Determination of the Orbital Elements for One Body under Parametrized Post Newtonian Metric

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Introduction

Although the title of this work is very general; the principal objective of this project is to determine the orbit followed by a particle around a slowly rotating weak gravitational source. By being specific, with "weak gravitational source" we mean that the central body generates a weak gravitational field, therefore is not necessary to work with the general Einstein equations and the Post-Newtonian (PN) approximation is enough to determine the motion of the particle. We work up to he 1PN order. [17, 23, 24]

In Chapter 1 we introduce the two-body problem, the perturbation theory, and the respective orbital elements for Classical Celestial Mechanics. The most relevant conclusion of this chapter is to introduce the general position in space of the orbit, determined by both the x, y, and z positions, and the time-dependent basis; finally, we give the derivative with respect to the true anomaly of the orbital elements. Continuing with the Linearized theory in Chapter 2 in which we introduce the principal characteristics of a gravitational source that can be considered as a weak field and some relevant aspects of the Energy-Momentum Tensor of a system modeled as a perfect fluid. Since we are working with a weak field, the metric is introduced as a perturbation to the Minkowski metric, and the small parameter that gives the order of expansion of the metric is also introduce. We write the Einstein Field Equations under this approximation and determine the potentials involved in these, in terms of the Energy-Momentum Tensor. The metric is expanded in terms of the potentials, and these are written in terms of the expanded Energy Momentum Tensor; all the expression are written considering the Standard Post-Newtonian gauge.[17]

The problem of one body following, the Schwarzschild metric, is developed in Chapter 3. We introduce the general form of the Lagrangian, the equations of motion under 1PN approximation and the concept of parametrize metric. The respective perturbing force and orbital elements are calculated.

The previous chapters have the purpose of introducing and provide enough resources to develop the main objective of the work. In Chapter 4 er introduce some relevant aspects of the Kerr Metric [2, 7, 18] and the Weyl Levi Civita Metric [25], with the latter determines the Newtonian potential for a non-spherically symmetric and axially symmetric body until quadrupole moment. Then we deduce the same potential but following the classical approach to determine the form of the quadrupole moment in terms of the inertia moments [8, 17]. The potentials needed to write the Post-Newtonian metric of this system are calculated considering a slowly rotating body; therefore, we only consider the first order of quadrupole moment and rotational velocity. The metric is parametrized following the one showed in [3]. We construct the perturbing force and its respective projections on the time dependent basis. The force is divided into four different types of contributions: the one corresponding to the Newtonian quadrupole, the Schwarzschild perturbation, the one due to the rotation of the central body, and the relativistic quadrupole perturbation [3]. The final part of this chapter consists in calculate the secular changes of the orbital elements. The secular changes corresponding to the Newtonian quadrupole correspond to the ones showed in the literature.

The project is concluded by showing, in the last Chapter, how the secular changes determine in the previous chapter, are applied to the system Earth + Satellite in two of the most famous types of orbit, the Geostationary and the Molniya. We plot the changes in terms of the orbital period and the 3D orbit for different angles of view.

Chapter 1

Celestial Mechanics

1.1 Two Body Problem

A two body system, as in Figure 1.1, can be described following the Newtonian formalism. This system is characterized by the newtonian potential,

$$\phi(r) = -\frac{GM}{r}, \tag{1.1}$$

where M is the total mass of the system.

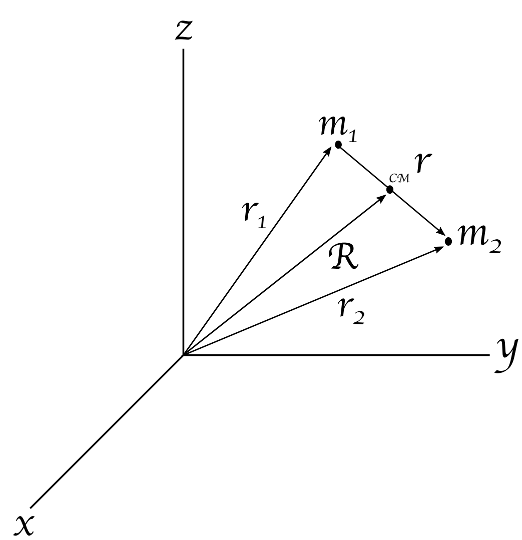

Figure 1.1: Two body system

The equation of motion is as usual

$$\ddot{\mathbf{x}} = -\frac{GM}{r^3}\mathbf{x}. \tag{1.2}$$

Note here that $\mathbf{x} = (x, y, z)$, and the equation of motion is of second order; therefore, to determine in a univocal form the orbit of the system, it is necessary to find six constants of motion.

Following the Euler-Lagrange formalism it is found that the angular momentum is a conserved quantity. The plane of motion; defined by the radial direction and the velocity, and according to the definition of angular momentum $\mathbf{L} = \mathbf{x} \times \dot{\mathbf{x}}$; is constant over time, then the motion of the system is fixed in a plane, and the angular momentum is perpendicular to that plane as shown in Figure 1.2.

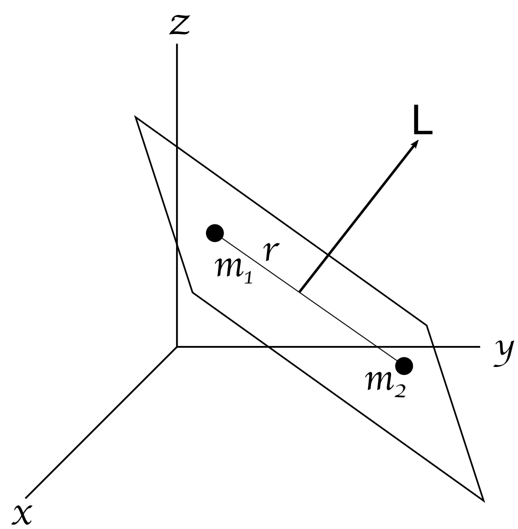

Figure 1.2: Plane of motion of the two body system

From the fact mentioned above, it is possible to introduce a new coordinate system in which the system lies on a $\mathcal{XY}$ plane and the angular momentum is on the $\mathcal{Z}$ direction, perpendicular to that plane. The motion is now described by the polar coordinates r and ϕ, where $\mathbf{r} = (r\cos\phi, r\sin\phi, 0)$. It is useful and more practical to describe the motion with a time dependent basis,

$$\mathbf{n} = (\cos\phi, \sin\phi, 0) = \frac{\mathbf{r}}{r} \tag{1.3a}$$

$$\boldsymbol{\lambda} = (-\sin\phi, \cos\phi, 0), \qquad \frac{d\mathbf{n}}{d\phi} = \boldsymbol{\lambda} \tag{1.3b}$$

$$\mathbf{k} = (0, 0, 1). \tag{1.3c}$$

The equation of the trajectory for the two body system is

$$u'' + u = \frac{GM}{\ell^2}, \qquad u(\phi) = \frac{1}{r(\phi)}, \tag{1.4}$$

where ℓ is the total angular momentum per unit mass.

The solution of this equation is the well know conic sections,

$$r(\phi) = \frac{p}{1 + e\cos(\phi + \omega)} \tag{1.5}$$

where e and ω are integration constants. Particularly, for elliptic orbits we have that

$$p = a(1 - e^2) = \frac{\ell^2}{GM}, \tag{1.6}$$

where a is a constant of integration, corresponding to the semi-major axis.

1.2 Orbital elements

A a general description of the motion of the system can be done by considering a coordinates (x, y, z), where the orbit is not longer defined in the $\mathcal{XY}$ plane. The orbit is completely defined by knowing the six orbital elements:

1. **a: semi-major axis,**
2. **e: eccentricity,**
3. **ω: argument of the pericenter,**
4. **ι: inclination,**
5. **Ω: longitude of the ascending node,**
6. **T: orbital period (in general a time constant).**

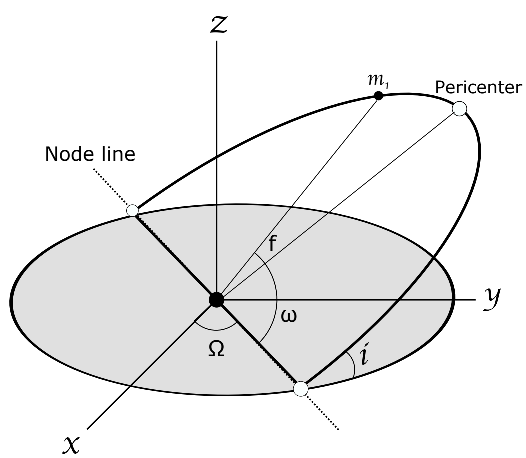

Figure 1.3: Configuration of the general coordinate system

The semi-major axis and eccentricity described the geometry of the orbit. For elliptical orbits, the eccentricity has values between 0 and 1. ω, ι and Ω describe the position of the orbit in the general system and a constant related to time.

The procedure to determine the connection between the $\mathcal{XY}$ and the general coordinates xyz is applying the following rotations:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\iota & -\sin\iota \\ 0 & \sin\iota & \cos\iota \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{Y}. \\ 0 \end{bmatrix} \tag{1.7}$$

The time dependent basis, in the general coordinate system, as shown in Figure 1.2 are:

In the radial direction, $\mathbf{n}$, where the angle ϕ in equations (1.3) is replace by the letter f, and is known as true anomaly,

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\iota & -\sin\iota \\ 0 & \sin\iota & \cos\iota \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos f \\ \sin f \\ 0 \end{bmatrix},$$

$$\begin{aligned} \mathbf{n} =& [\cos\Omega\cos(f+\omega) - \cos\iota\sin\Omega\sin(f+\omega)]\hat{\mathbf{e}}_{\mathbf{x}} \\ &+ [\sin\Omega\cos(f+\omega) + \cos\iota\cos\Omega\sin(f+\omega)]\hat{\mathbf{e}}_{\mathbf{y}} \\ &+ [\sin\iota\sin(f+\omega)]\hat{\mathbf{e}}_{\mathbf{z}}. \end{aligned} \tag{1.8}$$

The $\boldsymbol{\lambda}$ direction,

$$\begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\iota & -\sin\iota \\ 0 & \sin\iota & \cos\iota \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin f \\ \cos f \\ 0 \end{bmatrix},$$

$$\begin{aligned} \boldsymbol{\lambda} =& [-\cos\Omega\sin(f+\omega) - \cos\iota\sin\Omega\cos(f+\omega)]\hat{\mathbf{e}}_{\mathbf{x}} \\ &+ [-\sin\Omega\sin(f+\omega) + \cos\iota\cos\Omega\cos(f+\omega)]\hat{\mathbf{e}}_{\mathbf{y}} \\ &+ [\sin\iota\cos(f+\omega)]\hat{\mathbf{e}}_{\mathbf{z}}. \end{aligned} \tag{1.9}$$

The $\mathbf{k}$ direction,

$$\begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\iota & -\sin\iota \\ 0 & \sin\iota & \cos\iota \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{k} = \sin\iota\sin\Omega\hat{\mathbf{e}}_{\mathbf{x}} - \sin\iota\cos\Omega\hat{\mathbf{e}}_{\mathbf{y}} + \cos\iota\hat{\mathbf{e}}_{\mathbf{z}}. \tag{1.10}$$

The position is given by,

$$\begin{aligned} \mathbf{x} =& r[\cos\Omega\cos(f+\omega) - \cos\iota\sin\Omega\sin(f+\omega)]\hat{\mathbf{e}}_{\mathbf{x}} \\ &+ r[\sin\Omega\cos(f+\omega) + \cos\iota\cos\Omega\sin(f+\omega)]\hat{\mathbf{e}}_{\mathbf{y}} \\ &+ r[\sin\iota\sin(f+\omega)]\hat{\mathbf{e}}_{\mathbf{z}}, \end{aligned} \tag{1.11}$$

and its derivative with respect to time is

$$\begin{aligned}\dot{\mathbf{x}} = &-\sqrt{\frac{GM}{p}}[\cos\Omega(\sin(f+\omega)+e\sin\omega)+\cos\iota\sin\Omega(\cos(f+\omega)+e\cos\omega)]\hat{\mathbf{e}}_{\mathbf{x}} \\ &-\sqrt{\frac{GM}{p}}[\sin\Omega(\sin(f+\omega)+e\sin\omega)-\cos\iota\cos\Omega(\cos(f+\omega)+e\cos\omega)]\hat{\mathbf{e}}_{\mathbf{y}} \\ &-\sqrt{\frac{GM}{p}}\sin\iota[\cos(f+\omega)+e\cos\omega]\hat{\mathbf{e}}_{\mathbf{z}}.\end{aligned} \tag{1.12}$$

The gray plane in Figure 1.3, known as Reference plane, can be the Ecliptic plane (Sun at center), Equatorial plane (Earth at center) or Galactic plane. The x axis usually refers to the Υ: First point of Aries, which is the point, seen by an observer on the Earth, where the Sun crosses the equator from south to north. [19]

The orbital elements can be found with the quantities and vectors above as follows,

$$a = \frac{p}{1-e^2} \tag{1.13}$$

$$\cos\iota = \mathbf{k}\cdot\hat{\mathbf{e}}_{\mathbf{z}} \tag{1.14}$$

$$\sin\iota\sin\Omega = \mathbf{k}\cdot\hat{\mathbf{e}}_{\mathbf{x}} \tag{1.15}$$

$$\sin\iota\sin\omega = \frac{\mathbf{e}\cdot\hat{\mathbf{e}}_{\mathbf{z}}}{e}. \tag{1.16}$$

The eccentricity can be found via the Laplace - Runge -Lenz vector, defined as

$$\mathbf{e} = \frac{\dot{\mathbf{x}}\times\boldsymbol{\ell}}{GM} - \mathbf{n}. \tag{1.17}$$

1.3 Perturbation Theory

For the two body system, the orbital elements are constant over time. Introducing a perturbation to the system, such a third body, relativistic, quadrupole effects, etc., the elements are now dependent of time. A perturbing force, smaller than the newtonian force, must be added to the force given by the two body system in (1.2) [[3, 10]],

$$\ddot{\mathbf{x}} = -\frac{GM}{r^2}\mathbf{n} + \mathbf{F}. \tag{1.18}$$

The force can be decomposed in terms of the time dependent basis as

$$\mathbf{F} = \mathcal{R}\mathbf{n} + \mathcal{T}\boldsymbol{\lambda} + \mathcal{W}\mathbf{k}. \tag{1.19}$$

The first order correction to the orbital elements in terms of, the true anomaly are [3, 13]

$$\frac{da}{df} = \frac{2(1-e^2)}{n^2}\left[\frac{e\sin f}{(1+e\cos f)^2}\mathcal{R} + \frac{1}{1+e\cos f}\mathcal{T}\right] \tag{1.20a}$$

$$\frac{de}{df} = \frac{(1-e^2)^2}{n^2 a}\left[\frac{\sin f}{(1+e\cos f)^2}\mathcal{R} + \frac{2\cos f + e\left(1+\cos^2 f\right)}{(1+e\cos f)^3}\mathcal{T}\right] \tag{1.20b}$$

$$\frac{d\iota}{df} = \frac{(1-e^2)^2}{n^2 a}\frac{\cos(f+\omega)}{(1+e\cos f)^3}\mathcal{W} \tag{1.20c}$$

$$\frac{d\Omega}{df} = \frac{(1-e^2)^2}{n^2 a}\frac{\sin(f+\omega)}{(1+e\cos f)^3\sin\iota}\mathcal{W} \tag{1.20d}$$

$$\frac{d\omega}{df} = \frac{(1-e^2)^2}{n^2 a e}\left[-\frac{\cos f}{(1+e\cos f)^2}\mathcal{R} + \frac{2+e\cos f}{(1+e\cos f)^3}\sin f\mathcal{T} - e\frac{\sin(f+\omega)}{(1+e\cos f)^3}\cot\iota\mathcal{W}\right]. \tag{1.20e}$$

These derivatives can be written in terms of the coordinate time t, as shown in Chapter 3.

The changes in the orbital elements allow us to describe the orbit in which the system is moving. However, some of these changes are periodic, meaning that they average to zero after some period of time. On the other hand, other changes are not periodic, so they accumulate after each revolution and can produce considerable changes in the motion of the particles. These non-periodic contributions are called secular changes, and can be found by integrating over time, or over the true anomaly,

$$\Delta\vartheta = \int_0^{2\pi}\frac{d\vartheta}{df}df. \tag{1.21}$$

Chapter 2

Linearized gravity

2.1 Weak fields

The Einstein field equations are not linear. The physical reason is that the gravitational field of a body contains energy, therefore a mass and because of general relativity, creates a gravitational field; in conclusion, the field itself is in part, its own source and a feedback effect is created. The analysis of the gravitational field produced by many bodies is not the sum of the field produce by each body. If the body produces a weak gravitational field, the feedback effect can be ignored, and it is possible to make the Einstein field equations linear and easy to solve [1].

The strong gravitational field, such as the produced by black holes, is the one considered by the Einstein field equations. Nevertheless, the gravitational field produced by stars like the Sun, neutron stars and other astrophyisical objects are weak [17]. This characteristic allow us the study of the Einstein field equations and other relevant expressions for the general relativity as approximations. These expressions may describe the behavior of Newtonian gravity by imposing an asymptotic limit. When the consideration of the field is made, it is possible to linearize the field equations by means of introducing a perturbation to the metric Minkowski tensor [13].

Furthermore, most of the physic systems can be modeled as perfect fluids. An observer moving with the same velocity as the system (or fluid), sees this one as isotropic; it is said that the fluid can be described by its mass density and its isotropic pressure. On a perfect fluid, there is no shear stress. By means of the energy-momentum tensor $T_{\mu\nu}$, suppose the fluid is at some space-time point, the energy-momentum tensor in the observer's frame of reference can be written as

$$\tilde{T}^{ij} = p\delta_{ij} \tag{2.1}$$

$$\tilde{T}^{i0} = \tilde{T}^{0j} = 0 \tag{2.2}$$

$$\tilde{T}^{00} = \rho, \tag{2.3}$$

where p is the pressure [23].

Now suppose that the observer is in a reference frame at rest, by means of Lorentz transformation, the energy momentum tensor is [23]

$$T^{\alpha\beta} = \Lambda^{\alpha}_{\gamma}(\vec{v})\Lambda^{\beta}_{\delta}(\vec{v})\tilde{T}^{\gamma\delta} \tag{2.4}$$
$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+\rho)U^{\alpha}U^{\beta} \tag{2.5}$$

The four-velocity can be written as

$$U^{\alpha} = \gamma(c, \mathbf{v}) \tag{2.6}$$

With $|\mathbf{v}| << c$, the velocity of the gravitational source and $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

The systems are considered to be composed of particles moving at small velocities. Therefore, these are non-interacting particles, they are in rest with respect to each other. A small velocity means zero pressure for a perfect fluid. In terms of the energy momentum tensor,

$$T^{\alpha\beta} = \rho_0 U^{\alpha}U^{\beta}, \tag{2.7}$$

where ρ_0 is the rest mass density of the system.
Because of the velocity condition $v << c$, the velocity and energy momentum tensor, given in (2.7), can be written as

$$\begin{aligned} U^{\alpha} &= \gamma(c, \mathbf{v}) && (2.8)\\ &= \left(1 - \frac{v^2}{c^2}\right)^{-1/2} (c, \mathbf{v}) \\ &= (1 + \mathcal{O}(\epsilon^2))(c, \mathbf{v}) \\ &= (c, \mathbf{v}) + \mathcal{O}(\epsilon^2). && (2.9) \end{aligned}$$

The components of the energy momentum tensor are

$$T^{\alpha\beta} = \rho_0 U^{\alpha}U^{\beta} \tag{2.10a}$$
$$T^{00} = \rho_0 U^0 U^0 = \rho_0 c^2\gamma^2 = \rho_0 c^2 + \mathcal{O}(\epsilon^2) \tag{2.10b}$$
$$T^{0i} = \rho_0 U^0 U^i = \rho_0 c v^i \gamma^2 = \rho_0 c v^i + \mathcal{O}(\epsilon^3) \tag{2.10c}$$
$$T^{ij} = \rho_0 U^i U^j = \rho_0 v^i v^j \gamma^2 = \rho_0 v^i v^j + \mathcal{O}(\epsilon^4) \tag{2.10d}$$

2.2 Linearized theory

It is necessary to introduce a small parameter that gives the order of magnitude of all of the physical quantities. For the case of systems under the influence of gravitational forces, the small parameter is given by

$$\left(\frac{\phi}{c^2}\right)^{1/2} \sim \frac{v}{c} \sim \epsilon, \tag{2.11}$$

where ϕ is the Newtonian potential, c the velocity of light and v is the velocity of the particles under the influence of the gravitational source. ϵ is the perturbative order of the metric expansion.

When the gravitational field is weak, it is posible to introduce a small parameter, a perturbation to the Minkwoski metric tensor. In this way, the metric is approximately flat,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.12}$$
$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(\mathrm{h}^2), \tag{2.13}$$

where $g_{\mu\nu}$ is the metric tensor of general relativity, $\eta_{\mu\nu}$ is the metric for Minkoswki space, given by $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and $h_{\mu\nu}$ is the perturbation. The last one has the same symmetry properties as $\eta_{\mu\nu}$.

All the equations relevant to general relativity will be rewritten in term of the new-perturbed metric.

Connections

The connections are given by (2.14),

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\delta}(\partial_\nu g_{\delta\mu} + \partial_\mu g_{\delta\nu} - \partial_\delta g_{\mu\nu}). \tag{2.14}$$

Replacing (2.12) and neglecting terms of order h^2 or higher, gives

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} \eta^{\alpha\delta}(\partial_\nu h_{\delta\mu} + \partial_\mu h_{\delta\nu} - \partial_\delta h_{\mu\nu}) + \mathcal{O}(\mathrm{h}^2). \tag{2.15}$$

Riemann tensor

The Riemann tensor is given by (2.16),

$$R^{\alpha}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\beta}. \tag{2.16}$$

Replacing (2.12) and neglecting terms of order h^2 or higher, gives

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}\left(\partial_\mu\partial_\beta h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\mu\beta} - \partial_\mu\partial_\nu h_{\alpha\beta} - \partial_\alpha\partial_\beta h_{\mu\nu}\right) + \mathcal{O}(\mathrm{h}^2). \tag{2.17}$$

Ricci tensor

The Ricci tensor is given by (2.18),

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \tag{2.18}$$

Replacing (2.12) and neglecting terms of order h^2 or higher, gives

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \partial_\beta h^\beta_\nu + \partial_\alpha \partial_\nu h^\alpha_\mu - \partial_\mu \partial_\nu h^\alpha_\alpha - \eta^{\alpha\beta}\partial_\alpha \partial_\beta h_{\mu\nu}\right) + \mathcal{O}(\mathrm{h}^2) \tag{2.19}$$

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \partial_\beta h^\beta_\nu + \partial_\alpha \partial_\nu h^\alpha_\mu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu}\right) + \mathcal{O}(\mathrm{h}^2) \tag{2.20}$$

At this approximation order, the trace h is

$$h = h^\alpha_\alpha = \eta^{\alpha\beta} h_{\alpha\beta} \tag{2.21}$$

and the D'Alambert operator is

$$\Box = \eta^{\alpha\beta}\partial_\alpha \partial_\beta. \tag{2.22}$$

Curvature scalar

Following the same procedure, described for the previous quantities, the curvature scalar gives

$$R = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h + \mathcal{O}(\mathrm{h}^2). \tag{2.23}$$

2.2.1 Metric tensor of the linearized Field equations: First Post Newtonian Metric

The Einstein field tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{2.24}$$

With the results obtained, $g_{\mu\nu}$ given by (2.12), $R_{\mu\nu}$ given by (2.20) and R given by (2.23), it is found that

$$\frac{1}{2}(\partial_\sigma \partial_\mu h^\sigma_\nu + \partial_\sigma \partial_\nu h^\sigma_\mu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\mu \partial_\nu h^{\mu\nu} + \eta_{\mu\nu}\Box h) + \mathcal{O}(\epsilon^2) = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.25}$$

Now, we define the *trace-reversed* function for $h_{\mu\nu}$,

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{2.26}$$

Its principal characteristic is the property of the traces.

$$\overline{h} = -h. \tag{2.27}$$

The Einstein field equations are

$$-\frac{1}{2}(\Box \overline{h}_{\mu\nu} - \partial_\mu \partial_\sigma \overline{h}^\sigma_\nu - \partial_\nu \partial_\sigma \overline{h}^\sigma_\mu + \eta_{\mu\nu}\partial_\sigma \partial_\rho \overline{h}^{\sigma\rho}) + \mathcal{O}(\mathrm{h}^2) = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{2.28}$$

The *trace-reversed* function allows to use the Lorentz *gauge*

$$\partial_\nu \overline{h}^{\mu\nu} = 0. \tag{2.29}$$

Replacing the *gauge*, the field equations, given in (2.28), are reduced to

$$\Box \overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \tag{2.30}$$

where $\Box$ is known as D'Alambert operator, and was given in (2.22),

$$\Box = -\partial_t^2 + \nabla^2. \tag{2.31}$$

Taking into account the parameter ϵ introduced in (2.11),

$$\frac{\partial_0}{c} \sim \frac{1}{R}\frac{v}{c} \sim \frac{\epsilon}{R}, \qquad \partial_j \sim \frac{1}{R}, \tag{2.32}$$

The D' Alambertian can be written as

$$\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \sim (-\epsilon^2 \partial_j^2 + \partial_j^2) \tag{2.33}$$

$$\Box = (1 - \mathcal{O}(\epsilon^2))\nabla^2. \tag{2.34}$$

At this approximative order, it is found that

$$\Box = \nabla^2 \tag{2.35}$$

Dropping $\mathcal{O}(\epsilon^2)$ and higher orders, the field equations are reduced to

$$\nabla^2 \overline{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \tag{2.36}$$

The components of the energy momentum tensor $T_{\mu\nu}$ may be written as in (2.10d), (2.10c), (2.10b) and (2.10a).

