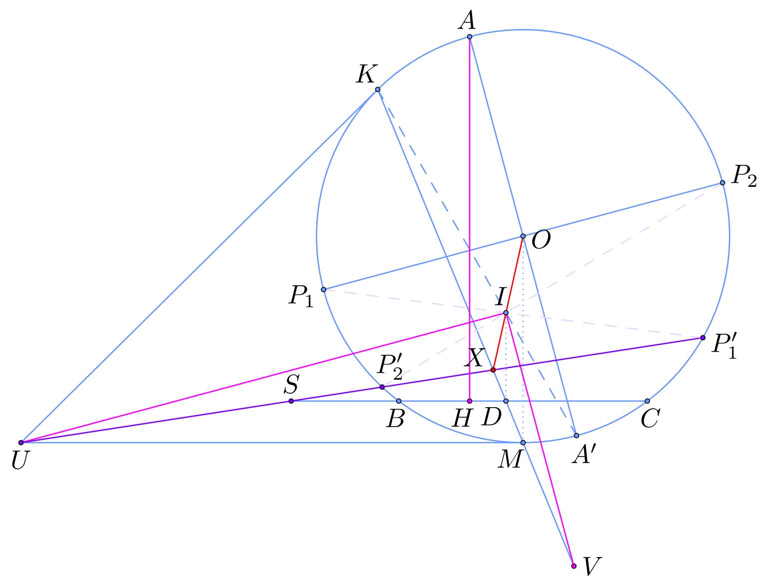
1 24usemo3

Let ABC be a triangle with incenter I. Two distinct points P and Q are chosen on the circumcircle of ABC such that

$$\angle API = \angle AQI = 45^\circ.$$

Lines PQ and BC meet at S. Let H denote the foot of the altitude from A to BC. Prove that $\angle AHI = \angle ISH$.



Define a multitude of points (and a circle):

- $\Omega = (ABC)$, O as its center, A' as the reflection of A in O;
- Choose points $P_1, P_2 \in \Omega$ such that $AP_1A'P_2$ is square. Then $\{\overline{P_1I} \cap \Omega, \overline{P_2I} \cap \Omega\} = \{P, Q\}$. For ease of reference refactor P, Q as $P_k' = \overline{P_kI} \cap \Omega$ in some order.
- $K = \overline{A'I} \cap \Omega$ as the so-called 'Sharky-devil point', M as the midpoint of arc BC exc. A, and D as the foot from I onto $\overline{BC}$;
- $(KM; P_1'P_2') \stackrel{I}{=} (AA'; P_1P_2) = -1$, the tangents to Ω at K, M meet $\overline{P_1'P_2'S}$ at some common point U.
- $V \in \overline{KM}$ as the pole of $\overline{P_1P_2S}$.

Claim 1 – $\overline{UI} \parallel \overline{P_1P_2}$, $\overline{VI} \parallel \overline{AA'}$.

Proof. Observe that $\overline{UI}$ is the polar of $\overline{AA'} \cap \overline{KM}$ which is evidently perpendicular to $\overline{AA'}$. Similarly $\overline{VI} \perp \overline{P_1P_2}$. □

Claim 2 – $\overline{P_1'P_2'}$, $\overline{KM}$, $\overline{OI}$ concurrent.

Proof. To see that $\overline{OI} \cap \overline{KM}$ lies on the polar of V in Ω, check that

$$-1 = (AA'; O\infty_{AA'}) \stackrel{I}{=} (K, M; \overline{OI} \cap \overline{KM}, V).$$ □

Call this last concurrency point X, which is well known to be the exsimilicenter of the circumcircle and incircle.

Observe that because (right) triangles SDI and UMO have two pairs of parallel sides and are perspective at X, they're homothetic. The remainder of the problem is a 'config':

Lemma – I is the Miquel point of $AKDH$, so $\triangle IDH \sim \triangle IKA$.

Proof. Consider (not shown) $R = \overline{AK} \cap \overline{BC}$.

- $\angle RKI = \angle AKI = 90^\circ \Rightarrow R \in (KID)$;
- Radical axis on (AI), (BIC), Ω implies that R lies on the radical axis of the first two, which are evidently tangent. As a result $\overline{RI}$ is the common tangent of said circles and $\overline{AM} \perp \overline{RI}$, and $\angle AIR = \angle AHR = 90^\circ$ means $R \in (AIH)$ as well.

