6 Geometry

Introduction

Geometry is one of the most important chapters for CAT and other MBA entrance exams. Approximately 20% of the CAT questions are asked from geometry alone. It is advisable to solve class 9th and 10th geometry from schoolbooks (NCERT), especially those questions in which we have to prove some results as these results can directly be used to solve CAT problems. In geometry, we will learn about different measures and properties of points, lines, surfaces, and solids.

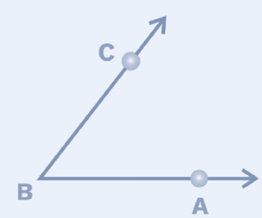
Basic Geometric Terms

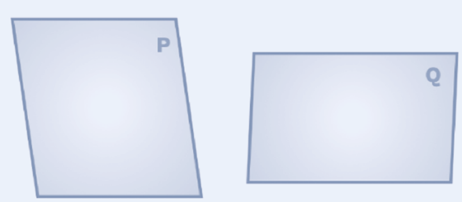
<table>
<tr><td>Point
Represented by a dot, it indicates an exact location in space. A point has no dimension.</td><td>

(Point A)</td></tr>
<tr><td>Line
A collection of points along a straight path that can extend endlessly in both directions.</td><td>

$\overleftrightarrow{PQ}$ (Line PQ)</td></tr>
<tr><td>Line Segment
A part of the line having two endpoints.</td><td>

$\overline{AB}$ (Line Segment PQ)</td></tr>
<tr><td>Ray
A part of the line having only one endpoint.</td><td>

$\overrightarrow{RS}$ (Ray RS)</td></tr>
<tr><td>Angle
Amount of turns between two rays that have a common endpoint is called the vertex of the angle.</td><td>

∠ABC (Angle ABC)
• The vertex is always the middle letter.
• It can also be written as ∠CBA or just ∠B.</td></tr>
<tr><td>Plane
A flat surface that extends endlessly in all directions.</td><td>

(Plane–P) (Plane–Q)</td></tr>
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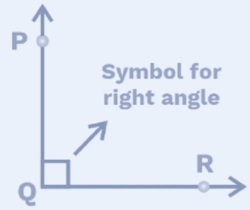
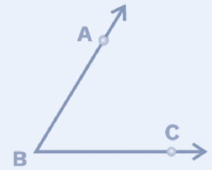
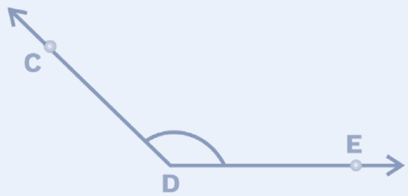
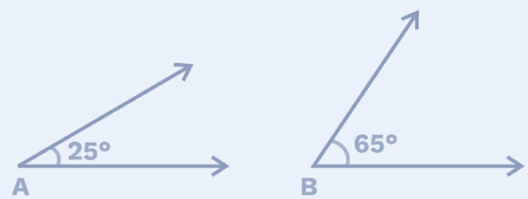
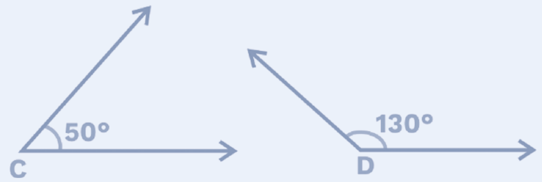
Straight Line An angle whose measure is 180°.	 ∠ABC is a straight angle.
Right Angle An angle whose measure is 90°.	 ∠PQR is a right angle.
Acute Angle An angle whose measure is less than 90°.	 ∠ABC is an acute angle.
Obtuse Angle An angle whose measure is more than 90° and less than 180°.	 ∠CDE is an obtuse angle.
Complementary Angles Two angles are complementary if the sum of their measures is 90°.	 m∠A = 25° m∠B = 65° m∠A + m∠B = 25° + 65° = 90° ∠A and∠B are complementary angles.
Supplementary Angles Two angles are supplementary angles if the sum of their measures is 180°.	 m∠C + m∠D = 50° + 130° = 180° ∠C and ∠D are supplementary angles.

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Intersecting Lines
Two lines that cross each other at some point.

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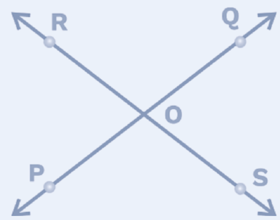

PQ and RS intersect at point O.

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Parallel Lines
If the perpendicular distance between the lines is constant, then the lines are said to be parallel.
OR
Two lines in the same plane, which do not intersect.

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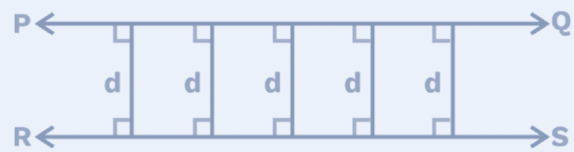

PQ || RS
(PQ is parallel to RS)

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Perpendicular Lines
Two lines that intersect to form right angles.

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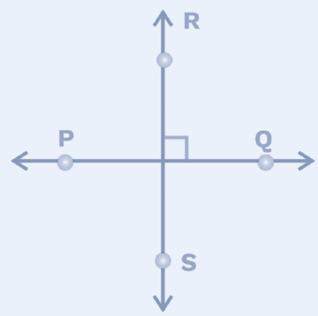

RS ⊥ PQ (RS is perpendicular to PQ)

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Vertical Angles
Two angles with equal measure formed by two intersecting lines.

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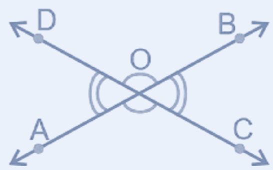

- ∠AOC and ∠DOB are vertically opposite angles.
- ∠BOC and ∠DOA are vertically opposite angles.

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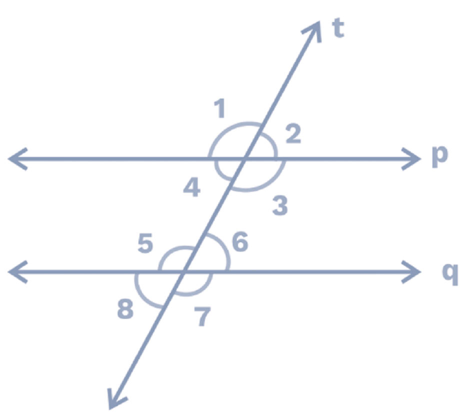
Angles Associated with Two or More Straight Lines

Let us consider two straight lines *p* and *q* that are intersected by a transversal line *t*.

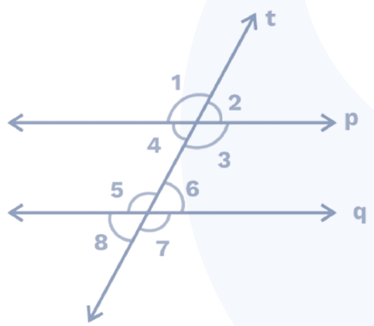


From the above figure, we can conclude the following:

Corres-ponding Pair of Angles	Alternate Interior Angles	Alternate Exterior Angles
∠1 and ∠5 ∠4 and ∠8 ∠2 and ∠6 ∠3 and ∠7	∠4 and ∠6 ∠3 and ∠5	∠1 and ∠7 ∠2 and ∠8

$\left.\begin{array}{l}\angle 4 \text{ and } \angle 5\\ \angle 3 \text{ and } \angle 6\end{array}\right\}$ Pair of interior angles on the same side of the transversal.

Note: If lines *p* and *q* are parallel in the above figure, then all the corresponding angles will be equal, pair of alternate interior and exterior angles will also be equal.



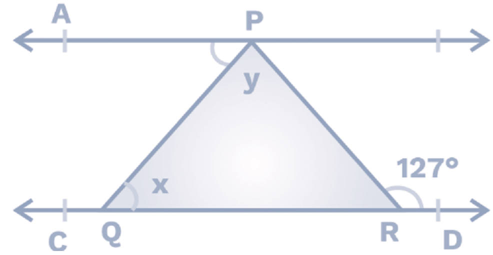
Vertically Opposite Angles	Corresponding Angles
∠1 = ∠3 ∠2 = ∠4 ∠5 = ∠7 ∠6 = ∠8	∠1 = ∠5 ∠4 = ∠8 ∠2 = ∠6 ∠3 = ∠7

Alternate Interior Angles	Alternate Exterior Angles
∠4 = ∠6 ∠3 = ∠5	∠1 = ∠7 ∠2 = ∠8

Also, ∠3 + ∠6 = 180° and ∠4 + ∠5 = 180° (interior angle on the same side of the transversal).

Example 1:

In the given figure, if AB || CD, ∠APQ = 33° and ∠PRD = 127°. Find *x* and *y*.



Solution: 33° and 94°

Since, AB || CD and PQ is acting as a transversal line.

Therefore, ∠APQ = ∠PQR (alternate interior angle)

Hence, ∠PQR = ∠x = 33°

Also, PR is acting as a transversal line.

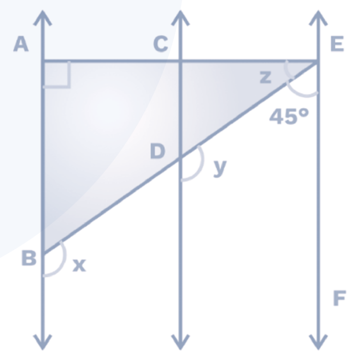
Therefore, ∠APR = ∠PRD

33° + ∠y = 127°

⇒ ∠y = 127° – 33° = 94°

Hence, ∠x = 33° and ∠y = 94°.

Example 2



In the given figure, AB || CD and CD || EF. Also, EA ⊥ AB. If ∠BEF = 45°. Find the values of *x*, *y*, and *z*, respectively.

Solution: 135°, 135°, and 45°

y + 45° = 180°

(Interior angles on the same side of the transversal line ED.)

Therefore, y = 180° – 45° = 135°

Again, (AB || CD, corresponding angles will be equal).

x = y = 135°

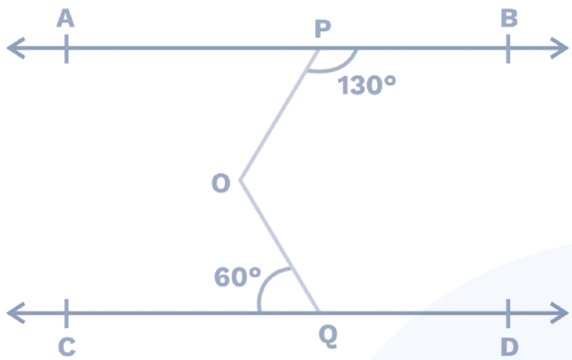
So, ∠EAB + ∠FEA = 180° (Interior angles on the side of transversal EA).

Therefore, $90° + z + 45° = 180°$

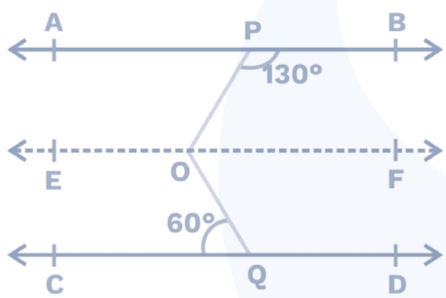
which gives, $z = 45°$

Example 3

In the given figure, AB || CD, ∠OPB = 130° and ∠OQC = 60°. Find ∠POQ.



Solution: 110°



Here, we need to draw a line EF parallel to line AB, through point O as shown in the figure. Now, AB || EF and AB || CD; therefore, EF || CD.

Now, ∠BPO + ∠POF = 180° (interior angle on the same side of transversal PO).

130° + ∠POF = 180°

∠POF = 180° – 130°

∠POF = 50° ...(i)

Also, ∠FOQ = ∠OQC (alternate interior angles)

∠FOQ = 60° ...(ii)

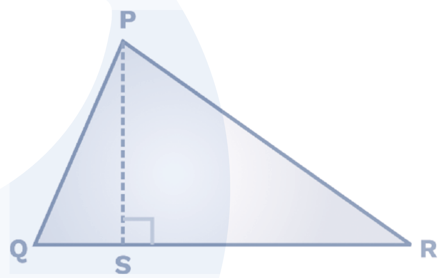
Adding equations (i) and (ii), we get

∠POF + ∠FOQ = 50° + 60°

That is, ∠POQ = 110°

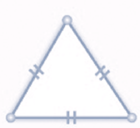
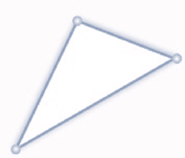
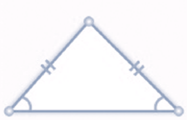
Triangle

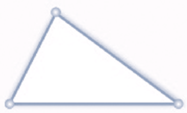
A triangle is a figure enclosed by three sides.



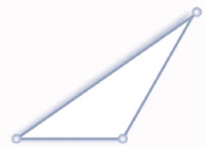
In the figure above, PQR is a triangle. Line PS represents the height of the triangle corresponding to the side QR.

Types of Triangles

By Sides		By Angles	
	**Equilateral triangle** **All three sides are equal and all interior angles are equal to 60° each**		**Acute triangle** **All three angles are less than 90°**
	**Isosceles triangle** **Exactly two sides are equal and angle opposite to these two sides are also equal.**		**Right triangle** **One of the angles is 90° (Right angle) and other two angles are less than 90°**



Scalene triangle
All the three sides are distinct.



Obtuse triangle
One of the angles is more than 90° and other two angles are less than 90°

Existence of Triangle

In any triangle, the sum of the lengths of any two sides has to be greater than the length of the third side. If a, b, and c are the sides of a triangle, then

$a + b > c$; $a + c > b$ and $b + c > a$

Example 4

Two sides of a triangle are 3 cm and 7 cm. If the length of the third side is an integral value, then find how many distinct triangles are possible?

Solution: Five distinct triangles

Let the length of the third side be x cm.

Case 1: x is the greatest side

$x < (3 + 7)$

$x < 10$ ...(i)

Case 2: 7 is the greatest side

$x + 3 > 7$

$x > 4$ ...(ii)

From equations (i) and (ii)

$4 < x < 10$

Therefore, $x = 5, 6, 7, 8$, and 9.

Hence, 5 distinct triangles are possible.

Example 5

Find the range of values of x, if $(x - 3)$, $(2x - 3)$, and $(4x - 22)$ are the three sides of a triangle.

Solution:

$$\frac{22}{3} < x < 22$$

We know that the sum of any two sides of a triangle must be greater than the third side.

Therefore,

Case 1: $(x - 3) + (2x - 3) > (4x - 22)$

$3x > 4x - 22$

$-x > -22$

$x < 22$ ....(i)

Case 2: $(x - 3) + (4x - 22) > (2x - 3)$

$5x - 25 > 2x - 3$

$3x > 22$

$x > 22/3$ ....(ii)

Case 3: $(2x - 3) + (4x - 22) > (x - 3)$

$6x - 25 > x - 3$

$5x > 22$

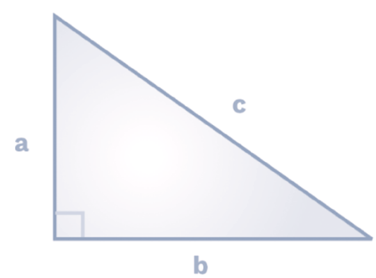
$x > 22/5$ ....(iii)

From equations (i), (ii), and (iii), we get:

$$\frac{22}{3} < x < 22$$

Relation between Sides of Acute, Obtuse, and Right-Angled Triangles

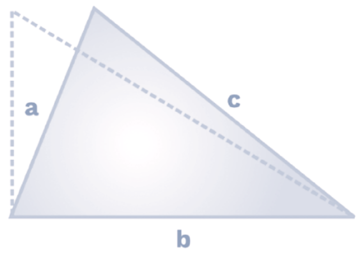
1. Consider a right-angled triangle, with c being the hypotenuse, i.e., the largest side, and the other two sides being a and b.



   For the right triangle, the relation is: $a^2 + b^2 = c^2$ (Pythagoras theorem).

2. Now, if the right angle is reduced a little, so that it becomes acute, and a and b are kept of the same length, it should be obvious that c will reduce.

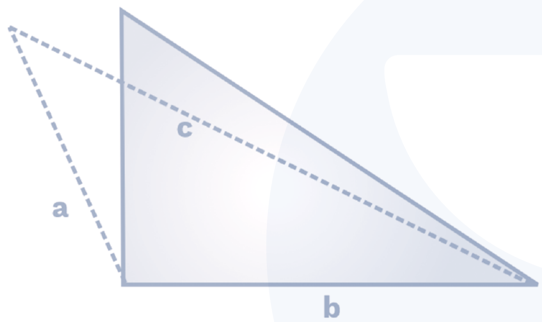
Now, $a^2 + b^2 > c^2$



So, in an acute angle triangle, the square of the largest side is less than the sum of squares of smaller sides.

3. If the right angle is increased a little, so that it becomes obtuse, and a and b are kept of the same length, then c has to increase and thus:

$a^2 + b^2 < c^2$



So, in an obtuse angle triangle, the square of the largest side is more than the sum of squares of smaller sides.

Example 6

What type of triangle with lengths 8, 15, and 12 units is formed (acute angle or right angle or obtuse angle triangle).

Solution: Obtuse angle triangle

This question is a straightforward application of what we just learnt.

Here, $a = 8$, $b = 12$ and $c = 15$

$a^2 + b^2 = 64 + 144 = 208$

$c^2 = 15^2 = 225$

Since, the square of the largest side (c^2) is more than the sum of squares ($a^2 + b^2$) of smaller sides.

Thus, the triangle is an obtuse-angled triangle.

Example 7

If 7 cm and 13 cm are the two sides of an obtuse-angled triangle, then find how many triangles with integral sides will exist?

(A) 8 (B) 9

(C) 10 (D) 12

Solution: (B)

Let the third side be x cm.

Now, possible range for length of the third side:

$(13 - 7) < x < (13 + 7)$

$6 < x < 20$

Case 1: If x is the greatest side:

Then $x^2 > 13^2 + 7^2$

$x^2 > 169 + 49$

$x^2 > 218$

Here x can be 15, 16, 17, 18, and 19.

Case 2: If 13 is the greatest side:

Then $(13)^2 > x^2 + 7^2$

$169 > x^2 + 49$

$120 > x^2$

Here x can be 7, 8, 9, and 10.

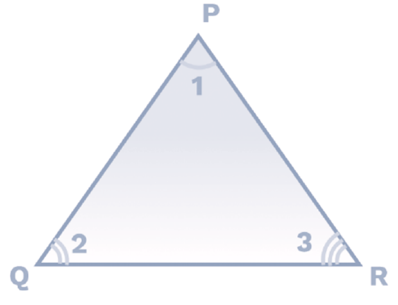
Therefore, a total of nine such triangles are possible.

Angle Sum Property of a Triangle

Theorem

The sum of the angles of a triangle is 180°.

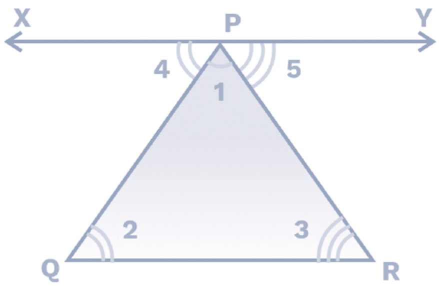
Proof



We are given in the statement above that PQR is a triangle and $\angle 1$, $\angle 2$, and $\angle 3$ are the angles of a ΔPQR. We need to prove that

$\angle 1 + \angle 2 + \angle 3 = 180°$ ...(i)

Let us draw a line XPY parallel to QR through vertex *P*.



Since, XPY is a line

Therefore, $\angle 4 + \angle 1 + \angle 5 = 180°$

But, XPY || QR and PQ, PR are transversals.

So, $\angle 4 = \angle 2$ and $\angle 5 = \angle 3$ (alternate interior angles).

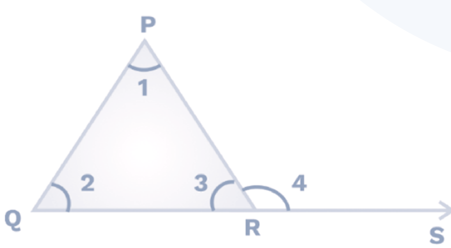
Substituting $\angle 4$ and $\angle 5$ in equation (i),

$\angle 4 + \angle 1 + \angle 5 = \angle 2 + \angle 1 + \angle 3 = 180°$ or $\angle 1 + \angle 2 + \angle 3 = 180°$.

Theorem

If a side of a triangle is produced, then the exterior angle so formed is equal to the sum of the two opposite interior angles.

Proof



We know that: $\angle 1 + \angle 2 + \angle 3 = 180°$ (angle sum property of triangle)

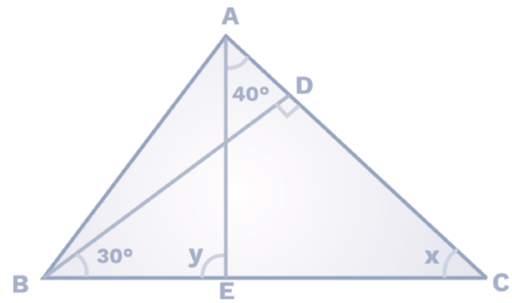
$\angle 1 + \angle 2 = 180° - \angle 3$ ...(i)

Also, $\angle 3 + \angle 4 = 180°$ (Linear pair)

$\angle 4 = 180° - \angle 3$ ...(ii)

From equations (i) and (ii), we can conclude that $\angle 4 = \angle 1 + \angle 2$.

Example 8



In the given figure, if BD is perpendicular to AC, $\angle DBC = 30°$ and $\angle EAC = 40°$. Find *x* and *y*.

Solution: $\boldsymbol{x = 60°, y = 100°}$

In ΔDBC, $90° + 30° + x = 180°$
(Angle sum property)

$\boxed{x = 60°}$

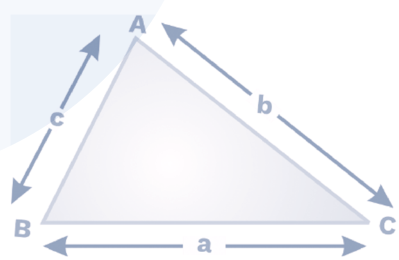
Now, $\angle y = 40° + x$ (Exterior angle property)

$\angle y = 40° + 60°$ $\boxed{\angle y = 100°}$

Sine Rule

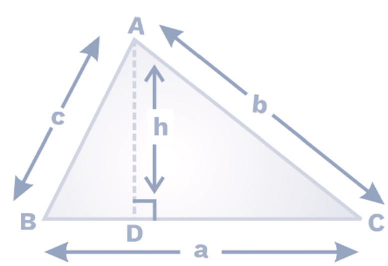
There is a direct relation between the sides and the opposite angles, sides are directly proportional to the sine of the angles opposite them.

Thus, $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$



Proof

Draw AD is perpendicular to BC.



In ΔADB, $\sin\angle B = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}}$

$$\sin\angle B = \frac{h}{c} \quad ...(i)$$

In ΔADC, $\sin\angle C = \frac{h}{b}$ ...(ii)

Dividing equations (i) and (ii),

$$\frac{\sin\angle B}{\sin\angle C} = \frac{(h/c)}{(h/b)} = \frac{b}{c}$$

$$\Rightarrow \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C}$$

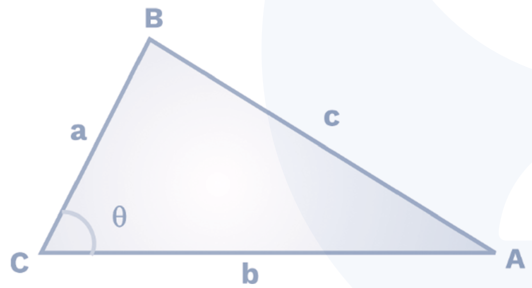
Similarly, we can prove for $\angle A$ also.

Hence, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine Rule

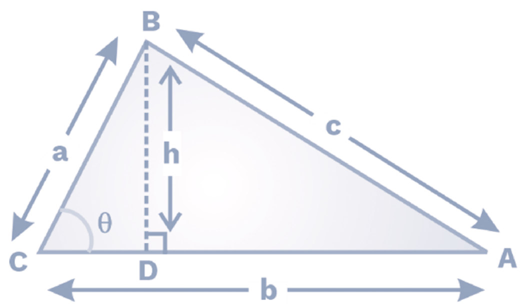
When two sides and the included angle are given, we should use the Cosine rule, which states:

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$$



Where θ is the included angle between lengths a and b, i.e., is the angle opposite to length c?

Proof



In ΔBDC, $\cos C = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{CD}{a}$

$CD = a \cos C$

$\Rightarrow$ $AD = AC - CD$

$\Rightarrow$ $AD = b - a \cos C$

Also, $\sin C = \dfrac{BD}{a} = \dfrac{h}{a}$

$BD = h = a \sin C$

Now, In ΔBDA, using Pythagoras theorem:

$c^2 = (BD)^2 + (DA)^2$

$c^2 = (a \sin C)^2 + (b - a\cos C)^2$

$c^2 = a^2 \sin^2 C + b^2 + a^2 \cos^2 C - 2ab\cos C$

$c^2 = a^2 (\sin^2 C + \cos^2 C) + b^2 - 2ab\cos C$

$c^2 = a^2 + b^2 - 2ab\cos C$

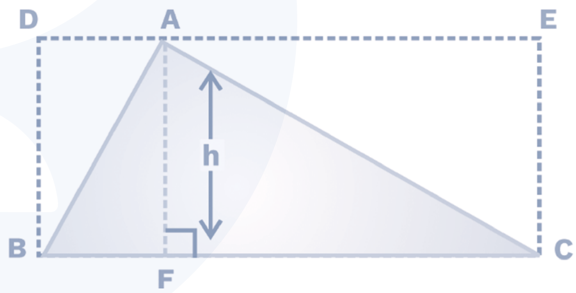
Or $2ab\cos C = a^2 + b^2 - c^2$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Area of Triangle

1. Area of triangle $= \dfrac{1}{2} \times$ Base $\times$ Height

Proof



Consider a triangle ABC.

