, I is the incenter and I_a is the A-excenter. Let $E = \overline{BI} \cap \overline{AC}$, $F = \overline{CI} \cap \overline{AB}$, $P = \overline{MI} \cap (ABC)$, and $S = (AEF) \cap (ABC)$ ($\neq A$). If X, Y are the reflections of I across $\overline{I_aE}, \overline{I_aF}$ respectively, prove that (BYF), (CXE), (PXY) and $\overline{PS}$ are concurrent.

Problem 3 (Brazil Revenge 2021/3). Let I, C, ω and Ω be the incenter, circumcenter, incircle and circumcircle, respectively, of the scalene triangle XYZ with $XZ > YZ > XY$. The incircle ω is tangent to the sides YZ, XZ and XY at the points D, E and F. Let S be the point on Ω such that XS, CI and YZ are concurrent. Let $(XEF) \cap \Omega = R$, $(RSD) \cap (XEF) = U$, $SU \cap CI = N$, $EF \cap YZ = A$, $EF \cap CI = T$ and $XU \cap YZ = O$.

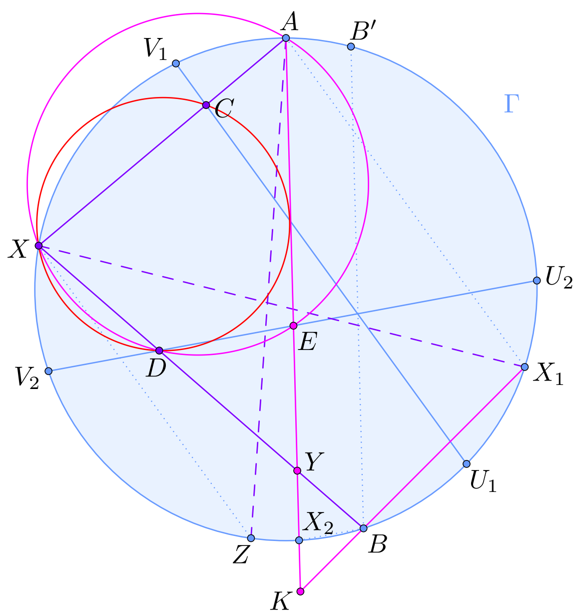
Prove that $NARUTO$ is cyclic.

1 Solutions

1.1 USA TST 2021/2, by Andrew Gu & Frank Han

Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.



Clearly, the problem statement should hold for any $X \in \Gamma$; here, all lengths are directed.
Let X_1, X_2 be the respective reflections of A, B in the perpendicular bisectors of $\overline{U_1V_1}$, $\overline{U_2V_2}$. We assert that $K = \overline{AX_2} \cap \overline{BX_1}$ fits the bill. For brevity, let '$\leftrightarrow$' denote 'is a constant multiple of', so '$x \leftrightarrow 1$' is a shorthand for 'x is constant'.

By Reim, $E = \overline{BX} \cap \overline{AX_2}$ lies on (ADX), so $\operatorname{Pow}(K, (ADX)) = KE \cdot KA \leftrightarrow 1$. Now, in the spirit of linpop, let $f(P) = \operatorname{Pow}(P, (ADX)) - \operatorname{Pow}(P, (XCD))$, so that because $f(Y) = 0$, we have

$$f(K) = -\frac{KY}{YA} f(A) = AC \cdot AX \frac{KY}{AY}.$$

The rest is a wild length chase; let B', Z be the respective reflections of B, X in the perpendicular bisector of $\overline{U_1V_1}$, so that $XX_1 = AZ$ and $\overline{AZ}, \overline{ACX}$ isogonal wrt $\angle U_1AV_1$. Then, observing that all lengths not involving X, C, D, Y are fixed,

$$\frac{KY}{AY} = (KA; Y\infty_{AK}) \overset{B}{=} (X_1A; XB') \leftrightarrow \frac{X_1X}{AX} = \frac{AZ}{AX};$$

$$\Rightarrow f(K) \leftrightarrow AC \cdot AZ = AU_1 \cdot AV_1 \leftrightarrow 1,$$

