Financial Mathematics

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By

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PREFACE

In the education of financiers and economists in all universities of the world, an important role belongs to mathematical disciplines. Among these disciplines, financial mathematics occupies a very serious place, because it is the base for disciplines such as corporate finance, financial management, investment, taxation, business valuation, ratings, etc.

This textbook is intended for both undergraduate and post-graduate students studying the course "Financial Mathematics".

It differs from other textbooks in a detailed and accessible presentation with the derivation and proof of all statements and theorems and a much broader consideration of the issues raised.

In each chapter of the textbook, detailed practical examples are given, and at the end of each chapter, questions and tasks are given to control the degree of assimilation of the material and consolidation of what has been studied.

Abstract

This textbook contains information on financial mathematics, knowledge of which is necessary not only for every financier, but also for any competent economist of a wide profile (and especially for financial analysts). It consists of seven chapters: "Interest theory", "Financial flows and rents", "Profitability and risk of financial transactions", "Portfolio analysis", "Bonds", "Capital structure: Modigliani-Miller theory" and "Capital structure: Brusov-Filatova-Orekhova theory". Each chapter contains many detailed practical examples, and at the end of each chapter questions and tasks for revision are given.

For undergraduate and graduate students of all financial and economic fields and profiles, including "Finance and Credit", "Accounting and Auditing", "Taxes and Taxation", "World Economy", etc. It will be useful for specialists of all financial and economic specialties (and especially for financial analysts) and for everyone who wants to master quantitative methods in finance and economics.

INTRODUCTION

In the education of financiers and economists in all universities of the world, an important role belongs to mathematical disciplines. Among these disciplines, financial mathematics occupies a very serious place, because it is the base for disciplines such as corporate finance, financial management, investment, taxation, business valuation, ratings, etc.

This textbook is intended for both undergraduate and post-graduate students studying the course “Financial Mathematics”. It differs from other textbooks in its detailed and accessible presentation with derivation and proofs of all statements and a much broader consideration of the issues raised. So, for example, if in all standard textbooks the “Rule of 70” (the term for doubling the deposit at a given interest rate) is given only for the case of compound interest, then this textbook considers an analogue of “the rule 70” for the case of simple interest, the so-called “rule 100” and moreover the cases of increasing the deposit an arbitrary number of times for all types of interest rates: complex, simple, continuous and with multiple accruals of interest.

In the chapter “Portfolio Analysis”, much attention is paid to the portfolio of two securities, the theory of which was developed by the authors specifically for this textbook. Mastering this simple Portfolio theory prepares students to study the more general portfolio theories of Markowitz and Tobin. Such a detailed and consistent presentation is aimed at the student’s conscious, creative assimilation of the course program so that they have the opportunity to independently solve a wide variety of tasks and problems arising in practice.

In each chapter of the textbook, detailed practical examples are given, and at the end of each chapter, questions and tasks are given to control the degree of assimilation of the material and consolidation of what has been studied.

The textbook is written using a competence-based approach based on the lectures given by the authors for more than 15 years at the Financial University under the Government of the Russian Federation (Moscow).

The authors are enthusiasts of the introduction of mathematical methods in economics and finance. The understanding that finance is essentially a quantitative science, and quantitative methods play a crucial role in the training of financiers and economists of all profiles, is increasingly spreading among the specialists responsible for their training. An example of this understanding is the introduction at the Financial University under the Government of the Russian Federation on the initiative of the authors of the course “Financial Mathematics” as a mandatory bachelor’s degree for students of all directions and profiles, which was an important step towards the integration of national financial and economic education into the global one, where the mathematical component of financial disciplines reaches 70% or more. Such extensive teaching of financial mathematics has led to the use of the knowledge gained during the study in the development of many special disciplines and ultimately to an improvement in the quality of education received by graduates.

The authors are planning to publish “Tasks on Financial Mathematics”, the use of which together with this textbook will allow the reader to creatively and firmly master this course.

The authors also plan to publish the second part of the textbook, intended for master’s students and including not general, but special questions for each specific master’s program. It sets out issues such as the cost and structure of capital, the company’s dividend policy, leasing and others for the “Financial Management” program, issues such as repayment of long-term loans, VaR and its application in banking and others for the “Banking industry” program, investments, modern models for evaluating the effectiveness of investment projects, financial markets and derivative financial instruments for the “Financial Markets” program, etc.

CHAPTER 1

THE THEORY OF INTEREST

Interest can be defined as compensation paid by the borrower to the lender for the use of capital. Therefore, interest can be considered as a rent that the borrower pays to the lender to compensate for losses from the latter's non–use of capital during the loan. In general, capital and interest do not necessarily represent the same commodity. However, we will consider capital and interest expressed in the same terms — in terms of money.

So, the lender provides the borrower with a certain amount of money; after the deadline, the borrower must repay the accrued amount equal to the amount of debt plus interest.

Effective interest rate is the amount paid to the borrower (investor) at the end of the accrual period for each unit amount borrowed (invested) at the beginning of the period.

Denoting the increased value of the unit amount at time t through a_t, the interest rate through i, and the increased value of the full amount through S_t, we have for the first accrual period

$$i_1 = \frac{(1+i)}{1} = \frac{a_1 - a_0}{a_0} = \frac{S_1 - S_0}{S_0}, \qquad (1.1)$$

for the n–th accrual period

$$i_n = \frac{a_n - a_{n-1}}{a_{n-1}} = \frac{S_n - S_{n-1}}{S_{n-1}}. \qquad (1.2)$$

From this formula it can be seen that the effective interest rate can change (and is changing) depending on the number of the accrual period, but, as will be shown below, in the very important and widely used case of compound interest, the effective interest rate for all accrual periods remains constant, i.e. for all $n \geq 1$.

1.1. Simple interest

Let S_0 be the initial amount of debt, i be the interest rate. In the simple interest scheme, S_0 will increase by iS_0 by the end of a single accrual period (usually a year), and the accrued amount of S_1 will be equal to

$$S_1 = S_0 + iS_0 = S_0(1 + i). \tag{1.3}$$

By the end of the second accrual period, the initial amount of debt S_0 will increase by another iS_0 and the accrued amount will become

$$S_2 = S_1 + iP = S_0(1 + 2i). \tag{1.4}$$

By the end of the n–th accrual interval, the accrued amount will be

$$S_n = S_0(1 + ni). \tag{1.5}$$

This formula is called **the simple interest formula**. The multiplier $(1 + ni)$ is called **the accrual coefficient (multiplier)**, and the value of ni is the interest rate for time n.

Thus, the sequence of incremented sums $S_1, S_2, \ldots, S_n$ is an arithmetic progression with the initial term S_0 and the difference iS_0.

The interest for n years can be represented as

$$I_n = S_0 in. \tag{1.6}$$

Effective interest rate in the simple interest scheme

$$i_n = \frac{a_n - a_{n-1}}{a_{n-1}} = \frac{S_n - S_{n-1}}{S_{n-1}} = \frac{(1+in)-(1+i(n-1))}{1+i(n-1)} = \frac{i}{1+i(n-1)} \tag{1.7}$$

decreases with the growth of n.

If different interest rates $i_1, i_2, \ldots, i_m$ are set at different intervals of interest accrual $n_1, n_2, \ldots, n_m$, then the accrued amount S_n for the time $n_1 + n_2 + \ldots + n_m$ will be equal to

$$S_n = S_0\left(1 + \sum_{k=1}^{m} n_k i_k\right). \tag{1.8}$$

The time of repayment of the loan may not be specified exactly, but may be a variable (for example, in the case of a cumulative deposit on demand). Then the formula of simple interest takes the following form:

$$S_t = S_0(1 + i(t - t_0)), \tag{1.9}$$

where t_0 — the moment when the loan was issued;

t — the moment of repayment of the debt with interest.

According to the formula (1.9), the accumulated sum is a linear function of time. The graph of this function on the "time—money" coordinate plane is a ray with a starting point (t_0, S_0) and an angular coefficient $S_0 i$. Obviously,

$$S_t{}' = S_0 i. \tag{1.10}$$

1.2. Compound interest

With the accrual of compound interest, **reinvestment, or capitalization of the interest received** occurs; thus, at the rate i, each subsequent accrued amount increases by a part i of the previous amount, which takes into account the interest accrued in previous periods.

In the S_0 compound interest scheme, by the end of a single interval, accruals will increase by iS_0, and the accrued amount of S_1 will be equal to

$$S_1 = S_0 + iS_0 = S_0(1 + i). \tag{1.11}$$

By the end of the second period, S_1 accruals will increase by iS_1 and the accrued amount will become

$$S_2 = S_1 + iS_1 = S_1(1 + i) = S_0(1 + i)^2. \tag{1.12}$$

By the end of the n–th accrual interval, the accrued amount will be

$$S_n = S_0(1 + i)^n. \tag{1.13}$$

The formula (1.13) is called **the compound interest formula**. Thus, the sequence of incremented sums S_1, $S_2,\ldots$, S_n is a geometric progression with the initial term S_0 and the denominator of the progression $q = (1 + i)$.

Effective interest rate in the compound interest scheme for the n–th accrual period

$$i_n = \frac{a_n - a_{n-1}}{a_{n-1}} = \frac{S_n - S_{n-1}}{S_{n-1}} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = i. \qquad (1.14)$$

does not depend on n and is equal to the nominal one.

The increased sum S_n is proportional to the initial sum S_0. The proportionality coefficient $(1 + i)^n$is called **the multiplicative factor**.

Note that any moment of time t_k can be taken as a “zero” one. In this case, the formula (1.13) takes the form:

$$S_n = S_k(1 + i_T)^{n-k}. \qquad (1.15)$$

где i_T — the interest rate for period T, constant for all periods.

Assuming $t = nT$, formula (1.15) can be rewritten as follows:

$$S(t) = S_0(1 + i_T)^{t/T}. \qquad (1.16)$$

Using the formula (1.16), it is possible to calculate the accrued amount at any time t (not necessarily a multiple of the accrual period T. In this case, it is said that **a continuous model** of a cumulative account is used in the compound interest scheme. In the future, unless otherwise specified, this particular model will be used.

The analogue of formula (1.15) in the continuous model is the following formula:

$$S(t) = S(\tau)(1 + i_T)^{(t-\tau)/T}. \qquad (1.17)$$

When interest is accrued once a year (or more generally, if the interest accrual period coincides with the main time unit), formulas (1.16) and

(1.17) are simplified:

$$S(t) = S_0(1+i)^t; \tag{1.18}$$

$$S(t) = S(\tau)(1+i)^{t-\tau}, \tag{1.19}$$

where i — annual interest rate.

The interest for n years can be represented as

$$I_n = S_0[(1+i)^n - 1]. \tag{1.20}$$

1.3. Multiple interest accrual

If compound interest accrual occurs several times a year (m) (quarterly, monthly, etc.), then after t years the accrued amount will become equal to:

a) in the case of simple interest:

$$S(t,m) = S_0\left(1 + \frac{i}{m}mt\right) = S_0(1+it),$$

that is, the accrued amount does not depend on the multiplicity of accrual. This conclusion will be used by us when considering the continuous accrual of interest in the case of simple interest;

b) in the case of compound interest:

$$S(t,m) = S_0\left(1 + \frac{i}{m}\right)^{mt}. \tag{1.21}$$

In the next paragraph it will be shown that the effective interest rate in the compound interest scheme increases with increasing multiplicity of accrual and reaches a maximum with continuous accrual of interest. At the same time, the effective interest rate practically reaches saturation at $m \geq 6 \div 10$, i.e. above this multiplicity of accrual, the growth of the effective interest rate slows down sharply.

1.4. Continuous interest accrual

If the frequency of accrual of compound interest m increases indefinitely, then there is **a continuous accrual of interest**. In this case, after t years, the accumulated amount will be equal to:

a) in the case of simple interest:

$$S(t,\infty) = \lim_{m\to\infty} S(t,m) = \lim_{m\to\infty} S_0\left(1+\frac{i}{m}mt\right) = S_0(1+it),$$

that is, the accrued amount remains the same as with a single interest charge. This conclusion was made by us in the case of multiple accrual of interest. Both conclusions are related to the fact that with any multiplicity of interest accrual, accrual is made on the initial amount in proportion to the time of the deposit;

b) in the case of compound interest:

$$S(t,\infty) = \lim_{m\to\infty} S(t,m) = \lim_{m\to\infty} S_0\left(1+\frac{i}{m}\right)^{mt} =$$

$$= \lim_{m\to\infty} S_0\left(1+\frac{i}{m}\right)^{mit/i} = S_0 e^{it}. \quad (1.22)$$

The interest rate i in the formula (1.22) is also called **the intensity of the growth rate** and is usually denoted by the letter δ. With this in mind, this formula can be written as:

$$S(t) = S_0 e^{\delta t}. \quad (1.23)$$

The intensity of the growth rate δ is characterized by the relative increase in the accrued amount over an infinitesimal period of time

$$S(t) = S_0 \cdot e^{\delta t} \cdot \delta = S(t) \cdot \delta, \quad (1.24)$$

or

$$\frac{dS(t)}{S(t)} = \delta \cdot dt. \quad (1.25)$$

If the intensity of the growth rate depends on time, then S(t) can be obtained as a solution of the differential equation (1.25). Finding the integral of both parts (1.25), we get

$$\ln S(t) - \ln S_0 = \int_0^t \delta \, dt. \qquad (1.26)$$

It means that

$$S(t) = S_0 e^{\int_0^t \delta \, dt}. \qquad (1.26)$$

Example 1.1. The bank has a deposit of 1000 $ at 10% per annum under the compound interest scheme. Find the amount of the deposit in three years when interest is accrued 1, 4, 6, 12 times a year and in the case of continuous interest accrual.

By the formula (1.21) we have

$$S_{3/1} = 1000(1 + 0.1)^3 = \$1{,}331,$$
$$S_{3/4} = 1000\left(1 + \frac{0.1}{4}\right)^{3\cdot 4} = \$1{,}344.9,$$
$$S_{3/6} = 1000\left(1 + \frac{0.1}{6}\right)^{3\cdot 6} = \$1{,}346.5,$$
$$S_{3/12} = 1000\left(1 + \frac{0.1}{12}\right)^{3\cdot 12} = \$1{,}348.2.$$

In the case of continuous accrual of interest, the formula (1.22) must be used

$$S_{3/\infty} = 1000e^{0.1\cdot 3} = \$1{,}349.6. \qquad (1.27)$$

Interest for three years amounted to ($):

— with a single accrual of interest — 331;
— with a four–time accrual — 344.9;
— with a six–time accrual — 346.5;
— at twelve–time accrual —348.2;
— with continuous accrual— 349.6.

We come to the conclusion that the accrued amount, as well as the amount of interest money, in the compound interest scheme increases with increasing multiplicity of accrual and reaches a maximum with continuous accrual of interest. Moreover, the growth rate of both values decreases with an increase in the multiplicity of accrual. (For proof of these facts, see paragraph 1.13.)

Example 1.2. An amount of $3,000 was put on the bank deposit on March 10 at 15% per annum under the compound interest scheme. What amount will the depositor receive on October 22?

We use the formula (1.13) for the accrual according to the scheme of compound interest:

$$S_n = S_0(1+i)^n$$

Duration of the financial transaction (in fractions of the period)

$$n = \frac{t}{T} = \frac{20 + 30 \cdot 6 + 22}{365} = 0.608$$

(it is assumed that there are 30 days in a month, 365 in a year), so we have

$$S_n = S_0(1+i)^n = 3{,}000(1+0.15)^{0.608} = \$3{,}266.07.$$

So, on October 22, the depositor will receive $3,266.07.

1.5. Equivalence of interest rates in the compound interest scheme

Let's consider interest rates, using which a model of the percentage growth of the accrual in the compound interest scheme can be described.

If the accrual rate i for the accrual period T is specified, then

$$S_t = S_0(1+i)^{\frac{t}{T}}. \tag{1.28}$$

If the annual rate j and the multiplicity of accrual (during the year) p are specified, then

$$S_t = S_0(1 + j/p)^{pt}. \quad (1.29)$$

In this case, it is said that j is **the nominal rate**.

With continuous accrual of interest

$$S_t = S_0 e^{\delta t}. \quad (1.30)$$

And the intensity of the growth rate δ is also called **the continuous nominal rate**.

Finally, if **the effective rate** i_{eff} is specified, the accrued amount is determined by the formula

$$S_t = S_0\left(1 + i_{eff}\right)^t. \quad (1.31)$$

Formulas (1.28)—(1.31) have the form:

$$S_t = S_0 a^t. \quad (1.32)$$

where a —the corresponding (normalized) **accrual coefficient**.

In each case, a is obtained as an annual accrual factor.

Rates are called **equivalent** if they have the same growth coefficients. This means that with the same initial amount, the amounts accumulated by any point in time t at equivalent rates are the same.

The growth coefficient a and the effective i_{eff} rate are related by a simple ratio

$$a = 1 + i_{eff} \quad (1.33)$$

With this in mind, we can say that the rates are equivalent if the effective rates equivalent to them coincide.

It is not difficult to specify the ratios that ensure the equivalence of rates of various types.

If j is the annual rate at the multiplicity of accrual p, then it is equivalent to the rate $iT = i_{1/p} = j/p$ for the period $T = 1/p$. The equivalent effective rate is determined by the formula

$$i_{eff} = \left(1 + \frac{j}{p}\right)^p - 1, \tag{1.34}$$

or

$$i_{eff} = (1 + i_T)^{1/T} - 1. \tag{1.35}$$

Accordingly,

$$j = p\left((1 + i_{eff})^{\frac{1}{p}} - 1\right); \tag{1.36}$$

$$i_T = \left(1 + i_{eff}\right)^T - 1. \tag{1.37}$$

With continuous accrual of interest, we get:

$$i_{eff} = e^{\delta} - 1, \tag{1.38}$$

$$\delta = \ln\left(1 + i_{eff}\right). \tag{1.39}$$

iT and *iT* interest rates with accrual periods T_1 and T_2, respectively, are equivalent if

$$\left(1 + i_{T_1}\right)^{\frac{1}{T1}} = \left(1 + i_{T_2}\right)^{\frac{1}{T2}}. \tag{1.40}$$

If different interest rates $i_1, i_2, \ldots, i_m$ are set at different intervals of interest accrual $n_1, n_2, \ldots, n_m$, then the accumulated amount Sn for the time $n_1 + n_2 + \ldots + n_m$ will be equal to

$$S_n = S_0(1 + i_1)^{n_1}(1 + i_2)^{n_2} \ldots (1 + i_m)^{n_m} = S_0 \prod_{k=1}^{m}(1 + i_k)^{n_k}. \tag{1.41}$$

1.6. Comparison of accruals at simple and compound interest rates

At the same interest rate, the increase according to the simple interest scheme is more advantageous for an accrual period of less than a year. For an accrual period of more than a year, it is more advantageous to be accrued according to the compound interest scheme (Fig. 1.1). For proof, it is sufficient to show that

$$f(t) = (1+i)^t < g(t) = 1 + ti, \quad if\ 0 < t < 1;$$

$$f(t) = (1+i)^t > g(t) = 1 + ti, \quad if\ t > 1$$

For the second order derivative of the function $f(t)$we have $f''(t) = =\ln^2(1+i)(1+i)^t > 0$, therefore, $f(t)$ is a convex down function at $t > 0$, and $g(t) = 1 + it$ is a chord to $f(t)$, since the equation $f(t) = g(t)$ or $(1 + i)^t = 1 + + ti$ has two solutions: $t = 0$ and $t = 1$. Hence $(1 + i)^t < 1 + ti$ if $0 < t < 1$, and $(1 + i)^t > 1 + ti$ if $t > 1$.

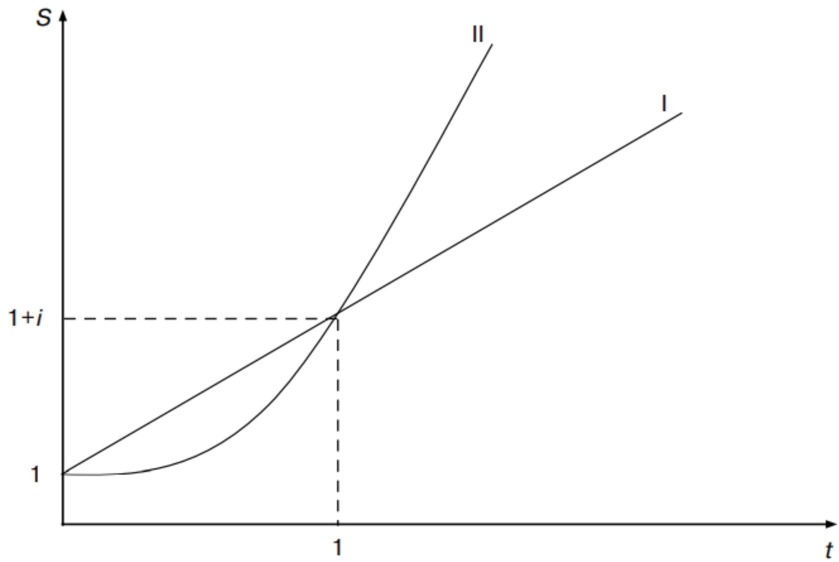

Figure 1.1. Accrual at simple (I) and complex (II) interest rates

Important notice 1

When interest is calculated once a year, simple interest is more effective than compound interest with a deposit term of up to one year, and compound interest is more effective with a deposit term of more than one year. This can be seen from Fig.1.1.

Does this condition change with multiple accrual percent and if it changes, then how. We leave readers to verify the following: with multiple accrual interest, simple interest is more effective than compound interest before the first accrual of interest. By other words at monthly accrual of interest, simple interest is more effective than compound interest during the first month; at quarterly accrual of interest, simple interest is more effective than compound interest during the first quarter; when interest is accrual semi–annually, simple interest is more effective than compound interest during the first half of the year, and so on.

Important notice 2

Concerning the continuous interest, they are more effective than simple or compound interest for any term of the deposit.

1.7. Discounting and interest deduction

Discounting and interest deduction are in a certain sense the reverse of interest accrual. There are *mathematical discounting* and *bank accounting*.

Mathematical discounting allows you to find out what initial amount S_0 needs to be invested in order to receive, after t years, the amount $S_t$when interest is accrued on S_0 at the rate i.

In the case of simple interest

$$S_o = S_t/(1 + ti). \qquad (1.42)$$

In the case of compound interest

$$S_o = S_t/(1 + i)^t. \qquad (1.43)$$

In the case of continuous accrual of interest

$$S_o = S_t/e^{\delta t}. \quad (1.43)$$

The value S_0 is called **the present value** of the value S_t. The values i and δ, which were previously called interest rates, now mean **discount rates**.

Bank accounting is the purchase by a bank of monetary obligations at a price less than the nominal amount specified in them.

An example of monetary obligations is **a promissory note** — a promissory note containing an obligation to pay a certain amount of money (nominal value) within a certain period.

In the case of a bank purchase of a bill, they say that the latter is *taken into account*, and the amount is paid to the client

$$S_n = S_o - I_n \quad (1.45)$$

where S_0 — nominal amount of the promissory note;

S_n — the purchase price of the promissory note by the bank for n years before maturity;

I_n — discount, or the bank's income (interest money).

$$I_1 = S_o d. \quad (1.46)$$

where d — discount rate (as a rule, through d we will further denote the discount rate).

The discount rate can be simple and complex, depending on which scheme is used — simple or compound interest. In the case of simple interest, the sequence of amounts remaining after the discount $\{S_n\}$ forms a decreasing arithmetic progression with a common term $S_n = S_0(1 - nd)$ equal to the amount that the client will receive n years before repayment.

In the case of compound interest, the sequence of amounts remaining after the discount $\{S_n\}$ forms a decreasing geometric progression with a common term $S_n = S_0(1 - d)^n$ equal to the amount that the client will receive *n* years before repayment.

1.7.1. Comparison of discounting at complex and simple discounting rates

For the bank, the discounting situation is the inverse of the accrual. For example, if the accounting period is less than one year, it is more profitable for the bank to discount at a complex discount rate (Figure 1.2) (the accrual — at a simple one (Figure 1.1)), and if the accounting period is more than one year — at a simple discount rate (Figure 1.2) (the accrual — at a complex one (see Figure 1.1)).

For proof, it is sufficient to show that

$$f(t) = (1 + d)^t < g(t) = 1 - td, \quad if\ 0 < t < 1;$$

$$f(t) = (1 + d)^t > g(t) = 1 - td, \quad if\ t > 1$$

For the second derivative of the function $f(t)$ we have f "(t) = $\ln^2(1–d)$• •$(1 - d)^t > 0$, hence $f(t)$ is a convex down function at $t > 0$, and $g(t) = 1–id$ is a chord to $f(t)$, since the equation $f(t) = g(t)$ or $(1 - d)^t = 1 - dt$ has two solutions: $t = 0$ and $t = 1$. Hence, $(1 - d)^t < 1 - dt$ if $0 < t < 1$, and $(1 - d)^t > 1 - dt$ if $t >1$.

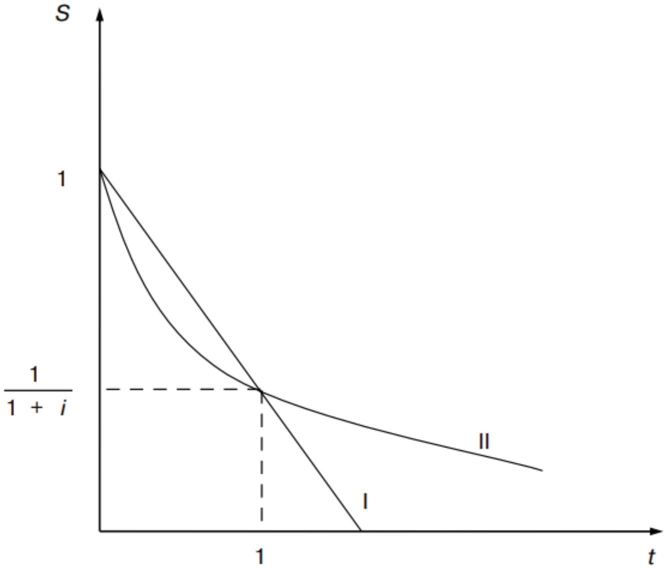

Figure 1.2. Discounting at simple (I) and complex (II) interest rates

1.7.2. Effective discount rate

Let d_{eff} be the annual (effective) discount rate (discount rate) with a multiplicity of accrual m. The equivalent effective discount rate is determined based on the equivalence principle

$$S_o\left(1 - d_{eff}\right)^n = S_o\left(1 - \frac{d}{m}\right)^{n \cdot m}, \qquad (1.47)$$

hence

$$1 - d_{eff} = \left(1 - \frac{d}{m}\right)^m, \qquad (1.48)$$

or

$$d_{eff} = 1 - \left(1 - \frac{d}{m}\right)^m, \qquad (1.49)$$

Inversely, the discount rate d is expressed in terms of the effective discount rate d_{eff}:

$$d = m\left(1 - \sqrt[m]{1 - d_{eff}}\right). \tag{1.50}$$

The discount rate *d* and the interest rate *i* lead to the same result over a period of time *t* if

$$S_0(1 + it) = S_t \text{ and } S_0 = S_t\,(1 - dt), \tag{1.51}$$

$$(1 + it)(1 - dt) = 1. \tag{1.52}$$

The last equality can be transformed as follows:

$$d = \frac{i}{1+it}; \;\; i = \frac{d}{1-dt}. \tag{1.53}$$

We can also write down the relationship between the nominal rates of increment and discounting

$$\left(1 + \frac{i}{m}\right)^m = \left(1 - \frac{d}{p}\right)^{-p}, \tag{1.54}$$

since both parts of the equation are equal to $(1 + i)$.

If $m = p$, we have

$$\left(1 + \frac{i}{m}\right)^m = \left(1 - \frac{d}{m}\right)^{-m}, \tag{1.55}$$

hence

$$\frac{i}{m} - \frac{d}{m} = \frac{i}{m} \cdot \frac{d}{m}. \tag{1.56}$$

If different discount rates $i_1, i_2, \ldots, i_m$ are set at different discount intervals $n_1, n_2, \ldots, n_m$, then Sn for the time $n_1 + n_2 + \ldots + n_m$ will be equal to

$$S_n = S_0(1 + i_1)^{-n_1}(1 + i_2)^{-n_2} \ldots (1 + i_m)^{-n_m} =$$

$$= S_0 \prod_{k=1}^{m}(1 + i_k)^{-n_k}, \tag{1.57}$$

Example 1.3. What amount should be deposited at 12% per annum in order to receive $ 500,000 in five years?

$$S_0 = \frac{S_n}{(1+i)^n} = \frac{500{,}000}{(1+0{,}12)^5} = \frac{500{,}000}{1.7623} = \$\ 283{,}713.43$$

Example 1.4. A promissory note worth $100,000 is accounted for four years before maturity at a compound discount rate of 15% per annum. It is required to find the amount received by the holder of the promissory note and the amount of the discount.