The solution of these equations are

$$\overline{h}_{00} = -\frac{4}{c^2}\phi + \mathcal{O}(\epsilon^4) \tag{2.37a}$$

$$\overline{h}_{0j} = \frac{4}{c^3}\zeta^j + \mathcal{O}(\epsilon^5) \tag{2.37b}$$

$$\overline{h}_{jk} = \mathcal{O}(\epsilon^2), \tag{2.37c}$$

where

$$\phi = \frac{-G}{c^2}\int \frac{T^{00}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' \tag{2.38a}$$

$$\zeta^j = \frac{-G}{c^2}\int \frac{T^{0j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \tag{2.38b}$$

These potentials satisfy the Poisson equations

$$\nabla^2\phi = 4\pi G\rho \tag{2.39}$$

$$\nabla^2\zeta^j = 4\pi G v^j. \tag{2.40}$$

With the inverse of the *trace-reversed* function given in (2.26), it is found that the $h_{\mu\nu}$ elements of the metric, according to (2.12), are

$$h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.41a}$$

$$h_{00} = -\frac{2}{c^2}\phi + \mathcal{O}(\epsilon^4) \tag{2.41b}$$

$$h_{0j} = \frac{4}{c^3}\zeta^j + \mathcal{O}(\epsilon^5) \tag{2.41c}$$

$$h_{jk} = -\frac{2}{c^2}\phi\delta_{jk} + \mathcal{O}(\epsilon^4). \tag{2.41d}$$

The metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.42a}$$

$$g_{00} = -1 - \frac{2}{c^2}\phi + \mathcal{O}(\epsilon^4) \tag{2.42b}$$

$$g_{0j} = \frac{4}{c^3}\zeta^j + \mathcal{O}(\epsilon^5) \tag{2.42c}$$

$$g_{jk} = \left(1 - \frac{2}{c^2}\phi\right)\delta_{jk} + \mathcal{O}(\epsilon^4). \tag{2.42d}$$

The metric tensor can be expanded to any approximation order, the cut off depends on the system that we want to described. Here, the idea is to analyze the interaction between a test particle and a gravitational source due to effects different from the Newtonian influence. The first term that induces a perturbation to the newtonian case, will give the cut off of the expansion.

The Lagrangian of a test particle of proper mass m_0 and proper time τ is

$$L = -m_0 c\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}. \tag{2.43}$$

Replacing the metric up to second order, given in (2.42a), the Lagrangian becomes

$$L = -m_0c^2 + \frac{1}{2}m_0 v^2 + m_0\phi + \mathcal{O}(\epsilon^3). \tag{2.44}$$

This Lagrangian shows only the newtonian case, therefore the metric until second order is not enough for the analysis and
higher perturbative orders are needed. The metric, up to second order, corresponds to the zero Post-Newtonian approximation, up to fourth order corresponds to first Post-Newtonian approximation and so on. The correct perturbative order for each term of the metric is given

	0PN	1PN	2PN
g_{00}	ϵ^2	ϵ^4	ϵ^6
g_{0j}	ϵ^1	ϵ^3	ϵ^5
g_{jk}	ϵ^0	ϵ^2	ϵ^4

Table 2.1: Post Newtonian approximation

in the Table (2.1).

For the purpose of this work, it is considered only the first Post-Newtonian approximation. The metric tensor, up to this order, can be written as

$$g_{00} = -1 + {}^{(2)}h_{00} + {}^{(4)}h_{00} + \mathcal{O}(\epsilon^6) \tag{2.45a}$$

$$g_{0j} = {}^{(3)}h_{0j} + \mathcal{O}(\epsilon^5) \tag{2.45b}$$

$$g_{jk} = \delta_{jk} + {}^{(2)}h_{jk} + \mathcal{O}(\epsilon^4). \tag{2.45c}$$

It is important to note that the terms g_{00} and g_{jk} only involve even order, and g_{0j} only odd order. The reason of this behavior lies in the fact that the terms g_{0j} must change sign under a $t \to -t$ transformation [23].

2.2.2 The solution of the Einstein field equations up to first Post-Newtonian approximation

It is possible to rewrite the connections given in (2.15) and the Ricci tensor given in (2.20) in terms of the metric tensor given in (2.45a), (2.45b) and (2.45c). The final purpose of rewriting these quantities in terms of the Post-Newtonian approximation is to find the field equations.

Taking into account the order of the derivatives given in (2.32), the connections up to 1PN order are

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}) \tag{2.46a}$$

$$\Gamma^0_{00} = {}^{(3)}\,\Gamma^0_{00} + \mathcal{O}(\epsilon^3/R) \tag{2.46b}$$

$$\Gamma^0_{0j} = {}^{(2)}\,\Gamma^0_{0j} + \mathcal{O}(\epsilon^4/R) \tag{2.46c}$$

$$\Gamma^0_{jk} = {}^{(3)}\,\Gamma^0_{jk} + \mathcal{O}(\epsilon^5/R) \tag{2.46d}$$

$$\Gamma^i_{00} = {}^{(2)}\,\Gamma^i_{00} + {}^{(4)}\,\Gamma^i_{00} + \mathcal{O}(\epsilon^6/R) \tag{2.46e}$$

$$\Gamma^i_{0j} = {}^{(3)}\,\Gamma^i_{0j} + \mathcal{O}(\epsilon^5/R) \tag{2.46f}$$

$$\Gamma^i_{jk} = {}^{(2)}\,\Gamma^i_{jk} + \mathcal{O}(\epsilon^4/R). \tag{2.46g}$$

Each term can be written as function of $h_{\mu\nu}$,

$$^{(3)}\Gamma^0_{00} = -\frac{1}{2}\partial_0^{(2)} h_{00} \tag{2.47a}$$
$$^{(2)}\Gamma^0_{0j} = -\frac{1}{2}\partial_j^{(2)} h_{00} \tag{2.47b}$$
$$^{(3)}\Gamma^0_{jk} = -\frac{1}{2}(\partial_j^{(3)} h_{0k} + \partial_k^{(3)} h_{0j} - \partial_0^{(2)} h_{jk}) \tag{2.47c}$$
$$^{(2)}\Gamma^i_{00} = -\frac{1}{2}\partial_i^{(2)} h_{00} \tag{2.47d}$$
$$^{(4)}\Gamma^i_{00} = -\frac{1}{2}\partial_i^{(4)} h_{00} + \partial_0^{(3)} h_{0i} - \frac{1}{2}(\partial_j^{(2)} h_{00})^{(2)} h_{ij} \tag{2.47e}$$
$$^{(3)}\Gamma^i_{0j} = \frac{1}{2}(\partial_0^{(2)} h_{ij} + \partial_j^{(3)} h_{0i} - \partial_i^{(3)} h_{0j}) \tag{2.47f}$$
$$^{(2)}\Gamma^i_{jk} = \frac{1}{2}(\partial_k^{(2)} h_{ij} + \partial_j^{(2)} h_{ik} - \partial_i^{(2)} h_{jk}). \tag{2.47g}$$

The Ricci tensor, can be expanded as

$$R_{00} =^{(2)} R_{00} +^{(4)} R_{00} + \mathcal{O}(\epsilon^6) \tag{2.48a}$$
$$R_{0j} =^{(3)} R_{0j} +^{(5)} R_{0j} + \mathcal{O}(\epsilon^7) \tag{2.48b}$$
$$R_{jk} =^{(2)} R_{jk} +^{(4)} R_{jk} + \mathcal{O}(\epsilon^6). \tag{2.48c}$$

Written as function of the connections terms, given in (2.46),

$$^{(2)}R_{00} = \partial_i^{(2)}\Gamma^i_{00} \tag{2.49a}$$
$$^{(4)}R_{00} = \partial_i^{(4)}\Gamma^i_{00} - \partial_0^{(3)}\Gamma^i_{0i} +^{(2)} \Gamma^i_{00}{}^{(2)}\Gamma^j_{ij} +^{(2)} \Gamma^0_{0i}{}^{(2)}\Gamma^i_{00} \tag{2.49b}$$
$$^{(3)}R_{0j} = \partial_0^{(2)}\Gamma^0_{0i} + \partial_j^{(3)}\Gamma^j_{0i} - \partial_i^{(3)}\Gamma^0_{00} - \partial_i^{(3)}\Gamma^j_{0j} \tag{2.49c}$$
$$^{(2)}R_{jk} = \partial_k^{(2)}\Gamma^k_{ij} - \partial_j^{(2)}\Gamma^0_{0i} - \partial_j^{(2)}\Gamma^k_{ik}. \tag{2.49d}$$

For the purpose of this work, the Standard Post-Newtonian gauge is used[1]. This gauge is like the Harmonic gauge, $g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$, but considering $\alpha = 0$ with the condition of null order $\mathcal{O}(\epsilon^3)$ and $\alpha = i$ and null order $\mathcal{O}(\epsilon^2)$, i.e.

$$\partial_j^{(3)} h_{0j} - \frac{1}{2}\partial_0^{(2)} h_{jj} = 0 \tag{2.50a}$$
$$\frac{1}{2}\partial_i^{(2)} h_{00} + \partial_j^{(2)} h_{ij} - \frac{1}{2}\partial_i^{(2)} h_{jj} = 0. \tag{2.50b}$$

[1]The election of this gauge has no physical meaning; it is just for convenience [24].

The Ricci tensor can be rewritten with the use of the Standard Post-Newtonian Gauge as

$$
\begin{aligned}
{}^{(2)}R_{00} &= -\frac{1}{2}\nabla^2[{}^{(2)}h_{00}] && (2.51a)\\
{}^{(4)}R_{00} &= -\frac{1}{2}\nabla^2[{}^{(4)}h_{00}] + \frac{1}{2}\partial_j^{(2)}h_{ij}\partial_i^{(2)}h_{00} + \frac{1}{2}{}^{(2)}h_{ij}\partial_i\partial_j^{(2)}h_{00} && (2.51b)\\
&\quad - \frac{1}{4}\partial_i^{(2)}h_{00}\partial_i^{(2)}h_{jj} - \frac{1}{4}\partial_i^{(2)}h_{00}\partial_i^{(2)}h_{00} &&\\
{}^{(3)}R_{0j} &= \frac{1}{2}\nabla^2[{}^{(3)}h_{0i}] + \frac{1}{2}\partial_0\partial_j^{(2)}h_{ij} - \frac{1}{4}\partial_0\partial_i^{(2)}h_{jj} && (2.51c)\\
{}^{(2)}R_{jk} &= -\frac{1}{2}\nabla^2[{}^{(2)}h_{ij}]. && (2.51d)
\end{aligned}
$$

The only element that is missing to rewrite the Einstein field equations is the energy-momentum tensor. Expanding as the Ricci tensor, we have

$$
\begin{aligned}
T_{00} &=^{(0)} T_{00} +^{(2)} T_{00} + \mathcal{O}(\epsilon^4) && (2.52a)\\
T_{0j} &=^{(1)} T_{0j} +^{(3)} T_{0j} + \mathcal{O}(\epsilon^5) && (2.52b)\\
T_{jk} &=^{(2)} T_{jk} +^{(4)} T_{jk} + \mathcal{O}(\epsilon^6). && (2.52c)
\end{aligned}
$$

The Einstein tensor can be rewritten as

$$
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{2.53}
$$

Taking the trace $R = g^{\mu\nu}R_{\mu\nu}$ y $T = g^{\mu\nu}T_{\mu\nu}$,

$$
R = -\left(\frac{8\pi G}{c^4}\right)T \tag{2.54}
$$

and replacing,

$$
\begin{aligned}
R_{\mu\nu} &= \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) && (2.55)\\
R_{\mu\nu} &= \frac{8\pi G}{c^4}S_{\mu\nu}. && (2.56)
\end{aligned}
$$

$S_{\mu\nu}$ can be expanded as $T_{\mu\nu}$ and Ricci tensor,

$$
\begin{aligned}
S_{00} &=^{(0)} S_{00} +^{(2)} S_{00} + \mathcal{O}(\epsilon^4) && (2.57a)\\
S_{0j} &=^{(1)} S_{0j} +^{(3)} S_{0j} + \mathcal{O}(\epsilon^5) && (2.57b)\\
S_{jk} &=^{(2)} S_{jk} +^{(4)} S_{jk} + \mathcal{O}(\epsilon^6). && (2.57c)
\end{aligned}
$$

Each term of $S_{\mu\nu}$ must be written with the energy momentum tensor

$$
S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T, \tag{2.58}
$$

where T is the trace of the energy momentum tensor, given by

$$T = -^{(0)}T^{00} -^{(2)} T^{00} +^{(2)} h_{00}T^{00} + \delta_{jk}T^{jk} + \mathcal{O}(\epsilon^4). \tag{2.59}$$

Finally

$$^{(0)}S_{00} = \frac{1}{2}{}^{(0)}T^{00} \tag{2.60a}$$

$$^{(2)}S_{00} = \frac{1}{2}\left({}^{(2)}T^{00} - 2^{(2)}h_{00}{}^{(0)}T^{00} + \delta_{ij}{}^{(2)}T^{ij}\right) \tag{2.60b}$$

$$^{(1)}S_{0j} = -^{(1)}T^{0j} \tag{2.60c}$$

$$^{(0)}S_{ij} = \frac{1}{2}\delta_{ij}{}^{(0)}T^{00}. \tag{2.60d}$$

Taking into account the form of $S_{\mu\nu}$ given in (2.58) and the order for the energy momentum tensor given in (2.10d)(2.10c)(2.10b)(2.10a), the field equations can be written as

$$^{(2)}R_{00} = \frac{8\pi G}{c^4}{}^{(0)}S_{00} \tag{2.61a}$$

$$^{(4)}R_{00} = \frac{8\pi G}{c^4}{}^{(2)}S_{00} \tag{2.61b}$$

$$^{(3)}R_{0i} = \frac{8\pi G}{c^4}{}^{(1)}S_{0i} \tag{2.61c}$$

$$^{(2)}R_{ij} = \frac{8\pi G}{c^4}{}^{(0)}S_{ij}. \tag{2.61d}$$

Combining the Ricci tensor after the use of the Standard post-Newtonian gauge given in (2.51a) and the $S_{\mu\nu}$ given in (2.60a) on the equations given above,

$$^{(2)}R_{00} : \nabla^2[{}^{(2)}h_{00}] = -\frac{8\pi G}{c^4}{}^{(0)}T^{\,00}. \tag{2.62}$$

Now, the way to resolve this differential equation is using a Green function. First, suppose that $^{(2)}h_{00}$ can be written as

$$^{(2)}h_{00} = \int \mathcal{G}(\mathbf{x}, \mathbf{x}')\left(-\frac{8\pi G}{c^4}{}^{(0)}T^{\,00}(t, \mathbf{x}')\right) d^{\,3}x'. \tag{2.63}$$

Applying ∇^2, gives

$$\nabla^2[{}^{(2)}h_{00}] = -\frac{8\pi G}{c^4}\int \nabla^2\mathcal{G}(\mathbf{x}, \mathbf{x}'){}^{(0)}T^{\,00}(t, \mathbf{x}')d^{\,3}x'. \tag{2.64}$$

For fullfilling the equation (2.62), ∇^2 applied to the Green function $\mathcal{G}(\mathbf{x}, \mathbf{x}')$ must give

$$\nabla^2\mathcal{G}(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'). \tag{2.65}$$

Solving, we get

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}. \tag{2.66}$$

The second order term to the tensor $h_{\mu\nu}$ is

$$^{(2)}h_{00} = \frac{2G}{c^4} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} {}^{(0)}T^{00}(t, \mathbf{x}') d^3x' \tag{2.67}$$

$$^{(2)}h_{00} = -\frac{2\phi}{c^2}. \tag{2.68}$$

where ϕ is the newtonian potential

$$\phi = -\frac{G}{c^2} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} {}^{(0)}T^{00}(t, \mathbf{x}') d^3x'. \tag{2.69}$$

Following the same procedure,

$$^{(4)}R_{00} : \nabla^2 \left[{}^{(4)}h_{00} + \frac{2}{c^4}\phi^2 \right] = -\frac{8\pi G}{c^4}[{}^{(2)}T^{00} +^{(2)} T^{jj}]. \tag{2.70}$$

Rewritten the left side of the equation by defining

$$^{(4)}h_{00} = -\frac{2}{c^2} \left(\frac{2}{c^2}\phi^2 + \psi \right), \tag{2.71}$$

we get

$$\nabla^2 \left[-\frac{2\psi}{c^2} \right] = -\frac{8\pi G}{c^4}[{}^{(2)}T^{00} +^{(2)} T^{jj}] \tag{2.72}$$

$$\nabla^2 \psi = \frac{4\pi G}{c^2}[{}^{(2)}T^{00} +^{(2)} T^{jj}], \tag{2.73}$$

where ψ is the potential

$$\psi = -\frac{G}{c^2} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|}[{}^{(2)}T^{00}(t, \mathbf{x}') +^{(2)} T^{jj}(t, \mathbf{x}')] d^3x'. \tag{2.74}$$

The next term is

$$^{(3)}R_{0i} : \nabla^2[{}^{(3)}h_{0i}] = \frac{16\pi G}{c^4} {}^{(1)}T^{0i} + \frac{1}{c^2}\partial_0\partial_i\phi. \tag{2.75}$$

Defining

$$\nabla^2 \zeta_i = \frac{4\pi G}{c} {}^{(1)}T^{0i} \tag{2.76}$$

$$\nabla^2 \chi = \frac{\phi}{c^2}, \tag{2.77}$$

$$\tag{2.78}$$

we get

$$
{}^{(3)}h_{0i} = \frac{4}{c^3}\zeta_i + \partial_0\partial_i\chi. \tag{2.79}
$$

Solving to find the potential ζ_i gives

$$
\zeta_i = -\frac{G}{c}\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}{}^{(1)}T^{\,0i}(t,\mathbf{x}')d^{\,3}x' \tag{2.80}
$$

and the superpotential χ

$$
\chi = -\frac{G}{2c^4}\int |\mathbf{x}-\mathbf{x}'|{}^{(0)}T^{\,00}(t,\mathbf{x}')d^{\,3}x'. \tag{2.81}
$$

χ is called super-potential because it depends on the Newtonian potential ϕ. The relation is defined through (2.77).

The term for ${}^{(2)}h_{ij}$ gives

$$
{}^{(2)}R_{ij} : \nabla^2[{}^{(2)}h_{ij}] = -\frac{8\pi G}{c^4}\delta_{ij}{}^{(0)}T^{\,00} \tag{2.82}
$$

$$
{}^{(2)}h_{ij} = -\frac{2}{c^2}\delta_{ij}\phi, \tag{2.83}
$$

$$
\phi = -\frac{G}{c^2}\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}{}^{(0)}T^{\,00}(t,\mathbf{x}')d^{\,3}x'. \tag{2.84}
$$

The metric given in (2.45c), (2.45b) and (2.45a) can be rewritten, using some of the results from above, given by equations (2.68), (2.71), (2.79) and (2.84). The following metric goes a little bit further than the given in (2.42a),

$$
g_{00} = -1 - \frac{2\phi}{c^2} - \frac{2}{c^2}\left(\frac{\phi^{\,2}}{c^2} + \psi\right) + \mathcal{O}(\epsilon^6) \tag{2.85a}
$$

$$
g_{0j} = \frac{4}{c^3}\zeta^j + \partial_0\partial_i\chi + \mathcal{O}(\epsilon^5) \tag{2.85b}
$$

$$
g_{jk} = \left(1 - \frac{2\phi}{c^2}\right)\delta_{jk} + \mathcal{O}(\epsilon^4). \tag{2.85c}
$$

Chapter 3

Static One Body Problem

The metric given in (2.85) is the general form of the Post-Newtonian metric, nevertheless a even more general metric is desired. The called Parametrized Post - Newtonian (PPN) is written in terms of a set of parameters can describe more than one theory.

The final purpose of this Chapter is to construct and determine the equation of motions for one static body system. The field produced by this body must meet with the conditions of a weak field, introduced in the Chapter 2. For the construction of these equations, the PPN metric is used.

3.1 Parametrized Metric

Let consider first the Schwarzschild metric. This is the solution of the Einstein field equations outside a spherically symmetric and static body, meaning the solution is independent of the time and invariant under the reflection $t \rightarrow -t$,

$$ds^2 = -\left(1 - \frac{2m}{r}\right) c^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1}$$

in geometric units, where $G = 1$, $c = 1$, and the mass is

$$m = \frac{GM}{c^2}. \tag{3.2}$$

All the results until now are based on the use of the Standard Post-Newtonian gauge, introduced in Chapter 2 in equation (2.50a),

$$\partial_\mu(\sqrt{-g} g^{\mu\nu}) = 0 \rightarrow g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0. \tag{3.3}$$

These conditions imply the use of a coordinate system that meets the following condition,

$$\Box z^\beta = 0 \tag{3.4}$$

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta z^\beta = 0. \tag{3.5}$$

For a scalar field, $\nabla_\beta z^\beta = \partial_\beta z^\beta$, and then

$$g^{\alpha\beta}\nabla_\alpha \partial_\beta z^\beta = \nabla_\alpha (g^{\alpha\beta}\partial_\beta z^\beta). \tag{3.6}$$

A covariant derivative is defined as

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\alpha} V^\alpha,$$

where (See [23])

$$\begin{aligned}
\Gamma^\mu_{\mu\alpha} &= \frac{1}{2} g^{\mu\kappa} \left(\partial_\alpha g_{\kappa\mu} + \partial_\mu g_{\kappa\alpha} - \partial_\kappa g_{\mu\alpha}\right) \\
&= \frac{1}{2} g^{\mu\kappa} \left(\partial_\alpha g_{\kappa\mu}\right) \\
&= \frac{1}{2} \partial_\alpha (lng) \\
&= \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g}).
\end{aligned}$$

Replacing this expression into $\nabla_\mu V^\mu$,

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g}) V^\alpha = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} V^\alpha),$$

replacing these expressions into (3.6)

$$\nabla_\alpha (g^{\alpha\beta} \partial_\beta z^\beta) = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{\alpha\beta} \partial_\beta z^\beta) = 0. \tag{3.7}$$

For $z^\beta = (ct, x, y, z)$,

$$\partial_\alpha (\sqrt{-g} g^{\alpha\beta}) = 0. \tag{3.8}$$

These type of coordinates are named quasi-harmonic, "quasi" because there are not exactly inertial coordinates due to the fact that it defines the position of a particle around a gravitational source.

The Schwarzschild metric given in (3.1) can be rewritten in terms of harmonic coordinates. Making a linear transformation of the r component, where r_h means for harmonic,

$$r = r_h + m,$$

We obtain,

$$ds^2 = -\left(\frac{1 - \frac{m}{r_h}}{1 + \frac{m}{r_h}}\right) c^2 dt^2 + \left(\frac{1 + \frac{m}{r_h}}{1 - \frac{m}{r_h}}\right) dr_h^2 + r_h^2 \left(1 + \frac{m}{r_h}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2).$$

In cartesian coordinates, this is

$$ds^2 = -\left(\frac{1 - \frac{m}{r_h}}{1 + \frac{m}{r_h}}\right) c^2 dt^2 + \left(1 + \frac{m}{r_h}\right)^2 \left[\delta_{ij} + \frac{\left(\frac{m}{r_h}\right)^2}{1 - \left(\frac{m}{r_h}\right)^2} \frac{x_i x_j}{r_h^2}\right] dx^i dx^j. \tag{3.9}$$

Making the approximation $\epsilon^2 \sim \frac{m}{r_h} << 1$,

$$\begin{aligned} ds^2 = &-\left[1 - \frac{2m}{r_h} + \frac{2m^2}{r_h^2} + \mathcal{O}(\epsilon^6)\right] c^2 dt^2 \\ &+ \left[\delta_{ij} + \frac{2m}{r_h}\delta_{ij} + \frac{m^2}{r_h^2}\left(\delta_{ij} + \frac{x_i x_j}{r_h^2}\right) + \mathcal{O}(\epsilon^6)\right] dx^i dx^j. \end{aligned} \tag{3.10}$$

As it was mentioned above, the metric given below is rewritten as a PPN metric [3],

$$g_{00} = -\left[1 - 2\kappa\frac{m}{r_h} + 2(\beta - \alpha)\frac{m^2}{r_h^2} + \mathcal{O}(\epsilon^6)\right] \tag{3.11a}$$

$$g_{0j} = 0 \tag{3.11b}$$

$$g_{ij} = \left[\delta_{ij} + \frac{2m}{r_h}\left[(\gamma - \alpha)\delta_{ij} + \alpha\frac{x_i x_j}{r_h^2}\right] + \mathcal{O}(\epsilon^4)\right], \tag{3.11c}$$

where [17],

- κ: Measures the newtonian contribution
- β y α: Measures the non-linear contribution
- γ: Measures the curvature because of the body that produces the gravitational field.