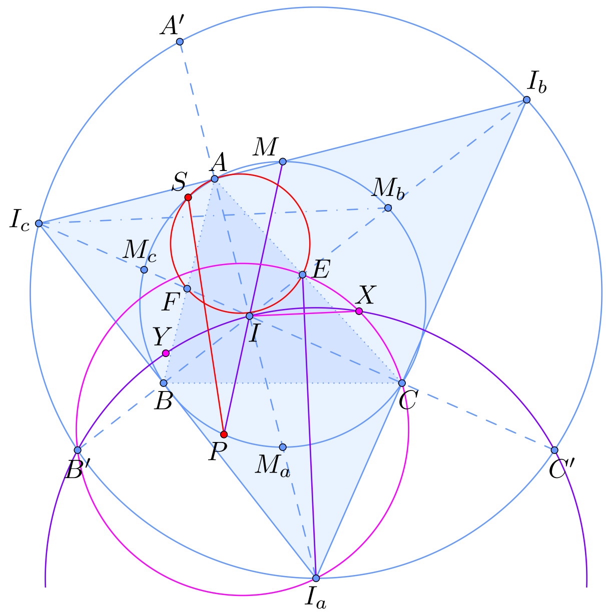
where the last equality follows because Z, C swapped by inversion at A with power $AU_1 \cdot AV_1$ composed with reflection in the angle bisector of $\angle U_1AV_1$, so we win.

> *Remark.* How on earth would someone find K? I considered the degenerate cases when (XCD) is a straight line (which occur when $X = X_1, X_2$, hence their names).

1.2 DIMO 2022/6

In triangle $\triangle ABC$, M is the midpoint of arc BAC, I is the incenter and I_a is the A-excenter. Let $E = \overline{BI} \cap \overline{AC}$, $F = \overline{CI} \cap \overline{AB}$, $P = \overline{MI} \cap (ABC)$, and $S = (AEF) \cap (ABC)$ $(\neq A)$. If X, Y are the reflections of I across $\overline{I_aE}, \overline{I_aF}$ respectively, prove that (BYF), (CXE), (PXY) and $\overline{PS}$ are concurrent.

Solved with **v4913**.



Let I_a, etc. and M_a, etc. be the excenters and minor arc midpoints, respectively, and A', etc. $\in (I_aI_bI_c)$ be the reflections of I in A, etc. Observe that $I_aI = I_aX = I_aY$.

> **Claim 1 –** (BYF), (CXE) pass through (I_a, C') and (I_a, B') respectively.

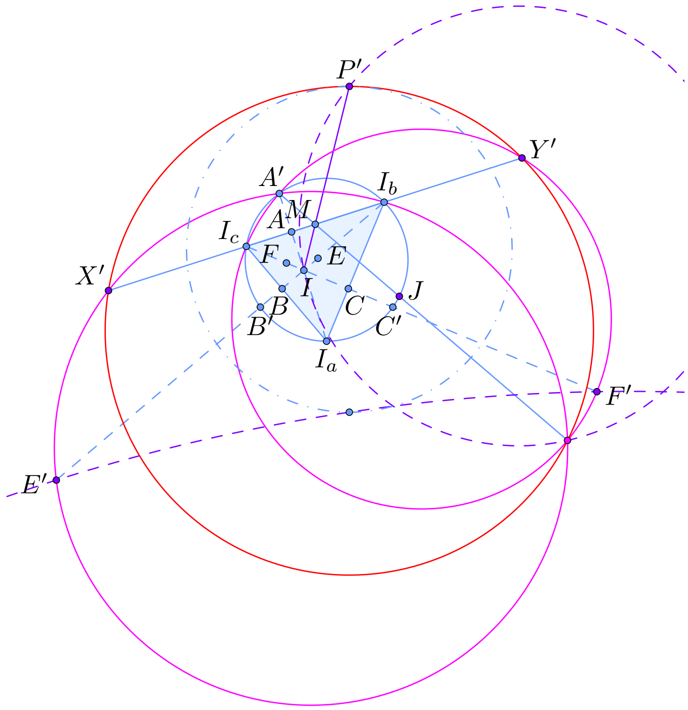
Proof. The cyclic quads involving I_a are handled as follows:

$$\measuredangle I_aYF = -\measuredangle I_aIF = \measuredangle CIA = \measuredangle I_aBF;$$

Meanwhile, since the tangent to $(I_aI_bI_c)$ at I_b is parallel to $\overline{AC}$, $B'I_aCE$ is cyclic by Reim, and similarly for its cyclic variant. This establishes the concyclicities. □

> **Claim 2 –** $I, X, 2I_c - I_b$ are collinear.

