A geodesic path is defined as the shortest path between two points in a given space. While a straight path is the shortest path between point A and point B on flat ground, on curved space straight lines are often simply not an option.

Another characteristic of these paths is that they are an inert trajectory: differential geometry focuses on spaces that may or may not be flat, finding the variations of curves in such a space must be done via a "covariant derivative". The method of a covariant derivative is:

$$\nabla_a A_n^m = A_{n,a}^m + \Gamma_{ia}^m A_n^i - \Gamma_{na}^i A_i^m$$

Here I use a comma notation where: $A_{i,j} = \frac{\partial A_i}{\partial x^j}$ as well as the summation convention where objects that are multiplied together are summed over their shared indices. Γ_{jk}^i are the Christoffel symbols, they account for the curvature of the space; among other properties, these objects are symmetric about the two down indices. In this example I use a simple (1,1) tensor A, but this can be done for a tensor of any dimensionality where terms that substitute up indices into the Cristoffel symbol are added and ones that substitute down indices are subtracted.

The Christoffel symbols can be derived from the metric tensor. As these symbols must account for the curvature of the space, the metric tensor – which defines the curvature of the space – must have a covariant derivative of 0 as given by the Christoffel symbols.

$$(1)\nabla_a g_{bc} = g_{bc,a} - \Gamma_{ab}^m g_{mc} - \Gamma_{ac}^m g_{bm} = 0$$

$$(2)\nabla_b g_{ca} = g_{ca,b} - \Gamma_{bc}^m g_{ma} - \Gamma_{ba}^m g_{cm} = 0$$

$$(3)\nabla_c g_{ab} = g_{ab,c} - \Gamma_{ca}^m g_{mb} - \Gamma_{cb}^m g_{am} = 0$$

$$(2) + (3) - (1) \Rightarrow g_{ab,c} + g_{ca,b} - g_{bc,a} - 2\Gamma_{bc}^m g_{ma} = 0$$

$$\Gamma_{bc}^m = \frac{1}{2} g^{ma}\left(g_{ab,c} + g_{ca,b} - g_{bc,a}\right)$$

We can now use these symbols to derive the geodesic equation. For this we use the covariant derivative of a curve's tangent with respect to itself: $\dot{\gamma}^m \frac{\partial}{\partial x^m}$ is the term for the tangent of curve γ along the *m*-th axis. The partial derivative term is simply a basis expression allowing this term to be compared to terms in other coordinate systems.

$$\left(\nabla_{\dot{\gamma}^m \frac{\partial}{\partial x^m}} \dot{\gamma}^n \frac{\partial}{\partial x^n}\right)^i = 0$$

The *i* over this expression is to represent that we want the resulting derivative in the *i*-th direction. We can solve this expression through the product rule and some properties of this covariant derivative; the $\dot{\gamma}^m$ term is not a function and therefore can be pulled to the far left of the derivative, however the derivative term is not a simple product on the subject of the function and therefore must be treated differently:

$$\left(\nabla_{\dot{\gamma}^m \frac{\partial}{\partial x^m}} \dot{\gamma}^n \frac{\partial}{\partial x^n}\right)^i = \dot{\gamma}^m \left(\nabla_{\frac{\partial}{\partial x^m}} \dot{\gamma}^n\right) \frac{\partial x^i}{\partial x^n} + \dot{\gamma}^m \dot{\gamma}^n \left(\nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^n}\right)^i = \dot{\gamma}^m \frac{\partial \dot{\gamma}^n}{\partial x^m} \delta_n^i + \dot{\gamma}^m \dot{\gamma}^n \Gamma_{mn}^q \delta_q^i$$

$$= \ddot{\gamma}^i + \Gamma_{mn}^i \dot{\gamma}^n \gamma^m = 0$$

$$\Rightarrow \ddot{\gamma}^i = -\Gamma_{mn}^i \dot{\gamma}^n \dot{\gamma}^m$$

Intuitively this makes sense to those familiar with the result. If one imagines walking across a rough landscape, they can notice their path curves with the terrain. We can apply this equation directly to that scenario: the $\dot{\gamma}$ terms here become the velocity, and just as the equation predicts, one's change in direction along this path is dependent on their velocity along with the curvature of the environment dictated by the Christoffel symbols.