Draw AF $\perp$ BC (height $= h$), BC as a base.

Draw rectangle AFBD and rectangle AECF.

Now, AB and AC are diagonal for rectangles AFBD and AFCE, respectively.

Area of $\Delta ABF = \dfrac{1}{2}(BF \times AF)$ ...(i)

Area of $\Delta ACF = \dfrac{1}{2}(FC \times AF)$ ...(ii)

Adding equations (i) and (ii)

Area of (ΔABF) + Area of (ΔACF)

$$= \frac{1}{2}(BF \times AF) + \frac{1}{2}(FC \times AF)$$

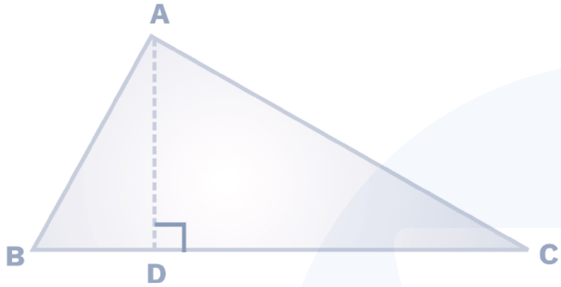
$$\text{Area of } \Delta ABC = \frac{1}{2} \times AF(BF + FC)$$

$$\text{Area of } \Delta ABC = \frac{1}{2} \times BC \times AF$$

2. Area of triangle = $\frac{1}{2} \times a \times b \times \sin\theta$, where θ is the angle between two sides.

Proof

Consider ΔABC in which AD represents its height and BC as its base.



In ΔABD, $\sin\angle B = \frac{AD}{AB}$.

$AD = AB \sin\angle B$

Now, we know that:

$\frac{1}{2} \times \text{Base} \times \text{Height} = \frac{1}{2} \times BC \times AD$

$\text{Area of triangle} = \frac{1}{2} \times BC \times AB \times \sin\angle B$

3. Heron's formula:

Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$,

where $s = \frac{a+b+c}{2}$ (semi-perimeter)

and a, b, c are sides.

4. Area of triangle = $r \times s$, where $r \to$ inradius, $s \to$ semi-perimeter.

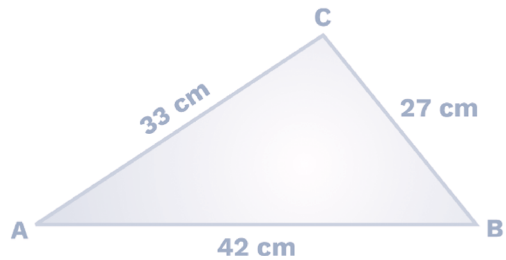
5. Area of triangle $= \frac{abc}{4R}$, where $R \to$ Circum-radius and a, b, c are sides of the triangle.

6. Area of equilateral triangle: Area of an equilateral triangle $= \frac{\sqrt{3}}{4} \times (\text{Side})^2$.

Example 9

In triangle ABC, AB = 42 cm, BC = 27 cm, and AC = 33 cm. Find the length of the altitude CD on side AB.

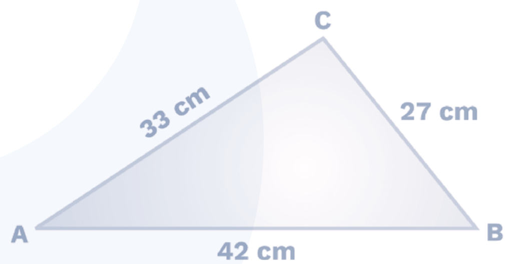
Solution: $x = \frac{36\sqrt{17}}{7}$ cm



$$S = \frac{a+b+c}{2} = \frac{33+27+42}{2} = \frac{102}{2} = 51 \text{ cm}$$

If the altitude to AB is x cm, then

$\text{Area of triangle} = \frac{1}{2} \times x \times AB$



Also, the area can be calculated using Heron's formula.

Equating the area is found by two methods.

Therefore,

$$\frac{1}{2} \times x \times AB = \sqrt{51 \times (51-33)(51-27)(51-42)}$$

$$\frac{1}{2} \times x \times AB = \sqrt{51 \times 18 \times 24 \times 9}$$

$$\frac{1}{2} \times x \times 42 = \sqrt{17 \times 3 \times 2 \times 3 \times 3 \times 3 \times 8 \times 3 \times 3}$$

$$x \times 21 = 3 \times 3 \times 3 \times 2 \times 2 \times \sqrt{17}$$

$$x = \frac{36\sqrt{17}}{7} \text{ cm}$$

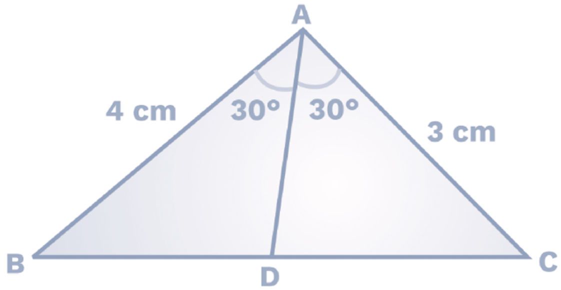
Example 10 **From Past Year Papers**

In a triangle ABC, the internal bisector of the angle A meets BC at D. If AB = 4 cm, AC = 3 cm, and $\angle A = 60°$, then the length of AD is?

(A) $2\sqrt{3}$ cm (B) $\frac{12\sqrt{3}}{7}$ cm

(C) $\frac{15\sqrt{3}}{8}$ cm (D) $\frac{6\sqrt{3}}{7}$ cm

Solution: (B)



From the figure, we can say:

Area of ΔBAD + Area of ΔDAC = Areas of ΔBAC.

$$\frac{1}{2} \times AB \times AD \times \sin 30° + \frac{1}{2} \times AD \times AC \times \sin 30°$$

$$= \frac{1}{2} \times BA \times AC \times \sin 60°$$

$$\frac{1}{2} \times AD \times \sin 30° (AB + AC) = \frac{1}{2} \times BA \times AC \times \sin 60°$$

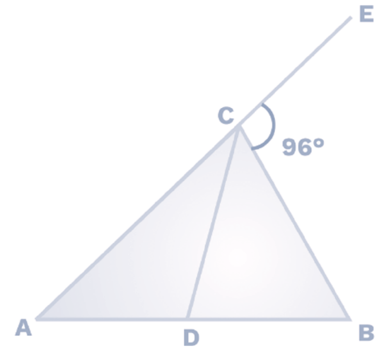
$$\frac{1}{2} \times AD \times \frac{1}{2} \times 7 = \frac{1}{2} \times 4 \times 3 \times \frac{\sqrt{3}}{2}$$

$$AD = \frac{12\sqrt{3}}{7} \text{ cm}$$

Hence, option (B) is the correct answer.

Example 11

In the figure given below, AD = CD = BC, what is the value of $\angle$CDB?



Solution: 64°

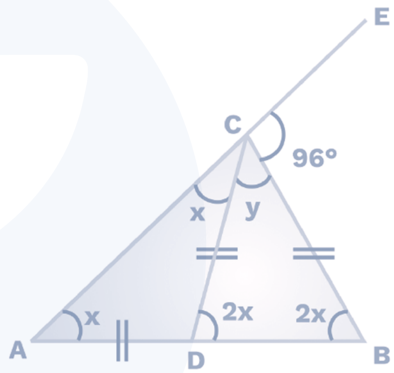
Since $x + y + 96° = 180°$

Therefore, $x + y = 84°$ ...(i)

Now, in ΔCDB

$\Rightarrow 4x + y = 180°$ ...(ii)

(Angle sum property)



Solving equations (i) and (ii)

$x = 32°$

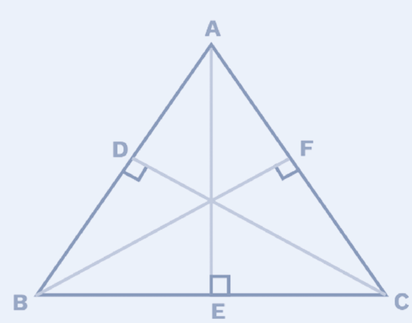
So, $2x = 2 \times 32° = 64°$

Hence, $\angle$CDB = 64°.

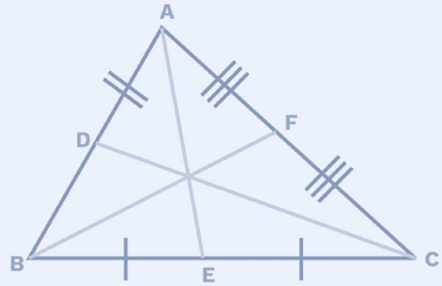
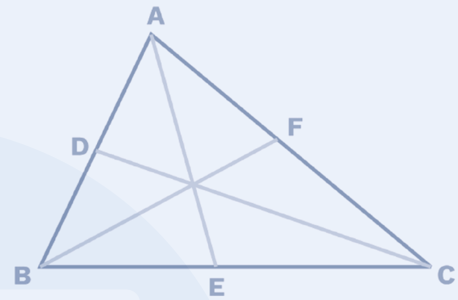
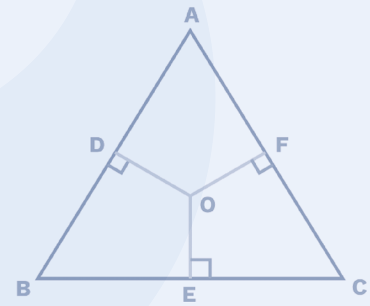
Special Line Segments Inside a Triangle

Altitude

The line segment that is perpendicular from the opposite vertex to the side of a triangle is called the altitude of the triangle. There are three altitudes in a triangle.



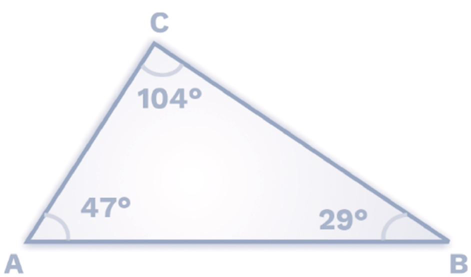
AE, BF, and CD are altitudes.

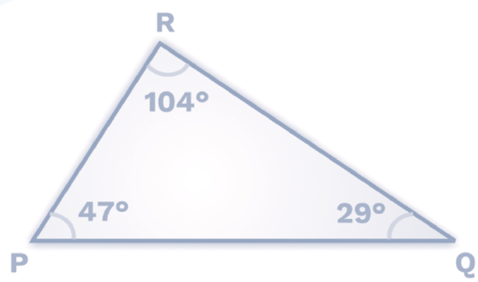
Median The line segment joining the midpoint of a side to the vertex opposite to the side is called a median. There are three medians in a triangle. A median divides the triangle into two triangles of equal area.	 AE, BF, and CD are medians BE = EC; FC = AF; AD = DB
Angle Bisector A line segment from a vertex that bisects the same angle is called an angle bisector. There are three angle-bisectors.	 AE, CD, and BD are angle bisectors.
Perpendicular Bisector A line that bisects a side and as well as perpendicular to the same side is called a perpendicular bisector. Three perpendicular bisectors are there in a triangle.	 OD, OE, and OF are perpendicular bisectors.

Congruent Figures

Two figures or objects are congruent if they have the same shape and size.

Example





The vertices and sides of the triangle ABC are matched by congruence with the vertices and the side of the triangle PQR in the following manner:

$A \leftrightarrow P$	$B \leftrightarrow Q$	$C \leftrightarrow R$
BC ↔ QR	CA ↔ RP	AB ↔ PQ

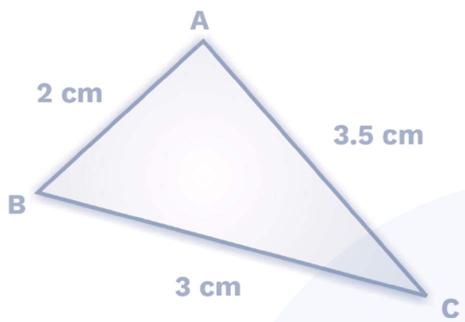
All of these can be expressed by using the symbol ≅ for *is congruent to*.

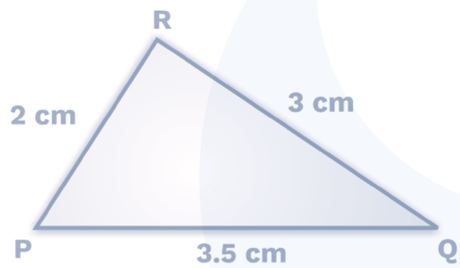
Hence, ΔABC ≅ ΔPQR

Congruence Tests for Triangles

1. SSS (Side–Side–Side):

The three sides of one triangle are respectively equal to the three sides of another triangle.



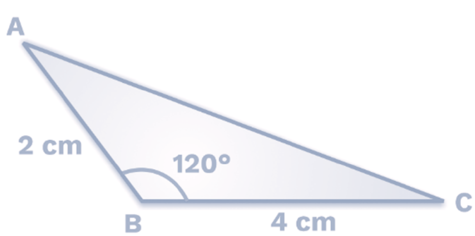


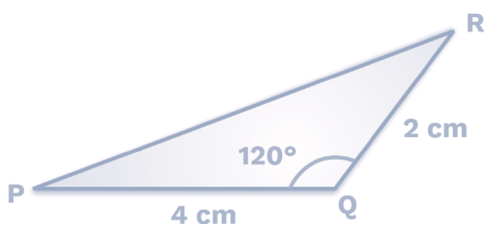
Here, AB = PR, BC = RQ, and AC = PQ

Therefore, ΔABC ≅ ΔPRQ

2. SAS (Side–Angle–Side):

Two sides and the included angles are respectively equal.



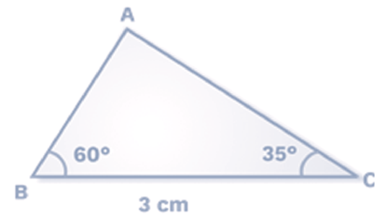


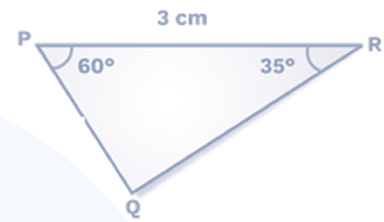
Here, AB = RQ, BC = PQ, and ∠ABC = ∠PQR

Therefore, ΔABC ≅ ΔRQP

3. AAS (Angle–Side–Angle):

Two angles and the included side are respectively equal.

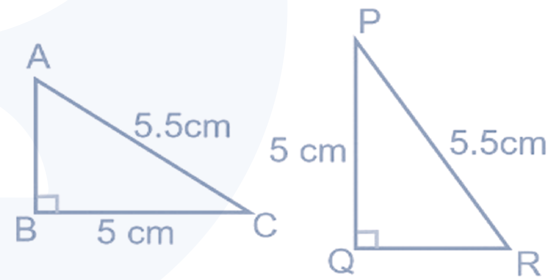




Here, BC = PR, ∠ABC = ∠QPR, and ∠ACB = ∠PRQ

Therefore, ΔABC ≅ DQPR

4. RHS (Right–Angled–Hypotenuse–Side):



Here, BC = PQ, AC = PR, and ∠ABC = ∠PQR

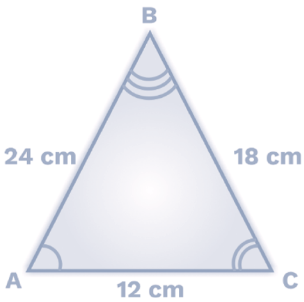
Therefore, ΔABC ≅ ΔRQP

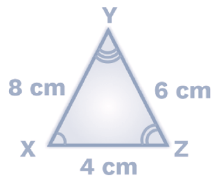
Similar Figures

Two figures are similar to each other when they have the same shape but not necessarily the same size.

Examples of Similar Figures from Geometry

Similarity in Triangles





Here, ΔABC is not congruent to ΔXYZ, but there is some relationship between these two triangles:

- Their corresponding angles are the same $\angle A = \angle X$, $\angle B = \angle Y$, and $\angle C = \angle Z$

  Also, lengths are just scaled-up version of each other.

  $AC = 3 \times XZ$, $BC = 3 \times YZ$, and $AB = 3 \times YX$

  Here, $\frac{AB}{XY} = 3$; $\frac{BC}{YZ} = 3$; $\frac{AC}{XZ} = 3$

  or, $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ} = K$ (which is 3 in the above figure).

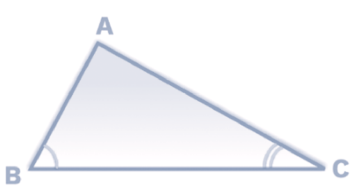
  Here, ΔABC is the enlargement of ΔXYZ.

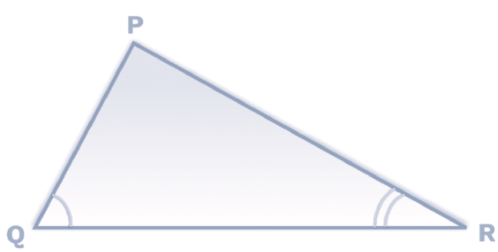
  Hence, $\Delta ABC \sim \Delta XYZ$ (where ~ is the symbol for *similar to*).

Similarity Tests for Triangles

1. A – A similarity test (Angle–Angle):

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.



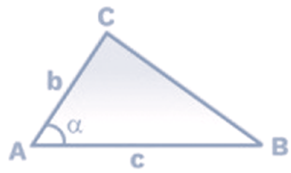


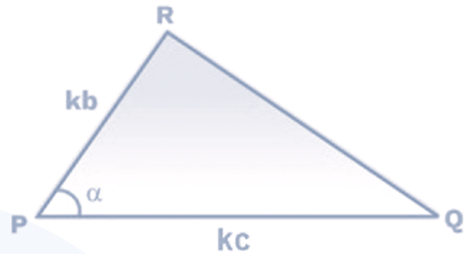
Here, $\angle B = \angle Q$, $\angle C = \angle R$

Therefore, $A = \angle P$ (angle sum property)

Hence, $\Delta ABC \sim \Delta PQR$

2. S–A–S similarity test (Side–Angle–Side):





If the ratio of the lengths of the two sides of one triangle is equal to the ratio of the lengths of the corresponding two sides of another triangle, and the included angles are equal, then the two triangles are similar.

In the figure, $\frac{AC}{AB} = \frac{b}{c}$

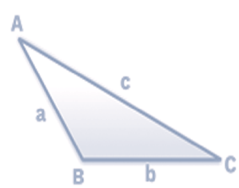
And $\frac{PR}{PQ} = \frac{Kb}{Kc} = \frac{b}{c}$

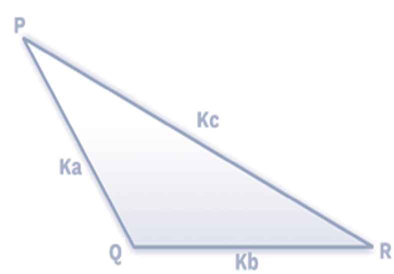
Which means $\frac{AC}{AB} = \frac{PR}{PQ}$

Also, $\angle CAB = \angle RPQ = \alpha$

Therefore, $ACB \sim \Delta PRQ$

3. S–S–S similarity test (Side–Side–Side):





If the ratio of the corresponding side of two triangles is constant or the same, then they are similar.

Here, $\frac{AB}{PQ} = \frac{a}{Ka} = \frac{1}{K}$

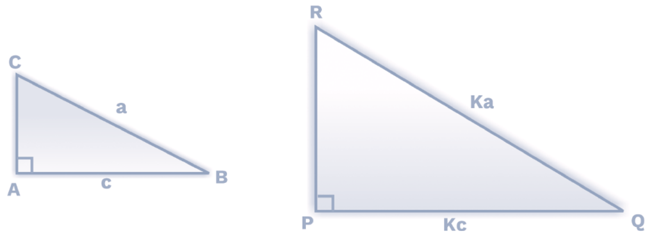
$\frac{BC}{QR} = \frac{b}{Kb} = \frac{1}{K}$

$$\frac{AC}{PR} = \frac{C}{Kc} = \frac{1}{K}$$

Since $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{1}{K}$

Therefore, ABC ~ ΔPQR

4. R–H–S similarity test (Right-Angle Hypotenuse Side):



If the ratio of the hypotenuse and one side of a right-angled triangle is equal to the ratio of the corresponding hypotenuse and one side of another right-angled triangle, then the two triangles are similar.

Here, ∠CAB = ∠RPQ = 90°

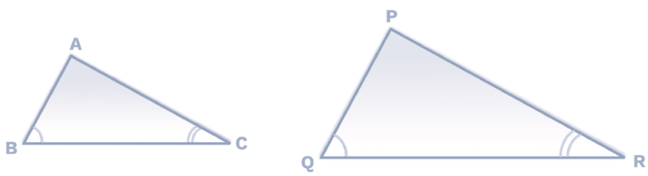
$$\frac{BC}{BA} = \frac{a}{c}$$

$$\frac{QR}{QP} = \frac{Ka}{Kc} = \frac{a}{c}$$

Hence, $\frac{BC}{BA} = \frac{QR}{QP}$

Therefore, ΔCAB ~ ΔRPQ.

Important Concepts Based on the Similarity of Triangles



If two triangles are similar, then:

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{h_1}{h_2} = \frac{r_1}{r_2} = \frac{R_1}{R_2}$$

$$= \frac{\text{Perimeter of}(\Delta ABC)}{\text{Perimeter of}(\Delta PQR)}$$

$$\frac{\text{Area}(ABC)}{\text{Area}(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{AC}{PR}\right)^2 = \left(\frac{h_1}{h_2}\right)^2$$

$$= \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{R_1}{R_2}\right)^2 = \left(\frac{P_1}{P_2}\right)^2$$

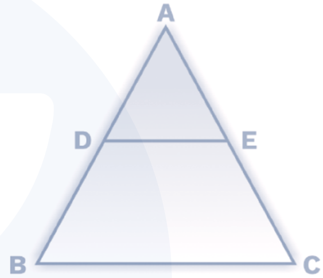
Here, h, r, R, and P are the altitude, inradius, circumradius, and perimeter of the triangle, respectively.

Cases Where Similarity Can Be Applied

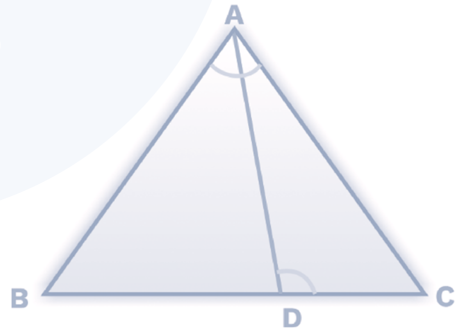
1. Parallel lines are drawn:

   DE || BC

   Then, ΔADE ~ ΔABC (by AA test of similarity)



2. If one angle of a triangle is specifically mentioned to be equal to one angle of another triangle.



   Given: ∠ADC = ∠BAC

   Then, ΔADC ~ ∠BAC

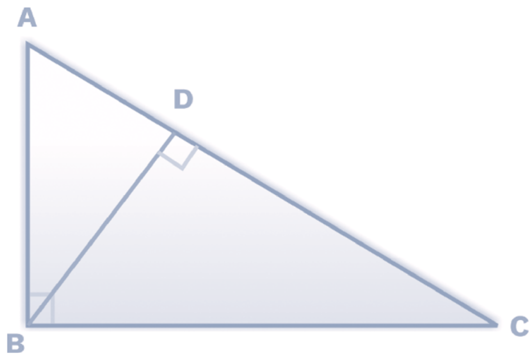
   (As C is a common angle)

3. Perpendicular dropped in a right-angled triangle.

   Given: right-angled triangle, the right angle at B.

   Then, ΔABC ~ ΔADB And ΔCDB ~ ΔCBA

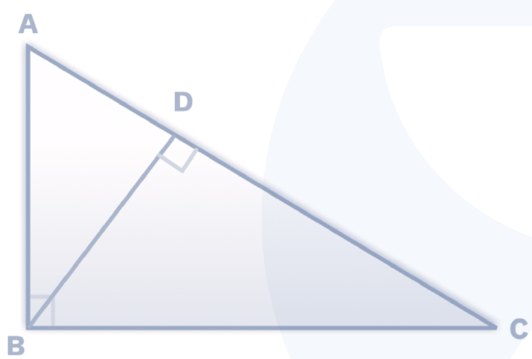
Hence, $\Delta ADB \sim \Delta BDC$.



Some Standard Results

In a right-angled triangle, if a perpendicular is drawn from point B to D, then

- $BD^2 = AD \times CD$
- $AB^2 = AD \times AC$
- $BC^2 = DC \times AC$



Proof

In ΔABC and ΔADB

$\angle A = \angle A$ (Common)

$\angle ABC = \angle ADB$ (Each 90°)

Therefore, $\angle ACB = \angle ABD$

(Angle sum property)

Hence, $\Delta ABC \sim \Delta ADB$

Therefore, $\frac{AB}{AD} = \frac{AC}{AB} \Rightarrow AB^2 = AD \times AC$

Now, In ΔBDC and ΔABC

$\angle C = \angle C$ (Common)

$\angle BDC = \angle ABC$ (Each 90°)

Therefore, $\angle CBD = \angle CBA$

(Angle sum property)

Hence, $\Delta BDC \sim \Delta ABC$

Therefore, $\frac{DC}{BC} = \frac{BC}{AC} \Rightarrow BC^2 = DC \times AC$

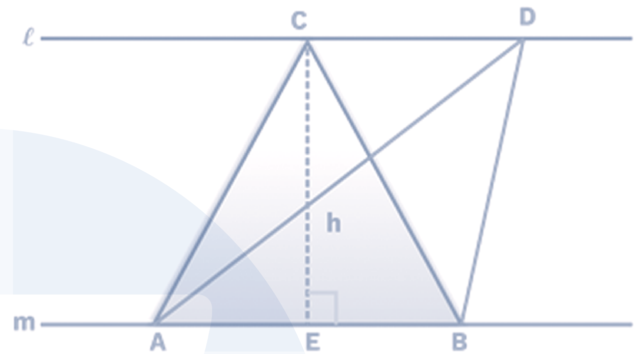
Since two Δ's (ΔADB and CDB) are similar to the same ΔABC, they are similar to themselves.