Proof. By Brocard on $AICI_b$, $\overline{I_aE} \perp \overline{M_bI_c}$, while we obviously have $\overline{IX} \perp \overline{I_aE}$. Since $\overline{IX}$ is just $\overline{M_bI_c}$ scaled at I_b by a factor of 2, the result follows. □



Next, we invert about I preserving $(I_aI_bI_c)$. Denote image/preimage by *.

- The circle at I_a through I, on which X, Y lie, gets mapped to $\overline{I_bI_c}$; hence $X^* = 2I_c - I_b$.
- $2P - I$ is an orthocenter Miquel point of triangle $I_aI_bI_c$, so $P* = 4M - 3I$.
- E, F get mapped to the images of $2B - I_b$, $2C - I_c$ when scaled at I by a factor of 2. (well-known)
- $A* = 2I_a - I$ and $I_a^* = A'$;

This is enough to finish off part of the concurrency: $(P^*X^*Y^*)$ is $(I_aI_bI_c)$ scaled at M by a factor of 3. If $J = \overline{A'M} \cap (I_aI_bI_c)$, then by power of a point at M, the first 3 circles all meet at $3J - 2M$.

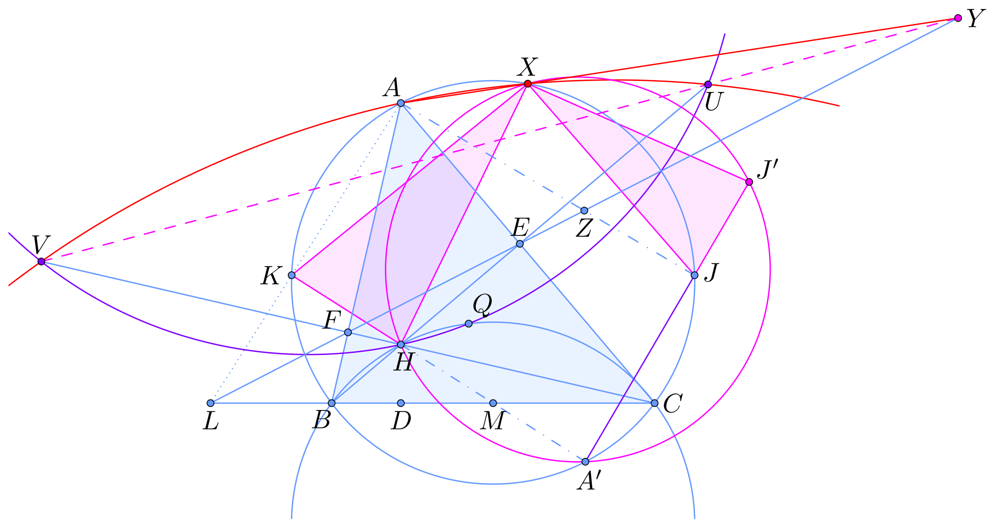
Next, to deal with the rest of the problem, we take a homothety at I with scale factor $1/2$ and refactor:

Second half refactored

In triangle ABC with orthocenter H, D, E, F are the feet of the altitudes, and M is the midpoint of $\overline{BC}$. Let A' be antipode of A on (ABC), and define $U = 2E - B$, $V = 2F - C$. Define J as the reflection of the A-orthocenter Miquel point in the perpendicular bisector of $\overline{BC}$ and

$$J' = \frac{H + 3J - 2M}{2} = \frac{3J - A'}{2}.$$

Prove that (AUV), (ABC), $(A'J'H)$ are concurrent.



