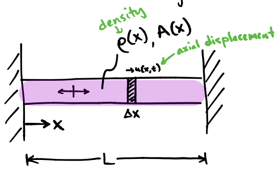
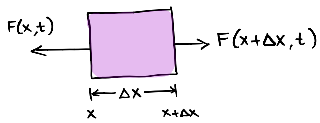
14 Nov

[Sound is a longitudinal wave]





Consider longitudinal/axial vibration in a beam

Causes compressive/expansive waves in the rod

↳ Can also describe air in a tube

axial stress $\sigma(x) = \frac{F(x)}{A(x)}$

Hooke's Law $\sigma(x) = E(x)\varepsilon(x)$

strain $\varepsilon(x) = \frac{du(x)}{dx}$

$$\therefore \; F(x,t) = EA(x)\frac{\partial u(x,t)}{\partial x}$$

FBD:

$$\Sigma f_x = \rho(x)A(x)\cancel{\Delta x}\,\frac{\partial^2 u(x,t)}{\partial t^2} = F(x+\Delta x,t) - F(x,t) \quad ; \quad F(x+\Delta x,t) = F(x,t) + \frac{\partial F(x,t)}{dx}\Delta x$$

$$= \frac{\partial F(x,t)}{\partial x}\cancel{\Delta x}$$

$$\rho(x)A(x)\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]$$

if rod is homogenous w/ cte x-section: $EA(x) = EA$; $A(x) = A$; $\rho(x) = \rho$

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\,\frac{\partial^2 u(x,t)}{\partial x^2}} \; ; \quad c^2 = \frac{E}{\rho}$$

~ "1D wave equation"

Utilizing seperation of variables:

$$u(x,t) = F(x)G(t)$$

$F \neq$ force; it has dropped from eqn

$\dot{} = \frac{\partial}{\partial t}$; $' = \frac{\partial}{\partial x}$

$$F(x)\ddot{G}(t) = c^2 F''(x)G(t)$$

$\frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = \frac{F''(x)}{F(x)}$; take $\frac{\partial}{\partial t}$ on both sides → RHS = 0 $\therefore \frac{1}{c^2}\frac{\ddot{G}(t)}{G(t)} = \text{cte} = K_1$

; take $\frac{\partial}{\partial x}$ on both sides → LHS = 0 $\therefore \frac{F''(x)}{F(x)} = \text{cte} = K_2$

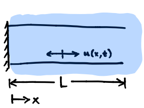
$$\therefore K_1 = K_2 = -\omega^2$$

$\boxed{\ddot{G}(t) + c^2\omega^2 G(t) = 0}$ ← Acoustic modes come from these → $\boxed{F''(x) + \omega^2 F(x) = 0}$

$F(x) = C_1\sin\omega x + C_2\cos\omega x$

*how to get C_1 & C_2?

Ex:



B.C.'s: $u(x=0) = 0$ → $F(0)G(t) = 0$ $\therefore F(0) = 0$

$P(L,t) = 0$ → $P(L,t) = E\frac{\partial u(L,t)}{\partial x} = 0$

$\boxed{\frac{\partial u(L,t)}{\partial x} = 0}$ $\therefore F'(L)G(t) = 0$ $\therefore F'(L) = 0$

Apply B.C.'s:

$F(0) = C_2 = 0$

$F'(L) = \omega C_1\cos\omega L = 0$

$\cos\omega L = 0$

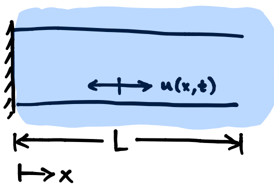
$\omega L = \frac{\pi}{2} + (n-1)\pi$; $n = 1, 2, 3, \ldots$

$\boxed{\omega_n = \frac{2n-1}{2}\frac{\pi}{L}}$ ~ eigenvalues of system

$F_n(x) = C_1\sin\frac{2n-1}{2}\frac{\pi}{L}x$ ~ eigenfunction — "mode shape"