$\Delta ADB \sim \Delta BDC$

Therefore, $\frac{AD}{BD} = \frac{DB}{DC} \Rightarrow BD^2 = AD \times CD$

Important Theorems

1. Triangles that are on the same base and between the same parallel lines are equal in area.



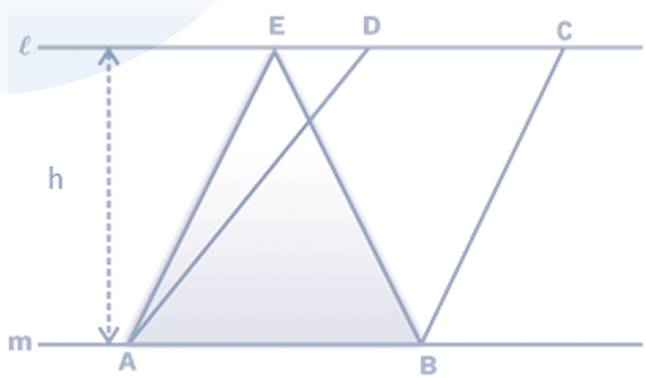
Proof

In the figure, $\ell \parallel m$ and two Δ's ABC and ABD are drawn on the same base AB.

Area of $\Delta ABC = \frac{1}{2} \times AB \times CE$

Area of $\Delta ABD = \frac{1}{2} \times AB \times CE$ (height is common for both triangles).

Therefore, $ar(\Delta ABC) = ar(\Delta ABD)$.



2. If a triangle and a parallelogram are drawn on the same base and between the same parallels, then the area of the triangle will be half of the area of the parallelogram.

Proof

Area of $\Delta ABE = \frac{1}{2} \times AB \times h$ ...(i)

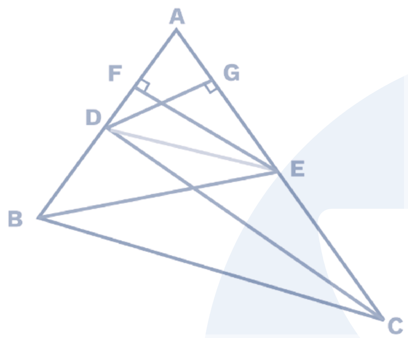
Area of parallelogram ABCD = AB × h ...(ii)

From equations (i) and (ii) We can conclude:

$$ar(\Delta ABE) = \frac{1}{2} ar(ABCD)$$

3. Basic proportionality theorem

If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other two sides are divided in the same ratio.



Proof

Here we have drawn DE parallel to BC

We need to prove: $\frac{AD}{DB} = \frac{AE}{EC}$

Draw EF ⊥ AD, DG ⊥ AE. Also join BE and CD.

Now, $ar(\Delta ADE) = \frac{1}{2} \times AD \times FE$ ...(i)

$ar(\Delta DEB) = \frac{1}{2} \times BD \times FE$ ...(ii)

Dividing equations (i) and (ii)

$$\frac{ar(\Delta ADE)}{ar(\Delta DEB)} = \frac{\frac{1}{2} \times AD \times FE}{\frac{1}{2} \times DB \times FE} = \frac{AD}{DB} \quad ...(iii)$$

Similarly, $\frac{ar(ADE)}{ar(DEC)} = \frac{AE}{EC}$ ...(iv)

Also, area (ΔDEB) and ar (ΔDEC) are on the same base and between the same parallels.

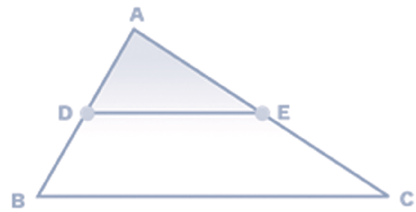
Therefore, ar(ΔDEB) = ar(ΔDEC) ...(v)

From equations (iii), (iv), and (v), we can conclude

$$\frac{AD}{DB} = \frac{AE}{EC}$$

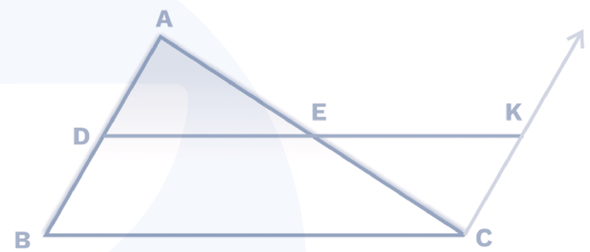
Midpoint Theorem

It states that the line segment joining the midpoints of any two sides of a triangle is parallel to the third side and equal to half of the third side.



If D and E are midpoints of AD and AC, respectively, in the above figure, then, $DE = \frac{1}{2} BC$ and DE || BC

Proof



Draw a line parallel to BD through C which cuts the line DE produced at point K.

In ΔAED and ΔCEK

ΔAED = ∠CEK (vertically opposite angles)

∠ADE = ∠CKE (alternate interior angles)

∠DAE = ∠ECK

AE = EC (E is midpoint)

Hence, ΔAED ≅ ΔCEK

Therefore, DE = EK

AD = KC

Also, AD = DB (as P is midpoint)

This implies BDKC is a parallelogram.

Therefore, DK = BC and DK || BC

$$DE = \frac{1}{2} BC \text{ and } DE \parallel BC$$

Angle Bisector Theorem

Angle bisector divides opposite sides in the ratio of adjacent sides, i.e., $\frac{AB}{AC} = \frac{BD}{DC}$

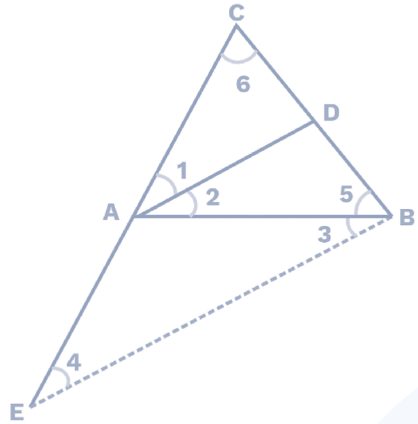
Proof

Extend CA and draw BE parallel to AD.

Now, $\angle 1 = \angle 4$ (corresponding angles)

$\angle 2 = \angle 3$ (alternate interior angles)

Also, we know that $\angle 1 = \angle 2$



Therefore, $\angle 4 = \angle 3$, which means AE = AB

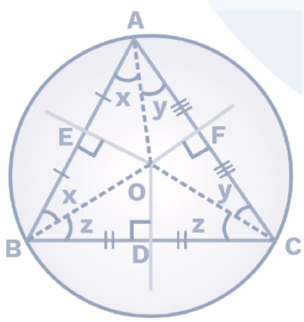
Using BPT, $\dfrac{AC}{AE} = \dfrac{CD}{DB}$

Or $\boxed{\dfrac{AC}{AB} = \dfrac{CD}{DB}}$

Centres of Triangle

Circumcentre

Circumcentre is the intersecting point of the perpendicular bisectors of the sides of the triangle.



OD, OE, and OF are perpendicular bisectors of sides.

The common point (intersecting point) is equidistant from A, B, and C, i.e.,
OA = OB = OC = circumradius of the circle.

Therefore, a circle can be drawn that passes through A, B, and C. The circle thus formed is a circumcircle and its centre, O, is known as the circumcentre.

Also, $\angle BOC = 180° - 2\angle Z$ ...(i)

In ΔABC, $2x + 2y + 2z = 180°$

$\angle BAC = x + y = 90 - \angle z$

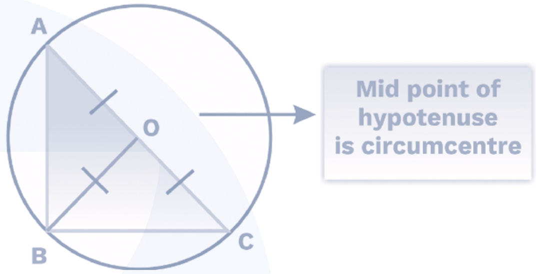
$2\angle BAC = 180° - 2\angle z$ ...(ii)

From equations (i) and (ii)

$\angle BOC = 2\angle BAC$

Positioning of the circumcentre

- If a triangle is an acute-angled triangle, the circumcentre will lie inside the triangle.
- If a triangle is a right-angled triangle, then the circumcentre will lie on the midpoint of the hypotenuse.

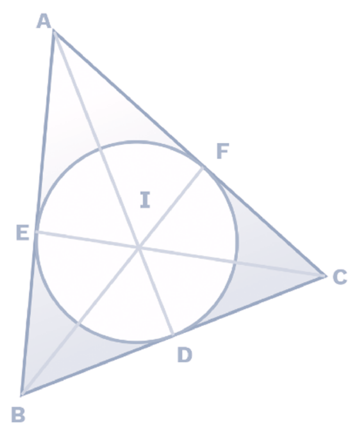


Circumradius is half the length of hypotenuse AC. BO will also be the median to the hypotenuse.

- If the triangle is an obtuse-angled triangle, then the circumcentre will lie outside the triangle.

Incentre

Incentre is the point of intersection of angle bisectors in a triangle.



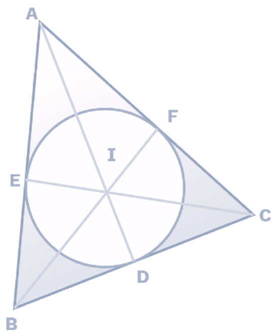
The incentre is equidistant from the three sides of the triangle.

Therefore, perpendiculars drawn from the incentre I are called the inradius of the triangle.

Hence, it is possible to draw a circle that is tangential to all three sides of the circle.

The circle so formed is called an incircle as it lies *in* the triangle. Point I is called incentre.

Important Results



If in the given figure, I is the incentre of triangle ABC, then

$$\angle BIC = 90° + \frac{\angle A}{2}$$

$$\angle AIC = 90° + \frac{\angle B}{2}$$

$$\angle AIB = 90° + \frac{\angle C}{2}$$

Proof

$$\angle BIC = 180° - (\angle IBC + \angle ICB)$$

$$= 180° - \left(\frac{1}{2}\angle ABC + \frac{1}{2}\angle ACB\right)$$

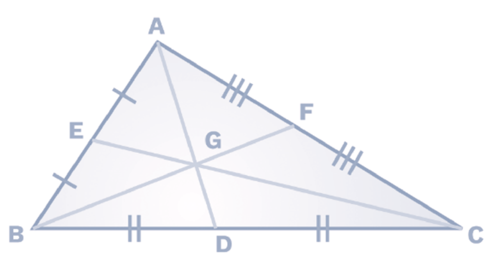
$$= 180° - \frac{1}{2}\left(\angle ABC + \angle ACB\right)$$

$$= 180° - \frac{1}{2}\left(180° - \angle A\right)$$

$$\angle BIC = 90° + \frac{\angle A}{2}$$

Centroid

A centroid is the intersection of medians in a triangle. It divides the medians in the ratio of 2:1, the part of the median towards the vertex being twice in length as the part towards the side.



Here, $\frac{AG}{GD} = \frac{BG}{GF} = \frac{CG}{GE} = \frac{2}{1}$

Since BD = DC

Therefore, area (ΔABD) = area (ΔADC)

(Ratio of areas of triangles, whose bases lie on the same line, is equal to the ratio of bases as height in that case will be equal for all triangles that can be formed with different bases on that line.)

Also, Area(ΔBGD) = $\frac{1}{2}$Area(ΔBGC)

$$= \frac{1}{2} \times \frac{2}{3} \times \text{area of}(\Delta BEC)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} \times \text{ area of } (\Delta ABC) \text{ [as BE = AB]}$$

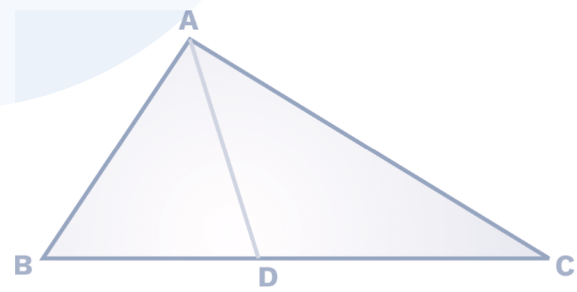
$$\text{Area of } (\Delta BGD) = \frac{1}{6} \times \text{ Area of } (\Delta ABC)$$

Also, centroid divides the triangle into six triangles of equal area.

So, ar(BGD) = ar(GDC) = ar(AGE) = ar(AGF) = ar(AGE) = ar(EGB) = ar(FGC)

Apollonius Theorem

Apollonius theorem relates medians and the sides of the triangle. It states that 'The sum of squares of two sides is equal to twice the sum of squares of median between them and half of third sides.'



According to Apollonius theorem:

$$AB^2 + AC^2 = 2 \times (AD^2 + BD^2)$$

Proof

Using the cosine rule

$AB^2 = BD^2 + AD^2 - 2BD \times AD \times \cos \angle ADB$ ...(i)

$AC^2 = CD^2 + AD^2 - 2CD \times AD \times \cos \angle ADC$

$= BD^2 + AD^2 + 2BD \times AD \times \cos \angle ADB$ ...(ii)

$(\angle ADB + \angle ADC = 180)$

Adding equations (i) and (ii)

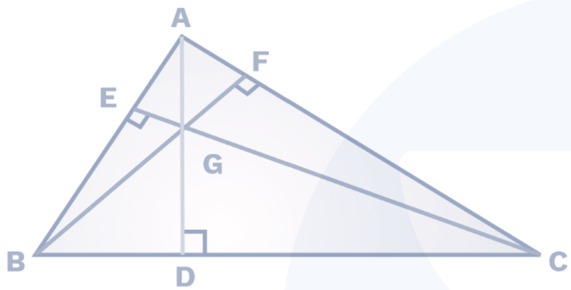
$AB^2 + AC^2 = 2 \times (BD^2 + AD^2)$

Orthocentre

Orthocentre is the intersection of altitudes.

Note:

- In an acute-angled triangle, the orthocentre lies inside the triangle.
- In a right-angled triangle, the orthocentre lies at the vertex of the triangle that forms a right angle.
- In an obtuse-angled triangle, the orthocentre lies outside the triangle.



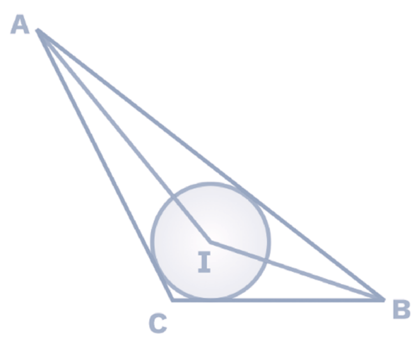
The angle formed by a side of the triangle at the orthocentre and the vertex angle are supplementary.

$\angle BGC + \angle A = 180°$

Let's Do Some Practice

Example 12

The centre of the incircle of triangle ABC is at a distance of 7 units and 3√3 units from the points A and B, respectively. Find the length of side AB, if the angle at the point C is 120°.



Solution: $\sqrt{139}$ **units**

In the figure, AI = 7 units, BI = $3\sqrt{3}$

Also, we know that $\angle AIB = 90° + \frac{1}{2}\angle ACB$

$\angle AIB = 90° + \frac{1}{2} \times 120 = 150°$

Using the cosine rule

$AB^2 = (AI)^2 + (IB)^2 - 2(AI) \times 2(AI) \times (IB) \times \cos 150°$

$= (7)^2 + (3\sqrt{3})^2 - 2 \times 7 \times 3\sqrt{3} \times \left(-\frac{\sqrt{3}}{2}\right)$

$= 49 + 27 + 63$

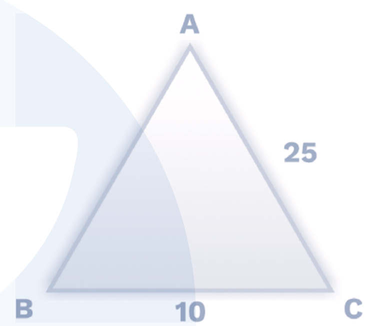
$AB^2 = 139$

$AB = \sqrt{139}$ units

Example 13

Given that the area of a triangle ABC = 100 cm², AC = 25 cm, BC = 10 cm. Find the length of AB.

Solution: $5\sqrt{17}$ **cm**



Area of triangle $= \frac{1}{2} \times 25 \times 10 \times \sin C$

$100 = 125 \sin C$

$\sin C = \frac{4}{5}$

Therefore, $\cos C = \frac{3}{5}$

Now, using the cosine rule for LC:

$$\cos C = \frac{(AC)^2 + (BC)^2 - (AB)^2}{2 \times AC \times BC}$$

$$\frac{3}{5} = \frac{(25)^2 + (10)^2 - (AB)^2}{2 \times 25 \times 10}$$

$300 = 625 + 100 - AB^2$

or $AB^2 = 725 - 300$

$AB^2 = 425$

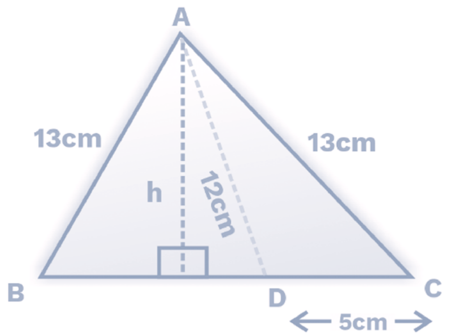
$AB = \sqrt{425}$

$AB = 5\sqrt{17}$ cm

Example 14

In a triangle ABC, AB = 13 cm, AC = 13 cm. Also D is a point on BC such that CD = 5 cm and AD = 12 cm. Find the area of triangle ABD.

Solution: 30 cm²



For ΔADC,

Semi-perimeter, $s = \dfrac{12+13+5}{2} = \dfrac{30}{2} = 15$ cm

Now, area of ΔADC $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{15 \times 2 \times 10 \times 3} = 30 \text{ cm}^2$

Also, area of $\Delta ADC = \dfrac{1}{2} \times DC \times h$

Therefore, $\dfrac{1}{2} \times 5 \times h = 30$

$h = 12$

This implies that AD is the height.

Since, ABC is an isosceles triangle and AD is height therefore,

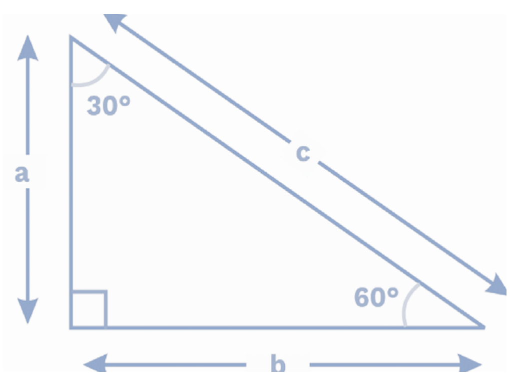
BD = DC = 5 cm.

Also, area of both the ΔABD and ΔADC will be equal.

Hence, area of $\Delta ABD = 30$ cm².

Important Facts Related to Triangles

1. 30° – 60° – 90° Triangle:



Using the sine rule: $\dfrac{\sin 90}{c} = \dfrac{\sin 30^\circ}{b} = \dfrac{\sin 60^\circ}{a}$

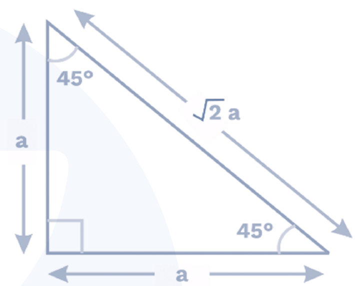
We can conclude: $\dfrac{1}{c} = \dfrac{1}{2b}$ and $\dfrac{1}{c} = \dfrac{\sqrt{3}}{2a}$

$b = \dfrac{1}{2}c$ and $a = \dfrac{\sqrt{3}}{2}c$

Therefore, the side opposite to 30° will be half of the hypotenuse. The side opposite to 60° will be $\dfrac{\sqrt{3}}{2}$ times of hypotenuse.

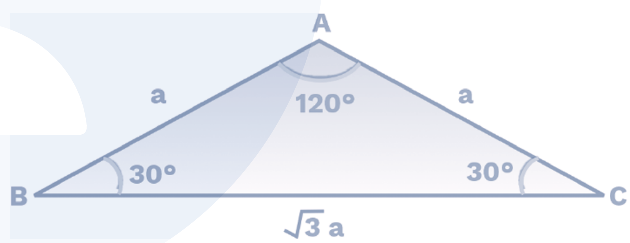
The required ratio of sides: $1 : \sqrt{3} : 2$

2. 45° – 45° – 90° Triangle:



The required ratio of sides: $1 : 1 : \sqrt{2}$

3. 30° – 30° – 120° Triangle:



Using the sine rule:

$\dfrac{\sin 30^\circ}{AC} = \dfrac{\sin 30^\circ}{AB} = \dfrac{\sin 120^\circ}{BC}$

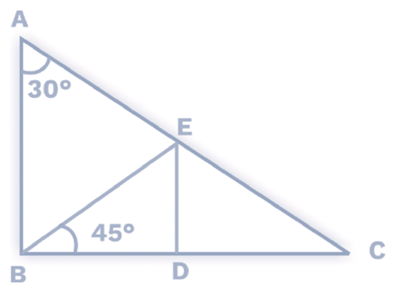
We can conclude: $\dfrac{1}{2}AC = \dfrac{1}{2}AB = \dfrac{\sqrt{3}}{2}BC$

If AC = AB = a, then $BC = \sqrt{3}\,a$

The required ratio of sides: $1 : 1 : \sqrt{3}$

Example 15

In a right-angled triangle ABC, the right angle at B, ED is parallel to AB. If the area of the triangle EDC is $8\sqrt{3}\,\text{cm}^2$, then find the area of triangles BED and ABC.



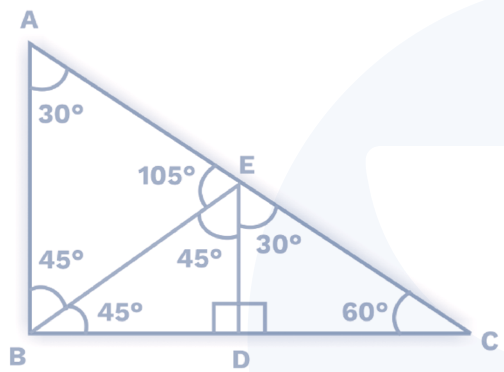
Solution:

Since $\angle A + \angle B + \angle C = 180°$

This gives us $\angle C = 60°$

Also $\angle BAE = \angle DEC = 30°$ (corresponding angles)

$\angle BED = 45°$ (DE||AB, $\angle BDE = 90°$)



Now, if $\angle EC = x$ cm

Then: $CD = \frac{x}{2}$ and $DE = \frac{\sqrt{3}}{2}x$ cm (30° – 60° – 90° triangle)

Since area of $\Delta EDC = 8\sqrt{3}\ cm^2$

$$\frac{1}{2}\times\frac{x}{2}\times\frac{\sqrt{3}}{2}x = 8\sqrt{3}$$

$$x = 8 \text{ cm}$$

Also, $BD = ED = \frac{\sqrt{3}}{2}\times(8) = 4\sqrt{3}$

Now, area of ΔBED $\frac{1}{2}\times(4\sqrt{3})\times(4\sqrt{3})$

Area of $\Delta BED = 24\ cm^2$

Also, $BC = BD + CD$

$$BC = 4\sqrt{3}+4 = 4(\sqrt{3}+1) \text{ cm}$$

Therefore, $AC = 8(\sqrt{3}+1)$ cm (30° – 60° – 90° angle)

And $AB = \frac{\sqrt{3}}{2}\times(8(\sqrt{3}+1))$

$$= (12+4\sqrt{3}) \text{ cm}$$

Now,

Area of $\Delta ABC = \frac{1}{2}\times BC\times AB$

$$= \frac{1}{2}\times 4(\sqrt{3}+1)(12+4\sqrt{3})$$

$$= 16(2\sqrt{3}+3)\ cm^2$$

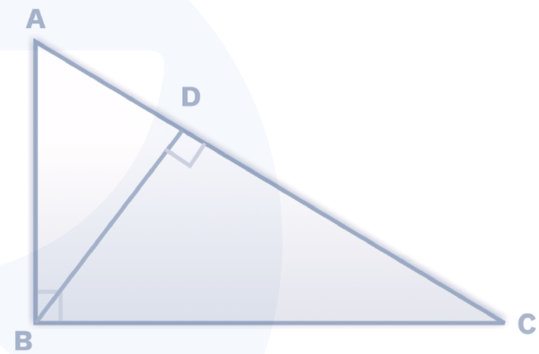
Example 16

In a triangle ABC, the right angle at *B*, also $BD \perp AC$. If AD = 2 cm and CD = 4 cm, then find the area of triangle ABC, ABD, and BDC.

