

# Geometry Favorites

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(Note: here  $\infty_{XY}$ ,  $\infty_{\perp XY}$  refer to the points  $\infty$  along in directions parallel and perpendicular to  $XY$ , respectively.)

## -1 Credits + remarks

Inspired by chapter 11 of EGMO book, and **Geometry At Its Best** by Eric Shen. These are roughly in order of difficulty, but no promises!  
Also thanks to collaborators...

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## 🌲 0 Problems

*Remark.* Some attempt has been made to deviate from the aforementioned two famous geometry papers.

**Problem 1 (SL 2009/G3).** Let  $ABC$  be a triangle. The incircle of  $\triangle ABC$  touches  $AB$  and  $AC$  at the points  $Z$  and  $Y$ , respectively. Let  $G = \overline{BY} \cap \overline{CZ}$ , and let  $R$  and  $S$  be points such that the two quadrilaterals  $BCYR$  and  $BCSZ$  are parallelograms. Prove that  $GR = GS$ .

**Problem 2 (SL 2015/G4).** Let  $ABC$  be an acute triangle and let  $M$  be the midpoint of  $AC$ . A circle  $\omega$  passing through  $B$  and  $M$  meets the sides  $AB$  and  $BC$  at points  $P$  and  $Q$  respectively. Let  $T$  be the point such that  $BPTQ$  is a parallelogram. Suppose that  $T$  lies on the circumcircle of  $ABC$ . Determine all possible values of  $\frac{BT}{BM}$ .

**Problem 3 (USEMO 2023/4).** Let  $ABC$  be an acute triangle with orthocenter  $H$ . Points  $A_1, B_1, C_1$  are chosen in the interiors of sides  $BC, CA, AB$ , respectively, such that  $\triangle A_1B_1C_1$  has orthocenter  $H$ . Define  $A_2 = \overline{AH} \cap \overline{B_1C_1}$ ,  $B_2 = \overline{BH} \cap \overline{C_1A_1}$ , and  $C_2 = \overline{CH} \cap \overline{A_1B_1}$ .

Prove that triangle  $A_2B_2C_2$  has orthocenter  $H$ .

**Problem 4 (SL 2016/G7).** Let  $I$  be the incentre of a non-equilateral triangle  $ABC$ ,  $I_A$  be the  $A$ -excentre,  $I'_A$  be the reflection of  $I_A$  in  $BC$ , and  $l_A$  be the reflection of line  $AI'_A$  in  $AI$ . Define points  $I_B, I'_B$  and line  $l_B$  analogously. Let  $P$  be the intersection point of  $l_A$  and  $l_B$ .

- (a) Prove that  $P$  lies on line  $OI$  where  $O$  is the circumcentre of triangle  $ABC$ .
- (b) Let one of the tangents from  $P$  to the incircle of triangle  $ABC$  meet the circumcircle at points  $X$  and  $Y$ . Show that  $\angle XIY = 120^\circ$ .

**Problem 5 (EGMO 2020/3).** Let  $ABCDEF$  be a convex hexagon such that  $\angle A = \angle C = \angle E$ ,  $\angle B = \angle D = \angle F$  and the (interior) angle bisectors of  $\angle A, \angle C, \angle E$  are concurrent. Prove that the (interior) angle bisectors of  $\angle B, \angle D, \angle F$  are also concurrent.

**Problem 6 (IMO 2008/6).** Let  $ABCD$  be a convex quadrilateral with  $BA \neq BC$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents to  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .

**Problem 7 (Iran TST 2018/1/4).** Let  $ABC$  be a triangle ( $\angle A \neq 90^\circ$ ), with altitudes  $\overline{BE}, \overline{CF}$ . The bisector of  $\angle A$  intersects  $\overline{EF}, \overline{BC}$  at  $M, N$ . Let  $P$  be a point such that  $\overline{MP} \perp \overline{EF}$  and  $\overline{NP} \perp \overline{BC}$ . Prove that  $\overline{AP}$  bisects  $\overline{BC}$ .

**Problem 8 (Eric Shen).** In  $\triangle ABC$ , let  $D, E, F$  be the feet of the altitudes from  $A, B, C$  respectively, and let  $O$  be the circumcenter. Let  $Z = \overline{AO} \cap \overline{EF}$ . There exists a point  $T$  such that  $\angle DTZ = 90^\circ$  and  $AZ = AT$ . If  $P = \overline{AD} \cap \overline{TZ}$ , and  $Q$  lies on  $\overline{EF}$  such that  $\overline{PQ} \parallel \overline{BC}$ , prove that  $\overline{AQ}$  bisects  $\overline{BC}$ .

**Problem 9 (SL 2018/G5).** Let  $ABC$  be a triangle with circumcircle  $\omega$  and incenter  $I$ . A line  $\ell$  meets the lines  $AI, BI, CI$  at points  $D, E, F$  respectively, all distinct from  $A, B, C, I$ . Prove that the circumcircle of the triangle determined by the perpendicular bisectors of  $\overline{AD}, \overline{BE}, \overline{CF}$  is tangent to  $\omega$ .

**Problem 10 (SL 2009/G6).** Let the sides  $AD$  and  $BC$  of the quadrilateral  $ABCD$  (such that  $AB$  is not parallel to  $CD$ ) intersect at point  $P$ . Points  $O_1$  and  $O_2$  are circumcenters and points  $H_1$  and  $H_2$  are orthocenters of triangles  $ABP$  and  $CDP$ , respectively. Denote the midpoints of segments  $O_1H_1$  and  $O_2H_2$  by  $E_1$  and  $E_2$ , respectively. Prove that the perpendicular from  $E_1$  on  $CD$ , the perpendicular from  $E_2$  on  $AB$  and the lines  $H_1H_2$  are concurrent.

**Problem 11 (MOP 2019 & USA TST 2019/6).** Let  $ABC$  be a triangle with incenter  $I$ , and let  $D$  be a point on line  $BC$  satisfying  $\angle AID = 90^\circ$ . Let the excircle of triangle  $ABC$  opposite the vertex  $A$  be tangent to  $\overline{BC}$  at  $A_1$ . Define points  $B_1$  on  $\overline{CA}$  and  $C_1$  on  $\overline{AB}$  analogously, using the excircles opposite  $B$  and  $C$ , respectively.

- (a) (MOP 2019) Let  $E, F$  be the feet of the altitudes from  $B, C$  respectively. Prove that if  $\overline{EF}$  touches the incircle, then quadrilateral  $AB_1A_1C_1$  is cyclic.

(b) (USA TST 2019/6) Prove that if quadrilateral  $AB_1A_1C_1$  is cyclic, then  $\overline{AD}$  is tangent to  $(DB_1C_1)$ .

**Problem 12 (ELMO SL 2024/G4).** In quadrilateral  $ABCD$  with incenter  $I$ , points  $W, X, Y, Z$  lie on sides  $AB, BC, CD, DA$  with  $AZ = AW, BW = BX, CX = CY, DY = DZ$ . Define  $T = \overline{AC} \cap \overline{BD}$  and  $L = \overline{WY} \cap \overline{XZ}$ . Let points  $O_a, O_b, O_c, O_d$  be such that  $\angle O_aZA = \angle O_aWA = 90^\circ$  (and cyclic variants), and  $G = \overline{O_aO_c} \cap \overline{O_bO_d}$ . Prove that  $\overline{IL} \parallel \overline{TG}$ .

**Problem 13 (APMO 2014/5).** Circles  $\omega$  and  $\Omega$  meet at points  $A$  and  $B$ . Let  $M$  be the midpoint of the arc  $AB$  of circle  $\omega$  ( $M$  lies inside  $\Omega$ ). A chord  $MP$  of circle  $\omega$  intersects  $\Omega$  at  $Q$  ( $Q$  lies inside  $\omega$ ). Let  $\ell_P$  be the tangent line to  $\omega$  at  $P$ , and let  $\ell_Q$  be the tangent line to  $\Omega$  at  $Q$ . Prove that the circumcircle of the triangle formed by the lines  $\ell_P, \ell_Q$  and  $AB$  is tangent to  $\Omega$ .

**Problem 14 (DeuX MO 2020/II/3).** In triangle  $ABC$  with circumcenter  $O$  and orthocenter  $H$ , line  $OH$  meets  $\overline{AB}, \overline{AC}$  at  $E, F$  respectively. Let  $\omega$  be the circumcircle of triangle  $AEF$  with center  $S$ , meeting  $(ABC)$  again at  $J \neq A$ . Line  $OH$  also meets  $(JSO)$  again at  $D \neq O$ . Define  $K = (JSO) \cap (ABC) (\neq J), M = \overline{JK} \cap \overline{OH}$ , and  $G = \overline{DK} \cap (ABC) (\neq K)$ . Prove that  $(GHM)$  and  $(ABC)$  are tangent to each other.

**Problem 15 (IMO 2021/3).** Let  $D$  be an interior point of the acute triangle  $ABC$  with  $AB > AC$  so that  $\angle DAB = \angle CAD$ . The point  $E$  on the segment  $AC$  satisfies  $\angle ADE = \angle BCD$ , the point  $F$  on the segment  $AB$  satisfies  $\angle FDA = \angle DBC$ , and the point  $X$  on the line  $AC$  satisfies  $CX = BX$ . Let  $O_1$  and  $O_2$  be the circumcenters of the triangles  $ADC$  and  $EXD$ , respectively. Prove that the lines  $BC, EF$ , and  $O_1O_2$  are concurrent.

**Problem 16 (USAMO 2021/6).** Let  $ABCDEF$  be a convex hexagon satisfying  $\overline{AB} \parallel \overline{DE}, \overline{BC} \parallel \overline{EF}, \overline{CD} \parallel \overline{FA}$ , and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let  $X, Y$ , and  $Z$  be the midpoints of  $\overline{AD}, \overline{BE}$ , and  $\overline{CF}$ . Prove that the circumcenter of  $\triangle ACE$ , the circumcenter of  $\triangle BDF$ , and the orthocenter of  $\triangle XYZ$  are collinear.

