MAE5005 / MAE403 Computational Fluid Dynamics, Spring 2024
Computer Project

Overall notes: Three of you will work as a group. The contributions of each student in the group should be coordinated and described in your group report. Your group report must be typed up neatly using Word, LaTex, or similar tool. Plots should be done properly using dimensionless quantities, at *professional quality.*

Schedule and grading:

1. Oral presentation and report for Part I: Thursday, May 16, 6 minutes / group.

2. Oral presentation and report for Part II: Thursday, May 30, 6 minutes / group.

3. Oral presentation and report for Part III: June 11-21 (TBD), 6 minutes / group.

General description. In this computer project, we consider the forced Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{1}$$

in *a periodic domain* of length $L = 1$ with the kinematic viscosity ν, where the flow velocity u is a function of x and t. This is the one-dimensional Navier-Stokes equation with a constant pressure, or the Burgers equation. The initial condition is $u(x, t = 0) = 0$ and a steady forcing

$$f(x,t) = \frac{u_0^2}{L}\left[\sin\left(\frac{2\pi x}{L}\right) + \cos\left(\frac{4\pi x}{L}\right)\right]. \tag{2}$$

The important governing parameter for this one-dimensional periodic flow is the flow Reynolds number defined as $Re = u_0 L/\nu$. By doing this project, you will learn: how the solution to the Burgers equation behaves *physically*, how different numerical methods can be used to solve this nonlinear flow, and what numerical issues might arise and how to resolve them?

I. Finite-Difference Method to Solve the Nonlinear Burgers Equation. Write a Fortran program to solve numerically $u(x,t)$ of the above Burgers equation as specified, for $Re = 1000$. The finite-difference scheme combines the Adams-Bashforth scheme for the nonlinear term / forcing term, and the Crank-Nicholson scheme for the viscous term, with the spatial derivatives treated by second-order central differencing. Follow the evolution up to $\tilde{t} \equiv t \cdot u_0/L = 100.0$. Your group should hand in the following:

1. A description of all the details you use to develop the scheme;

2. Explanations on how you decide the grid resolution you will need to obtain the converged numerical solution, by trying several different grid resolutions. The converged numerical solution will then be used as your benchmark. Discuss any issue you may encounter when the grid resolution is too coarse.

3. A description of how the solution evolve in time, and identify different stages of the flow evolution. For this purpose, plot the solution at 3 to 4 times for one selected grid resolution;

4. A plot of the solution at some specific times for at least four different grid resolutions;

5. A plot of the average kinetic energy (k), the average viscous dissipation rate (ε), and the rate of energy input by the forcing term (P) as a function of time, for at least four different grid resolutions;

$$k \equiv \left\langle \frac{u^2}{2} \right\rangle, \quad \varepsilon \equiv \nu \left\langle \left(\frac{\partial u}{\partial x} \right)^2 \right\rangle, \quad P \equiv \langle f(x) \cdot u(x,t) \rangle, \tag{3}$$

where $\langle ... \rangle$ indicates spatial averages.

6. A plot of the L1 and L2 errors, using the converged numerical solution from the fine resolution as the benchmark. For the convenience of computing the L1 and L2 errors, set the nodes points at $x_j = jL/N$, for $j = 0, 1, 2, ..., N$, where N is the grid resolution. Comment on the order of accuracy of your numerical method.

II. Finite-Volume Method to Solve the Forced Burgers Equation. Now repeat Part I at $Re = 1000$, but using a finite-volume formulation:

$$\frac{d}{dt}\overline{u}_j + \frac{1}{\Delta x}\left(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}\right) = \frac{1}{\Delta x}\left(\hat{D}_{j+\frac{1}{2}} - \hat{D}_{j-\frac{1}{2}}\right) + \overline{f}_j, \tag{4}$$

where $\hat{f}$ is the advection flux, and $\hat{D}$ is related to the diffusion flux,

$$\hat{D}_{j+\frac{1}{2}} = \left(\nu \frac{\partial u}{\partial x}\right)_{j+\frac{1}{2}} \tag{5}$$

The time integration will use the third-order Runge-Kutta scheme we discussed in the class. The fifth-order WENO as presented in the class shall be used to reconstruct the local values needed in the Lax-Friedrich flux formulation of $\hat{f}$.

1. The diffusion flux $\hat{D}_{j+\frac{1}{2}}$, which has no directional preference, will be treated by the reconstruction based on a 4-point stencil: $\overline{u}_{j-1}, \overline{u}_j, \overline{u}_{j+1}, \overline{u}_{j+2}$. Derive the reconstruction formula and comment on the order of accuracy.

2. Validate the results of your finite-volume code using the results from Part I.

3. Repeat all the parts in Part I, and make your own observations.

4. Also comment on the pros and cons of the finite-volume method in Part II, relative to the finite difference method in Part I.

III. Study of the physical problem.

Now that you have figured out two different numerical methods to obtain a converged numerical solution, we shall study the physical problem. Consider a range of Re numbers, from $Re = 100$ to $Re = 10000$. Discuss the following:

1. Discuss the grid resolution you would need to obtain the accurate (or converged) numerical solution, and how this required grid resolution changes with Re and time.

2. Analyze how the average kinetic energy, the average viscous dissipation rate, and the rate of energy input by the forcing term change with time, and how this evolution depends on flow Reynolds number.

3. Perform a Fourier analysis of the flow field, and study how the energy spectrum of this flow changes with time.

4. Finally, consider an unsteady forcing term, namely, adding phase jugglings to the forcing,
$$f(x,t) = \frac{u_0^2}{L}\left\{\sin\left[\frac{2\pi x}{L} + \phi_1(t)\right] + \cos\left[\frac{4\pi x}{L} + \phi_2(t)\right]\right\} \tag{6}$$
with
$$\phi_1(t) = \pi\sin(3\pi u_0 t/L), \quad \phi_2(t) = \pi\sin(5\pi u_0 t/L). \tag{7}$$
study how the various results change with this unsteady forcing.

5. Any other physical issues / numerical issues you think important.