The action describing a test particle of mass m_0 [17] is

$$\mathcal{S} = -\int_{t_1}^{t_2} L dt = -m_0 c^2 \int_{t_1}^{t_2} \frac{d\tau}{dt} dt,$$

where τ is the proper time of the particle and t is the time of an asymptotic observer.

$$\begin{aligned} ds^2 &= -c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \\ \left(\frac{d\tau}{dt}\right)^2 &= -g_{\mu\nu}\frac{\dot{x}^\mu \dot{x}^\nu}{c^2}. \end{aligned}$$

The metric tensor $g_{\mu\nu}$ is

$$g_{00} = -1 + {}^{(2)}h_{00} + {}^{(4)}h_{00} + \mathcal{O}(\epsilon^6) \tag{3.12a}$$

$$g_{0j} = {}^{(3)}h_{0j} + \mathcal{O}(\epsilon^5) \tag{3.12b}$$

$$g_{jk} = \delta_{jk} + {}^{(2)}h_{jk} + \mathcal{O}(\epsilon^4). \tag{3.12c}$$

Replacing above,

$$\left(\frac{d\tau}{dt}\right)^2 = 1 -^{(2)} h_{00} -^{(4)} h_{00} - 2^{(3)}h_{0j}\frac{\dot{x}^0 \dot{x}^j}{c^2} -^{(2)} h_{ij}\frac{\dot{x}^i \dot{x}^j}{c^2} - \frac{\dot{\mathbf{x}}^2}{c^2}. \tag{3.13}$$

3.2 Lagrangian and equations of motion

The Lagrangian for the test particle is

$$L = -m_0 c^2 \left(\frac{d\tau}{dt} - 1 \right),$$

where $\frac{d\tau}{dt} = \sqrt{\left(\frac{d\tau}{dt}\right)^2}$ is obtained from the equation (3.13) and considering the expansion $(1+x)^n$ when $x << 1$,

$$\begin{aligned} L = &\frac{1}{2} m_0 \dot{\mathbf{x}}^2 + \frac{1}{2} m_0 c^{2(2)} h_{00} + \frac{1}{8} m_0 \frac{(\dot{\mathbf{x}}^2)^2}{c^2} + \frac{1}{2} m_0 c^{2(4)} h_{00} + m_0 c^{(3)} h_{0j} \dot{x}^j \\ &+ \frac{1}{2} m_0^{(2)} h_{ij} \dot{x}^i \dot{x}^j + \frac{1}{8} m_0 c^2 \left({}^{(2)} h_{00}\right)^2 + \frac{1}{4} m_0^{(2)} h_{00} \dot{\mathbf{x}}^2 + \mathcal{O}(\epsilon^3). \end{aligned} \tag{3.14}$$

The associated equations of motion can be extracted in the usual way, with the Euler-Lagrange equations.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^j} \right) = \frac{\partial L}{\partial x^j},$$

$$\begin{aligned} \ddot{x}^j = &\frac{1}{2} c^2 \partial_j^{(2)} h_{00} + \frac{1}{2} c^2 \partial_j^{(4)} h_{00} - \frac{1}{2} c^{2(2)} h_{jk} \partial_k^{(2)} h_{00} - c^2 \partial_0^{(3)} h_{0j} - \frac{1}{2} c \dot{x}^j \partial_0^{(2)} h_{00} \\ &- c \dot{x}^k \partial_0^{(2)} h_{jk} - c \left(\partial_k^{(3)} h_{0j} - \partial_j^{(3)} h_{0k} \right) \dot{x}^k - \dot{x}^j \dot{x}^k \partial_k^{(2)} h_{00} \\ &- \left(\partial_k^{(2)} h_{jl} - \frac{1}{2} \partial_j^{(2)} h_{kl} \right) \dot{x}^k \dot{x}^l + \mathcal{O}(\epsilon^3). \end{aligned} \tag{3.15}$$

The Lagrangian in (3.14) and the equations of motion given in (3.15) can be rewritten in terms of the components of the tensor $h_{\mu\nu}$ given in (3.11), and (3.12) (See [3] with $\kappa = 1$).

The Lagrangian

$$\begin{aligned} L = &\frac{1}{2} m_0 \dot{\mathbf{x}}^2 + GM\kappa \frac{m_0}{r} + \frac{1}{8} m_0 \frac{(\dot{\mathbf{x}}^2)^2}{c^2} \\ &+ \frac{m m_0}{r} \left[\left(\gamma + \frac{1}{2}\kappa - \alpha \right) \dot{\mathbf{x}}^2 + \left(\alpha + \frac{1}{2}\kappa^2 - \beta \right) \frac{GM}{r} + \alpha \frac{(\mathbf{x} \cdot \dot{\mathbf{x}})}{r^2} \right] + \mathcal{O}(\epsilon^4) \end{aligned} \tag{3.16}$$

and the equations of motion

$$\begin{aligned} \ddot{x}^j + \frac{\kappa GM}{r^3} x^j = &\frac{m}{r^3} \left[2(\beta + \kappa\gamma - \alpha) \frac{GM}{r} - (\gamma + \alpha) \dot{x}_k \dot{x}^k + 3\alpha \frac{\left(x_k \dot{x}^k\right)^2}{r^2} \right] x^j \\ &+ \frac{m}{r^3} \left[2(\kappa + \gamma - \alpha) x_k \dot{x}^k \right] \dot{x}^j + \mathcal{O}(\epsilon^3). \end{aligned}$$

In vector notation, these are

$$\begin{aligned} \ddot{\mathbf{x}} + \frac{\kappa GM}{r^3} \mathbf{x} = &\frac{m}{r^3} \left[2(\beta + \kappa\gamma - \alpha) \frac{GM}{r} - (\gamma + \alpha) \dot{\mathbf{x}}^2 + 3\alpha \frac{(\mathbf{x} \cdot \dot{\mathbf{x}})^2}{r^2} \right] \mathbf{x} \\ &+ \frac{m}{r^3} \left[2(\kappa + \gamma - \alpha) \mathbf{x} \cdot \dot{\mathbf{x}} \right] \dot{\mathbf{x}} + \mathcal{O}(\epsilon^3). \end{aligned} \tag{3.17}$$

3.3 Conserved quantities

To determine the conserved quantities, transform to polar coordinates

$$
\begin{aligned}
\mathbf{x} &= \mathbf{r} = r\hat{n} \\
\dot{\mathbf{x}} &= \dot{\mathbf{r}} = \dot{r}\hat{n} + r\dot{\phi}\hat{\lambda} \\
\ddot{\mathbf{x}} &= \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\phi}^2)\hat{n} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})\hat{\lambda}.
\end{aligned}
$$

The Lagrangian is now written as

$$
\begin{aligned}
L =& \frac{1}{2}m_0(\dot{r}^2 + r^2\dot{\phi}^2) + GM\kappa\frac{m_0}{r} + \frac{1}{8}m_0\frac{(\dot{r}^2 + r^2\dot{\phi}^2)^2}{c^2} \\
&+ \frac{mm_0}{r}\left[\frac{GM}{r}\left(\frac{1}{2}\kappa^2 + \alpha - \beta\right) + \dot{r}^2\left(\gamma + \frac{1}{2}\kappa\right) + r^2\dot{\phi}^2\left(\gamma + \frac{1}{2}\kappa - \alpha\right)\right]
\end{aligned}
\tag{3.18}
$$

and

$$
\ddot{r} - r\dot{\phi}^2 = -\frac{\kappa GM}{r^2} + m\left[2(\beta + \kappa\gamma - \alpha)\frac{GM}{r^3} + (\gamma + 2\kappa)\frac{\dot{r}^2}{r^2} - (\alpha + \gamma)\dot{\phi}^2\right]. \tag{3.19}
$$

The Euler-Lagrange equations,

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0,
$$

give

$$
r^2\dot{\phi}\left[1 + \frac{1}{2}\frac{(\dot{r}^2 + r^2\dot{\phi}^2)}{c^2} + \frac{2m}{r}\left(\gamma + \frac{1}{2}\kappa - \alpha\right)\right] = \ell = \frac{P_\phi}{m_0}, \tag{3.20}
$$

where P_ϕ is the conjugate canonical momentum of ϕ. The angular momentum ℓ is a conserved quantity.

The hamiltonian can be written in terms of the Lagrangian,

$$
H = P_i x^i - L.
$$

The hamiltonian per unit mass (test particle's mass) is

$$
\begin{aligned}
\mathcal{H} =& \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{3}{8}\frac{(\dot{r}^2 + r^2\dot{\phi}^2)^2}{c^2} - \kappa\frac{GM}{r} \\
&+ \frac{m}{r}\left[\left(\gamma + \frac{1}{2}\kappa\right)\dot{r}^2 + \left(\gamma + \frac{1}{2}\kappa - \alpha\right)r^2\dot{\phi}^2 - \frac{GM}{r}\left(\frac{1}{2}\kappa^2 + \alpha - \beta\right)\right].
\end{aligned}
\tag{3.21}
$$

Rewriting the conjugate canonical momentum of the angular variable ϕ, given in (3.20), we obtain

$$
m_0 r^2\dot{\phi} = P_\phi\left[1 + \frac{1}{2}\frac{(\dot{r}^2 + r^2\dot{\phi}^2)}{c^2} + \frac{2m}{r}\left(\gamma + \frac{1}{2}\kappa - \alpha\right)\right]^{-1}. \tag{3.22}
$$

On the other hand, dropping orders of ϵ^4 and higher from the Hamiltonian, we obtain

$$\begin{aligned}\frac{\mathcal{H}}{c^2} &= \frac{1}{2c^2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{3}{8}\frac{(\dot{r}^2 + r^2\dot{\phi}^2)^2}{c^4} - \kappa\frac{GM}{rc^2} \\ &+ \frac{m}{rc^2}\left[\left(\gamma + \frac{1}{2}\kappa\right)\dot{r}^2 + \left(\gamma + \frac{1}{2}\kappa - \alpha\right)r^2\dot{\phi}^2 - \frac{GM}{r}\left(\frac{1}{2}\kappa^2 + \alpha - \beta\right)\right] \\ &= \frac{1}{2c^2}(\dot{r}^2 + r^2\dot{\phi}^2) - \kappa\frac{GM}{rc^2} + \mathcal{O}(\epsilon^4).\end{aligned} \tag{3.23}$$

Back to (3.22),

$$m_0 r^2\dot{\phi} = P_\phi\left[1 + \underbrace{\frac{1}{2}\frac{(\dot{r}^2 + r^2\dot{\phi}^2)}{c^2}}_{\frac{\mathcal{H}}{c^2} + \kappa\frac{GM}{rc^2}} + \frac{2m}{r}\left(\gamma + \frac{1}{2}\kappa - \alpha\right)\right]^{-1}.$$

Using $(1+x)^n$ when $x << 1$ and $\ell = {}^{P_\phi}/_{m_0}$, the angular momentum (See [3] with $\kappa = 1$) is

$$r^2\dot{\phi} = \ell\left[1 - \frac{\mathcal{H}}{c^2} - \frac{2m}{r}\left(\gamma + \kappa - \alpha\right)\right] + \mathcal{O}(\epsilon^4). \tag{3.24}$$

3.4 Trajectory equation

The differential equation for r is determined following the next steps

1. From (3.21), solve for $\dot{r}^2$
2. From (3.23), solve for $(\dot{r}^2 + r^2\dot{\phi})^2$ and replace in the expression for $\dot{r}^2$ obtained above.
3. From (3.24), square, drop orders of ϵ^2 and higher to find the angular momentum up to newtonian order.

$$r^2\dot{\phi}^2 = \frac{\ell^2}{r^2}(1 + \mathcal{O}(\epsilon^2)). \tag{3.25}$$

Replace this result in the expression for $\dot{r}^2$, obtained after step 2, only in the term $r^2\dot{\phi}^2$ due to the PPN metric (and due to the PN approximation).

The differential equation for r is (See [3] with $\kappa = 1$)

$$\begin{aligned}\dot{r}^2 &= 2\mathcal{H} - r^2\dot{\phi}^2 + 2\kappa\frac{GM}{r} - 3\frac{\mathcal{H}^2}{c^2} - 2\frac{m}{r}\left[2\mathcal{H}\left(\gamma + 2\kappa\right) - \alpha\frac{\ell^2}{r^2}\right] \\ &- 2\frac{m}{r}\left[\frac{GM}{r}\left(2\kappa^2 + 2\gamma\kappa + \beta - \alpha\right)\right] + \mathcal{O}(\epsilon^4).\end{aligned} \tag{3.26}$$

It is not of real interest the differential equation in terms of the time t; it is more common and more practical to write the differential equation in terms of the independent variable ϕ.

Making the substitution $u(\phi) = \frac{1}{r(\phi)}$,

$$
\begin{aligned}
\dot{r} &= \frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\dot{\phi} \\
u &= \frac{1}{r} \rightarrow \frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} \rightarrow \frac{dr}{d\phi} = -r^2\frac{du}{d\phi} \\
\dot{r} &= -r^2\frac{du}{d\phi}\dot{\phi}.
\end{aligned}
$$

$\dot{\phi}$ is given by (3.24),

$$
\begin{aligned}
\dot{r} &= -r^2\frac{du}{d\phi}\dot{\phi} \\
&= -r^2\frac{du}{d\phi}\frac{\ell}{r^2}\left[1 - \frac{\mathcal{H}}{c^2} - \frac{2m}{r}\left(\gamma + \kappa - \alpha\right)\right] + \mathcal{O}(\epsilon^4).
\end{aligned}
$$

Squaring,

$$
\dot{r}^2 = \left(\frac{du}{d\phi}\right)^2 \ell^2 \left[1 - \frac{\mathcal{H}}{c^2} - \frac{2m}{r}\left(\gamma + \kappa - \alpha\right)\right]^2 + \mathcal{O}(\epsilon^4).
$$

Solving for $\left(\frac{du}{d\phi}\right)^2 \ell^2$, replacing $\dot{r}^2$ with the expression given in (3.26) and changing all $r^2\dot{\phi}^2$ according to (3.25), the trajectory equation is (See [3] with $\kappa = 1$)

$$
\begin{aligned}
\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{2\mathcal{H}}{\ell^2} + 2\kappa\frac{GM}{\ell^2}u + m\Bigg[&\frac{\mathcal{H}^2}{\ell^2 c^2 m} + \frac{4\mathcal{H}}{\ell^2}u(\gamma + 2 - 2\alpha - \kappa) \\
&2\alpha u^3 + 2\frac{GM}{\ell^2}u^2(\alpha - \beta - 2\gamma\kappa - 2\kappa^2 + 4\gamma\kappa - 4\kappa\alpha + 4\kappa)\Bigg] + \mathcal{O}(\epsilon^4).
\end{aligned} \tag{3.27}
$$

The trajectory equation or the orbit's equation is obtained by differentiating with respect to ϕ (See [3] with $\kappa = 1$),

$$
\begin{aligned}
\frac{d^2u}{d\phi^2} + u =& \kappa\frac{GM}{\ell^2} + m\Bigg[\frac{2\mathcal{H}}{\ell^2}(\gamma + 2 - 2\alpha - \kappa) + 3\alpha u^2 \\
&+ \frac{2GM}{\ell^2}u(\alpha - \beta + 2\gamma\kappa - 2\kappa^2 - 4\kappa\alpha + 4\kappa)\Bigg] + \mathcal{O}(\epsilon^4).
\end{aligned} \tag{3.28}
$$

To recover the trajectory equation for the Schwarzschild case, we put $\gamma = 1$, $\beta = 1$, $\alpha = 1$ y $\kappa = 1$,

$$
\frac{d^2u}{d\phi^2} + u = \frac{GM}{\ell^2} + 3mu^2.
$$

3.4.1 Circular motion

The most simple case to consider is circular motion, where

$$r = r_0 \tag{3.29a}$$
$$\phi = nt + cte. \tag{3.29b}$$

Replacing in (3.19) to obtain the mean motion n (See [3] with $\kappa = 1$), we obtain

$$n = \sqrt{\frac{\kappa GM}{r_0^3}}\left[1 - \frac{m}{r_0\kappa}\left(\beta - \alpha\left(1 - \frac{\kappa}{2}\right) + \frac{\kappa\gamma}{2}\right)\right] + \mathcal{O}(\epsilon^4). \tag{3.30}$$

In the newtonian case, the mean motion is given by

$$n = \frac{2\pi}{T}, \tag{3.31}$$

where T is the period, according to the third law of Kepler,

$$T^2 = \frac{4\pi^2 a^3}{GM}$$
$$n = \sqrt{\frac{GM}{a^3}}, \tag{3.32}$$

where $a = r_0$.

3.4.2 Elliptic motion

For an elliptic orbit, the angular momentum and the energy are, respectively,

$$\ell^2 = GMa(1-e^2)(1+j) = GMa(1-e^2)(1+\mathcal{O}(\epsilon^2)) \tag{3.33}$$
$$\mathcal{H} = -\frac{GM}{2a}(1+g) = -\frac{GM}{2a}(1+\mathcal{O}(\epsilon^2)). \tag{3.34}$$

The first term in each equation corresponds to the newtonian contribution, and a is the major-axis of the ellipse and e its eccentricity.

Considering the equation (3.27), it can be arranged in such a way that it looks like a polynomial equation and replacing the terms given in (3.33) and (3.34),

$$\frac{\ell^2}{GM}\left(\frac{du}{d\phi}\right)^2 = Fu^3 + Au^2 + Bu + C + \mathcal{O}(\epsilon^4), \tag{3.35}$$

where

$$F = 2\alpha m a(1-e^2) \tag{3.36a}$$
$$A = -a(1-e^2)(1+j) + 2m(\beta - \alpha + 2\gamma\kappa - 2\kappa^2 - 4\kappa\alpha + 4\kappa) \tag{3.36b}$$
$$B = 2\kappa - \frac{2m}{a}(\gamma + 2 - 2\alpha - \kappa) \tag{3.36c}$$
$$C = -\frac{(1-g)}{a} + \frac{m}{4a^2}. \tag{3.36d}$$

The equation (3.35) is a third order polynomic equation that can be factorized as follows

$$\frac{\ell^2}{GM}\left(\frac{du}{d\phi}\right)^2 = Fu^3 + Au^2 + Bu + C + \mathcal{O}(\epsilon^4) = (u-u_1)(u-u_2)(Fu-u_3). \tag{3.37}$$

u_1, u_2 and u_3/F are zeros of $\frac{du}{d\phi}$; therefore u_1 and u_2 can be considered as the pericenter and apocenter because these are extreme points of an elliptic motion,

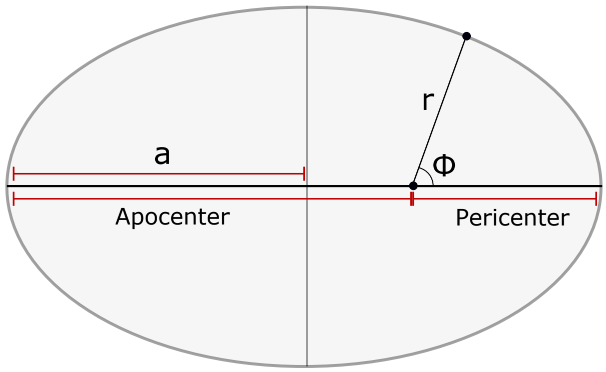

Figure 3.1: Ellipse

$$u_1 = \frac{1}{a(1-e)} : \text{Pericenter}$$
$$u_2 = \frac{1}{a(1+e)} : \text{Apocenter.}$$

Expanding the right side of equation (3.37) and comparing to find that

$$A = -u_3 - Fu_2 - Fu_1 \tag{3.38a}$$
$$B = u_2u_3 + u_1u_3 + Fu_1u_2 \tag{3.38b}$$
$$C = -u_1u_2u_3. \tag{3.38c}$$

Replacing the pericenter and apocenter and comparing with the coefficients A, B and C given in (3.36) to find the missing values of u_3, j and g, gives

$$u_3 = \left[\kappa - \frac{m}{a}(\gamma + 2 - 2\alpha - \kappa + \alpha)\right] a(1-e^2) \tag{3.39}$$
$$j = \kappa - 1 + \frac{m}{a}\left[-(\gamma + 2 - 2\alpha - \kappa + \alpha) + \frac{2}{(1-e^2)}(3\alpha - \beta + 2\gamma\kappa - 2\kappa^2 - 4\kappa\alpha + 4\kappa)\right] \tag{3.40}$$
$$g = \kappa - 1 - \frac{m}{a}\left(\gamma - \alpha - \kappa + \frac{7}{4}\right). \tag{3.41}$$

The equation of motion

$$
\begin{aligned}
\frac{\ell^2}{GM}\left(\frac{du}{d\phi}\right)^2 = &\left(u-\frac{1}{a(1-e)}\right)\left(u-\frac{1}{a(1+e)}\right) \\
&\times\left(2\alpha m u-\kappa(2-\kappa)+\frac{m}{a}(\gamma+2-\alpha-\kappa)(1-\kappa)\right. \\
&\left.+\frac{2\kappa m}{a(1-e^2)}\underbrace{(3\alpha-\beta+2\gamma\kappa-2\kappa^2-4\kappa\alpha+4\kappa)}_{Y}\right),
\end{aligned} \tag{3.42}
$$

replace j to find the angular momentum,

$$
\ell^2 = GMa(1-e^2)\left\{\kappa+\frac{m}{a}\left[-(\gamma+2-\alpha-\kappa)+\frac{2Y}{(1-e^2)}\right]\right\}, \tag{3.43}
$$

and g to find the energy,

$$
\mathcal{H} = -\frac{GM}{2a}\left[\kappa-\frac{m}{a}\left(\gamma-\alpha-\kappa+\frac{7}{4}\right)\right]. \tag{3.44}
$$

Now, considering the conic equation

$$
u(\phi) = \frac{1+e\cos f}{a(1-e^2)} \tag{3.45}
$$

$$
f = f(\phi) = \phi-\omega \quad \text{(See Figure (\textbf{??}))}, \tag{3.46}
$$

where f is the true anomaly and ω is the longitude of the pericenter.