Solution $\mathbf{6\sqrt{2}\ cm^2; 2\sqrt{2}\ cm^2; 4\sqrt{2}\ cm^2.}$

We know that for such a scenario:

$$BD^2 = AD \times CD$$



Therefore, $BD^2 = 2 \times 4$

$$BD = \sqrt{8} \text{ cm}$$

Now, area of $\Delta ADB = \frac{1}{2}\times AD\times BD$

$$= \frac{1}{2}\times 2\times\sqrt{8} = \sqrt{8}\ cm^2$$

$$= 2\sqrt{2}\ cm^2$$

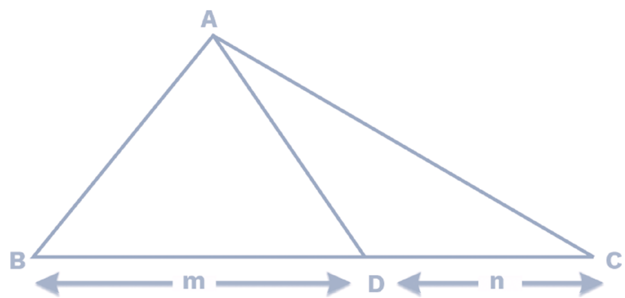
Area of $\Delta BDC = \frac{1}{2}\times CD\times BD = \frac{1}{2}\times 4\times\sqrt{8}$

$$= 4\sqrt{2}\ cm^2$$

Area of $\Delta ABC = \left(2\sqrt{2} + 4\sqrt{2}\right) cm^2 = 6\sqrt{2}\ cm^2.$

Things to Know

If the heights of triangles are the same, then areas will be proportional to these bases.



$$\frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ADC} = \frac{m}{n}$$

Example 17

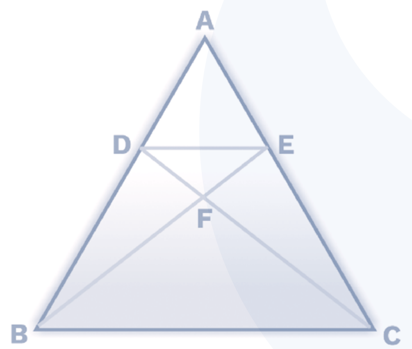
In the given figure, DE || BC and DE = 25% of BC. If the area of triangle ADE = 20 square units, then find the area of:

(A) ΔABC

(B) ΔDEB

(C) ΔDEF

(D) ΔBFC



Solution: (A) 320; (B) 60; (C) 12; (D) 192

(A) Since DE || BC

Therefore, $\Delta ADE \sim \Delta ABC$

Therefore, $\dfrac{ar(ADE)}{ar(ABC)} = \left(\dfrac{AE}{BC}\right)^2$

$$\frac{ar(ADE)}{ar(ABC)} = \frac{1}{16}$$

or $\quad ar(ABC) = 16 \times 20 = 320$ square units.

(B) Since $\dfrac{AE}{EC} = \dfrac{1}{3}$

Therefore, area of (ΔAEB)

$$= \frac{1}{4} \times \text{area}(\Delta ABC)$$

$$= \frac{1}{4} \times 320 = 80 \text{ square units.}$$

Now, ar (ΔDEB) = ar (ΔAEB) − ar (ΔADE)

$= 80 - 20 = 60$ square units.

(C) Since DE || BC

Therefore, $\Delta DEF \sim \Delta CBF \Rightarrow \dfrac{DE}{CB} = \dfrac{EF}{BF} = \dfrac{1}{4}$

Hence, area $(\Delta DEF) = \dfrac{1}{5} \times \text{area}(\Delta DEB)$

$= \dfrac{1}{5} \times 60 = 12$ square units.

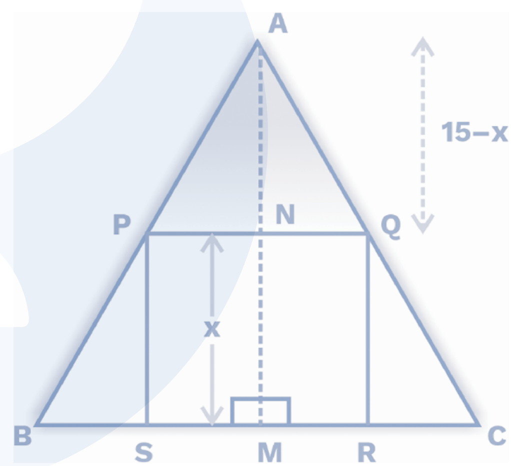
(D) $\dfrac{\text{Area }(\Delta DEF)}{\text{Area }(\Delta BFC)} = \left(\dfrac{DE}{BC}\right)^2$

$$\frac{12}{ar(BFC)} = \frac{1}{16}$$

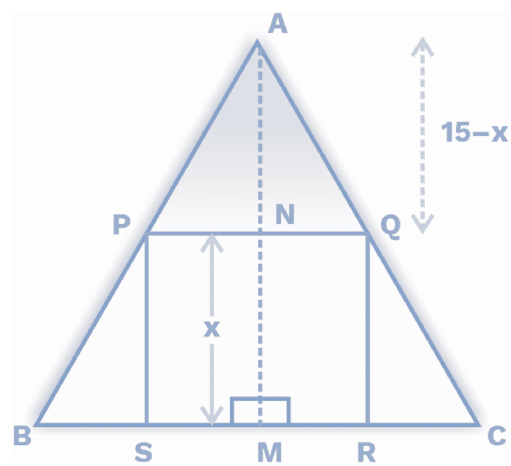
ar $(\Delta BFC) = 16 \times 12 = 192$ square units.

Example 18

In the figure, PQRS is a square and BC = 10 units. If the area of triangle ABC = 75 square units, then what is the area of square PQRS?



Solution: 36 square units



Draw AM $\perp$ BC

Now, area of ΔABC = 75 square units

$$\frac{1}{2} \times BC \times AM = 75$$

$$\frac{1}{2} \times 10 \times AM = 75$$

AM = 15 units.

Let the side of square = x unit.

Since PQ || BC

Therefore, APQ ~ ABC

Hence, $\frac{AN}{AM} = \frac{PQ}{BC}$

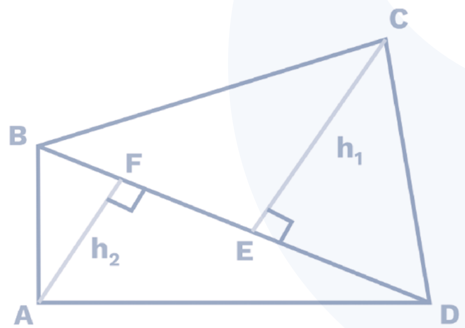
$$\frac{15 - x}{15} = \frac{x}{10}$$

Which implies that 150 – 10x = 15x or

25x = 150 x = 6 units.

Therefore, the area of square = (side)^2 = (6)^2 = 36 square units.

Quadrilaterals and Their Properties



A four-sided polygon is known as quadrilateral.

In the figure, BD = d, EC = h_1 and AF = h_2

Area (ΔBAD) = $\frac{1}{2} \times d \times h_2$

Area (ΔBDC) = $\frac{1}{2} \times d \times h_1$

- Area of quadrilateral ABCD = $\frac{1}{2} \times d \times (h_1 + h_2)$
- Area of quadrilateral ABCD = $\frac{1}{2}$ × Product of diagonal × Sine of angle between them.
- Area of the cyclic quadrilateral

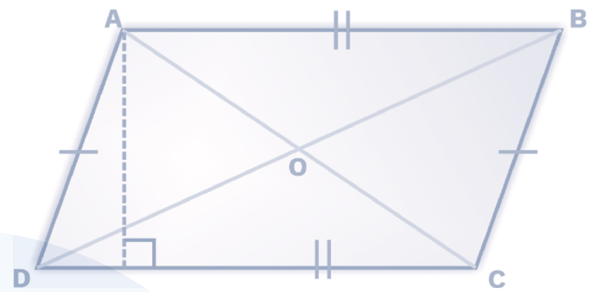
$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Where, a, b, c, and d are the sides of quadrilateral and

s = semi-perimeter = $\frac{a+b+c+d}{2}$

(Cyclic quadrilateral is a quadrilateral whose vertices lie on the circumference of the circle.)

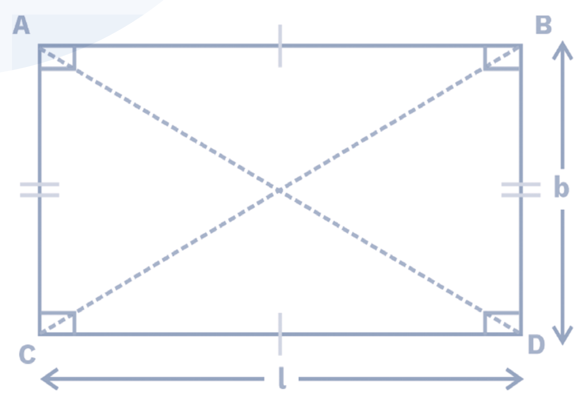
Parallelograms



- Area = base × height
- Opposite sides are parallel and equal.
- Opposite angles are equal, $\angle$A = $\angle$C and $\angle$B = $\angle$D.
- Diagonals of a parallelogram bisect each other.
- OB = OD and OA = OC
- The sum of the squares of the sides is equal to the sum of the squares of the diagonals.
- AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2

Rectangle

A quadrilateral whose opposite sides are equal, and each internal angle equals 90°.

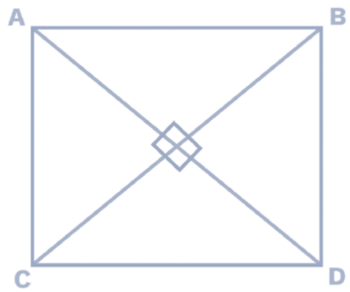


- Diagonals are equal and bisect each other.
- Area = length × breadth
- Perimeter = 2(length + breadth)
- Diagonal = $\sqrt{\ell^2 + b^2}$

Square

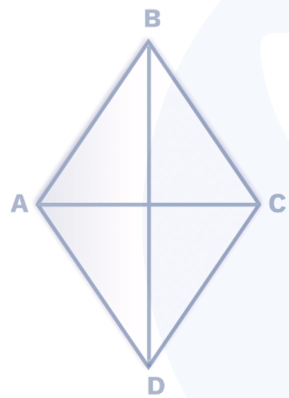
A quadrilateral whose all sides are equal, and each internal angle is 90°.

- Diagonals of a square are equal and bisect each other at 90°.
- Area = $(\text{Side})^2$
- Perimeter = $4 \times$ Side



Rhombus

A parallelogram whose all sides are equal.

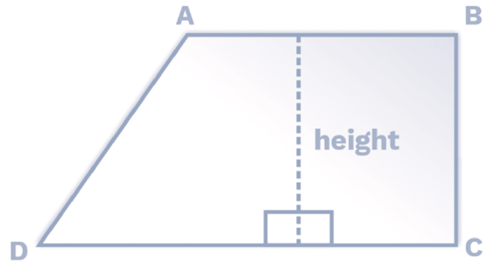


- Diagonals of a rhombus bisect each other at 90°.
- Area = $\frac{1}{2} \times$ Product of diagonals.

Trapezium

A quadrilateral that has one pair of parallel opposite sides.

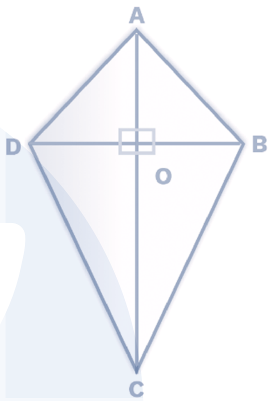
- Diagonals intersect each other proportionally in the ratio of lengths of parallel sides.
- Area = $\frac{1}{2}$ (sum of parallel sides) $\times$ height.
- If a trapezium is inscribed inside a circle, then it is an isosceles trapezium.



Kite

A quadrilateral whose two pairs of adjacent sides are equal and longer diagonal bisects the shorter diagonal at 90°.

In the figure:



- AD = AB and CD = CB
- $\angle AOD = \angle AOB = \angle BOC = \angle DOC = 90°$
- DO = OB
- Area of Kite = $\frac{1}{2} \times$ Product of diagonals.

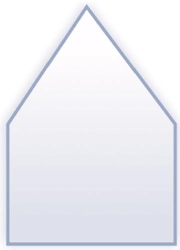
Polygons

Any closed plane figure with *n* sides is known as a polygon. If all the sides and angles of this polygon are equal, then the polygon is called a regular polygon.

Note: The word *polygon* is derived from the Greek *poly*, which means *many*, and *gonia*, which means *angle*.

Convex polygon

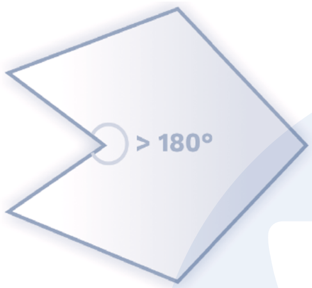
A convex polygon is a polygon in which every internal angle is < 180°. Hence, all the diagonals of the polygon remain inside its boundary.



(CONVEX POLYGON)

Concave polygon

If the polygon is not convex, it is called a concave polygon. At least one internal angle in a concave polygon is larger than 180°.



(CONCAVE POLYGON)

Polygons are named on the basis of the number of sides:

Number of Sides	Name of the Polygon
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon

Properties of a Polygon

1. Sum of all interior angles of any polygon $= (n - 2) \times 180°$.
2. Each interior angle of a Regular Polygon $= \dfrac{(n-2)\times 180^\circ}{n}$.
3. Sum of all exterior angles of any polygon $= 360°$.
4. Each exterior angle of a regular polygon $= \dfrac{360^\circ}{n}$.
5. The number of diagonals in a n-sided polygon $= \dfrac{n\times(n-3)}{2}$.

Circle

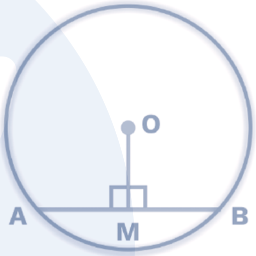
A circle is the locus of all points that are equidistant from a fixed point. The fixed point is known as the centre of the circle.

If r is the radius of the circle, then circumference $= 2\pi r$ and area $= \pi r^2$.

Chord

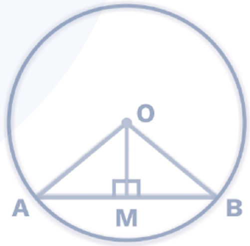
Line segment joining two points on the circumference is known as a chord. The longest chord is the diameter.

The perpendicular drawn from centre to the chord bisects the chord.



If OM $\perp$ AB, then AM = MB

Proof



Since OA = OB

Therefore, $\angle$OAM = $\angle$OBM (angle opposite to equal sides)

Now, in DOMA and in DOMB

$\angle$OMA = $\angle$OMB (Each 90°)

$\angle$OAM = $\angle$OBM

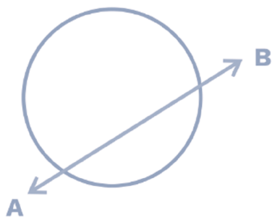
AO = BO

Therefore, DOMA $\cong$ DOMB

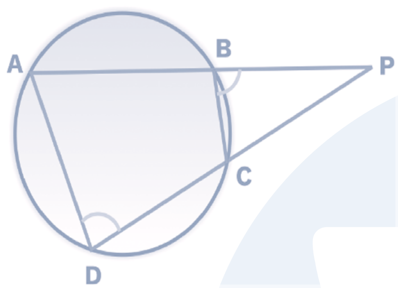
Hence, AM = MB

Secant

Secant is the line that intersects the circle at two distinct points. In the given figure, AB is a secant.



Important Result



$AP \times BP = DP \times CP$

Proof

Since ABCD is a cyclic quadrilateral:

Therefore, $\angle ADC + \angle ABC = 180°$

Also $\angle ABC + \angle PBC = 180°$ (Linear pair)

Hence, $\angle ADC = \angle PBC$

Now, In ΔADP and ΔCBP

$\angle ADP = \angle CBP$

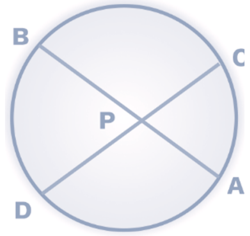
$\angle APD = \angle BPC$ (Common)

Therefore, $\Delta ADP \sim \Delta CBP$

Hence, $\frac{AP}{CP} = \frac{DP}{BP}$

or, $AP \times BP = DP \times CP$

Similarly, for the internal intersection of segments:

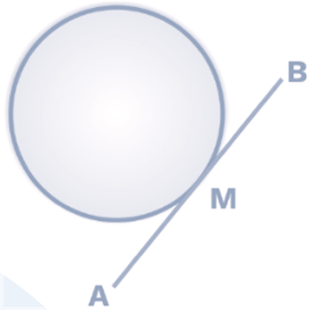


$PA \times PB = PC \times PD$

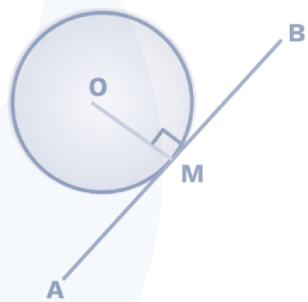
Tangent

A tangent is a line that touches the circle at only one point. The point where the tangent touches the circle is called the point of contact or point of tangency.

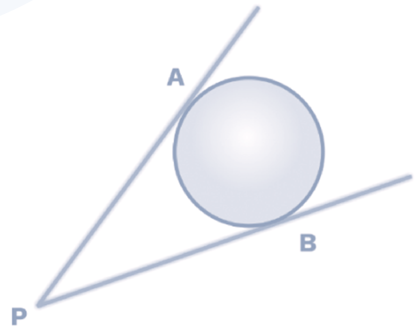
In the given figure, AB is the tangent to the circle and *M* is the point of contact or point of tangency.



- The line joining the centre and the point of tangency is perpendicular to the tangent.

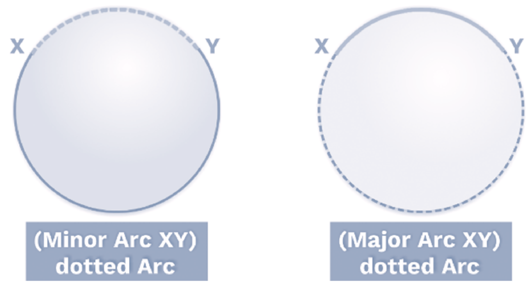


- The length of the two tangents that are drawn from an external point to the circle is equal.
  $PA = PB$

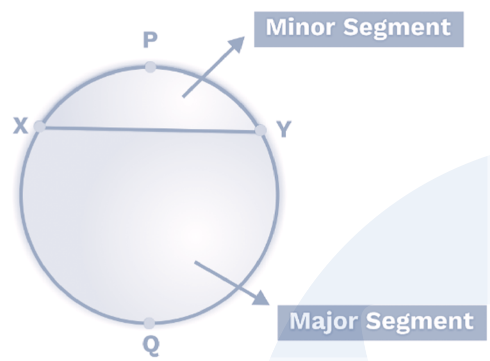


Arc

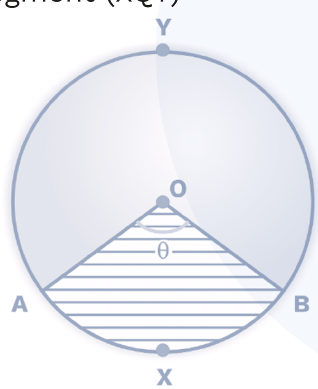
Any two points on the circle that divide the circle into two parts; the smaller part is called the minor arc and the larger part is called the major arc.



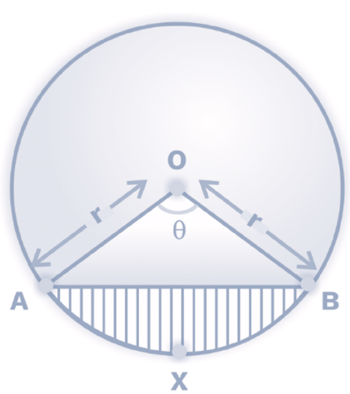
Note: If we join XY in the above figure, now chord XY is dividing the circle into two regions and they are called segments of a circle.



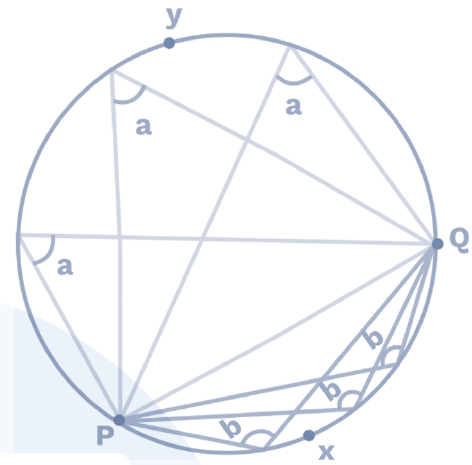
- Major segment (XPY)
- Minor segment (XQY)



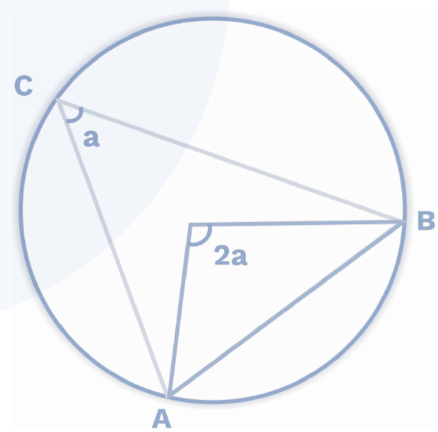
- Area of minor sector OAXB $= \dfrac{\theta}{360} \times \pi r^2$
- Area of major sector OAYB $= \left(\dfrac{360-\theta}{360}\right) \times \pi r^2$



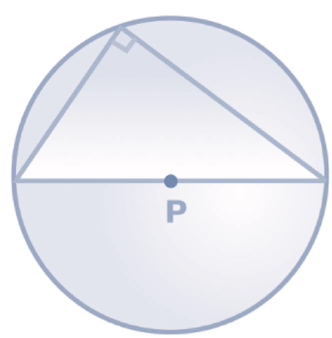
- Area of shaded region ABX = Area of sector OAXB – Area of DOAB
  $= \dfrac{\theta}{360} \times \pi r^2 - \dfrac{1}{2} \times r \times r \times \sin\theta$
- Angles subtended at the circumference of the circle in the same segment by the same arc or chord are equal.



- Angle subtended by the chord (or arc) at the centre is two times the angle subtended by the same chord (or arc) at the circumference of the circle in the same segment.

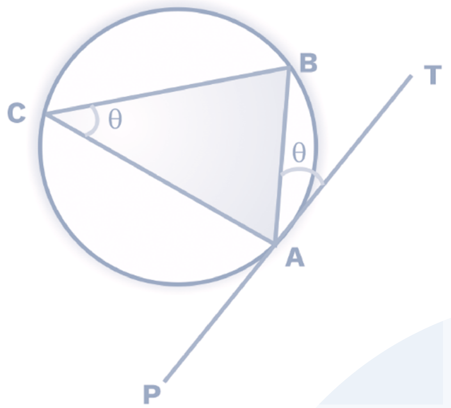


- Angle subtended by the diameter of the circle is a right angle.

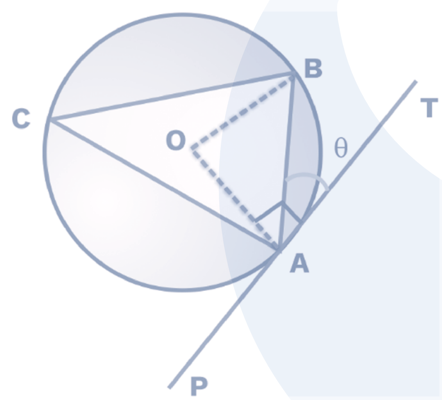


Tangent secant theorem (alternate segment theorem)

It states that. 'The angle between the tangent and a secant drawn at the point of the tangent is equal to the angle formed in the alternate segment'.



Proof



Let O be the centre of the circle.

Join OA and OB.

Since OA = OB (Radii of circle)

Therefore, $\angle OAB = \angle OBA$

Now, in DOAB

$\angle OAB + \angle OBA + \angle AOB = 180°$

$\angle AOB = 180° - 2\angle OAB$ ...(i)

Also, $\angle AOB = 2\angle ACB$ ...(ii)

(Angle subtended at centre)

From equations (i) and (ii):

$2\angle ACB = 180° - 2\angle OAB$

$2\angle OAB = 180° - 2\angle ACB$

$\angle OAB = 90° - \angle ACB$ ...(iii)

Also, OA ⊥ PT (As PT is tangent)

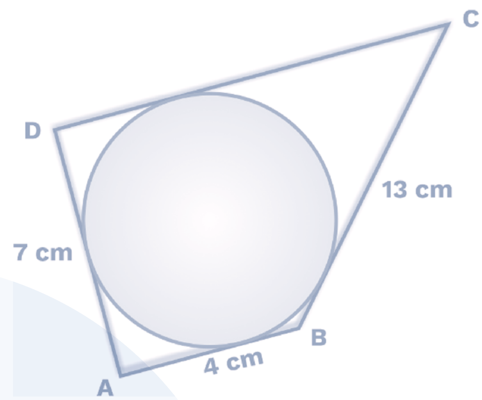
$\therefore$ $\angle OAB = 90° - \angle\theta$ ...(iv)

From equations (iii) and (iv):

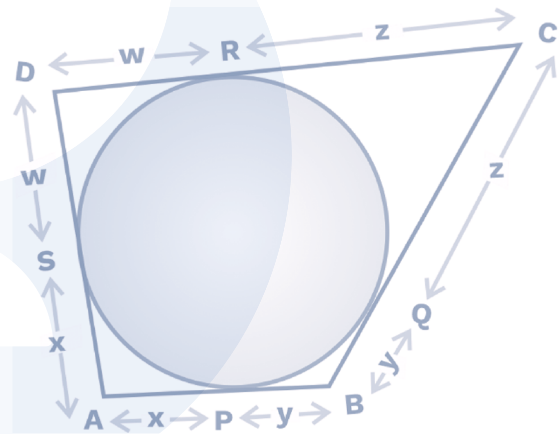
$\angle\theta = \angle ACB$ hence proved.

Example 19

In the given figure, find the length of CD.