The amount received by the holder of the promissory note is equal to

$$S_4 = S_0(1-d)^4 = 100{,}000(1-0.15)^4 = \$52{,}200.6$$

The discount value is equal to

$$I_4 = S_0 - S_4 = 100{,}000 - 52{,}200.6 = \$47{,}799.3$$

1.8. Multiplying and discounting multipliers

In practice, tables of multiplying and discounting multipliers are used for calculations with simple and complex interest rates. The multiplying multiplier shows how many times the initial amount deposited in the bank at i percent per annum will increase over n years

$$M(n,i) = (1+i)^n, \tag{1.58}$$

that is, it represents the future value of one monetary unit in n years at the interest rate i. The discounting multiplier shows what part will be the initial amount deposited in the bank at i percent per annum of the amount accrued by the end of the nth year:

$$D(n,i) = \frac{1}{M(n,i)} = (1+i)^{-n}, \tag{1.59}$$

that is, it represents the *present value* of one monetary unit in n years at the interest rate i.

1.9. "Rule of 70"

This rule allows you to answer the following question: in how many years will the deposit placed in the bank at i percent per annum double? Below we will consider this Rule in the case of complex, simple, continuous interest, as well as for multiple accruals of interest. We will also consider the term of increasing the contribution by an arbitrary number of times.

1.9.1. Compound interest

The doubling of capital in the compound interest scheme at the rate of i occurs approximately in

$$T = \frac{70}{i}\ years. \tag{1.60}$$

(The rate i in (1.60) is set as a percentage.) This rule is easily derived from the compound interest formula. Actually, $2S_0 = S_0(1 + i)^T$, hence, after taking the logarithm of both parts, one gets $ln2 = Tln(1+i)$. Decomposing $ln(1+i)$ by degrees i (at small i), we get $ln(1+i) \approx i$. Therefore, $ln2 \approx iT$, and $T \approx \frac{ln2}{i}$. Finally, we get $T \approx \frac{69,3}{i} \approx \frac{70}{i}$. In practice, the "Rule of 72" is more often used, since the number 72 has more divisors than 70.

Taking into account the next (quadratic) i term in the expansion $ln(1+$ $+\,i) \approx i - \frac{i^2}{2}$ gives a result $T \approx \frac{ln2}{i\left(1-\frac{i}{2}\right)}$ that increases the term of doubling of capital $T \approx \frac{ln2}{i}\left(1 - \frac{i}{2}\right)$ by $\Delta T \approx \frac{ln2}{i}$.

Example 1.5. In how many years will the capital double in the compound interest scheme at the rate of 18% per annum?

$T = \frac{70}{i} = \frac{70}{18} = 3.89$ years.

1.10. Generalization of "Rule of 70"

1.10.1. Simple interest ("Rule of 100")

In the case of simple interest, we have

$$2S_0 = S_0(1 + Ti),$$

hence $2 = 1 + Ti$, and $T = 1/i$, or (if i is expressed in percentage):

$$T = \frac{100}{i} \text{ years.} \tag{1.61}$$

Thus, the "Rule of 70" in the case of simple percentages is replaced by the "Rule of 100".

Example 1.6. In how many years will the capital double in the simple interest scheme at a rate of 18% per annum?

$$T = \frac{100}{i} = \frac{100}{18} = 5.56 \text{ years.}$$

In the scheme of compound interest, doubling under the same conditions occurred over 3.89 years.

1.10.2. Continuous interest

In the case of continuous accrual of interest, we have

$$2S_0 = S_0 e^{iT},$$

hence $\ln 2 = Ti$, and $T = \ln 2/i$. Finally, we get

$$T \approx \frac{69{,}3}{i} \approx \frac{70}{i}. \tag{1.62}$$

This formula formally coincides with the "Rule of 70" of the case of compound interest. Note, however, that in this case the formula $T = \ln 2/i$ is accurate, unlike the case of compound interest, where the exact formula for the term of doubling capital has the form $T = \ln 2/\ln(1+i)$, and the formula $T = \ln 2/i$ is obtained after decomposition in a series of small i functions $\ln(1 + i)$.

1.10.3. Multiple accrual of interest

In the case of accrual (m times) of interest for the period, we have

$$2S_0 = S_0\left(1+\frac{i}{m}\right)^{mT}.$$

Thus, in this case we have the exact formula

$$T = \frac{\ln 2}{m\ln\left(1+\frac{i}{m}\right)}. \tag{1.63}$$

Decomposing $ln(1+\frac{i}{m})$ by degrees *i,* we get $ln\left(1+\frac{i}{m}\right) \approx \frac{i}{m}$. Therefore, $T = \frac{ln2}{i}$. Finally, we get $T \approx \frac{69.3}{i} \approx \frac{70}{i}$. That is, with a multiple accrual of interest, we get, as in the case of a single accrual of compound interest, "Rule of 70". From the previous one, it is known that the accrued amount with a multiple of interest accrual increases with an increase in the multiplicity of accrual m, therefore, the period of capital doubling should decrease with an increase in m, which contradicts the "Rule of 70", since the formula $T \approx 70/i$ does not include m. This contradiction is connected with the use of only the first approximation in the decomposition $ln(1+\frac{i}{m})$ by degrees *i.* Already the consideration of the next (quadratic) *i*–term in the decomposition

$ln\left(1+\frac{i}{m}\right) \approx \frac{i}{m} - \frac{i^2}{2m^2}$ gives the result depending on m:

$$T \approx \frac{\ln 2}{m(\frac{i}{m}-\frac{i^2}{2m^2})} = \frac{\ln 2}{i\left(1-\frac{i}{2m}\right)}. \tag{1.64}$$

It is easy to see that with the growth of m, the term of doubling capital decreases from

$$T_1 \approx \frac{\ln 2}{i\left(1-\frac{i}{2}\right)} \tag{1.65}$$

(the period of doubling of capital in the case of a single accrual of interest) until

$T_\infty \approx \frac{\ln 2}{i}$ (1.66)

(the case of continuous accrual of interest).

1.11. Capital increase by an arbitrary number of times

Let's consider a more general problem about the term of increasing the deposit by an arbitrary number of times (n) at a given interest rate i.

Simple interest

In the case of simple interest, we have

$nS_0 = S_0(1 + Ti)$.

hence $n = 1 + Ti$, and

$$T = \frac{n-1}{i}. \quad (1.67)$$

For example, at a rate of 10% per annum, the deposit will grow 4 times for

$T = \frac{n-1}{i} = \frac{3}{0.1} = 30$ years.

Compound interest

Consider the problem of increasing capital by an arbitrary (n) number of times in the compound interest scheme at a given interest rate i. This rule is easily derived from the compound interest formula.

Actually, $nS_0 = S_0(1 + i)^T$, hence $\ln n = T\ln(1 + i)$. Decomposing $\ln(1 + i)$ by degrees i, we get $\ln(1 + i) \approx i$. Therefore, $\ln n \approx iT$, and

$$T \approx \ln n / i. \quad (1.68)$$

Taking into account the next (quadratic) by i term in decomposing of $\ln(1 + i) \approx i - i^2/2$ gives the result

$$T \approx \frac{\ln n}{i\left(1-\frac{i}{2}\right)}, \quad (1.66)$$

which increases the term of capital growth by n times $T \approx \frac{\ln n}{i}\left(1+\frac{i}{2}\right)$ by $\Delta T \approx \frac{\ln n}{2}$.

Thus, when considering the problem of increasing capital by an arbitrary number of times (n) in the scheme of compound interest at a given interest rate i, it is necessary in the "Rule of 70" only to make a substitution

$$\ln 2 \rightarrow \ln n. \qquad (1.70)$$

Example 1.7. For how many years, at a rate of 10% per annum, the deposit will grow 4 times in the scheme of simple interest?

$$T = \frac{100}{i} = \frac{100}{18} = 5.56 \text{ years}$$

$$T \approx \frac{\ln n}{i} \approx 10 \ln 4 \approx 13.86 \approx 14 \text{ years}$$

Continuous interest

In the case of the continuous accrual of interest, we have

$nS_0 = S_0 e^{iT}$, hence $\ln n = Ti$ and,

$$T = \ln n / i . \qquad (1.71)$$

We have obtained a formula that formally coincides with the case of compound interest. Note, however, that in this case the formula $T = \ln n/i$ is accurate, unlike the case of compound interest, where the exact formula for the term of capital increase by n times has the form $T = \ln n/\ln(1 + i)$, and the formula $T = \ln n/i$ is obtained after decomposition into a series of small i functions $\ln(1 + i)$. The situation is similar to the case of capital doubling discussed above.

Multiple interest accrual

With the interest accrual m times for the period, we have:

$$nS_0 = S_0\left(1+\frac{i}{m}\right)^{mT}.$$

Hence

$$\ln n = mT \ln\left(1 + \frac{i}{m}\right).$$

Thus, in this case we have the exact formula

$$T = \frac{\ln n}{m \ln\left(1+\frac{i}{m}\right)}. \quad (1.72)$$

Decomposing $\ln\left(1 + \frac{i}{m}\right)$ by degrees i, we get $\ln\left(1 + \frac{i}{m}\right) \approx \frac{i}{m}$. Therefore,

$$T = \frac{\ln n}{i}. \quad (1.73)$$

From the previous one, it is known that the accrued amount with a multiple of interest accrual increases with an increase in the multiplicity of accrual m, therefore, the term of capital increase by n times should decrease with an increase in m, which contradicts the formula (1.73), since it does not include m. This contradiction is connected with the use of only the first approximation in the decomposition of $\ln(1 + i/m)$ in powers of i. Already taking into account the next (quadratic) term in i in the decomposition of $\ln\left(1 + \frac{i}{m}\right) \approx \frac{i}{m} - \frac{i^2}{2m^2}$ gives a result depending on m:

$$T \approx \frac{\ln n}{m\left(\frac{i}{m} - \frac{i^2}{2m^2}\right)} = \frac{\ln n}{i\left(1 - \frac{i}{2m}\right)}. \quad (1.74)$$

It is easy to see that with the growth of m, the term of capital increase by n times decreases from $T_1 \approx \frac{\ln n}{i\left(1-\frac{i}{2}\right)}$. (the term of capital increase by n times in the case of one–time interest accrual) to $T_\infty = \frac{\ln n}{i}$. (the case of continuous interest accrual).

1.12. The impact of inflation on the interest rate

1.12.1. Fisher's Formula

It is said that inflation is a fraction of α per year if the cost of a product increases by $(1 + \alpha)$ per year times. Inflation reduces the real interest rate.

With inflation, money depreciates by 1+ α times, so the real equivalent of the amount accumulated over the year $S=S_0(1+i)$ will be (1 + α) times less:

$$S_\alpha = \frac{S}{1+\alpha} = \frac{S_0(1+i)}{1+\alpha} = \frac{S_0(1+\alpha-\alpha+i)}{1+\alpha} = S_0\left(1+\frac{i-\alpha}{1+\alpha}\right) = S_0(1+i_\alpha). \quad (1.75)$$

In (1.75), we denoted by i_α the inflation–adjusted interest rate (i is still the interest rate without inflation), for which we obtained the following expression (Fischer's formula):

$$i_\alpha = \frac{i-\alpha}{1+\alpha}. \quad (1.76)$$

With low inflation, the real interest rate is less than the nominal rate by about the amount of inflation. If inflation is high enough, i_α can become negative. In such a situation, the lender will work at a loss, and the borrower will enrich himself. To prevent this from happening, it is necessary to adjust the nominal interest rate i at which the increase occurs (it should at least exceed inflation: $i > \alpha$ and $i_\alpha > 0$). In order for the nominal interest rate i to provide the real interest rate i_α at annual inflation α, it must satisfy the equation $i = \alpha + i_\alpha(1 + \alpha)$. At low i and α cross member $\alpha \cdot i_\alpha$ can be neglected in this (often rough) approximation of the nominal interest rate i is equal to the sum of the effective interest rate i_α and the rate of inflation α: $i \approx \alpha + i_\alpha$. At the same time, the effective interest rate i_α is equal to the nominal interest rate i, reduced by the rate of inflation α:

$$i_\alpha \approx i - \alpha. \quad (1.77)$$

Example 1.8. What rate should the bank set so that with inflation of 8% per annum it could have a 10% yield?

Let 's solve the Fisher equation $i_\alpha = \frac{i-\alpha}{1+\alpha}$ with respect to i:

$$i = i_\alpha(1+\alpha) + \alpha = 0.1(1+0.08) + 0.08 = 0.188 = 18.8\%.$$

So, the answer of 18.8% is 0.8% higher than the simple answer of 18% obtained from (1.77) by simply adding the inflation rate and the effective interest rate.

1.12.2. Inflation rate for several periods

Let the inflation rates for consecutive time periods $t_1, t_2, \ldots, t_n$ be equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively. Let's find the inflation rate α for the period $t = t_1 + t_2 + \ldots + t_n$. Common sense suggests that the inflation rate is an additive quantity, so that α, at least approximately, is equal to the sum of the inflation rates $\alpha_1, \alpha_2, \ldots, \alpha_n$:

$$\alpha \approx \alpha_1 + \alpha_2 + \cdots + \alpha_n. \quad (1.78)$$

Below we will get an exact expression for the inflation rate for the total time period t and see how it differs from the intuitive result (1.78).

At the end of the first period, the accrued amount will be equal to $S_1 = S_0(1 + i)$, and with adjustment for inflation – $S_1 = S_0(1 + i)^{t_1}/(1 + +\alpha_1)$. At the end of the second period, the accrued amount will be equal to $S_2 = S_0(1 + i)^{t_1+t_2}$, and with adjustment for inflation – $S_2 = S_0(1 + + i)^{t_1+t_2}/(1 + \alpha_1)(1 + \alpha_2)$. At the end of the n–th period, the accrued amount will be equal to $S_\mathrm{n} = S_0(1 + i)^{t_1+t_2+\cdots+t_\mathrm{n}}$, and with adjustment for inflation:

$$S_\mathrm{n} = S_0(1 + i)^{t_1+t_2+\cdots+t_\mathrm{n}}/(1 + \alpha_1)(1 + \alpha_2) \cdot\ldots\cdot (1 + \alpha_\mathrm{n}). \quad (1.79)$$

On the other hand, at the rate of inflation α at the end of period t, the accumulated amount will be equal to

$$S_\mathrm{n} = S_0(1 + i)^t/(1 + \alpha). \ (1.80)$$

Equating the right parts of (1.79) and (1.80), we get

$$(1 + \alpha_1)(1 + \alpha_2) \cdot\ldots\cdot (1 + \alpha_\mathrm{n}) = 1 + \alpha. \quad (1.81)$$

Hence

$$\alpha = (1 + \alpha_1)(1 + \alpha_2) \cdot\ldots\cdot (1 + \alpha_\mathrm{n}) - 1. \quad (1.82)$$

A strict proof of this formula is not difficult to obtain by mathematical induction.

Note that the inflation rate for *n*–periods does not depend on the duration of the constituent periods, nor on the period *t*.

For equal inflation rates $\alpha_1 = \alpha_2 = \ldots = \alpha_n$ (while the time intervals may remain arbitrary and not equal to each other) we have

$$\propto = (1 + \alpha_1)^n - 1. \qquad (1.83)$$

Let's analyze the difference between the obtained results (1.82) and (1.83) from the intuitive expression (1.78) and the reason for this using the example of a time period consisting of two periods. Let the inflation rates for two consecutive time periods t_1, t_2 be equal to α_1, α_2 respectively. Then, according to the formula (1.82), the inflation rate α for the period $t = t_1 + t_2$ is equal to

$$\propto = (1 + \alpha_1)(1 + \alpha_2) - 1 = \alpha_1 + \alpha_2 + \alpha_1\alpha_2. \qquad (1.84)$$

As we can see, the difference from the sum of inflation rates is the appearance of the cross term $\alpha_1\alpha_2$. Although this term is a small quantity of a higher order of smallness compared to α_1 and α_2, provided that they are small, in practice it is necessary to take them into account.

1.12.3. Synergetic effect

We have obtained an example of the so–called synergetic effect (i.e., the effect (result) of two (several) parts is greater than the additive effect (simple summation)). Responsible for the synergetic effect is the emerging cross member $\alpha_1\alpha_2$. It leads to the fact that the inflation rate for two consecutive periods of time $t = t_1 + t_2$ turns out to be greater than the sum of the inflation rates.

Example 1.9. Let the inflation rates for two consecutive time periods t_1 and t_2 be 10 and 20%, respectively. Then, according to the formula (1.82), the inflation rate α for the period $t = t_1 + t_2$ is equal to:

$$\propto = (1 + \alpha_1)(1 + \alpha_2) - 1 = \alpha_1 + \alpha_2 + \alpha_1\alpha_2 = 0.1 + 0.2 + 0.1 \cdot 0.2 \\ = 0.32,$$

that is, 32%. Thus, the difference from the sum of the inflation rates is 2%.

Example 1.10. Let the inflation rate for the year α be 20%. Find the inflation rate for the quarter α_1, provided it is constant.

Let 's apply the formula

$\alpha = (1 + \alpha_1)^n - 1.$

We have

$\alpha + 1 = (1 + \alpha_1)^n, \qquad \alpha_1 + 1 = \sqrt[n]{1 + \alpha}\,,$

and, finally, $\alpha_1 = \sqrt[n]{1 + \alpha} - 1.$ Substituting α = 20% = 0,2, n = 4 into this expression, we get for the quarterly inflation rate

$\alpha_1 = \sqrt[4]{1 + \alpha} - 1 = \sqrt[4]{1.2} - 1 \approx 1.0466 - 1 = 0.0466 \approx 4.66\%$

As you can see, the inflation rate for the quarter turned out to be lower than the one obtained by simply dividing the annual inflation rate by four, i.e. 20%:4 = 5%. The difference is 0.36%.

Example 1.11. Let's solve the inverse problem. Let the inflation rate for the month α_1 be 2%. Find the rate of inflation for the year α provided that the rate of inflation is constant throughout the year. Let 's apply the formula $\alpha = (1 + \alpha_1)^n - 1.$ Substituting α = 2% = 0,02, *n* = 12 into it, we get for the annual rate of inflation

$\alpha = (1 + \alpha_1)^{12} - 1 = (1 + 0.02)^{12} - 1 =$

$= (1{,}02)^{12} - 1 \approx 1.268 - 1 = 0.268 = 26.8\%.$

We see that the inflation rate for the year turns out to be higher than the one obtained by simply multiplying the monthly inflation rate by 12, i.e. 2%·12 = 24%. The difference is 2.8%.

Two conclusions can be drawn from the last two examples:

1) the inflation rate for the total period exceeds the sum of the inflation rates for the constituent periods;

2) the rate of inflation for the constituent period turns out to be less than the corresponding share of the rate of inflation for the total period.

1.13. Effective interest rate

In paragraph 1.7.2, the effective discount rate was considered. Here, attention will be paid in detail to the effective interest rate.

The effective interest rate is the amount paid to the borrower (investor) at the end of the accrual period for each unit amount borrowed (invested) at the beginning of the period. Denoting the increased value of the unit amount at time t through a_t, the interest rate through i, and the increased value of the full amount through S_t, for the first period we have accruals

$$i_1 = \frac{(1+i)}{1} = \frac{a_1 - a_0}{a_0} = \frac{S_1 - S_0}{S_0}, \tag{1.85}$$

for the n–th accrual period

$$i_n = \frac{a_n - a_{n-1}}{a_{n-1}} = \frac{S_n - S_{n-1}}{S_{n-1}}. \tag{1.86}$$

It can be seen from formula (1.86) that the effective interest rate can and does vary depending on the accrual period number, however, as will be shown below, in the very important and widely used case of compound interest, the effective interest rate remains constant for all accrual periods, i.e. for all $n \geq 1$.

The effective interest rate may differ significantly from the nominal rate declared by the bank and appearing in the bank deposit or loan agreement. With deposits, it usually turns out to be less than the nominal rate (due to inflation, for example) or equal to it (however, it can also be higher than the nominal rate with a multiple of interest accrual), and when taking loans — higher. The effective rate depends on many factors: the multiplicity of interest accrual, the rate of inflation, the number of the accrual period, the availability and magnitude of transaction costs, taxes, and many others. Below we will consider almost all of the cases listed.

1.13.1. Simple and compound interest

Effective interest rate in the compound interest scheme for the n–th accrual period

$$i_n = \frac{S_n - S_{n-1}}{S_{n-1}} = \frac{S_0(1+i)^n - S_0(1+i)^{n-1}}{S_0(1+i)^{n-1}} = 1 + i - 1 = i \qquad (1.87)$$

does not depend on n and is equal to the nominal.

The effective interest rate in the simple interest scheme for the n–th accrual period

$$i_n = \frac{a_n - a_{n-1}}{a_{n-1}} = \frac{S_n - S_{n-1}}{S_{n-1}} = \frac{(1+in)-(1+i(n-1))}{1+i(n-1)} = \frac{i}{1+i(n-1)} \qquad (1.88)$$

decreases with the growth of n.

1.13.2. Multiple interest accrual

With the interest accrual m times, the amount accumulated over t years is equal to

$$S(t,m) = S_0\left(1 + \frac{i}{m}\right)^{mt}. \qquad (1.89)$$

We will find the effective interest rate in the case of multiple interest accrual. It can be defined as an interest rate that, with a single (for a period) interest accrual, leads to the same increased amount as with an m–time. Equating the increased values

$$S(t,m) = S_0\left(1 + \frac{i}{m}\right)^{mt} = S_0\left(1 + i_{eff}\right)^t, \qquad (1.90)$$

we will get an effective interest rate in case of multiple interest accrual

$$i_{eff} = \left(1 + \frac{i}{m}\right)^m - 1. \qquad (1.91)$$

We show that the effective interest rate in the compound interest scheme increases with increasing multiplicity of accrual and reaches a maximum with continuous accrual of interest. At the same time, the effective interest rate practically reaches saturation at $m \geq 6 \div 10$, i.e. above this multiplicity of accrual, the growth of the effective interest rate slows down sharply. To prove that the effective interest rate increases with an increase in the multiplicity of accrual, it is necessary to show that the derivative of the effective interest rate by the multiplicity of accrual

$$\frac{di_{eff}}{dm} > 0. \tag{1.92}$$

To prove that the growth of the effective interest rate slows down with the growth of m and reaches saturation, it is necessary to show that the second derivative of the effective interest rate by the multiplicity of accrual

$$\frac{d^2 i_{eff}}{dm^2} < 0 \; and \; m \rightarrow \; \infty \frac{d^2 i_{eff}}{dm^2} \rightarrow 0. \tag{1.93}$$

1) Let's show that the derivative of the effective interest rate by the multiplicity of accrual $\frac{di_{eff}}{dm} > 0$ (at least at low nominal rates $i \leq 1$).

$$i_{eff} = \left(1 + \frac{i}{m}\right)^m - 1; \tag{1.94}$$

$$\frac{di_{eff}}{dm} = \frac{d}{dm}\left[\left(1 + \frac{i}{m}\right)^m - 1\right] = \frac{d}{dm}\left(1 + \frac{i}{m}\right)^m. \tag{1.95}$$

Then we use the equation

$$\ln\left(1 + \frac{i}{m}\right)^m = m \ln\left(1 + \frac{i}{m}\right). \tag{1.96}$$

After that we differentiate both parts of the equation by m

$$\frac{\frac{d\left(1+\frac{i}{m}\right)^m}{dm}}{\left(1+\frac{i}{m}\right)^m} = \ln\left(1 + \frac{i}{m}\right) - \frac{i}{i+m}. \tag{1.97}$$

Hence

$$\frac{d\left(1+\frac{i}{m}\right)^m}{dm} = \left(1 + \frac{i}{m}\right)^m \left[\ln\left(1 + \frac{i}{m}\right) - \frac{i}{i+m}\right]. \tag{1.98}$$

If i is small, decomposing the terms in square brackets by degrees $\frac{i}{m}$ to terms of the second order $\left(\frac{i}{m}\right)^2$, we have

$$\ln\left(1+\frac{i}{m}\right) \approx \frac{i}{m} - \frac{i^2}{2m^2}, \quad \frac{i}{i+m} \approx \frac{i}{m} - \frac{i^2}{m^2} \tag{1.99}$$

Substituting the resulting expressions in (1.98), we get

$$\frac{di_{eff}}{dm} = \frac{d\left(1+\frac{i}{m}\right)^m}{dm} \approx \left(1+\frac{i}{m}\right)^m \left[\frac{i}{m} - \frac{i}{i+m}\right] = \left(1+\frac{i}{m}\right)^m \frac{i^2}{2m^2} > 0. \tag{1.100}$$

We have proved that the effective interest rate increases with increasing multiplicity of accrual. Note that in all other manuals on financial mathematics, this fact is only illustrated by examples, as is the slowdown of the growth rate of the effective interest rate with an increase in the multiplicity of accrual.

2. To prove the latter, we find the second order derivative $\frac{d^2 i_{eff}}{dm^2}$, differentiating both parts of equation (1.98) by m,

$$\frac{d^2\left(1+\frac{i}{m}\right)^m}{dm^2} = \left(\left(1+\frac{i}{m}\right)^m\right)' \left[\ln\left(1+\frac{i}{m}\right) - \frac{i}{i+m}\right] +$$

$$+\left(1+\frac{i}{m}\right)^m \left[\frac{i}{i+m}\left(-\frac{i}{m^2}\right) + \frac{i}{(i+m)^2}\right] =$$

$$= \left(1+\frac{i}{m}\right)^m \left\{\left[\ln\left(1+\frac{i}{m}\right) - \frac{i}{i+m}\right]^2 - \frac{i^2}{m(i+m)^2}\right\}. \tag{1.101}$$

If i is small, then

$$\frac{d^2 i_{eff}}{dm^2} \approx \frac{d^2\left(1+\frac{i}{m}\right)^m}{dm^2} \approx (1+i)\left\{\left[\frac{i^2}{2m^2}\right]^2 - \frac{1}{m}\left(\frac{i}{m}\right)^2\right\} \tag{1.102}$$

or finally, taking into account the smallness of $\frac{i}{m}$,

$$\frac{d^2 i_{eff}}{dm^2} \approx -(1+i)\frac{1}{m}\left(\frac{i}{m}\right)^2 < 0 \tag{1.103}$$

So, we have proved that the effective interest rate in the compound interest scheme increases with an increase in the multiplicity of accrual and reaches a maximum with continuous accrual of interest, as well as a slowdown in the rate of growth of the effective interest rate with an increase in the multiplicity of accrual. At the same time, the effective interest rate practically reaches saturation at $m \geq 6 \div 10$, i.e. above this multiplicity of accrual, the growth of the effective interest rate slows down sharply.

1.13.3. Adjustment for inflation

With inflation, money depreciates by $1 + \alpha$ times, so the real equivalent of the amount accumulated over the year $S = S_0(1 + i)$ will be $(1 + \alpha)$ times less

$$S_\propto = \frac{S}{1+\alpha} = S_0 \frac{1+i}{1+\alpha} = S_0 \frac{1+\alpha-\alpha+i}{1+\alpha} =$$

$$= S_0\left(1+\frac{i-\alpha}{1+\alpha}\right) = S_0(1+i_\alpha) \qquad (1.104)$$

Here we denoted by i_α the inflation–adjusted interest rate (i is still the interest rate without considering inflation), for which we obtained the Fischer formula, which determines the effective interest rate when taking into account inflation

$$i_{eff} = i_\alpha = \frac{i-\alpha}{1+\alpha}. \qquad (1.105)$$

1.13.4. Adjustment for taxes

A. Interest on a bank deposit is not taxed if it does not exceed the refinancing rate of the Bank of Russia +5% (currently $i_0 = 8.75\% + 5\% = 13.75\%$). Otherwise, a tax $t = 35\%$ is charged on interest exceeding the refinancing rate of the Bank of Russia. Interest on deposits in Russian banks currently reaches 15% or more, so knowledge of the real (effective) interest rate is extremely relevant.

Let's calculate the effective interest rate in this case. Consider one accrual period. The accrued value of the contribution at the end of the period is equal to:

$$S = S_0\big((1+i) - t(i - i_0)\big) = S_0(1 + i(1-t) + ti_0) = S_0\big(1 + i_{eff}\big). \qquad (1.106)$$

Hence, for the effective interest rate in the presence of taxes, we get

$$i_{eff} = i(1-t) + ti_0. \qquad (1.107)$$

Example 1.12. The deposit is placed in the bank at 17% per annum. Find the effective interest rate.

To find the effective interest rate, we use the formula (1.107):

$$i_{eff} = i(1-t) + ti_0 = 0.17(1+0.35) + 0.35 \cdot 0.1375 = 15.86\%$$

Thus, the real (effective) interest rate of 15.86% is 1.14% lower than the announced nominal (17%).

b. Interest on the loan is excluded from the tax base if it does not exceed the refinancing rate of the Bank of Russia plus a few points (for ruble loans) and a fixed amount for foreign currency loans (denote these values as i^*). Let's find the effective loan interest rate in this case.