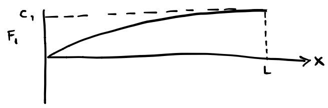
con't

↓

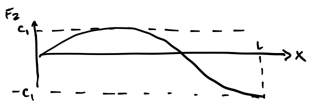


$$P(x,t) = E\,\frac{\partial u(x,t)}{\partial x}$$

$n=1: \frac{\pi}{2L}$; $F_1 = c_1 \sin \frac{\pi}{2L} x$



$n=2: \frac{3}{2}\frac{\pi}{2L}$; $F_2 = c_1 \sin \frac{3}{2}\frac{\pi}{L} x$



Displacement Mode Shapes: How the air is moving

· Shorthand Wave Eqn:

$u_{tt} = c^2 u_{xx}$

$c^2 u_{xx} - u_{tt} = 0 \longrightarrow$ consider soln: $u(x,t) = A\sin(ax+bt)$

$-c^2 a^2 \sin(ax+bt) + b^2 \sin(ax+bt) = 0$

solution if $b^2 = c^2 a^2$ or if $\frac{b}{a} = \pm c$

$\therefore u(x,t) = A \sin a(x \pm ct)$

$= A \cos a(x \pm ct)$

$= A e^{\pm ja(x \pm ct)}$

$= A \ln a(x \pm ct)$

$= A(x-ct)^n$; for any value n

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n (x-ct)^n$$ ~ Laurent Series → can represent any fnc

$\therefore u(x,t) = f(x-ct)$

any function whose argument is $(x-ct)$

Prove it! → $u_x(x,t) = f'(x-ct)(1)$ $\qquad u_t = f'(x-ct)(-c)$

→ $u_{xx}(x,t) = f''(x-ct)(1)$ $\qquad u_{tt} = f''(x-ct)(-c)(-c)$ ← plug into wave eqn

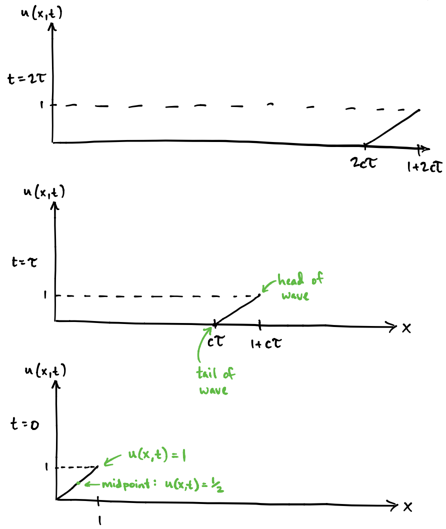
$c^2 f''(x-ct) - f''(x-ct)(c^2) = 0$ ✓

cont.

↓

· **Ex:**

$u(x,t) = A(x-ct)$, $A=1$; $0 < x-ct < 1$



$u(x,2\tau) = x-2c\tau$; $2c\tau < x < 1+2c\tau$

outgoing wave

$u(x,\tau) = x-c\tau$; $c\tau < x < 1+c\tau$

→ Shows progression of wave through time

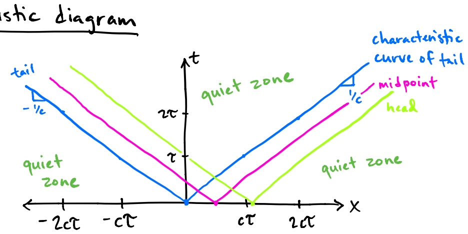
time elapsed = τ

distance moved = $c\tau$

velocity = $\dfrac{\text{distance moved}}{\text{time elapsed}} = \dfrac{c\tau}{\tau} = c$ ∴ if wave traveling through air, $c \equiv$ speed of sound