Solution: 16 cm



Since we know that tangents drawn from an external point to the circle are equal.

Therefore, AP = AS = x; BP = BQ = y; CQ = CR = z; DR = DS = w

Now, CD + AB = $w + z + x + y$ ...(i)

Also, BC + AD = $y + z + w + x$ ...(ii)

From equations (i) and (ii):

CD + AB = BC + AD

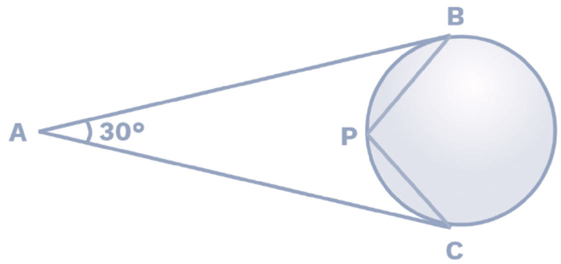
CD + 4 = 20

CD = 20 – 4 = 16 cm

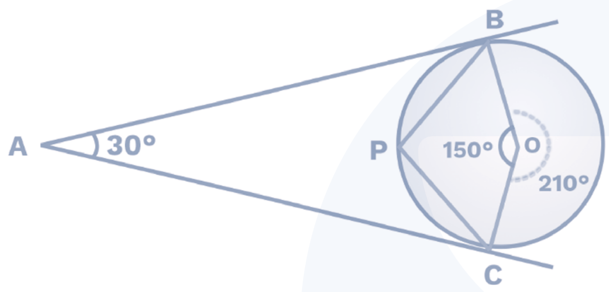
Note: If a circle is circumscribed in a quadrilateral such that the circle touches the four sides of the quadrilateral, then the sum of opposite sides of a quadrilateral is always equal.

Example 20

In the adjoining figure, AB and AC are tangents to the circle. If $\angle CAB = 30°$, find the measure of $\angle BPC$.



Solution: 105°



Let O be the centre of the circle.

Now, OB ⊥ AB and OC ⊥ AC.

Since ABOC is a quadrilateral.

Therefore, $\angle BAC + \angle BOC = 180°$

$\angle BOC = 180° - 30° = 150°$

Hence, the angle subtended by arc BC at the centre = 360° – 150° = 210°

Now, the angle subtended by the same arc BC at the circumference, i.e., $\angle BPC$ will be half of the angle subtended at the centre.

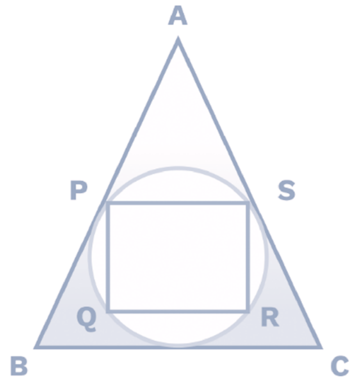
Therefore, $\angle BPC = \frac{1}{2} \times 210°$

$\angle BPC = 105°$

Example 21

A circle is inscribed in an equilateral triangle and a square is inscribed in the circle. What is the ratio of the area of the triangle to the area of the square?

Solution: $3\sqrt{3} : 2$



Let the side of the ΔABC is a.

Then, the radius of the circle (in radius)

$$r = \frac{a}{2\sqrt{3}} \text{ and area} = \frac{\sqrt{3}}{4}a^2$$

Then, diameter of the circle (PR) $= 2 \times \frac{a}{2\sqrt{3}} = \frac{a}{\sqrt{3}}$

Since PR is the diagonal of the square PQRS.

Then, side of the square $= \frac{a}{\sqrt{6}}$

So, area of the square $\frac{a^6}{6}$

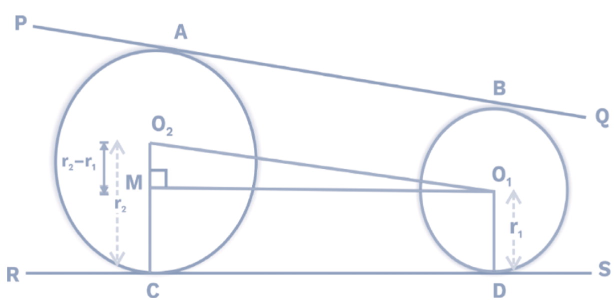
Then, ratio of $\frac{\text{Area of } \Delta ABC}{\text{Area of PQRS}} = \frac{\left(\frac{\sqrt{3}}{4}a^2\right)}{\left(\frac{a^2}{6}\right)} = \frac{3\sqrt{3}}{2}$

Therefore, ar(ΔABC): ar(PQRS) = $3\sqrt{3} : 2$

Direct Common Tangents and Transverse Common Tangents

Direct Common Tangent

AB and CD are direct common tangents in the figure.



A common tangent is said to be a direct common tangent if the two circles lie on the same side of the tangent.

In the figure: O_1C is the radius of a smaller circle (r_1)

O_2D is the radius of a larger circle (r_2)

O_1O_2 is the distance between the centres.

In $\Delta O_2 M O_1$: $(O_1O_2)^2 = (O_2M)^2 + (MO_1)^2$

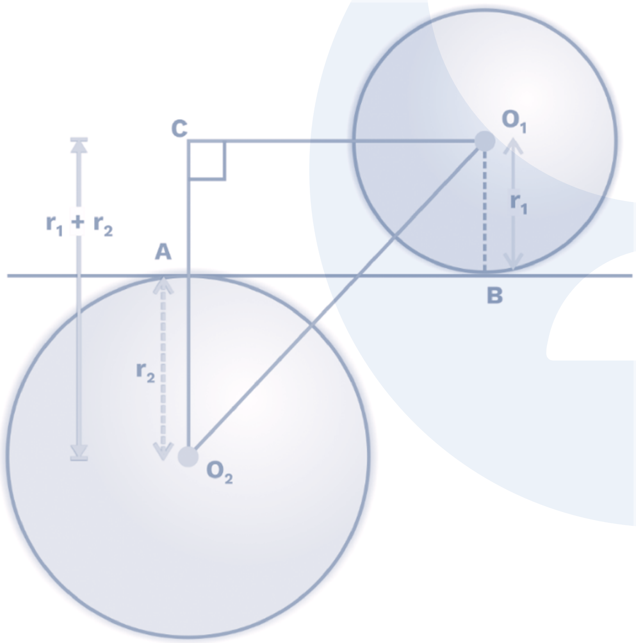
$(O_1O_2)^2 = (r_2 - r_1)^2 + (CD)^2$ [$\because MO_1 = CD$]

or, $CD = \sqrt{(O_1O_2)^2 - (r_2 - r_1)^2}$

Transverse Common Tangents

AB is a transverse common tangent in the figure.

A common tangent is said to be a transverse common tangent if two circles lie on the opposite sides of the tangent.



In the figure, in right $DO_1\ CO_2$

$(O_1O_2)^2 = (O_1C)^2 + (O_2C)^2$

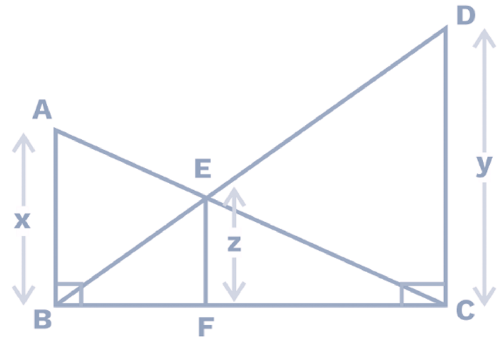
$(O_1O_2)^2 = (AB)^2 + (r_1 + r_2)^2$ [$O_1C = AB$]

$AB^2 = (O_1O_2)^2 - (r_1 + r_2)^2$

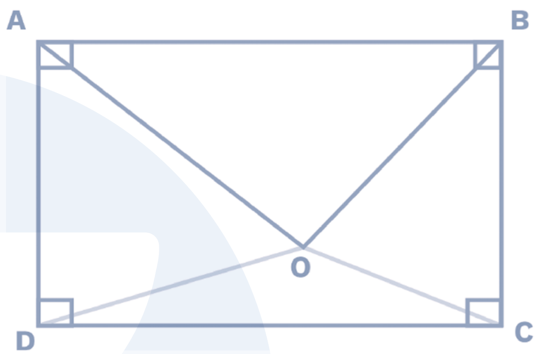
$AB = \sqrt{(O_1O_2)^2 - (r_1 + r_2)^2}$

Important results

1. $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$



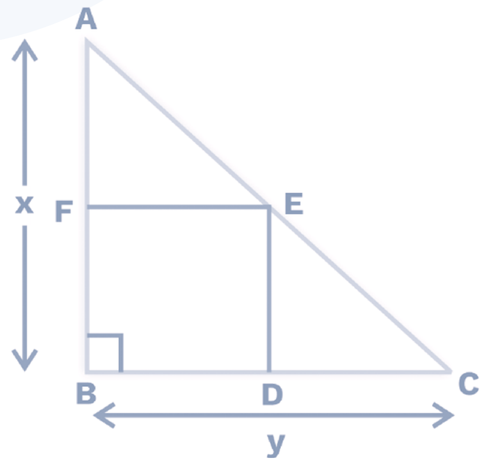
2.



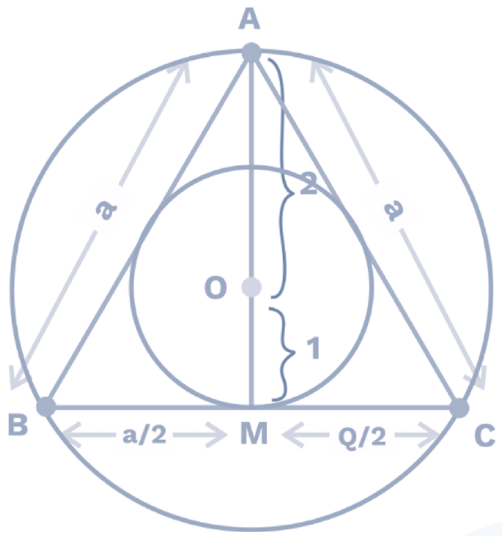
$(AO)^2 + (OC)^2 = (OB)^2 + (OD)^2$

3. If a square of maximum area (BDEF) needs to be inserted in a right-angled triangle with sides x unit and y unit, then the side of the square.

Side of square $= \frac{xy}{x+y}$



Case of Equilateral Triangle



ABC is an equilateral with a length of its side a units. This equilateral triangle is circumscribed by a circle, and also there is a circle inscribed in the equilateral triangle.

Note: In the case of an equilateral, centroid, circumcentre, incentre, and orthocenter all lie on the same point.

Since AM is median, AO: OM = 2: 1

Now, Height of equilateral triangle $= \frac{\sqrt{3}}{2} \times a$

AO (Circumradius, R) $= \frac{2}{3} \times$ height

$$= \frac{2}{3} \times \frac{\sqrt{3}}{2} \times a$$

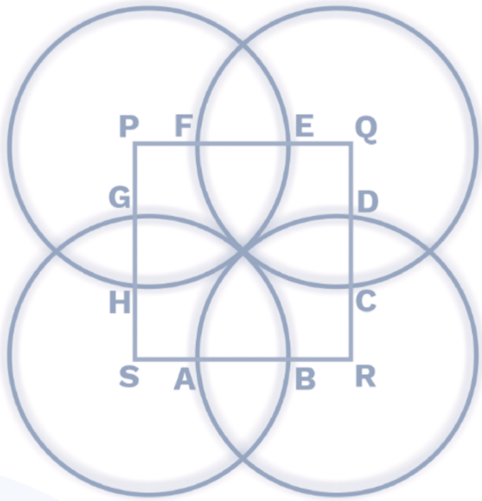
Circumradius, $R = \frac{a}{\sqrt{3}}$

OM (Inradius, r) $= \frac{1}{3} \times$ Height $= \frac{1}{3} \times \frac{\sqrt{3}}{2} \times a$

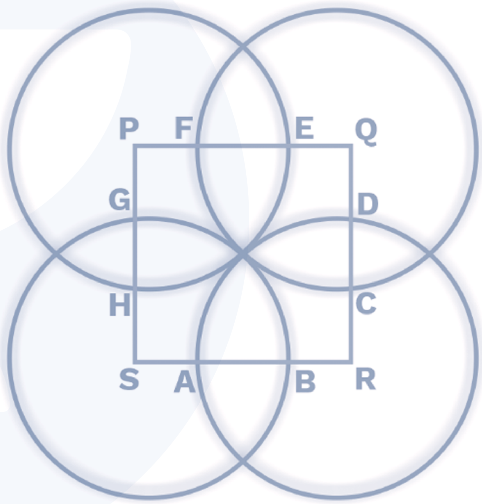
In radius, $r = \frac{a}{2\sqrt{3}}$

Example 22

P, *Q*, *R,* and *S* are the centres of four intersecting circles each having a radius of 12 cm. If AB = 8 cm, CD = 4 cm, EF = 7 cm, and GH = 9 cm, what is the perimeter of quadrilateral PQRS?



Solution: 68 cm



PQ = (PE + FQ) − FE = 12 + 12 − 7 = 24 − 7 = 17 cm

QR = (QC + DR) − DC = 12 + 12 − 4 = 24 − 4 = 20 cm

RS = (SB + AR) − AB = 12 + 12 − 8 = 24 − 8 = 16 cm

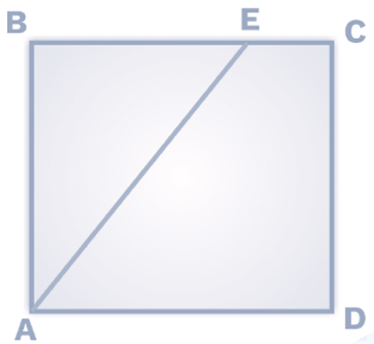
PS = (PH + GS) − GH = 12 + 12 − 9 = 24 − 9 = 15 cm

So, the perimeter = (17 + 20 + 16 + 15) cm = 68 cm

Practice Exercise – 1

Level of Difficulty – 1

1. Square ABCD has a side length of 10 cm, point E is on BC, and the area of ΔABE is 40 cm^2. What is BE?



(A) 4 cm
(B) 5 cm
(C) 8 cm
(D) 7 cm

2. The length of sides of a triangle is in the ratio $\frac{1}{3}:\frac{1}{4}:\frac{1}{5}$. If the perimeter is 188 cm, then what is the length of the smallest side?

(A) 48 cm
(B) 60 cm
(C) 80 cm
(D) 20 cm

3. The ratio of area of a square to that of the square drawn on its diagonal is:

(A) 1 : 4
(B) 2 : 1
(C) 1 : 2
(D) 1 : 3

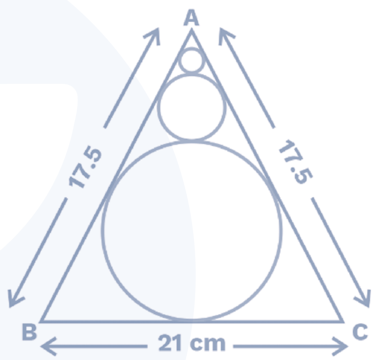
4. The supplementary angle of a given angle is five times the given angle. What is the measurement of the given angle?

(A) 30°
(B) 65°
(C) 40°
(D) 45°

5. The following figure is a parallelogram with two angles given in terms of x. Find the value of x.

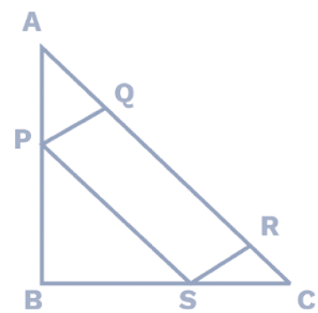
Level of Difficulty – 2

6. In an isosceles ΔABC, AB = AC = 17.5, BC = 21. Infinite circles are made inside this triangle as shown in the figure. Find the sum of the perimeter of all the circles.



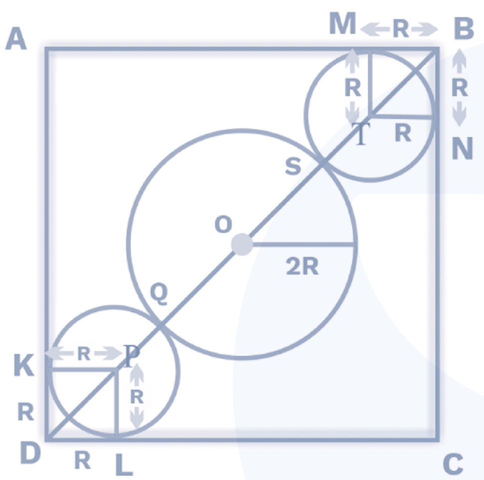
(A) 38.5
(B) 44
(C) 35
(D) 42

7. In the given diagram, ΔABC is an isosceles right-angled triangle, in which a rectangle is inscribed in such a way that the length of the rectangle is twice that of breadth. Q and R lie on the hypotenuse and points P and S lie on the two different smaller sides of the triangle. What is the ratio of the area of the rectangle to that of the area of the triangle?



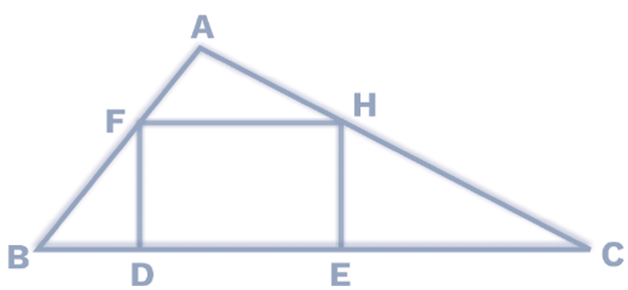
(A) $1:\sqrt{3}$
(B) $1:\sqrt{2}$
(C) $1:2$
(D) $1:3$

8. Inside a square ABCD, three circles are drawn touching one another as shown in the figure. The radius of two smaller circles is R units, and the radius of the bigger circle is 2R units. Diagonal BD of the square passes through the centre of all the circles. What is the ratio of the radius of the smaller circle to the side of the square?



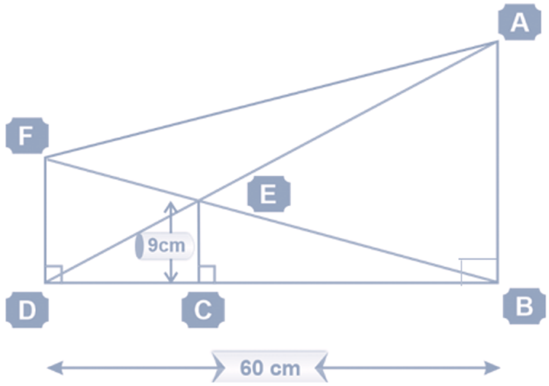
(A) 2 : 7
(B) 3 : 8
(C) 1 : 4
(D) None of these

9. Consider the following figure: AB = 10 cm, AC = 17 cm, BC = 21, cm and EHFD is a square. Find the length of the side of the square (in cm).



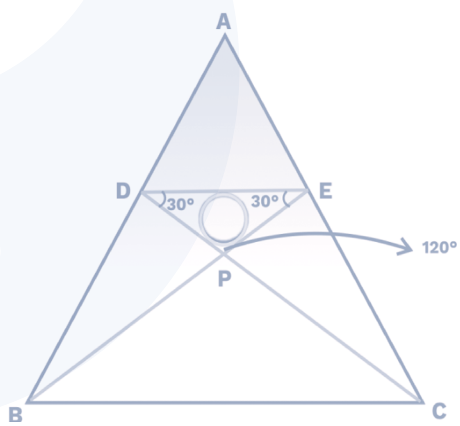
(A) 10.5 cm
(B) 12 cm
(C) 13.5 cm
(D) None of these

10. In the given figure, BC : CD = 2 : 1. Find the area of triangle AEF (in cm^2).



Level of Difficulty – 3

11. Triangle ABC is an equilateral triangle, points D and E are the midpoints of sides AB and CD that intersect each other at P. If the area of the ΔABC is $900\sqrt{3}$ m^2 then find the radius of the circle (in metres) inscribed in ΔDPE, line DE, EP, and ED are tangents to the circle.



(A) $\dfrac{75\sqrt{3}}{30+20\sqrt{3}}$ m

(B) $\dfrac{75\sqrt{3}}{80+20\sqrt{3}}$

(C) $\dfrac{225\sqrt{3}}{60-40\sqrt{3}}$

(D) $\dfrac{75\sqrt{3}}{30-20\sqrt{3}}$ m

12. The area of a rectangle is equal to the area of a triangle ABC with sides AB = 8 cm, BC = 6 cm, and $\angle ABC = 60°$. If the width of the rectangle is $2\sqrt{3}$ cm, find the ratio of the perimeter of the triangle to that of rectangle.

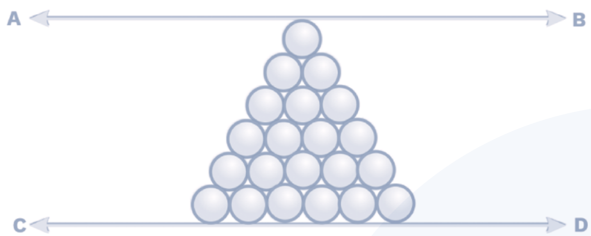
(A) $7+\sqrt{13}:6+\sqrt{12}$

(B) $7-\sqrt{13}:6+\sqrt{13}$

(C) $6+\sqrt{12}:7+\sqrt{12}$

(D) $6+\sqrt{12}:7+\sqrt{13}$

13. In the given figure, 21 identical circles are packed in between two parallel lines AB and CD whose distance between them is 10 cm. (in cm).



(A) $\frac{10}{74}(5\sqrt{3}-2)$ (B) $\frac{10}{5\sqrt{3}-2}$

(C) $\frac{10}{71}(5\sqrt{3}-2)$ (D) $\frac{10}{71}(5\sqrt{3}+2)$

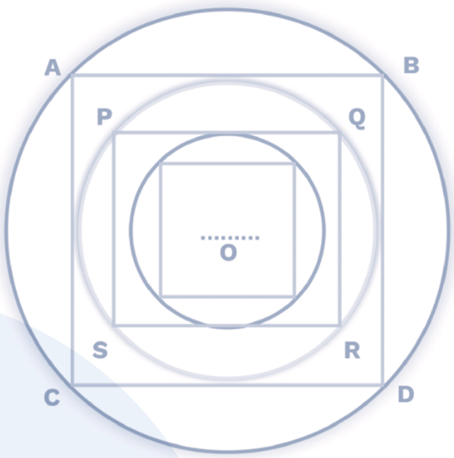
14. A square ABCD is inscribed in a circle. Now, a circle is inscribed in square ABCD, then again a square PQRS is inscribed in the second circle. Similarly, there are infinite squares and circles drawn in a similar way. Find the ratio of the sum of the area of all squares to the sum of axes of all circles.

(A) $3\sqrt{2}:\pi$

(B) $2:\pi$

(C) $\pi:3\sqrt{2}$

(D) $\pi:2$



15. PQRS is a quadrilateral whose diagonals are perpendicular to each other, intersecting at O. If PQ = 16 cm, QR = 12 cm, and RS = 20 cm, in square cm.

(A) 256π

(B) 512π

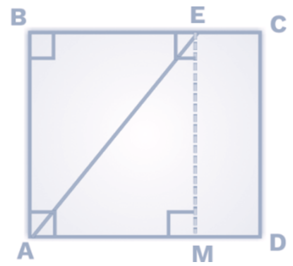
(C) 64π

(D) 128π

Solutions

1. (C)

Draw EM $\perp$ AD;



Now, area of rectangle ABEM = 2 × 40 = 80 cm^2

AB × AM = 80 cm^2

10 × x = 80 cm^2

x = 8 cm

Therefore, BE = AM = 8 cm

Hence, option (C) is correct choice.

2. (A)

Ratio of sides $=\frac{1}{3}:\frac{1}{4}:\frac{1}{5}$

$$=\frac{1\times 20}{3\times 20}:\frac{1\times 15}{4\times 15}:\frac{1\times 12}{5\times 12}=20:15:12$$

Let sides be $20x$, $15x$, $12x$

Therefore, $20x+15x+12x=188$

$47x=188$

$x=4$

Therefore, length of the smallest side $= 12x = 12 \times 4 = 48$ cm.

3. **(C)**

Let the side of the square be a unit.

Now, area of square, $A_1 = a^2$ square unit.

Also, we can find the diagonal of the square.

We know that: Diagonal of square $= a\sqrt{2}$ unit.

Therefore, the side of the square made on diagonal $= a\sqrt{2}$ unit.

Hence, the area of the new square $(A_2) = (a\sqrt{2} \times a\sqrt{2}) = 2a^2$ square units.

Hence, the required ratio $= \frac{A_1}{A_2} = \frac{a^2}{2a^2} = 1:2$.

Therefore, option (C) is the correct choice.

4. **(A)**

Let's assume the given angle $= K°$

Its supplementary angle $= (180 - K)°$

Now, A.T.Q., $\quad 180 - K = 5K°$

$6K = 180°$

$K = 30°$

Therefore, measurement of the given angle $= 30°$.

Hence, option (A) is the correct choice.

5. **147/8°**

Since ABCD is a parallelogram.

Therefore, CD || AB and BC is transversal

$\angle ABC + \angle BCD = 180°$.

(Interior angle on the same side of the quadrilateral.)

$(3x + 18°) + (5x + 15°) = 180°$

$8x + 33° = 180°$

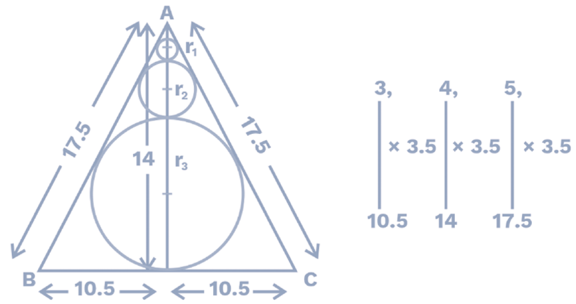
$8x = 180° - 33°$

$8x = 147°$

$x = \frac{147°}{8}$

6. **(B)**

By using the Pythagorean triplet of 3, 4, and 5, we can solve this problem easily.