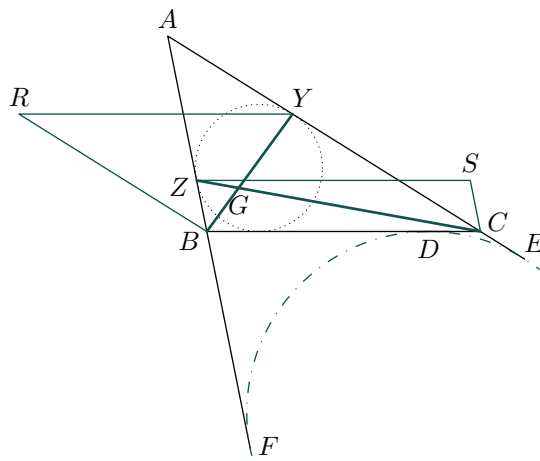
**Problem 17 (SL 2021/G8).** Let  $ABC$  be a triangle with circumcircle  $\omega$  and let  $\Omega_A$  be the  $A$ -excircle. Let  $X$  and  $Y$  be the intersection points of  $\omega$  and  $\Omega_A$ . Let  $P$  and  $Q$  be the projections of  $A$  onto the tangent lines to  $\Omega_A$  at  $X$  and  $Y$  respectively. The tangent line at  $P$  to the circumcircle of the triangle  $APX$  intersects the tangent line at  $Q$  to the circumcircle of the triangle  $AQY$  at a point  $R$ . Prove that  $\overline{AR} \perp \overline{BC}$ .

**Problem 18 (USEMO 2020/3).** Let  $ABC$  be an acute triangle with circumcenter  $O$  and orthocenter  $H$ . Let  $\Gamma$  denote the circumcircle of triangle  $ABC$ , and  $N$  the midpoint of  $OH$ . The tangents to  $\Gamma$  at  $B$  and  $C$ , and the line through  $H$  perpendicular to line  $AN$ , determine a triangle whose circumcircle we denote by  $\omega_A$ . Define  $\omega_B$  and  $\omega_C$  similarly. Prove that the common chords of  $\omega_A, \omega_B$  and  $\omega_C$  are concurrent on line  $OH$ .

# 1 Solutions

## 1.1 SL 2009/G3, by Hossein Karke Abadi

Let  $\triangle ABC$  be a triangle. The incircle of  $\triangle ABC$  touches  $AB$  and  $AC$  at the points  $Z$  and  $Y$ , respectively. Let  $G = \overline{BY} \cap \overline{CZ}$ , and let  $R$  and  $S$  be points such that the two quadrilaterals  $BCYR$  and  $BCSZ$  are parallelograms. Prove that  $GR = GS$ .



This is a very “troll” problem. Let  $(R)$ ,  $(S)$ ,  $\omega_a$  denote the point circles at  $R$ ,  $S$  (radius = 0) and the  $A$ -excircle respectively. Let  $\omega_a$  touch  $\overline{BC}$ ,  $\overline{CA}$ ,  $\overline{AB}$  at  $D$ ,  $E$ ,  $F$  respectively. Also, for brevity, let  $a = BC$ ,  $b = CA$ ,  $c = AB$ ,  $s = (a + b + c)/2$ .

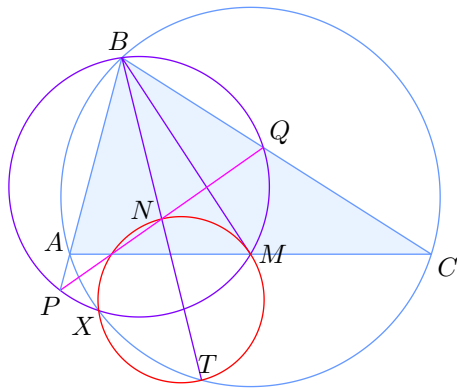
**Claim** –  $\overline{BY}$  is the radical axis of  $(R)$ ,  $\omega_a$ .

*Proof.*  $BD = BR = s - c$ , while  $YE = YR = a$ ; because  $\overline{BD}$ ,  $\overline{YE}$  touch  $\omega_a$ ,  $B$ ,  $Y$  have powers  $(s - c)^2$ ,  $a^2$  wrt each of  $(R)$ ,  $\omega_a$  as promised.  $\square$

By the claim,  $G = \overline{BY} \cap \overline{CZ}$  must be the radical center of  $(R)$ ,  $(S)$ ,  $\omega_a$ , implying the desired  $GR = GS$ .

## 1.2 SL 2015/G4

Let  $ABC$  be an acute triangle and let  $M$  be the midpoint of  $AC$ . A circle  $\omega$  passing through  $B$  and  $M$  meets the sides  $AB$  and  $BC$  at points  $P$  and  $Q$  respectively. Let  $T$  be the point such that  $BPTQ$  is a parallelogram. Suppose that  $T$  lies on the circumcircle of  $ABC$ . Determine all possible values of  $\frac{BT}{BM}$ .



The answer is  $\sqrt{2}$  only. Let  $X = (ABC) \cap (BPMQ)$  ( $\neq B$ ), and let  $N$  be the midpoint of  $\overline{BT}$ .

**Claim 1** –  $XNMT$  is cyclic, and  $\overline{BM}$  is tangent to this circle.

*Proof.* Since  $N$  is also the midpoint of  $\overline{PQ}$ , there is a spiral similarity at  $X$  sending  $PNQ$  to  $AMC$ . Thus, we have

$$\angle XMN = \angle XAP = \angle XTB,$$

proving the concyclicity. For the tangency, check that

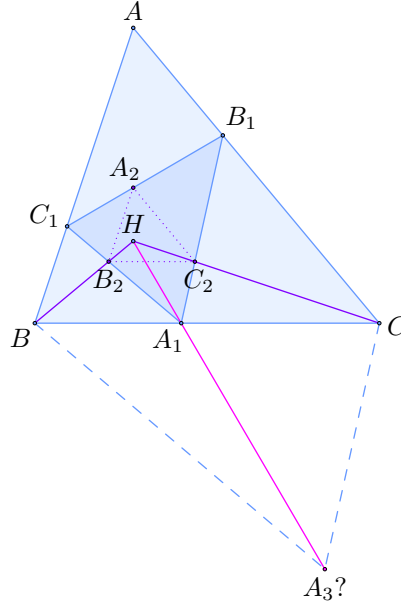
$$\angle XNM = \angle XPA = \angle XPB = \angle XMB. \quad \square$$

By power of a point,  $BM^2 = BN \cdot BT = \frac{BT^2}{2}$ , so  $\frac{BT}{BM} = \sqrt{2}$ .

### 1.3 USEMO 2023/4, by Ankan Bhattacharya

Let  $ABC$  be an acute triangle with orthocenter  $H$ . Points  $A_1, B_1, C_1$  are chosen in the interiors of sides  $BC, CA, AB$ , respectively, such that  $\triangle A_1B_1C_1$  has orthocenter  $H$ . Define  $A_2 = \overline{AH} \cap \overline{B_1C_1}$ ,  $B_2 = \overline{BH} \cap \overline{C_1A_1}$ , and  $C_2 = \overline{CH} \cap \overline{A_1B_1}$ .

Prove that triangle  $A_2B_2C_2$  has orthocenter  $H$ .



Working backwards: suffices to prove  $\overline{AA_2H} \perp \overline{B_2C_2}$ , then proceed as follows:

$$\begin{aligned} \overline{AA_2H} \perp \overline{B_2C_2} &\iff \overline{BC} \parallel \overline{B_2C_2} \iff HB_2/HB = HC_2/HC \\ &\iff \exists A_3 \in \overline{HA_1} \text{ with } \overline{BA_3} \parallel \overline{A_1C_1} \text{ and } \overline{CA_3} \parallel \overline{A_1B_1}. \end{aligned}$$

Indeed, this point would be chosen so that

$$\frac{HA_3}{HA} = \frac{HB_2}{HB} = \frac{HC_2}{HC},$$

lengths directed. This in turn equates to ' $\overline{HA_1}, \overline{B\infty_{A_1C_1}}, \overline{C\infty_{A_1B_1}}$  concurrent'.

Finish with cross-ratio chase:

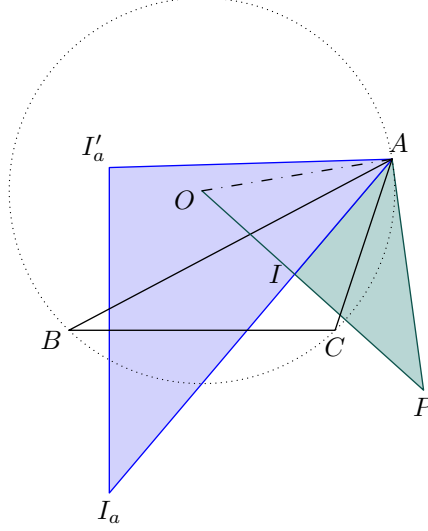
$$\begin{aligned} (\infty_{A_1C_1} \infty_{A_1B_1}; \infty_{\perp B_1C_1} \infty_{BC}) &\stackrel{\text{rotate } 90^\circ}{=} (\infty_{HB_1} \infty_{HC_1}; \infty_{B_1C_1} \infty_{HA}) \\ &\stackrel{H}{=} (B_1C_1; \infty_{B_1C_1} A_2) \\ &\stackrel{A}{=} (\overline{AC}, \overline{AB}; \overline{B_1C_1}, \overline{AH}) \\ &\stackrel{\text{rotate } 90^\circ}{=} (\overline{HB}, \overline{HC}; \overline{HA_1}, \overline{BC}) \\ &\stackrel{H}{=} (BC; A_1 \infty_{BC}) \end{aligned}$$

and the concurrence follows by prism lemma.

# 1.4 SL 2016/G7

Let  $I$  be the incentre of a non-equilateral triangle  $ABC$ ,  $I_A$  be the  $A$ -excentre,  $I'_A$  be the reflection of  $I_A$  in  $BC$ , and  $l_A$  be the reflection of line  $AI'_A$  in  $AI$ . Define points  $I_B, I'_B$  and line  $l_B$  analogously. Let  $P$  be the intersection point of  $l_A$  and  $l_B$ .

- Prove that  $P$  lies on line  $OI$  where  $O$  is the circumcentre of triangle  $ABC$ .
- Let one of the tangents from  $P$  to the incircle of triangle  $ABC$  meet the circumcircle at points  $X$  and  $Y$ . Show that  $\angle XIY = 120^\circ$ .



Redefine  $P$  as the inverse of  $I$  wrt  $(ABC)$ . For the first part we assert more strongly that:

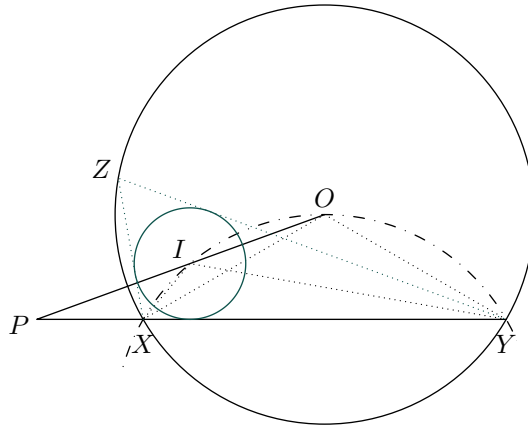
**Claim** –  $\triangle AI_A I'_A \sim \triangle API$ .