1. If the loan interest rate does not exceed i^*, then the real (effective) rate on the loan taken (D), taking into account the income tax of company (t), is as follows:

$$iD - tiD = iD(1-t) = i_{eff}D, \qquad (1.108)$$

hence

$$i_{eff} = i(1-t). \qquad (1.109)$$

Value (1 – t) is called **tax shield,** which shows the financial benefit of the company from the use of borrowed capital.

2. If the loan interest rate exceeds i^*, then only the value ti^*D is deducted from the loan fee iD, so we have

$$i_{eff}D = iD - ti^*D = iD - tiD + tiD - ti^*D =$$

$$= D[(1-t) + t(i-i^*)] = D[(1-t) + t\Delta i], \qquad (1.110)$$

hence

$$i_{eff} = i(1-t) + t\Delta i. \qquad (1.111)$$

Example 1.13. Find the effective loan interest rate if the income tax rate is 20% and the cut–off rate is i^* (8.75% x1.1 = 9.625%).

Using the formula (1.111), we have

$$i_{eff} = i(1-t) + t\Delta i = 0.2(1-0.2) + 0.2(0.2 - 0.09625) = 0.18075$$
$$= 18.08\%$$

Thus, the effective loan interest rate is 18.08% instead of 20%.

1.13.5. Equivalence of different interest rates

Equivalence of simple and compound interest rates

It is easy to obtain equivalence formulas for simple and compound percentages.

In the simplest case of a single accrual of interest, we have

$$S_0(1 + i_s n) = S_0(1 + i_c)^n, \qquad (1.112)$$

$$i_s = \frac{1}{n}[(1 + i_c)^n - 1], \ \ i_c = \sqrt[n]{1 + i_s n} - 1. \qquad (1.113)$$

In the case of m–multiple accrual of interest, we have for n–periods

$$S_0(1 + i_s n) = S_0\left(1 + \frac{i_c}{m}\right)^{m\cdot n}, \qquad (1.114)$$

hence

$$i_s = \frac{1}{n}\left[\left(1 + \frac{i_c}{m}\right)^{m\cdot n} - 1\right], i_c = m\left(\sqrt[m\cdot n]{1 + i_s n} - 1\right). \qquad (1.115)$$

Example 1.14. Find simple interest rate i_s, equivalent to a compound rate of 15% for a time interval of five years with monthly interest accrual.

Using the first formula from (1.115), we obtain

$$i_s = \frac{1}{n}\left[\left(1+\frac{i_c}{m}\right)^{m\cdot n} - 1\right] = \frac{1}{5}\left[\left(1+\frac{0.15}{12}\right)^{12\cdot 5} - 1\right] = \frac{1}{5}[(1.15)^{60} - 1]$$
$$= 0.2214$$

that is, the equivalent simple interest rate i_s= 22.14%.

Equivalence of simple and continuous interest rates

Similarly, we can consider the equivalence of other interest rates, for example, simple and continuous

$$S_0(1+i_s n) = S_0 e^{i_c\cdot n}, \tag{1.116}$$

$$i_s = \frac{1}{n}(e^{i_c\cdot n} - 1), i_c = \frac{1}{n}\ln(1+i_s n) \tag{1.117}$$

Example 1.15. Find a continuous interest rate i_c, equivalent to a simple rate of 15% for a time interval of five years.

Using the second formula from (1.117), we obtain

$$i_c = \frac{1}{n}\ln(1+i_s n) = \frac{1}{5}\ln(1+0.15\cdot 5) = \frac{1}{5}\ln 1.75 = \frac{1}{5}0.5596 = 0.119$$

that is, the equivalent continuous interest rate $\boldsymbol{i_c}$ = 11.19%.

Equivalence of compound and continuous interest rates

We equate the accrued amounts in the case of accrual of compound and continuous interest for n–periods

$$S_0(1+i_c)^n = S_0 e^{i_c\cdot n}, \tag{1.118}$$

here i_c — compound interest rate;

i_{cont} — continuous interest rate.

By reducing this equality by S_0 and extracting from both parts the root of degree n (to reduce n in the exponent), we obtain

$i_{cont} = \ln(1 + i_c)$, $i_c = e^{i_{cont}} - 1$. (1.119)

1.14. Internal rate of return

1.14.1. The concept of internal rate of return

Let us consider in detail one of the most important concepts in investment theory — the internal rate of return, and also investigate the dependence of net present income (*NPV*) on the discount rate.

The investment process described by the financial flow has the following form:

$CF = \{(t_0, C_0), (t_1, C_1), (t_2, C_2), \ldots, (t_n, C_n)\}$. (1.120)

The value of C_k (K = 0, 1, ..., n) represents the balance of investment costs and net income for the *k*–th period, updated at the end of this period. A negative payment in (1.120) means that investment costs exceeded net income, a positive payment means that net income exceeded investment costs.

By means of the reduction rate *i*, we calculate the current value of the flow (1.120), called in this context net present value (NPV):

$$NPV(i) = \frac{C_0}{(1+i)^{t_0}} + \frac{C_1}{(1+i)^{t_1}} + \frac{C_2}{(1+i)} + \cdots + \frac{C_n}{(1+i)^{t_n}} \quad (1.121)$$

If *NPV* has a negative value, it means that the income does not recoup the costs at the accepted rate of return *i*.

So, let's consider the dependence of *NPV* on the conversion rate (the accepted rate of return) *i*. First, let's pay attention to the simplest typical case when all costs are carried out at the initial moment, and then the investor begins to receive income. Let the financial flow have the form:

$CF = \{(0, -K), (1, C_1), (2, C_2), \ldots, (n, C_n)\}$, (1.122)

where $K > 0$ — initial investment, all payments C_k, $k = 1, 2, \ldots, n$ are non–negative, and there is at least one positive among them.

Then,

$$NPV(i) = -k + \sum\nolimits_{k=1}^{n} \frac{C_k}{(1+i)^k} \tag{1.123}$$

For $i > -1$, the net reduced income *NPV(i)* is a decreasing function of the reduction rate *i* (Fig. 1.3).

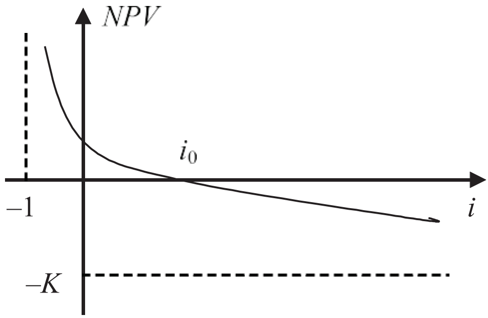

Figure 1.3. Dependence of net present income on the discount rate

On the one hand, with i tending to –1 (on the right), each summand $\frac{C_k}{(1+i)^k}$ with a positive C_k tends to infinity, which means that the sum (1.123) tends to infinity. Hence, *PV(i)* > 0 for *i* is sufficiently close to –1. On the other hand, with unlimited growth of *i*, all summands $\frac{C_k}{(1+i)^k}$ tend to zero, and the sum (1.123) tends to — *K*. Hence, *PV(i)* < 0 for sufficiently large *i*. Thus, for $i > -1$, the continuous function *PV(i),* decreasing, changes the sign from plus to minus. Therefore, *PV(i)* turns to zero for some $i = i_0$. The value of i_0 is called the internal rate of return of the payment flow (1.123).

The internal rate of return serves as the boundary of the interest rates for which the project has a positive and negative applied value: if $i >$ i0, then *PV(i)* < 0, if $i <$ i0, then *PV(i)* > 0.

Since $PV(0) = \sum_{k=1}^{n} C_k - K$, $i_0 > 0$ if and only if $\sum_{k=1}^{n} C_k > K$, that is the net amount of income exceeding the initial investment.

Example 1.16. Find the internal rate of return of the flow:

$CF = \{(0, –8000), (1, 6000), (2, 5000)\}$. (1.124)

Let 's make the equation:

$–8000 + 6000(1 + i)^{–1} + 5000(1 + i)^{–2} = 0$.

By replacing $x = (1 + i)^{–1}$, we get the quadratic equation:

$5000 \cdot x^2 + 6000 \cdot x – 8000 = 0$, or

$5 \cdot x^2 + 6 \cdot x – 8 = 0$.

Solving it, we receive

$$(1+i)^{-1} = \frac{-3 \pm \sqrt{9+40}}{5} = \frac{-3 \pm 7}{5}.$$

Of the two roots –2 and 0.8, the second one suits us. Solving the equation

$(1 + i)^{–1} = 0.8$,

we receive $i \approx 0.25$.

Consequently, the internal rate of return of the flow (1.124) is 25%.

In general, it is said that the financial flow (1.120) has an internal rate of return if the equation

$NPV(i) = 0$ (1.125)

has a unique solution $i_0 > –1$; the solution of equation (1.121) is called the internal rate of return of the flow (1.120).

Consider an arbitrary financial flow

$CF = \{(t_0, C_0), (t_1, C_1), (t_2, C_2), \ldots, (t_n, C_n)\}$, (1.126)

in which negative payments can alternate with positive ones, assuming, however, that it has an internal rate of return. We will additionally assume that C0 < 0 and Cn > 0. We will represent the CF stream as the difference of two non–negative streams A and B:

$$CF = A - B, \quad (1.127)$$

where

$A = \{(t_0, A_0), (t_1, A_1), (t_2, A_2), \ldots, (t_n, A_n)\}$,

$$B = \{(t_0, B_0), (t_1, B_1), (t_2, B_2), \ldots, (t_n, B_n)\}, \quad (1.128)$$

supposing $A_k = \max(0, C_k)$, $B_k = \max(0, -C_k)$. Let us consider τ_A и τ_B as average terms of flows A and B.

$$\frac{A_0}{(1+i)^{t_0}} + \frac{A_1}{(1+i)^{t_1}} + \cdots + \frac{A_2}{(1+i)^{t_n}} = \frac{A_0+A_1+\cdots+A_n}{(1+i)^{\tau_A}} \quad (1.129)$$

$$\frac{B_0}{(1+i)^{t_0}} + \frac{B_1}{(1+i)^{t_1}} + \cdots + \frac{B_2}{(1+i)^{t_n}} = \frac{B_0+B_1+\cdots+B_n}{(1+i)^{\tau_B}} \quad (1.130)$$

Therefore, the equation $PV(CF, i) = 0$ acquires the form:

$$= \frac{A_0+A_1+\cdots+A_n}{(1+i)^{\tau_A}} = \frac{B_0+B_1+\cdots+B_n}{(1+i)^{\tau_B}} \quad (1.131)$$

From here $(1+i)^{\tau_A-\tau_B} = \frac{A_0+A_1+\cdots+A_n}{B_0+B_1+\cdots+B_n}$ (1.132)

Consequently,

$$i = \left(\frac{A_0+A_1+\cdots+A_n}{B_0+B_1+\cdots+B_n}\right)^{\frac{1}{\tau_A-\tau_B}} - 1 \quad (1.133)$$

The formula (1.133) can be used to approximate the internal rate of return.

Example 1.17. Find the internal rate of return of the flow (1.124) using the formula (1.133).

Suppose,

$A = \{(0, 0), (1, 2000), (2, 3000)\}$,

$B = \{(0, 4000), (1, 0), (2, 0)\}$.

Evidently, $\tau_B = 0$. We will find approximately the average term of the flow A as a weighted sum of the moments of payments:

$$\tau_A = \frac{2000}{2000+3000} \cdot 1 + \frac{3000}{2000+3000} \cdot 2 = 1.6 .$$

Consequently,

$$i_0 \approx \left(\frac{2000+3000}{4000} \right)^{\frac{1}{1.6}} - 1 = 14.97\,\%$$

Recall that, directly solving the equation, we obtained earlier $i_0 = 15.14\%$.

1.14.2. The internal rate of return of typical investment flows

In the previous paragraph, it was actually established that for a financial flow of the type (1.122), the internal rate of return i_0 is determined, and $i_0 > 0$ if the net amount of income exceeds the initial investment.

This result allows generalization to the so–called typical investment flows . A **typical investment flow** is understood as a flow of payments in which all negative payments precede positive ones, i.e. investments prevail over return only at the initial stages of the investment process. Thus, a typical investment flow has the form:

$$CF = \{(t_0, -C_0), \ldots, (t_m, -C_m), (t_{m+1}, C_{m+1}), \ldots, (t_n, C_n)\}, \qquad (1.134)$$

where all C_k are positive, moreover C_0 and C_n are different from 0.

We prove that any typical investment flow has an unambiguously defined internal rate of return. Consider an arbitrary financial flow (1.134) satisfying the conditions of the theorem. The stream (1.134) can be represented as the difference of two streams:

$$CF = CF^{(+)} - CF^{(-)}, \qquad (1.135)$$

where

$CF^{(+)} = \{(t_{m+1}, C_{m+1}), \dots, (t_n, C_n)\}$ (1.136)

and

$CF^{(-)} = \{(t_0, C_0), \dots, (t_m, C_m)\}$. (1.137)

We shall bring the values of the flows (1.136) and (1.137) to a certain point in time τ between t_m and t_{m+1}. Let us suppose that

$$P^{(+)}(i) = PV_\tau\left(CF^{(+)}, i\right) = \frac{C_{m+1}}{(1+i)^{t_{m+1}-\tau}} + \dots + \frac{C_n}{(1+i)^{t_n-\tau}} \quad (1.138)$$

$$P^{(-)}(i) = PV_\tau\left(CF^{(-)}, i\right) = C_0(1+i)^{\tau-t_0} + \dots + C_m(1+i)^{\tau-t_m} \quad (1.139)$$

When i runs from –1 to +∞, the value of $P^{(+)}(i)$ decreases from +∞ to 0, and the value of $P^{(-)}(i)$ increases from 0 to +∞. Since

$PV_\tau(CF, i) = P^{(+)}(i) - P^{(-)}(i)$, (1.140)

$PV_\tau(CF, i)$ — decreases from +∞ to –∞. Hence, the equation

$PV_\tau(CF, i) = 0$ (1.141)

has a unique solution $i_0 > -1$. Since

$PV_\tau(CF, i) = (1+i)^\tau \cdot PV(CF, i)$, (1.142)

the equation

$PV(CF, i) = 0$ (1.143)

is equivalent to equation (1.141). Therefore, the equation $PV(CF, i) = 0$ for the flow (1.134) has a unique solution $i_0 > -1$, which is the internal rate of return of this flow.

The definition of the internal rate of return and the previous theorem can be extended to flows having a continuous component. So, if for the financial flow CF, the present value of $PV(CF, i)$ is defined and continuous with respect to i, and the equation

$PV(CF, i) = 0$ (1.144)

has a unique solution i_0 for $i > -1$, then i_0 serves as the internal rate of return of the CF flow.

Consider the financial flow CF in the time interval from 0 to T, represented as a sum

$$CF = CF^{(d)} + CF^{(c)}, \tag{1.145}$$

where

$$CF^{(d)} = \{(t_1, C_1), \ldots, (t_n, C_n)\}\ , 0 \leq t_1 \leq \ldots \leq t_n \leq T \tag{1.146}$$

is a discrete flow, $CF^{(c)}$ — flow with continuous density $\mu(t)$, $0 \leq t \leq T$.

In this case, it is assumed that for a period of time from 0 to T there is such moment τ that all negative payments precede the moment τ, and positive ones follow it. More precisely, we assume that τ is different from the moments of payments of the discrete flow $CF^{(d)}$ and

$C_k \leq 0$ at $t_k < \tau$; $C_k \geq 0$ at $t_k > \tau$;

$\mu(t) \leq 0$ at $t < \tau$; $\mu(t) \geq 0$ at $t > \tau$. (1.147) The equation $PV(CF, i) = 0$ has the form

$$\sum_{k=1}^{n} \frac{c_k}{(1+i)^{t_k}} + \int_0^T \frac{\mu}{(1+i)^t} dt = 0 \tag{1.148}$$

Discounting the cost of the flow to the moment τ and dividing the flows into positive and negative parts, we proceed from equation (1.148) to the equation

$$\sum_{t_k<\tau} |C_k|(1+i)^{\tau-t_k} + \int_0^T |\mu(t)|(1+i)^{\tau-t}\, dt =$$

$$= \sum_{t_k>\tau} \frac{C_k}{(1+i)^{t_k-\tau}} + \int_\tau^T \frac{\mu(t)}{(1+i)^{t-\tau}} dt \tag{1.149}$$

When i runs through the interval from –1 to +∞, the left side of equation (1.149) continuously increases from 0 to +∞, and the right side continuously decreases from +∞ to 0. Consequently, equation (1.149) has a unique solution: $i_0 > -1$, which is the internal rate of return of the financial flow CF.

Example 1.18. It is necessary to estimate the internal rate of return of the next project. One–time initial investments include expenses for the acquisition of the enterprise and the purchase of new equipment for it and amount to K. Over the next two years, there is a constant relative increase in daily income by q starting with daily income R_0. Production costs are constant for two years and amount to z per day. After T years, it is expected to sell the company for the amount of S.

It can be assumed that the financial flow associated with the described project contains discrete and continuous components. The discrete component has the form:

$CF^{(d)} = \{(0, -K), (T, S)\}$.

A continuous component is a financial flow with a density of

$\mu(t) = 365 \cdot (R_0 \cdot (1 + q)^t - z)$.

The internal rate of return is determined from the equation

$$-K + 365 \cdot \int_0^T \frac{R_0(1+q)^t - z}{(1+i)^t} dt + S = 0 \qquad (1.150)$$

Suppose $\alpha = \ln(1 + q)$ и $\delta = \ln(1 + i)$. Then

$$\int_0^T \frac{R_0(1+q)^t - z}{(1+i)^t} dt = R_0 \int_0^T e^{(\alpha-\delta)t}\, dt - z \int_0^T e^{-\delta t}\, dt =$$

$$= \frac{R_0}{\alpha - \delta}\left(e^{(\alpha-\delta)T} - 1\right) - \frac{z}{\delta}\left(e^{-\delta T} - 1\right) \cdot$$

Accordingly, equation (1.150) turns into the following equation with respect to δ:

$$-K + \frac{R_0}{\alpha - \delta}\left(e^{(\alpha-\delta)T} - 1\right) - \frac{z}{\delta}\left(e^{-\delta T} - 1\right) + S = 0.$$

1.14.3. Internal rate of return of financial flows with alternating positive and negative payments

For an arbitrary financial flow in which negative payments alternate with positive ones, the internal rate of return may be uncertain. Consider, for example, such a financial flow:

CF = {(0, –1000), (1, 3410), (2, –3856), (3, 1446)}. (1.151)

In this flow, the sum of negative payments is equal to the sum of positive ones, so that the present value is zero at $i_0 = 0$. In addition to the root i_0, the equation

$$-1{,}000 + \frac{3{,}410}{1+i} + \frac{3{,}856}{(1+i)^2} - \frac{1{,}446}{(1+i)^3} = 0$$

has two more roots: $i_1 = 0.1274$ и $i_2 = 0.2826$. Thus, two more rates claim to be the internal rate of return: 12.74% and 28.26%.

Another problem is related to the "instability" of the internal rate of return, even when it is clearly defined. So, relatively small changes in individual payments can lead to significant changes in the internal rate of return.

For example, consider along with the payment flow (1.151) flows

CF = {(0, –1,000), (1; 3,410), (2, –3,856), (3, 1,442)} (1.152)

CF = {(0, –1,000), (1; 3,410), (2, –3,856), (3, 1,452)}, (1.153)

obtained from (1.151) by change of last payment on 0.28% and 0.41% respectively. The internal rate of return is determined unambiguously for flows (1.152) and (1.153). For the flow (1.152) it is negative and equal to –6.15%, for the flow (1.153) it is positive and equal to 35,62%.

Nevertheless, the theorem on the internal rate of return of a typical investment flow can be extended to some financial flows with alternating positive and negative payments.

We will call the **net amount** of the payment flow

$CF = \{(t_0, C_0), (t_1, C_1), \ldots, (t_n, C_n)\}, t_0 \le t_1 \le \ldots \le t_n,$ (1.154)

to the time moment τ the value $S(\tau) = \sum C_k$ **.**

$tk \le \tau$

Let us prove that if in the payment flow (1.154) all net amounts $S(t_k)$, k = 1, ..., n, are non–negative, then it has a non–negative present value for any positive discount rate. If, in addition, the initial payment is positive, then the present value is positive for any positive discount rate.

As $C_0 = S_0$ and $C_k = S_k - S_{k-1}$, when $k = 1, 2, \ldots, n$, then

$$\frac{C_0}{(1+i)^{t_0}} \qquad (1.155)$$

If $i > 0$, then all the terms of the last sum are non–negative, and the first term is positive, so that $PV(i) > 0$.

In proving the previous theorem, a discrete analogue of integration in parts was used. In fact, let CF be a continuous flow with continuous density $\mu(t)$, $t \in [0, T]$. Suppose all net amounts $S(t) = \int_0^t \mu(\tau)dt$ are non–negative. Denote by $P(t)$ the present value of the flow CF over a period of time $[0, t]$. Then $dP(t) = e^{-\delta t}dS(t)$, where $\delta = \ln(1+i)$ is the intensity of growth rate. As $P(0) = 0$, then $P(t) = \int_0^t e^{-\delta\tau}\, dS\,(\tau)$ and $PV(i) = \int_0^T e^{-\delta\tau}\, dS\,(\tau)$. Integrating by parts, we get:

$$PV(i) = e^{-\delta\tau}S(\tau)|_0^T + \int_0^T \delta e^{-\delta\tau}S(\tau)\,d\tau. \qquad (1.156)$$

From (1.156) it follows that $PV(i) \ge 0$. Definitely, $e^{-\delta\tau}S(\tau)|_0^T = e^{-\delta T}S(T) \ge 0$, an integrable function $\delta e^{-\delta\tau}S(\tau)$ is non–negative as $\delta >$

0 at $i > 0$ and $S(\tau) \geq 0$ by assumption. Assuming additionally that at some initial interval $[0, t_0]$ the density function is non–negative and $S(t_0) > 0$, it can be proved that $PV(i) > 0$.

Below we will indicate the class of financial flows for which a positive internal rate of return is determined.

We will say that the financial flow (1.154) is an investment—type payment flow if it has the following properties:

— the initial payment is negative;
— net–the sum of the total flow is positive;
— there is a point from which all payments are positive, and all previous net amounts are negative.

We show that for an investment–type flow a positive interest rate, relative to which the present value of the flow is zero, exists and is determined.

Let (1.154) be an investment—type flow. According to the definition, this means that $C_0 < 0$, $S(t_n) > 0$ and there exists an m, such that $C_k > 0$ for $k \geq m + 1$, and $S(t_k) < 0$ when $k \leq m$. For sufficiently large values of the conversion rate i the sign of the value $PV(i)$ coincides with the sign C_0, so that $PV(i)$ is negative. At the same time, $PV(0) = S(t_n) > 0$. Consequently, $PV(i_0) = 0$ for some $i_0 > 0$. Let us show that i_0 is uniquely determined, and at point i_0 the value of $PV(i)$ changes sign from plus to minus. To do this, it is enough to establish that $PV(i) < 0$ for all $i > i_0$.

Let us introduce into consideration a payment flow

$$CF' = \{(t_0, B_0), (t_1, B_1), \ldots, (t_n, B_n)\}, \qquad (1.157)$$

in which

$$B_k = \frac{C_k}{(1+i_0)^{t_k}}, \quad k = 0,1, \ldots n. \qquad (1.158)$$

Let us denote by $S'(t_k)$ the net amount of payments of the flow CF' by the time t_k. Let us show that $S'(t_k) \leq 0$ for all $k = 0, 1, \ldots, n$. Definitely,

$$S'(t_k) = \sum_{l=1}^{k} \frac{C_l}{(1+i_0)^{t_l}}$$

is a discounted cost of payments of the *CF* stream up to and including the t_k moment (relative to the i_0 rate). According to the definition of i_0, we have $S'(t_n) = 0$. When $k > m$, C_k payments are positive, and B_k payments are positive along with them. So,, $S'(t_k) < 0$ when $k > m$. Finally, if $k \le m$, then all the net amounts of the payment flow

$$\{(t_0, C_0), (t_1, C_1), \ldots., (t_k, C_k)\} \tag{1.159}$$

are negative by assumption. Then, according to the theorem proved above, the flow (1.159) has a negative present value relative to any positive bid. In particular, $S'(t_k) < 0$.

Since all the net amounts of the payment flow (1.157) (except the last one equal to zero) are negative, the flow (1.159) has a negative present value relative to any positive rate. Then for any rate $i > i_0$ supposing that $j = \frac{1+i}{1+i_0} - 1$, then

$$PV(i) = \sum_{k=1}^{n} \frac{C_k}{(1+i)^{t_k}} = \sum_{k=1}^{n} \frac{B_k}{(1+i)^{t_k}} < 0,$$

which is what we needed to prove.

1.15. Currency transactions

Let us look at some operations with currency.

1.15.1. Deposits with and without currency conversion

The possibility of converting rubles into currency and back into rubles, as well as the possibility of placing both rubles and currencies on deposit, increase the number of schemes for generating income through deposits. Let's compare the income from the direct placement of available funds on deposit in the national currency (RR, Russian ruble) and through the conversion of the national currency into a foreign currency (FC, foreign currency), the placement of the latter on deposit with the subsequent reverse conversion of the accumulated amount in foreign currency into the national currency.

There are four possible income schemes:

1) $RR \to RR$;
2) $FC \to FC$;
3) $FC \to RR \to RR \to FC$;
4) $RR \to FC \to FC \to RR$.

The first and second schemes are not related to currency conversion and are described quite fully in the previous paragraphs, whereas the third and fourth schemes involve currency conversion at the beginning and end of a financial transaction.

Consider the third scheme $FC \to RR \to RR \to FC$.

Let us introduce the notations:

P_{FC} — the amount of the deposit in FC;
P_{RR} — the amount of the deposit in RR;
S_{FC} — the accrued amount in FC;
P_{RR} — the accrued amount in RR;
K_0 — exchange rate $FC \to RR$ at the beginning of the operation;
K_1 — exchange rate $FC \to RR$ at the end of the operation;
n — deposit term;
i — interest rate in RR;
j — interest rate in FC.

The operation consists of three stages — the conversion of foreign currency into national currency, the placement of rubles on deposit, followed by the reverse conversion of the accumulated amount into foreign currency.

As a result of all stages, we will receive the following increased amount in foreign currency:

— in the scheme of simple interest:

$$S_{FC} = P_{FC}K_0(1+in)/K_1; \quad (1.160)$$

— in the compound interest scheme:

$$S_{FC} = P_{FC}K_0(1 + i)^n/K_1. \quad (1.161)$$

The multiplier of the accrual *(M),* taking into account the double conversion, has the form:

— in the scheme of simple interest:

$$M = K_0(1 + in)/K_1 = \frac{(1+in)}{K_1/K_0} \quad (1.162)$$

— in the scheme of compound interest:

$$M = K_0(1 + i)^n/K_1 = \frac{(1+i)^n}{K_1/K_0} \quad (1.163)$$

The multiplier increases with an increase in the interest rate, the deposit term and the initial exchange rate and decreases with an increase in the final exchange rate.

Let us find the effective interest rate for the operation as a whole:

— in the scheme of simple interest:

$$S_{FC} = P_{FC}K_0(1 + in)/K_1 = P_{FC}(1+i_{\text{eff}}\,n), \quad (1.164)$$

from which

$$i_{eff} = \frac{K_0(1+in)/K_1 - 1}{n} = \frac{M-1}{n} \quad (1.165)$$

— in the scheme of compound interest:

$$S_{FC} = P_{FC}K_0(1+ i)^n/K_1 = P_{FC}(1+i_{\text{eff}})^n, \quad (1.166)$$

from which

$$i_{eff} = \frac{1+i}{\sqrt[n]{K_1 / K_0}} - 1 = \sqrt[n]{M} - 1 \qquad (1.167)$$

When $n = 1$

$$i_{eff} = \frac{1+i}{K_1/K_0} - 1 = M - 1. \qquad (1.168)$$

Now consider the fourth scheme $RR \to FC \to FC \to RR$. The operation consists of converting the national currency into a foreign currency, placing foreign currency on a deposit, followed by the reverse conversion of the accumulated amount into the national currency.

As a result, we will get the following increased amount in the national currency:

— in the scheme of simple interest:

$$S_{RR} = \frac{P_{RR}}{K_0}(1 + jn)K_1; \qquad (1.169)$$

— in the scheme of compound interest:

$$S_{RR} = \frac{P_{RR}}{K_0}(1 + j)^n K_1. \qquad (1.170)$$

The multiplier of the accrual (M), taking into account the double conversion, has the form:

— in the scheme of simple interest:

$$M = \frac{K_1}{K_o}(1 + jn) = \frac{(1+jn)}{K_0/K_1}; \qquad (1.171)$$

— in the scheme of compound interest:

$$M = \frac{K_1}{K_0}(1+j)^n = \frac{(1+j)^n}{K_0/K_1}. \qquad (1.172)$$

The multiplier increases with an increase in the interest rate, the deposit term, the final exchange rate and decreases with an increase in the initial exchange rate.