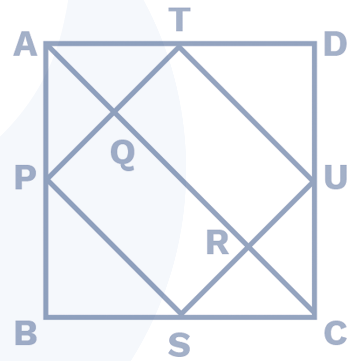
$2r_1 + 2r_2 + 2r_3 \ldots = 14$ cm

$\therefore$ Perimeter is $\pi(2r_1 + 2r_2 + 2r_3\ldots) = 14 \times \pi = 14 \times \frac{22}{7} = 44$ cm.

Hence, option (B) is the correct answer.

7. **(C)**

First, we will attach a congruent triangle ADC (congruent to given triangle ABC) to the existing triangle such that it becomes the square (as shown below)



Let the side of square ABCD be a.

Then the area of ABCD $= a^2$

ST = PU = diagonal of the square PTUS = side of the square ABCD $= a$

So, the side of square PTUS $= \left(\frac{a}{\sqrt{2}}\right)$

Area of PTUS $= \left(\frac{a}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\right) = \frac{a^2}{2}$

Ratio of area of (PTUS : ABCD) = 1 : 2

As the area of the rectangle, PQRS is half of square PTUS, and the area of ΔABC is half of that square ABCD. Therefore, the ratio of areas will remain the same.

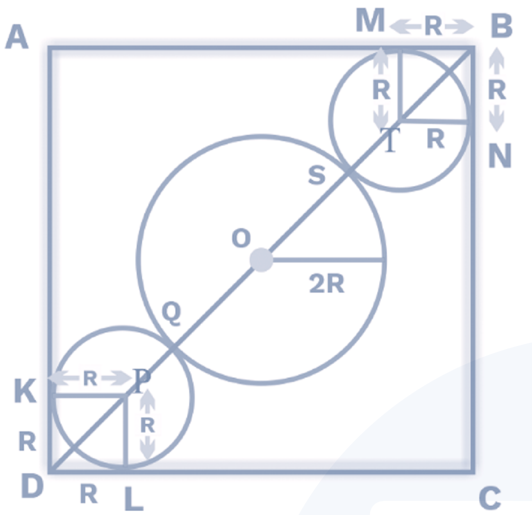
Area of rectangle PQRS: Area of triangle ABC = 1 : 2.

8. **(D)**

Let P, R, and T be the centres of the circle.

Now, KPDL and MBNT are squares.

Since DP is the diagonal of square KPLD.



Therefore, $DP = R\sqrt{2}$ unit

Similarly, $BT = R\sqrt{2}$ unit

Therefore, DB = DP + PQ + QS + ST + TB

$$= R\sqrt{2} + R + 4R + R + R\sqrt{2}$$

$$= (6R + 2R\sqrt{2}) \text{ unit}$$

$$= 2R(3+\sqrt{2}) \text{ unit}$$

Since BD is a diagonal for square ABCD.

So, if a the side of the square, then:

$$\sqrt{2}\,a = 2R(3+\sqrt{2})$$

$$a = \frac{2R(3+\sqrt{2})}{\sqrt{2}} \text{ unit}$$

Now the required ratio

$$= \frac{R}{\left(\frac{2R(3+\sqrt{2})}{\sqrt{2}}\right)} = \frac{\sqrt{2}\,R}{2R(3+\sqrt{2})} = \frac{1}{\sqrt{2}(3+\sqrt{2})}$$

$$= \frac{\sqrt{2}(3-\sqrt{2})}{2(3+\sqrt{2})(3-\sqrt{2})} = \frac{3\sqrt{2}-2}{14}$$

Therefore, option (D) is the correct choice.

9. **(D)**

We know the three sides of the triangle.

Therefore, semi-perimeter (S)

$$= \frac{21+10+17}{2} = 24$$

Therefore, the area of ΔABC

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{24(24-21)(24-17)(24-10)}$$

$$= \sqrt{24\times 3\times 7\times 14} = 84 \text{ cm}^2$$

Consider BC as base. Therefore, the area of $\Delta ABC = \frac{1}{2}\times BC \times h$

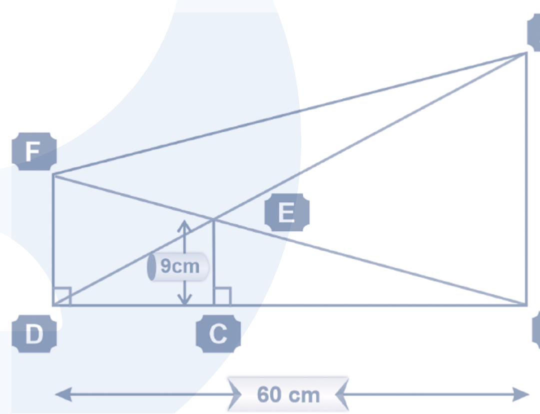
$$84 = h \times 21 \times \frac{1}{2}$$

$$x = 8 \text{ cm}$$

Since the side of the square must be smaller than the height.

Therefore, option (D) is the correct choice.

10. **270**



Since BC : CD = 2 : 1 and BD = 60 cm; therefore, BC = 40 cm and CD = 20 cm.

Here triangle BEC is similar to triangle BFD.

Therefore, $\frac{EC}{FD} = \frac{BC}{BD}$

$$\frac{9}{FD} = \frac{2}{3}$$

$$FD = 13.5 \text{ cm}$$

Also, triangle DEC is similar to triangle DAB.

Therefore, $\frac{EC}{AB} = \frac{CD}{BD}$

$$\frac{9}{AB} = \frac{1}{3}$$

$$AB = 27 \text{ cm}$$

Now, area of triangle AEF = (area of trapezium ABDF) – (area of trapezium FECD + area of trapezium AECB).

Area of DAEF $= \frac{1}{2}(13.5+27)\times 60 -$

$\left[\left\{\frac{1}{2}\times(13.5+9)\times 20\right\}+\left\{\frac{1}{2}\times(9+27)\times 40\right\}\right]$

$= (40.5 \times 30) - [(22.5 \times 10) + (36 \times 20)]$

$= 1215 - (225 + 720) = 1215 - 945$

$= 270\ cm^2$

Hence, 270 is the correct answer.

11. **(A)**

In the figure, D and E are midpoints.

Therefore, the area of $\Delta DPE = \frac{1}{3}\times Ar(\Delta BDE)$

(as BE is medians and BP : PE = 2 : 1)

Also, the area of $(\Delta DPE) = \frac{1}{3}\times\frac{1}{2}(Ar\ ABE)$

(ED is median for DABE)

$= \frac{1}{3}\times\frac{1}{2}\times\frac{1}{2}\ Ar(\Delta\ ABC)$

or $(\Delta DPE) = \frac{1}{3}\times\frac{1}{2}\times\frac{1}{2}\times 900\sqrt{3} = 75\sqrt{3}\ m^2$

Area of $(\Delta ABC) = \frac{\sqrt{3}}{4}(\text{side})^2$

$900\sqrt{3} = \frac{\sqrt{3}}{4}(\text{side})^2$

Side = 60 = AB = BC = AC

Therefore, AD = DE = AE = 30 m (as D in midpoints of AB).

We know that, in an equilateral triangle, median, altitude and perpendicular bisectors are the same.

Therefore, $\angle ADP = \angle AEP = 90°$.

Now, in quadrilateral ADPE

$\angle ADP + \angle AEP + \angle DAE + \angle DPE = 360°$

$90° + 90° + 60° + \angle DPE = 360°$

$\angle DPE = 120°$

Hence, $\angle PDE = \angle DEP = 30°$.

Now, in ΔDPE, using sine rule

$$\frac{\sin 120°}{DE} = \frac{\sin 30°}{DP}$$

$$\frac{\left(\frac{\sqrt{3}}{2}\right)}{30} = \frac{\left(\frac{1}{2}\right)}{DP}$$

or $DP = \frac{30}{\sqrt{3}}m$ or $DP = 10\sqrt{3}\,m$

Hence, $DP = PE = 10\sqrt{3}\ m$.

Now, the area of $\Delta DPE = r \times s$

$75\sqrt{3} = r\times(30+20\sqrt{3})$

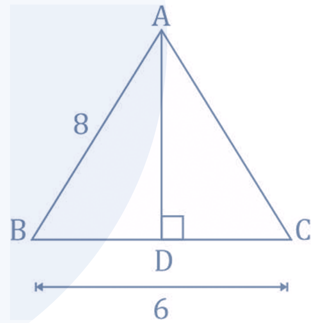
$r = \frac{75\sqrt{3}}{30+20\sqrt{3}}\ m$

Hence, radius, $r = \frac{75\sqrt{3}}{30+20\sqrt{3}}\ m.$

Therefore, option (A) is the correct choice.

12. **(A)**

Given: In ΔABC, AB = 8, BC = 6, and $\angle ABC = 60°$



and area of rectangle = Area of triangle

In ΔABD, $AD = AB \sin 60° = 8\times\frac{\sqrt{3}}{2} = 4\sqrt{3}$

$BD = AB \cos 60 = 8 \times \frac{1}{2} = 4$

$CD = BC - BD = 6 - 4 = 2$

$AD \perp BC$,

$AC^2 = AD^2 + DC^2$

$\Rightarrow AD^2 = AC^2 - DC^2$

$AC^2 = AD^2 + DC^2 = (4\sqrt{3})^2 + (2)^2$

$\Rightarrow AC^2 = 48 + 4 = 52 = (2\sqrt{13})^2$

$\Rightarrow AC = 2\sqrt{13}$

$\therefore$ Perimeter of

$\Delta ABC = AB + BC + AC = 8 + 6 + 2\sqrt{13}$ ... (i)

$= 2(7+\sqrt{13})$

Area of rectangle = Area of triangle

Length $\times$ Breadth

$= \frac{1}{2} \times 8 \times 6 \times \sin 60° = \frac{1}{2} \times 8 \times 6 \times \frac{\sqrt{3}}{2}$

$\Rightarrow$ Length $\times\ 2\sqrt{3} = 12\sqrt{3}$

$\Rightarrow$ Length $= \frac{12\sqrt{3}}{2\sqrt{3}} = 6$ units

Perimeter of rectangle = 2 (length + breadth)

$= 2(6+2\sqrt{3}) = 4(3+\sqrt{3})$ units

$\therefore$ The ratio of perimeter of triangle to perimeter of rectangle

$= \frac{2(7+\sqrt{13})}{4(3+\sqrt{3})} = \frac{7+\sqrt{13}}{2(3+\sqrt{3})}$

$= 7+\sqrt{13} : 6+2\sqrt{3} = 7+\sqrt{13} : 6+\sqrt{12}$

13. (C)

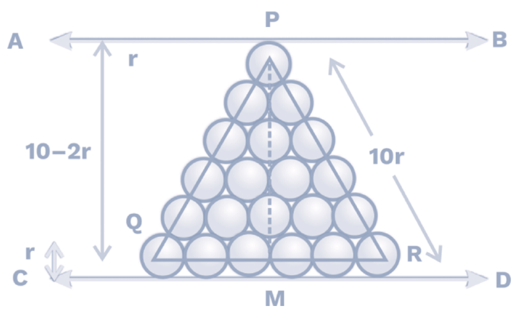
Let the radius of each circle = r units

As all circles are identical and on the side of the triangle PQR is passing from the centres of the circles lying outside:

Side of the equilateral triangle PQR = $10r$

Since we know that:

Height of equilateral triangle $= \frac{\sqrt{3}}{2} \times$ side



Therefore, $PM = \frac{\sqrt{3}}{2} \times 10r = 5\sqrt{3}\,r$

The length of PM can also be written as $10 - 2r$.

Hence, $10 - 2r = 5\sqrt{3}\,r$.

$r(5\sqrt{3}+2) = 10$

$r = \frac{10}{5\sqrt{3}+2}$

or, $r = \frac{10}{(5\sqrt{3}+2)} \times \frac{(5\sqrt{3}-2)}{(5\sqrt{3}-2)}$

$r = \frac{10(5\sqrt{3}-2)}{(5\sqrt{3})^2 - (2)^2}$

$r = \frac{10}{71}(5\sqrt{3}-2)$

Hence, option (C) is the correct choice.

14. (B)

Let r be the radius of the biggest circle ($r_1 = r$).

This means, diameter of largest circle = $2s$.

Now, the diameter of the largest circle = diagonal of the biggest square

$2r = a_1\sqrt{2}$

$a_1 = \frac{2r}{\sqrt{2}} = \sqrt{2}\,r$

As diagonals of a square bisect each other at 90°.

Therefore, area of right $\Delta AOB = \frac{1}{2} \times AO \times BO$

$\frac{1}{2} \times AB \times r_2 = \frac{1}{2} \times AO \times BO$

$\frac{1}{2} \times \sqrt{2}\,r \times r_2 = \frac{1}{2} \times r \times r$

$r_2 = \frac{r}{\sqrt{2}}$

Now, $r_2 = \frac{r}{\sqrt{2}}$ (a radius of the second largest circle).

$d_2 = \frac{2r}{\sqrt{2}} = \sqrt{2}\,r$ (diameter of the second largest circle).

Since the diameter of second largest circle = diagonal of the second largest square

(a_2 in the side of the second largest square).

$$\sqrt{2}\,r = a_2 \times \sqrt{2}$$

Therefore, $a_2 = r$

Note: Whenever a regular polygon is drawn inside a regular polygon, the length of the diameters always forms a geometric progression (GP).

Hence, we have:

For circle

$$r_1 = r;\ r_2 = \frac{r}{\sqrt{2}};\ r_3 = \frac{r}{2};\ r_4 = \frac{r}{2\sqrt{2}},\dots$$

$$A_1 = \pi r^2;\ A_2 = \frac{\pi r^2}{2},\ A_3 = \frac{\pi r^2}{4},\ A_4 = \frac{\pi r^2}{8},\dots$$

For square

$$a_1 = \sqrt{2}\,r,\ a_2 = r,\ a_3 = \frac{r}{\sqrt{2}},\dots$$

$$A_1' = 2r^2,\ A_2' = r^2,\ A_3' = = \frac{r^2}{2},\dots$$

Since we know that:

Sum of infinite G.P. $= \frac{a}{1-r}$ $(r < 1)$

($a \to$ first term, $r \to$ common ratio)

Now, if S_1 = Sum of the area of all squares

And S_2 = Sum of the area of all circles.

Therefore, $$\frac{S_1}{S_2} = \frac{\left(2r^2 + r^2 + \frac{r^2}{2} + \dots\right)}{\left(\pi r^2 + \frac{\pi r^2}{2} + \frac{\pi r^2}{4} + \dots\right)}$$

$$\frac{S_1}{S_2} = \frac{r^2\left(2 + 1 + \frac{1}{2} + \dots\right)}{\pi r^2\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right)}$$

$$\frac{S_1}{S_2} = \frac{r^2\left(\frac{2}{1-\frac{1}{2}}\right)}{\pi r^2\left(\frac{1}{1-\frac{1}{2}}\right)}$$

$$\frac{S_1}{S_2} = \frac{2}{\pi}$$

Hence, $S_1 : S_2 = 2 : \pi$.

Therefore, option (B) is the correct choice.

15. (D)

From the figure:

$(PQ)^2 = (PO)^2 + (OQ)^2$

(From right ΔPOQ) ...(i)

$(RS)^2 = (OR)^2 + (OS)^2$

(From right ΔROS) ...(ii)

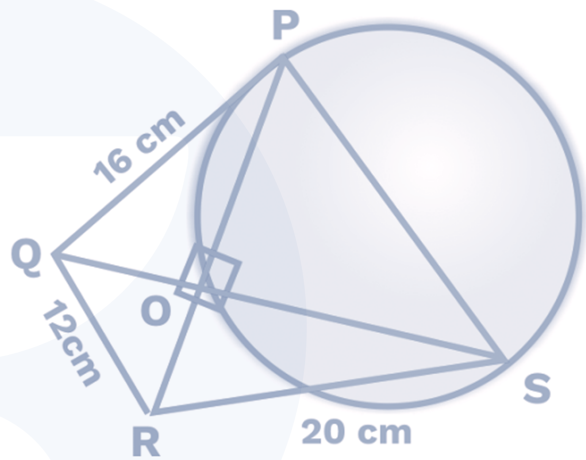
$(PS)^2 = (OP)^2 + (OS)^2$

(From right ΔPOS) ...(iii)

$(QR)^2 = (OQ)^2 + (OR)^2$

(From right ΔQOR) ...(iv)

From equations (i) to (iv)



We can conclude:

$(PQ)^2 + (RS)^2 = (QR)^2 + (PS)^2$

$(16)^2 + (20)^2 = (12)^2 + (PS)^2$

$656 = 144 + (PS)^2$

$PS = \sqrt{512}$

$PS = 16\sqrt{2}$ cm

Now, side PS will be the diameter of the circle circumscribing DPOS as $\angle POS = 90°$.

Now, radius of the circle, $r = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$.

Therefore, area of the circle $= \pi r^2 = \pi(8\sqrt{2})^2 = 128\,\pi$.

Hence, option (D) is the correct choice.

Practice Exercise – 2

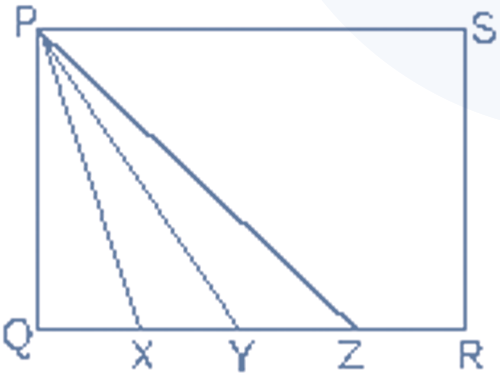
Level of Difficulty – 1

1. The value of an exterior angle of a regular polygon is 1/6 times its interior angle. Find the number of diagonals in this polygon.

2. What is the area of a triangle whose sides are 12 cm, 18 cm, and 26 cm, respectively?
   (A) $8\sqrt{35}$ cm^2
   (B) $16\sqrt{35}$ cm^2
   (C) $19\sqrt{35}$ cm^2
   (D) $10\sqrt{35}$ cm

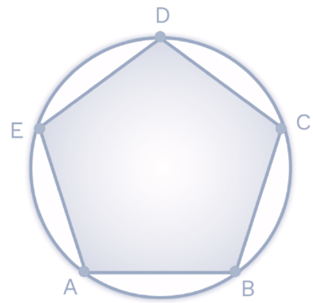
3. What is the ratio of the longest diagonal to the shortest diagonal of a regular octagon?
   (A) 3 : 1
   (B) 2 : 1
   (C) $\sqrt{3} : 1$
   (D) $\sqrt{2} : 1$

4. In the figure given below, PQRS is a rectangle with QX = XY = YZ = ZR. Find the ratio of the area of ΔPXY to the area of ΔPYZ.



   (A) 1 : 3
   (B) 1 : 2
   (C) 1 : 1
   (D) 2 : 3

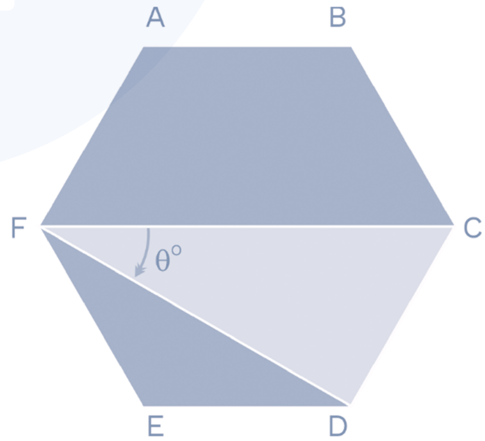
5. In the below given diagram, ∠DEA = ∠DCB = 140°. Then find ∠ADB.



   (A) 120°
   (B) 70°
   (C) 80°
   (D) 100°

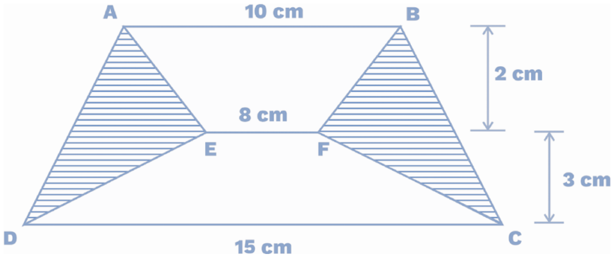
6. ABCDEF is a regular hexagon. Another hexagon is drawn by joining the mid-points of each side of the given regular hexagon. Find the ratio of the area of the smaller hexagon to the larger hexagon.
   (A) 1 : 2
   (B) 1 : 3
   (C) 2 : 3
   (D) 3 : 4

7. In a regular hexagon ABCDEF as shown in the figure, find the value of tan2θ.



   (A) $\frac{1}{\sqrt{3}}$
   (B) $\sqrt{3}$
   (C) $\frac{1}{2}$
   (D) 1

8. In the given trapezium ABCD, EF is parallel to CD. The area of the shaded region is:

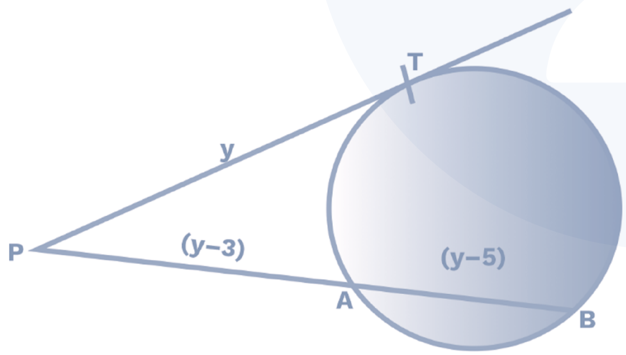


(A) 8 cm^2
(B) 10 cm^2
(C) 12 cm^2
(D) 15 cm^2

9. Two of the sides of an obtuse-angled triangle are 7 cm and 17 cm. How many such triangles are possible? (All the sides have integral values.)

(A) 8
(B) 9
(C) 10
(D) 12

10. In the figure given below, PT is the tangent to the circle and PAB is the secant to the circle. Find the value of *y* (an integer).



Level of Difficulty – 2

11. Two distinct circles of radius 5 cm and 3 cm are there such that the distance between their centres is 10 cm. Find the ratio of length of the direct common tangent to the length of the transverse common tangent between them.

(A) $\sqrt{9} : \sqrt{7}$
(B) $\sqrt{8} : \sqrt{3}$
(C) $\sqrt{5} : \sqrt{3}$
(D) $\sqrt{3} : \sqrt{2}$

12. 8, 15, and *K* are the sides of a triangle, given that *K* is an integer. Find the number of obtuse angle triangles that are possible.

(A) 5
(B) 10
(C) 15
(D) 12

13. PQRS is a trapezium with PQ parallel to RS. The diagonals PR and QS intersect at M. The area of the triangle PMQ is 48 cm^2 and the area of the triangle SMR is 75 cm^2. If PQ = 12 cm, find the measure of RS and the distance between the parallel sides of the trapezium.

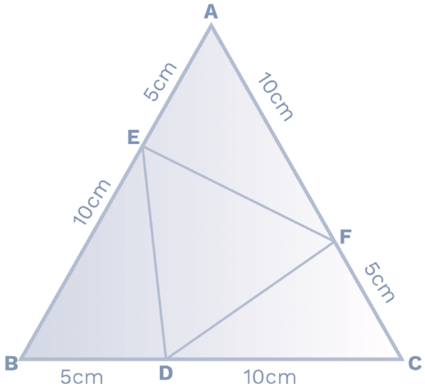
(A) 15 cm, 18 cm
(B) 15 cm, 21 cm
(C) 18 cm, 21 cm
(D) 21 cm, 24 cm

14. In a convex polygon with *n* sides, the number of points of intersection of the diagonals in the interior of the polygon is 70. Find the minimum value of *n*.

(A) 9
(B) 8
(C) 10
(D) 12

15. ABC is an isosceles triangle with AB = AC = 73 cm. From vertex A, a straight line is drawn such that it meets side BC at D and AD = 71. If CD = 9 cm, find the length of BD (in cm).

16. In the triangle ABC shown below, find the ratio of the area of triangle EFD to the area of triangle ABC.



(A) 1 : 4
(B) 2 : 5
(C) 2 : 3
(D) None of these

17. From a triangle PQR with its sides measuring 80 cm, 50 cm, and 70 cm, a triangular portion GQR is cut off, where G is the centroid of DPQR. The area in of the remaining portion of DPQR is:

(A) $\frac{2,000}{3}\sqrt{3}\text{ cm}^2$

(B) $\frac{1,000\sqrt{3}}{3}\text{ cm}^2$

(C) $587\sqrt{3}\text{ cm}^2$

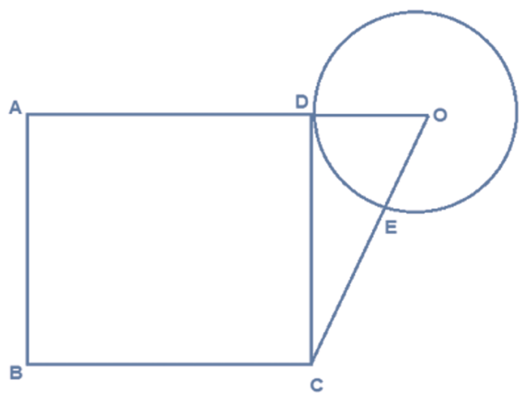
(D) $\frac{589\sqrt{3}}{3}\text{ cm}^2$

18. Given an equilateral triangle T_1 with each side measuring 32 cm, a second triangle T_2 is formed by joining the midpoints of the sides of T_1, then the third triangle is formed by joining the midpoints of the sides of T_2. If this process of forming a triangle is continued, the sum of the areas (in cm^2) of infinitely many such triangles T_1, T_2, T_3 ... will be:

(A) $\frac{1,024}{3}$ (B) $1,024\sqrt{3}$

(C) $\frac{1,024}{\sqrt{3}}$ (D) None of these

19. In the given figure, ABCD is a square. CD is a tangent to the circle with centre O at point D and DO = CE. Find the ratio of area of square ABCD to area of triangle DOC to the area of the circle.