$u(0) = x$; $0 < x < 1$

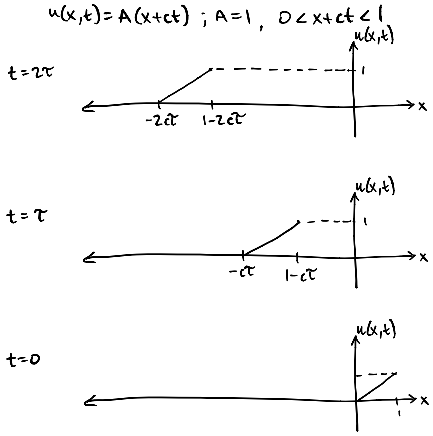
· **Characteristic diagram**



Define new variable

$u(x,t) = g(x+ct)$

$u(x,t) = A(x+ct)$; $A=1$, $0 < x+ct < 1$



$u(x,2\tau) = x+2c\tau$; $-2c\tau < x < 1-2c\tau$

incoming wave

$u(x,\tau) = x+c\tau$; $-c\tau < x < 1-c\tau$

$u(x,0) = x$; $0 < x < 1$

↳ Generally, $u(x,t) = f(x-ct) + g(x+ct)$ is the solution

existence of two waves

· Ex: infinitely-long string

$$u(x,0) = \begin{cases} 1 \;; & -a<x<a \\ 0 \;; & \text{elsewhere} \end{cases} \qquad u_t(x,0) = 0 \quad \text{velocity}$$

$= \text{rect}\left(\frac{x}{2a}\right)$; general form is $\text{rect}\left(\frac{x-x_0}{w}\right)$ ⇒ rectangular function centered @ x_0 $\bar{w}$ width w

General eqn is $u(x,t) = f(x-ct) + g(x+ct)$

IC's: $u(x,0) = f(x) + g(x) = \text{rect}\left(\frac{x}{2a}\right)$ (1)

$u_t(x,0) = -\cancel{c}f'(x) + \cancel{c}g'(x) = 0$

integrate both sides → cte. of integration C $-f(x) + g(x) = C$ (2)

Take (1) − (2): $2f(x) = \text{rect}\left(\frac{x}{2a}\right) - c$

(1) + (2): $2g(x) = \text{rect}\left(\frac{x}{2a}\right) + c$ → then divide everything by 2

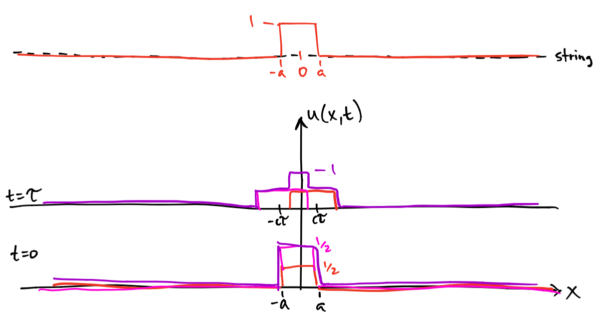
$$u(x,t) = \text{rect}\left(\frac{x-ct}{2a}\right) - \cancel{\frac{c}{2}} + \text{rect}\left(\frac{x+ct}{2a}\right) + \cancel{\frac{c}{2}}$$

$$\boxed{u(x,t) = \tfrac{1}{2}\text{rect}\left(\frac{x-ct}{2a}\right) + \tfrac{1}{2}\text{rect}\left(\frac{x+ct}{2a}\right)}$$

← how this specific wave will travel through time

outgoing wave

incoming wave



$u(x,\tau) = \frac{1}{2}\text{rect}\left(\frac{x-c\tau}{2a}\right) + \frac{1}{2}\text{rect}\left(\frac{x+c\tau}{2a}\right)$
① ②

$u(x,0) = \frac{1}{2}\text{rect}\left(\frac{x}{2a}\right) + \frac{1}{2}\text{rect}\left(\frac{x}{2a}\right)$
① ②

21 Nov

- Solution to the wave eqn:

$$u(x,t) = f(x-ct) + g(x+ct)$$

Given some IC's:

$$u(x,0) = P(x) \quad , \quad u_t(x,0) = Q(x)$$

deriv. WRT time

$u(x,0) = f(x) + g(x) = P(x)$ (1)

$u_t(x,0) = -cf'(x) + cg'(x) = Q(x) \longrightarrow \frac{1}{c}\int_{x_0}^{x} Q(x)\,dt = -[f(x) - g(x)] - K$ (2) ; $K \equiv g(x_0,0) - f(x_0,0)$