There is a simple proof that we can find the geodesic equations for a metric by applying the Euler-Lagrange equation. This proof will work for any metric g_{ij} that is constant in λ and uses an orthogonal coordinate system $\{x^a\}$. I will use an atypical use of a dot, instead of being a time derivative, it will act a as a derivative with respect to λ which will be a more general "time parameter".

$$\mathcal{L} = g_{ij}\dot{x}^i\dot{x}^j$$

We then simply solve for the "acceleration" (second derivative with respect to λ) of a test particle on this space via the Euler-Lagrange equation.

$$\frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right) - \frac{\partial\mathcal{L}}{\partial x^a} = 0$$

$$\frac{d}{d\lambda}\left(\frac{\partial}{\partial\dot{x}^a}\left(g_{ij}\dot{x}^i\dot{x}^j\right)\right) - \frac{\partial}{\partial x^a}\left(g_{ij}\dot{x}^i\dot{x}^j\right) = \frac{d}{d\lambda}\left(g_{ij}(\delta_a^j\dot{x}^j + \dot{x}^i\delta_a^i\right) - g_{ij,a}\dot{x}^i\dot{x}^j$$

$$= \frac{d}{d\lambda}\left(g_{aj}\dot{x}^j + g_{ia}\dot{x}^i\right) - g_{ij,a}\dot{x}^i\dot{x}^j$$

$$= g_{aj}\ddot{x}^j + g_{ia}\ddot{x}^i + g_{aj,i}\dot{x}^i\dot{x}^j + g_{ai,j}\dot{x}^i\dot{x}^j - g_{ij,a}\dot{x}^i\dot{x}^j$$

$$g^{am}\left(g_{aj}\ddot{x}^j + g_{ia}\ddot{x}^i\right) = -g^{am}\left(g_{aj,i} + g_{ai,j} - g_{ij,a}\right)\dot{x}^i\dot{x}^j$$

$$\ddot{x}^m = -\frac{1}{2}g^{am}\left(g_{aj,i} + g_{ai,j} - g_{ij,a}\right)\dot{x}^i\dot{x}^j = -\Gamma_{ij}^m\dot{x}^i\dot{x}^j$$

This is simply the geodesic equation derived earlier.

Our next step is to look at the Einstein field equation:

$$R_{ab} - \frac{1}{2}g_{ab}R + g_{ab}\Lambda = \frac{8\pi G}{c^4}T_{ab}$$

Where R_{ab} is the Ricci tensor (defined by $R_{ab} = \Gamma_{am,b}^m - \Gamma_{ab,m}^m + \Gamma_{an}^m\Gamma_{bm}^n - \Gamma_{mn}^m\Gamma_{ab}^n$), R is the Ricci scalar (defined by $R = g^{ab}R_{ab}$), G is the Newtonian gravitational constant, c is the speed of light, T_{ab} is the energy-momentum tensor, and Λ is the cosmological constant

We will be looking at a vacuum solution ($T_{ab} = 0$) over a scale smaller than a galaxy ($\Lambda \to 0$). This gives us the equation:

$$R_{ab} - \frac{1}{2}g_{ab}R = 0$$

To solve this, we must first set up a general metric. We will choose a metric describing the length of proper time. This solution should be radial, so we can work based on a spherical coordinate system with curvature on the coordinate time and radial space dimensions.

$$c^2 d\tau^2 = c^2 e^{A(r)} dt^2 - e^{B(r)} dr^2 - r^2 (d\theta^2 + \sin^2(\theta)\, d\varphi^2)$$

$A(r)$ and $B(r)$ are unknown functions defining the curvature on this space.

Now we can start finding these functions by trying to solve Einstein's field equation, the non-zero Christoffel symbols of this system are:

$\Gamma^0_{01} = A'$ $\quad \Gamma^1_{00} = c^2 A' e^{2(A-B)}$ $\quad \Gamma^1_{11} = B'$ $\quad \Gamma^1_{22} = -re^{-2B}$ $\quad \Gamma^1_{22} = -re^{-2B} \sin^2(\theta)$

$\Gamma^2_{12} = \frac{1}{r}$ $\quad \Gamma^2_{33} = -\sin(\theta)\cos(\theta)$ $\quad \Gamma^3_{13} = \frac{1}{r}$ $\quad \Gamma^3_{23} = \frac{\cos(\theta)}{\sin(\theta)}$

From here we can get the non-zero terms of the Ricci tensor

$$R_{00} = -c^2 \left(A'' + A'(A' - B') + \frac{2}{r} A' \right)$$

$$R_{11} = A'' + A'(A' + B') + \frac{2}{r} B'$$

$$R_{22} = -1 + e^{-2B} - 2rB' e^{-2B} + r(A' + B')e^{-2B}$$

$$R_{33} = R_{22} \sin^2(\theta)$$

Now we can plug this into Einstein's field equation. First we should enforce a boundary condition. This is a radial vacuum solution, as such we must require $R_{22} = R_{33} = 0$. From here we can begin to solve this:

$$R_{00} - g_{00} R = 0 \Rightarrow R_{00} - c^2 e^{2A} \left(\frac{1}{c^2} e^{-2A} R_{00} - e^{-2B} R_{11} \right)$$

$$\therefore A' + B' = 0$$

This result combined with the condition that the curvature tends to 0 over very large distances allows us to conclude that $A = -B$.