U, V are just the reflections of B, C over their respective opposite sides. We add in the $X = (AUV) \cap (ABC)$, and K as the A-orthocenter Miquel point. Since (A', H, K) and (A', J, J') are collinear, $A'J'HX$ being cyclic is equivalent to showing that a spiral similarity at X takes KH to JJ', or $\frac{XK}{XJ} = \frac{KH}{JJ'}$. Let r be the reflection in the angle bisector of $\angle BAC$ composed with a dilation, sending $A' \to H$, so that $KH/JJ' = 2KH/AJ = 2AH/AA'$.

Claim 3 – AX, UV, EF are concurrent.

Proof. Let Q be the A-Humpty point. Consider the spiral similarity s at Q mapping $(B, E) \to (C, F)$ and thus $U \to V$ as well. As a result, Q is also the Miquel point of $BCVU$, and thus lies on (HUV). It suffices to prove that $\overline{EF}$ is the radical axis of (ABC), $(UVHQ)$, since the claim will then follow from radical axis theorem on (ABC), $(AXUV)$, $(UVHQ)$. Indeed, this is obvious- $AE \cdot CE = HE \cdot UE$ and similarly for F. □

At this point, we add in a few intersections: $L = \overline{AK} \cap \overline{EF} \cap \overline{BC}$, $Z = \overline{AJ} \cap \overline{EF} = r(L)$, and Y as the concurrency point from claim 3.

Claim 4 – Z is the midpoint of $\overline{LY}$.

Proof. Instead we'll show $U, V, 2Z - L$ are collinear, which is again by spiral similarity:

- Let s' be the spiral similarity at Q sending $B, C \to E, F$; then $BL/CL \overset{r}{=} EZ/FZ$, so s' also sends L to Z.
- Another one sends $B, C, L \to E, F, Z \to U, V, 2Z - L$, so the last three are indeed collinear. □

Finally,

$$\frac{1}{2} = (\infty_{EF} Y; LZ) \overset{A}{=} (AX; KJ), \text{ so that } \frac{XK}{XJ} = \frac{2AK}{AJ} \overset{r}{=} \frac{2AH}{AA'},$$

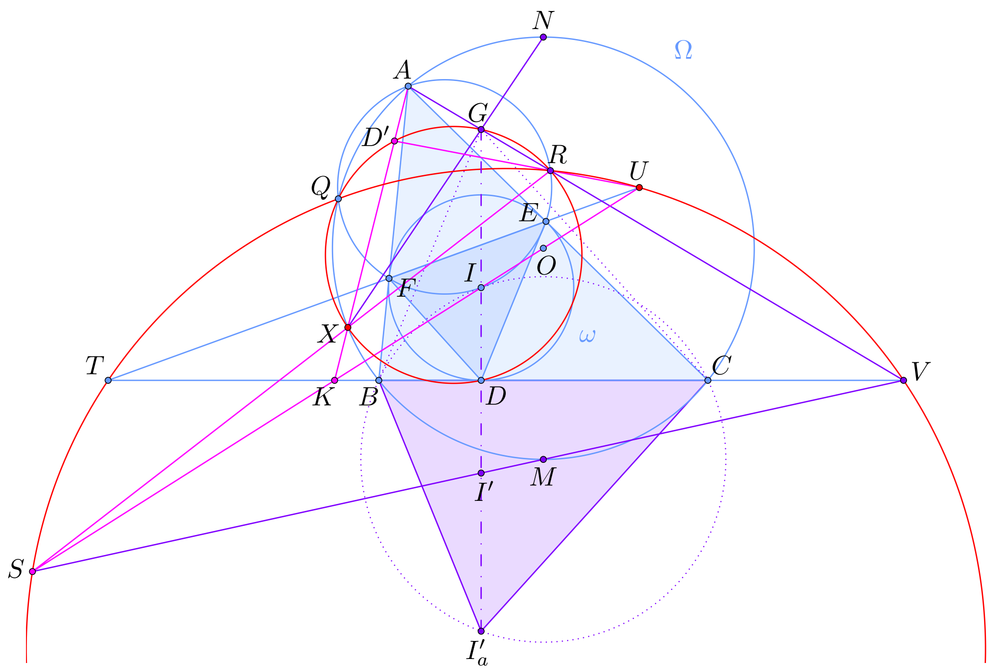
and we're done!