Find the effective interest rate for the operation as a whole:

— in the scheme of simple interest:

$$S_{RR} = P_{RR}K_1(1 + in)/K_0 = P_{RR}(1+i_{eff}n), \qquad (1.173)$$

from where

$$i_{eff} = \frac{\frac{K_1(1+in)}{K_0}-1}{n} = \frac{M-1}{n}. \qquad (1.174)$$

— in the scheme of compound interest:

$$S_{RR} = P_{RR}K_1(1 + i)^n/K_0 = P_{RR}(1 + i_{\text{eff}})^n, \qquad (1.175)$$

from which

$$i_{eff} = \frac{1+i}{\sqrt[n]{K_0/K_1}} - 1 = \sqrt[n]{M} - 1 \qquad (1.176)$$

When $n = 1$

$$i_{eff} = \frac{1+i}{K_0/K_1} - 1 = M - 1. \qquad (1.177)$$

Example 1.19. We will place \$2,000 after conversion on a deposit at simple interest (i = 15%) for a period of two years. The dollar sale rate at the beginning of the deposit period was 26 rubles, the dollar purchase rate at the end of the operation was 34 rubles. The rate for a dollar deposit (j) is 5%. It is necessary to compare the effectiveness of this operation with the effectiveness of directly placing dollars on a foreign currency deposit.

$$S_{FC} = \frac{P_{FC}K_0(1+in)}{K_1} = 2{,}000 \cdot \frac{26}{34}(1 + 0.15 \cdot 2) = \$1{,}988.24$$

Direct placement of dollars on a foreign currency deposit will give an accrual amount

$S_{FC} = P_{FC}(1+jn) = 2{,}000(1 + 0{,}05\cdot 2) = \$2{,}200$

Thus, it is more profitable to place dollars directly on a foreign currency deposit.

1.15.2. Dual currency basket

The dual currency basket is an operational benchmark of the exchange rate policy of the Central Bank of the Russian Federation (CBR), introduced on February 1, 2005 to determine the real exchange rate of the ruble in relation to the main currencies: the dollar and the euro. At the time of introduction, the dual currency basket consisted of 10% euro and 90% USD. The current values were set on February 8, 2007; the dual currency basket consists of 45% euro and 55% USD. The Central Bank of the Russian Federation establishes a corridor of permissible fluctuations of the dual currency basket, intends to gradually expand the boundaries of the corridor of the dual currency basket, approach the free exchange rate of the ruble and the process of inflation targeting. At the beginning of February 2009, the cost of a dual currency basket was 41 rubles. The calculation of the cost of a dual currency basket is given in example 1.20.

Example 1.20. Let's calculate the cost of a dual currency basket (DCB) on 21.10.2009. The dollar exchange rate was 29.19 rubles, the euro exchange rate was 43.69 rubles. Taking into account the structure of the basket, we get

$\mathrm{DCB} = 29.19\cdot 0.55 + 43.69\cdot 0.45 = 35.72.$

So, the cost of a dual currency basket is 35.72 rubles.

Example 1.21. A multi–currency deposit is opened in the bank: 100,000 rubles at 16% per annum, 10,000 dollars at 6% per annum and 5,000 euro at 5% per annum. Find the effective interest rate of a multi–currency deposit, if the exchange rates at the beginning and end of the (annual) term of the deposit are equal to 29 and 34, 43 and 46 rubles, respectively.

After a year , the accrued amounts will amount to:

$S_{rub} = P_{rub}(1+i) = 100{,}000(1+0.16) = 116{,}000$ RUB

$S_{\$} = P_{\$}(1+i) = 10{,}000(1+0.06) = 10{,}600$ USD

$S_{\text{euro}} = P_{\text{euro}}(1+i) = 5{,}000(1+0.05) = 5{,}250$ EUR.

By converting the currency at the rates of the end of the deposit period, we will get the full accrued amount

$S = 116{,}000 + 10{,}600\cdot 34 + 5{,}250\cdot 46 = 717{,}900$ RUB

We shall find the initial deposit amount in rubles

$S_0 = 100{,}000 + 10{,}000\cdot 29 + 5{,}000\cdot 43 = 605{,}000.$

The effective interest rate of a multicurrency deposit is found from the formula

$S = S_0(1 + i_{eff}),$

from which

$$i_{eff} = \frac{S}{S_0} - 1 = \frac{S - S_0}{S_0} = \frac{717{,}900 - 605{,}000}{605{,}000} = 18.67\%$$

So, with rates for individual currencies of 16, 6 and 5%, the effective interest rate turned out to be 18.67%. The fact, strange at first glance, is explained by a significant increase in conversion rates over the year (by 17.24% for the dollar and 6.98% for the euro).

If we assume that the exchange rates have remained unchanged, we will get a more understandable result – 7.3%:

$S = 116{,}000 + 10{,}600\cdot 29 + 5{,}250\cdot 43 = 649{,}150$ RR,

$$i_{eff} = \frac{S - S_0}{S_0} = \frac{649{,}150 - 605{,}000}{605{,}000} = 7.3\%.$$

The same result will be obtained if calculating the weighted average interest rate

$i_{eff} = i_{rub} \cdot w_{rub} + i_{\$} \cdot w_{\$} + i_{euro} \cdot w_{euro} =$

$$= 0.16 \cdot \frac{100{,}000}{605{,}000} + 0.06 \cdot \frac{10{,}000 \cdot 29}{605{,}000} + 0.05 \cdot \frac{5{,}000 \cdot 43}{605{,}000} = \frac{44{,}150}{605{,}000} \approx 7.3\%.$$

Control questions and tasks

1. Derive the formula for the effective interest rate in the case of simple interest (three cases).

2. Derive the formula for the effective interest rate in the case of compound interest (three cases).

3. Derive the formula for the effective interest rate in the presence of taxes (two cases).

4. Derive the formula for the accrued amount with continuous accrual of interest in the case of simple interest.

5. A deposit of 2000 rubles was placed at the bank for three years at 16% per annum under the simple interest scheme. Find the accrued amount after six years for two cases:
   1) the deposit is extended for three years at a rate of 10% per annum;
   2) the deposit is closed after three years and the deposit is placed for three years at 10% per annum.

6. Derive the formula for the accrued amount with continuous accrual of interest in the case of compound interest.

7. What is mathematical discounting and bank accounting?

8. The nominal discount rate is 10%. Interest is accrued quarterly. Find an effective discount rate.

9. Compare the accrual at simple and compound interest rates.

10. What are multiplier and discounting factors?

11. Derive the “Rule of 70” in case of compound interest accrual.

12. Derive the "Rule of 70" in the case of simple interest. What can it be called?

13. Derive the "Rule of 70" in the case of multiples of interest.

14. Derive the "Rule of 70" in the case of continuous interest accrual.

15. Derive the Fisher’s formula.

16. Derive a formula for the inflation rate over several periods.

17. The inflation rate for the year is 24%. Find the inflation rate for the month, assuming that it is constant.

18. The inflation rate for the quarter is 3%. Find the inflation rate for the month, assuming that it is constant.

19. What is the internal rate of return?

20. Investigate the relationship between the net present value (NPV) and the rate of return (accepted rate of return) i. Give a qualitative graph of this relationship.

21. Find the internal rate of return of the flow

    $CF = \{(0, -500), (1, 250), (2, 300), (3, 400)\}$.

22. Find the internal rate of return of the flow from Task 21, using the division of the flow into 'positive' and 'negative' (formula (1.125)).

23. Prove that any typical investment flow has a single–digit internal rate of return.

24. Prove that the effective interest rate in a compound interest scheme increases with the multiplicity of accrual and reaches a maximum with continuous interest accrual.

25. Prove that the rate of increase of the effective interest rate in a compound interest scheme decreases with increasing multiplicity of accrual and goes to zero in case of continuous accrual of interest.

26. Let us deposit 120,000 rubles after conversion into euro at compound interest (j = 6%) for a period of three years. The euro selling rate at the beginning of the deposit period is 34 rubles, the euro buying rate at the end of the transaction is 44 rubles. Rate for ruble deposit (i = 14%). Compare the efficiency of this transaction with the efficiency of direct deposit of rubles.

27. On February 14, 2022 a deposit of $5,000 is deposited in the bank at 8% per annum with compound interest. How much will the depositor receive on September 20, 2023? In case of simple interest scheme with the same rate?

28. Prove, that with multiple accrual interest, simple interest is more effective than compound interest before the first accrual of interest. By other words at monthly accrual of interest, simple interest is more effective than compound interest during the first month; at quarterly accrual of interest, simple interest is more effective than compound interest during the first quarter; when interest is accrual semi–annually, simple interest is more effective than compound interest during the first half of the year, and so on.

29. Prove, that the continuous interest are more effective than simple or compound interest for any term of the deposit.

CHAPTER 2

FINANCIAL FLOWS, ANNUITIES

2.1. Financial flows

Financial flows are quite widespread in practice. Examples of financial flows are: wage payments, utility payments, rent payments, payments to repay consumer loans, bank loans, company tax payments, regular contributions to pension and other funds, interest payments on securities (shares, bonds, etc.), etc. Practically any regular (and irregular) payments represent financial flows. So the importance of studying them cannot be over emphasized.

The theory of financial flows is set out in many books on financial mathematics [1-4]; this book greatly expands and deepens materials on financial mathematics and makes them more detailed.

A payment P made at time t is called a financial event, i.e. a **financial event** is an ordered pair *(P, t)*, or *(t, P),* consisting of the amount of payment P and the moment of payment t. Payments can be positive (receipts) or negative (payments).

A finite or infinite sequence of financial events

$$(P_0, t_0), (P_1, t_1), (P_2, t_2), \ldots, (P_n, t_n), \tag{2.1}$$

is called a (finite or infinite) **discrete financial flow**. It is assumed that $t_0 < t_1 < t_2 < \ldots$ In the case of infinite flow, it is additionally assumed that t_k increases infinitely with increasing k.

Financial flows are denoted by the symbol CF (cash flow). For example, an n-payment flow $(P_0, t_0), (P_1, t_1), (P_2, t_2), \ldots, (P_n, t_n)$ is written as:

$CF = \{(P_0, t_0), (P_1, t_1), (P_2, t_2), \ldots, (P_n, t_n)\}$.

Financial flow can be represented graphically in different ways. One of the simplest and most practical is as points on the time axis indicating payment amounts (Figure 2.1).

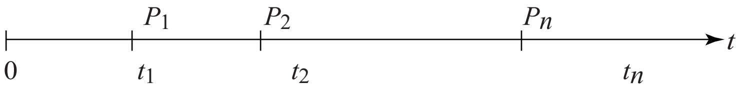

Figure 2.1. Graphical representation of the financial flow

The financial flow can be characterized by a so-called **payment function**, which maps a sum of money $C(t)$ to each time point t such that $C(t_k) = C_k$ and $C(t) = 0$, if t does not coincide with any of the times t_k, $k = 0, 1, \ldots$.

2.2. Current, present, future, present value and final value of the financial flow

Let the financial flow be

$$CF = \{(P_0, t_0), (P_1, t_1), (P_2, t_2), \ldots, (P_n, t_n), \ldots\}. \qquad (2.2)$$

Recall that money has a time value (see Chapter 1). This makes it impossible to directly sum up the payments from different time moments. In order to calculate *the value of the flow* at some point in time t, each payment has to be discounted to that point in time at some interest rate i, which is assumed to be known and constant for the entire flow, and then the discounted payments have to be added up. Usually, the discounting is done according to the compound interest scheme.

The sum of all the cash flow payments reduced to a point in time t is called the **current**, or **discounted value** of the flow (at point in time t) and is denoted by $PV_t\,(CF, i)$ (*present value*), or simply PV_t.

$$PV_t = \frac{P_0}{(1+i)^{t_0-t}} + \frac{P_1}{(1+i)^{t_1-t}} + \ldots + \frac{P_n}{(1+i)^{t_n-t}} \qquad (2.3)$$

In the case of infinite flow, the current value is only considered as certain when the series on the right-hand side of (2.3) converges.

If $t_0 = 0$, the current value of the flow at the initial time is called the **present value of the flow** and is denoted simply by PV:

$$PV_0 = P_0 + \frac{P_1}{(1+i)^{t_1}} + \ldots + \frac{P_n}{(1+i)^{t_n}}$$

For the moment $t > t_n$ the value of the flow is

$$P = P_0(1+i)^{t-t_0} + P_1(1+i)^{t-t_1} + \cdots + P_n(1+i)^{t-t_n} = \sum_{k=0}^{n} P_k(1+i)^{t-t_k}. \qquad (2.4)$$

The value (2.4) is called **the future accrued flow value** (2.2) and is denoted by FV_t (CF, i) (*future value*), or simply FV_t. In the case of finite flow

$$CF = \{(P_0, t_0), (P_1, t_1), (P_2, t_2), \ldots, (P_n, t_n)\} \qquad (2.5)$$

its value at the time of the last payment $t = t_n$, denoted as $FV(CF, i)$ or FV, is called **the final value of the flow**. It is equal to

$$FV = P_0(1+i)^{t_n-t_0} + P_1(1+i)^{t_n-t_1} + \cdots + P_{n-1}(1+i)^{t_n-t_{n-1}} + P_n. \qquad (2.6)$$

Replacing $t - t_k$ by $t - t_n + t_n - t_k$ and taking out the common factor $(1+ i)^{t - tn}$ in (2.6), we obtain the relation between flow values (2.5) at time moments t and t_n (when $t > t_n$)

$$FV_t = FV(1+ i)^{t-tn}. \qquad (2.7)$$

For the finite flow (2.5) and time moments τ and $t \geq t_n$ the present value PV_τ and future value FV_t related as follows:

$$FV_t = PV_\tau(1 + i)^{t-\tau}. \qquad (2.8)$$

2.3. Average term of financial flow

The average term of the financial flow

$$CF = \{(P_0, t_0), (P_1, t_1), (P_2, t_2), \ldots, (P_n, t_n)\} \quad (2.9)$$

in relation to the discount rate i is the time moment t for which

$$PV_t\,(CF) = P_1 + P_2 + \ldots + P_n. \qquad (2.10)$$

The latter means that the flow (2.9) and the flow consisting of a single payment $P = P_1 + P_2 + \ldots + P_n$ at the time moment t have the same current value. Equality (2.10) can be rewritten as follows:

$$\frac{P_1}{(1+i)^{t_1}} + \frac{P_2}{(1+i)^{t_2}} + \cdots + \frac{P_n}{(1+i)^{t_n}} = \frac{P_1+P_2+\cdots+P_n}{(1+i)^t}. \qquad (2.11)$$

Decomposing $(1 + i)^{-x}$ by powers i (when $|i| < 1$), we get

$$(1 + i)^{-x} = 1 - xi + \frac{x(x+1)}{2} i^2 + \cdots.$$

Equality (2.11) with accuracy up to the terms of the second order of smallness (with respect to i) will be written in the form:

$$P_1(1 - t_1 i) + \ldots + P_n(1 - t_n i) = (P_1 + \ldots + P_n)(1 - ti).$$

From which

$$t = \frac{P_1 \cdot t_1 + P_2 \cdot t_2 + \ldots + P_n \cdot t_n}{P_1 + P_2 + \ldots + P_n} \tag{2.12}$$

Example 2.1. Find the average cash flow term.

$CF = \{(0, 100), (1, 200), (2, 400), (3, 100)\}$.

According to formula (2.12) we have

$$t = \frac{P_1 t_1 + \cdots + P_n t_n}{P_1 + \cdots + P_n} = \frac{100 \cdot 0 + 200 \cdot 1 + 400 \cdot 2 + 100 \cdot 3}{100 + 200 + 400 + 100} = \frac{1300}{800} = 1.625,$$

meaning that $t = 1.625$.

If all payments are positive, then $t_1 < t < t_n$, i.e. t lies between the starting and ending time moments. In the general case (when the payments may be of different signs), the average flow term may lie outside the payment time horizon.

2.4. Continuous financial flows

2.4.1. Accrued and discounted values of continuous financial flows

Continuous financial flows are used to simulate flows that consist of payments with short intervals between them. Examples of such flows are the financial flow of a large bank, the flow of utility payments, tax payments, etc. When describing a continuous flow, instead of a payment at a fixed point in time t, it is necessary to consider the sum of payments received during the period from t_1 to t_2. Alternatively, one can consider the sum of all payments received at time t, starting from some fixed point.

Let us assume that the financial flow CF is given by the function $C(t)$, defined as the sum of the payments over the period from the starting point at time 0 to the t.

The value $C(t_2) - C(t_1)$ is the sum of the payments over time from moment

t_1 till moment t_2, and the time derivative of the flow $C'(t)$ is called **the flow density** $C(t)$ at the time moment t.

If the financial flow $C(t)$ has a density $\mu(t)$ at each time moment t and the function $\mu(t)$ is continuous, then

$$C(t_2) - C(t_1) = \int \mu(t)dt. \tag{2.13}$$

Besides,

$$C'(t) = \mu(t). \tag{2.14}$$

Thus, a continuous financial flow can be stated by its density.

Consider now how the cumulative amount of cash flow $C(t)$ with continuous density is calculated $\mu(t)$.

Let δ represent the intensity of growth rate in case of continuous accrual of interest. Suppose that by time t the accumulated amount is $S(t)$. The accrued amount by time $t+\Delta t$ is composed of two components: interest on the accrued amount $S(t)$ and the money attracted by the flow $C(t)$. If Δt is a short time period, the interest on the amount $S(t)$ over it will approximate $\delta S(t)\,\Delta t$; in time Δt the cash flow $C(t)$ will bring the amount roughly equal to $\mu(t) \cdot \Delta t$ (the approximation is taken here to the accuracy of the terms of the order of smallness above the first). Thus, the linear part of the increment $\Delta S(t)$ is $\delta S(t)\cdot\Delta t + \mu(t)\cdot\Delta t$. Turning to differentials, we get the equation

$$dS(t) = \delta S(t) \cdot dt + \mu(t) \cdot dt,$$

or

$$S'(t) = \delta S(t) + \mu(t). \tag{2.15}$$

The general solution of equation (2.15) can be found, for instance, by the method of constant variation. First consider the corresponding homogeneous equation

$$S(t) = \delta S(t). \tag{2.16}$$

The general solution of equation (2.16) is

$S(t) = Ae^{\delta t}$.

The solution to equation (2.15) will be found in the form $S(t) = A(t)e^{\delta t}$, considering A to be a function of t. We have:

$S'(t) = A'(t)e^{\delta t} + \delta A(t)e^{\delta t}$.

Substituting in (2.15), we obtain:

$A'(t)e^{\delta t} + \delta A(t)e^{\delta t} = \delta A(t)e^{\delta t} + \mu(t)$.

From this

$A'(t) = \mu(t)e^{-\delta t}$
and

$A(t) = \int \mu(\tau)e^{-\delta\tau}d\tau + \text{const}$,
where const is arbitrary constant;

t_0 is arbitrary fixed time moment.

Thus, the general solution to equation (2.15) is

$$S(t) = \left(\int_{t_0}^{t} \mu(\tau)e^{-\delta\tau}d\tau + \text{const}\right) e^{\delta\tau}. \qquad (2.17)$$

Assuming that at time moment t_0 the accrued sum is equal to S_0, i.e. $S(t_0) = S_0$, formula (2.17) can be written as follows:

$$S(t) = S_0 e^{\delta(t-t_0)} + \int \mu(\tau)e^{\delta(t-\tau)}d\tau. \qquad (2.18)$$

In particular, if $t_0 = 0$ and $S_0 = 0$, then

$$S(t) = \int_0^t \mu(\tau)e^{\delta(t-\tau)}d\tau. \qquad (2.19)$$

According to (2.19), the present value of the financial flow with density $\mu(t)$ over the time period from 0 to T amounts to the value

$$A = e^{-\delta T} \int_0^T \mu(\tau) e^{\delta(T-\tau)} d\tau = \int_0^T \mu(\tau) e^{-\delta\tau} d\tau. \quad (2.20)$$

In order to calculate the present value and the accrued value of a continuous stream of payments, it is necessary to know its density $\mu(t)$. Below we consider two special cases: 1) density $\mu(t)$ is a linear function of time; 2) density $\mu(t)$ is an exponential function of time.

2.4.2. Linearly varying financial flows

For a linearly varying financial flow, its density is:

$\mu = R_0 + \alpha t,$

where R_0 is an initial payment amount.

Let us find the present value of the financial flow over time from 0 to t:

$$A = \int_0^t (R_0 + \alpha\tau) e^{-\delta\tau}\, d\tau = R_0 \int_0^t e^{-\delta\tau}\, d\tau + \alpha \int_0^t \tau e^{-\delta\tau}\, d\tau =$$

$$= R_0 a_{\bar{n};\delta}^{(\infty)} + \frac{1}{\delta}\left(a_{\bar{n};\delta}^{(\infty)} - te^{-\delta\tau}\right)\alpha = \left(R_0 + \frac{\alpha}{\delta}\right) a_{\bar{n};\delta}^{(\infty)} - \frac{\alpha}{\delta} te^{-\delta\tau}. \quad (2.21)$$

The formula (2.21) shows the dependence of the present value on the initial payment and the growth rate of payments.

The accrued value of the flow is found from the ratio

$$S = Ae^{\delta \cdot t}. \quad (2.22)$$

2.4.3. Exponentially varying financial flow

For an exponentially varying financial flow the density is equal to: $\mu = R_0 e^{\beta \cdot t}$, where β is the continuous rate of payments accrual.

Find the present value of the financial flow for time from 0 to t.

$$A = \int_0^t R_0 e^{\beta t} e^{-\delta\tau} d\tau = R_0 \int_0^t e^{(\beta-\delta)\tau} d\tau = R_0 \frac{e^{(\beta-\delta)t} - 1}{\beta - \delta}. \quad (2.23)$$

Herewith, the difference β – δ can be expressed in terms of the discrete growth rate of payments q:

$$\beta - \delta = \ln[(1+q)/(1-q)]. \qquad (2.24)$$

The accrued value of the flow, as before, is found from the ratio

$$S = Ae^{\delta \cdot t}. \qquad (2.25)$$

2.5. Regular financial flows

2.5.1. Ordinary annuities

A stream of positive payments separated by equal time intervals is called a **financial rent**, or simply a **rent**. The period of time between two consecutive payments is called a **rent period** (*payment period*). It is assumed that each payment is made either at the beginning of the period that corresponds to it or at the end. In the first case, the rent is called **annuity due**; in the second case, it is called **ordinary annuity** or *post-annuity*. Rents with a finite number of payments are called **finite annuities**. The interval of time between the beginning of the first period and the end of the last period is called the **term** of the finite annuity. Annuities with an infinite number of payments are called **perpetual**, or **perpetuity**. If all payments are equal, the rent is called **constant**. (A large part of Chapter 2 deals with **constant** annuities.)

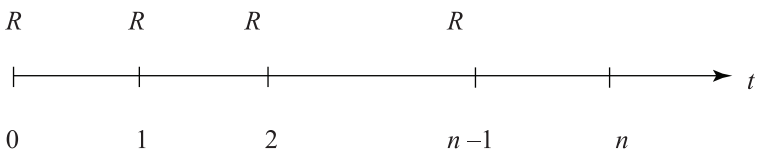

Figure 2.2. Finite annual constant annuity due

Figure 2.3. Finite annual constant ordinary annuity

The rent is described by the following parameters: the size of the single payment (annuity term), the period and term of the rent, the interest rate, the number of payments per year (p – fixed-term rents, continuous rents ($p \to \infty$)), and the method (simple, compound and continuous interest) and the frequency of interest charges (annuity with annual interest charges, with k times a year (k – multiples), with continuous interest charges).

When the period of a constant rent is one year, i.e., payments are made once a year, the rent is called an annuity (Figures 2.2, 2.3). In Russian-language literature, annuity is also called a constant annuity with an arbitrary period. Hereafter, unless otherwise specified, rents will be referred to as annuities.

2.5.2. Coefficients of annuity discounting and accrual

2.5.2.1. Ordinary annuity

Find the present value A of ordinary annuity

$$\{(0, 0), (R, 1), (R, 2), \ldots, (R, n)\} \tag{2.26}$$

with interest rate i. According to the definition

$$A = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \cdots + \frac{R}{(1+i)^n}. \tag{2.27}$$

On the right we have the sum of the terms of a geometric progression with denominator $(1+i)^{-1}$ and the first term $R/(1+i)$. Summing up using the formula for the sum of n-members of a geometric progression,

$S_n = \dfrac{a_1\left(1-q^n\right)}{1-q}$, we get:

$$A = R\frac{1-(1+i)^{-n}}{i}. \qquad (2.28)$$

The multiplier at R in the right-hand side of (2.28) is called the ordinary annuity discounting coefficient. In financial calculations it is usually denoted $a_{\overline{n}|i}$ the symbol. Thus,

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}. \qquad (2.29)$$

The ordinary annuity discounting coefficient shows how many times the present value of the rent is greater than the value of the (annual) payment.

The present value of the annuity $A = R \cdot a_{\overline{n}|i}$.

The accrued amount S of the annuity is determined by the following equation:

$$S = R(1+i)^{n-1} + R(1+i)^{n-2} + \ldots + R. \qquad (2.30)$$

On the right we have the sum of the terms of a geometric progression with denominator $q = (1 + i)$ and the first term R (considering the sum from right to left). Summing up, we obtain:

$$S = R\frac{(1+i)^{n}-1}{i}. \qquad (2.31)$$

The multiplier at R in the right-hand side of (2.31) is called the annuity coefficient of accrual and is denoted by the symbol $s_{\overline{n}|i}$. Thus,

$$s_{\overline{n}|i} = \frac{(1+i)^{n}-1}{i}. \qquad (2.32)$$

The annuity coefficient of accrual has a similar meaning to the annuity discounting coefficient: it shows by how many times the accrued value of the rent is greater than the value of the payment.

It follows from (2.32) that the annuity accrual coefficient depends only on the term (number of members) and the interest rate. The relationship between the annuity accrual coefficient and the discounting of the annuity is as follows:

$$s_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^n. \qquad (2.33)$$

The coefficients $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ represent the present value and accrued amount of a single annuity, respectively

$$H = \{(0, 0), (1, 1), (1,2), \ldots, (1, n)\}. \qquad (2.34)$$

A similar relationship exists between the present value and the accrued value of ordinary annuity

$$S = A(1 + i)^n. \qquad (2.35)$$

2.5.2.2. The accrual and discounting coefficients for several consecutive periods

We first obtain expressions for the accrual and discounting coefficients for several consecutive periods.

If the total period in question is equal to $n = n_1 + n_2$, then, discounting the rent for each of the periods to the beginning of the first period and using the possibility of adding up the values reduced to one point in time, we obtain:

$$A = A_1 + A_2, \qquad (2.36)$$

$$Ra_{\overline{n}|i} = Ra_{\overline{n_1}|i} + Ra_{\overline{n_2}|i}(1+i)^{-n_1}, \qquad (2.37)$$

from where, reducing both parts of the equation by R, we have

$$a_{\overline{n}|i} = a_{\overline{n_1}|i} + a_{\overline{n_2}|i}\nu^{n_1} = a_{\overline{n_1}|i} + a_{\overline{n_2}|i}(1+i)^{-n_1}, \qquad (2.38)$$

where $v = (1+i)^{-1}$.

In deriving formula (2.38) we used the fact that in order to bring the payment (in this case the second period rent, already reduced to the end of the first period) to the beginning of the first period, it must be discounted with period n_1.

We now obtain an expression for the accrual coefficient for two consecutive periods. By discounting the rent for each of the periods to the end of the second period and using the ability to sum the values discounted to a single point in time, we obtain

$$S = S_1 + S_2, \qquad (2.39)$$

$$Rs_{\overline{n}|i} = Rs_{\overline{n_2}|i} + Rs_{\overline{n_1}|i}(1+i)^{n_2}, \qquad (2.40)$$

from where, reducing both parts of the equation by R, we have

$$s_{\overline{n}|i} = s_{\overline{n_1}|i}(1+i)^{n_2} + s_{\overline{n_2}|i} = s_{\overline{n_1}|i} \cdot \nu^{-n_2} + s_{\overline{n_2}|i}. \qquad (2.41)$$

In deriving formula (2.41) we used the fact that to increase the payment (in this case the first period's rent, already accrued by the beginning of the second period) by the end of the second period, it must be accrued with the period n_2.