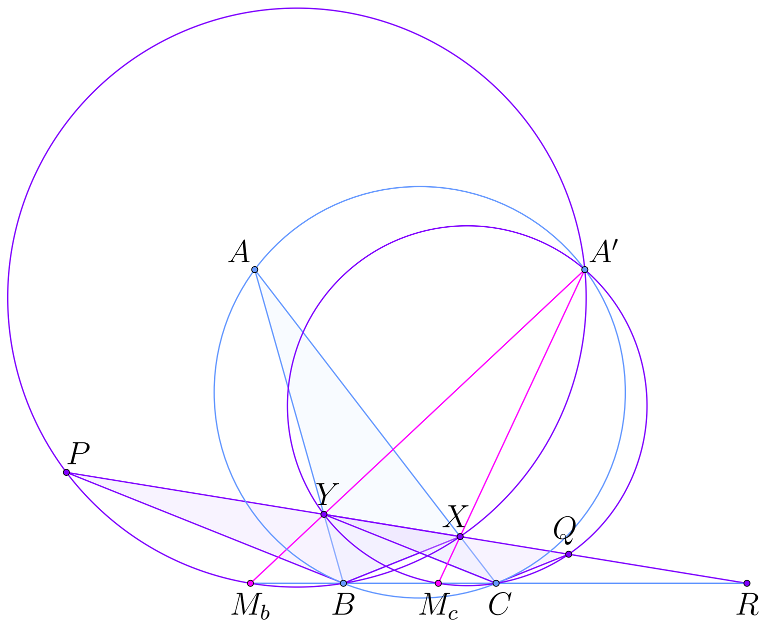
□

Putting everything together gives this similar triangle chain, which implies the problem:

$$IDH \sim IKA \sim UMO \sim SDI.$$

2 24rmm5

What a problem.



Define:

- $M_b = (BXP) \cap \overline{BC}$ and M_c similarly. These are the centers of the desired circles because M_b (by design) is the midpoint of one of the arcs PX, and similarly for M_c.
- γ_b be the circle at M_b through X and similarly for γ_c.
- $M = (B + C)/2$. The wording of the problem statement implies that M should have a fixed power wrt. each of these γ's.
- $R = \overline{XYPQ} \cap \overline{BC}$.

Indeed, I assert this fixed power is $\boxed{\frac{3}{4}a^2}$. (Here, we use $a = BC$, etc. for ease of computation.)

Claim 1 – A' is the Miquel point of $BCXY$.

Proof. Check that $A'B/BY = a^2/bc = A'C/CX$, triangle similarity follows using equal angles. □

Claim 2 – Define A' as the reflection of A in the perpendicular bisector of $\overline{BC}$. Then $A' \in \overline{M_bY}, \overline{M_cX}$.

Proof. We assert that $CXYM_b$ is cyclic, so that the claim will follow by Reim converse:

$$\measuredangle RCQ = \measuredangle M_bBX = \measuredangle M_bPX.$$ □

It immediately follows (by angles, say) that $A' \in (BXP), (CXQ)$. As a result, using power of a point and the so-called 'shooting lemma' we may obtain

$$M_bX^2 = M_bY \cdot M_bA' = M_bM_c \cdot M_bC.$$

The rest of the problem is computation, but we make use of directed lengths as well as $BM_bY \overset{+}{\sim} AA'Y$ and the similar $CM_cX \overset{+}{\sim} AA'X$:

$$\begin{aligned}
& M_bM^2 - M_bM_c \cdot M_bC = -3a^2/4 \\
\iff & \frac{1}{2}\left(M_bB^2 + M_bC^2\right) - MB^2 - M_bM_c \cdot M_bC = -3a^2/4 \\
\iff & M_bB^2 + (M_bB + BC)^2 - (M_bB + BC + CM_c)(M_bB + BC) + BC^2 = 0 \\
\iff & \frac{AA'}{M_bB} + \frac{AA'}{BC} + \frac{AA'}{CM_c} = 0 \\
\iff & \frac{AA'}{BC} = \frac{AY}{YB} - \frac{AX}{XC} = \frac{AB}{YB} - \frac{AC}{XC} = \frac{b^2 - c^2}{a}
\end{aligned}$$

which is evident.