1.3 Brazil Revenge 2021/3, by Joao P.R. Viana Costa

Let I, C, ω and Ω be the incenter, circumcenter, incircle and circumcircle, respectively, of the scalene triangle XYZ with $XZ > YZ > XY$. The incircle ω is tangent to the sides YZ, XZ and XY at the points D, E and F. Let S be the point on Ω such that XS, CI and YZ are concurrent. Let $(XEF) \cap \Omega = R$, $(RSD) \cap (XEF) = U$, $SU \cap CI = N$, $EF \cap YZ = A$, $EF \cap CI = T$ and $XU \cap YZ = O$.

Prove that $NARUTO$ is cyclic.

Colloquially known as "Naruto".



Solved with **CyclicISLscelesTrapezoid** and help from **Eyed, v4913**. We do a massive refactoring and simplification; consider the following equivalent problem, a breakdown of the given, despite being longer:

Naruto simplified

In triangle ABC with circumcircle Ω centered at O, the incircle ω centered at I touches the sides at D, E, F. Let I', I_a' be the respective reflections of I and the orthocenter of $\triangle BIC$ in $\overline{BC}$, and M the midpoint of arc BC on Ω. Further define:

- S as the intersection of the Euler lines $\overline{OI}$ of $\triangle DEF$, $\overline{MI'}$ of $\triangle I_a'BC$;
- $T = \overline{EF} \cap \overline{BC}$, $U = \overline{EF} \cap \overline{OI}$, $V = \overline{MI'} \cap \overline{BC}$, $R = \overline{AV} \cap (AI)$;
- $K = \overline{OI} \cap \overline{BC}$;

Prove that (a) Q, R, S, T, U, V are concyclic, and (b) $\overline{AK}$, Ω, (QRD), $\overline{RS}$ concurrent;

(a) The concyclicity Let the spiral similarity s at Q with (directed) angle θ map $E, F \to C, B$ and thus D, I and the orthocenter of $\triangle DEF$ to I', M, I_a' respectively. Clearly, S is the intersection of the Euler lines of two triangles related by s: DEF, $I_a'CB$.

By design, we have $U \xrightarrow{s} V$, so

$$\measuredangle VQU = \theta = \measuredangle(\overline{BC}, \overline{EF}) \stackrel{s}{=} \measuredangle(\overline{MI'}, \overline{OI}) = \measuredangle VSU,$$

whence Q, S, T, U, V concyclic. To see that the last point is also concyclic with the other five, let N be the midpoint of $\widehat{BAC}$, so that $\overline{NA}$ touches (AI). Indeed, then

$$\measuredangle QRV = \measuredangle QRA = \measuredangle QAN \stackrel{s}{=} \measuredangle QUV$$

as needed.

Remark. In fact, by design, S is the exsimilicenter of the incircle and the circle at O with radius half that of Ω, so it's actually the inverse of I wrt Ω.

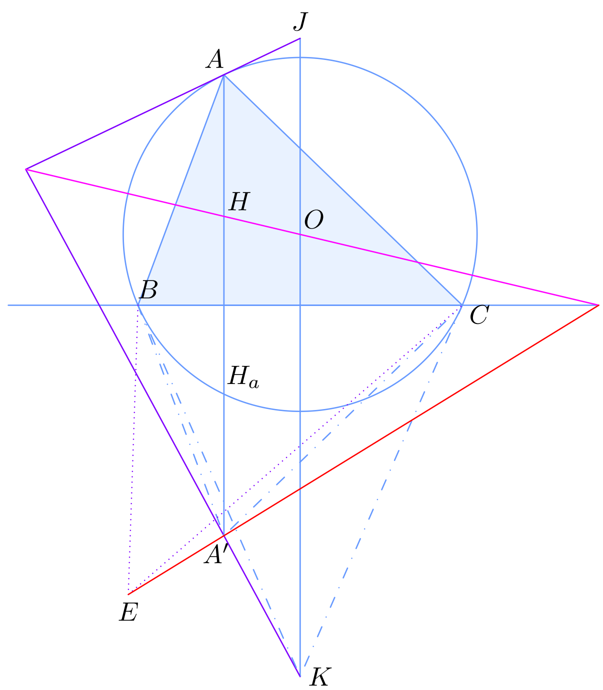
(b) The concurrence Let D' be the reflection of D in $\overline{EF}$, and G the orthocenter of $\triangle BIC$, so that $D' \xrightarrow{s} G$. We easily have $DD'GQ$ cyclic. As $\measuredangle(\overline{AD'}, \overline{NG}) = \theta$, the point $X = \overline{AD'} \cap \overline{NG}$ lies on both $(DD'GQ)$, Ω. We require the following result(s):