We can now look at R_{22} to get:

$$-1 + e^{-2B} - 2rB'e^{-2B} = 0$$

$$\frac{d}{dr}(re^{-2B}) = 1$$

$$e^{2A} = 1 + \frac{k}{r}$$

Where k is an integration constant.

Now we have the general solution to the Einstein field equation:

$$c^2 d\tau^2 = c^2\left(1 + \frac{k}{r}\right)dt^2 - \frac{dr^2}{1 + \frac{k}{r}} - r^2(d\theta^2 + \sin^2(\theta)\, d\varphi^2)$$

We can analyze this by looking at the Lapacian and comparing it to that of the Newtonian gravitational energy formula. Through this we can enforce the boundary conditions that as $m \to 0$ and $r \to \infty$ the two solutions must converge, or else Einstein's formula would be useless to us as Newton's models have been observed to be almost correct. In this limit the time derivatives are 1 as Newton equations have no expression for time dilation. The Laplacian, as its derivatives are used for equations of motion, is not really effected by transformations such as multiplication or addition by constants, because of this we are allowed to write the equations as:

$$\mathcal{L}_{Schwarzchild} = \frac{\dot{r}^2}{1 + \frac{k}{r}} + r^2\left(\dot{\theta}^2 + \sin^2(\theta)\, d\varphi^2\right) - \frac{c^2 k}{r}$$

$$\mathcal{L}_{Newton} = \dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2(\theta)\, d\varphi^2\right) + \frac{2GM}{r}$$

Comparing Newton's and Schwarzschild's solutions we can see that, to give them similar results, we must say $k \ll r$ and c must be very big. This allows the radial kinetic energy terms to tend toward each other.

The last terms force us to conclude:

$$k = -\frac{2GM}{c^2}$$

There are no other terms to correct adequately for this, any other solution would mean that the two Lagrangians vary on all scales. (this term happens to be equal in magnitude to the distance at which Newtonians equations predict escape velocity to be the speed of light).

Now we can look at our solution:

$$d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \frac{1}{1 - \frac{2GM}{rc^2}} \frac{dr^2}{c^2} + \frac{r^2}{c^2} (d\theta^2 + \sin^2(\theta)\, d\varphi^2)$$

On inspection, we can see a rather odd property. If $M = 0$ and we set the velocity of our test particle to the speed of light, it’s perceived time, $d\tau$, goes to 0; time dilates to nothing as we tend toward the speed of light. Accepting this, we can see a stranger phenomenon: if we introduce mass we can set the time to 0 without even moving our particle. At $r = \frac{2GM}{c^2}$ time goes to 0, any such state constitutes a “light like” vector. This means that light will simply remain stationary at this point known as the Schwarzchild Radius.

Let’s look more closely at how light actually moves around a black hole. As mentioned before, a light like vector is any path through spacetime in which $d\tau = 0$. So we can simply impose this requirement and get the optical metric:

$$dt^2 = \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)^2} \frac{dr^2}{c^2} + \frac{r^2}{\left(1 - \frac{2GM}{rc^2}\right) c^2} (d\theta^2 + \sin^2(\theta)\, d\varphi^2)$$

In the interest of simplicity I will now convert my units to Planck units, setting $c = 1$ and $G = 1$ ($\hbar = 1$ also, though this does not concern us now). I will also set $\theta = \frac{\pi}{2}$ (this will eliminate the term and its derivatives, the radial nature of the metric means that this will have no qualitative difference in result) and find the Lagrangian:

$$\mathcal{L} = \frac{1}{\left(1 - \frac{2m}{r}\right)^2}\dot{r}^2 + \frac{r^2}{\left(1 - \frac{2m}{r}\right)}\dot{\varphi}^2$$

$$\ddot{r} = \frac{2m}{r - 2m}\dot{r}^2 + (r - 3m)\dot{\varphi}^2$$

$$\frac{\partial}{\partial \lambda}\left(\frac{r^3}{r - 2m}\dot{\varphi}\right) = 0;\ \frac{r^3}{r - 2m}\dot{\varphi} = L$$

Because of the conservation of angular momentum, L, we can now write the velocities involved in this equation purely in terms of the radius, its derivatives, and constants:

$$\ddot{r} = \frac{2m}{r - 2m}\dot{r}^2 + \frac{(r - 3m)(r - 2m)^2}{r^6}L^2$$

$$\dot{\varphi} = \frac{r - 2m}{r^3}L$$

There are various properties that can be seen here, such as the photon sphere at $r = 3m$ where light can perform an orbit around the black hole. This is the formula that I used in the very basic light deflection simulation I made on scratch:

https://scratch.mit.edu/projects/166018435/