Rewriting the equation (3.42), where

$$
\frac{du}{d\phi} = -\frac{e\sin f}{a(1-e^2)}\frac{df}{d\phi},
$$

to obtain

$$
\begin{aligned}
\frac{df}{d\phi} = \sqrt{\kappa(2-\kappa)}&\left\{1-\frac{m}{a\kappa(2-\kappa)}\left[(1-\kappa)(\gamma+2-\alpha-\kappa)+\frac{\kappa Y}{(1-e^2)}+\frac{\alpha}{1-e^2}\right]\right\} \\
&\times\left[1-\frac{m\alpha e\cos f}{a\kappa(2-\kappa)(1-e^2)}\right].
\end{aligned} \tag{3.47}
$$

All terms inside the curly brackets are constants, so we write

$$
\frac{df}{d\phi} = \nu\left[1-\frac{m\alpha e\cos f}{a\kappa(2-\kappa)(1-e^2)}\right], \tag{3.48}
$$

where

$$
\nu = \sqrt{\kappa(2-\kappa)}\left\{1-\frac{m}{a\kappa(2-\kappa)}\left[(1-\kappa)(\gamma+2-\alpha-\kappa)+\frac{\kappa Y}{(1-e^2)}+\frac{\alpha}{1-e^2}\right]\right\}
$$

This can be written as an expansion,

$$\frac{d(^{(0)}f + ^{(2)}f)}{d\phi} = \nu \left[1 - \frac{m\alpha e \cos^{(0)} f f}{a\kappa(2-\kappa)(1-e^2)} \right].$$

The zeroth order,

$$\begin{aligned} \frac{d^{(0)}f}{d\phi} &= \nu \\ ^{(0)}f &= \nu(\phi - \omega). \end{aligned}$$

The second order,

$$\begin{aligned} \frac{d^{(2)}f}{d\phi} &= -\nu \left[\frac{m\alpha e \cos^{(0)} f}{a\kappa(2-\kappa)(1-e^2)} \right] \\ ^{(2)}f &= -\frac{m\alpha e \sin[\nu(\phi-\omega)]}{a\kappa(2-\kappa)(1-e^2)}. \end{aligned}$$

The true anomaly is

$$f = \nu(\phi - \omega) - \frac{m\alpha e \sin[\nu(\phi-\omega)]}{a\kappa(2-\kappa)(1-e^2)} \tag{3.49}$$

To recover the newtonian case we fix the constants as $\gamma = 0$, $\beta = 0$, $\alpha = 0$ y $\kappa = 1$, then

$$f = \phi - \omega.$$

Up to now, there are only differential equations with ϕ dependence, but it is possible to recover the time dependence in the equations. From (3.24) and (3.44),

$$r^2 d\phi = \ell \left[1 - \frac{\mathcal{H}}{c^2} - \frac{2m}{r} (\gamma + \kappa - \alpha) \right] dt$$

and

$$\frac{\mathcal{H}}{c^2} = -\frac{m}{2a}\kappa + \mathcal{O}(\epsilon^4).$$

Replacing ${}^{\mathcal{H}}\!/\!{}_{c^2}$ in the expression of $r^2 d\phi$, ℓ as given by (3.43),

$$\ell = \sqrt{GMa(1-e^2)\kappa} \left\{ 1 - \frac{m}{2a\kappa} \left[(\gamma + 2 - \alpha - \kappa) - \frac{2Y}{(1-e^2)} \right] \right\}, \tag{3.50}$$

and the term $\sqrt{GM\kappa}$ rewritten in terms of the mean anomaly given in (3.30), the resultant equation is,

$$ndt = \frac{r^2 d\phi}{a^2\sqrt{1-e^2}}\left\{1 + \frac{2m}{r}(\gamma - \alpha + \kappa) + \frac{m}{2a}\left[(2-\alpha-2\kappa) - \frac{2Y}{1-e^2}\right] + \frac{m}{2a}\left[\alpha\left(1+\frac{\kappa}{2}\right) - \beta\right]\right\}. \tag{3.51}$$

From the equation (3.47), solve for $d\phi$ in terms of df and replace in the equation ndt above. Finally a equation in terms of df and dt is obtained,

$$\begin{aligned}\frac{df}{dt} =& \frac{na^2\sqrt{1-e^2}\sqrt{\kappa(2-\kappa)}}{r^2}\left\{1 - \frac{2m}{r}(\gamma-\alpha+\kappa) - \frac{m}{2a}\left(2-\alpha-2\kappa - \frac{2Y}{1-e^2}\right)\right. \\ & -\frac{m}{a\kappa}\left[\alpha\left(1+\frac{\kappa}{2}\right) - \beta\right] - \frac{\alpha m}{\kappa r(2-\kappa)} \\ & \left. -\frac{m}{2a\kappa(2-\kappa)}\left[(1-\kappa)(\gamma+2-\alpha+\kappa) + \frac{2\kappa Y}{1-e^2}\right].\right\}\end{aligned} \tag{3.52}$$

3.4.3 Perturbing Force

The change of the orbital elements can be determined on account of the perturbative force. It can be identified by comparing with the equation of motion

$$\ddot{\mathbf{x}} = -\frac{GM}{r^2}\mathbf{x} + \mathbf{F}.$$

The perturbative force is $\mathbf{F}$

$$\begin{aligned}\mathbf{F} =& \left\{(1-\kappa)GM + m\left[2(\beta+\kappa\gamma-\alpha)\frac{GM}{r} - (\gamma+\alpha)\dot{\mathbf{x}}^2 + 3\alpha\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{r^2}\right]\right\}\frac{\mathbf{x}}{r^3} \\ & + \frac{m}{r^3}\left[2(\kappa+\gamma-\alpha)\mathbf{x}\cdot\dot{\mathbf{x}}\right]\dot{\mathbf{x}} + \mathcal{O}(\epsilon^3).\end{aligned} \tag{3.53}$$

This force can be written as

$$\mathbf{F} = \mathcal{R}\mathbf{n} + \mathcal{T}\boldsymbol{\lambda} + \mathcal{W}\mathbf{k} \tag{3.54}$$

with

$$\mathbf{n} = \frac{\mathbf{x}}{x} \tag{3.55a}$$

$$\boldsymbol{\lambda} = \frac{d\mathbf{n}}{d\phi} \tag{3.55b}$$

$$\mathbf{k} = \mathbf{n}\times\boldsymbol{\lambda}. \tag{3.55c}$$

Each coefficient is

$$\mathcal{R} = \mathbf{F}\cdot\mathbf{n}, \quad \mathcal{T} = \mathbf{F}\cdot\boldsymbol{\lambda}, \quad \mathcal{W} = \mathbf{F}\cdot\mathbf{k} \tag{3.56}$$

The radial component is

$$\mathcal{R} = \frac{n^2a^2}{\kappa r^2}\left\{(1-\kappa)a + m\left[\frac{2a}{r}(\beta-\alpha+2\kappa\gamma+2\kappa^2) - (2\kappa+\gamma)\kappa \right.\right.$$
$$\left.\left. -(2\gamma+2\kappa+\alpha)a^2\frac{(1-e^2)\kappa}{r^2} + (1-\kappa)\frac{2}{\kappa}\left(\beta-\alpha\left(1+\frac{\kappa}{2}\right)+\frac{\kappa\gamma}{2}\right)\right]\right\}, \quad (3.57a)$$

the component ortogonal to $\mathbf{n}$ is

$$\mathcal{T} = 2n^2a^3e\sqrt{\kappa(2-\kappa)}(\kappa+\gamma-\alpha)\frac{m}{r^3}\sin f \quad (3.57b)$$

and the $\mathbf{k}$ component is

$$\mathcal{W} = 0. \quad (3.57c)$$

3.4.4 Orbital elements

The change of the orbital elements in term of the coefficients of the perturbative force are [3, 13]

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[e\sin f\mathcal{R} + (1+e\cos f)\mathcal{T}\right] \quad (3.58a)$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[\sin f\mathcal{R} + \frac{2\cos f + e\left(1+\cos^2 f\right)}{1+e\cos f}\mathcal{T}\right] \quad (3.58b)$$

$$\frac{d\iota}{dt} = \frac{\sqrt{1-e^2}}{na}\frac{\cos(f+\omega)}{1+e\cos f}\mathcal{W} \quad (3.58c)$$

$$\frac{d\Omega}{dt} = \frac{\sqrt{1-e^2}}{na}\frac{\sin(f+\omega)}{(1+e\cos f)\sin\iota}\mathcal{W} \quad (3.58d)$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left[-\cos f\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\mathcal{T} - e\frac{\sin(f+\omega)}{1+e\cos f}\cot\iota\mathcal{W}\right]. \quad (3.58e)$$

(3.58f)

using the coefficients given in (3.57), we can write

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[e\sin f\mathcal{R} + (1+e\cos f)\mathcal{T}\right] \quad (3.59a)$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[\sin f\mathcal{R} + \frac{2\cos f + e\left(1+\cos^2 f\right)}{1+e\cos f}\mathcal{T}\right] \quad (3.59b)$$

$$\frac{d\iota}{dt} = 0 \quad (3.59c)$$

$$\frac{d\Omega}{dt} = 0 \quad (3.59d)$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae}\left[-\cos f\mathcal{R} + \frac{2+e\cos f}{1+e\cos f}\sin f\mathcal{T}\right], \quad (3.59e)$$

(3.59f)

where p the *semi-latus rectum*

$$r = \frac{p}{1 + e \cos f} = \frac{a(1 - e^2)}{1 + e \cos f}. \tag{3.60}$$

Up to this approximative order, the perturbative force $\mathbf{F}$ does not have a component in the $\mathbf{k}$ direction.

Chapter 4

Stationary One Body Problem

In Chapter 3 we worked the problem of a test particle under the influence of a Schwarzschild gravitational field, means a spherically symmetric and static body. Now we will consider the same test particle but under the influence of a weak gravitational field, generated by a slowly rotating source.

The first step to develop this problem is to understand, in a general way, the Kerr metric and Weyl Levi Civita metric [18, 7, 25, 5].

4.1 Metric of slowly rotating spheroid

4.1.1 Kerr metric

The Kerr solution is the exact solution of the Einstein field equations; outside a stationary body. In the proper reference frame with no time dependence; Kerr metric, just as the Schwarzschild metric, tends to the Lorentz frame at infinity, i.e., the field is asymptotically ($r \to \infty$) flat [3, 7, 18].

Boyer - Lindquist coordinates (t, r, θ, ϕ) give the asymptotically behavior, using these coordinates, usually called "Schwarzschild-Like" coordinates,

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 + \frac{4ma}{\rho^2} r \sin^2\theta d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{A \sin^2\theta}{\rho^2} d\phi^2 \tag{4.1}$$

$$\Delta = r^2 - 2mr + a^2 \tag{4.2a}$$
$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{4.2b}$$
$$A = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta. \tag{4.2c}$$

The metric depends on two parameters m and a. At large distances, $r \to \infty$, the metric given in (4.1) is

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{4ma}{r} \sin^2\theta d\phi dt + \left(1 + \frac{2m}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \qquad (4.3)$$

Compare this with the metric in the weak field, to find that $j = am$, where j is the angular momentum in geometric units ($[j] = [length^2]$). Without rotation $a = 0$, the metric in (4.3) is the Schwarzschild metric, therefore m is the mass of the gravitational source in geometric units.

The event horizon

The coefficient g_{11} diverges when $\Delta = 0$, i.e., when

$$\begin{aligned} &r^2 - 2mr + a^2 = 0 \\ &r_\pm = m \pm \sqrt{m^2 - a^2}. \end{aligned} \qquad (4.4)$$

For the purpose of this work, there is no real interest in the event horizon of a black hole. Nevertheless, to know the event horizon radius allows to determine that for physical solutions, $m > a$. This implies that the rotation of a black hole is slow.

Constants of motion

As usual the Lagrangian is defined as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \\ \dot{x}^\nu &= \frac{dx^\mu}{d\tau}. \end{aligned}$$

To determine the conserved quantities, the Euler-Lagrange equations for the temporal and the ϕ components [2] give

$$\begin{aligned} &\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) - \frac{\partial \mathcal{L}}{\partial t} = 0 \\ &\frac{d}{d\tau}\left(g_{00}\frac{dt}{d\tau} + g_{03}\frac{d\phi}{d\tau}\right) = 0 \\ &g_{00}\frac{dt}{d\tau} + g_{03}\frac{d\phi}{d\tau} = -E = const \end{aligned} \qquad (4.5)$$

$$\begin{aligned} &\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \\ &\frac{d}{d\tau}\left(g_{03}\frac{dt}{d\tau} + g_{33}\frac{d\phi}{d\tau}\right) = 0 \\ &g_{03}\frac{dt}{d\tau} + g_{33}\frac{d\phi}{d\tau} = L_z = const. \end{aligned} \qquad (4.6)$$

E is the energy of the test particle per unit mass and L_z is the projection of angular momentum of the particle per unit mass along the rotation axis of the gravitational body, measured by a distant stationary observer [14].

Only for orbits in the equatorial plane, L_z is the total angular momentum per unit mass that is, only for orbits in the equatorial plane ($\theta = \pi/2$) the total angular momentum is a constant of motion [18].

Unlike the Schwarzschild solution, where the total angular momentum is conserved, and the motion is restricted to a plane, the motion around a Kerr solution is, in general, not restricted to a plane. This fact is the reason why studying the motion of test particles under the influence of a rotating gravitational body is a hard task.

Solving the Euler-Lagrange equations for $\dot{\phi}$ and $\dot{t}$ and replacing the coefficients of the metric,

$$\frac{dt}{d\tau} = \frac{EA}{\rho^2\Delta} - \frac{2mar}{\rho^2}L_z \tag{4.7}$$

$$\frac{d\phi}{d\tau} = -\frac{2marE}{\rho^2\Delta} - \left(\frac{\Delta - a^2\sin^2\theta}{\Delta\rho^2\sin^2\theta}\right)L_z. \tag{4.8}$$

There is a third constant, defined by the θ component, that can be obtained with the Hamilton - Jacobi method [2, 21].

The Hamiltonian is given by

$$H = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu. \tag{4.9}$$

Transforming the Hamiltonian in such a way that the “new” Hamiltonian is a constant, gives

$$K = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \frac{dS}{d\tau} = 0,$$

where the canonical momentum is defined as $p_\mu = \frac{\partial S}{\partial x^\mu}$.

It is possible to propose a solution for S, where the dependence with the parameter τ is separate from the coordinates x^μ [21],

$$S = W(x^\mu) + T(\tau).$$

Replacing in K

$$\frac{1}{2}g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}\right)\left(\frac{\partial S}{\partial x^\nu}\right) + \frac{\partial S}{\partial \tau} = 0$$

$$\frac{1}{2}g^{\mu\nu}\left(\frac{\partial W}{\partial x^\mu}\right)\left(\frac{\partial W}{\partial x^\nu}\right) = -\frac{dT}{d\tau} = \Lambda.$$

Therefore,

$$H\left(x^\mu, \frac{\partial W}{\partial x^\mu}\right) - \Lambda = 0 \tag{4.10}$$

$$\frac{1}{2}g^{\mu\nu}p_\mu p_\nu - \Lambda = 0. \tag{4.11}$$

Defining $\Lambda = -\frac{1}{2}$,

$$g^{\mu\nu}\left(\frac{\partial W}{\partial x^\mu}\right)\left(\frac{\partial W}{\partial x^\nu}\right) + 1 = 0$$

expanding

$$g^{00}\left(\frac{\partial W}{\partial x^0}\right)^2 + 2g^{03}\left(\frac{\partial W}{\partial x^0}\right)\left(\frac{\partial W}{\partial x^3}\right) + g^{33}\left(\frac{\partial W}{\partial x^3}\right)^2 + g^{11}\left(\frac{\partial W}{\partial x^1}\right)^2 + g^{22}\left(\frac{\partial W}{\partial x^2}\right)^2 + 1 = 0.$$

The coefficients $g^{\mu\nu}$ can be found in [7, 18]. Since these are independent of ϕ and t, we propose the form of W [2] as

$$W = -Et + L_z\phi + S_r(r) + S_\theta(\theta)$$

Replacing in the equation above,

$$\frac{-(r^2+a^2)^2 + a^2\sin^2\theta}{\Delta}E^2 + \frac{4marEL_z}{\Delta} + \Delta\left(\frac{dS_r}{dr}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} - \frac{a^2L_z^2}{\Delta} + \rho^2 = 0$$

and collecting on the left hand side all the terms containing θ-dependence and on the right hand side the r-dependence and constants, each side must be equal to a constant Q.

For θ we have that,

$$\left(\frac{dS_\theta}{d\theta}\right)^2 + \cos^2\theta\left[a^2(1-E^2) + \frac{L_z^2}{\sin^2\theta}\right] = Q \tag{4.12}$$

$$p_\theta = \frac{dS_\theta}{d\theta} = g_{22}\frac{d\theta}{d\tau}, \qquad \frac{d\theta}{d\tau} = \frac{dS_\theta}{d\theta}g^{22} = \left(\frac{dS_\theta}{d\theta}\right)\left(\frac{1}{\rho^2}\right)$$

where Q is called the Carter integral,

$$\rho^4\left(\frac{d\theta}{d\tau}\right)^2 + \cos^2\theta\left[a^2(1-E^2) + \frac{L_z^2}{.}\sin^2\theta\right] = Q \tag{4.13}$$

For r,

$$\begin{aligned}
\Delta\left(\frac{dS_r}{dr}\right)^2 &= \frac{1}{\Delta}\left[(r^2+a^2)^2E^2 - 4marEL_z + a^2L_z^2\right] - (a^2E^2 + L_z^2) - (Q + r^2) \\
\Delta\left(\frac{dS_r}{dr}\right)^2 &= \frac{1}{\Delta}\left[(r^2+a^2)E - aL_z\right]^2 - (L_z - aE)^2 - (Q + r^2).
\end{aligned} \tag{4.14}$$

The Carter integral can be interpreted by considering first the total angular momentum per unit mass,

$$L^2 = r^4 \left[\left(\frac{d\theta}{d\tau} \right)^2 + \sin^2\theta \left(\frac{d\phi}{d\tau} \right)^2 \right]. \tag{4.15}$$

For a particle at a large distance from the gravitational body, the angular momentum can be rewritten by replacing (4.13) and (4.8) evaluated at $r \to \infty$,

$$\begin{aligned} L^2 &= Q - \cos^2\theta \left[a^2(1-E^2) + \frac{L_z^2}{\sin^2\theta} \right] + \frac{L_z^2}{\sin^2\theta} &(4.16)\\ &= Q - a^2(1-E^2)\cos^2\theta + L_z^2. &(4.17) \end{aligned}$$

The angular momentum for the particle is not conserved, it has a dependence on θ. These analysis support the fact that for Kerr space-time the motion is not restricted to a plane unless the plane is located at the equator in $\theta = \pi/2$.

Innermost Stable Circular Orbit (ISCO)

For a circular orbit, the radius is constant, $\frac{dr}{d\tau} = 0$ and $\frac{d^2r}{d\tau^2} = 0$. The differential equation for the component r, can be found with (4.14)[18],

$$p_r = \frac{dS_r}{dr} = g_{11}\frac{dr}{d\tau}, \qquad \frac{dr}{d\tau} = \frac{dS_r}{dr} g^{11} = \left(\frac{dS_r}{dr} \right) \left(\frac{-\Delta}{\rho^2} \right)$$

$$\rho^4 \left(\frac{dr}{d\tau} \right)^2 = \left[(r^2+a^2)E - aL_z \right]^2 - \Delta \left[(L_z - aE)^2 + (Q + r^2) \right]. \tag{4.18}$$

The radial equation can be found with the Euler Lagrange equations [2]. For a test particle in equatorial circular orbit $\dot{r} = 0, \ddot{r} = 0, \dot{\theta} = 0$, we obtain

$$\begin{aligned} &\frac{d}{d\tau}(g_{\mu\nu}\dot{x}^\nu) - \frac{1}{2}(\partial_\mu g_{\alpha\beta})\dot{x}^\alpha \dot{x}^\beta = 0 \\ &\frac{d}{d\tau}(\underbrace{g_{00}\dot{t} + g_{03}\dot{\phi}}_{E=const} + \underbrace{g_{30}\dot{t} + g_{33}\dot{\phi}}_{L=const}) - \frac{1}{2}(\partial_\mu g_{\alpha\beta})\dot{x}^\alpha \dot{x}^\beta = 0 \\ &\partial_1 g_{00} + 2\partial_1 g_{03}\frac{\dot{\phi}}{\dot{t}} + \partial_1 g_{33}\left(\frac{\dot{\phi}}{\dot{t}} \right)^2 = 0. \end{aligned}$$

For a circular orbit and replacing the coefficients, this is

$$\left(\frac{\dot{\phi}}{\dot{t}} \right)_\pm = \frac{\pm\sqrt{m}}{r^{3/2} \pm a\sqrt{m}}. \tag{4.19}$$

The energy E and angular momentum L, given in (4.5) and (4.6) respectively, can be rewritten with the equation above. Both expression have the same denominator, circular orbits can exist only when that denominator is real $\sqrt{r^{3/2} - 3mr^{1/2} \pm 2am^{1/2}} \geq 0$ [18].

From the condition of circular orbit [18],

$$\left(r \frac{dr}{d\tau} \right)^2 = R(E, L, r) = 0 \tag{4.20}$$

where

$$R(E, L, r) = \left(r^2 + a^2 + \frac{2ma^2}{r} \right) (E^2 - 1) - \frac{4maEL}{r} + L^2 \left(\frac{2m}{r} - 1 \right) + 2m \left(\frac{a^2}{r} - r \right). \tag{4.21}$$

The condition $\frac{d^2 r}{d\tau^2} = 0$ is given for the marginally stable case, then

$$\begin{aligned} &2\frac{dR}{dr} + r\frac{d^2R}{dr^2} = 6r^2(E^2 - 1) + 4m = 0 \\ &E^2 = 1 - \frac{2m}{3r_{ms}}. \end{aligned} \tag{4.22}$$

Eliminating E given by (4.19) and E in (4.22) gives

$$r_{ms}^2 - 6mr_{ms} \pm 8am^{1/2} r_{ms}^{1/2} - 3a^2 = 0. \tag{4.23}$$

For $a = 0$, $r_{ms} = 6m$, the ISCO for Schwarzschild. The positive $(+)$ sign is for co-rotating test particle and $(-)$ for counter-rotating test particle.

4.1.2 Weyl Levi Civita metric

The metric of a nonrotating mass with quadrupole moment was determined by Erez and Rosen in 1959 [5]. Nevertheless this metric had a wrong asymptotic behavior. The metric with the respective corrections were given by John and Coulter in 1969 [25]. The latter is the one we will use here.

The line element for vacuum space, outside a static and axially symmetric body, is given in cylindrical coordinates by

$$ds^2 = -e^{2\psi} dt^2 + e^{2\gamma - 2\psi}(d\rho^2 + dz^2) + \rho^2 e^{-2\psi} d\phi^2. \tag{4.24}$$

Changing the coordinates to

$$\begin{aligned} \lambda &= \frac{r_+ + r_-}{2m} \\ \mu &= \frac{r_+ - r_-}{2m} \\ r_\pm^2 &= \rho^2 + (z \pm m)^2 \end{aligned}$$

and then to

$$
\begin{aligned}
r &= m(\lambda + 1) \\
\theta &= \cos^{-1}\mu
\end{aligned}
$$

gives the line element as

$$
\begin{aligned}
ds^2 = &-e^{2\psi}dt^2 + e^{2\gamma-2\psi}\left[\left(1+\frac{m^2\sin^2\theta}{r^2-2mr}\right)d\rho^2 + (r^2-2mr+m^2\sin^2\theta)d\theta^2\right] \\
&+ e^{-2\psi}(r^2-2mr)\sin^2\theta d\phi^2.
\end{aligned} \tag{4.27}
$$

The solution for ψ is given in terms of the Legendre polynomials. It is possible to write the solution as a multipole expansion as $\psi = \psi_0 + \bar{q}_l\psi_l$, where $\bar{q}_l$ is the multipole moment of order l. For a quadrupole moment, $l = 2$,

$$
\begin{aligned}
\psi = &\frac{1}{2}\ln\left(1-\frac{2m}{r}\right) + \frac{\bar{q}_l}{2}\frac{r^2}{m^2}(3\cos^2\theta - 1)\times \\
&\left[\left(\frac{3}{4} - \frac{3}{2}\frac{m}{r} + \frac{1}{2}\frac{m^2}{r^2}\right)\ln\left(1-\frac{2m}{r}\right) + \frac{3}{2}\frac{m}{r} - \frac{3}{3}\frac{m^2}{r^2}\right].
\end{aligned} \tag{4.28}
$$

The full expression of γ is given in [3, 25].

The expression for ψ can be expanded in powers of ${}^{m}/_{r}$, ($m = {}^{GM}/_{c^2}$),

$$
\ln\left(1-\frac{2m}{r}\right) = -\frac{2m}{r} - \frac{2m^2}{r^2} - \frac{8m^3}{3r^3} - \frac{4m^4}{r^4} - \frac{32m^5}{5r^5} - \frac{64m^6}{6r^6} + ... + ... \tag{4.29}
$$

Replacing,

$$
\psi = \frac{1}{2}\ln\left(1-\frac{2m}{r}\right) + \frac{\bar{q}_l}{2}(3\cos^2\theta - 1)\left[-\frac{2}{15}\frac{m^3}{r^3} - \frac{2}{5}\frac{m^4}{r^4} + ...\right]. \tag{4.30}
$$

The g_{00} component is then,

$$
g_{00} = -e^{2\psi} = -\left(1-\frac{2m}{r}\right) + \frac{2\bar{q}_l}{15}\frac{m^3}{r^3}(3\cos^2\theta - 1) + \mathcal{O}(\epsilon^4) \tag{4.31}
$$

According to (2.37a), (2.45a) and the expression above, the Newtonian potential is

$$
\phi = -\frac{GM}{r} + \frac{\bar{q}_l}{15}\frac{m^3c^2}{r^3}(1 - 3\cos^2\theta). \tag{4.32}
$$

4.1.3 Newtonian Quadrupole Moment

In this section, we will obtain the same expression for the potential ϕ given by expanding the Weyl Levi Civita Metric; but following a Newtonian approach. The newtonian potential is given by

$$
\phi = -G\int\frac{\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{4.33}
$$

where the domain of integration extends over the volume occupied by the source and the non-prime coordinates refers to the point, outside the source, where the potential is evaluated.

Expanding the expression in the denominator in terms of spherical harmonics,

$$\phi = -G \int \rho(t\,, \mathbf{x}') \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{l\,m}(\theta', \varphi') Y_{l\,m}(\theta, \varphi) d^3x'$$

where $0 < \theta < \pi$, $0 < \varphi < 2\pi$, and $r_<$ stands for the $\min(r', r)$, and $r_>$ stands for the $\max(r', r)$.

Outside the source, $r_< = r'$, and $r_> = r$, the potential is

$$\phi = -G \int \rho(t\,, \mathbf{x}') \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{4\pi}{2l+1} \frac{r'^{\,l}}{r^{\,l+1}} Y^*_{l\,m}(\theta', \varphi') Y_{l\,m}(\theta, \varphi) d^3x'. \tag{4.34}$$

Rewritten the potential as

$$\phi = -G \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{4\pi}{2l+1} \frac{I_{lm}(t)}{r^{\,l+1}} Y_{l\,m}(\theta, \varphi), \tag{4.35}$$

where I_{lm} are the multipolar moments, defined as [17]

$$I_{lm} = \int \rho(t\,, \mathbf{x}') r'^{\,l} Y^*_{l\,m}(\theta', \varphi') d^3x'. \tag{4.36}$$

The origin of coordinates is the center of mass of the source. Then, the first two elements, the monopole and the dipole, are

$$I_{00} = \frac{M}{\sqrt{4\pi}} \tag{4.37}$$

$$I_{1m} = 0, \tag{4.38}$$

where M is the total mass of the source.

