Geometry At Its Worst

Nyan

2022-4

Eric Shen's 'Geometry At Its Best' compilation shows the most elegant geo solutions he's seen. Well, instead of emulating him, why not go in the opposite direction? That is, problems which are just purely beyond reasonability… ~~Also this definitely doesn't show off the power of Geogebra.~~

Contents

0 Problems **2**

1 Solutions **3**

1.1 USA TST 2021/2, by Andrew Gu & Frank Han . . . 3

1.2 DIMO 2022/6 . . . 4

1.3 Brazil Revenge 2021/3, by Joao P.R. Viana Costa . . . 7

0 Problems

Problem 1 (USA TST 2021/2). Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.

Problem 2 (DIMO 2022/6). In triangle $\triangle ABC$, M is the midpoint of arc BAC, I is the incenter and I_a is the A-excenter. Let $E = \overline{BI} \cap \overline{AC}$, $F = \overline{CI} \cap \overline{AB}$, $P = \overline{MI} \cap (ABC)$, and $S = (AEF) \cap (ABC)$ $(\neq A)$. If X, Y are the reflections of I across $\overline{I_aE}, \overline{I_aF}$ respectively, prove that (BYF), (CXE), (PXY) and $\overline{PS}$ are concurrent.

Problem 3 (Brazil Revenge 2021/3). Let I, C, ω and Ω be the incenter, circumcenter, incircle and circumcircle, respectively, of the scalene triangle XYZ with $XZ > YZ > XY$. The incircle ω is tangent to the sides YZ, XZ and XY at the points D, E and F. Let S be the point on Ω such that XS, CI and YZ are concurrent. Let $(XEF) \cap \Omega = R$, $(RSD) \cap (XEF) = U$, $SU \cap CI = N$, $EF \cap YZ = A$, $EF \cap CI = T$ and $XU \cap YZ = O$.

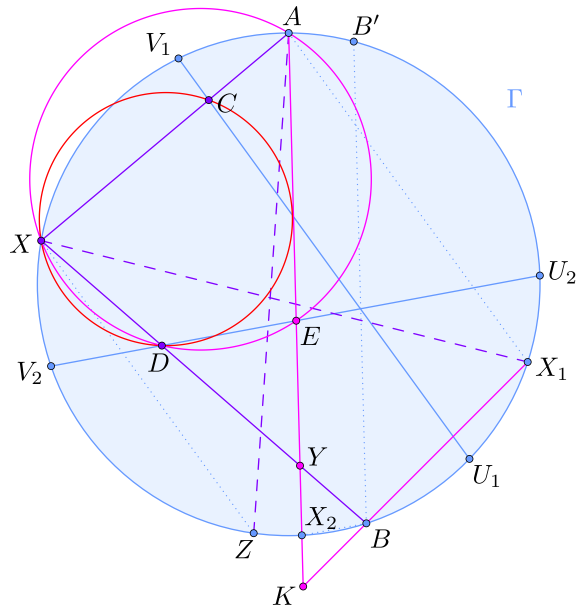
Prove that $NARUTO$ is cyclic.

1 Solutions

1.1 USA TST 2021/2, by Andrew Gu & Frank Han

Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ, in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$. Let X be a variable point on the arc V_1V_2 of Γ not containing A or B. Line XA meets line U_1V_1 at C, while line XB meets line U_2V_2 at D.

Prove there exists a fixed point K, independent of X, such that the power of K to the circumcircle of $\triangle XCD$ is constant.



Clearly, the problem statement should hold for any $X \in \Gamma$; here, all lengths are directed.
Let X_1, X_2 be the respective reflections of A, B in the perpendicular bisectors of $\overline{U_1V_1}$, $\overline{U_2V_2}$. We assert that $K = \overline{AX_2} \cap \overline{BX_1}$ fits the bill. For brevity, let '$\leftrightarrow$' denote 'is a constant multiple of', so '$x \leftrightarrow 1$' is a shorthand for 'x is constant'.