*Proof.* By angle chasing,  $\angle I_A = \angle P$  follows easily. We contend that  $I_A I'_A / I_A A = IP / AP$ ; indeed, the first ratio equals  $2 \cos \angle B I_A C = 2 \sin \frac{A}{2}$  because of similar triangles  $I_A BC \sim \triangle I_A I_B I_C$ , while

$$\frac{IP}{AP} = \frac{OP}{AP} - \frac{OI}{OA} \frac{OA}{AP} = \frac{OA}{AI} - \frac{OI^2}{OA \cdot AI} = \frac{R}{AI} - \frac{R^2 - 2rR}{R \cdot AI} = \frac{R - (R - 2r)}{AI} = 2 \sin \frac{A}{2},$$

so the ratios are equal. The similarity follows by SAS. □

The claim clearly implies the isogonality.

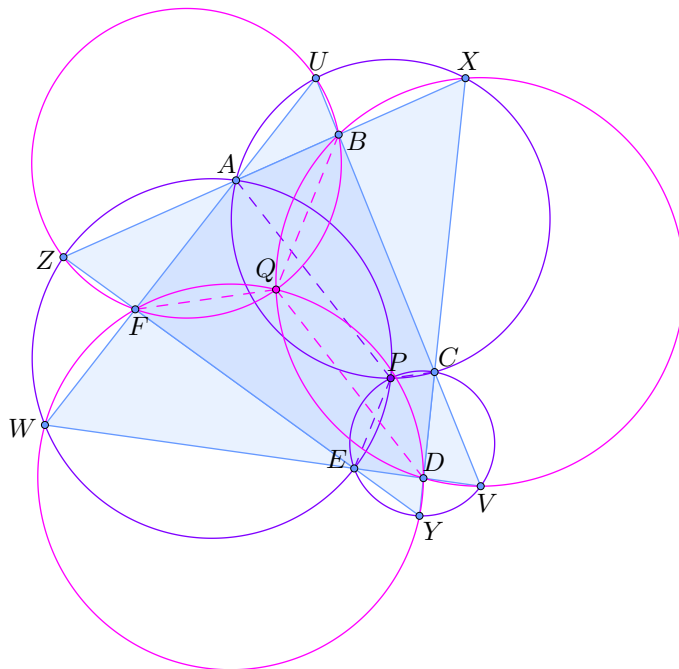


For the second part, using Poncelet, let  $Z \in (ABC)$  be the unique point so that  $\triangle XYZ, ABC$  share a incircle and circumcircle. Inverting “ $P, X, Y$  collinear” wrt the circumcircle gives  $O, I, X, Y$  concyclic, or  $\angle XOY = \angle XIY$ . As it’s well-known that  $\angle XOY = 2\angle Z$  and  $\angle XIY = (\pi + \angle Z)/2$ , we must have  $\angle Z = \pi/3 \Rightarrow \angle XIY = 2\pi/3$  as needed.



### 1.5 EGMO 2020/3

Let  $ABCDEF$  be a convex hexagon such that  $\angle A = \angle C = \angle E$ ,  $\angle B = \angle D = \angle F$  and the (interior) angle bisectors of  $\angle A$ ,  $\angle C$ ,  $\angle E$  are concurrent. Prove that the (interior) angle bisectors of  $\angle B$ ,  $\angle D$ ,  $\angle F$  are also concurrent.



Since  $\angle A + \angle B = 240^\circ$  and cyclic variants,  $\overline{AB}$ ,  $\overline{CD}$ ,  $\overline{EF}$  form an equilateral triangle, as do  $\overline{BC}$ ,  $\overline{DE}$ ,  $\overline{FA}$ . Label them  $UVW$ ,  $XYZ$  as shown, and let the given concurrency point be  $P$ . By an angle chase,  $P \in (ACXU)$ ,  $(CEYV)$ ,  $(EAWZ)$ , so it's the center of the spiral similarity  $s_1$  mapping  $U, V, W \rightarrow X, Y, Z$ .

**Claim** –  $\triangle UVW \cong \triangle XYZ$ .

*Proof.* Recall that  $s_1$  maps  $\overline{UV} \rightarrow \overline{XY}$ , but the fact that  $P$  lies on the bisector of  $\angle C$  means that  $P$  is equidistant from these lines.

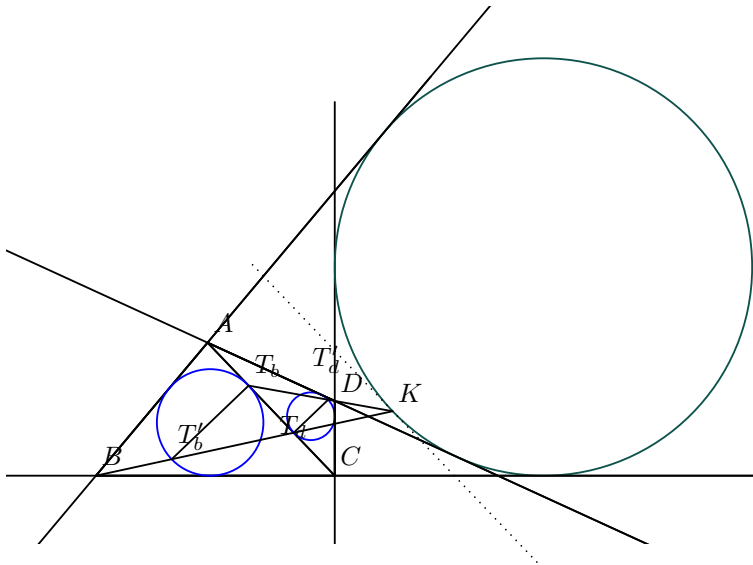
As this means that the spiral similarity above is in fact a rotation, we win.  $\square$

To finish the problem, note that the center  $Q = (BDVX) \cap (DFWY) \cap (FBUZ)$  of the rotation  $s_2$  mapping  $U, V, W \rightarrow Z, X, Y$  is equidistant from the pairs of sides  $(\overline{UV}, \overline{XZ})$  and cyclic variants, so it lies on the bisectors of the angles  $\angle B$ ,  $\angle D$ ,  $\angle F$  formed by those pairs of lines.

*Remark.* I wish I'd seen this problem before failing **USEMO 2020/5** in-contest...

### 1.6 IMO 2008/6, by Vladimir Shmarov

Let  $ABCD$  be a convex quadrilateral with  $BA \neq BC$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents to  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .



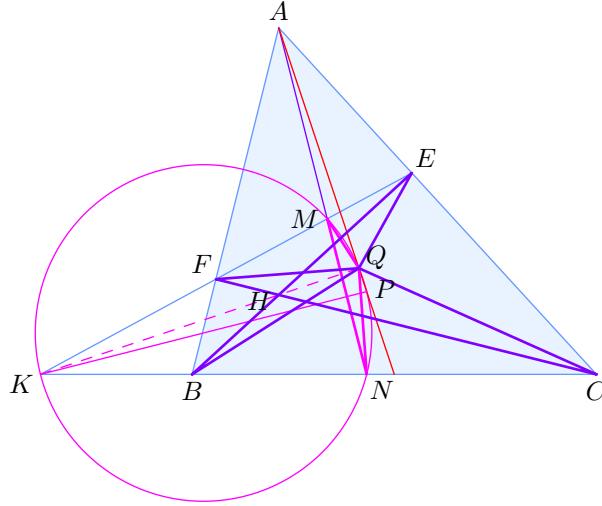
Rename  $\omega_1, \omega_2$  to  $\omega_b, \omega_d$ ; by Pitot-like reasoning we have  $AB + AD = CB + CD$ ; let  $T_b, T_d$  be the intouch points on  $\overline{AC}$ ; then  $T_b, T_d$  are isotomic by the obtained length condition.

If we let  $T'_b, T'_d$  be the antipodes of  $T_b, T_d$  on their respective circles, then an EGMO lemma (ch4) implies that  $B, T_d, T'_b$  and sym variant are collinear.

Construct the point  $K'$  on the "closer" side to the rest of the figure so that the tangent to  $\omega$  at  $K$  is parallel to  $\overline{AC}$ . Then by homothety  $K' \in \overline{BT_d}, \overline{DT_b}$ , so this is the desired exsimilicenter.

### 1.7 Iran TST 2018/1/4, by Iman Maghsoudi & Hooman Fattahi

Let  $\triangle ABC$  be a triangle ( $\angle A \neq 90^\circ$ ), with altitudes  $\overline{BE}$ ,  $\overline{CF}$ . The bisector of  $\angle A$  intersects  $\overline{EF}$ ,  $\overline{BC}$  at  $M$ ,  $N$ . Let  $P$  be a point such that  $\overline{MP} \perp \overline{EF}$  and  $\overline{NP} \perp \overline{BC}$ . Prove that  $\overline{AP}$  bisects  $\overline{BC}$ .



Construct  $K = \overline{EF} \cap \overline{BC}$ ,  $Q$  as the  $A$ -Humpty point,  $H$  as the orthocenter of  $\triangle ABC$ , and  $\omega = (KMN)$ , so that the  $P$  given is the antipode of  $K$  on it. Let spiral similarity  $s$  at  $Q$  take  $(E, F) \rightarrow (B, C)$ . The main point of the problem is then:

**Claim** –  $MKQN$  cyclic. In other words,  $Q \in \omega$ .

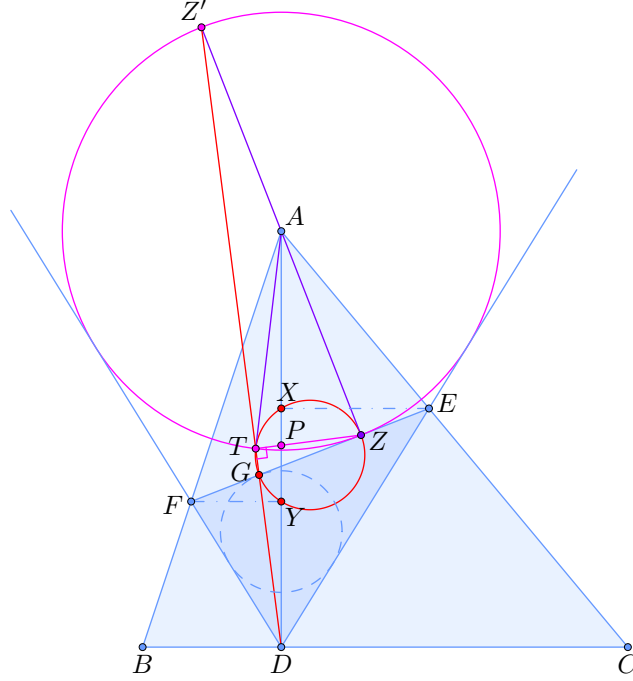
*Proof.* From angle bisector theorem,

$$\frac{EM}{MF} = \frac{EA}{AF} = \frac{BA}{AC} = \frac{BN}{NC} \Rightarrow (M \xrightarrow{s} N) \Rightarrow \angle MQN = -\arg(s) = \angle(\overline{EF}, \overline{BC}) = \angle MKN. \quad \square$$

Since  $P$  is the antipode of  $K$  on  $\omega$ ,  $\angle KQP = 90^\circ = \angle KQA$ , implying that  $P \in \overline{AQ}$ , the  $A$ -median.