(1)-(2): $f(x) = \frac{1}{2}P(x) - \frac{1}{2c}\int_{x_0}^{x} Q(y)\,dy - \frac{K}{2}$

(1)+(2): $g(x) = \frac{1}{2}P(x) + \frac{1}{2c}\int_{x_0}^{x} Q(y)\,dy + \frac{K}{2}$

Plug these into general solution

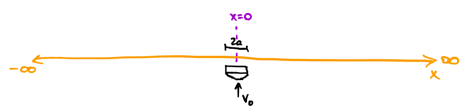
$$u(x,t) = \frac{1}{2}P(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} Q(y)\,dy - \cancel{\frac{K}{2}} + \frac{1}{2}P(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} Q(y)\,dy + \cancel{\frac{K}{2}}$$

factor $\frac{1}{2}$...

$$\boxed{u(x,t) = \frac{1}{2}\left[P(x-ct) + P(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} Q(y)\,dy\right]}$$

d'Alembert's solution

Ex: Piano string struck by a hammer



$u(x,0) = P(x) = 0$

$u_t(x,0) = V_0\,\text{rect}\left(\frac{x}{2a}\right) = Q(x)$

$= V_0\left[H(x+a) - H(x-a)\right]$

starts @ $-a$ starts @ a

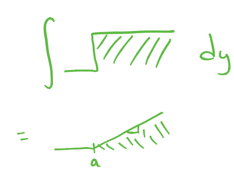
$\text{rect}\left(\frac{x-x_0}{w}\right)$

Heaviside step function: $H(x-x_0) = \begin{cases} 1 & x \geq x_0 \\ 0 & x < x_0 \end{cases}$

P(x) drops out...

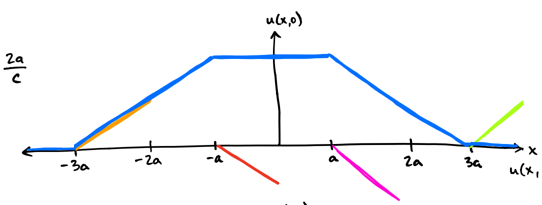
$\hookrightarrow u(x,t) = \frac{V_0}{2c}\int_{x-ct}^{x+ct}\left[H(y+a) - H(y-a)\right]dy$

$= \frac{V_0}{2c}\left[R(x+ct+a) - R(x+ct-a) - R(x-ct+a) + R(x-ct-a)\right]$



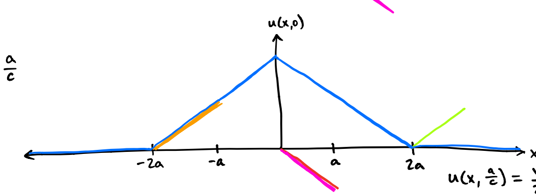
Ramp function $R(x-a)$

$t = \frac{2a}{c}$



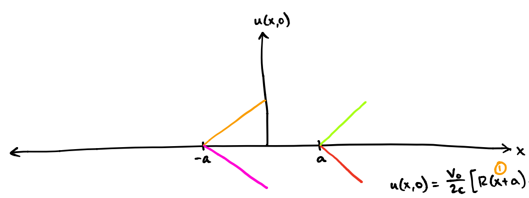
$u\left(x, \frac{2a}{c}\right) = \frac{V_0}{2c}\left[R(x+3a) - R(x+a) - R(x-a) + R(x-3a)\right]$ (①②③④)

$t = \frac{a}{c}$



$u\left(x, \frac{a}{c}\right) = \frac{V_0}{2c}\left[R(x+2a) - R(x) - R(x) + R(x-2a)\right]$ (①②③④)

$t = 0$



$u(x,0) = \frac{V_0}{2c}\left[R(x+a) - R(x-a) - R(x+a) + R(x-a)\right] = 0$ (①②③④)

[Assignment 6 discussion] - due 12 Dec.