The potential becomes

$$\phi = -\frac{GM}{r} - G \sum_{l=2}^{\infty} \sum_{m=-l}^{m=l} \frac{4\pi}{2l+1} \frac{I_{lm}(t)}{r^{\,l+1}} Y_{l\,m}(\theta, \varphi). \tag{4.39}$$

Defining the source as an axially symmetric body, and the z-direction as the symmetry axis, we do not longer have a dependence on the φ coordinate the expression for the potential becomes

$$\phi = -\frac{GM}{r} - G \sum_{l=2}^{\infty} \sqrt{\frac{4\pi}{2l+1}} \frac{I_{l\,0}}{r^{\,l+1}} P_l(\cos\theta). \tag{4.40}$$

Now, introducing the quantity,

$$J_l = \sqrt{\frac{4\pi}{2l+1}} I_{l\,0}, \tag{4.41}$$

the potential takes the following form, up to $l = 2$,

$$\begin{aligned}\phi &= -\frac{GM}{r} + \frac{GJ_2}{2r^3}(1 - 3\cos^2\theta) + ... \\ &= -\frac{GM}{r} + \frac{GJ_2}{2r^3}\left(1 - \frac{3z^2}{r^2}\right) + ... \quad .\end{aligned}$$

The quantity J_2 can be written in terms of the components of the inertia moment[8],

$$\mathcal{I}^{km} = \int \rho(\mathbf{x})(r^2\delta^{km} - x^k x^m) d^3x \tag{4.42}$$

This tensor can be written as

$$\mathcal{I}^{km} = \delta^{jk} q - q^{km} \tag{4.43}$$

where $q = q^j_j$ and the non-traceless inertia moment is [3]

$$q^{km} = \int \rho(\mathbf{x}) x^k x^m d^3x. \tag{4.44}$$

For an axially symmetric body and a rotation around the z-axis, the moments of inertia in the x and y axis are equal $\mathcal{I}^x = \mathcal{I}^y$.

The non-traceless inertia moments can be written in terms of the inertia moments $\mathcal{I}^{km}$.

Define

$$\mathcal{I}_z = C, \qquad \mathcal{I}_x = \mathcal{I}_y = A. \tag{4.45}$$

According to (4.43)

$$q_z = -\mathcal{I}_z + q, \qquad q_y = q_x = -\mathcal{I}_x + q$$

where

$$q = -\mathcal{I} + 3q \rightarrow q = \frac{\mathcal{I}}{2} = \frac{2A + C}{2}.$$

Then,

$$q_z = A - \frac{C}{2} \tag{4.46a}$$

$$q_y = q_x = \frac{C}{2}. \tag{4.46b}$$

On the other hand, J_2 can be written as

$$
\begin{aligned}
J_2 &= \sqrt{\frac{4\pi}{5}}\, I_{2\,0} = \sqrt{\frac{4\pi}{5}} \int \rho(\mathbf{x}')r'^2 Y^*_{20}(\theta', \varphi')d^3x' \\
&= \frac{1}{2}\int \rho(\mathbf{x}')(3r' \cos^2\theta' - r'^2)d^3x' = \frac{1}{2}\int \rho(\mathbf{x}')(3z'^2 - r'^2)d^3x' \\
&= \frac{1}{2}\int \rho(\mathbf{x}')(-x'^{f\,2} - y'^2 + 2z'^2)d^3x'.
\end{aligned}
$$

The inertia moment in (4.42) is

$$
\mathcal{I}^z = \int \rho(\mathbf{x}')(x'^2 + y'^2)d^3x' \tag{4.47}
$$

$$
\mathcal{I}^x = \int \rho(\mathbf{x}')(y'^2 + z'^2)d^3x' = \mathcal{I}^y. \tag{4.48}
$$

Then, J_2 is

$$
\begin{aligned}
J_2 &= \frac{1}{2}\int \rho(\mathbf{x}')(x'^2 + z'^2)d^3x' - \int \rho(\mathbf{x}')(x'^2 + y'^2)d^3x' + \int \rho(\mathbf{x}')(y'^2 + z'^2)d^3x' \\
&= \frac{1}{2}\mathcal{I}^x - \mathcal{I}^z + \frac{1}{2}\mathcal{I}^z = \mathcal{I}^x - \mathcal{I}^z.
\end{aligned}
$$

Finally, the quantity J_2 is

$$
J_2 = A - C. \tag{4.49}
$$

Going back to the Newtonian potential

$$
\phi = -\frac{GM}{r} + \frac{GJ_2}{2r^3}\left(1 - \frac{3z^2}{r^2}\right)
$$

$$
\phi = -\frac{GM}{r} + \frac{Q}{2r^3}\left(1 - \frac{3z^2}{r^2}\right). \tag{4.50}
$$

Note here that this form is equivalent to the one given in (4.32).

It is possible to write the metric for the slowly rotating spheroid with the general post newtonian metric given in (2.85) as

$$
h_{00} = {}^{(2)}h_{00} + {}^{(4)}h_{00} + \mathcal{O}(\epsilon^6) = -\frac{2}{c^2}\phi - \frac{2}{c^2}\left(\frac{\phi^2}{c^2} + \psi\right) + \mathcal{O}(\epsilon^6) \tag{4.51a}
$$

$$
h_{0j} = {}^{(3)}h_{0j} + \mathcal{O}(\epsilon^5) = \frac{4}{c^3}\zeta^j + \partial_0\partial_k\chi + \mathcal{O}(\epsilon^5) \tag{4.51b}
$$

$$
h_{jk} = {}^{(2)}h_{jk} + \mathcal{O}(\epsilon^4) = \left(1 - \frac{2}{c^2}\phi\right)\delta_{jk} + \mathcal{O}(\epsilon^4) \tag{4.51c}
$$

The gravitational field of a slowly rotating spheroid can be characterized by the potential described above (4.50), the vector potential ζ^j described in Chapter 2 in (2.38b) and the ψ potential given in (2.74).

4.1.4 Vector potential

The vector potential is defined as

$$\zeta^j = -\frac{G}{c^2}\int \frac{T^{0j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x', \tag{4.52}$$

where $\mathbf{x}$ is the vector between the center of mass of the gravitational body and the test particle, and the domain of integration extends over the volumen occupied by the source. Expand the term $|\mathbf{x}-\mathbf{x}'|^{-1}$ in a Taylor series

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} - x'^l\partial_l\left(\frac{1}{r}\right) + \frac{1}{2}x'^l x'^m \partial_l\partial_m\left(\frac{1}{r}\right) + ... \tag{4.53}$$

the energy momentum component T^{0j} in (2.10c), then the vector potential in terms of the velocity of the source is

$$\zeta^j = -G\int \frac{\rho(\mathbf{x}')v'^j}{|\mathbf{x}-\mathbf{x}'|}d^3x' = -\frac{G}{r}\int \rho(\mathbf{x}')v'^j d^3x' - \frac{Gx^m}{r^3}\int \rho(\mathbf{x}')x'^m v'^j d^3x', \tag{4.54}$$

where the velocity of the rotating spheroid is

$$v^j = \epsilon_{ijk}\omega^j x^k \tag{4.55}$$

The first term is null, due to the definition of center of mass,

$$\int \rho(\mathbf{x}')v'^j d^3x' = \frac{d}{dt}\int \rho(\mathbf{x}')x'^j d^3x' = 0.$$

For the second term,

$$-\frac{Gx^m}{r^3}\int \rho(\mathbf{x}')x'^m v'^j d^3x' = -\epsilon_{jik}\frac{Gx^m\omega^i}{r^3}\int \rho(\mathbf{x}')x'^m x'^k d^3x'$$

Since the term inside the integral is the non-traceless inertia moment given in (4.44), the vector potential is

$$\zeta^j = -\frac{G}{r^3}\epsilon_{jik}\omega^i q^{km}x^m. \tag{4.56}$$

For a axially symmetric body and a rotation around the z-axis, the angular velocity is just

$$\omega = \omega^3. \tag{4.57}$$

The components of the vector potential, in terms of (4.46), are

$$\zeta^1 = \frac{GC\omega}{2r^3}y \tag{4.58a}$$

$$\zeta^2 = -\frac{GC\omega}{2r^3}x \tag{4.58b}$$

$$\zeta^3 = 0. \tag{4.58c}$$

4.1.5 Potential ψ

The potential ψ is given in (2.74),

$$\psi = -\frac{G}{c^2}\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}[^{(2)}T^{\,00}(t,\mathbf{x}') +^{(2)} T^{\,jj}(t,\mathbf{x}')]d^{\,3}x'. \tag{4.59}$$

The energy-momentum components $^{(2)}T^{\,00}$ and $^{(2)}T^{\,jj}$ are given in (2.10c) by the expansion of the Lorentz factor γ,

$$\gamma(v) = U^0 = \frac{dt}{d\tau}, \tag{4.60}$$

where U^0 is the temporal component of the four-velocity and $\frac{dt}{d\tau}$ can be written as given in (3.13). Dropping $\mathcal{O}(\epsilon^4)$ and higher orders,

$$\gamma(v) = 1 + \frac{v^2}{2c^2} + \frac{1}{2}{}^{(2)}h_{00} + \mathcal{O}(\epsilon^4). \tag{4.61}$$

Then,

$$^{(2)}T^{\,00} = \rho c^2\left(\frac{v^2}{c^2} - \frac{2\phi}{c^2}\right), \tag{4.62}$$

where the term v^2 is neglected,

$$v^2 = (\epsilon_{ijk}\omega^j x^k)\cdot(\epsilon_{ijk}\omega^j x^k) \sim \omega^2.$$

The $^{(2)}T^{\,jj}$ term is neglected,

$$c^{-2(2)}T^{\,jj} = \rho v^j v^j \tag{4.63}$$

$$v^j v^j = (\epsilon_{jik}\omega^i x^k)\cdot(\epsilon_{jik}\omega^i x^k) \sim \omega^2.$$

The ψ potential up to first order of approximation in ω is

$$\begin{aligned}\psi &= -G\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')\left(-\frac{2\phi}{c^2}\right)d^{\,3}x' \\ &= \frac{2G}{c^2}\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')\phi(\mathbf{x}')d^{\,3}x'.\end{aligned} \tag{4.64}$$

Expand the term $|\mathbf{x}-\mathbf{x}'|^{-1}$ as in (4.53)

$$\begin{aligned}\psi &= \frac{2G}{c^2}\left[\frac{1}{r}\int \rho(\mathbf{x}')\phi(\mathbf{x}')d^{\,3}x' - \partial_m\left(\frac{1}{r}\right)\int \rho(\mathbf{x}')\phi(\mathbf{x}')x'^{m}d^{\,3}x'\right] + \ldots \\ &= \frac{2G}{c^2}\left[\frac{1}{r}\int \rho(\mathbf{x}')\phi(\mathbf{x}')d^{\,3}x' + \frac{x^m}{r^3}\int \rho(\mathbf{x}')\phi(\mathbf{x}')x'^{m}d^{\,3}x'\right] + \ldots \quad .\end{aligned}$$

Expand the potential $\phi(\mathbf{x}')$

$$\phi(\mathbf{x}') = \phi(\mathbf{x}) + (x'^j - x^j)\partial_j\phi(\mathbf{x}) + ... \tag{4.65}$$

to obtain

$$\psi = \frac{2G}{c^2}\left[\frac{1}{r}\phi(\mathbf{x})\int\rho(\mathbf{x}')d^3x' + \frac{1}{r}\partial_k\phi(\mathbf{x})\int\rho(\mathbf{x}')(x'^k - x^k)d^3x'\right.$$
$$\left.+\frac{x^m}{r^3}\phi(\mathbf{x})\int\rho(\mathbf{x}')x'^m d^3x' + \frac{x^m}{r^3}\partial_k\phi(\mathbf{x})\int\rho(\mathbf{x}')(x'^k - x^k)x'^m d^3x'\right] + ... \quad .$$

As usual, the terms $\int\rho(\mathbf{x})xd^3x = 0$ and $\int\rho(\mathbf{x})d^3x = M$,

$$\begin{aligned}\psi &= \frac{2G}{c^2}\left[\frac{M}{r}\phi(\mathbf{x}) - \frac{Mx^k}{r}\partial_k\phi(\mathbf{x}) + \frac{x^m}{r^3}\partial_k\phi(\mathbf{x})\underbrace{\int\rho(\mathbf{x}')x'^k x'^m d^3x'}_{q^{km}}\right] \\ &= \frac{2m}{r}\left[\phi(\mathbf{x}) - x^k\partial_k\phi(\mathbf{x}) + \frac{x^m}{Mr^2}\partial_k\phi(\mathbf{x})q^{km}\right].\end{aligned} \tag{4.66}$$

The second term is

$$\begin{aligned}x^k\partial_k\phi(\mathbf{x}) &= x^k\left[\frac{GM}{r^3}x^k - \frac{3Q}{2r^5}x^k\left(1 - \frac{3z^2}{r^2}\right) + \frac{Q}{2r^3}\left(-\frac{6z}{r^2}\delta^{kz} + \frac{6z^2}{r^4}x^k\right)\right] \\ &= \frac{GM}{r} - \frac{3Q}{2r^3}\left(1 - \frac{3z^2}{r^2}\right)\end{aligned} \tag{4.67}$$

and the third term is

$$\begin{aligned}\frac{x^m}{Mr^2}\partial_k\phi(\mathbf{x})q^{km} &= \frac{x^m}{Mr^2}q^{km}\left[\frac{GM}{r^3}x^k + Q(r,M,G)\right] \\ &= \frac{q^{km}x^m}{r^2}\frac{G}{r^3}x^k + (Q^2) \\ &= \frac{G}{r^5}\left(x^2q^{11} + y^2q^{22} + z^2q^{33}\right) = \frac{G}{r^5}\left[\frac{C}{2}(x^2 + y^2 - z^2) + Az^2\right] \\ &= \frac{G}{r^5}\left[\frac{C}{2}(r^2 - 2z^2) + Az^2\right] \\ &= \frac{GC}{2r^3} + \frac{Qz^2}{r^5},\end{aligned} \tag{4.68}$$

where $Q(r, M, G)$ refers to the term of the derivative $\partial_k\phi(\mathbf{x})$ that has dependence on Q, and Q^2 to the term involving multiplication of the inertia moments.

Replace (4.67), (4.68) and the Newtonian potential in (4.66),

$$\begin{aligned}\psi &= \frac{2m}{r}\left[-\frac{GM}{r}+\frac{Q}{2r^3}\left(1-\frac{3z^2}{r^2}\right)-\frac{GM}{r}-\frac{3Q}{2r^3}\left(1-\frac{3z^2}{r^2}\right)+\frac{GC}{2r^3}+\frac{Qz^2}{r^5}\right]\\ &= -\frac{4GMm}{r^2}+\frac{2Qm}{r^4}\left(2-\frac{5z^2}{r^2}\right)+\frac{GmC}{r^4}.\end{aligned} \tag{4.69}$$

4.1.6 Parametrized Metric

With all the necessary elements, the metric given in (4.51) can be written, in components as

$$^{(2)}h_{00} = \frac{2m}{r}\frac{Q}{c^2r^3}\left(1-\frac{3z^2}{r^2}\right) \tag{4.70a}$$

$$\begin{aligned}^{(4)}h_{00} &= \frac{6m^2}{r^2}-\frac{2Qm}{c^2r^4}\left(3-\frac{7z^2}{r^2}\right)-\frac{2GmC}{c^2r^4}\\ &= \frac{6m^2}{r^2}-\frac{6Qm}{c^2r^4}\left(1-\frac{3z^2}{r^2}\right)-\frac{4Qm}{c^2r^4}\frac{z^2}{r^2}-\frac{2GmC}{c^2r^4}\end{aligned} \tag{4.70b}$$

$$^{(3)}h_{01} = \frac{2GC\omega}{c^3r^3}y \tag{4.70c}$$

$$^{(3)}h_{02} = -\frac{2GC\omega}{c^3r^3}x \tag{4.70d}$$

$$^{(3)}h_{03} = 0 \tag{4.70e}$$

$$^{(2)}h_{ij} = \frac{2m}{r}\delta_{ij}-\frac{Q}{c^2r^3}\left(1-\frac{3z^2}{r^2}\right)\delta_{ij}. \tag{4.70f}$$

It is convenient to parameterize this metric, just as did in Chapter 3,

$$^{(2)}h_{00} = \frac{2m}{r}\frac{Q}{c^2r^3}\left(1-\frac{3z^2}{r^2}\right) \tag{4.71a}$$

$$\begin{aligned}^{(4)}h_{00} &= 2[3+(\alpha-4)\beta]\frac{m^2}{r^2}-[6+(3\alpha-8)\beta]\frac{Qm}{c^2r^4}\left(1-\frac{3z^2}{r^2}\right)\\ &\quad -4(1-\beta)\frac{Qm}{c^2r^4}\frac{z^2}{r^2}-2(1-\beta)\frac{GmC}{c^2r^4}\end{aligned} \tag{4.71b}$$

$$^{(3)}h_{01} = \frac{2GC\omega}{c^3r^3}y \tag{4.71c}$$

$$^{(3)}h_{02} = -\frac{2GC\omega}{c^3r^3}x \tag{4.71d}$$

$$^{(3)}h_{03} = 0 \tag{4.71e}$$

$$^{(2)}h_{ij} = 2(1-\alpha\beta)\frac{2m}{r}\delta_{ij}+2\alpha\beta\frac{m}{r^3}x^ix^j-\frac{Q}{c^2r^3}\left(1-\frac{3z^2}{r^2}\right)\delta_{ij}. \tag{4.71f}$$

Note here that with $\beta = 0$ the metric in (4.71) recovers (4.70), and with $\beta = 1$ the metric reduces to that found in [3].

4.2 Construction of the Perturbing Force

4.2.1 Lagrangian

Consider the Lagrangian described in the equation (3.14) and replace the metric given in (4.71),

$$
\begin{aligned}
L = & \frac{1}{2}\dot{\mathbf{x}}^2 + \frac{1}{2}c^2\left[\frac{2m}{r} - \frac{Q}{r^3c^2}\left(1-\frac{3z^2}{r^2}\right)\right] + \frac{1}{8}\frac{(\dot{\mathbf{x}}^2)^2}{c^2} + \frac{1}{2}c^2\left[2[3+(\alpha-4)\beta]\frac{m^2}{r^2}\right.\\
& \left. -[6+(3\alpha-8)\beta]\frac{Qm}{c^2r^4}\left(1-\frac{3z^2}{r^2}\right) - 4(1-\beta)\frac{Qm}{c^2r^4}\frac{z^2}{r^2} - 2(1-\beta)\frac{GmC}{c^2r^4}\right]\\
& + c\left[-\frac{2GC\omega}{c^3r^3}(x\dot{y}-y\dot{x})\right] + \frac{1}{2}\left[2(1-\alpha\beta)\frac{m}{r}\delta_{jk} + 2\alpha\beta\frac{m}{r^3}x^jx^k - \frac{Q}{r^3c^2}\left(1-\frac{3z^2}{r^2}\right)\delta_{jk}\right]\dot{x}^j\dot{x}^k\\
& + \frac{1}{8}c^2\left[\frac{4m^2}{r^2} - \frac{4Qm}{c^2r^4}\left(1-\frac{3z^2}{r^2}\right)\right] + \frac{1}{4}\left[\frac{2m}{r} - \frac{Q}{r^3c^2}\left(1-\frac{3z^2}{r^2}\right)\right]\dot{\mathbf{x}}^2 + \mathcal{O}(\epsilon^3, Q^2).
\end{aligned}
\tag{4.72}
$$

Grouping terms,

$$
\begin{aligned}
L_2 = & \frac{1}{8}\frac{(\dot{\mathbf{x}}^2)^2}{c^2} + [3+(\alpha-4)\beta]c^2\frac{m^2}{r^2} + (1-\alpha\beta)\frac{m}{r}\delta_{jk}\dot{x}^j\dot{x}^k + \alpha\beta\frac{m}{r^3}x^jx^k\dot{x}^j\dot{x}^k + \frac{1}{2}\frac{m^2c^2}{r^2} + \frac{m}{2r}\dot{\mathbf{x}}^2\\
= & \frac{1}{8}\frac{(\dot{\mathbf{x}}^2)^2}{c^2} + \dot{\mathbf{x}}^2\left[\frac{m}{2r} + (1-\alpha\beta)\frac{m}{r}\right] + \frac{c^2m^2}{r^2}\left[(3+(\alpha-4)\beta) + \frac{1}{2}\right] + \frac{\alpha\beta m}{r^3}(\mathbf{x}\cdot\dot{\mathbf{x}})^2\\
= & \frac{1}{8}\frac{(\dot{\mathbf{x}}^2)^2}{c^2} + \frac{m}{r}\left[\left(\frac{3}{2}-\alpha\beta\right)\dot{\mathbf{x}}^2 + c^2\frac{m}{r}\left(\frac{7}{2}+(\alpha-4)\beta\right) + \alpha\beta\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{r^2}\right]
\end{aligned}
\tag{4.73a}
$$

$$
L_3 = c\left[-\frac{2GC\omega}{c^3r^3}(x\dot{y}-y\dot{x})\right]
\tag{4.73b}
$$

$$
\begin{aligned}
L_4 = & -[6+(3\alpha-8)\beta]\frac{Qm}{2r^4}\left(1-\frac{3z^2}{r^2}\right) - 2(1-\beta)\frac{Qm}{r^4}\frac{z^2}{r^2} - \frac{Q}{2r^3c^2}\left(1-\frac{3z^2}{r^2}\right)\dot{\mathbf{x}}^2\\
& - \frac{Qm}{2r^4}\left(1-\frac{3z^2}{r^2}\right) - \frac{Q}{4r^3c^2}\left(1-\frac{3z^2}{r^2}\right)\dot{\mathbf{x}}^2 - (1-\beta)\frac{GmC}{r^4}\\
= & \frac{Q}{2c^2r^3}\left(1-\frac{3z^2}{r^2}\right)\left[-\frac{3}{2}\dot{\mathbf{x}}^2 - [7+(3\alpha-8)\beta]\frac{mc^2}{r}\right] - 2(1-\beta)\frac{Qm}{r^4}\frac{z^2}{r^2} - (1-\beta)\frac{GmC}{r^4}
\end{aligned}
$$

The Lagrangian is given by

$$L = L_0 + L_1 + L_2 + L_3 + L_4 \tag{4.74}$$

where

$$L_0 = \frac{1}{2}\dot{\mathbf{x}}^2 + \frac{GM}{r} \tag{4.75a}$$

$$L_1 = -\frac{Q}{2r^3}\left(1 - \frac{3z^2}{r^2}\right) \tag{4.75b}$$

$$L_2 = \frac{1}{8}\frac{(\dot{\mathbf{x}}^2)^2}{c^2} + \frac{m}{r}\left[\left(\frac{3}{2} - \alpha\beta\right)\dot{\mathbf{x}}^2 + c^2\frac{m}{r}\left(\frac{7}{2} + (\alpha-4)\beta\right) + \alpha\beta\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{r^2}\right] \tag{4.75c}$$

$$L_3 = -\frac{2GC\omega}{c^2r^3}(x\dot{y} - y\dot{x}) \tag{4.75d}$$

$$L_4 = \frac{Q}{2c^2r^3}\left(1 - \frac{3z^2}{r^2}\right)\left[-\frac{3}{2}\dot{\mathbf{x}}^2 - [7 + (3\alpha - 8)\beta]\frac{mc^2}{r}\right] \tag{4.75e}$$

$$-2(1-\beta)\frac{Qm}{r^4}\frac{z^2}{r^2} - (1-\beta)\frac{GmC}{r^4} \tag{4.75f}$$

4.3 Perturbing Force

With the Lagrangian given above calculate the perturbing force, via the Euler Lagrange equations. The term L_0 is the Newtonian contribution, i.e.,it is not part of the perturbation. Although the perturbing forces do not arise from a potential, those forces can be found by

$$F = \frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}}\right). \tag{4.76}$$

For L_1, the Euler Lagrange equations give

$$\mathbf{F_1} = \frac{\partial}{\partial \mathbf{x}}\left[-\frac{Q}{2r^3}\left(1 - \frac{3z^2}{r^2}\right)\right] - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{\mathbf{x}}}\left(-\frac{Q}{2r^3}\left(1 - \frac{3z^2}{r^2}\right)\right)\right]$$

$$\begin{aligned}
\mathbf{F_1} &= -\frac{Q}{2}\left[-\frac{3}{r^5}\left(1 - \frac{3z^2}{r^2}\right)\mathbf{x} + \frac{1}{r^3}\left(-\frac{6z}{r^2}\hat{\mathbf{e}}_{\mathbf{k}} + \frac{6z^2}{r^4}\mathbf{x}\right)\right] \\
&= \frac{3Q}{r^5}\left[\frac{1}{2}\left(1 - \frac{3z^2}{r^2}\right)\mathbf{x} + z\hat{\mathbf{e}}_{\mathbf{k}} - \frac{z^2}{r^2}\mathbf{x}\right] \\
&= \frac{3Q}{r^5}\left[\frac{1}{2}\left(1 - \frac{5z^2}{r^2}\right)\mathbf{x} + z\hat{\mathbf{e}}_{\mathbf{k}}\right].
\end{aligned}$$