The resulting expressions (2.38) and (2.41) can easily be generalised to the case of several (k) consecutive periods $n = n_1 + n_2 + \ldots + n_k$:

$$a_{\overline{n}|i} = a_{\overline{n_1}|i} + a_{\overline{n_2}|i}\nu^{n_1} + a_{\overline{n_3}|i}\nu^{n_1+n_2} + \ldots + a_{\overline{n_k}|i}\nu^{n_1+n_2+\ldots+n_{k-1}}; \qquad (2.42)$$

$$s_{\overline{n}|i} = s_{\overline{n_k}|i} + s_{\overline{n_{k-1}}|i} \cdot \nu^{-n_k} + s_{\overline{n_{k-2}}|i} \cdot \nu^{-n_k-n_{k-1}} + \ldots + s_{\overline{n_1}|i} \cdot \nu^{-n_k-n_{k-1}-\ldots-n_2}. \qquad (2.43)$$

Example 2.2. Find the discounting coefficient for three consecutive periods of 1, 2 and 3 respectively at a rate of 10%.

First determine the discounting coefficients for each of the periods using the formula (2.29)

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}.$$

We obtain

$$a_{\overline{n_1}|i} = \frac{1-(1+0{,}1)^{-1}}{0{,}1} = 0{,}909,\ a_{\overline{n_2}|i} = \frac{1-(1+0{,}1)^{-2}}{0{,}1} = 1{,}736,$$

$$a_{\overline{n_3}|i} = \frac{1-(1+0{,}1)^{-3}}{0{,}1} = 2{,}487.$$

The following is based on the formula (2.42)

$$a_{\overline{n}|i} = a_{\overline{n_1}|i} + a_{\overline{n_2}|i}\nu^{n_1} + a_{\overline{n_3}|i}\nu^{n_1+n_2} + \ldots + a_{\overline{n_k}|i}\nu^{n_1+n_2+\ldots+n_{k-1}}$$

and we receive

$$a_{\overline{n}|i} = 0{,}909 + 1{,}736\cdot\frac{1}{1{,}1^1} + 2{,}487\cdot\frac{1}{1{,}1^{1+2}} = 0{,}909 + 1{,}578 + 1{,}869 = 4{,}356.$$

The result is easily verified by directly calculating the coefficient $a_{\overline{n}|}$

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} = \frac{1-(1+0{,}1)^{-6}}{0{,}1} = 4{,}355.$$

2.5.2.3. Annuity due

Let's find the present value $\ddot{A}$ of annuity due

$\{(R, 0), (R, 1), (R, 2), \ldots, (R, n-1), (0, n)\}$ (2.44)

with the interest rate i. According to the definition,

$$A = R + \frac{R}{(1+i)} + \cdots + \frac{R}{(1+i)^{n-1}}. \qquad (2.45)$$

On the right we have the sum of the terms of a geometric progression with denominator $(1+i)^{-1}$ and the first term R. Summing up using the formula for the sum of n-members of a geometric progression

$S_n = \frac{a_1(1-q^n)}{1-q}$, we get

$$\ddot{A} = R\frac{1-(1+i)^{-n}}{i}(1+i). \qquad (2.46)$$

The multiplier at R in the right-hand side of (2.46) is called the discounting coefficient of annuity due. In financial calculations it is usually denoted by the symbol $\ddot{a}_{n\rceil i}$. Thus,

$$\ddot{a}_{\overline{n|}i} = \frac{1-(1+i)^{-n}}{i}(1+i). \qquad (2.47)$$

The discounting coefficient of annuity due shows how many times the present value of the rent is greater than the (annual) payment.

The discounted value of annuity due is equal to $\ddot{A} = R \cdot \ddot{a}_{n\rceil i}$.

The accrued amount $\ddot{S}$ of the annuity due is determined by the following equation

$$\ddot{S} = R(1+i)^n + R(1+i)^{n-1} + \ldots + R(1+i). \qquad (2.48)$$

On the right we have the sum of n-members of a geometric progression with denominator $q = (1+i)$ and the first term $R(1+i)$ (considering the sum from right to left). Summing up, we obtain:

$$\ddot{S} = R\frac{(1+i)^n - 1}{i}(1+i) \qquad (2.49)$$

The multiplier at R in the right-hand side of (2.49) is called the annuity due **accrual coefficient** and is denoted by $\ddot{s}_{n\rceil i}$. Thus,

$$\ddot{s}_{\overline{n|}i} = \frac{(1+i)^n - 1}{i}(1+i). \qquad (2.50)$$

The annuity due accrual coefficient has the same meaning as the annuity due discounting coefficient: it shows how many times the accrued value of the prenuptial annuity is greater than the payment value.

It follows from (2.50) that the accrual coefficient of an annuity due, like an ordinary one, depends only on its term (number of members) and the interest rate. The relationship between the annuity accrual and the rent discounting coefficient is similar to the case of the ordinary annuity

$$\ddot{s}_{\overline{n|}i} = \ddot{a}_{\overline{n|}i}(1+i)^n. \qquad (2.51)$$

Coefficients $\ddot{a}_{\overline{n}|i}$: and $\ddot{s}_{\overline{n}|i}$ represent the discounted value and the accrued amount, respectively, of a single annuity:

$H = \{(1, 0), (1, 1), (1,2), \ldots, (0, n)\}$. (2.52)

A similar relationship exists between the accrued and discounted values of annuity due

$$\ddot{S} = \ddot{A} \cdot (1+i)^n .$$

It can be seen that the formulas for calculating the discounted value and the accrued amount (at the end of year n) of the annual advance rent (prenumerando) having the form of

$\{(R, 0), (R, 1), \ldots, (R, n-1), (0, n)\}$ (2.53)

are obtained from (2.46) and (2.49) by multiplying by coefficient $(1 + i)$.

2.5.2.4. The relationship between the discounted value and the accrued annuity amount

For ordinary annuities (as will be shown below, and for p-term annuities) with a single year-end interest charge, there is a relationship between the discounted value and the annuity's accrued amount:

$$A(1+i)^n = R\frac{1-(1+i)^{-n}}{i}(1+i)^n = R\frac{(1+i)^n-1}{i} = S, \qquad (2.54)$$

i.e.

$S = A(1 + i)^n = v^{-n}A$. (2.55)

Here $v = (1 + i)^{-1}$.

From this

$A = S(1 + i)^{-n} = v^n S$. (2.56)

If interest is accrued i times a year, we get

$S = A(1 + i/k)^{kn}$; (2.57)

$A = S(1 + i/k)^{-kn}$. (2.58)

A similar relationship, as noted above, exists between the discounting and accrual coefficients

$$s_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^n = a_{\overline{n}|i}\nu^{-n},\ a_{\overline{n}|i} = s_{\overline{n}|i}(1+i)^{-n} = s_{\overline{n}|i}\nu^n. \qquad (2.59)$$

2.5.2.5. The relationship between ordinary annuity and annuity due discounting and accrual coefficients

The discounting and accrual coefficients of the annuity due are denoted by the symbols $\ddot{a}_{\overline{n}|i}$: and $\ddot{s}_{\overline{n}|i}$. The following relations linking the annuity due and ordinary annuity discounting and accrual coefficients are derived directly from the definitions:

$$\ddot{a}_{\overline{n}|i} = 1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{n-1}} = (1+i)a_{\overline{n}|i}; \qquad (2.60)$$

$$\ddot{s}_{\overline{n}|i} = (1+i)^n + (1+i)^{n-1} + \ldots + (1+i) = (1+i)s_{\overline{n}|i}. \qquad (2.61)$$

2.5.3. Calculation of annuity parameters

Consider the parameters that characterize the annuity: the size of the individual payment R, the term of the rent n, the interest rate i, the accrued amount S, and the present value of the rent A. These quantities are dependent, so some of them can be expressed in terms of the others. Similar calculations are used to find the unknown annuity parameters. Different cases are possible.

1. Let n, i, R be known. Then the accrued sum S and the present value of the rent A can be found using the formulas

$A = R \cdot a(n, i)$, $S = R \cdot s(n, i)$, (2.62)

where $a(n, i)$ and $s(n, i)$ are, respectively, the rent discounting and accrual rates.

2. If A, i, R are known, then n is found from the equation

$$A = R\frac{1-(1+i)^{-n}}{i} \tag{2.63}$$

and equals to

$$n = \left[-\frac{ln(1-Ai/R)}{ln(1+i)}\right]. \tag{2.64}$$

where [...] represents the integer part.

An alternative method for determining n is to solve the equation

$A = R \cdot a(n, i)$ with respect to n. We have $A/R = a(n, i)$, then we find n from the rent conversion factor table.

3. Similar to the previous case, if S, i, R are known, then n is found from the equation

$$S = R\frac{(1+i)^{n}-1}{i} \tag{2.65}$$

and equals to

$$n = \left[\frac{ln(1-Si/R)}{ln(1+i)}\right]. \tag{2.66}$$

n can also be found from the equation $S = R \cdot s(n, i)$. We have

$S/R = s(n, i)$, then we find n from the table of coefficients of rent accrual.

4. If n, i, A are known, $R = A/a(n, i)$.

5. If n, i, S are known, $R = S/s(n, i)$.

6. If n, R, A are given, the interest rate i is determined from the following equation:

$$A = R\frac{1-(1+i)^{-n}}{i} \tag{2.67}$$

7. If n, R, S are given, the interest rate i is determined from the following equation

$$S = R\frac{(1+i)^n-1}{i}. \quad (2.68)$$

Equations in items 6 and 7 cannot be solved analytically, they can only be solved approximately. To find the interest rate *i*, a linear approximation or an iterative method can be used.

In linear approximation, knowing *R* and *A* (item 6) or *R* and *S* (item 7), we first find the discounting factor (item 6) or the accrual factor (item 7) by (2.67) or (2.68)

$$a\,(n,i) = A\,/\,R;\ s\,(n,i) = S/R. \quad (2.69)$$

Then we find the interest rate *i* using the interpolation formula

$$\frac{i-i_1}{i_2-i_1} = \frac{a-a_1}{a_2-a_1} \quad (2.70)$$

where a_1 и a_2 are the values of the discounting or accrual factors at the minimum and maximum interest rate (i_1 and i_2 respectively);

a is the value of the discounting or accrual coefficient at the assumed interest rate *i*.

The interest rate estimate *i* in formula (2.70) is overestimated when using a discounting factor and underestimated when using an accrual factor, and the accuracy of the estimate increases as the interval $i_2 - i_1$ decreases.

Example 2.3. Find the term of the ordinary annuity if S = 2000, I = 15%, R = 100.

Let us use the formula (2.66)

$$n = \left[\frac{\ln(1+Si/R)}{\ln(1+i)}\right] = \left[\frac{\ln(1+2000\cdot 0{,}15/100)}{\ln(1+0{,}15)}\right] =$$
$$= \left[\frac{\ln 4}{\ln 1{,}15}\right] = \left[\frac{1{,}386}{0{,}140}\right] = [9{,}9] = 9.$$

So, the term of the annuity is 9 years (if one takes the whole part of it), 9.9 years to be exact.

2.5.4. Perpetual, *p*-term and continuous annuities

Consider perpetual annuity

$\{(0, 0), (R, 1), (R, 2), \ldots\}$.(2.71)

Its present value A is defined as the sum of a series of

$$A = \frac{R}{1+i} + \frac{R}{(1+i)^2} \ldots \tag{2.72}$$

By summing up an infinitely decreasing geometric progression according to the formula $S = \frac{a_1}{1-q}$ with $a_1 = \frac{R}{1+i}$, $q = \frac{1}{1+i}$, we receive:

$$A = \frac{R}{1+i} \cdot \frac{1}{1-\frac{1}{1+i}} = \frac{R}{i}. \tag{2.73}$$

It is evident that

$$\lim_{n\to\infty} a_{\overline{n}|i} = \lim_{n\to\infty} \frac{1-(1+i)^{-n}}{i} = 1/i, \tag{2.74}$$

which is consistent with (2.73). With this in mind, we assume

$a\,(\infty, i) = 1/i.$

Equality (2.73) written as

$$R = Ai, \tag{2.76}$$

can be interpreted as follows: by paying (lending "forever") amount A, the owner of the perpetual rent is entitled to receive rent payments equal to the interest on the amount A.

The accrued value of the perpetual annuity and the accrual factor are equal to infinity. For the latter we have

$$s_{\overline{\infty}|i} = \lim_{n\to\infty} a_{\overline{n}|i} \left(1+i\right)^n = a_{\overline{\infty}|i} \cdot \infty = \infty.$$

Example 2.4. Find the size of the deposit that ensures that at the end of each year \$2,000 will be received in perpetuity at a compound interest rate of 14% per annum. Using the formula (2.73),

$$A = \frac{R}{i} = \frac{2000}{0.14} = \$14\,285.71\ .$$

So, having deposited \$14,285.71 at 14% per annum, the depositor (and his heirs) will receive \$2,000 at the end of each year in perpetuity.

2.5.5. *P*-term annuity

When the rent payment R is not made in a lump sum (once at the end of the annual period), but is divided into p equal payments evenly distributed throughout the year, the corresponding stream of payments has the form:

$$CF = \{(\ R/p,\ 1/p),\ (R/p,\ 2/p),\ \ldots,\ (R/p,\ n - 1/p)\ ),\ (R/p,\ n)\} \qquad (2.77)$$

and is called **a *p*-term annuity** (Figure 2.4). Let interest accrue k times a year in this case. Consider the following cases:

$k = 1,\ k = p,\ k \neq p.$

Figure 2.4. *p*-term ordinary annuity

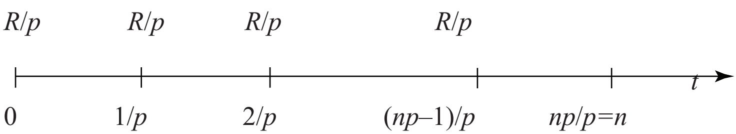

Figure 2.5. *p*-term annuity due

2.5.5.1. *P*-term annuity (case of *k* = 1)

Let's find the present value of the p-term ordinary rent. A total of np payments of R/p each are made over n years. Converting them to $t = 0$, we have

$$A^{(p)} = \frac{R}{p(1+i)^{1/p}} + \frac{R}{p(1+i)^{2/p}} + \ldots + \frac{R}{(1+i)^{np/p}}. \qquad (2.78)$$

Summing up the geometric progression with $a_1 = R/p$, $q = \frac{1}{(1+i)^{1/p}}$,

and $n \to np$, we get **the present value of the p-term rent**:

$$A^{(p)} = \frac{R}{p} \cdot \frac{1-(1+i)^{-np/p}}{1-(1+i)^{-1/p}} = \frac{R}{p} \cdot \frac{1-(1+i)^{-n}}{(1+i)^{1/p}-1}. \qquad (2.79)$$

The multiplier

$$a_{\overline{n}|i}^{(p)} = \frac{1}{p} \cdot \frac{1-(1+i)^{-n}}{(1+i)^{1/p}-1} \qquad (2.80)$$

is called the **p-term annuity discounting factor**.

Now calculate the accrued value of the p-term rent. Over n years, np payments of R/p each are made. The rent is a geometric progression with the first term R/p и and denominator $(1+i)^{1/p}$ (starting from the last payment)

$$S^{(p)} = \frac{R}{p} + \frac{R}{p}(1+i)^{1/p} + \cdots + \frac{R}{p}(1+i)^{\frac{np}{p}}. \qquad (2.81)$$

By finding its amount, we get for the **accrued value of the p-term annuity**

$$S^{(p)} = \frac{R}{p} \cdot \frac{(1+i)^{n}-1}{(1+i)^{1/p}-1} = Rs_{\overline{n}|i}^{(p)}. \qquad (2.82)$$

The multiplier

$$s_{\overline{n}|i}^{(p)} = \frac{1}{p} \cdot \frac{(1+i)^{n}-1}{(1+i)^{1/p}-1} \qquad (2.83)$$

is called the ***p*-term annuity accrual coefficient**.

2.5.5.2. The relationship between the present value and the accrued value of the *p*-term rent

Let us establish the relationship between the present value and the accrued value of the *p*-term rent. This is easily obtained from formulae (2.80), (2.83) and has the same form as for the usual annual rent

$$S^{(p)} = A^{(p)}\cdot(1+i)^n. \qquad (2.84)$$

2.5.5.3. Continuous annuity

Moving to the limit at $p \to \infty$, we obtain a continuous financial flow with a constant density $\mu(t) = R$, the so-called *continuous annuity*.

Let us calculate the limit $A^{(p)} = \frac{R}{p}\cdot\frac{1-(1+i)^{-n}}{(1+i)^{1/p}-1}$ in case of $p\to\infty$. Using the Lopital's rule, let's calculate the limit

$$\lim_{p\to\infty}\frac{(1+i)^{1/p}-1}{1/p} = \lim_{p\to\infty}\frac{(1+i)^{1/p}\left(-1/p^2\right)}{-1/p^2}\ln(1+i) = \ln(1+i).$$

Using it, we obtain an expression for the present value of the continuous rent

$$A^{(\infty)} = \lim_{p\to\infty} A^{(p)} = \frac{R}{p}\cdot\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{p}}-1} = R\cdot\frac{1-(1+i)^{-n}}{\ln(1+i)}. \qquad (2.85)$$

The discount factor is equal to

$$a_{\overline{n}|i}^{(\infty)} = \frac{1-(1+i)^{-n}}{\ln(1+i)}. \qquad (2.86)$$

For the accrued sum and the continuous annuity accrual factor, we easily obtain the formulas from (2.85) and (2.86):

$$S = R\cdot\frac{1-(1+i)^{-n}}{\ln(1+i)}\cdot(1+i)^n = R\cdot\frac{(1+i)^n-1}{\ln(1+i)};\cdot$$

$$s_{\overline{n}|i}^{(\infty)} = \frac{(1+i)^n-1}{\ln(1+i)}. \qquad (2.87)$$

The formulas show that the transition from discrete to continuous annuities leads to an $i/ln(1+i)$ times increase in the discounting and accrual coefficients, i.e. we have the following relationship between the coefficients

$$a_{\overline{n}|i}^{(\infty)} = \frac{i}{\ln(1+i)} a_{\overline{n}|i},\ s_{\overline{n}|i}^{(\infty)} = \frac{i}{\ln(1+i)} s_{\overline{n}|i}. \tag{2.88}$$

2.5.5.4. *P* – term annuity (case of *k* ≠ *p*)

Consider the most general case, a *p*-term annuity with interest accruing *k* times a year. The number of annuity terms is *np*, payments of *R/p* each. The rent is a geometric progression with the first term *R/p* and the denominator $(1+i/k)^{k/p}$. By calculating its sum, we obtain for the accrued value of the *p*-term rent

$$S^{(p)} = S^{(p)} = \frac{R}{p} \cdot \frac{(1+i/k)^{(k/p)(np)} - 1}{(1+i/k)^{k/p} - 1} = R \cdot \frac{(1+i/k)^{kn} - 1}{p\left[(1+i/k)^{k/p} - 1\right]} = Rs_{\overline{kn}|i/k}^{(p)} \tag{2.89}$$

For the present value of the annuity, we have

$$\begin{aligned} A^{(p)} &= \frac{R}{p} \cdot \left[\frac{1}{(1+i/k)^{k/p}} + \frac{1}{(1+i/k)^{2k/p}} + \ldots \right] = \\ &= \frac{R}{p} \cdot \frac{(1+i/k)^{-k/p}\left[1-(1+i/k)^{-(k/p)np}\right]}{1-(1+i/k)^{-(k/p)}} = \\ &= R \cdot \frac{1-(1+i/k)^{-nk}}{p\left[(1+i/k)^{k/p} - 1\right]} = Ra_{\overline{k\overline{n}}|i/k}^{(p)}. \end{aligned} \tag{2.90}$$

2.5.5.5. The relationship between the discounted value and the accrued value p-term annuity with k-fold interest accrual

Finally, let us establish the relationship between the discounted and the accrued values of the p-term annuity with k-fold interest. This is easily obtained from formulae (2.89) and (2.90)

$$S_k^{(p)} = A_k^{(p)}(1 + \tfrac{i}{k})^{kn}, \quad A_k^{(p)} = S_k^{(p)}(1 + \tfrac{i}{k})^{-kn}. \qquad (2.91)$$

2.5.5.6. *P*-term annuity (case *k* = *p*)

The number of members of the annuity is equal to the number of interest accruals, payments of R/k each. This case is the most common in practice. From (2.90), assuming $p = k$, we obtain for the present value of the annuity

$$A^{(p)} = \frac{R}{k} \cdot \frac{1-(1+i/k)^{-np}}{i/k} = R \cdot \frac{1-(1+i/k)^{-np}}{i} \qquad (2.92)$$

Multiplier

$$a_{ni}^{(p)} = \frac{1-(1+i/p)^{-np}}{i} \qquad (2.93)$$

is the discounting coefficient of the p-term annuity in the case $k = p$.

For the accrual value of the *p-fold* annuity we obtain

$$S^{(p)} = \frac{R}{k} \cdot \frac{\left(1+\frac{i}{k}\right)^{nk} - 1}{\frac{i}{k}} = R \cdot \frac{\left(1+\frac{i}{k}\right)^{nk} - 1}{\mathrm{i}}. \qquad (2.94)$$

2.5.5.7. *P*-term annuity with continuous accrual of interest

Using the formula (2.90)

$$A_k^{(p)} = R \cdot \frac{1-(1+i/k)^{-nk}}{p\left[(1+i/k)^{k/p} - 1\right]}$$

and passing to the limit at $k \to \infty$, we obtain for the present value of the annuity

$$A_{\infty}^{(p)} = \lim_{k\to\infty} A_k^{(p)} = \lim_{k\to\infty} R\cdot\frac{1-(1+i/k)^{-n\kappa}}{p\left[(1+i/k)^{k/p}-1\right]} = \frac{R}{p}\cdot\frac{1-e^{-ni}}{e^{i/p}-1},$$

$$A_{\infty}^{(p)} = \frac{R}{p}\cdot\frac{1-e^{-ni}}{e^{i/p}-1}. \qquad (2.95)$$

Let us show that the relationship between the present value and the accrued value of the annuity with continuous accrual of interest is as follows:

$$S_{\infty}(p) = A_{\infty}(p)\cdot e^{in} \qquad (2.96)$$

From this we get an expression for the accrued value of the p-term annuity with continuous interest accrual

$$S_{\infty}^{(p)} = \frac{R}{p}\cdot\frac{1-e^{-ni}}{e^{i/p}-1}\cdot e^{ni} = \frac{R}{p}\cdot\frac{e^{ni}-1}{e^{i/p}-1},$$

$$S_{\infty}^{(p)} = \frac{R}{p}\cdot\frac{e^{ni}-1}{e^{i/p}-1}. \qquad (2.97)$$

2.5.5.8. Continuous annuity with *k*-fold interest rate

Let's find the present value of continuous annuity with k-fold interest accrual

$$A_{\infty,k} = \lim_{p\to\infty} R\frac{1-(1+i/k)^{-nk}}{p\left[(1+i/k)^{k/p}-1\right]}. \qquad (2.98)$$

Using Lopital's rule, we obtain

$$A_{\infty,k} = \lim_{p\to\infty} R\left[1-(1+i/k)^{-nk}\right]\frac{1/p}{\left[(1+i/k)^{k/p}-1\right]} =$$
$$= \lim_{p\to\infty} R\left[1-(1+i/k)^{-nk}\right]\frac{-(1/p^2)}{(1+i/k)^{k/p}\,k(-1/p^2)\ln(1+i/k)} =$$
$$= R\frac{1-(1+i/k)^{-nk}}{k\ln(1+i/k)}. \tag{2.99}$$

So, for present value of continuous annuity with k-fold interest, we have

$$S^{(p)} = R\cdot\frac{1-(1+i/k)^{-nk}}{k\cdot\ln(1+i/k)} \tag{2.100}$$

Similarly, we find the accrued value of continuous annuity with k-fold accrual of interest. Using the formula

$$S^{(p)} = R\cdot\frac{(1+i/k)^{kn}-1}{p\left[(1+i/k)^{k/p}-1\right]} \tag{2.101}$$

and passing to the limit at p→ ∞, we have

$$S_{\infty,k} = \lim_{p\to\infty} R\cdot\frac{(1+i/k)^{kn}-1}{p\left[(1+i/k)^{k/p}-1\right]}. \tag{2.102}$$

Using Lopital’s rule, we obtain

$$S_{\infty,k} = \lim_{p\to\infty} R\left[(1+i/k)^{nk}-1\right]\frac{1/p}{\left[(1+i/k)^{k/p}-1\right]} =$$
$$= \lim_{p\to\infty} R\cdot\left[(1+i/k)^{nk}-1\right]\frac{-(1/p^2)}{(1+i/k)^{k/p}\,k(-1/p^2)\ln(1+i/k)} =$$
$$= R\cdot\frac{(1+i/k)^{nk}-1}{k\ln(1+i/k)}. \tag{2.103}$$

So, for the accrued value of continuous annuity with k-fold accrual of interest, we have

$$S_{\infty,k} = R \cdot \frac{(1+i/k)^{nk} - 1}{k \cdot \ln(1+i/k)} \qquad (2.104)$$

2.5.5.9. The relationship between the present value and the accrued value of the continuous annuity with *k*-fold interest accrual

Finally, let us establish the relationship between the present value and the accrued value of the continuous annuity with k-fold interest accrual. It is easily obtained from formulas (2.100) and (2.104)

$$S_{\infty,k} = A_{\infty,k} \cdot (1+i/k)^{nk} \qquad (2.105)$$

Example 2.5. Calculate the accrued value of 8-year 15% continuous annuity with 12 times accrual interest and annuity payment R = 150.

According to formula (2.104)

$$S_{\infty,k} = R \cdot \frac{(1+i/k)^{nk} - 1}{k \cdot \ln(1+i/k)}.$$

We have

$$S_{\infty,12} = 150 \cdot \frac{(1+0{,}15/12)^{8\cdot 12} - 1}{12\ln(1+0{,}15/12)} = 150 \cdot \frac{2{,}296}{0{,}149} = 2311{,}41.$$

2.5.5.10. Continuous annuity with continuous accrual of interest

From formula (2.100)

$$A_{\infty,k} = R \cdot \frac{1-(1+i/k)^{-nk}}{k \cdot \ln(1+i/k)}$$

it is easy to obtain the present value of the continuous annuity with continuous interest by going to the limit of $k \to \infty$

$$A_{\infty,\infty} = \lim_{k\to\infty} R \cdot \frac{1-(1+i/k)^{-nk}}{k\ln(1+i/k)} = \lim_{k\to\infty} R \cdot \frac{1-(1+i/k)^{\frac{k}{i}(-nk)\frac{i}{k}}}{\ln(1+i/k)^{\frac{k}{i}i}} = R \cdot \frac{1-e^{-ni}}{i}. \quad (2.106)$$

So, for the present value of the continuous annuity with continuous interest, we obtain the expression

$$A_{\infty,\infty} = R \cdot \frac{1-e^{-ni}}{i} \quad (2.107)$$

Similarly, we find the accrued value of the continuous annuity with continuous accrual of interest.

In the formula

$$S_{\infty,k} = R \cdot \frac{(1+i/k)^{nk}-1}{k \cdot \ln(1+i/k)} \quad (2.108)$$

proceeding to the limit $k \to \infty$, we get

$$S_{\infty,\infty} = \lim_{k\to\infty} R \cdot \frac{(1+i/k)^{nk}-1}{k\ln(1+i/k)} = \lim_{k\to\infty} R \cdot \frac{(1+i/k)^{\frac{k}{i}(nk)\frac{i}{k}}-1}{\ln(1+i/k)^{\frac{k}{i}i}} = R \cdot \frac{e^{ni}-1}{i}. \quad (2.109)$$

So, for the present value of the continuous annuity with continuous accrual of interest, the expression is obtained

$$S_{\infty,\infty} = R \cdot \frac{e^{ni}-1}{i}. \quad (2.110)$$

2.5.5.11. The relationship between the present value and the accrued value of a continuous annuity with continuous interest accrual

In conclusion, let us note the relationship between the present value and the accrued value of the continuous annuity with continuous interest. This can be easily obtained from formulae (2.107) and (2.110):

$$S_{\infty,\infty} = A_{\infty,\infty} \cdot e^{ni} . \qquad (2.111)$$

2.5.5.12. The relationship between the present value and the accrued value of arbitrary annuities

The analysis of all cases of connection between the present value and the accrued value of annuities shows that the connection coefficient depends only on the multiplicity of interest accrual and does not depend on the term of the annuity or any of its other parameters.

Thus, we have:

- in case of a one-time interest accrual:

$$S = A \cdot (1+ i)n \text{ , } A = S \cdot (1+ i)^{-n} ; \qquad (2.112)$$

- with k-fold accrual of interest:

$$S = A \cdot (1 + i / k)^{nk} ; A = S \cdot (1 + i / k)^{-nk} \qquad (2.113)$$

- in the case of continuous accrual of interest:

$$S = A \cdot e^{ni}, \ A = S \cdot e^{-ni} . \qquad (2.114)$$

Example 2.6. Calculate the present value and the accrued value of a 10-year noncontinuous annuity with continuous interest with the annuity payment R = 300 at the rate of 14% per annum.

