Kernel Bayes' Rule: Toy Example

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In this short note, I derive the analytic solution to the problem presented in 5.1 of Fukumizu et al., 2011.

For random variables $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^d$, we are given the joint distribution $P(\boldsymbol{X}, \boldsymbol{Y}) = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{bmatrix} \mathbf{0}_d \\ \mathbf{1}_d \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{XX} & \boldsymbol{\Sigma}_{XY} \\ \boldsymbol{\Sigma}_{YX} & \boldsymbol{\Sigma}_{YY} \end{bmatrix}. \tag{1}$$

This means we can easily write down the likelihood

$$P(\boldsymbol{Y}|\boldsymbol{X}) = \mathcal{N}\left(\mathbf{1}_d + \boldsymbol{\Sigma}_{YX}\boldsymbol{\Sigma}_{XX}^{-1}\boldsymbol{X}, \boldsymbol{\Sigma}_{YY} - \boldsymbol{\Sigma}_{YX}\boldsymbol{\Sigma}_{XX}^{-1}\boldsymbol{\Sigma}_{XY}\right). \tag{2}$$

The aim is to find the posterior mean of $Q(\boldsymbol{X}|\boldsymbol{Y})$ with likelihood $P(\boldsymbol{Y}|\boldsymbol{X})$ and prior $\Pi(\boldsymbol{X}) = \mathcal{N}(\mathbf{0}_d, \boldsymbol{\Sigma}_{XX}/2)$. We can easily write down the joint distribution $Q(\boldsymbol{X}, \boldsymbol{Y}) = P(\boldsymbol{Y}|\boldsymbol{X})\Pi(\boldsymbol{X})$ as a multivariate Gaussian distribution given by:

$$Q(\boldsymbol{X}, \boldsymbol{Y}) = \mathcal{N}(\boldsymbol{\mu}_Q, \boldsymbol{\Sigma}_Q) \tag{3}$$

where

$$\boldsymbol{\mu}_Q = \begin{bmatrix} \mathbf{0}_d \\ \mathbf{1}_d \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_Q = \begin{bmatrix} \boldsymbol{\Sigma}_{XX}/2 & \boldsymbol{\Sigma}_{XX}^{-1}\boldsymbol{\Sigma}_{XY} \\ \boldsymbol{\Sigma}_{YX}\boldsymbol{\Sigma}_{XX}^{-1} & \boldsymbol{\Sigma}_{YY} - \boldsymbol{\Sigma}_{YX}\boldsymbol{\Sigma}_{XX}^{-1}\boldsymbol{\Sigma}_{XY}/2 \end{bmatrix}. \tag{4}$$

And thus the posterior mean of $Q(\boldsymbol{X}|\boldsymbol{Y})$ is given by

$$\boldsymbol{\Sigma}_{XX}^{-1}\boldsymbol{\Sigma}_{XY}\left(\boldsymbol{\Sigma}_{YY} - \boldsymbol{\Sigma}_{YX}\boldsymbol{\Sigma}_{XX}^{-1}\boldsymbol{\Sigma}_{XY}/2\right)^{-1}(\boldsymbol{Y} - \mathbf{1}_d). \tag{5}$$

References

Kenji Fukumizu, Le Song, and Arthur Gretton. Kernel bayes' rule. *Advances in neural information processing systems*, 24, 2011.