For L_2, the Euler Lagrange equations give

$$\begin{aligned}
\mathbf{F_2} =& \frac{\partial}{\partial \mathbf{x}}\left\{\frac{m}{r}\left[\left(\frac{3}{2} - \alpha\beta\right)\dot{\mathbf{x}}^2 + c^2\frac{m}{r}\left(\frac{7}{2} + (\alpha-4)\beta\right) + \alpha\beta\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{r^2}\right]\right\} \\
&- \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\mathbf{x}}}\left(\frac{1}{8}\frac{(\dot{\mathbf{x}}^2)^2}{c^2} + \frac{m}{r}\left[\left(\frac{3}{2} - \alpha\beta\right)\dot{\mathbf{x}}^2 + \alpha\beta\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{r^2}\right]\right)\right\}.
\end{aligned}$$

The derivative with respect to $\mathbf{x}$ is

$$
\begin{aligned}
&\frac{\partial}{\partial \mathbf{x}}\left\{\frac{m}{r}\left[\left(\frac{3}{2}-\alpha\beta\right)\dot{\mathbf{x}}^2+c^2\frac{m}{r}\left(\frac{7}{2}+(\alpha-4)\beta\right)+\alpha\beta\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{r^2}\right]\right\}\\
&\quad=-\frac{m}{r^3}\left(\frac{3}{2}-\alpha\beta\right)\dot{\mathbf{x}}^2\mathbf{x}-\frac{2GMm}{r^4}\left(\frac{7}{2}+(\alpha-4)\beta\right)\mathbf{x}\\
&\quad=-\frac{3m\alpha\beta}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})^2\mathbf{x}+\frac{2m\alpha\beta}{r^3}(\mathbf{x}\cdot\dot{\mathbf{x}})\mathbf{x}.
\end{aligned}
\tag{4.77}
$$

For the derivative with respect to t, let see the first term,

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\mathbf{x}}}\left(\frac{1}{8}\frac{(\dot{\mathbf{x}}^2)^2}{c^2}\right)\right\}=\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\mathbf{x}}}\left(\frac{1}{8}\frac{(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})^2}{c^2}\right)\right\}=\frac{d}{dt}\left\{\frac{2}{8}\frac{(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})(\dot{\mathbf{x}}+\dot{\mathbf{x}})}{c^2}\right\}\\
&\frac{d}{dt}\left\{\frac{1}{2}\frac{(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}}{c^2}\right\}=\frac{1}{2c^2}\left\{2(\ddot{\mathbf{x}}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}+(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\ddot{\mathbf{x}}\right\}.
\end{aligned}
$$

The other terms in the temporal derivative,

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{m}{r}\left[\left(\frac{3}{2}-\alpha\beta\right)\dot{\mathbf{x}}^2+\alpha\beta\frac{(\mathbf{x}\cdot\dot{\mathbf{x}})^2}{r^2}\right]\right\}\\
&\quad=\underbrace{\frac{1}{2c^2}\left\{2(\ddot{\mathbf{x}}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}+(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\ddot{\mathbf{x}}\right\}}_{1.\mathbf{F_4}}-\frac{2m}{r^3}\left(\frac{3}{2}-\alpha\beta\right)(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}+\underbrace{\frac{2m}{r}\left(\frac{3}{2}-\alpha\beta\right)\ddot{\mathbf{x}}}_{2.\mathbf{F_4}}-\frac{6m\alpha\beta}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})^2\mathbf{x}\\
&\quad+\frac{2m\alpha\beta}{r^3}\left[(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\mathbf{x}+\underbrace{(\mathbf{x}\cdot\ddot{\mathbf{x}})\mathbf{x}}_{3.\mathbf{F_4}}+(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}\right].
\end{aligned}
\tag{4.78}
$$

The second derivative $\ddot{\mathbf{x}}$ in the equations above, according to (3.15), can be written as

$$
\ddot{\mathbf{x}}=-\nabla\phi,
$$

where ϕ is given by (4.50)

$$
\ddot{\mathbf{x}}=-\nabla\left[-\frac{mc^2}{r}+\frac{Q}{2r^3}\left(1-\frac{3z^2}{r^2}\right)\right]=-\frac{GM}{r^3}\mathbf{x}-\nabla\left[\frac{Q}{2r^3}\left(1-\frac{3z^2}{r^2}\right)\right].
\tag{4.79}
$$

To preserve the order of this contribution (without Q), we only consider the first term of $\ddot{\mathbf{x}}$, then (4.78) becomes

$$\frac{1}{2c^2}\left\{-2\frac{GM}{r^3}(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}-\frac{GM}{r^3}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\mathbf{x}\right\}-\frac{2m}{r^3}\left(\frac{3}{2}-\alpha\beta\right)(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}-\frac{2GMm}{r^4}\left(\frac{3}{2}-\cancel{\alpha\beta}\right)\mathbf{x}$$

$$-\frac{6m\alpha\beta}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})^2\mathbf{x}+\frac{2m\alpha\beta}{r^3}\left[(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\mathbf{x}-\cancel{\frac{GM}{r^3}(\mathbf{x}\cdot\mathbf{x})\mathbf{x}}+(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}\right]$$

$$=\frac{1}{2c^2}\left\{-2\frac{GM}{r^3}(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}-\frac{GM}{r^3}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\mathbf{x}\right\}-\frac{2m}{r^3}\left(\frac{3}{2}-\alpha\beta\right)(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}-\frac{3GMm}{r^4}\mathbf{x}$$

$$-\frac{6m\alpha\beta}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})^2\mathbf{x}+\frac{2m\alpha\beta}{r^3}\left[(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\mathbf{x}+(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}\right]$$

$$=-\frac{m}{r^3}(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}-\frac{m}{2r^3}\dot{\mathbf{x}}^2\mathbf{x}-\frac{3GMm}{r^4}\mathbf{x}+\frac{2m\alpha\beta}{r^3}\dot{\mathbf{x}}^2\mathbf{x}-\frac{6m\alpha\beta}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})^2\mathbf{x}$$

$$-\frac{2m}{r^3}\left(\frac{3}{2}-2\alpha\beta\right)(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}. \tag{4.80}$$

The second term in (4.79) (with Q) is going to be used for the $\mathbf{F}_4$ contribution.

Adding (4.77) and (4.80)

$$\mathbf{F_2}=-\frac{m}{r^3}\left(\frac{3}{2}-\alpha\beta\right)\dot{\mathbf{x}}^2\mathbf{x}-\frac{2GMm}{r^4}\left(\frac{7}{2}+(\alpha-4)\beta\right)\mathbf{x}-\frac{3m\alpha\beta}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})^2\mathbf{x}+\frac{2m\alpha\beta}{r^3}(\mathbf{x}\cdot\dot{\mathbf{x}})\mathbf{x}$$

$$+\frac{m}{r^3}(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}+\frac{m}{r^3}\dot{\mathbf{x}}^2\mathbf{x}-\frac{2m}{r^3}\left(\frac{3}{2}-2\alpha\beta\right)(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}+\frac{3GMm}{r^4}\mathbf{x}$$

$$+\frac{6m\alpha\beta}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})^2\mathbf{x}-\frac{2m\alpha\beta}{r^3}(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\mathbf{x}$$

$$\mathbf{F_2}=-\frac{m}{r^3}(1+\alpha\beta)\dot{\mathbf{x}}^2\mathbf{x}+\frac{3m\alpha\beta}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})^2\mathbf{x}+\frac{2m}{r^3}(2-\alpha\beta)(\mathbf{x}\cdot\dot{\mathbf{x}})\mathbf{x}+\frac{2GMm}{r^4}(-4+8\beta-2\alpha\beta)\mathbf{x}$$

$$\mathbf{F_2}=\frac{m}{r^3}\left\{\left[\frac{3\alpha\beta}{r^2}(\mathbf{x}\cdot\dot{\mathbf{x}})^2+\frac{GM}{r}[-2\beta(\alpha-4)-4]-(1+\alpha\beta)\dot{\mathbf{x}}^2\right]\mathbf{x}+(4-2\alpha\beta)(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}\right\}.$$

For L_3, the Euler Lagrange equations give

$$\mathbf{F_3} = \frac{d}{d\mathbf{x}}\left\{-\frac{2GC\omega}{c^2r^3}(x\dot{y} - y\dot{x})\right\} - \frac{d}{dt}\left\{\frac{d}{d\dot{\mathbf{x}}}\left(-\frac{2GC\omega}{c^2r^3}(x\dot{y} - y\dot{x})\right)\right\}.$$

Calculate the first term

$$\mathbf{x} \times \dot{\mathbf{x}} = (y\dot{z} - z\dot{y})\hat{\mathbf{i}} - (x\dot{z} - z\dot{x})\hat{\mathbf{j}} + (x\dot{y} - y\dot{x})\hat{\mathbf{e}}_{\mathbf{k}}$$

and then, the equation can be written as

$$\mathbf{F_3} = \frac{\partial}{\partial\mathbf{x}}\left\{-\left[\frac{2GC\omega}{c^2r^3}(\mathbf{x} \times \dot{\mathbf{x}}) \cdot \hat{\mathbf{e}}_{\mathbf{k}}\right]\right\} - \frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\mathbf{x}}}\left(-\left[\frac{2GC\omega}{c^2r^3}(\mathbf{x} \times \dot{\mathbf{x}}) \cdot \hat{\mathbf{e}}_{\mathbf{k}}\right]\right)\right\}.$$

To solve the derivatives, lets introduce the cross product in the matrix form,

$$\mathbf{x} \times \dot{\mathbf{x}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -z\dot{y} + y\dot{z} \\ z\dot{x} - x\dot{z} \\ -y\dot{x} + x\dot{y} \end{bmatrix}.$$

This is notated as

$$\mathbf{x} \times \dot{\mathbf{x}} = [\mathbf{x}] \cdot \dot{\mathbf{x}}. \tag{4.81}$$

Hence,

$$\mathbf{F_3} =\frac{\partial}{\partial \mathbf{x}}\left\{-\left[\frac{2GC\omega}{c^2r^3}([\mathbf{x}]\cdot\dot{\mathbf{x}})\cdot\hat{\mathbf{e}}_\mathbf{k}\right]\right\}-\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\mathbf{x}}}\left(-\left[\frac{2GC\omega}{c^2r^3}([\mathbf{x}]\cdot\dot{\mathbf{x}})\cdot\hat{\mathbf{e}}_\mathbf{k}\right]\right)\right\}$$

$$=\frac{2GC\omega}{c^2}\left\{\frac{3}{r^5}\mathbf{x}(\mathbf{x}\times\dot{\mathbf{x}})\cdot\hat{\mathbf{e}}_\mathbf{k}-\frac{1}{r^3}[\dot{\mathbf{x}}]\cdot\hat{\mathbf{e}}_\mathbf{k}\right\}+\frac{2GC\omega}{c^2}\frac{d}{dt}\left\{-\frac{1}{r^3}[\mathbf{x}]\cdot\hat{\mathbf{e}}_\mathbf{k}\right\}$$

$$=\frac{2GC\omega}{c^2r^3}\left\{\frac{3}{r^2}\mathbf{x}(\mathbf{x}\times\dot{\mathbf{x}})\cdot\hat{\mathbf{e}}_\mathbf{k}-(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k})\right\}+\frac{2GC\omega}{c^2}\left\{\frac{3}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})(\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{k})-\frac{1}{r^3}(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k})\right\}$$

$$=\frac{2GC\omega}{c^2r^3}\left\{\frac{3}{r^2}\mathbf{x}(\mathbf{x}\times\dot{\mathbf{x}})\cdot\hat{\mathbf{e}}_\mathbf{k}-2(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k})+\frac{3}{r^2}(\mathbf{x}\cdot\dot{\mathbf{x}})(\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{k})\right\}$$

$$\begin{aligned}=\frac{2GC\omega}{c^2r^3}\Big\{&-2(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k})+\frac{3}{r^2}x(x\dot{y}-y\dot{x})\hat{\mathbf{e}}_\mathbf{x}+\frac{3}{r^2}(x\dot{x}+y\dot{y}+z\dot{z})y\hat{\mathbf{e}}_\mathbf{x}\\&+\frac{3}{r^2}y(x\dot{y}-y\dot{x})\hat{\mathbf{e}}_\mathbf{y}-\frac{3}{r^2}(x\dot{x}+y\dot{y}+z\dot{z})x\hat{\mathbf{e}}_\mathbf{y}+\frac{3z}{r^2}(x\dot{y}-y\dot{x})\hat{\mathbf{e}}_\mathbf{z}\Big\}\end{aligned}$$

$$=\frac{2GC\omega}{c^2r^3}\left\{-2(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k})+\frac{3}{r^2}(x^2+y^2)\dot{y}\hat{\mathbf{e}}_\mathbf{x}+\frac{3}{r^2}yz\dot{z}\hat{\mathbf{e}}_\mathbf{x}-\frac{3}{r^2}(x^2+y^2)\dot{x}\hat{\mathbf{e}}_\mathbf{y}-\frac{3}{r^2}xz\dot{z}\hat{\mathbf{e}}_\mathbf{y}+\frac{3z}{r^2}(x\dot{y}-y\dot{x})\hat{\mathbf{e}}_\mathbf{z}\right\}$$

$$=\frac{2GC\omega}{c^2r^3}\left\{-2(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k})+\frac{3}{r^2}(r^2-z^2)(\dot{y}\hat{\mathbf{e}}_\mathbf{x}-\dot{x}\hat{\mathbf{e}}_\mathbf{y})+\frac{3z}{r^2}y\dot{z}\hat{\mathbf{e}}_\mathbf{x}-\frac{3x}{r^2}x\dot{z}\hat{\mathbf{e}}_\mathbf{y}+\frac{3z}{r^2}(x\dot{y}-y\dot{x})\hat{\mathbf{e}}_\mathbf{z}\right\}$$

$$=\frac{2GC\omega}{c^2r^3}\left\{-2(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k})+3(\dot{y}\hat{\mathbf{e}}_\mathbf{x}-\dot{x}\hat{\mathbf{e}}_\mathbf{y})-\frac{3z}{r^2}(z\dot{y}\hat{\mathbf{e}}_\mathbf{x}-z\dot{x}\hat{\mathbf{e}}_\mathbf{y})+\frac{3z}{r^2}y\dot{z}\hat{\mathbf{e}}_\mathbf{x}-\frac{3x}{r^2}x\dot{z}\hat{\mathbf{e}}_\mathbf{y}+\frac{3z}{r^2}(x\dot{y}-y\dot{x})\hat{\mathbf{e}}_\mathbf{z}\right\}$$

$$\mathbf{F_3} =\frac{2GC\omega}{c^2r^3}\left\{(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k})+\frac{3z}{r^2}\left[(y\dot{z}-z\dot{y})\hat{\mathbf{e}}_\mathbf{x}+(z\dot{x}-x\dot{z})\hat{\mathbf{e}}_\mathbf{z}+(x\dot{y}-y\dot{x})\hat{\mathbf{e}}_\mathbf{z}\right]\right\}$$

$$\mathbf{F_3} =\frac{2GC\omega}{c^2r^3}\left\{(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k})+\frac{3z}{r^2}(\mathbf{x}\times\dot{\mathbf{x}})\right\}.$$

For L_4, the Euler Lagrange equations are

$$\begin{aligned}\mathbf{F_4} =&\frac{\partial}{\partial\mathbf{x}}\left\{\frac{Q}{2c^2r^3}\left(1-\frac{3z^2}{r^2}\right)\left[-\frac{3}{2}\dot{\mathbf{x}}^2-\frac{GM}{r}[7+(3\alpha-8)\beta]\right]\right\}+\frac{\partial}{\partial\mathbf{x}}\left\{-2(1-\beta)\frac{Qm}{r^4}\frac{z^2}{r^2}-\frac{GmC}{r^3}(1-\beta)\mathbf{x}\right\}\\&-\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{\mathbf{x}}}\left(\frac{Q}{2c^2r^3}\left(1-\frac{3z^2}{r^2}\right)\left[-\frac{3}{2}\dot{\mathbf{x}}^2-\frac{GM}{r}[7+(3\alpha-8)\beta]\right]\right)\right\}.\end{aligned}$$

where the gray term is going to be include at the end of all the calculations.

The derivative with respect to $\mathbf{x}$ is

$$\frac{\partial}{\partial \mathbf{x}}\left\{\frac{Q}{2c^2r^3}\left(1-\frac{3z^2}{r^2}\right)\left[-\frac{3}{2}\dot{\mathbf{x}}^2-\frac{GM}{r}[7+(3\alpha-8)\beta]\right]\right\}$$

$$\begin{aligned}
&=\frac{Q}{2c^2}\left\{\left[-\frac{3\mathbf{x}}{r^5}-\frac{6z\hat{\mathbf{e}}_{\mathbf{k}}}{r^5}+\frac{15z^2\hat{\mathbf{e}}_{\mathbf{k}}}{r^7}\right]\left[-\frac{3}{2}\dot{\mathbf{x}}^2-\frac{GM}{r}[7+(3\alpha-8)\beta]\right]\right.\\
&\qquad\left.+\frac{1}{r^3}\left(1-\frac{3z^2}{r^2}\right)\left[[7+(3\alpha-8)\beta]\frac{GM}{r^3}\mathbf{x}\right]\right\}
\end{aligned}$$

$$\begin{aligned}
&=\frac{Q}{2c^2}\left\{\frac{3\mathbf{x}}{r^6}GM[7+(3\alpha-8)\beta]+\frac{9\dot{\mathbf{x}}^2\mathbf{x}}{2r^5}+\frac{6z\hat{\mathbf{e}}_{\mathbf{k}}}{r^6}GM[7+(3\alpha-8)\beta]+9z\dot{\mathbf{x}}^2\hat{\mathbf{e}}_{\mathbf{k}}\right.\\
&\qquad\left.-\frac{15z^2\mathbf{x}}{r^8}GM[7+(3\alpha-8)\beta]-\frac{45z^2\dot{\mathbf{x}}^2\mathbf{x}}{2r^7}+\frac{\mathbf{x}}{r^6}GM[7+(3\alpha-8)\beta]-\frac{3z^2\mathbf{x}}{r^8}GM[7+(3\alpha-8)\beta]\right\}.
\end{aligned}$$

The derivative with respect to $\dot{\mathbf{x}}$ is

$$\frac{\partial}{\partial \dot{\mathbf{x}}}\left\{\frac{Q}{2c^2r^3}\left(1-\frac{3z^2}{r^2}\right)\left[-\frac{3}{2}\dot{\mathbf{x}}^2-\frac{GM}{r}[7+(3\alpha-8)\beta]\right]\right\}=-\frac{3Q}{2c^2}\frac{1}{r^3}\left(1-\frac{3z^2}{r^2}\right)\dot{\mathbf{x}}.$$

The derivative with respect to t gives

$$\begin{aligned}
&\frac{d}{dt}\left\{-\frac{3Q}{2c^2}\frac{1}{r^3}\left(1-\frac{3z^2}{r^2}\right)\dot{\mathbf{x}}\right\}\\
&\quad=\frac{Q}{2c^2}\left\{\left[-\frac{3}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})-\frac{6z\dot{z}}{r^5}+\frac{15z^2}{r^7}(\mathbf{x}\cdot\dot{\mathbf{x}})\right](-3\dot{\mathbf{x}})+\frac{1}{r^3}\left(1-\frac{3z^2}{r^2}\right)(-3\ddot{\mathbf{x}})\right\}.
\end{aligned}$$

The contributions due to $\ddot{\mathbf{x}}$ given in (4.79) must be taking into account for the first (without Q) and the second term. Let first see the contribution related to the newtonian potential without quadrupole,

$$\begin{aligned}
&\frac{Q}{2c^2}\left\{\left[-\frac{3}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})-\frac{6z\dot{z}}{r^5}+\frac{15z^2}{r^7}(\mathbf{x}\cdot\dot{\mathbf{x}})\right](-3\dot{\mathbf{x}})+\frac{1}{r^3}\left(1-\frac{3z^2}{r^2}\right)(-3\ddot{\mathbf{x}})\right\}\Big|_{\ddot{\mathbf{x}}:\text{without Q}}\\
&=\frac{Q}{2c^2}\left\{\frac{9}{r^5}(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}+\frac{18z\dot{z}}{r^5}\dot{\mathbf{x}}-\frac{45z^2}{r^7}(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}+\frac{3GM}{r^6}\mathbf{x}-\frac{9GMz^2}{r^8}\mathbf{x}\right\}.
\end{aligned}$$

Adding the derivative with respect to $\mathbf{x}$ and minus the derivative with respect to t and grouping the terms gives

$$\frac{Q}{2c^2}\frac{GM}{r}\mathbf{x}\left\{\frac{25}{r^5}+\frac{4(3\alpha-8)\beta}{r^5}-\frac{117z^2}{r^7}-\frac{18(3\alpha-8)\beta z^2}{r^7}\right\} \tag{4.82a}$$

$$+\frac{Q}{2c^2}\dot{\mathbf{x}}^2\mathbf{x}\left\{\frac{9}{2r^5}\left(1-\frac{5z^2}{r^2}\right)\right\} \tag{4.82b}$$

$$+\frac{Q}{2c^2}z\hat{\mathbf{e}}_{\mathbf{k}}\left\{6[7+(3\alpha-8)\beta]\frac{GM}{r^6}+\frac{9\dot{\mathbf{x}}^2}{r^5}\right\} \tag{4.82c}$$

$$-\frac{Q}{2c^2}\dot{\mathbf{x}}\left\{\frac{9}{r^5}\left[\left(1-\frac{5z^2}{r^2}\right)(\mathbf{x}\cdot\dot{\mathbf{x}})+2z\dot{z}\right]\right\}. \tag{4.82d}$$

The second term in (4.79) corresponds to the first perturbation calculated, given in (4.89a),

$$\ddot{\mathbf{x}}=-\frac{GM}{r^3}\mathbf{x}\underbrace{-\nabla\left[\frac{Q}{2r^3}\left(1-\frac{3z^2}{r^2}\right)\right]}_{\mathbf{F_1}}.$$

Due to the order of approximation ($Q^2\sim 0$), we only take into account the contributions due to F_2. From (4.78), consider first the term (2.$\mathbf{F_4}$) and (3.$\mathbf{F_4}$) with the respective minus sign,

$$-\left\{\frac{2m}{r}\left(\frac{3}{2}-\alpha\beta\right)\ddot{\mathbf{x}}+\frac{2m\alpha\beta}{r^3}(\mathbf{x}\cdot\ddot{\mathbf{x}})\mathbf{x}\right\}$$

$$=-\left\{\frac{2GM}{c^2r}\left(\frac{3}{2}-\alpha\beta\right)\left[\frac{3Q}{r^5}\left[\frac{1}{2}\left(1-\frac{5z^2}{r^2}\right)\mathbf{x}+z\hat{\mathbf{e}}_{\mathbf{k}}\right]\right]+\frac{2GM\alpha\beta}{c^2r^3}\frac{3Q}{r^5}\mathbf{x}\left[\frac{1}{2}\left(1-\frac{5z^2}{r^2}\right)(\mathbf{x}\cdot\mathbf{x})+(\mathbf{x}\cdot z\hat{\mathbf{e}}_{\mathbf{k}})\right]\right\}$$

$$=-\left\{\frac{9mQ}{2r^6}\mathbf{x}-\frac{45mQz^2}{2r^8}\mathbf{x}+\frac{12m\alpha\beta Qz^2}{2r^8}\mathbf{x}+z\hat{\mathbf{e}}_{\mathbf{k}}\left(\frac{9mQ}{r^6}-\frac{6m\alpha\beta Q}{r^6}\right)\right\}$$

$$=-\frac{Q}{2c^2}\frac{GM}{r}\mathbf{x}\left\{\frac{9}{r^5}-\frac{45z^2}{r^7}+\frac{12\alpha\beta z^2}{r^7}\right\} \tag{4.83a}$$

$$-\frac{Q}{2c^2}z\hat{\mathbf{e}}_{\mathbf{k}}\left\{\frac{18GM}{r^6}-\frac{12GM\alpha\beta}{r^6}\right\}. \tag{4.83b}$$

Now, consider the term ($1.\mathbf{F_4}$) from (4.78).

$$
\begin{aligned}
&-\frac{1}{2c^2}\left\{2(\ddot{\mathbf{x}}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}+(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}})\ddot{\mathbf{x}}\right\}\\
&\quad=-\frac{1}{2c^2}\left\{\frac{6Q}{r^5}\dot{\mathbf{x}}\left[\left(\frac{1}{2}-\frac{5z^2}{2r^2}\right)(\mathbf{x}\cdot\dot{\mathbf{x}})+\underbrace{(z\hat{\mathbf{e}}_{\mathbf{k}}\cdot\dot{\mathbf{x}})}_{z\dot{z}}\right]+\frac{3Q}{r^5}\dot{\mathbf{x}}^2\left[\left(\frac{1}{2}-\frac{5z^2}{2r^2}\right)\mathbf{x}+z\hat{\mathbf{e}}_{\mathbf{k}}\right]\right\}\\
&\quad=-\frac{Q}{2c^2}\dot{\mathbf{x}}\left\{\frac{3}{r^5}\left[\left(1-\frac{5z^2}{r^2}\right)(\mathbf{x}\cdot\dot{\mathbf{x}})+2z\dot{z}\right]\right\} &\text{(4.84a)}\\
&\qquad-\frac{Q}{2c^2}\dot{\mathbf{x}}^2\mathbf{x}\left\{\frac{3}{2r^5}\left(1-\frac{5z^2}{r^2}\right)\right\} &\text{(4.84b)}\\
&\qquad-\frac{Q}{2c^2}z\hat{\mathbf{e}}_{\mathbf{k}}\left\{\frac{3}{r^5}\dot{\mathbf{x}}^2\right\}. &\text{(4.84c)}
\end{aligned}
$$