## 1.8 Mock AIME 2019/15', by Eric Shen & Raymond Feng

In  $\triangle ABC$ , let  $D, E, F$  be the feet of the altitudes from  $A, B, C$  respectively, and let  $O$  be the circumcenter. Let  $Z = \overline{AO} \cap \overline{EF}$ . There exists a point  $T$  such that  $\angle DTZ = 90^\circ$  and  $AZ = AT$ . If  $P = \overline{AD} \cap \overline{TZ}$ , and  $Q$  lies on  $\overline{EF}$  such that  $PQ \parallel \overline{BC}$ , prove that  $\overline{AQ}$  bisects  $\overline{BC}$ .



Construct points  $X, Y$  as the projections of  $E, F$  onto  $\overline{AD}$  respectively.\*

After drawing a diagram on Geogebra, we obtain:

### Characterization of T

$T$  is the harmonic conjugate of  $Z$  wrt  $XY$  – i.e. it lies on  $\omega = (XYZ)$  so that the resulting quadrilateral is harmonic.

In American style, we show that this choice of  $T$  lies on  $\omega_a$  (defined as the circle at  $A$  thru  $Z$ ) and  $(DZ)$ ,

### Verification (inspired by USA TST 2015/1)

For  $AZ = AT$ , we use power of a point / length chase to get  $AZ^2 = AX \cdot AY$  whence  $\overline{AZ}$  touches  $\omega$ . Hence, by harmonics  $\overline{AT}$  is also tangent to  $\omega$ , so this property follows.

$\angle DTZ = 90^\circ$  is much less straightforward. We define  $Z' = 2A - Z$  and  $G = E + F - Z$  as the antipodes of  $Z$  on the circle at  $A$  through  $Z$ . By a well-known lemma,  $D, Z', G$  collinear (along the cevian through the intouch point in  $\triangle DEF$ ).

But also at the same time,  $T$  is on  $\omega, \omega_a \Rightarrow \angle ZTG = \angle ZTZ' = \pi/2$  due to antipodes. Hence,  $\angle DTZ = \pi/2$ , completing the verification.

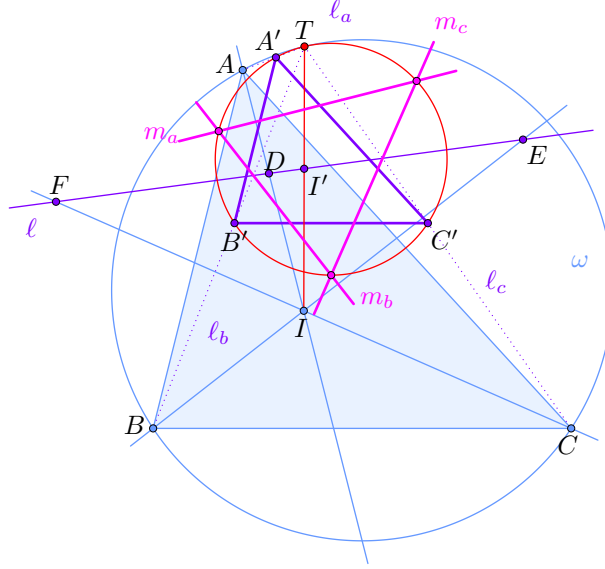
By this definition, we clearly have  $(AP; XY) = -1$ . From here (the chase is best discovered backwards), harmonic chasing suffices. Define  $K = \overline{EF} \cap \overline{AO}_{BC}$ . Then the bisection is established by

$$(\overline{AQ} \cap \overline{BC}, \infty_{BC}; B, C) \stackrel{A}{=} (QK; FE) \stackrel{\infty_{BC}}{=} (PA; YX) = -1.$$

\*Eric Shen originally included these points in the problem statement (as seen in the 2019 version of “Geometry At Its Best”), but I guess the problem’s made harder by deleting them. For me, thinking about their relevance/use was important in making nonzero progress on the problem.

### 1.9 SL 2018/G5, by Denmark

Let  $ABC$  be a triangle with circumcircle  $\omega$  and incenter  $I$ . A line  $\ell$  meets the lines  $AI$ ,  $BI$ ,  $CI$  at points  $D$ ,  $E$ ,  $F$  respectively, all distinct from  $A$ ,  $B$ ,  $C$ ,  $I$ . Prove that the circumcircle of the triangle determined by the perpendicular bisectors of  $\overline{AD}$ ,  $\overline{BE}$ ,  $\overline{CF}$  is tangent to  $\omega$ .



Solution by **TheUltimate123**.

Let  $\ell_a$  and cyclic variants be the reflections of  $\ell$  in the perpendicular bisectors  $x_a$  of  $\overline{AD}$ , etc.

**Claim** –  $\ell_a, \ell_b, \ell_c, \omega$  concur at a point  $T$ .

*Proof.* Because

$$\angle(\ell_b, \ell_c) = 2\angle(x_b, x_c) = 2\angle BIC = \angle BAC,$$

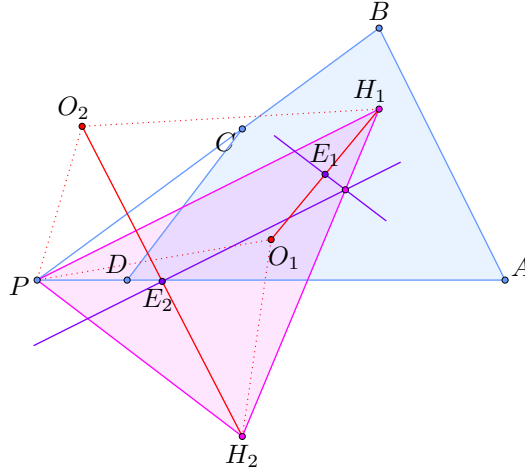
$\ell_b \cap \ell_c \in \omega$ ; the result follows by symmetry.  $\square$

Let  $I' = \overline{TI} \cap \ell$ , and consider the homothety  $h$  at  $T$  mapping  $I \rightarrow I'$ . Let  $P'$  denote the image of point  $P$  under  $h$ , so  $I'$  is the incenter of  $\triangle A'B'C'$ . Since  $\overline{A'T'} \parallel \overline{ADI}$  while  $A' \in \ell_a$  and  $I' \in \ell$ ,  $m_a$  is also the perpendicular bisector of  $\overline{AI'}$ .

From here it follows that the pairwise intersections of  $m_a, m_b, m_c$  are just the arc midpoints in  $(A'B'C')$ . By  $h$ ,  $(A'B'C')$ ,  $(ABC)$  tangent at  $T$ , hence done.

### 1.10 SL 2009/G6, by Eugene Bilopitov (Ukraine)

Let the sides  $AD$  and  $BC$  of the quadrilateral  $ABCD$  (such that  $AB$  is not parallel to  $CD$ ) intersect at point  $P$ . Points  $O_1$  and  $O_2$  are circumcenters and points  $H_1$  and  $H_2$  are orthocenters of triangles  $ABP$  and  $CDP$ , respectively. Denote the midpoints of segments  $O_1H_1$  and  $O_2H_2$  by  $E_1$  and  $E_2$ , respectively. Prove that the perpendicular from  $E_1$  on  $CD$ , the perpendicular from  $E_2$  on  $AB$  and the lines  $H_1H_2$  are concurrent.



Trying not to bash excessively...consider the problem wrt  $\triangle PH_1H_2$ . Observe that by isogonals,  $\angle O_2PH_1 = \angle H_1PO_2$ , so they've equal sines and

$$\frac{PH_1}{PO_1} = 2 \cos P = \frac{PH_2}{PO_2} \Rightarrow [PO_2H_1] = [PO_1H_2] \Rightarrow b_1(O_1) = -b_2(O_2) \xrightarrow{\text{linearity}} \boxed{b_1(E_1) + b_2(E_2) = 1}$$

in barycentrics wrt  $\triangle PH_1H_2$ , where  $p(X)$  denotes the  $P$ -coordinate of  $X$ , and similarly for the  $H_k$ . This means that the three desired lines (which can be defined as those through  $E_1, E_2$  parallel to  $\overline{PH_2}, \overline{PH_1}$  respectively) concur at

$$\boxed{0P + b_1(E_1) \cdot H_1 + b_2(E_2) \cdot H_2} \in \overline{H_1H_2}$$

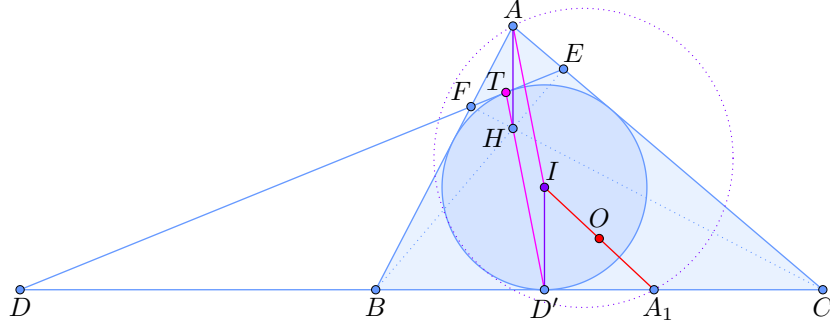
which is a valid barycentric point because of the first boxed equation.

### 🌲 1.11 MOP + USA TST, by Ankan Bhattacharya

Let  $ABC$  be a triangle with incenter  $I$ , and let  $D$  be a point on line  $BC$  satisfying  $\angle AID = 90^\circ$ . Let the excircle of triangle  $ABC$  opposite the vertex  $A$  be tangent to  $\overline{BC}$  at  $A_1$ . Define points  $B_1$  on  $\overline{CA}$  and  $C_1$  on  $\overline{AB}$  analogously, using the excircles opposite  $B$  and  $C$ , respectively.