Use formula (2.101) to find the present value of the annuity

$$A_{\infty,\infty} = R \cdot \frac{1-e^{-ni}}{i} = 300 \cdot \frac{1-e^{-10 \cdot 0.14}}{0.14} = 300 \cdot 5.38 = 1614$$

The accrued value of the annuity can be found either by the formula (2.110)

$$A_{\infty,\infty} = R \cdot \frac{e^{ni}-1}{i} = 300 \cdot \frac{e^{10 \cdot 0.14}-1}{0.14} = 300 \cdot 21.823 = 6{,}546.9 \text{ or}$$

using the relationship between the present value and the accrued value of the continuous annuity with continuous interest (2.114)

$S\infty,\infty = A\ \infty,\infty \cdot e^{ni} = 1614 \cdot e^{10\ \cdot\ 0.14} = 1614 \cdot 4.055 = 6544.8$.

The small difference (2 : 6545 = 0.03%) is due to the approximate calculations from the two different annuity schemes.

2.5.6. Other types of annuity

2.5.6.1. Annuity due

As mentioned above, due annuities are annuities with payments at the beginning of periods. Compared to the ordinary annuity, the accrual for each annuity member (except for the last one) is *(1 + i)* times higher in this case due to the accrual for the first period.

Therefore, the accrued amount of due annuity $\ddot{S}$ is (1 + i) times is greater than the accrued sum of the ordinary annuity

$\ddot{S} = S\,(1+i)$. (2.115)

Similar relations take place for the given values of the annuities

$\ddot{A} = A\,(1+i)$. (2.116)

As noted in the previous sections, the conversion and increment coefficients of the due and ordinary annuities are related by the relations

$\ddot{a}_{\overline{n}|i} = (1+i)\,a_{\overline{n}|i},\ \ddot{s}_{\overline{n}|i} = (1+i)\,s_{\overline{n}|i}$. (2.117)

$$\ddot{S} = S(1+i/k)^k,\ \ddot{A} = A(1+i/k)^k. \qquad (2.118)$$

For an annual annuity with interest accrued k times a year we have

For a p-term annuity with interest accruing k times a year we have

$$k = 1:\ \ddot{S} = S(1+i)^{1/p},$$
$$k \neq p:\ \ddot{S} = S(1+i/k)^{k/p}. \qquad (2.119)$$

Turning in each of the equations (2.119) to the limit at $p \to \infty$, we find that for continuous annuities (at any multiplicity of interest accrual) the relations are satisfied

$$\ddot{S} = S,$$
$$\ddot{A} = A. \qquad (2.120)$$

That is, for continuous annuities the concepts “due” and “ordinary” do not exist (or coincide) due to the fact that the interval between payments tends to zero.

Formulas (2.120) are easily obtained from the expressions for the present value and the accrued sum of p-terms of due and ordinary annuities, passing to the limit at $p \to \infty$.

2.5.6.2. Annuities with payments in the middle of periods

If the payments are distributed more or less evenly, but their receipt does not fall on the beginning or the end of the period, it is possible to attribute the total payments for the period to the middle of periods. In this case the annuities reduced and accrued are equal to the corresponding ordinary annuities accrued for half of the period:

$S_{1/2} = S(1 + i)^{1/2}$, $A_{1/2} = A(1 + i)^{1/2}$ при $p = 1$, $k = 1$; (2.121)

$S_{1/2} = S(1 + i)^{1/2p}$, $A_{1/2} = A(1 + i)^{1/2p}$ при $p > 1$, $k = 1$; (2.122)

$S_{1/2} = S(1 + i/k)^{k/2}$, $A_{1/2} = A(1 + i/k)^{k/2}$, при $p = 1$, $k > 1$; (2.123)

$S_{1/2} = S(1 + i/k)^{k/2p}$, $A_{1/2} = A(1 + i/k)^{k/2p}$, при $p > 1$, $k > 1$. (2.124)

2.5.6.3. Immediate and deferred annuities

Immediate annuities are annuities whose payments are made now (at the beginning or end of periods). Deferred annuity is an annuity whose beginning of payments has been postponed for some time t. Deferred annuity does not affect its accrued value, but the present value of annuity ${}_tA$ changes

$$ {}_tA = A\nu^t = Ra_{\overline{n}|i}\nu^t. \tag{2.125}$$

Tables 2.1 and 2.2 summarize the present and accrued values of the ordinary and due annuities.

Table 2.1. Present and accrued values of ordinary annuities

ANNUITY TYPE		The present value of A	The accrued value S
Annual		$R\cdot\frac{1-(1+i)^{-n}}{i}$	$R\cdot\frac{(1+i)^{n}-1}{i}$
Perpetual		R/i	∞
p-term	$k=1$	$R\cdot\frac{1-(1+i)^{-n}}{(1+i)^{-1/p}-1}$	$R\cdot\frac{(1+i)^{n}-1}{(1+i)^{-1/p}-1}$
	$k\neq p$	$R\cdot\frac{1-(1+i/k)^{-nk}}{p\left[(1+i/k)^{k/p}-1\right]}$	$R\cdot\frac{(1+i/k)^{kn}-1}{p\left[(1+i/k)^{k/p}-1\right]}$
	$k=p$	$R\cdot\frac{1-(1+i/k)^{-nk}}{i}$	$R\cdot\frac{(1+i/k)^{nk}-1}{i}$
Continuous		$R\cdot\frac{1-(1+i)^{-n}}{\ln(1+i)}$	$R\cdot\frac{(1+i)^{n}-1}{\ln(1+i)}$
p-term with continuous accrual of interest		$\frac{R}{p}\cdot\frac{1-e^{-ni}}{e^{i/p}-1}$	$\frac{R}{p}\cdot\frac{e^{ni}-1}{e^{i/p}-1}$
Continuous with k-fold accrual of interest		$R\cdot\frac{1-(1+i/k)^{-nk}}{k\ln(1+i/k)}$	$R\cdot\frac{(1+i/k)^{nk}-1}{k\ln(1+i/k)}$
Continuous with continuous accrual of interest		$R\cdot\frac{1-e^{-ni}}{i}$	$R\cdot\frac{e^{ni}-1}{i}$

Table 2.2. Present and accrued values of due annuities

ANNUITY TYPE		The present value A	The accrued value S
Annual		$R\frac{1-(1+i)^{-n}}{i}(1+i)$	$R\frac{(1+i)^{n}-1}{i}(1+i)$
Perpetual		$R(1+i)/i$	∞
p-term	$k=1$	$\frac{R}{p}\cdot\frac{1-(1+i)^{-n}}{(1+i)^{1/p}-1}(1+i)^{1/p}$	$\frac{R}{p}\cdot\frac{(1+i)^{n}-1}{(1+i)^{1/p}-1}(1+i)^{1/p}$
	$k\neq p$	$R\cdot\frac{1-(1+i/k)^{-nk}}{p\left[(1+i/k)^{k/p}-1\right]}\cdot(1+i/k)^{k/p}$	$R\cdot\frac{(1+i/k)^{kn}-1}{p\left[(1+i/k)^{k/p}-1\right]}\cdot(1+i/k)^{k/p}$
	$k=p$	$R\cdot\frac{1-(1+i/k)^{-nk}}{i}\cdot(1+i/k)$	$R\cdot\frac{(1+i/k)^{nk}-1}{i}\cdot(1+i/k)$
Continuous		$R\cdot\frac{1-(1+i)^{-n}}{\ln(1+i)}$	$R\cdot\frac{(1+i)^{n}-1}{\ln(1+i)}$
p-term with continuous accrual of interest		$\frac{R}{p}\cdot\frac{1-e^{-ni}}{e^{i/p}-1}\cdot e^{i/p}$	$\frac{R}{p}\cdot\frac{e^{ni}-1}{e^{i/p}-1}\cdot e^{i/p}$
Continuous with k-fold accrual of interest		$R\cdot\frac{1-(1+i/k)^{-nk}}{k\ln(1+i/k)}$	$R\cdot\frac{(1+i/k)^{nk}-1}{k\ln(1+i/k)}$
Continuous with continuous accrual of interest		$R\cdot\frac{1-e^{-ni}}{i}$	$R\cdot\frac{e^{ni}-1}{i}$

2.5.7. Arithmetic and geometric annuities

Below we will consider annuities with constant absolute and constant relative change in payments over time – *arithmetic and geometric* annuities. In the arithmetic annuity the values of periodic payments change linearly, in the geometric annuity – exponentially.

2.5.7.1. Arithmetic annuities

In the arithmetic annuity each new payment differs from the previous one by the same value Q. The financial flow of annual arithmetic annuity for n years looks like:

$$C^{(n)} = \{(R, 1), (R + Q, 2), \dots, (R + (n - 1)Q, n)\}. \quad (2.126)$$

This flow can be represented as a linear combination of two annuities $G^{(n)}$ and $I^{(n)}$:

$$C^{(n)} = R \cdot G^{(n)} + Q \cdot I^{(n)}. \quad (2.127)$$

Here the ordinary unit annuity:

$$G^{(n)} = \{(1, 1), (1, 2), \dots, (1, n)\}; \quad (2.128)$$

unit arithmetic annuity:

$$H^{(n)} = \{(0, 1), (1, 2), \dots, (n - 1, n)\}; \quad (2.129)$$

Then

$$PV(C^{(n)}) = R \cdot PV(G^{(n)}) + Q \cdot PV(H^{(n)}). \quad (2.130)$$

Here are several ways to calculate the present value of a unit arithmetic annuity.

1. for $v = (1 + i)^{-1}$ we have:

$$PV(H^{(n)}) = v^2 + 2v^3 + \dots + (n - 1)v^n = v^2(1 + 2v + \dots + (n - 1)v^{n-2}).$$

Let's put,

$f(x) = 1 + x + x^2 + \ldots + x^{n-1}$.

Then

$f'(x) = 1 + 2x + \ldots + (n-1)x^{n-2}$.

The value of $f(x)$ can be found as the sum of the terms of a geometric progression:

Hence,

$$f'(x) = \frac{1-x^n}{(1-x)^2} - \frac{nx^{n-1}}{1-x} \cdot f(x) = \frac{1-x^n}{1-x}$$

Since

We get:

$$PV\left(H^{(n)}\right) = v^2 \cdot \frac{1-v^n}{(1-v)^2} - v\frac{nv^n}{1-v} = \frac{1-(1+i)^{-n}}{i^2} - \frac{n(1+i)^{-n}}{i}.$$

$$PV(H^{(n)}) = v^2(1 + 2v + \ldots + nv^{n-1}) = v^2 \cdot f'(v), \tag{2.131}$$

Using (2.29), we can rewrite (2.100) as:

$$PV\left(I^{(n)}\right) = \frac{a_{\overline{n}|i} - n(1+i)^{-n}}{i}, n \geq 2. \quad i \tag{2.132}$$

From this we obtain the formula for the future value of stream I (accumulated value at time $t = n$):

$$FV\left(H^{(n)}\right) = \frac{s_{\overline{n}|i} - n}{i}.$$

Now write down the formulas for the present and future value of the arithmetic annuity:

$$PV\left(C^{(n)}\right) = R \cdot a_{\overline{n}|i} + Q\frac{a_{\overline{n}|i} - n(1+i)^{-n}}{i}; \tag{2.133}$$

$$FV\left(C^{(n)}\right) = R \cdot s_{\overline{n}|i} + Q\frac{s_{\overline{n}|i} - n}{i}. \tag{2.134}$$

$$PV\left(C^{(\infty)}\right) = \lim_{n\to\infty} PV\left(C^{(n)}\right) = R \cdot \frac{1}{i} + Q \cdot \frac{1}{i^2}. \tag{2.135}$$

Going to the limit in (2.133) when n tends to infinity, we obtain the present value of the infinite arithmetic annuity $C^{(\infty)}$. As $\lim_{n\to\infty} n \cdot (1+i)^{-n} = 0$, then

2. Just as for the ordinary annuity, formulas (2.133) and (2.124) for the finite arithmetic annuity can be obtained by determining the value of the infinite annuity. In fact, the financial flow of the infinite arithmetic annuity

$H = \{(0, 1), (1, 2), (2, 3), \ldots\}$

can be represented as an amount of one year's deferred ordinary perpetual annuity with single payments

$G_1 = \{(0, 1), (1, 2), (1, 3), \ldots\}$

and infinite arithmetic annuity

$H_1 = \{(0, 1), (0, 2), (1, 3), (2, 4), \ldots\}$.

$$PV(H) = \frac{1}{1+i} \cdot \left(\frac{1}{i} + PV(H)\right).$$

By reducing the values of all annuities to the initial point in time, we obtain the equation

From here $PV(H) = 1/i^2$, which gives us formula (2.135). Finally, representing a finite arithmetic annuity as the difference of two infinite annuities (starting respectively at times t=0 and t=n) and applying formula (2.135), we obtain

$$PV\left(C^{(n)}\right)=R\cdot\frac{1}{i}+Q\cdot\frac{1}{i^2}-\frac{1}{(1+i)^n}\left((R+nQ)\cdot\frac{1}{i}+Q\frac{1}{i^2}\right).$$

Here is another way to calculate the present value of the arithmetic annuity:

$$A = R\nu + (R+Q)\nu^2 + \ldots + (R+(n-1)Q)\nu^n. \qquad (2.136)$$

here $v = (1+i)^{-1}$.

Multiplying equality (2.136) by (1+i) and subtracting equality (2.135), we obtain

$$Ai = R + Qv + Qv^2 + \ldots + Qv^{n-1} - [R+(n-1)Q]\cdot v^n =$$
$$= R(1-v^n) + Q\cdot\sum_{j=1}^{n-1} v^j - nQv^n + Qv^n.$$

From here

$$A = R\left(\frac{1-\nu^n}{i}\right)+\frac{Qa_{\overline{n}|i}-nQ\nu^n}{i}=\left(R+\frac{Q}{i}\right)a_{\overline{n}|i}-\frac{nQ\nu^n}{i}. \qquad (2.137)$$

2.5.7.2. *P*-term arithmetic annuity

Consider the p-term arithmetic annuity

$C^{(n)} = \{(R, 1), (R + Q/p, 2), \ldots, (R + (n-1)Q/p, n)\}$.

The current payment of this annuity is $R + (j - 1)Q/p, j = 1, 2, \ldots, pn$. For the present value and the accrued value of the ordinary annuity we have, respectively.

$$A=\sum_{j=1}^{pn}\left(R+\frac{Qj}{p}\right)\cdot \nu^{j/p}; \qquad (2.138)$$

$$S=\sum_{j=1}^{pn}\left(R+\frac{Q(j-1)}{p}\right)\cdot \nu^{j/p-n}. \qquad (2.139)$$

2.5.7.3. Continuous arithmetic annuities

A continuous analogue of the arithmetic annuity is a financial flow with density $\mu(t) = R + \gamma t$. The present value of such flow for the time interval from 0 to T is

$$A = \int_{0} (R + \gamma t)e^{-\delta t}dt, \qquad (2.140)$$

where $\delta = \ln(1 + i)$ — intensity of growth rate.

Calculating the integral, we find:

$$A = -\left(\frac{\gamma}{\delta^2} + \frac{R}{\delta} + \frac{\gamma}{\delta}t\right)e^{-\delta t}\Big|_0^T = \left(\gamma\delta^2 + r\delta\right)\left(1 - e^{-\delta T}\right) - \gamma\delta T e^{-\delta T} =$$

$$= R \cdot a^{(\infty)}_{\overline{T}|i} + \gamma \cdot \frac{a^{(\infty)}_{\overline{T}|i} - Te^{-\delta T}}{\delta} \qquad (2.141)$$

Here

$$a^{(\infty)}_{\overline{T}|i} = \frac{1 - e^{-\delta T}}{\delta}$$

2.5.7.4. Geometric annuities postnumerando

A geometric annuity is an annuity in which payments change over time with a constant relative increase q, i.e., each successive payment differs from the previous one by the same number of pro-cents q. In other words, q is the growth rate of payments. The financial flow of annual geometric annuity for n years looks like:

$$E^{(n)} = \{(1, R), (2, R(1 + q)), (3, R(1+ q)^2), \ldots, (n, R(1 + q)^{n-1})\}. \quad (2.142)$$

Calculation of the present value of geometric annuity is reduced to the sum of discounted payments, i.e. to the calculation of the sum of geometric progression with the first term Rv and the denominator $k = 1 + q$:

$$A = \sum_{t=1}^{n} \frac{R(1+q)^{t-1}}{(1+i)^t} = \frac{R}{1+i} \cdot \frac{1 - \left(\frac{1+q}{1+i}\right)^n}{1 - \frac{1+q}{1+i}} = R \cdot \frac{1 - \left(\frac{1+q}{1+i}\right)^n}{i - q}. \qquad (2.143)$$

Let's point out that the increment can be both positive ($q > 0$), and negative ($q < 0$).

For the incremental value of the geometrical annuity we obtain

$$S = A \cdot (1+i)^n = R \cdot \frac{k^n - (1+i)^n}{k - (1+i)} = R \cdot \frac{(1+q)^n - (1+i)^n}{q - i} \tag{2.144}$$

2.5.7.5. *P*-term geometric annuity

If payments are made p times a year, and interest (at rate i) is charged once a year postnumerando, then the payments are a geometric progression

$$C^{(n)} = \{(R, 1), (Rk, 2), \dots, (Rk^{np-1}, n)\}. \tag{2.145}$$

Here $k = 1 + q$ — growth rate for the period. Discounting and summing up the terms of the progression, we obtain for the accrued value of the annuity [2, 4]

$$S = R \cdot \frac{k^{np} - (1+i)^n}{k - (1+i)^{1/p}} = R \cdot \frac{(1+q)^{np} - (1+i)^n}{1+q-(1+i)^{1/p}} \tag{2.146}$$

For the given value of the annuity we have

$$A = R \cdot \frac{k^{np} \cdot v^n - 1}{k - (1+i)^{1/p}} = R \cdot \frac{(1+q)^{np} \cdot v^n - 1}{1+q-(1+i)^{1/p}} \tag{2.147}$$

2.5.7.6. Geometric annuity due

By performing similar calculations for the geometric due annuity or by using the relations between the values of the reduced and accrued annuities of the ordinary and due, specified in Chapter 2, we obtain the following expressions for its reduced and accrued annuities

$$\ddot{A} = R\frac{(k\nu)^n - 1}{k\nu - 1}(1+i) = R\frac{1-\left(\frac{1+q}{1+i}\right)^n}{q-i}(1+i); \qquad (2.148)$$

$$\ddot{S} = R\frac{(k\nu)^n - 1}{k\nu - 1}(1+i)^n = R\frac{1-\left(\frac{1+q}{1+i}\right)^n}{q-i}(1+i)^{n+1}. \qquad (2.149)$$

2.5.8. Comparison of financial flows and annuities

2.5.8.1. The general principle of comparing financial flows and annuities

Quite often it is necessary to choose between several annuities with different parameters. In order to be able to make an informed choice, it is necessary to be able to compare annuities. The same problem arises when comparing financial flows of a more general type. If the terms of the annuities or financial flows being compared are the same, it is necessary to compare the accrued values of the annuities (flows) and choose the annuity (flow) whose accrued value is greater.

Alternative ways to choose the annuity (stream) are to compare their present values (discounted to the initial point in time) or present values (discounted to some point in time between the initial and final points).

Let us consider two flows:

$CF_1 = \{(P_0, t_0), (P_1, t_1), \ldots, (P_n, t_n)\}$ (2.150)

and

$CF_1 = \{(Q_0, t_0), (Q_1, t_1), \ldots, (Q_n, t_n)\}$,

which differ only in the size of payments P_i and Q_i. If we compare the present values of these flows, the result of the comparison will generally depend on the discount rate, i.e. at one interest rate the first flow may be preferable, and at another – the second. Under certain conditions the present value of the first stream (annuity) will be greater than the present value of the second stream (annuity) at any discount rate.

An obvious sufficient condition for this is the fulfillment of the inequalities $Pi \geq Qi$ for all i. Another sufficient condition for the preference of the first stream (annuities) is $P \geq Q$, where $P = \sum_{i=0}^{n} P_i$, Q=$\sum_{i=0}^{n} Q_i$

There are other weaker sufficient conditions.

2.5.8.2. Comparison of annual and fixed-term annuities

As mentioned in the previous paragraph, when choosing annuities, it is necessary to compare the accrued annuities and choose the one with the higher value. The magnitude of the accrued annuity depends on the period of the annuity and the frequency of interest accrual. If these parameters are entered as arguments of the incremental annuity, it can be denoted as $S(p, k)$. Thus, $S(p, k)$ — is the accrued amount of a *p-fold* annuity with interest accrued k times a year.

For annuities with the same terms, members and interest rates, differing only by two characteristics – the multiplicity of the annuity and the frequency of interest accrual from the above formulas, we can obtain a number of ratios useful in the preliminary evaluation of the annuity agreement:

$$S(1,1) < S(1,k) < S(1,\infty) < S(p,1) < S(p,k) < S(p,k) < S(p,k) < S(p,\infty);$$

$$k > 1 \quad p > 1 \quad p > k > 1 \quad p = k > 1 \quad k > p > 1 \qquad (2.151)$$

From these ratios we can immediately estimate that if all other annuity parameters are equal, the *p-fold* annuity is preferable to the annual annuity with interest accrued once a year. It is also clear that if k, $p > 1$ annuity with $k > p$ will be more preferable to the annuity with $p > k$, т.е. annuity с $k = 5$ and $p = 3$ will be more preferable to the annuity with $k = 3$ и $p = 5$.

An annuity with a rate of interest equal to a multiple is preferable to an annuity with a multiple greater than the rate of interest ($p > k$), but is less preferable than an annuity with a multiple greater than the frequency of interest ($k > p$).

2.5.9. Annuity conversion

There are situations when it is necessary to change the conditions of annuity payment, to replace one annuity with another one-time payment or, on the contrary, to replace one-time payment with an annuity, as well as to replace several annuities with different parameters of one annuity. In all of the above cases, the annuities are converted, subject to a simple rule: the present values of the old (old) and new (new) annuities must be equal. This follows from the assumption that the conversion of the annuities should not change the financial situation of the parties, i.e. the principle of financial equivalence should be observed (the principle of financial justice). The algorithm for calculating the parameters of the new annuity is as follows:

1. the current value of the old (old) annuities is determined.

2. If the annuities are combined, these values are added up to give the present value of the new annuity.

3. Knowing the present value of the new annuity, by the method described above, we calculate the parameters of the new annuity, such as the size of a separate payment R, the duration of the annuity n and the interest rate i.

Let us consider such kinds of annuity conversion as change of annuity parameters, replacement of one annuity with another, annuity buyout (replacement of annuity with single payment), installment payment (replacement of single payment with annuity) as well as annuity consolidation (merger) (replacement of several annuities with different parameters with one annuity).

2.5.9.1. Substitution of one annuity for another

Change of annuity parameters. In practice it is often necessary to change the parameters of the annuity. For example, it is necessary to change the term of the annuity or the value of the annuity payment, or to change the frequency of payments (term of the annuity), etc.

The algorithm for calculating the parameters of a new annuity is the same as above: the old annuity value is determined, which will be equal to the value of the new annuity. Then set all but one of the new annuity parameters and find the missing parameter of the new annuity from the equivalence equation *A1* = *A2* (see paragraph 2.5.3).

If we set such parameters of the new annuity as the size of a separate payment *R* and the term of the annuity *n*, we can find the interest rate *i* from the equivalence equation or, given the term of the annuity n and the interest rate *i*, determine the value of the annuity payment *R*, etc. Even more complicated cases are possible, in particular, it may be necessary to change the annual annuity p for the term or vice versa. Consider these and other cases in more detail.

Replacing an annuity with a term annuity. Here are three examples of replacing one annuity with another. As the first example, let us consider the replacement of an annual annuity with parameters R_1, n_1 by a *p-term* annuity with parameters R_2, n_2, p. Let us equate the present values of the old and new annuities:

$$R_1 a_{\overline{n_1}|i} = R_2 a^{(p)}_{\overline{n_2}|i}. \tag{2.152}$$

From this equation we can either find the value of the term annuity payment R_2, if its term n_2 and term p are given, or determine the term annuity n_2, if the value of the payment R_2 and term annuity p are given.

In the first case

$$R_2 = R_1 \frac{a_{\overline{n_1}|i}}{a^{(p)}_{\overline{n_2}|i}}. \tag{2.153}$$

If the terms of both annuities, as well as the interest rates, are the same and differ only in the periodicity of annuity payments (one payment per year for the first annuity and p payments per year for the second annuity), then the payments of these annuities are related by the relation

$$R_2 = R_1 \frac{p\left[(1+i)^{1/p} - 1\right]}{i}. \tag{2.154}$$

In the second case, to find the annuity term *n2* we first find the reduction factor for the *p-term* annuity

$$a^{(p)}_{\overline{n_2}|i} = \frac{A}{R_2} = \frac{R_1}{R_2} a_{\overline{n_1}|i}. \qquad (2.155)$$

$$n_2 = \frac{\ln\left[1 - \frac{A}{R}\left[(1+i)^{1/p} - 1\right]\right]^{-1}}{\ln(1+i)}. \qquad (2.156)$$

Solving this equation with respect to n_2, we determine the term of the *p-term* annuity

Example 2.7. To replace the usual (annual) annuity with the parameters $R_1 = 200$, $n = 5$, $i = 10\%$ by a term (quarterly) annuity with the parameters $R_2 = 100$, $i = 10\%$.

Let us first find the present value of the annual annuity

$$A = R_1 \cdot \frac{1-(1+i)^{-n}}{i} = 200 \cdot \frac{1-(1+0.1)^{-5}}{0.1} = 200 \cdot 3.79 = 758$$

Then according to the formula (2.145) we find the term of 4 – term annuity

$$n_2 = \frac{\ln\left[1 - \frac{A}{R_2}\left[(1+i)^{1/p} - 1\right]\right]^{-1}}{\ln(1+i)} = \frac{\ln\left[1 - \frac{758}{100}\left[(1+0.1)^{1/4} - 1\right]\right]^{-1}}{\ln(1+0.1)} =$$

$$= \frac{1.699}{0.0953} = 17.83\, years$$

Replacement of the immediate rent by the deferred rent. As a second example, consider the replacement of the immediate annuities with parameters *R1, n1* by the deferred rent with parameters R_2, n_2, t. Let us equate the present values of the old and new rents:

$$R_1 a_{\overline{n_1}|i} = R_2 a_{\overline{n_2}|i} \nu^t, \qquad (2.157)$$

where

$$v^t = (1 + i)^{-t}. \qquad (2.158)$$

From this equation we can either find the value of the deferred annuities payment R_2, if its term n_2 and the duration of deferment t are given, or determine the term of the rent n_2, if the value of the payment R_2 and the duration of deferment t are given.

In the first case, the value of payment R_2 equals:

$$R_2 = R_1 \frac{a_{\overline{n_1}|i}}{a^{(p)}_{\overline{n_2}|i}} (1+i)^t. \qquad (2.159)$$

If the terms of both annuities are equal, their payments are related by the ratio

$$R_2 = R_1(1 + i)^t, \qquad (2.160)$$

i.e., the term of the deferred annuities is equal to the accrued during the deferment t is equal to the term of the immediate annuity.

In the second case, from the equality

$$R_1 a_{\overline{n_1}|i} = R_2 a_{\overline{n_2}|i} \nu^t \qquad (2.161)$$

given *R2* and *t*, we find the term of the new annuity (R1 and n1 are known). If the size of the annuity term is conserved (*R2* = *R1*) it is defined by the relation

$$n_2 = \frac{\ln\left[1 - \left[1 - (1+i)^{-n_1}\right](1+i)^t\right]}{\ln(1+i)}. \qquad (2.162)$$

2.5.9.2. Consolidation of annuities

When several annuities are replaced by a single annuity, the equality of the present values of the old and new annuities looks like:

$$A = \sum_i A_i \tag{2.163}$$

This equation allows us to find only one parameter of the consolidating annuity (the annuity term or its term), while all its other parameters must be specified. If the annuity term is unknown, it is determined for an ordinary annuity with term n by the formula

$$R = \frac{\sum_i A_i}{a_{\overline{n}|i}}. \tag{2.164}$$

If the term of consolidating rents is unknown, then first we find the coefficient of the reduction.

$$a_{\overline{n}|i} = \frac{\sum_i A_i}{R} = \frac{1-(1+i)^{-n}}{i}, \tag{2.165}$$

from where we already find the term of the annuity

$$n = \frac{-\ln\left(1 + i\sum_i A_i \middle/ R\right)}{\ln(1+i)}. \tag{2.166}$$

An important special case of rent consolidation is the situation when the consolidating rent term is equal to the sum of the terms of the rents being replaced. If the interest rate of all rents is the same, from the condition of financial equivalence we get:

$$R\frac{1-(1+i)^{-n}}{i} = \frac{\sum_j R_j\left[1-(1+i)^{-n_j}\right]}{i}, \tag{2.167}$$

where we find the term of "annuity"

$$n = \frac{\ln R - \ln \sum_j R_j (1+i)^{-n_j}}{\ln(1+i)}. \qquad (2.168)$$

Annuity redemption

The redemption of the annuity is the replacement of the annuity with a lump-sum payment. The principle of financial equivalence here is reduced to the fact that the lump sum payment P must be equal to the present value of the annuity to be redeemed A:

$$A = R\frac{1-(1+i)^{-n}}{i} = P. \qquad (2.169)$$

This formula is used to determine the value of the lump sum payment for the known parameters of the rent to be redeemed: the size of the separate payment R, the term of the rent n and the interest rate i.