By Reim, $E = \overline{BX} \cap \overline{AX_2}$ lies on (ADX), so $\operatorname{Pow}(K, (ADX)) = KE \cdot KA \leftrightarrow 1$. Now, in the spirit of linpop, let $f(P) = \operatorname{Pow}(P, (ADX)) - \operatorname{Pow}(P, (XCD))$, so that because $f(Y) = 0$, we have

$$f(K) = -\frac{KY}{YA} f(A) = AC \cdot AX \frac{KY}{AY}.$$

The rest is a wild length chase; let B', Z be the respective reflections of B, X in the perpendicular bisector of $\overline{U_1V_1}$, so that $XX_1 = AZ$ and $\overline{AZ}, \overline{ACX}$ isogonal wrt $\angle U_1AV_1$. Then, observing that all lengths not involving X, C, D, Y are fixed,

$$\frac{KY}{AY} = (KA; Y\infty_{AK}) \stackrel{B}{=} (X_1A; XB') \leftrightarrow \frac{X_1X}{AX} = \frac{AZ}{AX};$$

$$\Rightarrow f(K) \leftrightarrow AC \cdot AZ = AU_1 \cdot AV_1 \leftrightarrow 1,$$

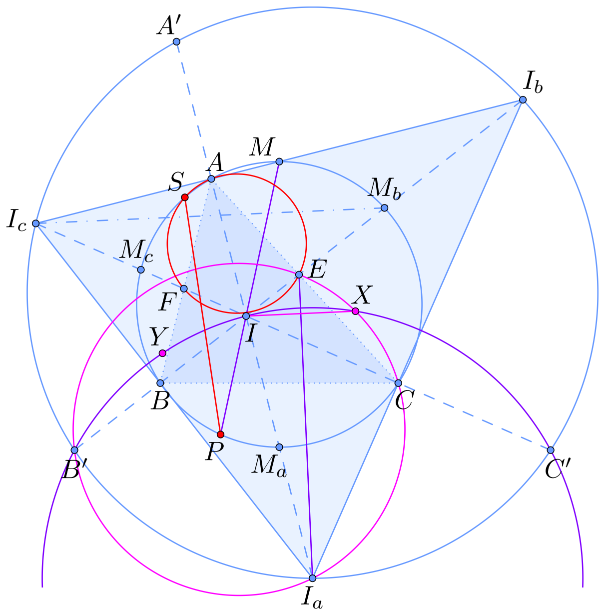
where the last equality follows because Z, C swapped by inversion at A with power $AU_1 \cdot AV_1$ composed with reflection in the angle bisector of $\angle U_1AV_1$, so we win.

> *Remark.* How on earth would someone find K? I considered the degenerate cases when (XCD) is a straight line (which occur when $X = X_1, X_2$, hence their names).

1.2 DIMO 2022/6

In triangle $\triangle ABC$, M is the midpoint of arc BAC, I is the incenter and I_a is the A-excenter. Let $E = \overline{BI} \cap \overline{AC}$, $F = \overline{CI} \cap \overline{AB}$, $P = \overline{MI} \cap (ABC)$, and $S = (AEF) \cap (ABC)$ $(\neq A)$. If X, Y are the reflections of I across $\overline{I_aE}, \overline{I_aF}$ respectively, prove that (BYF), (CXE), (PXY) and $\overline{PS}$ are concurrent.

Solved with **v4913**.



Let I_a, etc. and M_a, etc. be the excenters and minor arc midpoints, respectively, and A', etc. $\in (I_aI_bI_c)$ be the reflections of I in A, etc. Observe that $I_aI = I_aX = I_aY$.

> **Claim 1 –** (BYF), (CXE) pass through (I_a, C') and (I_a, B') respectively.

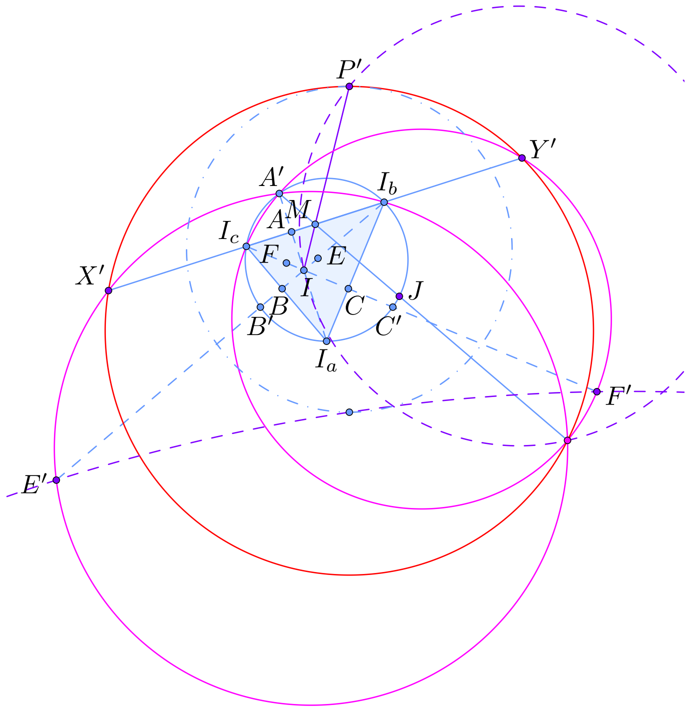
Proof. The cyclic quads involving I_a are handled as follows:

$$\measuredangle I_aYF = -\measuredangle I_aIF = \measuredangle CIA = \measuredangle I_aBF;$$

Meanwhile, since the tangent to $(I_aI_bI_c)$ at I_b is parallel to $\overline{AC}$, $B'I_aCE$ is cyclic by Reim, and similarly for its cyclic variant. This establishes the concyclicities. □

> **Claim 2 –** $I, X, 2I_c - I_b$ are collinear.

Proof. By Brocard on $AICI_b$, $\overline{I_aE} \perp \overline{M_bI_c}$, while we obviously have $\overline{IX} \perp \overline{I_aE}$. Since $\overline{IX}$ is just $\overline{M_bI_c}$ scaled at I_b by a factor of 2, the result follows. □



Next, we invert about I preserving $(I_aI_bI_c)$. Denote image/preimage by *.

- The circle at I_a through I, on which X, Y lie, gets mapped to $\overline{I_bI_c}$; hence $X^* = 2I_c - I_b$.
- $2P - I$ is an orthocenter Miquel point of triangle $I_aI_bI_c$, so $P* = 4M - 3I$.
- E, F get mapped to the images of $2B - I_b$, $2C - I_c$ when scaled at I by a factor of 2. (well-known)
- $A* = 2I_a - I$ and $I_a^* = A'$;

This is enough to finish off part of the concurrency: $(P^*X^*Y^*)$ is $(I_aI_bI_c)$ scaled at M by a factor of 3. If $J = \overline{A'M} \cap (I_aI_bI_c)$, then by power of a point at M, the first 3 circles all meet at $3J - 2M$.

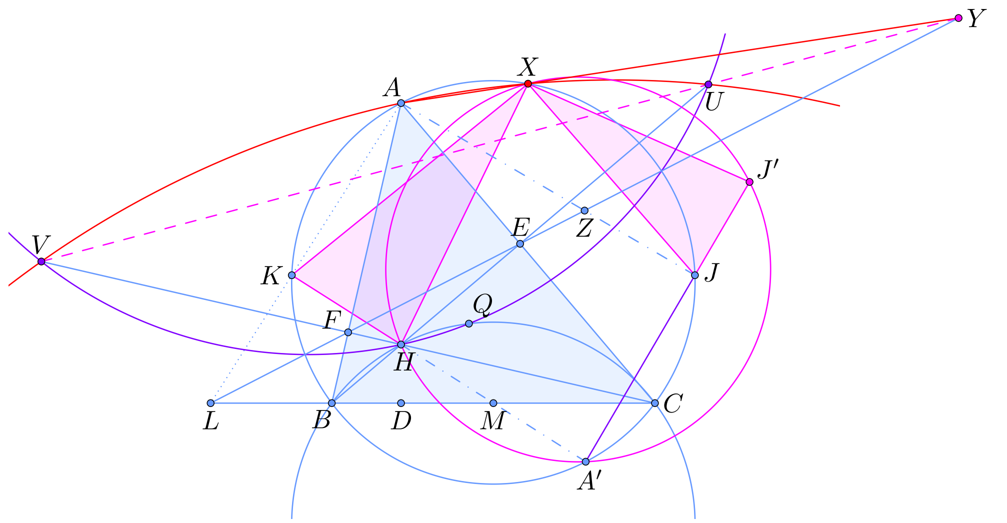
Next, to deal with the rest of the problem, we take a homothety at I with scale factor $1/2$ and refactor:

Second half refactored

In triangle ABC with orthocenter H, D, E, F are the feet of the altitudes, and M is the midpoint of $\overline{BC}$. Let A' be antipode of A on (ABC), and define $U = 2E - B$, $V = 2F - C$. Define J as the reflection of the A-orthocenter Miquel point in the perpendicular bisector of $\overline{BC}$ and

$$J' = \frac{H + 3J - 2M}{2} = \frac{3J - A'}{2}.$$

Prove that (AUV), (ABC), $(A'J'H)$ are concurrent.