#### 🌲 1.11.1 MOP 2019/(?)

Let  $E, F$  be the feet of the altitudes from  $B, C$  respectively. Prove that if  $\overline{EF}$  touches the incircle, then quadrilateral  $AB_1A_1C_1$  is cyclic.



Call the incircle  $\omega$ .

**Claim 1** –  $D, E, F$  are collinear.

*Proof.* We will prove that the tangent line from  $D$  is antiparallel to  $\overline{BC}$  wrt  $\angle A$ . Indeed, this line is found by reflecting  $\overline{DBC}$  over  $\overline{DI}$ , a line perpendicular to  $\overline{AI}$ , so we win.  $\square$

Let  $\omega$  touch  $\overline{DEF}$  at a point  $T$ , and let  $D'$  denote the  $A$ -intouch point.

**Claim 2** –  $\overline{AI} \parallel \overline{HD'}$ ; hence  $AID'H$  is a parallelogram and  $AH = r$ , the inradius of  $\triangle ABC$ .

*Proof.* Because  $BCEF$  is tangential, it follows by degenerate Brianchon that lines  $BE, CF, DT'$  concur, i.e.  $H \in \overline{TD'}$ . Observe that  $DT = DD'$ ; then  $\overline{THD'} \perp \overline{DI}$  by symmetry, while  $\overline{AI} \perp \overline{DI}$  is given; the lines are thus parallel as claimed.  $\square$

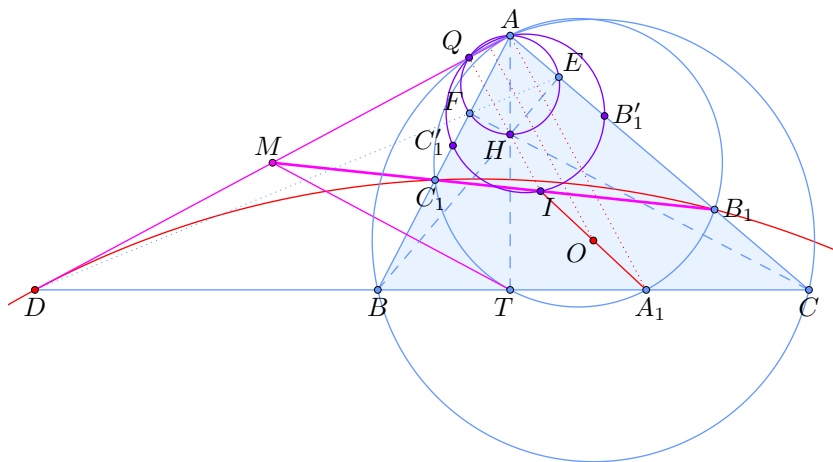
Now, let  $\omega_a$ , etc denote  $(AB_1C_1)$ , etc, respectively. We observe that because the perpendicular from  $A_1$  to  $\overline{BC}$  and its cyclic variants all concur at the point  $2O - I$ , it follows that all three circles must concur at this point by Miquel spam.

But because  $r/2 = AH/2$  is the distance from  $O$  to  $\overline{BC}$ , we actually have  $2O - I = A_1$  (also because of their feet onto  $\overline{BC}$ ). Hence  $A_1 \in \omega_a$  as desired.

**Remark.** I know for a fact that this was a problem during the Red 2019 tests- Eric Shen seems to have thoroughly enjoyed it in contest.

### 1.11.2 USA TST 2019/6

Prove that if quadrilateral  $AB_1A_1C_1$  is cyclic, then  $\overline{AD}$  is tangent to the circumcircle of  $\triangle DB_1C_1$ .



From MOP 2019, we make the following observations:

- By its converse,  $D, E, F$  collinear; then, if  $T$  is the foot from  $A$  to  $\overline{BC}$ , we have  $(TD; BC) = -1$ .
- As  $A_1$  is the Bevan point  $2O - I$ , its projections onto  $\overline{AC}, \overline{AB}$  are  $B_1, C_1$  respectively. It follows that  $A, A_1$  are antipodes on  $\omega_a$ .
- Since  $BCEF$  is bicentric, if the incircle touches  $\overline{AC}, \overline{AB}$  at  $B'_1, C'_1$ , then  $BC'_1/FC'_1 = CB'_1/EB'_1$ , so the  $A$ -incenter and orthocenter Miquel points coincide, say at  $Q \in (ABC)$ .

From the last item,  $\angle AQI = \angle AQH = 90^\circ$ .

**Claim -  $\overline{AD}$  touches  $\omega_a$ .**

*Proof.* Since  $(ABC) \cap (AH) = \{A, Q\}$ , the projection of  $O$  onto  $\overline{AQD}$  is  $\frac{A+Q}{2}$ . At the same time, the above implies  $Q$  is the projection of  $I$  onto  $\overline{AQD}$ . By linearity the projection of  $A_1 = 2O - I$  onto  $\overline{AD}$  is  $2\frac{A+Q}{2} - Q = A$ —in other words,  $\angle A_1AD = 90^\circ$ . This proves the tangency as  $\overline{AA_1}$  is a diameter of  $\omega_A$ .  $\square$

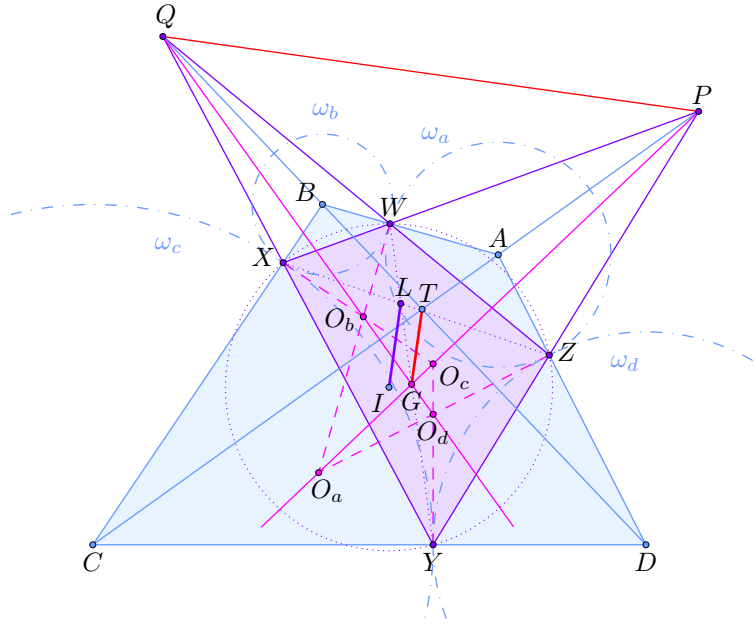
Let  $M = \frac{A+D}{2}$ , so  $\overline{MT}$  touches  $\omega_a$  as well by symmetry in the perpendicular bisector  $M\infty_{BC}$  of  $\overline{AT}$ . Now,  $(AT; B_1C_1) \stackrel{A}{=} (DT; CB) = -1$  means  $M \in \overline{B_1C_1}$ .

Finish by power of a point converse:  $MD^2 = MA^2 = MB_1 \cdot MC_1$  gives the needed tangency.



### 1.12 ELMO SL 2024/G4 (Nyan)

In quadrilateral  $ABCD$  with incenter  $I$ , points  $W, X, Y, Z$  lie on sides  $AB, BC, CD, DA$  with  $AZ = AW, BW = BX, CX = CY, DY = DZ$ . Define  $T = \overline{AC} \cap \overline{BD}$  and  $L = \overline{WY} \cap \overline{XZ}$ . Let points  $O_a, O_b, O_c, O_d$  be such that  $\angle O_aZA = \angle O_aWA = 90^\circ$  (and cyclic variants), and  $G = \overline{O_aO_c} \cap \overline{O_bO_d}$ . Prove that  $\overline{IL} \parallel \overline{TG}$ .



Draw the circle at  $A$  through  $W, Z$  and its cyclic variants, which we respectively call  $\omega_a, \dots, \omega_d$ . Then  $IW = IX = IY = IZ$  follows by symmetry about  $\overline{AI}$  and its cyclic variants.

**Claim 1** –  $\overline{O_aO_c}$  is the radical axis of  $\omega_b, \omega_d$ .

*Proof.*  $O_a$  has power  $O_aW^2 = O_aZ^2$  wrt  $\omega_b, \omega_a, \omega_b$  so it's their radical center. □

The crux of the problem is:

**Claim 2** – Let  $P, Q$  be the exsimilicenters of  $(\omega_b, \omega_d), (\omega_a, \omega_c)$ . Then  $P \in \overline{WX}, \overline{YZ}, \overline{O_aO_c}, \overline{BD}$  and similarly  $Q \in \overline{WZ}, \overline{XY}, \overline{O_bO_d}, \overline{AC}$ .

*Proof.* We show that the first four lines pass through the exsimilicenter of  $\omega_b$  and  $\omega_d$ .

- $P = \overline{WX} \cap \overline{YZ}$  and  $Q = \overline{WZ} \cap \overline{XY}$  follow by Monge on all sets of 3 circles;
- $P \in \overline{AC}$  by design;
- $P \in \overline{O_aO_c}$  are obtained from radical axis theorem on  $(WXYZ), \omega_b, \omega_d$  in conjunction with claim 1;

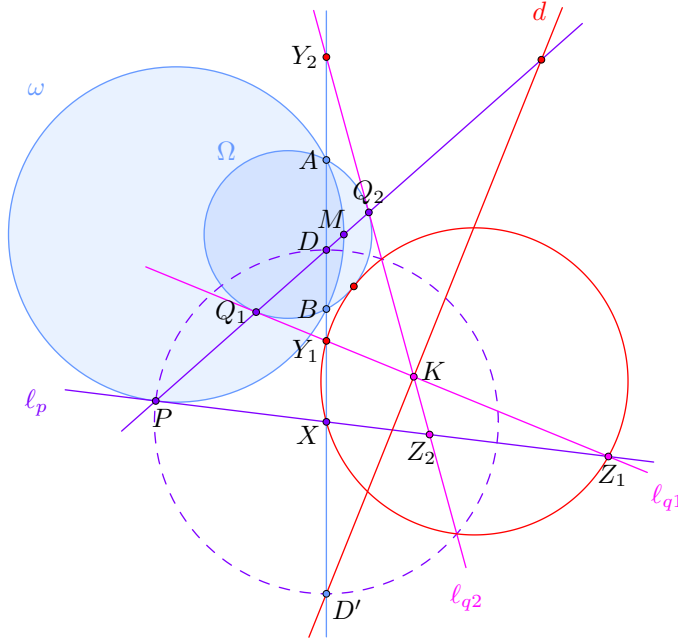
□

To finish, note that we have the orthocentric systems:

- $PQIL$  via Brocard on  $WXYZ$ ;
- $PQTG$  because  $\overline{O_aO_c} \perp \overline{BD}$  and  $\overline{O_bO_d} \perp \overline{AC}$ .