Example 2.8. Replace the two ordinary annuities with the parameters

$R_1 = 200$, $n_1 = 4$, $i_1 = 12\%$, $R_2 = 250$, $n_2 = 6$, $i_2 = 14\%$

by a single payment at a point in time $n = 4$, $t = 15\%$.

First, let's find the discounted values of both annuities:

$$A_1 = R_1 \cdot \frac{1-(1+i)^{-n_1}}{i_1} = 200 \cdot \frac{1-(1+0.12)^{-4}}{0.12} = 200 \cdot 3.037 = 607.47;$$

$$A_2 = R_2 \cdot \frac{1-(1+i)^{-n_2}}{i_2} = 250 \cdot \frac{1-(1+0.14)^{-6}}{0.14} = 250 \cdot 3.889 = 972.17;$$

Now let's determine the sum of the sums of discounted values of both annuities:

$A = A_1 + A_2 = 607.47 + 972.17 = 1579.64$.

This amount should be equal to the lump sum payment discounted to the initial point in time:

$$A = \frac{P}{(1+i)^n}.$$

From here

$$P = A \cdot (1+i)^n = 1{,}579.64 \cdot (1+0.15)^4 = 2{,}762.80$$

$P = 2762.80$.

2.5.9.3. Installment payment

The installment payment is the replacement of the debt (lump sum payment) by the annuity. In this case, all the rent parameters are set except one, and this unknown parameter is determined from the condition of equality of the debt to the present value of the introduced annuity:

$$P = A = R \cdot \frac{1-(1+i)^{-n}}{i} \qquad (2.170)$$

Control questions and tasks

1. What is a financial event and financial flow?
2. Define and derive a formula for the average term of the financial flow.
3. Define annuity, annuity due, and ordinary annuity.
4. What parameters are used to describe annuities?
5. Derive formulas for the conversion and accretion coefficients of ordinary annuities.
6. Derive formulas for the conversion and accretion coefficients of annuity due.
7. Derive the relationship between the quantities $\ddot{s}_{\overline{n}|i}$ и A.
8. Derive the relationship between the quantities $\ddot{a}_{\overline{n}|i}$ и S.

9. Derive the relationship between the quantities S и A.

10. Derive the relationship between the quantities S и $\ddot{s}_{\overline{n}|i}$.

11. Derive the relationship between the quantities $\ddot{S}$ и A.

12. Output the relationship between the present value and the accrued annuity amount.

13. Derive the relation between the conversion and accretion coefficients of due and ordinary annuities.

14. Let n, i, R be known. Find the incremental sum S and the present value of the rent A.

15. Let us know A, i, R. Find the term of the annuity n.

16. Let us know S, i, R. Find the term of the annuity n.

17. Let us know n, i, A. Find the annuity payment R.

18. Let us know n, i, S. Find the annuity payment R.

19. Let us know n, R, A. Find the interest rate i.

20. Let us know n, R, S. Find the interest rate i.

21. What are called perpetual, term, and continuous annuities?

22. Find the present value and the accrued amount of the perpetual rent.

23. Derive the flow
$CF = \{(0, 600), (1, 250), (2, 350), (3, 600)\}$
by the time $t = 2$.

24. Find the present value and accrued amount of the p-term ordinary annuity (case $k = 1$).

25. Find the present value and accrued amount of the p-term ordinary annuity (case $k \neq p$).

26. Find the present value and accrued amount of the p-term ordinary annuity (case $k = p$).

27. Find the present value and accrued amount of the p-term ordinary annuity (case $k = 1$).

28. Find the present value and accrued amount of the p-term ordinary annuity (case $k \neq p$).

29. Find the present value and accrued amount of the p-term ordinary annuity (case $k = p$).

30. Define a continuous annuity and derive formulas for its conversion and increment coefficients.

31. Find the present value and the accrued amount of a continuous annuity with k-fold accrual of interest.

32. Find the present value and accrued amount of a continuous interest annuity.

33. Define immediate and deferred annuities. What is the relationship between the present values and the accrued amounts of immediate and deferred annuities?

34. Define arithmetic and geometric annuities. Find the present value and accrued amount of arithmetic annuity.

35. Define arithmetic and geometric annuities. Find the present value and the accrued amount of the geometric annuity.

36. How is the general principle of comparing financial flows and annuities formulated?

37. Replace the annual annuity with the parameters R_1, n_1 p-term annuity with the parameters R_2, n_2, p.

38. Replace two ordinary annuities with parameters R_1 = 2000, n_1 = 3, i_1 =
= 10%, R_2 = 2500, n_2 = 5, i_2 = 15% with a single payment at time n = 4, i = 12%.

39. Consolidate three annuities postnumerando with parameters R_1 = 1000, n_1 = 3, R_2 = 1500, n_2 = 5, R_3 = 2000, n_3 = 7, i =10% 4-year ordinary annuity with i = 15%.

40. How to replace the immediate annuity with the parameters R_1, n_1 by the delayed annuity with the parameters R_2, n_2, t?

41. Define and give an example of an annuity buyout.

42. Define and give an example of annuity consolidation.

43. Give an example of replacing an immediate annuity with a deferred annuity.

44. Define and give an example of installment payments.

45. Replace the annual annuity R_1 = 2, n_1 = 3, i = 20% by p-term (quarterly) annuity n = 4, i = 20%.

46. Obtain expressions for the conversion and accretion coefficients of the ordinary annuity for two adjacent periods.

47. Obtain expressions for the conversion and accretion coefficients of ordinary annuities for several ($k > 2$ adjoining periods.

48. Obtain expressions for the conversion and accretion coefficients of annuities due for two adjoining periods.

49. Obtain expressions for the conversion and accretion coefficients of annuities due over several ($k > 2$) adjoining periods.

50. Compare the two flows in terms of average time
CF_1 = {(0, 500), (1, 300), (2, 450), (3, 100)},
CF_2 = {(0, 600), (1, 250), (2, 350), (4, 50)}.

51. Find the increment rate for three consecutive periods of 1, 2 и 3 respectively, at the rate of 10%.

52. Find the term of the annuity due if you know $S = 4000$, $i = 12\%$, $R = 100$.

53. Find the term of the annuity due if you know $A = 3000$, $i = 11\%$, $R = 200$.

54. Find the term of the ordinary annuity if you k$A = 2500$, $i = 13\%$, $R = 250$.

55. Make the relationship between the present value and the accrued value of a continuous annuity with continuous accrual of interest.

56. Determine the relationship between the present value and the accrued value of p-term annuity with continuous accrual of interest.

57. Determine the relationship between the present value and the accrued value of p-term annuity and k- multiple accruals of interest.

58. Determine the relationship between the present value and the accrued value of k- multiple accruals of interest.

59. On which rent parameters does the relationship between the present value and the accrued value of the annuity depend?

60. Replace the lump-sum payment at time $t = 5$ by p-term ordinary annuity with the parameters $R_1 = 400$, $n_1 = 8$, $i_1 = 13\%$, $p = 12$.

61. Derive formulas for the conversion and accretion coefficients of a postnumerando p-term annuity with continuous accrual of interest.

CHAPTER 3

PROFITABILITY AND RISK OF FINANCIAL OPERATION

3.1. Income and profitability of the financial transaction

Any operation that has an initial and final state that have a financial (monetary) expression (evaluation) (P and P') is called a ***financial operation***. One of the main goals of any financial operation is to gain maximum profit ($P' - P$), that is why profit is one of the main characteristics of a financial operation, along with the resulting income (P'). More precisely, a financial operation is characterized by its profitability (or efficiency) ($(P' - P)/P$). Under deterministic conditions, discussed in the previous chapters, the profitability is a well-defined value, depending on the interest rate, inflation rate and other factors that we assumed to be known.

3.1.1. Multi–period returns

Let us find the yield for several periods, if the yield for each period is known. Let the returns for consecutive periods of time $t_1, t_2, ..., t_n$ be equal $\mu_1, \mu_2, ..., \mu_n$ respectively. Let us find the yield μ for the period $t = t_1 + t_2 + ... + t_n$. Common sense suggests that the yield is additive, so that μ, at least approximately, it is equal to the sum of returns for each period $\mu_1, \mu_2, ..., \mu_n$:

$$\mu = \mu_1 + \mu_2 + ... + \mu_n \quad (3.1)$$

Below we get the exact expression for the total time period t and see how different it is from the intuitive result (3.1).

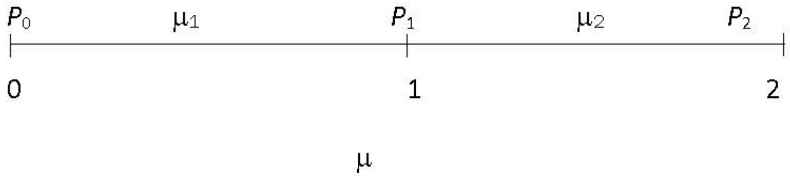

Figure 3.1. To the derivation of the formula for the return over two periods

Let us first consider two periods t_1 и t_2. Denoting the value of the asset at the moments $t = 0$, $t = t_1$, $t = t_2$ by P_0, P_1, P_2 respectively (Fig. 3.1), we have the following expressions for the first (μ_1) and second (μ_2) periods of returns:

$$\mu_1 = \frac{P_1 - P_0}{P_0}; \; \mu_2 = \frac{P_2 - P_1}{P_1} \tag{3.2}$$

The yield μ for the period $t = t_1 + t_2$ is equal

$$\mu = \frac{P_2 - P_0}{P_0}. \tag{3.3}$$

Dividing the divisions of (3.2) and (3.3), we obtain

$$\mu_1 = \frac{P_1}{P_0} - 1; \; \mu_2 = \frac{P_2}{P_1} - 1; \; \mu = \frac{P_2}{P_0} - 1 \tag{3.4}$$

Transferring –1 to the left parts, we have

$$\mu_1 + 1 = \frac{P_1}{P_0}; \; \mu_2 + 1 = \frac{P_2}{P_1}; \; \mu + 1 = \frac{P_2}{P_0} \tag{3.5}$$

By multiplying the first two expressions we obtain

$$(\mu_1 + 1)(\mu_2 + 1) = \frac{P_2}{P_0}. \tag{3.6}$$

The right–hand side of (3.6) is equal to the right-hand side of the third equation in (3.5).

By equating them, we obtain

$$(\mu_1 + 1)(\mu_2 + 1) = \mu + 1. \tag{3.7}$$

or finally

$$\mu = (\mu_1 + 1)(\mu_2 + 1) - 1. \tag{3.8}$$

Generalizing (3.8) to the case of n–periods (Fig. 3.2), for the yield for the period $t = t_1 + t_2 + ... + t_n$ we have

$$\mu = (\mu_1 + 1)\cdot(\mu_2 + 1)\cdot...\cdot(\mu_n + 1) - 1. \tag{3.9}$$

The exact proof of formula (3.9) is easy to obtain by the method of mathematical induction.

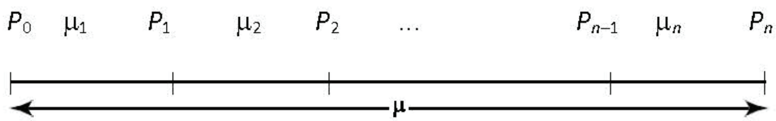

Figure 3.2. To the derivation of the formula for multi–period returns

Note that the yield for n–periods does not depend on the duration of the constituent periods as well as on the period t.

The result obtained for the yield for several periods is completely similar to the result we obtained earlier for the inflation rate for several periods.

For equal returns in separate periods $\mu_1 = \mu_2 = ... = \mu_n$ time intervals may remain arbitrary and not equal to each other) we have

$$\mu = (\mu_1 + 1)^n - 1. \tag{3.10}$$

Let us analyze the difference between the obtained results (3.9) and (3.10) and the intuitive expression (3.1) and the reason for this on the example of a time interval consisting of two periods.

Let the returns for two consecutive time periods t_1, t_2 be equal to μ_1, μ_2 respectively. Then according to formula (3.9) the return μ for the period $t = t_1 + t_2$ is equal

$$\mu = (1+\mu_1)(1+\mu_2) - 1 = \mu_1 + \mu_2 + \mu_1\mu_2. \tag{3.11}$$

As we can see, the difference from the sum of returns is the appearance of the cross term μ_1, μ_2. Although it is a small value of a higher order of smallness compared to μ_1, μ_2 provided that they are small, in practice it is necessary to take them into account.

3.1.2. Synergistic effect

Here, as in the case of the inflation rate, we have an example of a synergistic effect (i.e., the effect (result) of two (several) parts is greater than the additive effect (simple summation)). As in the case of the inflation rate, the cross term μ_1, μ_2 is responsible for the synergistic effect. It leads to the fact that the returns for two consecutive periods of time $t = t_1 + t_2$ are greater than the sum of returns.

Example 3.1. Let the yield for two consecutive periods of time t_1, t_2 be 20 and 30% respectively. Then, according to formula (3.11), yield for period $t = t_1 + t_2$ is equal to

$$\mu = (1+\mu_1)(1+\mu_2) - 1 = \mu_1 + \mu_2 + \mu_1\mu_2 = 0.2 + 0.3 + 0.2 \cdot 0.3 = 0.56,$$

i.e. 56%. Thus, the difference from the sum of returns is 6%.

Example 3.2. The return on the asset for the year is 20%. We need to find the return on the asset for quarter μ_1 if it is constant.

Let's apply the formula

$$\mu = (1+\mu_1)^n - 1.$$

We have $\mu+1=(\mu_1+1)^n$, $\mu_1+1=\sqrt[n]{\mu_1+1}$

Finally

$$\mu_1=\sqrt[n]{\mu_1+1}\ -1$$

By substituting in this formula μ = 20% = 0,2, n = 4, we obtain for the quartile yield

$$\mu_1=\sqrt[n]{1+\mu}-1=\sqrt[4]{1.2}-1=1.0466-1=0.0466\approx 4.66\%.$$

We see that the yield for the quarter is lower than the one obtained by simply dividing the annual yield by four, i.e. 20 : 4 = 5%. The difference is 5% – 4.66% = 0.36%.

Example 3.3. Let's solve the inverse problem. Let the return on the asset in month μ_1 is equal to 2%. Find the return on the asset for the year assuming that the monthly return is constant throughout the year μ.

Let's apply the formula $\mu=(\mu_1+1)^n-1.$

Substituting here μ = 2% = 0.02, n = 12, we get for the annual income

$\mu=(1+\mu_1)^{12}-1=(1.02)^{12}-1\approx 1.268-1=0.268=26.8\%.$

It turns out that the yield for the year is higher than the monthly yield obtained by simply multiplying it by twelve, i.e. 2%·12 = 24%. The difference is 2.8%.

From the last two examples we can conclude that, firstly, the total period yield is greater than the sum of returns for the periods which make it up;

secondly, the yield for the period which makes it up is smaller than its corresponding proportion of the total period yield.

3.2. Financial operation risk

However, as a rule, most of financial operations are conducted under uncertainty, when the above-mentioned and other factors are either unknown or are random values, which leads to uncertainty and profitability of the financial operation.

In this situation, a financial transaction is characterized in addition to profitability by another value closely related to profitability and determining the degree of uncertainty of this financial transaction, namely the risk of a financial transaction.

The term “risk” is not understood unambiguously. Even leaving aside the fact that there are different types of financial risks (banking, credit, currency, investment, deposit, insurance, inflation, price, asset liquidity risk, etc.), we note that even the general definition of this concept is unclear, ambiguous and contradictory.

Intuitively, risk is understood as possible losses connected with conducting a financial operation in conditions of uncertainty. The presence of uncertainty does not allow predicting the result of financial transaction in advance, so when it is carried out, both profit and loss (or less profit compared to what could have been) are possible. Under this understanding of risk, the probability of a loss or a smaller profit is considered a risk, while the probability of a larger profit is not considered a risk.

A distinction is often made between risk and uncertainty. It is considered that risk takes place when probabilities of different outcomes of financial operation are known. If the probabilities of outcomes are unknown, then the corresponding situation is considered uncertainty. From our point of view, risk exists in both cases, and they differ in the completeness of information characterizing the risk. Let us consider different cases of uncertainty.

Another aspect of the concept of “risk” is the presence of a risk-taker (investor). A number of authors call a transaction risky if it can have several outcomes that are not equivalent for the investor. Thus, in their perception the concept of risk necessarily presupposes a risk-taker – the one to whom this risk relates, who is concerned about the result of an operation. Risk itself, in their view, arises, “only if the transaction is likely to end in outcomes that are not equivalent for him, despite, perhaps, his best efforts to manage that transaction.”

Although this definition of risk is acceptable at first sight, it does not seem quite justified for at least two reasons: first, it obscures the objective nature of risk, which is often caused by external circumstances beyond the control of the investor, and second, it unnecessarily complicates the situation by introducing yet another degree of freedom conditioned by the presence of the investor.

The presence of an investor should be taken into account only where it is really necessary, namely when considering an individual's preference system and utility function. This is where a third characteristic of a financial transaction will appear, which is related to the presence of an investor (a utility function, a pleasure function, or some other similar value).

In all other cases, we will proceed from the following definition of financial transaction risk. The risk of financial transaction under uncertainty is the deviation of profitability from the average value. Thus, the possibility of deviation of profitability in any direction (profit or loss) is regarded as risk.

So, in conditions of uncertainty a financial transaction acquires one more characteristic: risk, and thus is characterized by two values: profitability and risk.

Let us proceed to quantitative assessment of risk.

3.2.1. Quantitative assessment of financial operation risk

To quantify the risk, it is necessary to know the probabilities of different outcomes of financial transaction, and consequently, the probabilities P_i of its different returns q_i. We have a random variable, the yield Q, with the distribution law $p_i = p(q_i)$. **The expected mean** return of a financial operation is the mathematical expectation (mean value) of a random variable

$$Q : M(Q) = \sum_i q_i p_i.$$

The variance of the *Q* yield of a financial operation is the mathematical expectation of the squared deviation of the yield from its average value, i.e. the average value of a random variable $(q-M(q))^2$:

$$D(Q)=M\left[\left(q-M(q)\right)^2\right].$$

The risk of a financial operation is the root mean square (standard deviation) of yield

$$r(q)=\sigma(q)=\sqrt{D(q)}.$$

In theory and practice, the mean square deviation of income *D* is sometimes used to determine risk. This definition of risk does not fully characterize the risk of a financial transaction, because it does not relate it to its average return. Clearly, the root mean square deviation of income per \$10 for two operations with \$50,000 and \$1000,000 means a completely different risk, which is large in the first operation and negligibly small in the second. It makes much more sense, of course, not to use the standard deviation of the mean square deviation of the return as a measure of risk $r(d)=\sigma(d)=D(d)$, but rather the relative root-mean-square deviation of income (the ratio of the mean square deviation of income to the of the income (the ratio of the mean square deviation of income to mean income)

$$r(d)=\frac{\sigma(d)}{M(d)}=\frac{\sqrt{D(d)}}{M(d)},$$

or as it is done above, the mean square deviation of returns. Below we will use both alternative measures of risk, preferring the definition of risk based on returns (rather than income).

If the scale of the operation is increased by *c* times, i.e. if all random income values increase by *c* times, the efficiency of the operation increases by *c* times, the risk increases by |c| times, and the average return does not change. The first property follows from the fact that the constant multiplier can be taken out from under the sign of the mean, the second property follows from the fact that the constant multiplier is taken out from under the sign of the variance squared. Let us prove the second property:

$$D(cX) = M\left[\left(cX - M(cX)\right)^2\right] = M\left[c^2\left(X - M(X)\right)^2\right] =$$
$$= c^2 M\left[\left(X - M(X)\right)^2\right] = c^2 D(X). \qquad (3.12)$$

From here

$$r(cX) = \sigma(cX) = \sqrt{D(cX)} = \sqrt{c^2 D(X)} = c\sqrt{D(X)} = cr(X).$$

If you change all income by the same constant number, the efficiency of the operation also changes by that number, and the risk does not change.

Note one general formula known in probability theory and useful in calculating variance

$$D(X) = M(X^2) - m^2. \qquad (3.13)$$

It is easy to derive the above formula:

$$D(X) = M\left[(X - m)^2\right] = M\left(X^2 - 2mX + m^2\right) =$$
$$= M\left(X^2\right) - 2mM(X) + M\left(m^2\right) =$$
$$= M\left(X^2\right) - 2m^2 + m^2 = M\left(X^2\right) - m^2.$$

The average expected return on an operation $M(q)$ and its risk $r(q)$ are related by the Chebyshev inequality

$$P\,(q - M\,(q\,)) > \delta) \le r_q{}^2 \,/\, \delta^2, \text{ or } P\,(q - M\,(q\,)) < \delta) > 1 - r_q{}^2 \,/\, \delta^2. \quad (3.14)$$

The meaning of Chebyshev's inequality is the statement that the probability that the deviation of the return on the operation from the mean value exceeds a given number is bounded from above by the number

$$r_q{}^2/\delta^2 = D/\delta^2,$$

or, respectively, that the probability that the deviation of the operation’s yield from the mean will not exceed a given number δ, is bounded from below by the number $1- r_q{}^2/\delta^2 = 1- D/\delta^2$. Thus, the importance of the introduction of the mean square deviation is due to the fact that it defines the limits within which a random variable's value is to be expected with a given probability.

From the Chebyshev inequality $P(\ |X - m|\ < \varepsilon) > 1 - D/\varepsilon^2$ follows «*rule* 3σ»: for any random variable X the following inequality holds

$$P\left(\left|X-m\right|<3\sigma\right)>1-D/9\sigma^2=8/9. \qquad (3.15)$$

This means that if the mean value of a random variable and its standard deviation are known, there is a probability greater than 8/9 (89%) that the value of the random variable will be in the range $(m - 3\sigma, m + 3\sigma)$. That is, values of a random variable outside this interval can be disregarded in practice.

Moreover, in reality, for most random variables encountered in practice, this probability is much closer to 1 than 8/9. Thus, if the distribution of a random variable is close to normal, we can say with 68% probability that the value of the random variable X (in our case *Q yield*) is within $M(q) \pm \sigma$, with 95% probability it is within $M(q) \pm 2\sigma$, and with 99.7% probability it is within $M(q) \pm 3\sigma$ and etc.

For the distribution of random variables, uniform and normal distributions play a special role in economics and finance (as well as in the natural sciences).

3.3. The role of uniform and normal distributions

3.3.1. The role of even distribution

The important role of the uniform distribution is due to two factors:

1) it is the simplest of all distributions, and in situations where the true probability distribution is unknown, the uniform distribution is used for a primary (albeit rough) estimate of the numerical characteristics of random variables;

2) a number of situations have symmetries that make the uniform distribution a good approximation of the real distribution, so that

calculations using this distribution in such situations are quite justified.

Here are some basic formulas for the uniform (one-dimensional) distribution (on the interval [a, b]):

— distribution density:

$$f(x)=\begin{cases}\frac{1}{b-a}, & x\in[a,b]\\ 0, & x\notin[a,b]\end{cases}; \qquad (3.16)$$

— distribution function:

$$F(x)=\begin{cases}0, & x<a\\ \frac{x-a}{b-a}, & x\in[a,b];\\ 1, & x>b\end{cases} \qquad (3.17)$$

— expected value (mean):

$$M(x)=\frac{a+b}{2} \qquad (3.18)$$

— dispersion:

$$D(x)=\frac{(b-a)^2}{12} \qquad (3.19)$$

— standard deviation:

$$\sigma(x)=\sqrt{D(x)}=\frac{(b-a)}{2\sqrt{3}}. \qquad (3.20)$$

3.3.2. The highlighted role of the normal distribution

The special role of normal distribution is theoretically substantiated by the central limit theorem, which ideologically can be formulated as follows: the law of distribution of arithmetic mean of a large number of random variables under sufficiently general conditions is close to normal. General conditions are as follows: individual deviations of each random quantity must be of the same order of smallness and small in comparison to the total deviation (deviation of the sum of random quantities). Since economic and financial applications quite often deal with arithmetic averages (or sums) of a large number of random variables, the importance of normal distribution cannot be overestimated: financial flows, company profits that depend on a large number of factors, measurement errors of various variables, etc. are all distributed according to this law.

Here are some basic formulas for the normal (univariate) distribution that may be useful:

— distribution density:

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-(x-m)^2/2\sigma^2\right]; \qquad (3.21)$$

— distribution function:

$$F(x)=0.5+\Phi(x-m); \qquad (3.22)$$

— expected (mean) value:

$$M(x)=m; \qquad (3.23)$$

— dispersion

$$D(x)=\sigma^2. \qquad (3.24)$$

Here $\Phi\left(\frac{x-m}{\sigma}\right)$ — Laplace function, σ — standard deviation.

3.4. Correlation of financial operations

The concepts of correlation, interrelation, and interdependence of financial transactions are among the most important in financial analysis. This is due to the fact that in real business, correlated, interrelated financial transactions occur much more frequently than independent and uncorrelated ones. In hedging, for example, it is necessary to select only correlated transactions, and those correlated with the underlying transaction negatively. In diversification, either independent (uncorrelated) operations or negatively correlated ones should be selected. Below we will study the main properties of correlated financial operations.

Random variables *X* and *Y* are called correlated if their correlation moment (or covariance)

$$K_{xy} = Cov(X,Y) = M\left[(x - M(x))(y - M(y)) = M(XY) - M(X)\cdot M(Y)\right]$$

is different from zero, and uncorrelated if it is equal to zero. The correlation momentum K_{xy} and the correlation coefficient ρ_{xy} are related by the following relation $Kxy = \sigma_x\sigma_y\rho_{xy}$; the independent cases are uncorrelated; the converse is not correct.

Let operations $Q^{(1)}$ and $Q^{(2)}$ be uncorrelated, then the variance of their sum is equal to the sum of variances, so the risk of the sum operation is

$$r = \sqrt{r_1^2 + r_2^2}.$$

In the general case, i.e. for two arbitrary financial transactions $Q^{(1)}$ and $Q^{(2)}$, the risk of a total transaction is equal to:

$$r = \sqrt{r_1^2 + 2r_1r_2\rho_{12} + r_1^2},$$

where

ρ_{12} — is the correlation coefficient of the random income of operations.

This follows from the dispersion property of the sum of random variables

$$D(X+Y) = D(X) + D(Y) + 2\mathrm{Cov}(X,Y). \tag{3.25}$$

Let us prove that $\rho_{12} \leq 1$. Consider the obvious inequality

$$M\left[\left(\frac{X - m_X}{\sigma_X} \pm \frac{Y - m_Y}{\sigma_Y}\right)^2\right] > 0. \tag{3.26}$$

Squaring the expression under the expectation sign, we obtain

$$
\begin{aligned}
&M\left[\left(\frac{X-m_X}{\sigma_X}\pm\frac{Y-m_Y}{\sigma_Y}\right)^2\right]=\\
&=M\left[\left(\frac{X-m_X}{\sigma_X}\right)^2\pm 2\frac{X-m_X}{\sigma_X}\frac{Y-m_Y}{\sigma_Y}+\left(\frac{Y-m_Y}{\sigma_Y}\right)^2\right]=\\
&=\frac{1}{\sigma_X^2}M\left(X-m_X\right)^2\pm\frac{2}{\sigma_X\sigma_Y}M\left[\left(X-m_X\right)\left(Y-m_Y\right)+\frac{1}{\sigma_Y^2}M\left(X-m_Y\right)\right]=\\
&=\frac{D(X)}{\sigma_X^2}\pm\frac{2\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}+\frac{D(Y)}{\sigma_Y^2}=\\
&=2\pm\frac{2\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y}=2\pm 2\rho_{12}=2(1\pm\rho_{12})>0.
\end{aligned}
\tag{3.27}
$$

It follows that $\pm\rho_{12} < 1$, i.e.

$$\left|\rho_{12}\right| \le 1 \tag{3.28}$$

From the formula (3.28) follows that the risk of total operation can be greater than $r=\sqrt{r_1^2+r_2^2}$ (if $\rho_{12} > 0$ — at the so-called positive correlation of transaction returns), and less than this value (if $\rho_{12} < 0$ — at the negative correlation of transaction returns). Generally speaking, the risk of total transaction is within the limits of

$$\left|r_1-r_2\right| \le r \le r_1+r_2 \,. \tag{3.29}$$

In this case the boundary values ($\left|r_1-r_2\right|$, $r_1 + r_2$) are achieved with a complete negative ($\rho_{12} = -1$) and a complete positive ($\rho_{12} = 1$) correlation of operations, respectively.
These extreme cases are referred to as cases of full anticorrelation and full correlation, respectively. Consider two important and illustrative correlated financial transactions. Let's find the correlation coefficient of random variables X and $Y = \alpha X$.

$$Cov\,(X,Y) = Cov\,(X,\alpha X) = M\,(\alpha X^2) - M\,(X)M\,(\alpha X) =$$

$$=\alpha\,[M(X^2) - M^2\,(X)] = \alpha D(X). \tag{3.30}$$

From this we find:

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\alpha D(X)}{|\alpha| \sigma_X^2} = \text{sign}\alpha, \tag{3.31}$$

where

$$\text{sign}\,\alpha = \begin{cases} 1, \alpha > 0 \\ -1, \alpha < 0. \end{cases} \tag{3.32}$$

It can be seen that operations X and $Y = \alpha X$ at $\alpha > 0$ are positively correlated with the correlation coefficient $\rho_{XY} = 1$, and at $\alpha < 0$ financial operations are negatively correlated with the correlation coefficient $\rho_{XY} = -1$.

