Part I

Methods of Econometrics

Chapter 1

Consistency and Asymptotic Normality

1.1 Convergence of real sequences

Real numbers are numbers that have points on the number line $\mathbb{R}$. Let a be a real number and $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers:

$$(a_n)_{n=1}^{\infty} = a_1,\ a_2,\ a_3, \ldots$$

For example, a sequence

$$a_1 = 1,\ a_2 = 1.4,\ a_3 = 1.41,\ \cdots,\ a_n = 1.4142\ldots, \cdots$$

gets closer and closer to $\alpha = \sqrt{2}$ $(= 1.414213\ldots)$ as n increases. The number α is called the **limit** of the sequence $(a_n)_{n=1}^{\infty}$. We say that the sequence $(a_n)_{n=1}^{\infty}$ **converges** to α, and write

$$\lim_{n\to\infty} a_n = \alpha$$

$$\text{or equivalently } \ a_n \to \alpha \ \ (n \to \infty).$$

For other examples, the limit of the sequence $a_1 = 1,\ a_2 = 1/2,\ a_3 = 1/3,\ \cdots,\ a_n = 1/n, \cdots$ is clearly $\lim_{n\to\infty} a_n = 0$. It is well-known that the sequence

$$a_1 = (1 + (1/1))^1,\ a_2 = (1 + (1/2))^2,\ \cdots,\ a_n = (1 + (1/n))^n, \cdots$$

converges to the Napier's constant: $\lim_{n\to\infty} a_n = e$ $(= 2.718\cdots)$. On the other hand, the limit of the sequence $a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ \cdots,\ a_n = n, \cdots$ is infinite: $\lim_{n\to\infty} a_n = \infty$. When the limit of a sequence is infinite, we say that the sequence is a **divergent**. There are sequences that are neither convergent nor divergent. For example, the sequence

$$a_1 = -1,\ a_2 = 1,\ a_3 = -1,\ \cdots,\ a_n = (-1)^n, \cdots$$

alternates between 1 and -1.

The statement "the sequence $(a_n)_{n=1}^{\infty}$ gets closer to α as n increases" is rather informal and mathematically unclear. A formal definition of convergence is as follows.