*Doing same problem as last week, just a different initial shape

$F'(L) = \omega C_1 \cos \omega L = 0$

$\cos \omega L = 0$

$\omega L = \frac{\pi}{2} + (n-1)\pi \quad ; n = 1, 2, 3, \ldots$

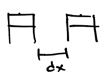
$\omega_n = \frac{2n-1}{2}\frac{\pi}{L}$ ~ eigenvalues of system

See 14 Nov notes!

· Measurement of Sound

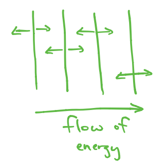
– L : measure of pressure, no direction

– IL: sound intensity; pressure + direction



– PWL: power level; single microphone, power over area

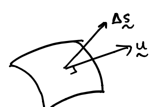
· Sound Intensity $IL = \frac{\text{average energy flow}}{\text{unit area}}$



energy flow = energy/unit time $\equiv$ Power

mechanical power : $\underset{\sim}{f} \cdot \underset{\sim}{u}$

acoustic power : $\underbrace{p\underset{\sim}{u}}_{\underset{\sim}{i}} \cdot \Delta \underset{\sim}{S}$ (particle velocity)



$\underset{\sim}{i}$ – instantaneous energy flow per unit area

$\underset{\sim}{I} = \frac{1}{T}\int_0^T p\underset{\sim}{u}\, dt$ (Watts)

$\underset{\sim}{I} = \frac{1}{\rho_0 c_0}\frac{1}{T}\int_0^T p^2\, dt \; ; \; P_{rms} = \sqrt{\frac{1}{T}\int_0^T p^2 dt}$

$= \frac{P_{rms}^2}{\rho_0 c_0}$

For progressive waves (no interference or feedback),

acoustic impedance $Z_0 = \frac{P}{u} = \rho_0 c_0$

$\rho_0 \equiv$ density of medium (air, water, etc.)

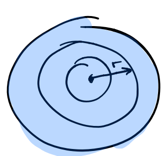
$c_0 \equiv$ speed of sound in medium

$\therefore u = \frac{P}{\rho_0 c_0}$

· Sound Power $W = \int_S \underset{\sim}{I} \cdot \Delta \underset{\sim}{S}$

For progressive waves, $W = IS$

For an omnidirectional outgoing spherical wave @ a distance r from the source (assuming lossless fluid)



$W = I(4\pi r^2) \qquad I \sim \frac{1}{r^2}$

· Sound pressure level : $SPL = 20 \log_{10} \frac{P}{P_{ref}}$; $P_{ref} = 20\,\mu Pa$ (quietest pressure a youngster can hear in the 1-4 kHz range)

Ordinary speech has $P_{rms} \cong 0.1\,Pa$

↳ $SPL = 20 \log_{10}\left(\frac{0.1\,Pa}{20\,\mu Pa}\right) = 74\,dB$

SPL near jet engine during takeoff $\cong$ 120dB

$P_{rms} = 20\,\mu Pa \cdot 10^{120/20} = 20\,Pa$

Rocket engine $P_{rms} = 10^9 \cdot P_{ref}$

$SPL = 20 \log_{10} 10^9 = 180\,dB$

• Sound Intensity level: $SIL = 10\log_{10}\frac{I}{I_{ref}}$ dB; $I_{ref} = 10^{-12}\ W/m^2$ for air

For progressive waves, $I = \frac{P_{rms}^2}{\rho_o c_o} \longrightarrow SIL = 10\log_{10}\left(\frac{P_{rms}^2}{\rho_o c_o I_{ref}} \cdot \frac{P_{ref}^2}{P_{ref}^2}\right) = \underbrace{20\log_{10}\frac{P_{rms}}{P_{ref}}}_{SPL} + 10\log_{10}\frac{P_{ref}^2}{\rho_o c_o I_{ref}} = SPL - 0.16\,dB$

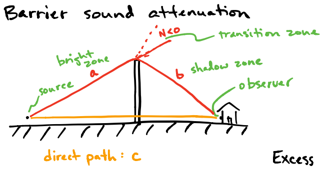
@STP:
$\rho_o = 1.41\ kg/m^3$
$c_o = 343\ m/s$