### 1.13 APMO 2014/5, by Ilya Bogdanov & Medeubek Kungozhin

Circles  $\omega$  and  $\Omega$  meet at points  $A$  and  $B$ . Let  $M$  be the midpoint of the arc  $AB$  of circle  $\omega$  ( $M$  lies inside  $\Omega$ ). A chord  $MP$  of circle  $\omega$  intersects  $\Omega$  at  $Q$  ( $Q$  lies inside  $\omega$ ). Let  $\ell_P$  be the tangent line to  $\omega$  at  $P$ , and let  $\ell_Q$  be the tangent line to  $\Omega$  at  $Q$ . Prove that the circumcircle of the triangle formed by the lines  $\ell_P$ ,  $\ell_Q$  and  $AB$  is tangent to  $\Omega$ .



We'll consider both  $Q$ 's at once, the one inside and outside. Call them  $Q_1, Q_2$  in any order. Define (here  $k = 1, 2$ ):

- $X = \ell_P \cap \overline{AB}$ ,  $Y_k = \ell_{Q_k} \cap \overline{AB}$ ,  $Z_k = \ell_{Q_k} \cap \ell_P$ ;
- $D$  and  $D' = 2X - D$  as the intersections of the internal and external bisectors of  $\angle APB$  with  $\overline{AB}$ , respectively, so that  $XP = XD = XD'$ ;
- $K = \ell_{Q_1} \cap \ell_{Q_2}$  as the pole of  $\overline{Q_1 Q_2}$  wrt  $\Omega$ , so that  $KQ_1 = KQ_2$ .

**Claim 1** –  $Y_1 Y_2 Z_1 Z_2$  is cyclic.

*Proof.* Note that triangles  $PXD, KQ_1 Q_2$  are both isosceles. Then

$$\angle(\ell_P, \ell_{Q_1}) = \angle XPD + \angle PQ_1 K \stackrel{\text{isosceles}}{=} -\angle XDP - \angle PQ_2 K = -\angle(\overline{AB}, \ell_{Q_2}),$$

whence the quadrilateral formed by  $\ell_P, \ell_{Q_1}, \overline{AB}, \ell_{Q_2}$  (in order) is cyclic.  $\square$

Let  $i$  denote inversion at  $X$  with power  $XP^2 = XD^2 = XA \cdot XB$  (last equality by midpoints of harmonic bundles lemma).

**Claim 2** –  $i$  swaps  $Y_1, Y_2$  as well.

*Proof.* Consider the polar  $\overline{KD'}$  of  $D$  wrt  $\Omega$ , which we call  $d$ . Then

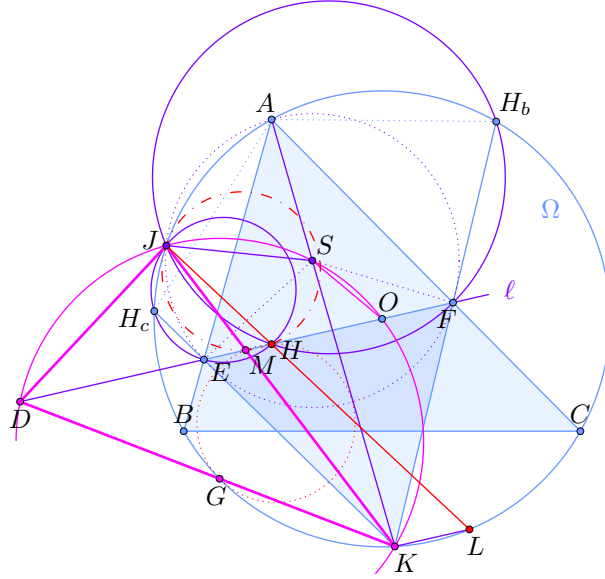
$$(Y_1 Y_2; DD') \stackrel{K}{=} (Q_1, Q_2; D, d \cap \overline{Q_1 D Q_2}) = -1,$$

the last harmonic bundle holding by definition of polar. The claim follows by another application of midpoints of harmonics bundles lemma.  $\square$

By the previous two claims and power of a point at  $X$ ,  $i$  also swaps  $(Z_1, Z_2)$ . Applying  $i$  to the given “ $\overline{Y_2 Z_2}$  touches  $\Omega$ ” yields  $(XY_1 Z_1)$  also tangent to  $\Omega$ , concluding the proof.

### 1.14 DeuX MO 2020/II/3, by Hao Minyan (China)

In triangle  $ABC$  with circumcenter  $O$  and orthocenter  $H$ , line  $OH$  meets  $\overline{AB}, \overline{AC}$  at  $E, F$  respectively. Let  $\omega$  be the circumcircle of triangle  $AEF$  with center  $S$ , meeting  $(ABC)$  again at  $J \neq A$ . Line  $OH$  also meets  $(JSO)$  again at  $D \neq O$ . Define  $K = (JSO) \cap (ABC) (\neq J), M = \overline{JK} \cap \overline{OH}$ , and  $G = \overline{DK} \cap (ABC) (\neq K)$ . Prove that  $(GHM)$  and  $(ABC)$  are tangent to each other.



Solution by **crazyeyemoody907, v4913**.

Let  $\Omega = (ABC)$ ,  $H_b, H_c$  be the respective reflections of  $H$  in  $\overline{AC}, \overline{AB}$ , and  $\ell = \overline{EFOH}$ . Redefine  $K = \overline{H_cE} \cap \overline{H_bF}$  (we'll see this is an equivalent definition). As  $\overline{EA}, \overline{FA}$  are external angle bisectors wrt  $\triangle KEF$ , we have  $\angle EKF = \pi - 2A$ .

**Claim 1** –  $J \in (HEH_c), (HFH_b)$ .

*Proof.* Let  $J' = (HEH_c) \cap (HFH_b) (\neq H)$ . Then:

$$\angle H_c J' H_b = \angle H_c J' H + \angle H J' H_b = \angle H_c E H + \angle H F H_b = \angle (\overline{H_c E}, \overline{H_b F}) = \angle H_b K H_c = \angle H_b A H_c \Rightarrow J' \in \Omega.$$

The construction of  $J'$  implies that  $\overline{J'E}, \overline{J'F}$  respectively bisect  $\angle H_c J' H, \angle H_b J' H$ , and thus

$$\angle E J' F = \frac{1}{2} \angle H_b J' H_c = \angle BAC = \angle EAF \Rightarrow J' \in (AEF),$$

finishing the claim. □

Let  $L = \overline{JH} \cap \Omega (\neq J)$ ; then, as  $JH_c K L, JH_c E H$  cyclic,  $\ell \parallel \overline{KL}$  by Reim. By homothety,  $(JHM)$  touches  $(JKL) = \Omega$ .

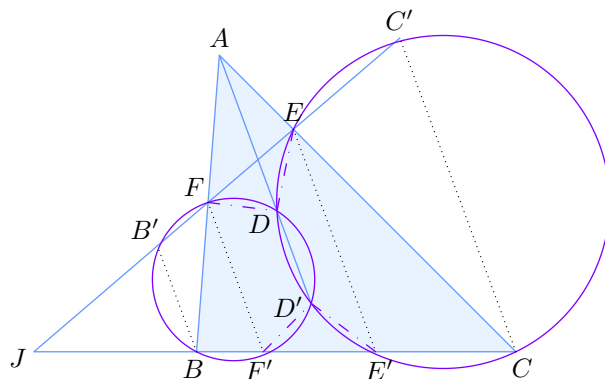
**Claim 2** – For the  $K$  defined in solution,  $K \in \overline{AS}, (JSO)$ .

*Proof.* Since  $\angle ESF = 2\angle BAC = \angle EKF$ , we have  $KESF$  cyclic; as  $SE = SF, AH_b = AH_c, A, S$  both lie on bisector of  $\angle EKF$ . Next, we prove that  $O$  is the midpoint of  $\widehat{JSK}$  on  $(JSK)$ . Because  $\overline{OS}$  is the perpendicular bisector of  $\overline{AJ}$  by symmetry, it externally bisects  $\angle JSK$  as  $K \in \overline{AS}$ . At the same time,  $OJ = OK$  means  $O$  is on the perpendicular bisector of  $\overline{JK}$ . These two properties imply that  $O$  is the claimed arc midpoint. □

From here, as  $DJKO$  cyclic and  $OJ = OK$ ,  $\overline{DO}$  bisects  $\angle JDK$ , and  $G = \overline{DK} \cap \Omega$  is the reflection of  $J$  in  $\ell$  by symmetry. Reflecting “ $(JHM)$  touches  $\Omega$ ” over  $\ell$  completes the proof.

### 1.15 IMO 2021/3

Let  $D$  be an interior point of the acute triangle  $ABC$  with  $AB > AC$  so that  $\angle DAB = \angle CAD$ . The point  $E$  on the segment  $AC$  satisfies  $\angle ADE = \angle BCD$ , the point  $F$  on the segment  $AB$  satisfies  $\angle FDA = \angle DBC$ , and the point  $X$  on the line  $AC$  satisfies  $CX = BX$ . Let  $O_1$  and  $O_2$  be the circumcenters of the triangles  $ADC$  and  $EXD$ , respectively. Prove that the lines  $BC$ ,  $EF$ , and  $O_1O_2$  are concurrent.