(A) $6:\sqrt{3}:\pi$

(B) $2\sqrt{3}:\sqrt{3}:2\pi$

(C) $6:2:2\pi$

(D) $6:\sqrt{3}:2\pi$

20. P is the centre of a square constructed on the hypotenuse AC of a right-angled triangle ABC. Find $\angle ABP$ (in degrees).

Level of Difficulty – 3

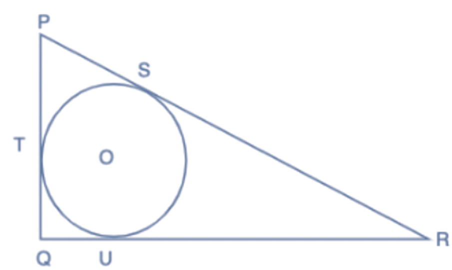
21. Two circles, each of radius of 12 cm, touch externally. Each of these two circles is touched externally by a third circle. If these three circles have a common tangent, then what is the diameter of the third circle in cm?

22. A rectangular field is 80 m long and 60 m wide. Draw diagonals on this field and then draw circles of radius 2 m, with centres only on the diagonals. Each circle must fall completely within the field. Any two circles can touch each other but should not overlap. What is the maximum number of such circles that can be drawn in the field?

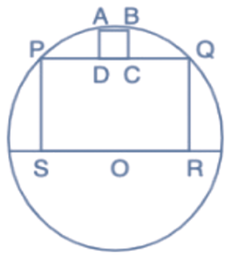
(A) 50
(B) 49
(C) 48
(D) 47

23. A polygon is made of vertical and horizontal lines. The sum of all the convex angles is 9,000 degrees. Find the sum of all the concave angles (in degrees).

24. Triangle PQR is right angled at Q. S is the point of contact of the incircle of PQR with PR and PS : SR = 3 : 10. If the perimeter of PQR is 300 cm, what is the inradius of PQR (in cm)?



25. In the given figure, O is the centre of the circle. Both ABCD and PQRS are squares. What is the ratio of the area of ABCD to that of PQRS?



(A) 4 : 1

(B) $(37 - 8\sqrt{21}) : 25$

(C) $(18 + 8\sqrt{2}) : 15$

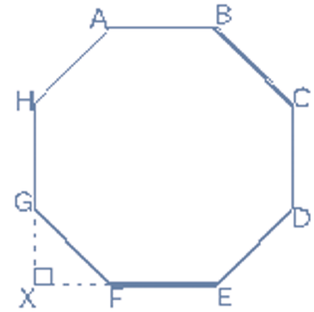
(D) $(\sqrt{21} - 4) : 5$

26. There are five parallel lines that intersect a set of five parallel lines, such that a grid of 16 congruent squares is formed. Now, a third set of seven parallel lines is drawn such that each of the 16 squares formed has a diagonal drawn in it. If the area of each square 1 square units how many triangles formed have an area of 0.5 each?

27. A person of height 2 m is travelling towards a lamp post of height 5 m as shown below. At the given point of time, the length of the shadow of the person formed on the road is 3 m. He travels x metres forward and the length of the shadow becomes 1 m. Which of the following can be the value of x?

(A) 3 m
(B) 4 m
(C) 6 m
(D) Cannot be determined

28. In the figure given below, ABCDEFGH is a regular octagon and ΔFXG is the right-angled triangle. Find the ratio of area of ΔFXG to the area of octagon ABCDEFGH.



(A) $\dfrac{\sqrt{2}-1}{8}$

(B) $\dfrac{1}{8}$

(C) $2\sqrt{2}$

(D) $(1+\sqrt{2})$

29. Two circles are drawn in a rectangle such that each circle touches three sides of the rectangle and passes through the centre of the other circle. If the area of the rectangle is 864 cm^2, then find the area of the region that is common to both the circles.

(A) $288\left(\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{2}\right)\text{cm}^2$

(B) $144\left(\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{2}\right)\text{cm}^2$

(C) $288\left(\dfrac{\pi}{3}-\dfrac{\sqrt{3}}{4}\right)\text{cm}^2$

(D) $144\left(\dfrac{\pi}{3}-\dfrac{\sqrt{3}}{4}\right)\text{cm}^2$

30. A rectangular piece of paper is folded in such a way that one pair of diagonally opposite vertices coincide. If the dimensions of the rectangle are 40 cm $\times$ 30 cm, what is the length (in cm) of the fold?

(A) 36
(B) 36.5
(C) 37
(D) 37.5

Solutions

1. 77

Let the number of sides of the polygon be n.

So, the exterior angle and the interior angle of the polygon become $360/n$ and $[(n-2)180]/n$, respectively.

As the exterior angle is 1/6th of the interior angle,

$360/n = 1/6 \times [(n-2)180]/n$

$\Rightarrow n = 14$

So, the number of diagonals of the polygon $= {}^{14}C_2 - 14 = 91 - 14 = 77$

2. (B)

We can solve this problem by using Heron's formula.

Area of $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

Where S = Semi-perimeter of a Δ.

a, b, c are the respective sides of a Δ.

$\therefore S = \frac{P}{2} = \frac{12+18+26}{2} = 28\text{cm}$

Area of the

$\Delta = \sqrt{28(28-12)(28-18)(28-26)}$

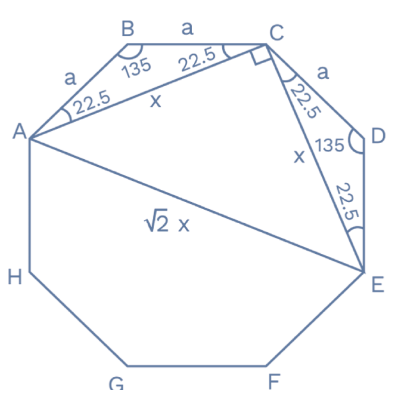
$= \sqrt{28 \times 16 \times 10 \times 2}$

$= \sqrt{2 \times 2 \times 7 \times 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 2}$

$= 2 \times 2 \times 2 \times 2\sqrt{35} = 16\sqrt{35}\ \text{cm}^2$

Hence, option (B) is the correct answer.

3. (D)



$\therefore \frac{AE}{AC} = \frac{\sqrt{2}x}{x} = \frac{\sqrt{2}}{1}$

4. (C)

Area ΔPXY = area of ΔPQY – area of ΔPQX

$= \frac{1}{2}QY \times PQ - \frac{1}{2}QX \times PQ$

$= \frac{1}{2}PQ(QY - QX)$

$= \frac{1}{2}PQ \times XY$ ...(i)

Area of ΔPYZ = area of ΔPQZ – area of ΔPQY

$= \frac{1}{2}QZ \times PQ - \frac{1}{2}QY \times PQ$

$= \frac{1}{2}PQ(QZ - QY) = \frac{1}{2}PQ \times YZ$

Required ratio $= \frac{\text{area of } \Delta PXY}{\text{area of } \Delta PYZ}$

$= \frac{\frac{1}{2}(PQ) \times (XY)}{\frac{1}{2}(PQ) \times (YZ)}$

$= \frac{XY}{YZ}$ ...(ii)

Now it is given $XY = YZ$

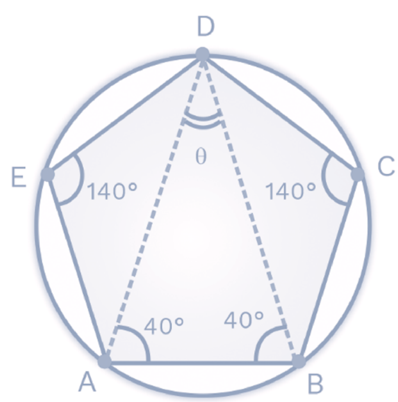
$\Rightarrow$ Required ratio = 1 : 1

Alternate explanation

Triangles with common height have areas always in the ratio of their base.

Ratio = 1 : 1.

5. (D)

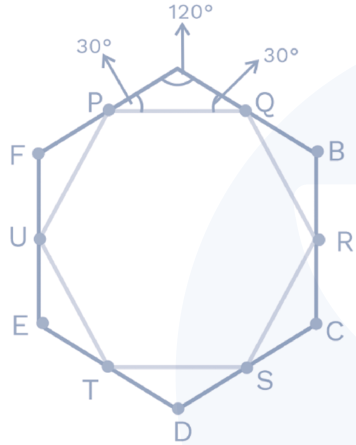


Quadrilateral ABDE and quadrilateral ADCB are cyclic quadrilaterals.
In cyclic quadrilateral ABDE

$\angle DEA + \angle ABD = 180°$
$140° + \angle ABD = 180°$
$\angle ABD = 40°$
Similarly, in quadrilateral ADCB
$\angle DCB + \angle DAB = 180°$
$140° + \angle DAB = 180°$
$\boxed{\angle DAB = 40°}$

Now, we have to find $\angle ADB$.
Since DAB is a triangle:
$\angle ADB + \angle DAB + \angle ABD = 180°$
$\theta + 40° + 40° = 180$
$\theta = 100°$
Hence, option (D) is the correct answer.

6. (D)



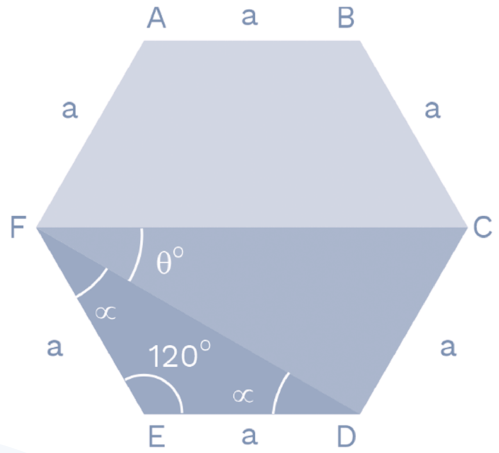
Let's assume the side of the hexagon $= x$ units
In triangle APQ, angles are 30°, 30°, and 120°; so sides would be in the ratio $1 : 1 : \sqrt{3}$

Here AP = AQ = $x/2$ PQ $= \sqrt{3} \times \frac{x}{2}$

$$\frac{\text{Area of small hexagon}}{\text{Area of large hexagon}} = \frac{\frac{\sqrt{3}}{4} \times \left(\frac{\sqrt{3}x}{2}\right)^2}{\frac{\sqrt{3}}{4} \times (x)^2} = \frac{3}{4}$$

7. (B)

All the sides of a regular hexagon are equal. Then ΔEFD is an isosceles Δ.



Since $\angle FED = 120°$ (each Internal angle of regular hexagon = 120°).
Then $2 \propto = 60°$

$\propto = \angle EFD = \angle EDF = 30°$

Then, $\angle CFE = \frac{1}{2} \angle AFE$

$\angle CFE = \frac{1}{2} \times 120° = 60°$

$\therefore \propto + \theta = 60°$

$30° + \theta = 60°$

$\theta = 30°$

Hence, $\tan 2\theta = \tan(2 \times 30°) = \tan 60° = \sqrt{3}$.
Option (B) is the correct answer.

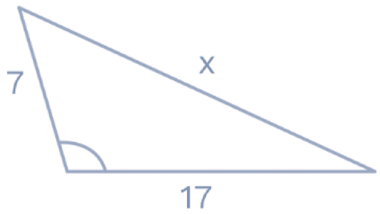
8. (B)

As EF is parallel to CD which is parallel to AB, both ABEF and CDEF will be trapezium.
Now, area of the shaded region = area of trapezium ABCD – area of trapezium ABEF – area of trapezium CDEF area of the shaded region $= \frac{1}{2}(10 + 15) \times 5 -$

$\frac{1}{2}(10 + 8) \times 2 - \frac{1}{2}(8 + 15) \times 3 = 10$ cm^2.

9. (C)

Let the third side be x cm.
Case 1: When x is the longest side.



Now, $x > 17$

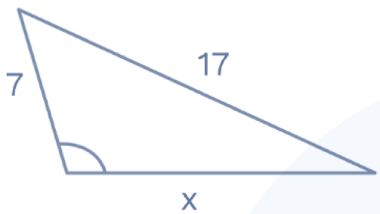
Also, $x < 24$ (sum of other two sides).

Also, $a^2 + b^2 < c^2$ (where c is the longest side).

$\Rightarrow 49 + 289 < x^2 \Rightarrow x^2 > 338$ or $x > 18.38$

$\Rightarrow 18 < x < 24 \Rightarrow x$ can be 19, 20, 21, 22, 23

Case 2: When 17 is the longest side.



Now, $x + 7 > 17 \Rightarrow x > 10$

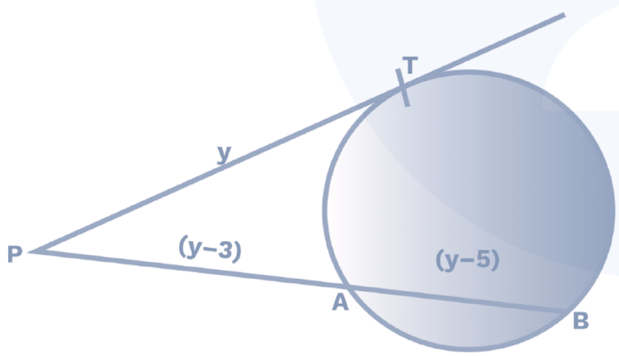
Also, $x < 17$

Also, $7^2 + x^2 < 17^2 \Rightarrow x < 15.49$

$10 < x < 15.49 \Rightarrow x$ can be 11, 12, 13, 14, 15

Therefore, total possible cases 10.

10. 12



According to the tangent–secant theorem

$PT^2 = PA \times PB$

$y^2 = (y - 3)(2y - 8)$

$\Rightarrow y^2 = 2y^2 - 14y + 24$

$\Rightarrow y^2 - 14y + 24 = 0$

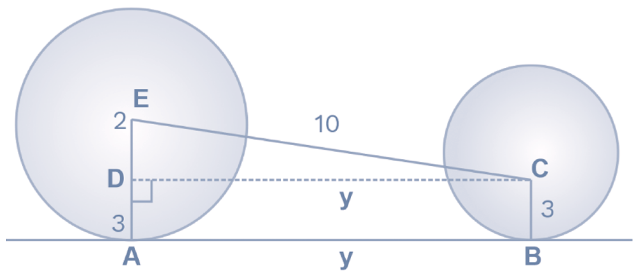
$\Rightarrow (y - 12)(y - 2) = 0$

$\Rightarrow y = 12, 2$

$y = 2$ is not possible (because then $(y - 3)$ and $(y - 5)$ will be negative).

$\Rightarrow y = 12$ is the correct answer.

11. (B)

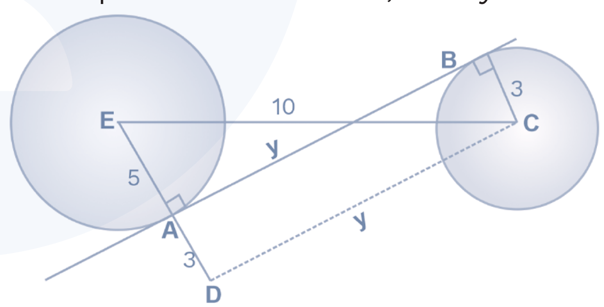


In the above diagram AB = y is the direct common tangent and we have drawn a perpendicular from point C to AE, meeting AE at D. Now ABCD would be a rectangle as radius is always perpendicular to the tangent at the point of contact. So, CD = y

Now triangle CDE is a right-angled triangle and according to the Pythagoras theorem.

$CE^2 = ED^2 + CD^2$ or $10^2 = 2^2 + y^2$ or $y = \sqrt{96}$

Similarly, in the diagram given below, AB = y is the transverse common tangent and we have drawn a perpendicular from point C to AE, meeting AE at D. Now ABCD would be a rectangle as radius is always perpendicular to the tangent at the point of contact. So, CD = y.



Now triangle CDE is a right-angled triangle, and according to the Pythagoras theorem $CE^2 = ED^2 + CD^2$

$10^2 = 8^2 + y^2$ or $y = \sqrt{36}$

Hence, the required ratio

$= \sqrt{96} : \sqrt{36} = \sqrt{8} : \sqrt{3}$

12. (B)

To form a triangle, sum of two smaller sides must be greater than the larger/largest side.

So possible values of K are from 8 to 22, that is 15 values.

So, 15 total triangles are possible.

Now for obtuse-angled triangles

(largest side)^2 > (sum of squares of other 2 sides).

Case 2 when 15 is the largest side

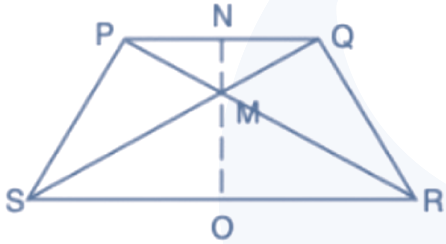
For obtuse triangle, $15^2 > 8^2 + K^2$, now possible values of K are 8, 9, 10, 11, and 12.

Case II when K is the largest side

For obtuse triangle, $K^2 > 8^2 + 15^2$, now possible values of K are 18, 19, 20, 21, and 22.

Hence following 10 obtuse triangles are possible: (8, 15, 8), (8, 15, 9), (8, 15, 10), (8, 15, 11), (8, 15, 12), (8, 15, 18), (8, 15, 19) , (8, 15, 20), (8, 15, 21), and (8, 15, 22).

13. (A)



Clearly, $\Delta PMQ \sim \Delta RMS$

$$\therefore \frac{(SR)^2}{(PQ)^2} = \frac{\text{area of}(\Delta RMS)}{\text{area of}(\Delta PMQ)}$$

$$\therefore \frac{(SR)^2}{(12)^2} = \frac{75}{48}$$

Solving the above equation, we will get SR = 15 cm.

Let $\overline{NO}$ be perpendicular to $\overline{PQ}$ and $\overline{RS}$ through the point M.

$$\therefore \frac{1}{2}(MN)(PQ) = 48$$

$$\Rightarrow \frac{1}{2}(MN)(12) = 48$$

$$\Rightarrow (MN)(6) = 48$$

$$\Rightarrow MN = \frac{48}{6}$$

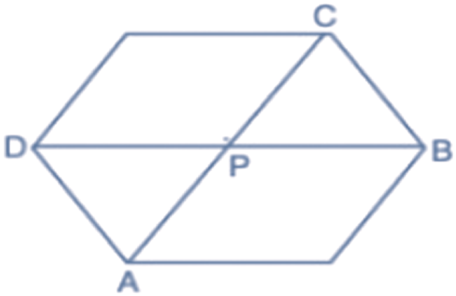
$$\Rightarrow MN = 8$$

$$\frac{1}{2}(MO)(RS) = 75$$

Or, MO = 10 cm

$\therefore$ Distance between the parallel sides = 18 cm.

14. (B)



In the polygon shown in the figure, the two diagonals AC and BD intersect in the interior of the region. That is, every selection of four distinct vertices corresponds to one point of intersection of the diagonals in the interior of the region.

Different sets of four vertices may produce the same point. If they all produce different points, we would get the maximum number of points of intersection (in the interior). This maximum number is ${}^nC_4$ [where n is the number of vertices (or sides)].

From the given data ${}^nC_4 = 70$.

$= n(n - 1)(n - 2)(n - 3)/24 = 70$

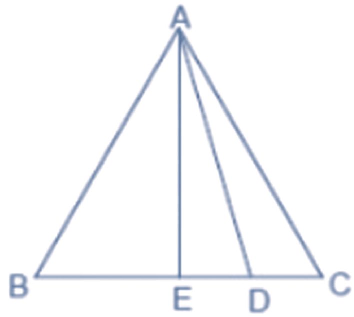
$= n(n - 1)(n - 2)(n - 3) = 70 \times 24 = 8 \times 7 \times 6 \times 5$

$n \geq 8$,

If $n = 8$, the maximum value of p (the number of points of intersection of the diagonals lying in the interior) is 70.

If $p = 70$, the minimum value of n is 8.

15. 32



AB = AC = 73 cm

AD = 71 cm

Drop the perpendicular AE to BC.

Since ΔABC is an isosceles triangle, BE = EC

In ΔAEB, $AB^2 = AE^2 + BE^2$ ...(i)

In ΔAED, $AD^2 = AE^2 + ED^2$ ...(ii)

Subtracting equation (ii) from (i), we get

$AB^2 - AD^2 = BE^2 - ED^2$

$73^2 - 71^2 = (BE + ED)(BE - ED)$. Since BE = CE

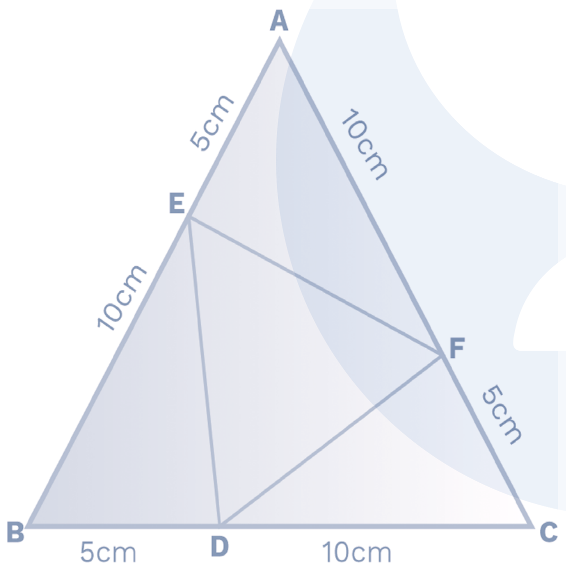
$73^2 - 71^2 = (BE + ED)(CE - ED)$

$73^2 - 71^2 = (BD)(CD)$

2 (144) = BD (9)

Therefore, BD = 32 cm.

16. (D)



D is an equilateral D with each side measuring 15 cm.

$\Rightarrow$ area of triangle ABC

$$= \frac{\sqrt{3}}{4} \times (15)^2 = \frac{225\sqrt{3}}{4}$$

Now, triangle AEF, triangle BED, and triangle CFD are congruent triangles and their areas will be equal.

(As they have common sides of 5 cm and 10 cm with angle between the sides = 60°.)

Now, area of triangle

$$AEF = \frac{1}{2} \times 10 \times 5 \times \sin 60 = \frac{50\sqrt{3}}{4}$$

Area of triangle AEF + area triangle BED

$$+ \text{area of triangle CFD} = 3 \times \frac{50\sqrt{3}}{4} = \frac{150\sqrt{3}}{4}$$

$$\Rightarrow \text{area of } \Delta EFD = \frac{225\sqrt{3}}{4} - \frac{150\sqrt{3}}{4}$$

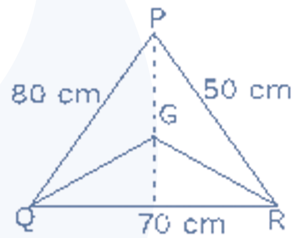
$$= \frac{75 \times \sqrt{3}}{4}$$

$\Rightarrow$ Required ratio

$$= \frac{\text{area of } \Delta EFD}{\text{area of } \Delta ABC}$$

$$= \frac{75 \times \frac{\sqrt{3}}{4}}{225 \times \frac{\sqrt{3}}{4}} = \frac{1}{3}$$

17. (A)



First, we have to find the area of the ΔPQR.

Since the lengths of the side of the triangle, PQR is given, so that by using Heron's formula we can get the area.

$\therefore$ Perimeter of the triangle PQR = 80 + 70 + 50.

= 200 cm.

Therefore, semi-perimeter (s)

$$= \frac{\text{Perimeter}}{2} = \frac{200}{2} = 100 \text{ cm}$$

$\therefore$ Area of the ΔPQR

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{100 \times (100-80)(100-70)(100-50)}$$

$$= \sqrt{100 \times 20 \times 30 \times 50}$$

$$= \sqrt{100 \times 100 \times 100 \times 3}$$

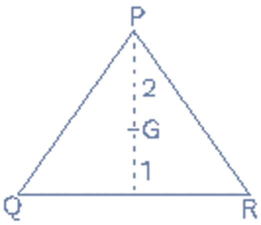
$$= 100 \times 10\sqrt{3}$$

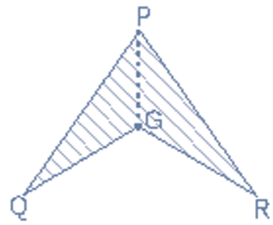
$$= 1{,}000\sqrt{3} \text{ cm}^2$$

The triangle portion GQR is cut off, where G is the centroid.

We also know that the centroid divides the median into 2 : 1, with the larger side towards the vertices..

$\therefore$ The remaining portion of the ΔPQR is



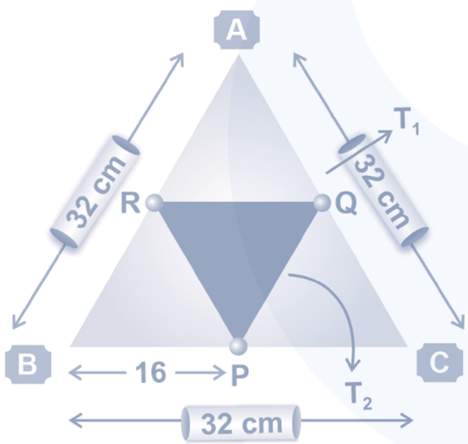


$\therefore$ Area of the remaining portion of the $\Delta PQR = \frac{2}{3} \times$ area of the $\Delta PQR = \frac{2}{3} \times 1{,}000\sqrt{3}$

$= \frac{2{,}000}{3}\sqrt{3}$ cm^2.