• Sound Power Level: $PWL = 10\log_{10}\frac{W}{W_{ref}}$; $W_{ref} = 10^{-12}\ W$

very soft whisper — $PWL = 30\,dB \rightarrow W = 10^{-9}\ W$ Saturn rocket — $PWL = 195\,dB \rightarrow W = 30\,MW$

• Sound Solutions

– Barrier sound attenuation



Fresnel #: $N = \frac{2(A+B-c)}{\lambda} = \frac{2(A+B-c)}{\lambda}f$

λ = wavelength $= \frac{c_o}{f}$
f – freq. (Hz)
c_o – speed of sound

Excess Attenuation, A'

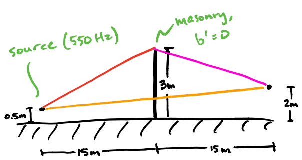
$A' = 0$ for $N < -0.916 - 0.0635b'$ $b' = 0$ for walls (masonry), $= 1$ for berms (natural)

$A' = 5(1+0.6b') + 20\log_{10}\left(\frac{\sqrt{2\pi|N|}}{\tanh\sqrt{2\pi|N|}}\right)$ $-0.1916 - 0.0635b' < N < 5.03$

$A' = 20(1+0.15b')$ for $N \geq 5.03$

20dB is the best reduction that can be done

Ex:



A = 15.207m
B = 15.033m
C = 30.037m
c_o = 343.1 m/s @ 20°C
∴ N = 0.649

$A' = 5 + 20\log_{10}\left(\frac{\sqrt{2\pi(0.649)}}{\tanh\sqrt{2\pi(0.649)}}\right)$

$A' = 11.41\,dB$

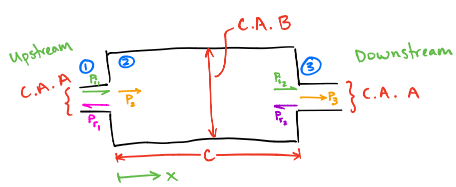
∴ if noise was 74 dB before wall, noise is (74 − 11.41) dB after wall

28 Nov

Mufflers & Silencers

• Mufflers — exhaust gas silencers on I.C. engines
• Silencers — noise suppression in ducts & intakes

Dissipative muffler — use sound absorption material
Reactive muffler — use destructive interference



$i \sim$ incident waves (outgoing)
$r \sim$ reflective wave (incoming)

Wave Number $\equiv k = \frac{\omega}{c_o}$ (units: 1/m) — speed of sound

Wave Length $\equiv \lambda = \frac{2\pi}{k} = \frac{2\pi c_o}{\omega} = \frac{c_o}{f}$ (units: m) since $\omega = 2\pi f$

Wave Eqn: $\varphi_{xx} - \frac{1}{c_o^2}\varphi_{tt} = 0$; $\varphi = \Phi(x)e^{j\omega t}$

$\Phi_{xx}e^{-j\omega t} - \frac{1}{c_o}\Phi(-\omega^2)e^{j\omega t} = 0$

$\Phi_{xx} + \frac{\omega^2}{c_o^2}\Phi = 0 \longrightarrow \Phi_{xx} + k^2\Phi = 0$

Helmholtz Equation

$\varphi \sim$ velocity potential
$u \sim$ particle velocity $= \varphi_x$
pressure $p = -\rho_o\varphi_t$

Solution: $\Phi = A_1 e^{jKx} + A_2 e^{-jKx}$

Recall: $u(x,t) = f(x-ct) + g(x+ct)$

Wave eqn soln becomes $\varphi = \overbrace{A_I e^{j(\omega t - Kx)}}^{\text{eqn. for outgoing wave}} + \overbrace{A_R e^{j(\omega t + Kx)}}^{\text{eqn for incoming wave}}$

$u = \varphi_x = -jK\left[A_I e^{j(\omega t - Kx)} - A_R e^{j(\omega t + Kx)}\right]$ in terms of particle velocity