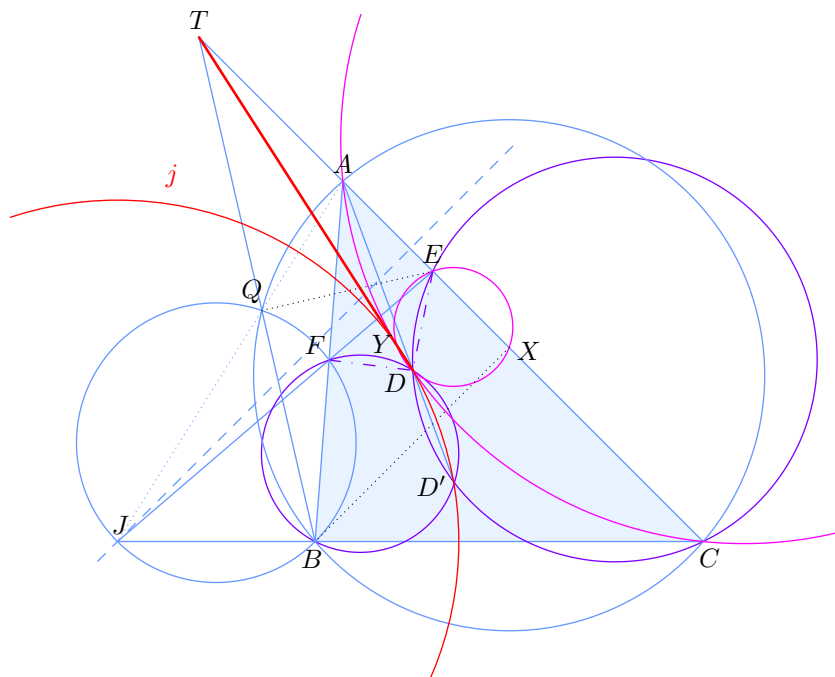
Solution by **v4913**.

Let  $J = \overline{EF} \cap \overline{BC}$ , and  $D' \in \overline{AD}$  be the isogonal conjugate of  $D$  wrt  $\triangle ABC$ . The given angle conditions imply that  $BDD'E$ ,  $CDD'E$  are cyclic, while power of a point at  $A$  implies  $BCEF$  cyclic as well.

**Claim 1** –  $J$  is the exsimilicenter of  $(EDC)$ ,  $(FDB)$ ; hence,  $JD = JD'$  by symmetry.

*Proof.* Construct  $E_1 = (CDD'E) \cap \overline{BC}$  ( $\neq C$ ),  $F_1 = (BDD'F) \cap \overline{BC}$  ( $\neq B$ ). By isogonality,  $DF = D'F'$  and  $DE = D'E'$  whence  $DD'E'E$ ,  $DD'F'F$  are both cyclic isosceles trapezoids.  $\overline{DD'}$ ,  $\overline{EE'}$ ,  $\overline{FF'}$  share a perpendicular bisector  $b$ , and in fact, this is the bisector of  $\angle J$ , i.e.  $JE = JE'$ ,  $JF = JF'$ .

Reflect  $B, C$  over  $b$  to obtain  $B', C'$ ; then, because  $JB/JF' = JB/JF = JE/JC = JE'/JC$ , there is a homothety at  $J$  mapping  $(B, B', F, F') \rightarrow (E', E, C', C)$  and thus their circumcircles  $(BB'DD') \rightarrow (CC'DD')$  as well.  $\square$



Let  $Y = (ADC) \cap (EXD)$  ( $\neq D$ ),  $Q$  be the Miquel point of  $ABCJEF$ , and  $j$  the circle at  $J$  through  $D, D'$ . Observing that  $\overline{O_1 O_2}$  is the perpendicular bisector of  $\overline{DY}$ , it remains to prove  $Y \in j$ .

**Claim 2** –  $XQEB$  is cyclic.

*Proof.* This is a simple angle chase: using cyclic quadrilaterals  $(ABCQ)$ ,  $(JFBQ)$ ,  $(ECJQ)$ , and  $(AEFQ)$ , we obtain

$$\angle EQB = \angle EQA + \angle AQB = \angle ACB + \angle EFA = 2\angle ACB = \angle EXB \quad \square$$

Next, we characterize the radical axis of  $j$ ,  $(JBF)$  – it's perpendicular to the line of centers and through  $A$ :

**Claim 3** – The line through  $B$  and the center of  $(JBF)$  is perpendicular to  $\overline{AC}$ .

*Proof.* This is equivalent to “ $t_b$ , the tangent to  $(JBF)$  at  $J$ , is parallel to  $\overline{AC}$ ”. Because  $\angle(\overline{BJC}, t_b) = \angle BFJ = \angle JCE$ , the result follows.  $\square$

Because  $\text{Pow}(A, j) = AD \cdot AD' = AQ \cdot AJ = \text{Pow}(A, (JBQF))$ ,  $A$  is on the radical axis of  $j$ ,  $(JBF)$ . By the previous claim, it follows that  $\overline{AC}$  is the radical axis of  $j$ ,  $(JBF)$ .

To finish, define  $T = \overline{DY} \cap \overline{AC} \cap \overline{BQ}$  as the radical center of  $(JBF)$ ,  $(ABC)$ ,  $(EXD)$ ,  $(ADC)$ , and the phantom point  $Y' = \overline{TD} \cap j$  ( $\neq D$ ). Because  $T$  is on  $\overline{AC}$ , the radical axis of  $j$ ,  $(JBF)$ , we have (lengths directed)

$$TY' \cdot TD = \text{Pow}(T, j) = \text{Pow}(T, (JBF)) = \text{Pow}(T, (ABCQ)) = TA \cdot TC = TY \cdot TD \Rightarrow Y = Y',$$

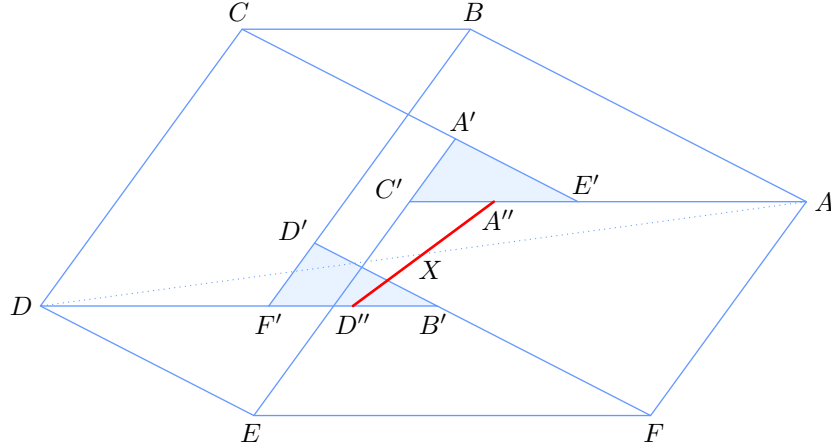
the end!

### 1.16 USAMO 2021/6, by Ankan Bhattacharya

Let  $ABCDEF$  be a convex hexagon satisfying  $\overline{AB} \parallel \overline{DE}$ ,  $\overline{BC} \parallel \overline{EF}$ ,  $\overline{CD} \parallel \overline{FA}$ , and

$$AB \cdot DE = BC \cdot EF = CD \cdot FA.$$

Let  $X$ ,  $Y$ , and  $Z$  be the midpoints of  $\overline{AD}$ ,  $\overline{BE}$ , and  $\overline{CF}$ . Prove that the circumcenter of  $\triangle ACE$ , the circumcenter of  $\triangle BDF$ , and the orthocenter of  $\triangle XYZ$  are collinear.



Construct parallelogram  $CDEA'$  and cyclic variants:  $A' = C + E - D$ , etc. We may compute using vectors that  $\triangle B'D'F'$  is a translation of  $\triangle A'C'E'$  by the vector  $(B + D + F) - (A + C + E)$ . In particular, they're congruent.

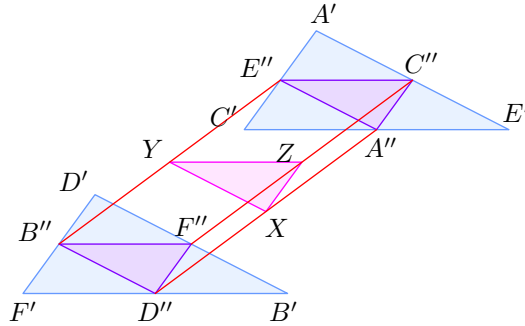
**Claim 1** –  $A, C, E$  have same power wrt  $(A'C'E')$ ; in other words,  $\triangle ACE, A'C'E'$  share a circumcenter.

*Proof.* Observing that  $\text{Pow}(A, (A'C'E')) = AC' \cdot AE' = BC \cdot EF$  by parallelograms, this claim follows by the given length condition.  $\square$

Next, construct  $A'' = \frac{C' + E'}{2}$  and cyclic variants. The circumcenter of  $\triangle A'C'E'$  is then the orthocenter of  $\triangle A''C''E''$ .

**Claim 2** –  $X = \frac{A'' + D''}{2}$ .

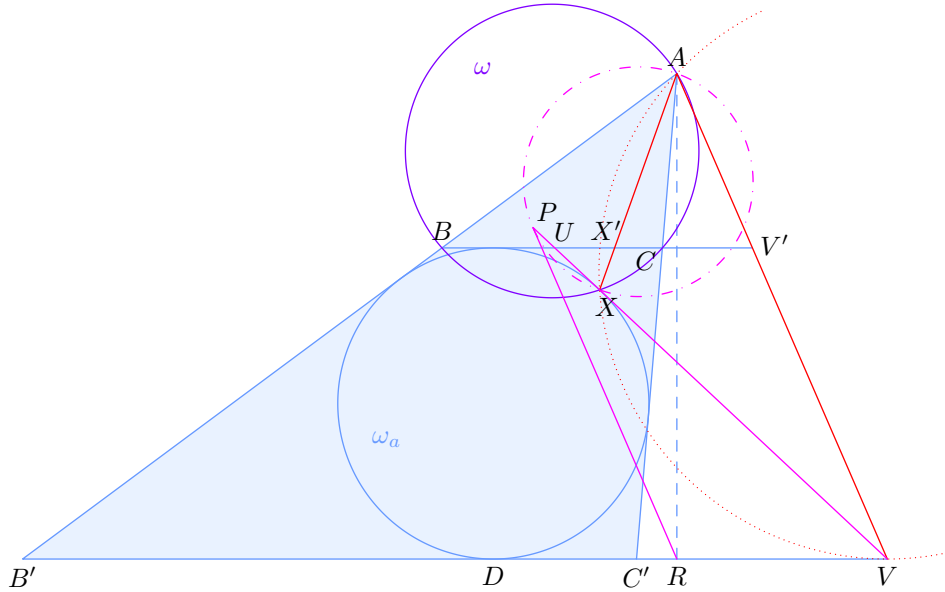
*Proof.* Using vectors,  $B' + C' = E' + F' = A + D \Rightarrow \frac{A + D}{2} = \frac{B' + C' + E' + F'}{4} = \frac{A'' + D''}{2}$ .  $\square$



By claim 2 + symmetry,  $\triangle XYZ$  is the vector average of (congruent) triangles  $A''C''E'', B''D''F''$ , so their orthocenters are collinear.