Combining the terms given in (4.82a) and (4.83a),

$$
\frac{Q}{c^2}\frac{GM}{r^6}\mathbf{x}\left\{8+6\alpha\beta-16\beta+\frac{3z^2}{r^2}[-12-11\alpha\beta+24\beta]\right\}, \tag{4.85}
$$

(4.82b) and (4.84b),

$$
\frac{Q}{c^2r^5}\dot{\mathbf{x}}^2\mathbf{x}\left\{\frac{3}{2}\left(1-\frac{5z^2}{r^2}\right)\right\}, \tag{4.86}
$$

(4.82c), (4.83b) and (4.84c),

$$
\frac{3Q}{c^2r^5}z\hat{\mathbf{e}}_{\mathbf{k}}\left\{[4+(5\alpha-8)\beta]\frac{GM}{r}+\dot{\mathbf{x}}^2\right\}, \tag{4.87}
$$

(4.82d) and (4.84a),

$$
-\frac{6Q}{c^2r^5}\dot{\mathbf{x}}\left\{\left(1-\frac{5z^2}{r^2}\right)(\mathbf{x}\cdot\dot{\mathbf{x}})+2z\dot{z}.\right\} \tag{4.88}
$$

Finally $\mathbf{F_4}$ is

$$
\begin{aligned}
\mathbf{F_4}=\frac{Q}{c^2r^5}&\left\{\frac{GM}{r}\mathbf{x}\left[8+6\alpha\beta-16\beta+\frac{3z^2}{r^2}[-12-11\alpha\beta+24\beta]\right]+\dot{\mathbf{x}}^2\mathbf{x}\left[\frac{3}{2}\left(1-\frac{5z^2}{r^2}\right)\right]\right.\\
&\left.+3z\hat{\mathbf{e}}_{\mathbf{k}}\left[[4+(5\alpha-8)\beta]\frac{GM}{r}+\dot{\mathbf{x}}^2\right]-6\dot{\mathbf{x}}\left[\left(1-\frac{5z^2}{r^2}\right)(\mathbf{x}\cdot\dot{\mathbf{x}})+2z\dot{z}\right]\right\}\\
&-\frac{4(1-\beta)Qm}{r^6}\left[z\hat{\mathbf{e}}_{\mathbf{k}}-\frac{3z^2}{r^2}\mathbf{x}\right]+\frac{GmC}{r^6}4(1-\beta)\mathbf{x}.
\end{aligned}
$$

Summarizing, the perturbing force arising from the perturbative Lagrangian given in (4.75) is given by

$$\mathbf{F_1} = \frac{3Q}{r^5}\left[\frac{1}{2}\left(1-\frac{5z^2}{r^2}\right)\mathbf{x} + z\hat{\mathbf{e}}_\mathbf{k}\right] \tag{4.89a}$$

$$\begin{aligned}\mathbf{F_2} =& \frac{m}{r^3}\left\{\left[\frac{3\alpha\beta}{r^2}(\mathbf{x}\cdot\dot{\mathbf{x}})^2 + \frac{GM}{r}[-2\beta(\alpha-4)-4] - (1+\alpha\beta)\dot{\mathbf{x}}^2\right]\mathbf{x}\right.\\ &\left.+(4-2\alpha\beta)(\mathbf{x}\cdot\dot{\mathbf{x}})\dot{\mathbf{x}}\right\}\end{aligned} \tag{4.89b}$$

$$\mathbf{F_3} = \frac{2GC\omega}{c^2r^3}\left\{(\dot{\mathbf{x}}\times\hat{\mathbf{e}}_\mathbf{k}) + \frac{3z}{r^2}(\mathbf{x}\times\dot{\mathbf{x}})\right\} \tag{4.89c}$$

$$\begin{aligned}\mathbf{F_4} =& \frac{Q}{c^2r^5}\left\{\frac{GM}{r}\mathbf{x}\left[8+6\alpha\beta-16\beta+\frac{3z^2}{r^2}[-12-11\alpha\beta+24\beta]\right] + \dot{\mathbf{x}}^2\mathbf{x}\left[\frac{3}{2}\left(1-\frac{5z^2}{r^2}\right)\right]\right.\\ &\left.+3z\hat{\mathbf{e}}_\mathbf{k}\left[[4+(5\alpha-8)\beta]\frac{GM}{r}+\dot{\mathbf{x}}^2\right] - 6\dot{\mathbf{x}}\left[\left(1-\frac{5z^2}{r^2}\right)(\mathbf{x}\cdot\dot{\mathbf{x}}) + 2z\dot{z}\right]\right\}\\ &-\frac{4(1-\beta)Qm}{r^6}\left[z\hat{\mathbf{e}}_\mathbf{k} - \frac{3z^2}{r^2}\mathbf{x}\right] + \frac{GmC}{r^6}4(1-\beta)\mathbf{x}.\end{aligned} \tag{4.89d}$$

The term $\mathbf{F_1}$ is the newtonian perturbation due to the quadrupole moment. $\mathbf{F_2}$ is the relativist perturbation for one body problem, note here that this terms is exactly, less than parameters, as the equation of motion given in (3.53). $\mathbf{F_3}$ is the perturbation due to the rotation of the gravitational source, note here the presence of the angular velocity ω and the inertia moment in the z-direction C. The final term, $\mathbf{F_4}$, is the relativistic perturbation due to the quadrupole moment.

The perturbative Lagrangian in (4.75) and the perturbing force in (4.89) are the same given in [3], by defining $\beta = 1$.

4.3.1 Components of the perturbing force

The components in the direction of the time dependent basis of the perturbing force can be found as in (3.56). Where $\mathbf{x}$, $\dot{\mathbf{x}}$, $\mathbf{n}$, $\boldsymbol{\lambda}$ and $\mathbf{k}$ are the ones given in (1.11), (1.12), (1.8), (1.9), (1.10) respectively. The final components are in terms of the orbital elements and the true anomaly. The argument of the pericenter is now notated as $\bar{\omega}$.

$$\mathcal{R} = \frac{3Q}{2r^4} + \frac{mQ}{r^5}(8 + 6\alpha\beta - 16\beta) - \frac{2GMm}{r^3}[\beta(\alpha - 4) + 2] + \frac{GMm}{r^2 p}(4 + \alpha\beta)e^2 \sin^2 f$$

$$- \frac{GMm}{r^2 p}(1 + \alpha\beta)(1 + e^2 + 2e\cos f) + \frac{3Qm}{2r^4 p}(1 + e^2 + 2e\cos f) - \frac{9Q}{2r^4}\sin^2 \iota \sin^2(f + \bar{\omega})$$

$$+ \frac{2mQ}{r^5}(-8 - 9\alpha\beta + 20\beta)\sin^2 \iota \sin^2(f + \bar{\omega})$$

$$- \frac{9Qm}{2r^4 p}\sin^2 \iota \sin^2(f + \bar{\omega})(1 + e^2 + 2e\cos f) - \frac{6Qm}{r^4 p}e^2 \sin^2 f[1 - 5\sin^2 \iota \sin^2(f + \bar{\omega})]$$

$$- \frac{12Qm}{r^4 p}e \sin f \sin^2 \iota \sin(f + \bar{\omega})\left[e\cos\bar{\omega} + \cos(f + \bar{\omega})\right] + \frac{2GC\omega}{c^2 r^3}\sqrt{\frac{GM}{p}}\cos\iota(1 + e\cos f)$$

$$+ \frac{4GCm}{r^5}(1 - \beta)$$

$$\mathcal{T} = \frac{3Q}{2r^4}\sin^2 \iota \sin(2f + 2\bar{\omega}) + \frac{3Qm}{2r^5}\left[\frac{8}{3} + \left(5\alpha - \frac{20}{3}\right)\beta\right]\sin^2 \iota \sin(2f + 2\bar{\omega})$$

$$+ \frac{3Qm}{2r^4 p}\sin^2 \iota \sin(2f + 2\bar{\omega})(1 + e^2 + 2e\cos f) + \frac{2GMm}{r^2 p}(2 - \alpha\beta)e\sin f(1 + e\cos f)$$

$$- \frac{6Qm}{r^4 p}e\sin f(1 + e\cos f)[1 - 5\sin^2 \iota \sin^2(f + \bar{\omega})] - \frac{2GC\omega}{c^2 r^3}\sqrt{\frac{GM}{p}}e\cos\iota \sin f$$

$$- \frac{12Qm}{r^4 p}\sin^2 \iota \sin(f + \bar{\omega})(1 + e\cos f)\left[e\cos\bar{\omega} + \cos(f + \bar{\omega})\right]$$

$$\mathcal{W} = \frac{3Q}{2r^4}\sin(2\iota)\sin(f + \bar{\omega}) - \frac{2GC\omega}{c^2 r^3}\sqrt{\frac{GM}{p}}\sin\iota[e\sin\bar{\omega} + \sin(f + \bar{\omega})]$$

$$+ \frac{6GC\omega}{c^2 r^3}\sqrt{\frac{GM}{p}}\sin\iota \sin(f + \bar{\omega})(1 + e\cos f) + \frac{3Qm}{2r^5}\left[\frac{8}{3} + \left(5\alpha - \frac{20}{3}\right)\beta\right]\sin(2\iota)\sin(f + \bar{\omega})$$

$$+ \frac{3Qm}{2r^4 p}\sin(2\iota)\sin(f + \bar{\omega})(1 + e^2 + 2e\cos f).$$

4.4 Osculating Elements

The derivative with respect to the true anomaly of the orbital elements in terms of the components of the perturbative force are given in (1.20). Each one of those expressions are integrated, as showed in (1.21), to find the osculating elements,

$$\begin{aligned}\Delta a =& \frac{15Qm\pi e^4 \sin(2\bar{\omega})}{n^2 a^5 (1-e^2)^4} \\ &+ \frac{3Qm\pi e^2}{4n^2 a^5 (1-e^2)^4} \sin^2 \iota \sin(2\bar{\omega})[-60 + 24\alpha\beta + e^2(4\alpha\beta - 25)]\end{aligned} \tag{4.90}$$

$$\begin{aligned}\Delta e =& \frac{15Qm\pi e^3 \sin(2\bar{\omega})}{2n^2 a^6 (1-e^2)^3} \\ &+ \frac{Qm\pi e}{8n^2 a^6 (1-e^2)^3} \sin^2 \iota \sin(2\bar{\omega}) \left[-22 + 180e + 30e^3 + 2\beta(20 + 21\alpha) + e^2[-239 - 2\beta(20 - 21\alpha)]\right]\end{aligned} \tag{4.91}$$

$$\Delta \iota = \frac{3Qm\pi e^2}{8n^2 a^6 (1-e^2)^3} \sin(2\iota) \sin(2\bar{\omega}) \left[\frac{14}{3} + \left(5\alpha - \frac{20}{3}\beta\right)\right] \tag{4.92}$$

$$\begin{aligned}\Delta \Omega =& \frac{3Q\pi \cos \iota}{n^2 a^5 (1-e^2)^2} + \frac{6GC\omega\pi}{nc^2 a^3 (1-e^2)^{3/2}} \\ &+ \frac{3Qm\pi \cos \iota}{2n^2 a^6 (1-e^2)^3} \left\{2(5 + 4e^2) + (2 + e^2)(5\alpha - 8)\beta - e^2 \cos(2\bar{\omega}) \left[3 + \left(\frac{5\alpha}{2} - 4\right)\beta\right]\right\}\end{aligned} \tag{4.93}$$

$$\Delta\bar{\omega} = \frac{3Q\pi(3+5\cos(2\iota))}{4n^2a^5(1-e^2)^2} - \frac{2m\pi(4\beta-7)}{a(1-e^2)} - \frac{12GC\omega\pi\cos\iota}{na^3c^2(1-e^2)^{3/2}}$$
$$- \frac{Qm\pi}{4n^2a^6(1-e^2)^3}\left\{e^2\left[97+8(6\alpha-11)\beta\right]+4\left[53+(33\alpha-68)\beta\right]\right\}$$
$$+ \frac{Qm\pi e^2\cos(2\bar{\omega})}{4n^2a^6(1-e^2)^3}[14+5(3\alpha-4)\beta]$$
$$+ \frac{Qm\pi\sin^2\iota}{8n^2a^6(1-e^2)^3}\left\{496+16(21\alpha-40)\beta+e^2\left[227+2\beta(57\alpha-100)\right]\right\}$$
$$+ \frac{Qm\pi\sin^2\iota\cos(2\bar{\omega})}{8n^2a^6(1-e^2)^3}\left\{-82+40\beta+42\alpha\beta+e^2\left[-88+100\alpha-51\alpha\beta\right]\right\}$$
$$- \frac{6(1-\beta)GCme\pi}{n^2a^6(1-e^2)^3}. \tag{4.94}$$

Chapter 5

Applications

The osculating elements found at the end of Chapter 4 can be used to determine the changes in the orbit described by different types of objects that follow the approximations considered along this project. A proper candidate is to consider the system Earth + Satellite. The Earth is a celestial body that generates a quadrupole moment due to its rotation around the z-axis and deformed shape.

5.1 Artificial Satellites

Artificial Satellites are objects created by humans that are put in orbit around a central body by being launched with a speed that allows the radial and tangential be equal. Artificial satellites are commonly used for communication, navigation, Earth sensing, investigation, observation, research, etc [15].

The orbits of the satellites around the Earth can be divided into four groups [15]:

- Low Earth Orbit: Between 180 km and 2000 km above the surface of the Earth.
- Medium Earth Orbit: Between 2000 km and $35,786$ km above the surface of the Earth.
- Geosynchronous or Geostationary: Exactly $35,786$ km above the surface of the Earth.
- Super - synchronous: Above the geosynchronous orbit, but bellow the Lunar orbit.

Earth Properties

Here we introduce the multipolar moments as (4.41); however, in the literature is usual to define the multipolar moments as an a-dimensional quantity [17],

$$J_l = \sqrt{\frac{4\pi}{2l+1}}\frac{I_{l0}}{MR^2} \tag{5.1}$$

where M and R are the mass and the radius, respectively, of the central body.

[1] [9]

Earth Properties	
Mass	5.972×10^{27} g
Radius	6.3781×10^{6} m
Angular Velocity [1] (ω)	7.73141×10^{-12} rad/s
Quadrupole Moment[1] (J_2)	1.0826×10^{-3}
Quadrupole Moment (Q)	$(GMR^2)1.0826 \times 10^{-3}$
Inertia Momento z-axis	8.034×10^{40} g/m^2
Inertia Momento x-axis	8.008×10^{40} g/m^2

Table 5.1: Earth Properties

5.1.1 Geostationary Orbit

A Geostationary orbit is the one described by an object who look stationary from a given point on the Earth. Such as orbit has the initial parameters given in Table 5.2 [10, 16].

Geostationary Satellite: Initial Properties	
Semi-mayor axis (a)	4.2164×10^{7} m
Eccentricity (e)	0.01
Inclination (ι)	$0.01°$
Argument of the ascending node (Ω)	$0°$
Argument of the pericenter $(\bar{\omega})$	$0°$

Table 5.2: Geostationary Satellite: Initial Properties

By an iterative process, where each step is one period T, and the change of each orbital element is given by the expressions (4.90), (4.91), (4.92), (4.93) and (4.94), where we used the information given in Table 5.1. We find the orbit for each period by considering the perturbations. In some cases, the change is so small, that we plot in the y-axis the difference between the value of the orbital elements in the last period and the initial value.

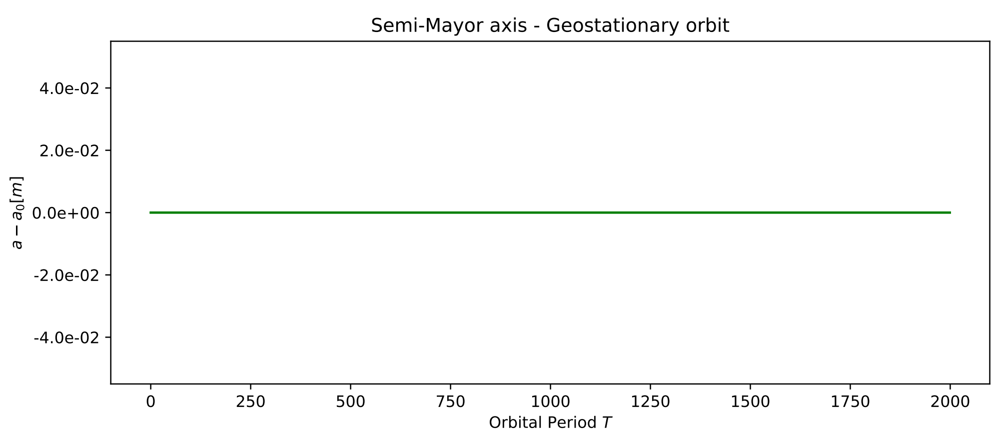

Figure 5.1: Semi-mayor axis for geostationary orbit considering relativistic and slowly rotation contributions.

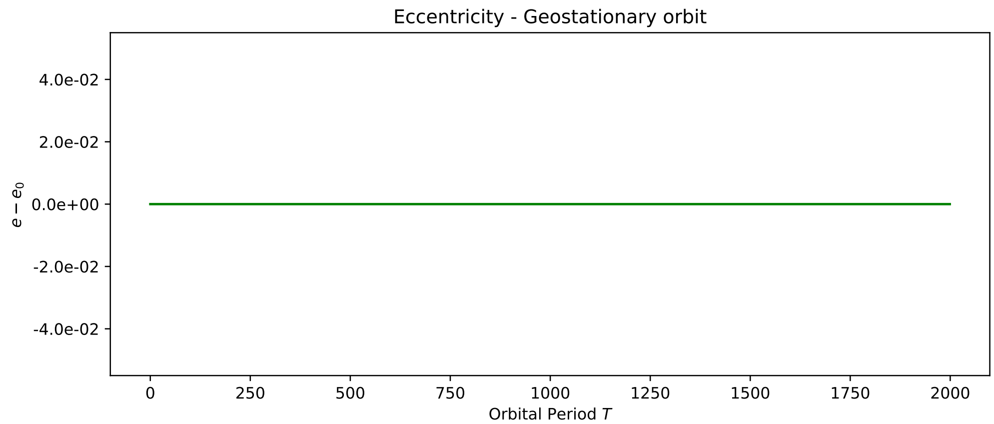

Figure 5.2: Eccentricity axis for geostationary orbit considering relativistic and slowly rotation contributions.

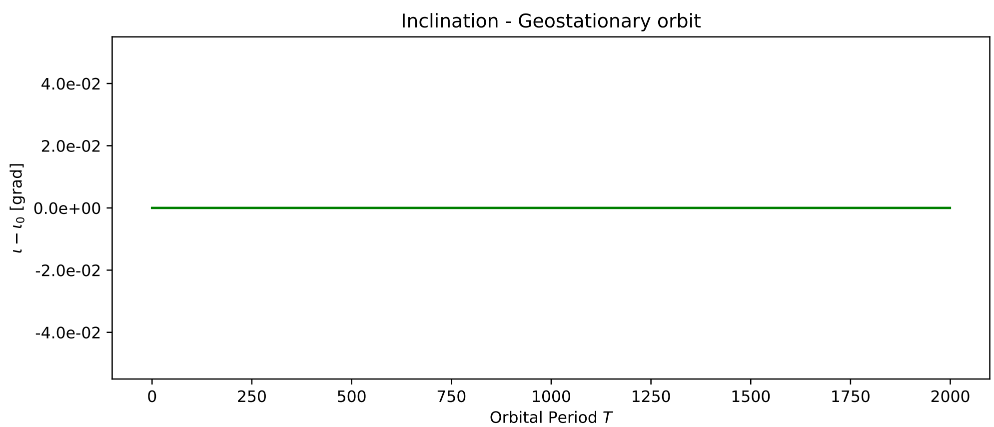

Figure 5.3: Inclination for geostationary orbit considering relativistic and slowly rotation contributions.

Until this degree of approximation there is no change for the semi-mayor axis in Figure 5.1, eccentricity in Figure 5.2 and inclination in Figure 5.3. Which agree with the fact that the parameters for this kind of orbit are determined by establish that the satellite must be always the equator. The principal applications for this orbits are communication and navigation satellites.

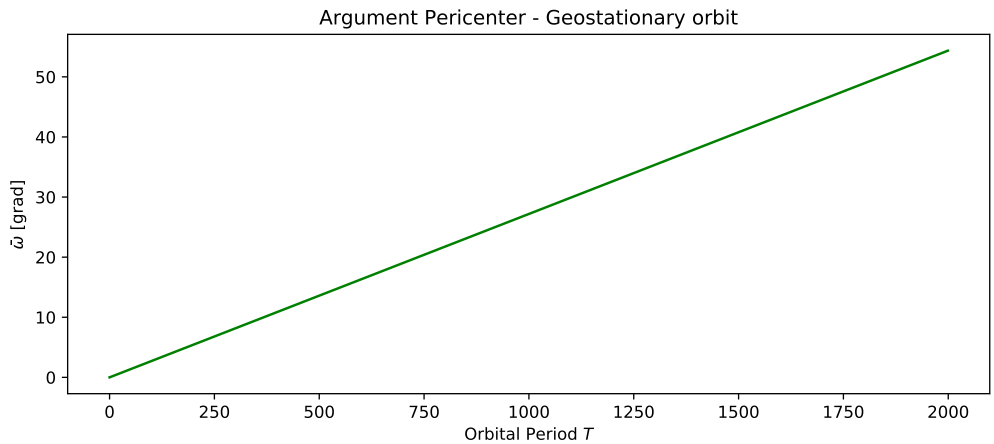

Figure 5.4: Argument of the Pericenter for geostationary orbit considering relativistic and slowly rotation contributions.

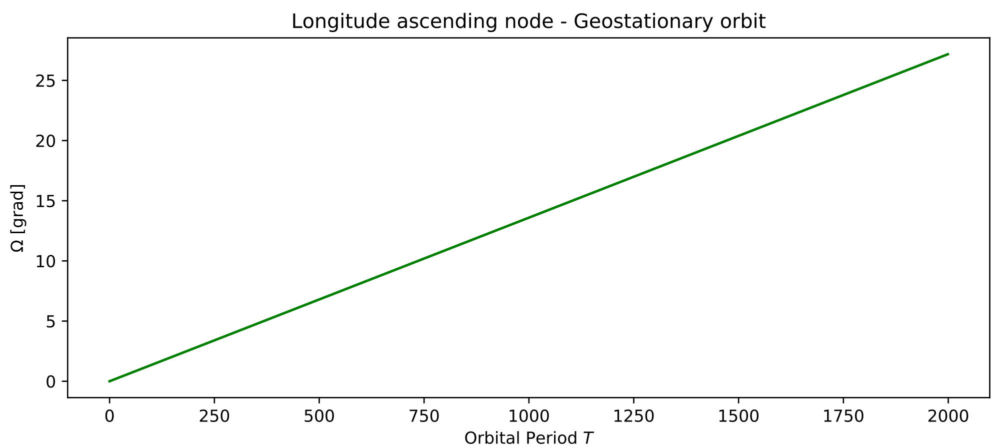

Figure 5.5: Argument of the Ascending Node for geostationary orbit considering relativistic and slowly rotation contributions.

For the argument of the pericenter and the argument of the ascending node in Figure 5.4 and 5.5 respectively, there is a constant increase; however, by analyzing that the inclination and eccentricity have small values ($<< 1$), it is clear that the orbit is almost a circular one, and the plane of the orbit coincide with the equator plane; therefore, these changes do not affect in a notorious way the movement of the satellite in the orbit.

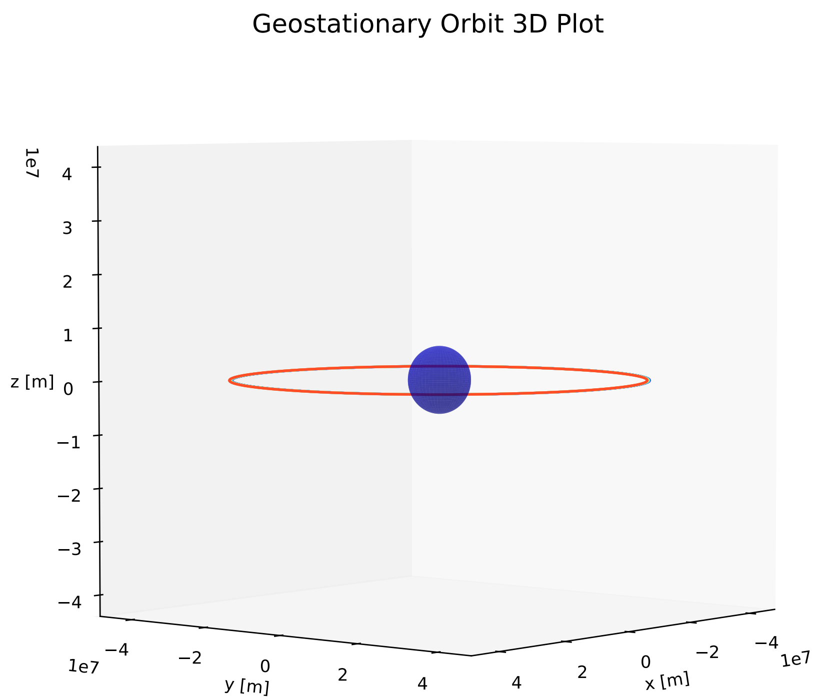

Figure 5.6: Geostationary orbit considering relativistic and slowly rotation contributions.