U, V are just the reflections of B, C over their respective opposite sides. We add in the $X = (AUV) \cap (ABC)$, and K as the A-orthocenter Miquel point. Since (A', H, K) and (A', J, J') are collinear, $A'J'HX$ being cyclic is equivalent to showing that a spiral similarity at X takes KH to JJ', or $\frac{XK}{XJ} = \frac{KH}{JJ'}$. Let r be the reflection in the angle bisector of $\angle BAC$ composed with a dilation, sending $A' \to H$, so that $KH/JJ' = 2KH/AJ = 2AH/AA'$.

> **Claim 3 –** AX, UV, EF are concurrent.

Proof. Let Q be the A-Humpty point. Consider the spiral similarity s at Q mapping $(B, E) \to (C, F)$ and thus $U \to V$ as well. As a result, Q is also the Miquel point of $BCVU$, and thus lies on (HUV). It suffices to prove that $\overline{EF}$ is the radical axis of (ABC), $(UVHQ)$, since the claim will then follow from radical axis theorem on (ABC), $(AXUV)$, $(UVHQ)$. Indeed, this is obvious- $AE \cdot CE = HE \cdot UE$ and similarly for F. □

At this point, we add in a few intersections: $L = \overline{AK} \cap \overline{EF} \cap \overline{BC}$, $Z = \overline{AJ} \cap \overline{EF} = r(L)$, and Y as the concurrency point from claim 3.

> **Claim 4 –** Z is the midpoint of $\overline{LY}$.

Proof. Instead we'll show $U, V, 2Z - L$ are collinear, which is again by spiral similarity:

- Let s' be the spiral similarity at Q sending $B, C \to E, F$; then $BL/CL \stackrel{r}{=} EZ/FZ$, so s' also sends L to Z.
- Another one sends $B, C, L \to E, F, Z \to U, V, 2Z - L$, so the last three are indeed collinear. □

Finally,

$$\frac{1}{2} = (\infty_{EF} Y; LZ) \stackrel{A}{=} (AX; KJ), \text{ so that } \frac{XK}{XJ} = \frac{2AK}{AJ} \stackrel{r}{=} \frac{2AH}{AA'},$$

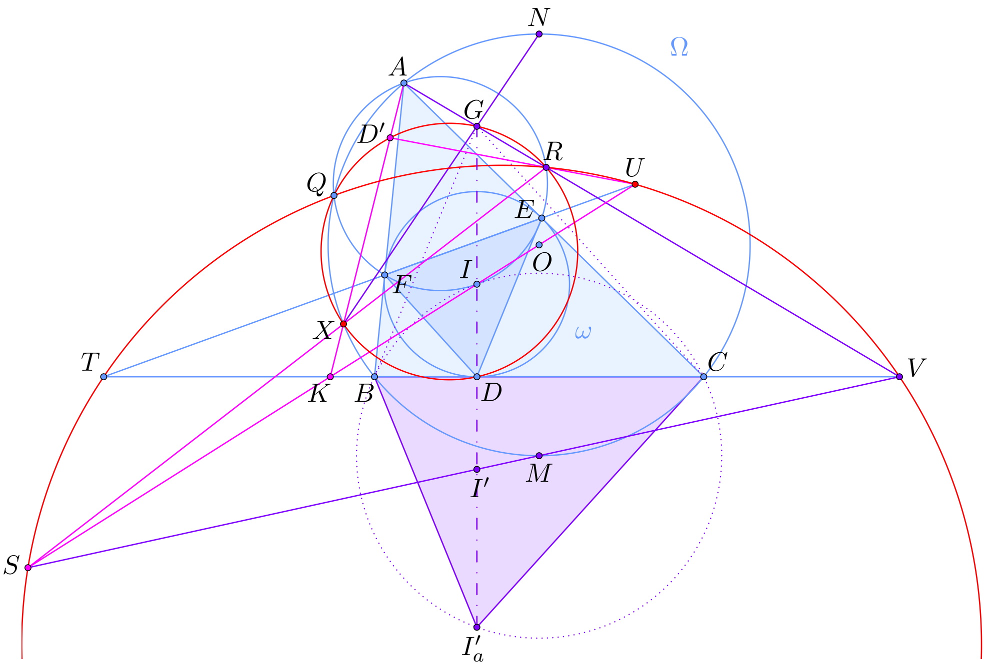
and we're done!