Theorem: weird concurrences
In a scalene triangle ABC with circumcenter O, circumcircle Ω, and orthocenter H.

(a) let K be the polar of $\overline{BC}$ wrt Ω, and A' be the reflection of A in $\overline{BC}$. Then $\overline{OH}$, $\overline{A'K}$ and the tangent to Ω at A are concurrent.

(b) Let E be the reflection of the point E_0 (such that A is the incenter or excenter of $\triangle E_0BC$) in the perpendi-cular bisector of $\overline{BC}$. Then $\overline{OH}, \overline{BC}, \overline{EA'}$ are also concurrent.

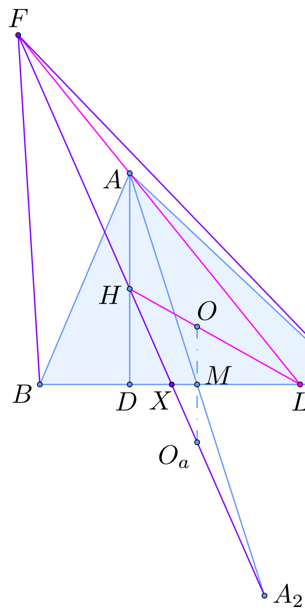
(parentheses used above for easier grammatical parsing)



Proof. These two parts actually aren't connected at all...

Part (a), by CyclicISLscelesTrapezoid Let J be the intersection of the tangent to Ω at A with the perpendicular bisector of $\overline{BC}$, and $H_a \in \Omega$ be the reflection of H in $\overline{BC}$. We contend that the triples (A, H, A'), (J, O, K) are homothetic. Indeed, they lie on parallel lines. To finish, check that (if R denotes the radius of Ω)

$$JO = \frac{R}{\cos(B-C)}, OK = \frac{R}{\cos A}, AH = 2R\cos A, HA' = AH_a = 2R\cos(B-C) \Rightarrow \frac{JO}{OK} = \frac{AH}{HA'}.$$



Part (b), by crazyeyemoody907 Let $F = B + C - E_0$, and $A_2 = B + C - A$, so that A_2 is an incenter or excenter of $\triangle FBC$. Since H is the antipode of A_2 on (BA_2C), it is another incenter / excenter. To prove that A, L, F collinear where $X = \overline{FHA_2} \cap \overline{BC}$, $L = \overline{OH} \cap \overline{BC}$, verify that (where $O_a \in \overline{H_a A_2}$ is the reflection of O in $\overline{BC}$)

$$(\overline{AF} \cap \overline{BC}, X; D, M) \stackrel{A}{=} (FX; HA_2) = -1 \text{ while } (DM; XL) \stackrel{H}{=} (\infty_{\perp BC} M; O_a O) = -1. \qquad \square$$

Returning to the problem, applying respective parts of the theorem to $\triangle DEF, I'_a BC$, we obtain (A, D', K) and (A, G, V) collinear. Since $R \in (UVQ), \overline{GV}$, and Q is the Miquel point of $D'GVU$, we must have $R = \overline{D'U} \cap \overline{GV}$ – an intersection of opposite sides. Hence, by definition of Miquel point, $R \in (QD'G)$.

It remains to prove that R, X, S collinear. In fact, there is a spiral similarity at Q mapping $D', X \longrightarrow U, S$ since $Q \in (URS), (D'XR)$, so we're done!