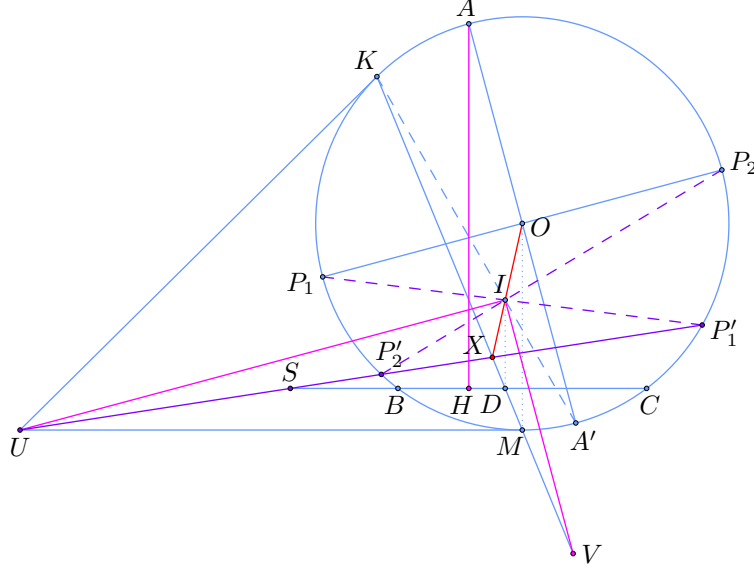


# 1 24usemo3

Let  $ABC$  be a triangle with incenter  $I$ . Two distinct points  $P$  and  $Q$  are chosen on the circumcircle of  $ABC$  such that

$$\angle API = \angle AQI = 45^\circ.$$

Lines  $PQ$  and  $BC$  meet at  $S$ . Let  $H$  denote the foot of the altitude from  $A$  to  $BC$ . Prove that  $\angle AHI = \angle ISH$ .



Define a multitude of points (and a circle):

- $\Omega = (ABC)$ ,  $O$  as its center,  $A'$  as the reflection of  $A$  in  $O$ ;
- Choose points  $P_1, P_2 \in \Omega$  such that  $AP_1A'P_2$  is square. Then  $\{\overline{P_1I} \cap \Omega, \overline{P_2I} \cap \Omega\} = \{P, Q\}$ . For ease of reference refactor  $P, Q$  as  $P'_k = \overline{P_kI} \cap \Omega$  in some order.
- $K = \overline{A'I} \cap \Omega$  as the so-called 'Sharky-devil point',  $M$  as the midpoint of arc  $BC$  exc.  $A$ , and  $D$  as the foot from  $I$  onto  $\overline{BC}$ ;
- $(KM; P'_1P'_2) \stackrel{I}{=} (AA'; P_1P_2) = -1$ , the tangents to  $\Omega$  at  $K, M$  meet  $\overline{P'_1P'_2S}$  at some common point  $U$ .
- $V \in \overline{KM}$  as the pole of  $\overline{P_1P_2S}$ .

**Claim 1** -  $\overline{UI} \parallel \overline{P_1P_2}, \overline{VI} \parallel \overline{AA'}$ .

*Proof.* Observe that  $\overline{UI}$  is the polar of  $\overline{AA'} \cap \overline{KM}$  which is evidently perpendicular to  $\overline{AA'}$ . Similarly  $\overline{VI} \perp \overline{P_1P_2}$ .  $\square$

**Claim 2** -  $\overline{P'_1P'_2}, \overline{KM}, \overline{OI}$  concurrent.

*Proof.* To see that  $\overline{OI} \cap \overline{KM}$  lies on the polar of  $V$  in  $\Omega$ , check that

$$-1 = (AA'; O\infty_{AA'}) \stackrel{I}{=} (K, M; \overline{OI} \cap \overline{KM}, V).$$

$\square$

Call this last concurrency point  $X$ , which is well known to be the exsimilicenter of the circumcircle and incircle.

Observe that because (right) triangles  $SDI$  and  $UMO$  have two pairs of parallel sides and are perspective at  $X$ , they're homothetic.

The remainder of the problem is a 'config':

**Lemma** –  $I$  is the Miquel point of  $AKDH$ , so  $\triangle IDH \sim \triangle IKA$ .

*Proof.* Consider (not shown)  $R = \overline{AK} \cap \overline{BC}$ .

- $\angle RKI = \angle AKI = 90^\circ \Rightarrow R \in (KID)$ ;
- Radical axis on  $(AI)$ ,  $(BIC)$ ,  $\Omega$  implies that  $R$  lies on the radical axis of the first two, which are evidently tangent.  
As a result  $\overline{RI}$  is the common tangent of said circles and  $\overline{AM} \perp \overline{RI}$ , and  $\angle AIR = \angle AHR = 90^\circ$  means  $R \in (AIH)$  as well.

□

Putting everything together gives this similar triangle chain, which implies the problem:

$$IDH \sim IKA \sim UMO \sim SDI.$$