1.3 Brazil Revenge 2021/3, by Joao P.R. Viana Costa

Let I, C, ω and Ω be the incenter, circumcenter, incircle and circumcircle, respectively, of the scalene triangle XYZ with $XZ > YZ > XY$. The incircle ω is tangent to the sides YZ, XZ and XY at the points D, E and F. Let S be the point on Ω such that XS, CI and YZ are concurrent. Let $(XEF) \cap \Omega = R$, $(RSD) \cap (XEF) = U$, $SU \cap CI = N$, $EF \cap YZ = A$, $EF \cap CI = T$ and $XU \cap YZ = O$.

Prove that $NARUTO$ is cyclic.

Colloquially known as "Naruto".



Solved with **CyclicISLscelesTrapezoid** and help from **Eyed**, **v4913**. We do a massive refactoring and simplification; consider the following equivalent problem, a breakdown of the given, despite being longer:

Naruto simplified

In triangle ABC with circumcircle Ω centered at O, the incircle ω centered at I touches the sides at D, E, F. Let I', I'_a be the respective reflections of I and the orthocenter of $\triangle BIC$ in $\overline{BC}$, and M the midpoint of arc BC on Ω. Further define:

- S as the intersection of the Euler lines $\overline{OI}$ of $\triangle DEF$, $\overline{MI'}$ of $\triangle I'_aBC$;
- $T = \overline{EF} \cap \overline{BC}$, $U = \overline{EF} \cap \overline{OI}$, $V = \overline{MI'} \cap \overline{BC}$, $R = \overline{AV} \cap (AI)$;
- $K = \overline{OI} \cap \overline{BC}$;

Prove that (a) Q, R, S, T, U, V are concyclic, and (b) $\overline{AK}$, Ω, (QRD), $\overline{RS}$ concurrent;

(a) The concyclicity Let the spiral similarity s at Q with (directed) angle θ map $E, F \to C, B$ and thus D, I and the orthocenter of $\triangle DEF$ to I', M, I'_a respectively. Clearly, S is the intersection of the Euler lines of two triangles related by s: DEF, I'_aCB.
By design, we have $U \xrightarrow{s} V$, so

$$\measuredangle VQU = \theta = \measuredangle(\overline{BC}, \overline{EF}) \stackrel{s}{=} \measuredangle(\overline{MI'}, \overline{OI}) = \measuredangle VSU,$$

whence Q, S, T, U, V concyclic. To see that the last point is also concyclic with the other five, let N be the midpoint of $\widehat{BAC}$, so that $\overline{NA}$ touches (AI). Indeed, then

$$\measuredangle QRV = \measuredangle QRA = \measuredangle QAN \stackrel{s}{=} \measuredangle QUV$$

as needed.

> *Remark.* In fact, by design, S is the exsimilicenter of the incircle and the circle at O with radius half that of Ω, so it's actually the inverse of I wrt Ω.

(b) The concurrence Let D' be the reflection of D in $\overline{EF}$, and G the orthocenter of $\triangle BIC$, so that $D' \xrightarrow{s} G$. We easily have $DD'GQ$ cyclic. As $\measuredangle(\overline{AD'}, \overline{NG}) = \theta$, the point $X = \overline{AD'} \cap \overline{NG}$ lies on both $(DD'GQ), \Omega$. We require the following result(s):