Figure 5.6 shows the 3D orbit for 2000 periods T and there is not a relevant change in the orbit due to the rotation of the Earth.

In Figure 5.7 and 5.8 we show two views of the plot in Figure 5.6. Particularly in Figure 5.8, where are plot the $x - z$ axis, it is a notorious change between the orbit for the initial period (the purple one) and the red orbit, corresponding to 1800 periods after. The apparently change in the eccentricity is because we are trying to plot a 3D orbit in a 2D space. On the other hand, it is important to note the scale of the z axis (1×10^3 m) and the x axis (1×10^7 m); although we are seeing a notorious change, the change is actually small compared to the scale of the orbit and the scale of the Earth. However, the difference seeing here is due to the changes in the argument of the pericenter and the argumente of the ascending node.

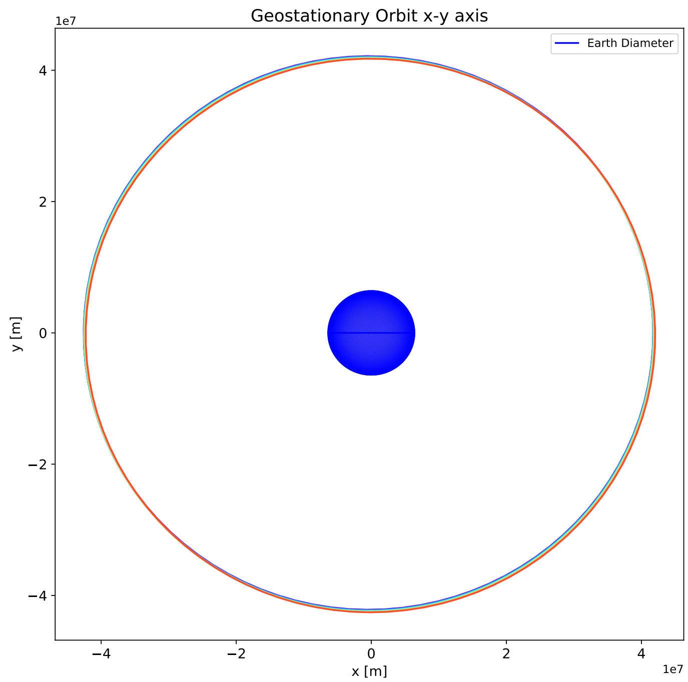

Figure 5.7: Geostationary orbit x-y view considering relativistic and slowly rotation contributions.

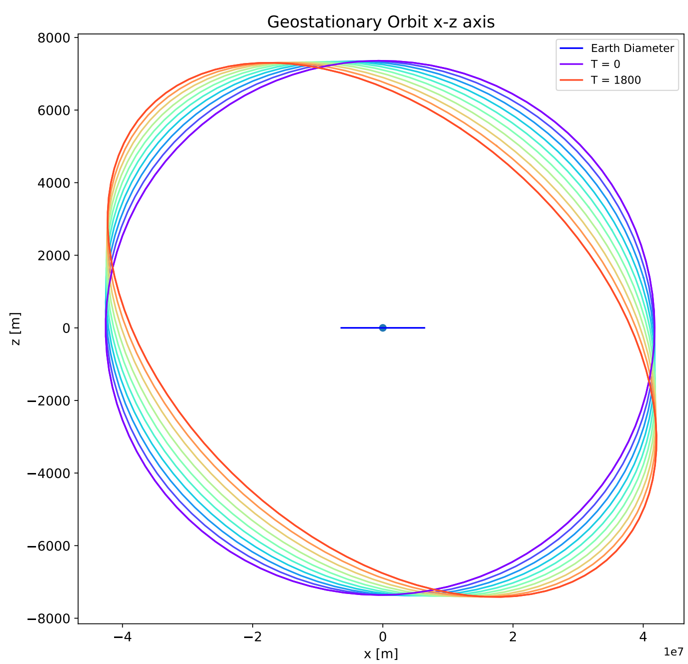

Figure 5.8: Geostationary orbit x-z view considering relativistic and slowly rotation contributions.

5.1.2 Molniya Orbit

A Molniya Orbit is one type of orbit that has a high eccentricity, in such a way that most of the time the satellite is in the north hemisphere of the Earth, meaning the pericenter distance is above this hemisphere. The inclination of this is determine so that the perturbation due to the quadrupole Newtonian perturbation is negligible, in this way the argument of the pericenter is almost the same after each period T, this guarantee the pericenter distance of the satellite does not change, and start to move away from the north hemisphere. See the secular change of $\bar{\omega}$ in (4.94), the first term is the quadrupole Newtonian perturbation, by making this term zero, the inclination must be $\iota = 63.435°$ [10, 16].

Molniya Satellite: Initial Properties	
Semi-mayor axis (a)	2.66×10^7 m
Eccentricity (e)	0.74
Inclination (ι)	63.435°
Argument of the ascending node (Ω)	270°
Argument of the pericenter ($\bar{\omega}$)	310.3°

Table 5.3: Molniya Satellite: Initial Properties

Below we plot the orbital elements considering the inclination given in Table 5.3 plus the elements considering a small change in the inclination, $\iota = 63°$ and preserving the rest of the initial parameters.

For the semi-mayor axis, eccentricity and inclination, corresponding to the Figure 5.9, 5.10 and 5.11 respectively; the y axis correspond to the difference between the current value of the orbital element and the initial value. In this way, and because the changes are small, there is a easy observation of the changes in the orbital elements.

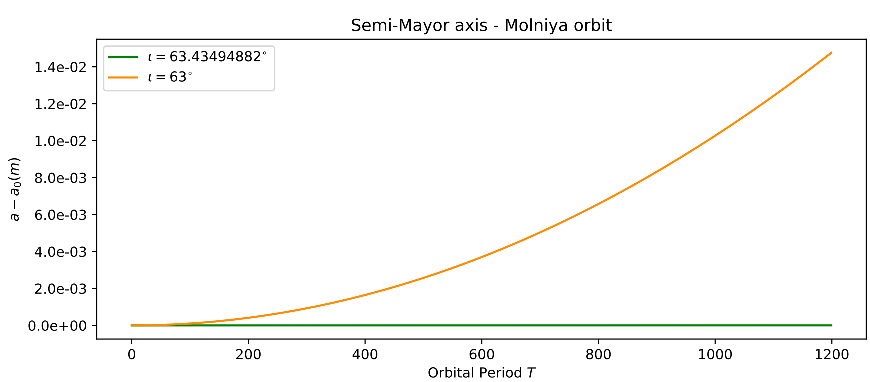

Figure 5.9: Semi-mayor axis for Molniya orbit considering relativistic and slowly rotation contributions.

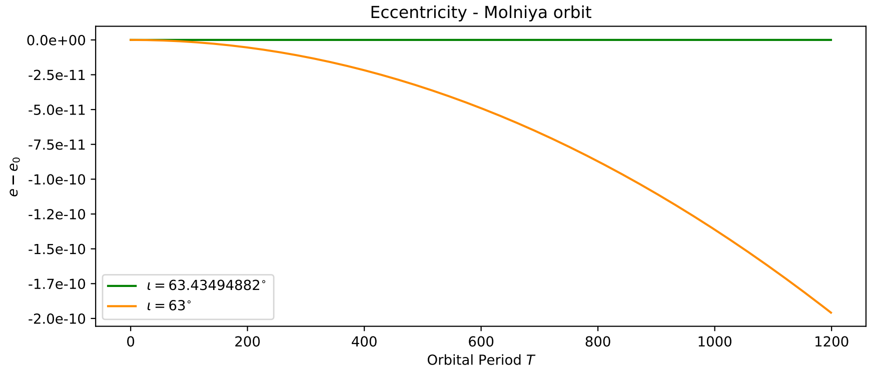

Figure 5.10: Eccentricity axis for Molniya orbit considering relativistic and slowly rotation contributions.

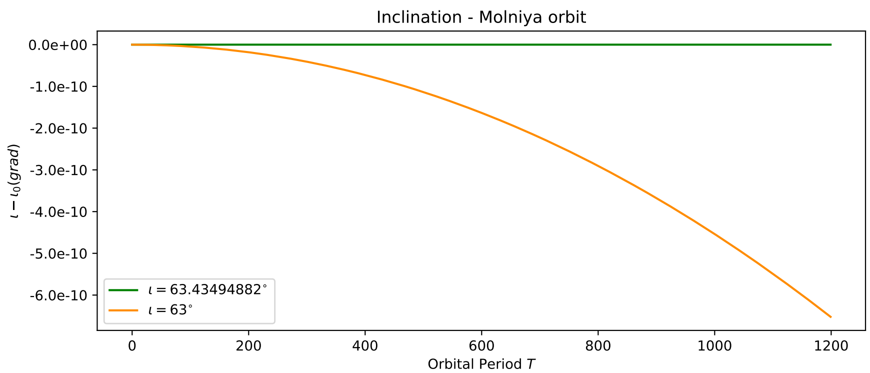

Figure 5.11: Inclination for Molniya orbit considering relativistic and slowly rotation contributions.

The change in this three orbital elements is notorious by setting the initial inclination as $\iota = 63^\circ$, seeing in the orange curves; for this value, the principal contribution for the argument of the pericenter given by the first term in (4.94) is no longer a quantity tending to zero. This behavior revels the importance of the accumulate small changes due to perturbations worked in this project.

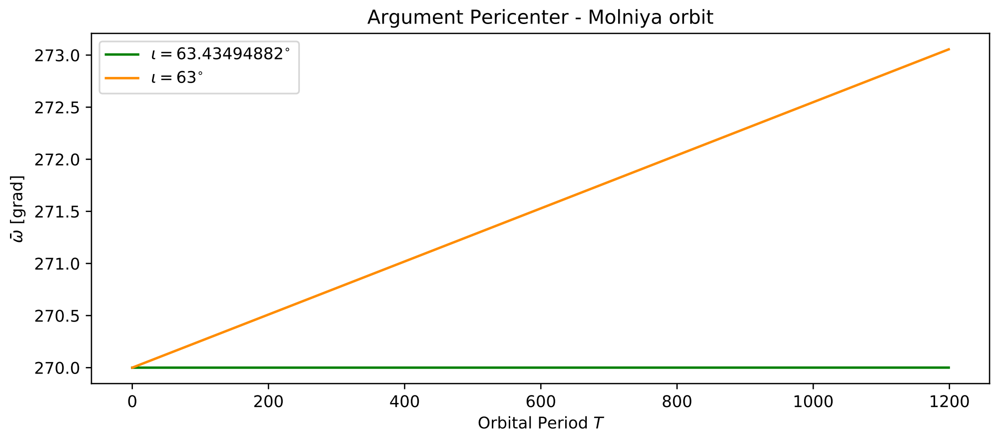

Figure 5.12: Argument of the Pericenter for Molniya orbit considering relativistic and slowly rotation contributions.

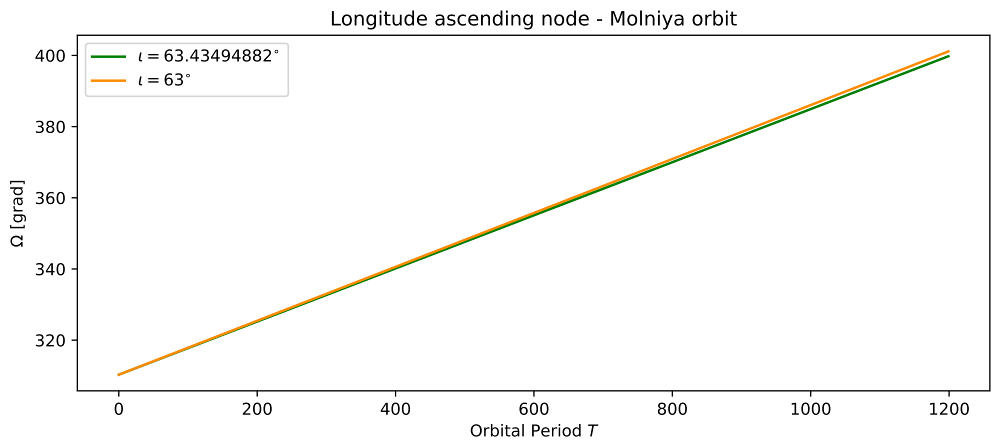

Figure 5.13: Argument of the Ascending Node for Molniya orbit considering relativistic and slowly rotation contributions.

The argument of the pericenter, plotted in Figure 5.12 for $\iota = 63.435°$, shows the expected behavior. $\bar{\omega}$ does not change across the orbital periods and this repercute in the other orbital elements plotted above, causing a very stable orbit shape. The argument of the ascending node, plotted in Figure 5.13, shows a constant increase in the angle over the orbital periods; however, the purpose of this. satellite (spend most of the time above the north hemisphere) is still fulfilled, the change in Ω only means that for example, instead of being above Russia, is going to be above Canada.

Figure 5.14 shows the 3D orbit for 1100 periods T, where the purple orbit correspond to $T = 0$ and the red to $T = 1100$. This plot reflects the change in the argument of the pericenter $\bar{\omega}$, and the argument of the ascending node Ω.

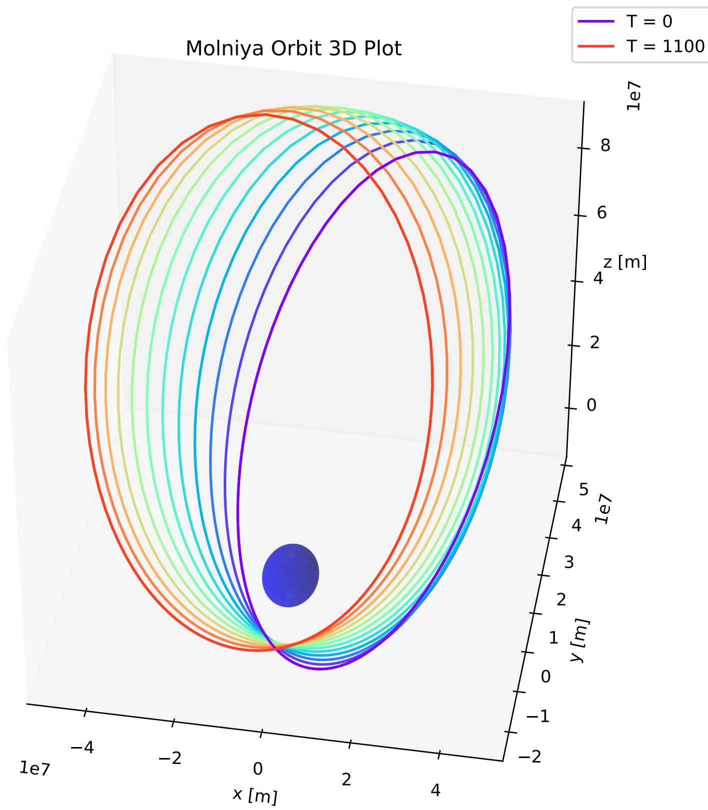

Figure 5.14: Molniya orbit considering relativistic and slowly rotation contributions for $\iota = 63.435°$
.

Figure 5.15 shows $x - z$ view, where is clear the change in the argument of the pericenter $\bar{\omega}$, and the argument of the ascending node Ω.

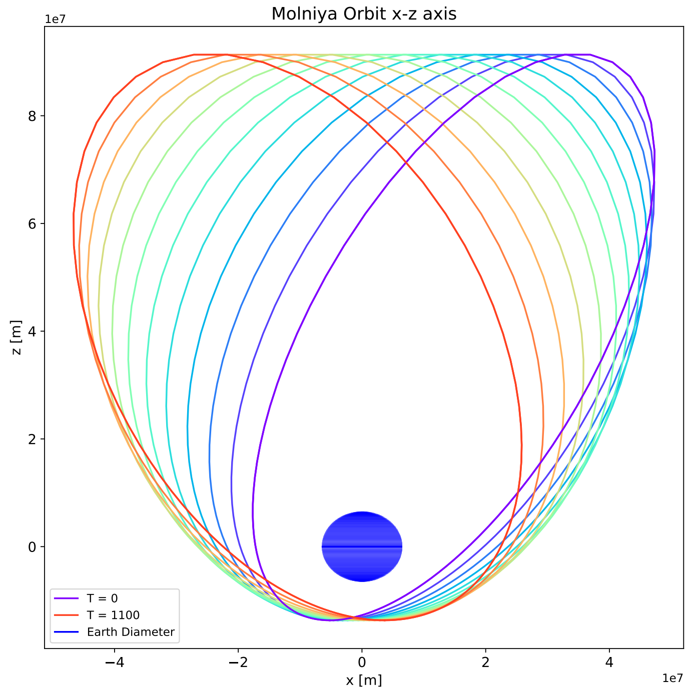

Figure 5.15: Molniya orbit for $\iota = 63.435°$ x-z view considering relativistic and slowly rotation contributions .

In Figure 5.16 we plot the initial and $T = 1100$ Molniya orbit for $\iota = 63.435°$ and $\iota = 63°$ plus the geostationary orbit. To have a general view of the BIG difference between this two types of orbits.

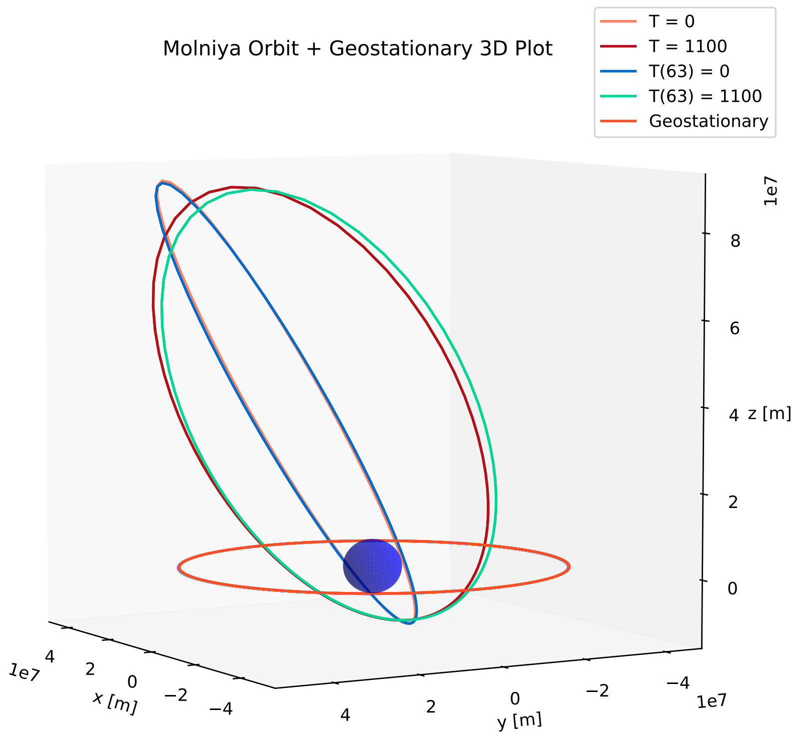

Figure 5.16: Molniya orbit + Geostationary Orbit view considering relativistic and slowly rotation contributions.

5.2 Conclusions

The construction of the metric for a slowly rotational gravitational body following the approach given in [3] differs from the result obtained in the book in the ${}^{(4)}h_{00}$ term in (4.70). The disagreement arises because the book implements the harmonic gauge; however, in this work, we used the standard post-Newtonian gauge, as mentioned in Chapter 2 and 3. The difference can be easily managed by writing the metric in the parametric form, as in (4.71); for a given value for β and α the metric in [3] is recovered.

The perturbing force determined by the metric gives reason for all the contributions that we were expecting; the Newtonian quadrupole perturbation, Schwarzchild-like relativistic perturbation, perturbation due to the rotation of the body, and the relativistic quadrupole perturbation. The perturbations allow determining the secular changes in all the six orbital elements. The changes in the eccentricity, semi-mayor axis, and inclination arise only for the relativistic quadrupole perturbations. The argument of the ascending node change due to the rotation of the central body, and the perturbations related to the quadrupole, both Newtonian and relativistic. Finally, the argument of the pericenter change due to the contribution of all the four perturbing forces considered here, note here that is the only orbital elements affected by the Schwarzchild-like relativistic perturbation.

The plot of the secular changes over the elliptical period replicate the good know behavior of the geostationary and Molniya orbit for satellites around the Earth. The order of each on of the terms of the secular changes correspond to perturbing orders, i.e., for the argument of the pericenter, given in (4.94), the contributions and the respective order are in Table 5.4.

Perturbation	Order - Geo	Order - Molniya
Newtonian quadrupole	5×10^{-4}	3×10^{-13}
Schwarzchild-like	2×10^{-9}	9×10^{-9}
Rotation	3×10^{-11}	9×10^{-11}
Relativistic quadrupole (1)	2×10^{-11}	9×10^{-12}
Relativistic quadrupole (2)	9×10^{-19}	4×10^{-13}
Relativistic quadrupole (3)	6×10^{-21}	1×10^{-11}
Relativistic quadrupole (4)	1×10^{-34}	1×10^{-19}

Table 5.4: Order of the terms in the secular change for $\bar{\omega}$ for the geostationary and Molniya orbit.

From Table 5.4 we see that the perturbations for the Molniya orbit are greater than for the Geostationary orbit; the reason is that the Molniya orbit is not located at the equator of the Earth, then the satellite is under the influence of an effect called Frame-Dragging, which was not covered in this work. Another interesting aspect about the Molniya contributions is the fact that the relativistic contributions are greater than the Newtonian, nevertheless, all the contributions are pretty small and the orbit is almost unaffected by them.

Bibliography

[1] Adler, R., Bazin, M., and Schiffer, M. *Introduction to General Relativity.* McGraw-Hill Kogakusha, 1975.

[2] Bambi, C. *Black Holes: A Laboratory for Testing Strong Gravity.* Springer, 2017.

[3] Brumberg, V. A. *Essential Relativistic Celestial Mechanics.* Adam Hilger, 1991.

[4] Carrol, S. *Spacetime and Geometry: An Introduction to General Relativity.* Addison Wesley, Pearson Education, 2004.

[5] Erez, G., and Rosen, N. The gravitational field of a particle possessing a multipole moment. *Bull. Research Council Israel 8F*, 47 (1959).

[6] Fitzpatrick, R. *Newtonian Dynamics.* The University of Texas at Austin, Lecture Notes.

[7] Frolov, V. P., and Novikov, I. D. *Black Holes Physics: Basic Concepts and New Developments.* Springer, 1997.

[8] Goldstein, H., Poole, C., and Safko, J. *Classical Mechanics. Third Edition.* Addison Wesley, 2001.

[9] Gurfil, P. Effect of equinoctial precession on geosynchronous earth satellites. *Journal of Guidance, Control, and Dynamics 30*, 1 (2007).

[10] Klioner, S. A. Lectures notes on basic celestial mechanics. *arXiv e-prints* (Sept. 2016), arXiv:1609.00915.

[11] Kopeikin, S., Efroimsky, M., and Kaplan, G. *Relativistic Celestial Mechanics of the Solar System.* Wiley-VCH, 2006.

[12] Landau, L. D., and Lifshitz, E. M. *The Classical Theory of Fields.* Pergamon Press, 1971.

[13] Larrañaga, E. A. *An Introduction to Relativistic Celestial Mechanics.* Notas, 2020.

[14] Misner, C. W., Thorne, K. S., and Wheeler, J. A. *Gravitation.* W.H Freeman and Company, 1973.

[15] Pelton, J. N., and Howkins, J. *Satellites International.* Stockton Press, 1988.

[16] Pelton, J. N., and Madry, S. *Handbook of Satellite Applications.* Springer International Publishing, 2017.

[17] Poisson, E., and Will, C. M. *Gravity: Newtonian, Post-Newtonian, Relativistic.* Cambridge University Press, 2014.

[18] Raine, D., and Thomas, E. *Black Holes: An Introduction 2nd ed.* Imperial College Press, 2009.

[19] Roy, A., and Clarke, D. *Astronomy: Principles and Practice, Fourth Edition.* Taylor & Francis, 2003.

[20] Soffel, M. H. *Relativity in Astrometry, Celestial Mechanics and Geodesy.* Springer, 1989.

[21] Straumann, N. *General Relativity. 2nd ed.* Springer, 2013.

[22] Wald, R. M. *General Relativity.* The University of Chicago Press, 1984.

[23] Weinberg, S. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* John Wiley & Sons, 1972.

[24] Will, C. M. *Theory and Experiment in Gravitational Physics.* Cambridge University Press, 2018.

[25] Young, J. H., and Coulter, C. A. Exact metric for a nonrotating mass with a quadrupole moment. *Physical Review 184*, 5 (1969), 1313–1315.