### 1.17 SL 2021/G8

Let  $ABC$  be a triangle with circumcircle  $\omega$  and let  $\Omega_A$  be the  $A$ -excircle. Let  $X$  and  $Y$  be the intersection points of  $\omega$  and  $\Omega_A$ . Let  $P$  and  $Q$  be the projections of  $A$  onto the tangent lines to  $\Omega_A$  at  $X$  and  $Y$  respectively. The tangent line at  $P$  to the circumcircle of the triangle  $APX$  intersects the tangent line at  $Q$  to the circumcircle of the triangle  $AQY$  at a point  $R$ . Prove that  $\overline{AR} \perp \overline{BC}$ .



Solution by [crazyeyemoody907](#).

Let the antipode of the  $A$ -extouch point be  $D$ , and the tangent to  $\omega_a$  at  $D$  intersect  $\overline{AB}, \overline{AC}$  at  $B', C'$  respectively. Also, construct the tangent line to  $\omega_a$  at  $X$ , meeting  $\overline{BC}, \overline{B'C'}$  at  $U, V$  respectively. Finally, let  $X' = \overline{AX} \cap \overline{BC}$ ,  $V' = \overline{AV} \cap \overline{B'C'}$ .

**Claim 1** –  $AXUV'$  cyclic.

*Proof.* Apply DDIT to  $A, UXV \infty_{BC}$  (inconic  $\omega_a$ ), and project onto  $\overline{BC}$ , to obtain an involutive pairing  $(BC; UV'; \infty_{BC} X')$  – or equivalently,  $X'B \cdot X'C = X'U \cdot X'V'$ . By power of a point,  $X'B \cdot X'C = X'A \cdot X'X$ , so the claim follows from  $X'U \cdot X'V' = X'A \cdot X'X$ .  $\square$

**Claim 2** –  $\overline{DV}$  is tangent to  $(AXV)$ .

*Proof.* Angle chase using previous claim, and the fact that  $\overline{BC} \parallel \overline{B'C'}$ :

$$\angle XAV \stackrel{\text{claim 1}}{=} \angle XUV' = \angle XVD. \quad \square$$

Redefine  $R$  as the foot from  $A$  to  $\overline{B'C'}$ . It remains to show,

**Claim 3** –  $\overline{PR}$  touches  $(APX')$ .

*Proof.* Since  $\angle VPA = \angle VRA = 90^\circ$ ,  $APRV$  cyclic, so we may anglechase as follows:

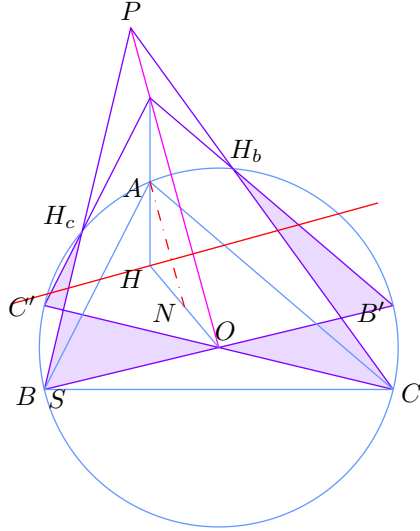
$$\angle APR = \angle AVR \stackrel{\text{claim 2}}{=} \angle AXV = \angle AXP. \quad \square$$

### 1.18 USEMO 2020/3, by Anant Mudgal

Let  $ABC$  be an acute triangle with circumcenter  $O$  and orthocenter  $H$ . Let  $\Gamma$  denote the circumcircle of triangle  $ABC$ , and  $N$  the midpoint of  $\overline{OH}$ . The tangents to  $\Gamma$  at  $B$  and  $C$ , and the line through  $H$  perpendicular to line  $AN$ , determine a triangle whose circumcircle we denote by  $\omega_A$ . Define  $\omega_B$  and  $\omega_C$  similarly.

Prove that the common chords of  $\omega_A$ ,  $\omega_B$ , and  $\omega_C$  are concurrent on line  $OH$ .

Let  $H_a, A'$  denote the respective reflections of  $H$  in  $\overline{BC}$ ,  $A$  in  $O$ , and their symmetric variants.



**Claim 1** – The polar  $\ell_a$  of  $\overline{BH_c} \cap \overline{CH_b}$  passes through  $H$  and is perpendicular to  $\overline{AN}$ .

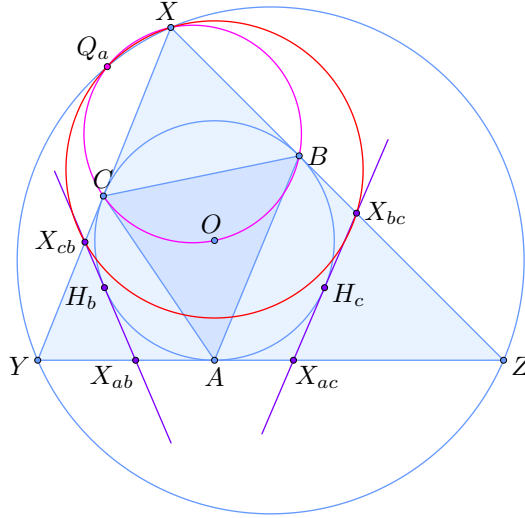
*Proof.* Let  $P = \overline{BH_c} \cap \overline{CH_b}$  and  $S = 2A - H$ .  $H \in \ell_a$  is just Brocard, so it suffices to prove  $\overline{AN} \parallel \overline{OP}$ . By Pascal on  $BB'H_bCC'H_c$ , we have  $P, O, S$  collinear. Taking a homothety at  $H$  with scale factor  $\frac{1}{2}$  maps the latter two points to  $N, A$ , which implies the required parallel lines.  $\square$

In  $\triangle ABC$ , let  $X_{bc}$  be the pole of  $\overline{BH_c}$  wrt  $\Gamma$  (and 5 other variants),  $X, Y, Z$  be the poles of the sides,  $D, E, F$  be the feet of the altitudes. Clearly,  $\ell_a = \overline{X_{bc}X_{cb}}$ .

**Note.** Here, the condition  $\triangle ABC$  acute comes in:  $\Gamma$  is the incircle, not excircle, of  $\triangle XYZ$ .

We'll show that  $\overline{XD}$  is the radical axis of  $\omega_b, \omega_c$ . (By a somewhat-known configuration (say, **Brazil 2013/6**),  $\overline{XD} \cap \overline{YE} \cap \overline{ZF}$  lies on the Euler line.) Also let  $Q_a, Q_b, Q_c$  be the SD points of  $\triangle XYZ$ .





**Claim 2** –  $Q_a$  lies on  $\omega_a$ .

*Proof.* By spiral similarity, it suffices to prove  $YX_{bc}/YC = ZX_{cb}/ZB$ . By antiparallel lines,  $\triangle XYZ \sim \triangle X_{ab}YX_{cb}$ ,  $X_{ac}X_{bc}Z$ . But since  $\Gamma$  is the  $Y$ -excircle of  $\triangle X_{ab}YX_{cb}$ , we have  $YX_{cb}/YC = a/s$ . Similarly  $ZX_{bc}/ZB = a/s$  as well.

(In some awful notation,  $a = YZ$ ,  $b = ZX$ ,  $c = XY$  and  $s = \frac{a+b+c}{2}$ .) □

Let  $L = \overline{YQ_b} \cap \overline{ZQ_c}$ .

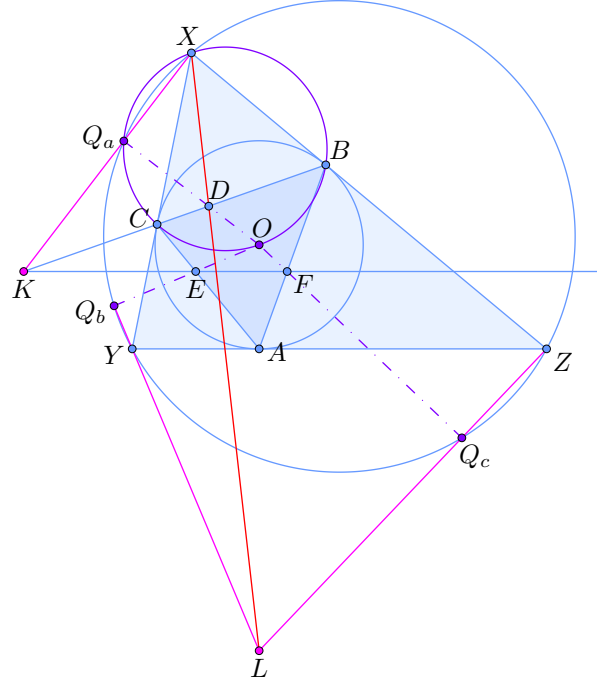
**Claim 3** –  $\overline{XL}$  is the radical axis of  $\omega_b, \omega_c$ .

*Proof.* By antiparallel lines again,  $YZX_{ba}X_{ca}$  cyclic, so that

$$\text{Pow}(X, \omega_b) = XX_{ca} \cdot XY = XX_{ba} \cdot XZ = \text{Pow}(X, \omega_c), \text{ while}$$

$$\text{Pow}(L, \omega_b) = LY \cdot LQ_b = LZ \cdot LQ_c = \text{Pow}(L, \omega_c). \quad \square$$

It remains to prove  $X, D, L$  collinear.



**Claim 4** –  $L$  is the pole of  $\overline{EF}$  wrt  $\Gamma$ .

*Proof.* Since  $Q_a$  is the inverse of  $D$  wrt  $\Gamma$  and  $\angle OQ_aX = 90^\circ$ ,  $\overline{XQ_a}$  is the polar of  $D$  wrt  $\Gamma$ . Similarly,  $\overline{YQ_b}$ ,  $\overline{ZQ_c}$  are the respective polars of  $E, F$  wrt  $\Gamma$ . The claim is then established by la Hire.  $\square$

**Claim 5** –  $\overline{BC}, \overline{EF}, \overline{XQ_a}$  concurrent.

*Proof.* Let  $K = \overline{EF} \cap \overline{BC}$  so that  $(KD; BC) = -1$ . Because  $\overline{Q_aO}$  bisects  $\angle BQ_aC$ ,  $\angle KQ_aO = 90^\circ = \angle AQ_aO \Rightarrow X, Q_a, K$  collinear.  $\square$

Taking poles wrt  $\Gamma$  in the last claim gives the desired collinearity.

**Remark.** The problem can be bary'd wrt  $\triangle XYZ$  after the first claim, but it's monstrous from my experience a long time ago, oops