$p = -\rho_o \varphi_t = -\rho_o j\omega\left[A_I e^{j(\omega t - Kx)} + A_R e^{j(\omega t + Kx)}\right]$ in terms of pressure

Define some variables:

$-\rho_o j\omega A_I e^{j(\omega t - Kx)} \equiv p_i$

$-\rho_o j\omega A_R e^{j(\omega t + Kx)} \equiv p_r$

$-jKA_I e^{j(\omega t - Kx)} \equiv u_i$

$-jKA_r e^{j(\omega t + Kx)} \equiv u_r$

$\hookrightarrow$ $u = u_i - u_r$ $\quad p = p_i - p_r$

Pressure continuity:

① @ $x=0$: $p_{i_1} + p_{r_1} = p_2$

within ②: $p_2 = p_{i_2} + p_{r_2}$

$\Big\} \rightarrow p_{i_1} + p_{r_1} = p_{i_2} + p_{r_2} \rightarrow A_{I_1} + A_{R_1} = A_{I_2} + A_{R_2}$

(A_{I_1}: ampl. of incident pressure going into chamber)

@ $x=c$: $p_{i_2} + p_{r_2} = p_3$; $A_{I_2} e^{-jKc} + A_{R_2} e^{jKc} = A_3 e^{-jKc}$

Continuity of flow volume:

@ $x=0$: $A(u_{i_1} - u_{r_1}) = B u_2$

(A: x-sect. area of inlet/outlet pipes; B: x-sect. area of muffler body)

within ②: $Bu_2 = B(u_{i_2} - u_{r_2})$

$\Big\} \rightarrow A(u_{i_1} - u_{i_2}) = B(u_{i_2} - u_{r_2})$

$A(A_{I_1} - A_{R_1}) = B(A_{I_2} - A_{R_2})$

@ $x=c$: $B(u_{i_2} - u_{r_2}) = Au_3 \rightarrow B(A_{I_2} e^{-jKc} - A_{R_2} e^{jKc}) = A A_3 e^{-jKc}$

5 unknowns, four eqns $\therefore$ solve for:

$$\frac{A_{I_1}}{A_3} = \frac{(1+m)(1+{}^1\!/_m)e^{jKc} + (m-1)({}^1\!/_m - 1)e^{jKc}}{4}$$

$\rightarrow m \equiv \frac{B}{A}$; $K = \frac{\omega}{c_o}$

$$\frac{A_{I_1}}{A_3} = \cos(Kc) + \frac{j}{2}\left(m + \frac{1}{m}\right)\sin(Kc)$$

Sound Intensity $I = \frac{P_{rms}^2}{\rho_o c_o} \longrightarrow \frac{I_{I_1}}{I_3} = \frac{P^2_{rms}(I_1)}{P^2_{rms}(I_3)} = \left|\frac{A_{I_1}^2}{A_3^2}\right| = 1 + \frac{1}{4}\left(m - \frac{1}{m}\right)\sin^2(Kc)$

Transmission loss (ratio of power incident on the muffler to the power transmitted):

$TL = 10\log\left(\frac{W_{I_1}}{W_3}\right)$; $W = IS$

$= 10\log\left(\frac{I_{I_1} \cdot A}{I_3 \cdot A}\right) = 10\log\left[1 + \frac{1}{4}\left(m - \frac{1}{m}\right)\sin^2(Kc)\right]$

$TL = 0$ when $\sin Kc = 0 \rightarrow$ when $Kc = n\pi$, when $n = 0, 1, 2, \ldots$

$TL = TL_{max}$ " $\sin Kc = 1 \rightarrow$ " $Kc = \frac{2n+1}{2}\pi$, $n = 0, 1, 2, \ldots$

$K = \frac{\omega}{c_o} = \frac{2\pi}{\lambda}$ $\therefore$ muffler works best when $c = \frac{2n+1}{4}\lambda$ (odd # of quarter-wave lengths)

" ineffective " $c = \frac{n}{2}\lambda$ (integer # of half-wavelengths)

C dictates what frequencies we can cancel; $m = \frac{B}{A}$ dictates how much

[Don't do last problem on Project 2]