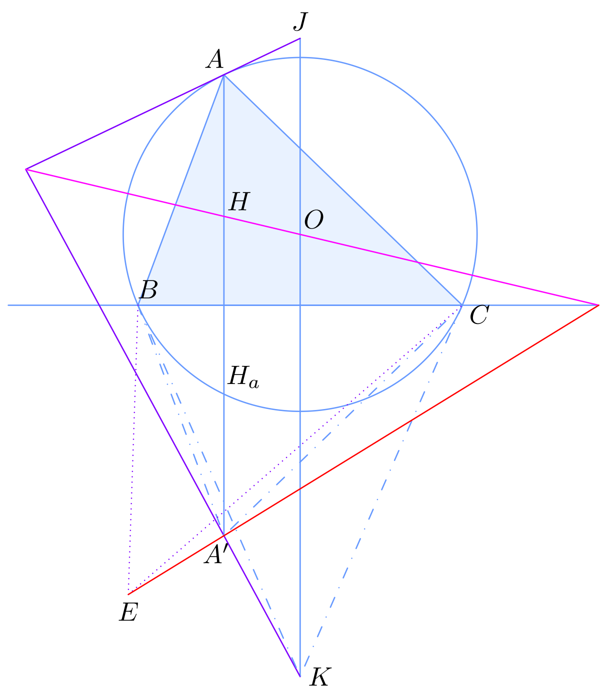
Theorem: weird concurrences

In a scalene triangle ABC with circumcenter O, circumcircle Ω, and orthocenter H.

(a) let K be the polar of $\overline{BC}$ wrt Ω, and A' be the reflection of A in $\overline{BC}$. Then $\overline{OH}, \overline{A'K}$ and the tangent to Ω at A are concurrent.

(b) Let E be the reflection of the point E_0 (such that A is the incenter or excenter of $\triangle E_0BC$) in the perpendi-cular bisector of $\overline{BC}$. Then $\overline{OH}, \overline{BC}, \overline{EA'}$ are also concurrent.

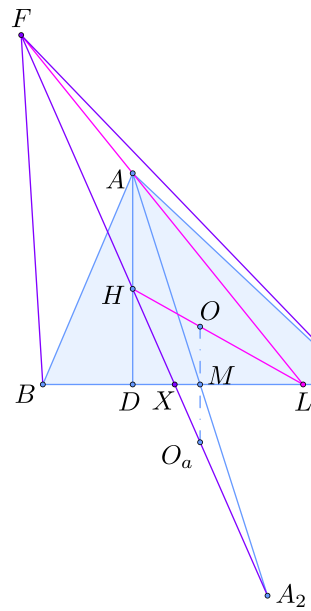
(parentheses used above for easier grammatical parsing)



Proof. These two parts actually aren't connected at all...

Part (a), by CyclicISLscelesTrapezoid Let J be the intersection of the tangent to Ω at A with the perpendicular bisector of $\overline{BC}$, and $H_a \in \Omega$ be the reflection of H in $\overline{BC}$. We contend that the triples (A, H, A'), (J, O, K) are homothetic. Indeed, they lie on parallel lines. To finish, check that (if R denotes the radius of Ω)

$$JO = \frac{R}{\cos(B-C)}, OK = \frac{R}{\cos A}, AH = 2R\cos A, HA' = AH_a = 2R\cos(B-C) \Rightarrow \frac{JO}{OK} = \frac{AH}{HA'}.$$



Part (b), by crazyeyemoody907 Let $F = B + C - E_0$, and $A_2 = B + C - A$, so that A_2 is an incenter or excenter of $\triangle FBC$. Since H is the antipode of A_2 on (BA_2C), it is another incenter / excenter. To prove that A, L, F collinear where $X = \overline{FHA_2} \cap \overline{BC}, L = \overline{OH} \cap \overline{BC}$, verify that (where $O_a \in \overline{H_aA_2}$ is the reflection of O in $\overline{BC}$)

$$(\overline{AF} \cap \overline{BC}, X; D, M) \overset{A}{=} (FX; HA_2) = -1 \text{ while } (DM; XL) \overset{H}{=} (\infty_{\perp BC}M; O_aO) = -1. \qquad \square$$

Returning to the problem, applying respective parts of the theorem to $\triangle DEF, I_a'BC$, we obtain (A, D', K) and (A, G, V) collinear. Since $R \in (UVQ), \overline{GV}$, and Q is the Miquel point of $D'GVU$, we must have $R = \overline{D'U} \cap \overline{GV}$ – an intersection of opposite sides. Hence, by definition of Miquel point, $R \in (QD'G)$.

It remains to prove that R, X, S collinear. In fact, there is a spiral similarity at Q mapping $D', X \to U, S$ since $Q \in (URS), (D'XR)$, so we're done!