Hence, option (A) is the correct answer.

18. (C)



$\therefore$ Area of the equilateral $\Delta T_1 = \frac{\sqrt{3}}{4} \times (32)^2$

Since we know that if we join the mid-points of an equilateral triangle, then another equilateral triangle will form, but the area will become 1/4th the original triangle.

$\therefore$ Area of the equilateral $\Delta T_2 = \dfrac{\frac{\sqrt{3}}{4} \times (32)^2}{4}$

If we again join the midpoints of the ΔT_2 then we will get another equilateral ΔT_3 whose area becomes 1/4th the area of ΔT_2.

Therefore,

$T_1 + T_2 + T_3 + \ldots\ldots\ldots\ldots \infty$

$$= \frac{\sqrt{3}}{4}a^2 + \frac{\frac{\sqrt{3}}{4}a^2}{4} + \frac{\frac{\sqrt{3}}{4}a^2}{16} + \ldots\ldots\ldots\ldots\infty$$

or we can say that

$$T_1 + \frac{T_1}{4} + \frac{T_1}{4^2} + \frac{T_1}{4^3} + \frac{T_1}{4^4} + \ldots\infty$$

Since the above series is in GP.

$$\text{Common ratio } (r) = \frac{\frac{T_1}{4}}{T_1} = \frac{1}{4}$$

Also, the sum of infinite terms

$$= \frac{a}{1-r} \to \text{First term}$$

$$= \frac{T_1}{1-\frac{1}{4}} = \frac{4T_1}{3} = \frac{4}{3} \times \frac{\sqrt{3}}{4}(32)^2$$

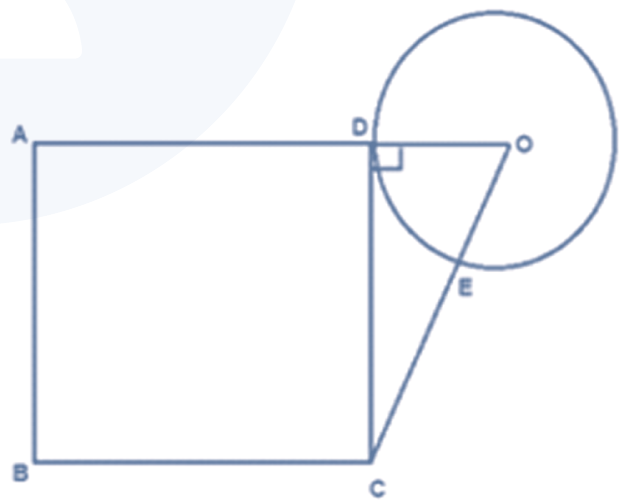
$$= \frac{1}{\sqrt{3}}(32)^2 = \frac{1{,}024}{\sqrt{3}} = \frac{1{,}024\sqrt{3}}{3}$$

Hence, the sum of the areas of infinitely many such triangles

$(T_1, T_2, T_3, \ldots\infty) = \frac{1{,}024}{\sqrt{3}}$.

Hence, option (C) is the correct answer.

19. (D)



Since CD is the tangent drawn to the circle.

So, $\angle ODC = 90°$

Let the radius of the circle be r.

Then, OD = OE = r

And, OD = CE

So, OE = CE

Therefore, OC = $2r$

In triangle DOC,

$CD^2 = (2r)^2 - r^2$

$CD^2 = 3r^2$

$CD = \sqrt{3}r$

Area of square $=$ side$^2 = 3r^2 = 3r^2$ sq unit

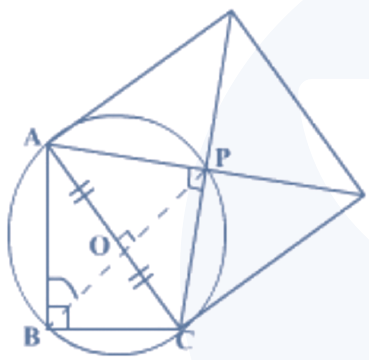
Area of circle $= \pi r^2$

And, area of triangle $= \frac{1}{2} \times \text{OD} \times \text{CD}$

$= \frac{1}{2} \times r \times \sqrt{3}r = \frac{\sqrt{3}}{2} r^2$ sq unit

Therefore, ratio of area of square ABCD : area of triangle DOC : area of circle $= 6 : \sqrt{3} : 2\pi$

20. 45



The diagonals of a square are perpendicular.

$\therefore \angle APC = 90°$

$\therefore$ *A*, *B*, *C*, and *P* are concyclic points.

Hypotenuse AC is the diameter of the circumcircle of quadrilateral APCB.

So, the midpoint of AC will also be the centre of the circumcircle O.

The line joining the centre of the square (i.e., point *P*) and the midpoint of side AC of the square will always be perpendicular to side AC.

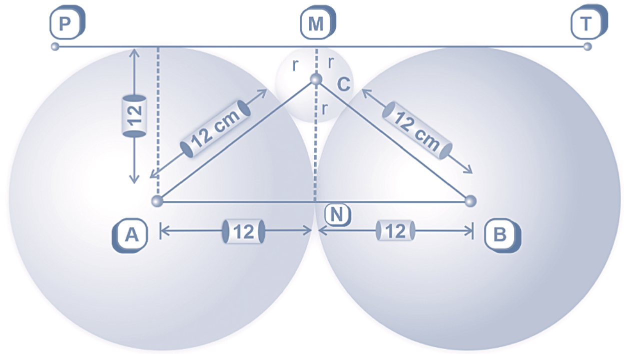
Hence, the central angle subtended by minor arc AP is $\angle AOP = 90°$.

Hence, by the inscribed angle theorem, $\angle ABP = (1/2) \times$ Central angle subtended by minor arc AP $= 90/2 = 45°$.

$\therefore \angle ABP = 45°$

21. 6

Let the radius of the third (smaller circle) is *r* cm and the height of the ΔABC be *h* cm.



Semi-perimeter of the $\Delta ABC = (S)$

$= \frac{AB + BC + CA}{2}$

$= \frac{24 + 12 + r + 12 + r}{2}$

$= \frac{48 + 2r}{2} = (24 + r)$ cm.

Therefore, the area of the ΔABC

$= \sqrt{S(s-a)(S-b)(S-c)}$

$= \sqrt{(24+r)\,(24+r-12-r) \times (24+r-12-r) \times (24+r-24)}$

$= \sqrt{(24+r) \times 12 \times 12 \times r}$

$= 12\sqrt{24r + r^2}$

Also, the area of the ΔABC in terms of height

$= \frac{1}{2} \times AB \times h = \frac{1}{2} \times 24 \times h$

$\therefore 12\sqrt{24r + r^2} = \frac{1}{2} \times 24 \times h$

$\sqrt{24 + r^2} = h$

MN = radius of the larger circle = 12 cm $= h + r$.

So, height (h) of the triangle ABC $= 12 - r$

$\therefore 12 - r = \sqrt{24r + r^2}$

$(12 - r)^2 = 24r + r^2$

$144 + r^2 - 24r = 24r + r^2$

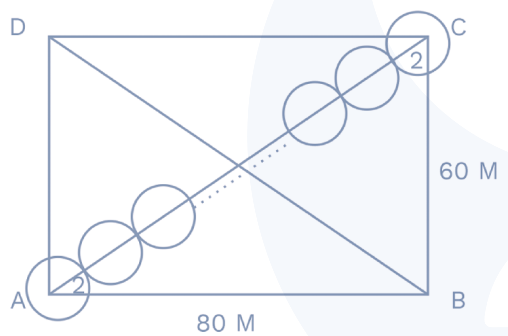
$48r = 144$

$r = \frac{144}{48} = 3$ cm

Therefore, the diameter of the smaller circle $= 2r = 2 \times 3 = 6$ cm.

22. (D)

By Pythagoras theorem, AC = BD = 100 m. A maximum number of circles with a radius of 2 m that we can draw on diagonal AC = 24, as a circle with centre A and *C* will lie outside the field.



Similarly, the maximum number of circles with a radius of 2 m on BD = 24.

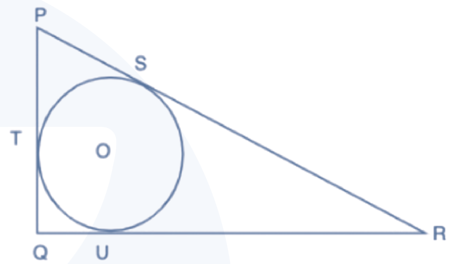
Total circles on both diagonals $= (24 \times 2) - 1 = 47$.

(As circle at the intersection point of diagonal will be counted twice.)

23. 25,920

As we know that:
Convex angles < 180 degrees
Concave angles > 180 degrees
For polygons made of vertical and horizontal lines, then
Each convex angle = 90 degrees
Each concave angle = 270 degrees
The polygon made up of vertical and horizontal lines, with the least number of sides, is a rectangle (squares included).
For a square, the number of convex angles is equal to four, and the number of concave angles is zero.
Difference between the number of concave and convex angles $= 4 - 0 = 4$.
This difference will remain the same for all polygons with more than four sides.
According to the question, the sum of the convex angles (i.e., 90 degrees) = 9,000 degrees, so there will be 100 such angles. The number of concave angles would be equal to $100 - 4 = 96$ angles.
Sum of 96 concave angles $= 96 \times 270 = 25{,}920$.

24. 20



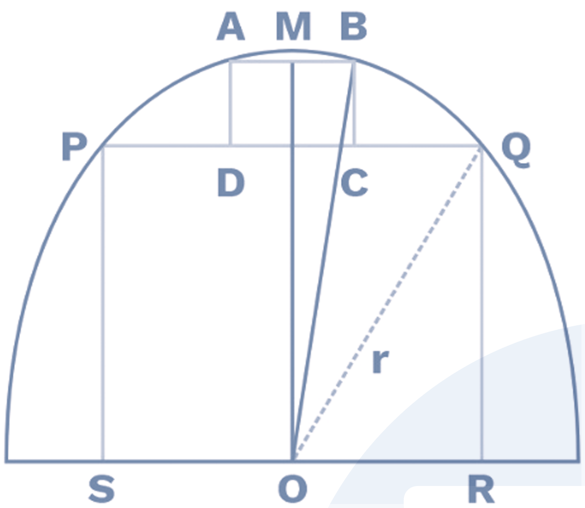
Let *T* and *U* be the points of contact of the incircle with PQ and QR, respectively. Let *O* be the centre of the incircle. Let the inradius be *r*.
$\angle Q = 90°$
$\angle OTQ = \angle OUQ = 90°$ ($\because$ PQ and QR are tangents to the incircle at *T* and *U*, respectively.)
$\angle TOU = 90°$
TOUQ is a rectangle. Also, OT = OU = radius.
$\therefore$ TOUQ is a square.
$QT = QU = r$
PS : SR = 3 : 10
By property of tangents, PT = PS and RU = RS.
Let PS = 3k and SR = 10k
Perimeter of PQR = PQ + QR + PR = (PT + TQ) + (QU + UR) + (PS + SR)
$= (3k + r) + (r + 10k) + 13k$
$2r + 26K = 300$
$PQ^2 + QR^2 = PR^2$
$(3k + r)^2 + (r + 10k)^2 = (13k)^2$
$r^2 + 13rk - 30k^2 = 0$
$(r + 15k)(r - 2k) = 0$

$r = 2k\ (r > 0)$

$$2r + 26\left(\frac{r}{2}\right) = 300$$

$\Rightarrow r = 20.$

25. (B)



Let r be the radius of the circle and x be the length of each side of PQRS.

$$x^2 + \left(\frac{x}{2}\right)^2 = r^2 \Rightarrow x = \frac{2}{\sqrt{5}}r$$

Let $r = \sqrt{5}$ units $\therefore$ $x = 2$ units.

Let the length of each side of ABCD by y cm.

$$(2+y)^2 + \left(\frac{y}{2}\right)^2 = r^2 = 5$$

$$\frac{5}{4}y^2 + 4y - 1 = 0$$

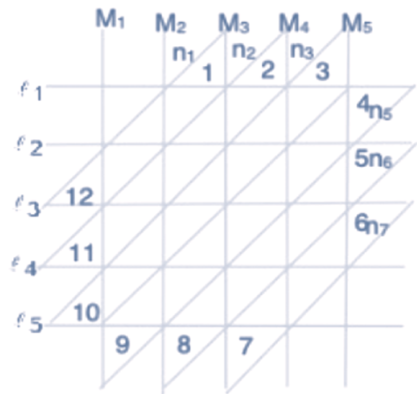
$5y^2 + 16y - 4 = 0$

$$\Rightarrow y = \frac{2\sqrt{21}-8}{5}$$

$$\text{Ratio of areas} = \frac{y^2}{x^2} = \left(\frac{2\sqrt{21}-8}{5(2)}\right)^2$$

$$= \frac{37-8\sqrt{21}}{25}$$

26. 44



The first set of parallel lines are $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$

The second set of parallel lines are M_1, M_2, M_3, M_4, M_5.

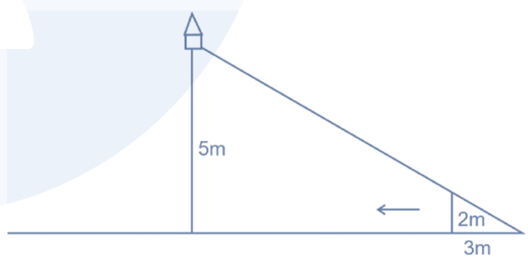
We get a 4×4 grid.

The third set of parallel lines are $n_1, n_2, n_3, n_4, n_5, n_6, n_7$.

These seven lines provide a diagonal for each of the 16 cells.

We can see that there are 16(2) + 12 triangles that are formed, i.e., 44 triangles.

27. (D)



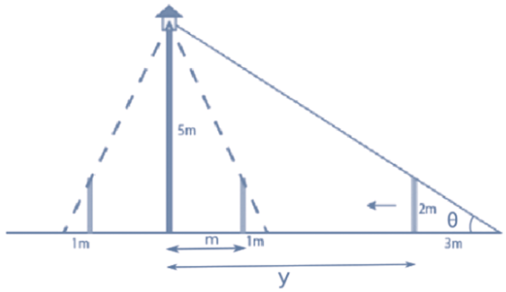
Let the distance of the pole from the person = y.

$$\tan\theta = \frac{2}{3} = \frac{5}{y+3}$$

$y = 4.5$ m

Let the person be m metres away from the pole when its shadow's length is 1 m.

We will get equal shadows on both sides of the lamp.



Now, $\frac{2}{1} = \frac{5}{1+m}$

$M = 1.5$ m

But there will be two positions where the man will form a shadow of 1 m, on both sides of the pole.

$x = 4.5 - 1.5 = 3$ m or $x = 4.5 + 1.5 = 6$ m.

i.e., x can be 3 m or 6 m.

28. (A)

Let each side of regular octagon be x

$\Rightarrow$ GF $= x$, GX = XF (because angle GFX = angle FGX = 45 degrees)

$\Rightarrow GX^2 + XF^2 = GF^2$

$\Rightarrow 2GX^2 = x^2$

$\Rightarrow$ GX $= \frac{x}{\sqrt{2}}$ = XF

Area of octagon ABCDEFGH = $2(\text{side})^2 (1+\sqrt{2})$ (formula to find the area of the regular octagon)

Area of octagon ABCDEFGH $= 2x^2(1+\sqrt{2})$

Area of ΔFXG $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times \frac{x}{\sqrt{2}} \times \frac{x}{\sqrt{2}} = \frac{x^2}{4}$

Required ratio =

$\frac{\text{Area of } \Delta\text{FXG}}{\text{Area of octagon ABCDEFGH}}$

$= \frac{\frac{x^2}{4}}{2x^2(1+\sqrt{2})}$

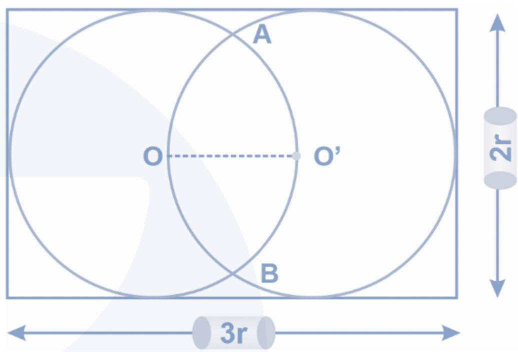
$= \frac{1}{8(1+\sqrt{2})} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$

$= \frac{\sqrt{2}-1}{8}$

29. (C)

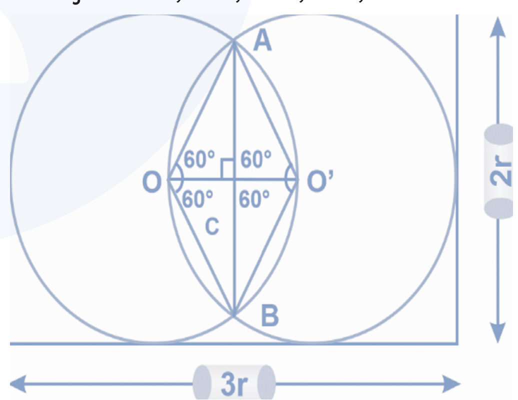
According to the question, the diagram would be as given below, where O and O' are centres of the circles and A and B are the points where circles intersect each other.

Let's assume the radius of each circle $= r$ cm.



Area of rectangle $= 3r \times 2r = 6r^2 = 864 \rightarrow r = 12$ cm.

Let's join AB, OA, O'A, OB, and O'B.



Now in triangle OAO' OA = O'A = OO' $= r$

$\Rightarrow$ triangle OAO' is an equilateral triangle.

Similarly, triangle OBO' is also an equilateral triangle.

$\Rightarrow \angle AOB = \angle AO'B = 120°$

Now area common to both the circles

= Area of minor segment AOB + Area of minor segment AO'B.

Also, area of minor segment AOB = Area of minor segment AO'B

$\Rightarrow$ Area common to both the circles = 2 (area of minor segment AOB).

Area minor segment AOB = (Area of sector AO'BO) – (Area of triangle AO'B).

$$= \left(\frac{120}{360} \times \pi(12)^2\right) - \left(\frac{1}{2} \times (12)^2 \times \sin 120\right)$$

$$144\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) \text{ cm}^2$$

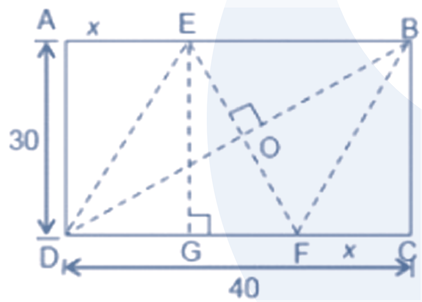
Area common to both the circles = 2 (area of minor segment AOB)

$$= 288\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) \text{cm}^2$$

Hence, option (C) is the correct answer.

30. (D)

Consider the following figure of a rectangular paper ABCD.



The line EF is the fold, which is made such that corner *D* meets the diagonally opposite corner *B*.

As DF coincides with FB upon making the fold DF = FB. Similarly, DE = EB.

Also, as DF is the part of the length of the rectangle that is being folded so that *D* coincides with the opposite vertex and BE is the part of the length, that is being folded so that *B* coincides with *D*, DF = BE (from symmetry), i.e., quadrilateral DFBE is a rhombus ...... (i)

Now assume, FC = x cm

In the right-angled triangle ΔBFC, BF

$$= \sqrt{BC^2 + FC^2} = \sqrt{(30)^2 + x^2}$$

Since EDBF is a rhombus, BF = DF

$$\text{Hence } = \sqrt{30^2 + x^2} = 40 - x$$

$$\Rightarrow 900 + x^2 = 1{,}600 + x^2 - 80x$$

$$\Rightarrow x = \frac{700}{80} = \frac{35}{4}$$

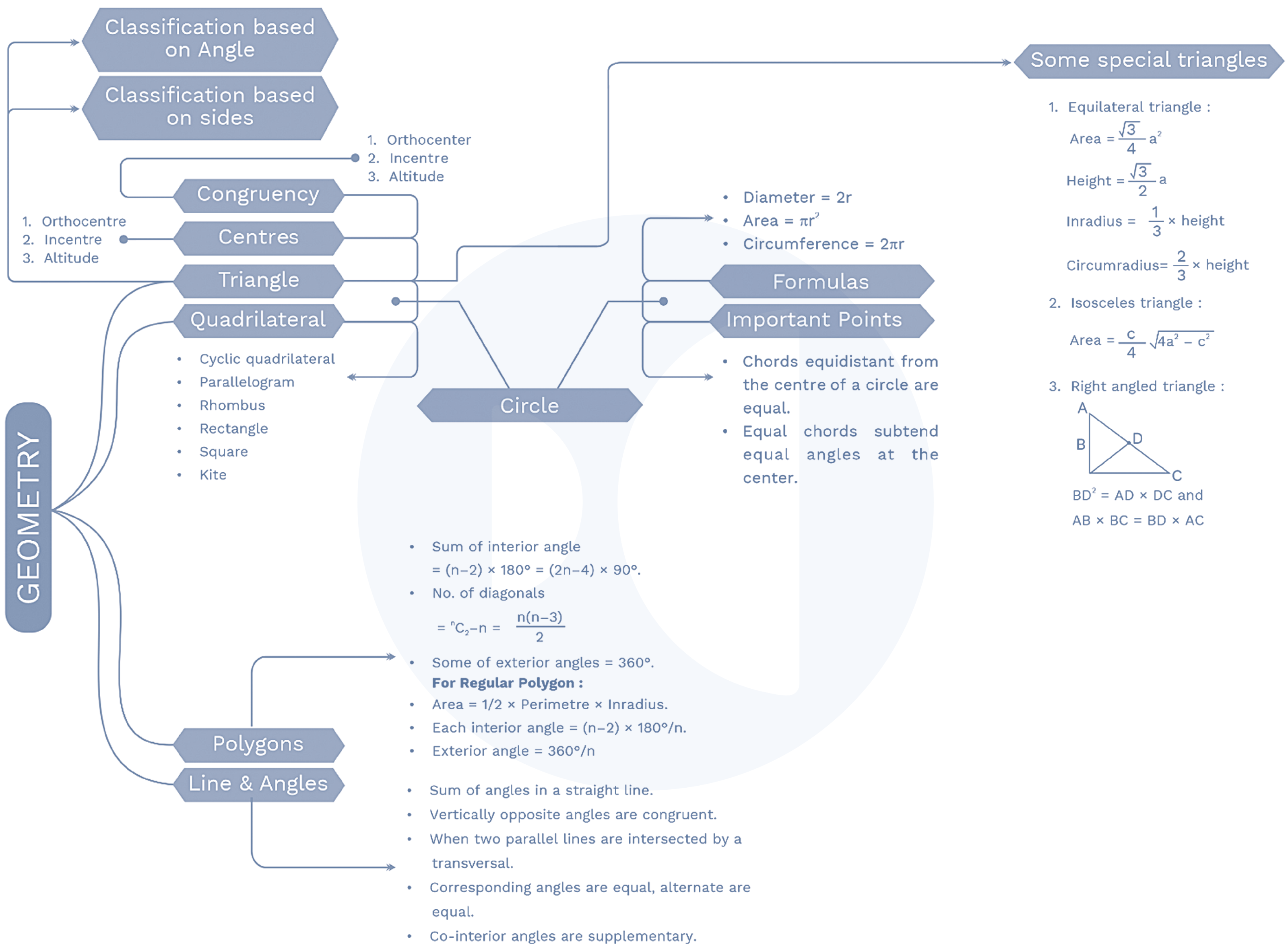
Now, consider *G* on DC, such that EG $\perp$ DC.

In ΔEGF, GF = 40 – 2x (as AE = FC = x), EG = 30 and EF is the length of the fold.

$$\therefore GF = 40 - \frac{70}{4} = \frac{90}{4} = \frac{45}{2}$$

$$\Rightarrow EF = \sqrt{\left(\frac{45}{2}\right)^2 + 30^2} = 37.5 \text{ cm.}$$

Mind Map


GEOMETRY
Classification based on Angle
Classification based on sides
Congruency
1. Orthocenter
2. Incentre
3. Altitude
Centres
1. Orthocentre
2. Incentre
3. Altitude
Triangle
Quadrilateral
• Cyclic quadrilateral
• Parallelogram
• Rhombus
• Rectangle
• Square
• Kite
Circle
Formulas
• Diameter = 2r
• Area = πr^2
• Circumference = $2\pi r$
Important Points
• Chords equidistant from the centre of a circle are equal.
• Equal chords subtend equal angles at the center.
Some special triangles
1. Equilateral triangle :
Area = $\frac{\sqrt{3}}{4}a^2$
Height = $\frac{\sqrt{3}}{2}a$
Inradius = $\frac{1}{3} \times$ height
Circumradius= $\frac{2}{3} \times$ height
2. Isosceles triangle :
Area = $\frac{c}{4}\sqrt{4a^2 - c^2}$
3. Right angled triangle :
A
B
D
C
$BD^2 = AD \times DC$ and
$AB \times BC = BD \times AC$
Polygons
• Sum of interior angle
= $(n-2) \times 180° = (2n-4) \times 90°$.
• No. of diagonals
= ${}^nC_2 - n = \frac{n(n-3)}{2}$
• Some of exterior angles = 360°.
For Regular Polygon :
• Area = 1/2 × Perimetre × Inradius.
• Each interior angle = $(n-2) \times 180°/n$.
• Exterior angle = $360°/n$
Line & Angles
• Sum of angles in a straight line.
• Vertically opposite angles are congruent.
• When two parallel lines are intersected by a transversal.
• Corresponding angles are equal, alternate are equal.
• Co-interior angles are supplementary.