The values $\rho_{XY} = 1$ and $\rho_{XY} = -1$ mean the strongest correlation and anti–correlation, which is known to occur in a linear relationship between random variables.

3.5. Other risk measures

The root mean square deviation is the best way to quantify the risk of a financial transaction. However, risks can also be measured by other quantities. In most cases these values are probabilities of undesirable events. Here are some examples.

If the distribution function $F(d)$ of the random income D of the operation is known, it follows from its definition that the probability that the income of the operation will be less than a given d, is equal $P\,(D < d) = F(d)$. The probability that the income will be less than the average expected income m is also equal to $F(m)$. The probability of loss is $F(0)$, their average value is equal to $\int_{-\infty}^{0} x dF(x) = \int_{-\infty}^{0} x f(x) dx$ (here $f(x)$ — the density of income distribution), and the ratio of average expected losses to average expected income is equal. The lower this ratio, the lower the risk of ruin.

In Chapter 5, we will look at risk measures such as duration and bulge. Below we will focus on another one, the so-called “Value at Risk” (VaR), which has been the most important measure of risk for more than two decades.

3.5.1. Value at Risk

Value at Risk (*VaR*) is recommended by the Basel Committee on Banking Supervision and is the most common method of measuring and controlling market and credit risks under normal business conditions today.

Value–at–Risk is the absolute maximum amount of losses that can be expected from holding a financial instrument (or portfolio of financial instruments) over some fixed (given) period of time (time horizon) in normal market conditions at a given level of confidence probability.

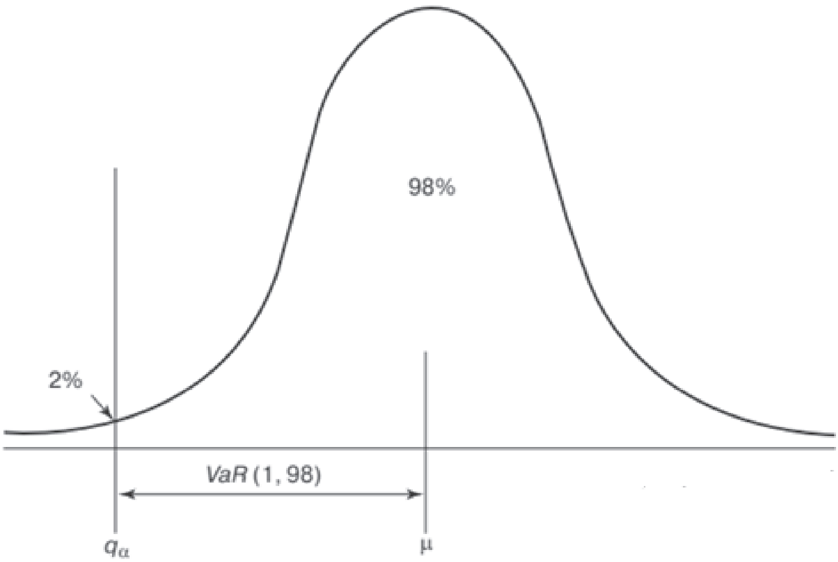

Fig. 3.3. *VaR* (1, 98)

Thus, *VaR* — is:

1) the largest expected loss from a fluctuation in the value of a portfolio of assets of a given structure that can occur over a given period

of time with a given probability of occurrence. More formally:

$P(X \leq VaR) = 1 - \alpha,$

where α — specified confidence level;

2) the amount of loss, which can be exceeded with probability not more than $1-\alpha$ within next t days (time horizon).

VaR is not used for markets that are in a state of shock. Stress-testing methodology is used for this purpose.

Necessary capital reserves are calculated on the basis of VaR. Reserve capital requirements are calculated as a maximum of two values: the current estimate of unexpected losses, defined as an estimate of the maximum possible loss from adverse changes in market prices, and the average value over the preceding 60 days, multiplied by a factor λ, of between 3 and 4. The value of the factor λ depends on the accuracy of one-day model prediction for the previous periods.

Methods used to estimate VaR can be classified according to assumptions regarding probability distribution of risk factors and regarding the type of functional dependence of changes in portfolio value on changes in risk factors. In these methods the main role is played by a statistical measure – quantile (median, decile, percentile).

Currently, the models used to calculate VaR are classified as follows:

1) parametric models (otherwise known as the variation-covariance method, or delta-normal method), which include:

– constant covariance method;

– exponential-weighted covariance method;

2) semi-parametric models;

3) nonparametric models including:

– the method of modeling by historical data (historical simulation);

– Monte Carlo method;

4) models using the theory of extreme values;

5) models using scenario analysis.

Parametric and nonparametric models are distinguished as follows:

1) parametric models of VaR calculation are characterized by the fact that assumptions are made that the distribution function belongs to any family of analytic distribution functions, which are distinguished by the following assumptions

– about a multivariate normal distribution – method of variation – covariance. Various methods, both parametric and nonparametric, are used to estimate the distribution parameters – the covariance matrix;

– The distribution is different from the normal distribution. For example, it can be any of distributions suitable for describing "heavy tails" of distributions. In this case, the methods of extreme value theory are used, for example;

2) The nonparametric method of calculating *VaR* is characterized by the fact that no assumptions are made about the type of distribution, but an empirical distribution function is used. An example of such method is historical modeling.

3.6. Types of financial risks

Currently, there are the following financial risks.

Banking risk: banking risks are divided into external and internal. External risks are risks not related to the activity of the bank or a specific client: political, economic and other. Internal risks in their turn are divided into losses on the main and auxiliary activity of the bank. The first ones represent the most widespread group of risks – credit, interest, currency and market risks. The latter include losses on formation of deposits, risks of new types of activity and risks of bank abuse.

Credit risk: the risk of failure to repay a loan taken on time.

Foreign exchange risk: the risk of foreign exchange losses caused by fluctuations in the exchange rate of foreign currency in relation to the national currency during foreign trade operations.

Investment risk: the risk of depreciation of capital investments as a result of actions of public authorities.

Inflation risk: the possibility of depreciation of monetary assets, income and company profits due to growing inflation. One of the methods of inflation risk insurance is the inclusion of an inflation premium into the future income.

***Deposit risk*:** the possibility of early withdrawal of the deposit.

***Insured risk*:** an estimated event for the occurrence of which insurance is carried out. An event considered as an insured risk must have signs of probability and randomness of its occurrence.

***Price risk*:** The risk of loss due to future changes in the market price of a commodity or financial instrument. There are three types of price risks: currency risk, interest rate risk, and market risk.

***Asset liquidity risk*:** the inability of the bank to ensure the payment of funds to its customers who have invested funds on a short-term basis.

***Risk of ruin*:** the probability of large losses leading to the ruin of the investor.

3.7. Methods of reducing the risk of financial transactions

3.7.1. Diversification

The diversification method (as applied to uncorrelated financial transactions) is based on the following statement: the risk of a (composite) financial transaction consisting of n uncorrelated financial transactions to its average income is inversely proportional to $\sqrt{n}$ and, consequently, with an increase *in n*, the relative risk of a composite financial transaction decreases. Let's prove it. Let the income of a financial transaction,

$$X = \sum_i X_i \text{ , then } \quad M(X) = \sum_i M(X_i) \infty n,$$

$$D(X) = M\left[(X - M(X))^2\right] = M\left(\sum_i X_i - M\left(\sum_i X_i\right)\right)^2 =$$
$$= M\left(\sum_i (X_i - M(X_i))\right)^2 = M\left(\sum_i \Delta X_i\right)^2 =$$
$$= M\left(\sum_i (\Delta X_i)^2 + 2\sum_{i \neq j} \Delta X_i \Delta X_j\right). \qquad (3.33)$$

Due to the uncorrelated nature of the financial transactions that make up the financial transaction *X*,

$$M\left(\sum_{i \neq j} \Delta X_i \cdot \Delta X_j\right) = 0,$$

therefore: $D(X) = M\left(\sum_i (\Delta X_i)^2\right) \propto n$

and $r(X) = \sigma(X) = \sqrt{D(X)} \propto n$.

Hence follows:

$$\frac{r(X)}{M(X)} \infty \frac{\sqrt{n}}{n} \infty \frac{1}{\sqrt{n}}. \qquad (3.34)$$

Thus, the relative risk of a composite financial transaction decreases with the growth of n. When proving the statement, it was assumed that the income of the financial transactions that make up operation X are of the same order, as well as their risks.

We come to a similar conclusion, considering the financial operation as the "arithmetic mean" of several uncorrelated operations: $Q=(Q_1+Q_2+...+Q_n)/n$. In this case, the efficiency of the composite operation is equal to $\mu=(\mu_1+\mu_2+\ldots+\mu_n)/n$, that is, it remains approximately equal to the effectiveness of a single operation, and its risk $r=\sqrt{r_1^2+r_2^2+\ldots+r_n^2}\,/n$ and thus (as $r_1^2+r_2^2+\ldots+r_n^2\propto n$ and $\sqrt{r_1^2+r_2^2+\ldots+r_n^2}\propto\sqrt{n}$ it turns out to be inversely proportional to $\sqrt{n}$ and, consequently, as n grows, it decreases.

So, with an increase in the number of uncorrelated operations, their arithmetic mean has an efficiency of the order of the efficiency of each of the operations, and the risk decreases. In the case of an operation equal to the sum of the original operations, as shown above, with an increase in the number of uncorrelated operations with an increase in n, the relative risk decreases.

This effect is called **the diversification effect**. It means that you need to carry out a variety of unrelated operations. (It's also known as the "don't put all your eggs in one basket" principle.) With this strategy, the effectiveness of the "arithmetic mean" operation is averaged, and the risk is reduced.

Let's consider another example of the impact of the scale of diversification on the risk of a composite financial transaction. Let a financial transaction be a linear combination of a number of financial transactions, and profitability is a random variable

$$X=\sum_i a_i X_i.$$

The variance of the income of a composite financial transaction is generally equal to

$$D(X)=D\left(\sum_i a_i X_i\right)=\sum_i a_i^2 D(X_i)+2\sum_{i\neq j} a_i a_j r_{ij}\sigma_i\sigma_j. \qquad (3.35)$$

If financial transactions are uncorrelated,

$$D(X) = D\left(\sum_i a_i X_i\right) = \sum_i a_i^2 D(X_i). \tag{3.36}$$

In case of equal weights and variances of individual operations

($a_i = 1/n$ = const и $\sigma_i = \sigma$= const):

$$D(X) = a_i^2 \sum_i D(X_i) = \frac{1}{n^2} \sum_i D(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}, \tag{3.37}$$

where *n* is the number of financial transactions that make up the composite one.

For one financial transaction $D_1 = \sigma^2$, for two $D_2 = \sigma^2/2$, $\sigma_2 = \sigma/\sqrt{2} = 0.71\sigma$, for three $D_3 = \sigma^2/3, \sigma_3 = \sigma/\sqrt{3} = 0.58\sigma$

It can be seen that with an increase in the number of financial transactions that make up a composite and are carried out simultaneously, the risk of a composite operation decreases even with the same risks of individual transactions.

As can be seen from the conclusions of all three examples, the requirement of uncorrelated operations is essential and necessary to reduce the risk of "arithmetic mean" operation (or the risk of composite operation). It is easy to show that if there are positive correlations between the operations that make up the "arithmetic mean" (composite) (correlation coefficients are positive), then the risk of the latter does not decrease. So, if the same operations are selected as the operations that make up the "arithmetic mean", then the risk of the "arithmetic mean" operation will be equal to the risk of a separate operation (in this case, the scale of the operation increases once, therefore, the risk increases once, i.e. does not change).

Thus, the diversification effect takes place in the absence of positive correlations (correlation coefficients must be either negative or equal to zero).

The effect of diversification is manifested when averaging operations, not only carried out simultaneously, but also carried out in different places (averaging in space) and at different times (averaging over time). It is positive, since the efficiency is averaged, and the risk is reduced. However, the effect of diversification may become insignificant if the costs of diversification itself (associated with conducting a large number of operations, tracking their results, etc.) are too high. The effect of diversification is discussed in more detail in portfolio theory in Chapter 4.

3.7.2. Hedging

To use the diversification effect, a composite operation is made up of several existing operations. The essence of hedging consists in the selection or special design of such new operations, which, when carried out together with the main one, reduce the risk.

When diversifying, it is necessary to carry out independent (uncorrelated) operations or negatively correlated ones. When hedging, only operations that are negatively correlated with the main operation should be selected**.**

Consider the variance of a financial transaction consisting of two operations: the main Q_1 and the auxiliary Q_2. Then the variance of a composite operation equal to the sum of two operations $Q_1 + Q_2$ is

$$D(Q_1+Q_2)=r_1^2+r_2^2+2\rho_{12}r_1r_2,$$

where ρ_{12} is the correlation coefficient of the main and auxiliary operations.

It can be seen that the variance can be less than the variance of the main operation only if the correlation coefficient is negative. In this case, the condition must be met

$$r_2^2+2\rho_{12}r_1r_2<0, \text{ или } \rho_{12}<-r_2/2r_1. \qquad (3.38)$$

It is not always easy to choose an auxiliary operation that is negatively correlated with the main and zero efficiency. Usually a small negative efficiency of the auxiliary operation is allowed, because of which the efficiency of the composite operation becomes somewhat less than the main one. How acceptable it is to reduce the effectiveness of a composite operation while reducing its risk depends on the investor's attitude to the efficiency — risk ratio.

Varieties of hedging are options and insurance.

Options are a type of exchange transaction with a premium paid for the right to sell or buy a commodity, currency, security in a certain quantity at the price and within the terms specified in the option contract.

Insurance is a financial transaction aimed at compensating (in monetary form) for possible losses in the event of an insured event.

3.8. Financial transactions in conditions of uncertainty

3.8.1. Matrices of consequences and risks

The degree of uncertainty of the situation may be different. If there is no information, the situation is completely uncertain. If, say, the probabilities of different outcomes are known, the situation is probabilistic and only partially uncertain. Let's consider both situations and the possible behavior of the investor in them. Suppose the issue of conducting a financial transaction is being considered. The result is unclear, so several possible solutions and their consequences are analyzed. The situation is uncertain, it is only known that one of the options under consideration is being implemented. If the *i* decision is made, and the situation is j, then the investor will receive income qij. The matrix ||qij|| is called the matrix of consequences (possible solutions). (An alternative to the matrix of consequences made up of possible returns is the matrix of consequences made up of possible returns.) What decision should an investor make? In this uncertain situation, only some preliminary recommendations can be made. They will not necessarily be accepted by the investor. A lot will depend, for example, on his propensity to risk. Let's assess the risk in this scheme. Let's start with the risk that the *i* solution carries. The real situation is unknown, but if the investor knew it, he would choose the solution that brings the greatest income. If the situation is *j*, then a decision would be made giving income $q_j = \max_i q_{ij}$. So, making the *i* decision, we risk getting not q_j, but only q_{ij}, that is, making the i decision carries the risk of not getting enough $r_{ij}=q_j-q_{ij}$. The matrix $R = ||r_{ij}||$ is called the risk matrix.

Financial transactions under conditions of uncertainty

Example 3.4. Let the matrix of consequences be

$$Q = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 10 & 6 & 7 & 8 \\ 2 & 5 & 9 & 1 \end{pmatrix}.$$

Let's make a risk matrix by subtracting this element from the maximum in each column. For the maximum element in each column we have:

$q_1 = \max q_{i1} = 10$; $q_2 = \max q_{i2} = 6$; $q_3 = \max q_{i3} = 9$; $q_4 = \max q_{i4} = 8$.

Now we can write the risk matrix as

$$R = \begin{pmatrix} 7 & 2 & 4 & 2 \\ 0 & 0 & 2 & 0 \\ 8 & 1 & 0 & 7 \end{pmatrix}.$$

3.8.2. Decision-making in conditions of complete uncertainty

A situation of complete uncertainty is characterized by the absence of any additional information (for example, about the probabilities of certain options, the real situation). At the same time, there are rules and recommendations for decision-making in this situation as well.

3.8.2.1. Wald's rule (the rule of extreme pessimism)

Considering the i solution, we will assume that the situation is the worst, that is, bringing the smallest income: $ai = \min_j qij$ (working with the matrix of consequences). But now let's choose the solution $i0$ with the largest $ai0$. So, Wald's rule recommends taking a decision $i0$ such that $i_0 = \max_i \left(\min_j q_{ij}\right)$. So, in our example, we have $a_1 = 3$, $a_2 = 6$, $a_3 = 1$. Now from the numbers 3, 6, 1 we find the maximum: 6. So, Wald's rule recommends taking the second decision.

3.8.2.2. Savage rule (minimum risk rule)

When applying this rule, the risk matrix $R = \|r_{ij}\|$ is analyzed. Considering the first solution, we will assume that in fact there is a situation of maximum risk

$$b_i = \max_j r_{ij}$$

But now let's choose the solution i_0 with the smallest b_{i0}. So, Savage's rule recommends making a decision i_0 such that $b_{i_0} = \min_i b_i \min_i \left(\max_j r_{ij}\right)$

So, in our example, we have $b_1 = 7$, $b_2 = 2$, $b_3 = 8$. Now from the numbers 7, 2, 8 we find the minimum: 2. So, Savage's rule recommends taking the second decision.

3.8.2.3. Hurwitz Rule (weighing pessimistic and optimistic approaches to the situation). The decision i is made, at which the maximum is reached

$$\left[\lambda \min_j q_{ij} + (1-\lambda) \max_j q_{ij}\right]$$

where $0 \leq \lambda \leq 1$.

Meaning λ selected for subjective reasons. If λ approaches 1, then Hurwitz's rule approaches Wald's rule; when approaching λ to 0, the Hurwitz rule approaches the rule of "pink optimism".

In our example:

1. When $\lambda = 1/2$ we have:

$$c_1 = \frac{1}{2}(3+6) = 4.5;\ c_2 = \frac{1}{2}(6+10) = 8;\ c_3 = \frac{1}{2}(1+9) = 5.$$

Choosing the maximum value of c_i equal to 8, we come to the conclusion that the Hurwitz rule recommends the second solution.

2. If we choose $\lambda = {}^1/_4$, that:

$$c_1 = \frac{1}{4}\cdot 3 + \frac{3}{4}\cdot 6 = 5.25;\ c_2 = \frac{1}{4}\cdot 6 + \frac{3}{4}\cdot 10 = 9;\ c_3 = \frac{1}{4}\cdot 1 + \frac{3}{4}\cdot 9 = 7.$$

Choosing the maximum value of c_i equal to 9, we come to the conclusion that the Hurwitz rule also recommends the second solution in this case.

If we choose $\lambda={}^3/_4$, we will get:

$$c_1 = \frac{3}{4}\cdot 3 + \frac{1}{4}\cdot 6 = 3.75;\ c_2 = \frac{3}{4}\cdot 6 + \frac{1}{4}\cdot 10 = 7;\ c_3 = \frac{3}{4}\cdot 1 + \frac{1}{4}\cdot 9 = 3.$$

Choosing the maximum value of c_i equal to 7, we come to the conclusion that the Hurwitz rule also recommends the second solution in this case.

So, all three rules (and the Hurwitz rule for all three values of λ) recommend the second solution, so we accept it.

3.9. Decision-making in conditions of partial uncertainty

Suppose, in the scheme under consideration, the probabilities are known that the real situation develops according to option j. It is this situation that is called partial uncertainty. The decision in such a situation is made in accordance with one of the following rules.

3.9.1. The rule of maximizing the average expected income

The income received by the firm during the implementation of the i solution is a random variable Q_i with a co-distribution series $p_j(q_{ij})$. The mathematical expectation $M(Q_i)$ is the average expected income, also denoted by Q_i. So, the rule recommends making a decision that brings the maximum average expected income.

Suppose in our example the probabilities are 1/5, 4/15, 4/15, 4/15. Then the average expected income for each solution is equal to

$$M(Q_1) = \frac{1}{5}\cdot 3 + \frac{4}{15}\cdot (4+5+6) = 4.6;$$

$$M(Q_2) = \frac{1}{5} \cdot 10 + \frac{4}{15} \cdot (6 + 7 + 8) = 7.6;$$

$$M(Q_3) = \frac{1}{5} \cdot 2 + \frac{4}{15} \cdot (5 + 9 + 1) = 4.4.$$

The maximum average expected income is 7.6 and you meet the requirements of the second solution.

3.9.2. The rule of minimizing the average expected risk

The company's risk in implementing the *i* solution is a random variable R_i with a distribution series $p_j(r_{ij})$. The mathematical expectation $M(R_i)$ is the average expected risk, also denoted by R_i. The rule recommends making a decision involving the minimum average expected risk. Let's calculate the average expected risks with the above probabilities:

$$M(R_1) = \frac{1}{5} \cdot 7 + \frac{4}{15} \cdot (2 + 4 + 2) = 3.5(3) \ ;$$

$$M(R_2) = \frac{1}{5} \cdot 0 + \frac{4}{15} \cdot (0 + 2 + 0) = 0.5(3) \ ;$$

$$M(R_3) = \frac{1}{5} \cdot 8 + \frac{4}{15} \cdot (1 + 0 + 7) = 3.7(3).$$

We get

$$M(R_1) = 3.5(3) \ ; \ M(R_2) = 0.5(3) \ ; \ M(R_3) = 3.7(3).$$

The minimum average expected risk is 0.5(3) and corresponds to the same second solution. The difference between partial (probabilistic) uncertainty and full uncertainty is very significant. Of course, making decisions according to the rules of Wald, Savage, Hurwitz is not final nor necessarily the best (the example given is an exception). This is just the first step, some preliminary considerations. Then they try to find out something about the variants of the real situation, first of all about the possibility of this or that option, about its probability. But when we begin to evaluate the probability of an option, it already assumes the repeatability of the decision-making scheme under consideration: this has already happened in the past, or it will happen in the future, or it is repeated somewhere, for example, in the branches of the company.

3.9.3. Optimal (Pareto) financial transaction

So, when trying to choose the best solution, we are faced with the fact that each solution has two characteristics — the average expected income and the average expected risk. Now we have an optimization two-criterion problem for choosing the best solution. There are several ways to set such optimization tasks.

Let's consider such a problem in general form. Let A be a set of operations, each operation a has two numerical characteristics $E(a)$, $r(a)$ (efficiency and risk, for example) and different operations necessarily differ by at least one characteristic. When choosing the best operation, it is desirable that E be greater and r less.

We will say that operation a dominates operation b, and denote $a > b$, if $E(a) \geq E(b)$ and $r(a) \leq r(b)$ and at least one of these inequalities is strict. In this case, operation a is called dominant, and operation b is called dominant. It is clear that under no reasonable choice of the best operation, the dominant operation cannot be recognized as such. Therefore, the best operation should be sought among the dominated operations. The set of these operations is called the Pareto set, or the Pareto optimality set.

On the Pareto set, each of the characteristics E, r is a (one-to-one) function of the other. In other words, if an operation belongs to the Pareto set, then one of its characteristics can uniquely determine the other. Let's prove it. Let a, b be two operations from the Pareto set, then $r(a)$, $r(b)$ are numbers. Suppose $r(a) \leq r(b)$, then $E(a)$ cannot be equal to $E(b)$, since both points a, b belong to the Pareto set. It is proved that the characteristic r can be used to determine the characteristic E. It is also simply proved that the characteristic E can be used to determine the characteristic r.

Let's continue the analysis of the given example. Let's consider a graphic illustration (Fig. 3.4). We mark each operation (solution) (Q, R) with a point on the plane — we postpone income along the abscissa axis, and risk along the ordinate axis. We get three points and continue to analyze the example. The higher the point (Q, R), the riskier the operation, the point to the right, the more profitable it is. So, you need to choose a point lower and to the right. In our case, the Pareto set consists of only one second operation.

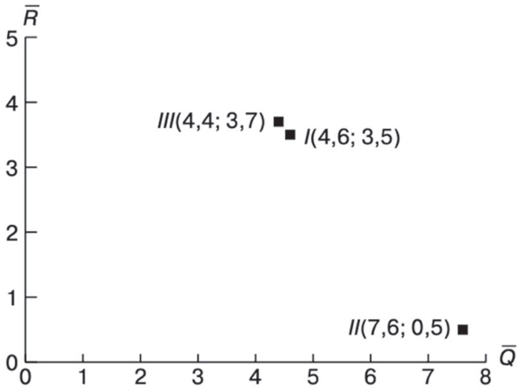

Fig. 3.4. Averaged characteristics (Q, R) of three operations (solutions)

To find the best operation, a suitable weighing formula is sometimes used, which expresses the investor's attitude to income and risk. For operation Q with characteristics *($\overline{R}$, $\overline{Q}$)*. The weighing formula gives one number by which the best operation is determined. For example, let the weighing formula be $f\,(Q) = 2\overline{Q} - \overline{R}$. This means that the investor agrees to increase the risk of the operation by two units, if the income of the operation increases at the same time by at least one unit. Then for the financial transactions of our example we have:

$f(Q_1) = 2 \cdot 4.6 - 3.5 = 5.7;$
$f(Q_2) = 2 \cdot 7.6 - 0.5 = 14.7;$
$f(Q_3) = 2 \cdot 4.4 - 3.7 = 5.1.$

It can be seen that the second operation is the best, and the third is the worst.

3.9.4. Laplace's rule of equal opportunity

This rule is applied under conditions of complete uncertainty: all unknown probabilities p_j are considered equal. After that, you can choose one of the two above rules — recommendations for decision-making, that is, the rule of maximizing the average expected income or the rule of minimizing the average expected risk.

Control questions and tasks

1. Define the income, profitability and risk of a financial transaction.
2. Express the profitability of the asset for two periods through the profitability of the asset for each of the periods.
3. Express the profitability of the asset for three periods as a whole through the profitability of the asset for each of the periods.
4. Express the profitability of the asset for several periods in general through the profitability of the asset for each of the periods (use the method of mathematical induction).
5. What is the synergetic effect when considering the profitability of an asset over several periods?
6. What is the highlighted role of uniform and normal distributions?
7. Prove that $|p_{12}| \leq 1$.
8. How is the correlation of financial transactions measured?
9. Give the risk measures you know.
10. List the types of financial risks, give them a definition and a brief description.
11. Define *VaR*.

12. List the methods known to you to reduce the risk of financial transactions, give them a definition and a brief description.

13. Define diversification and give an example.

14. Define hedging and give an example.

15. Define the matrix of consequences and risks.

16. Select a 3 x 4 impact matrix, find the risk matrix and conduct a full analysis of the situation.

17. Formulate a decision-making algorithm under conditions of complete uncertainty.

18. Formulate the rules of Wald, Savage, Hurwitz. Give examples.

19. Formulate a rule for maximizing the average expected income. Give an example.

20. Formulate a rule to minimize the average expected risk. Give an example.

21. Formulate the Laplace rule of equal opportunity. Give an example.

Chapter 4

Portfolio Analysis

The main goal of any investor is to ensure maximum return on investment. When implementing this goal, at least two main problems arise: the first is in which assets from the available and in what proportions to invest? The second problem is that in practice, as is known, a higher level of profitability is associated with a higher risk. Therefore, an investor can choose an asset with a high yield and high risk or a more or less guaranteed low yield. The two selection problems described above constitute the problem of forming an investment portfolio, the solution of which is provided by portfolio theory.

4.1. The yield of the security and portfolio

We will consider the securities market as static and follow its functioning at a fixed interval of time, during which the investor owns a security, the value of which is designated p_0 at the beginning of the interval, p_1 at the end of the interval. Let d be the dividends paid for the time period under consideration. Then the yield of a security for this time interval is called the value

$$r = (p_1 + d - p_0)/p_0. \tag{4.1}$$

If we do not consider dividends and other values on which profitability depends (inflation, etc.), then the formula (4.1) takes the simplest possible form: $r = (p_1 - p_0)/p_0$.

A portfolio consisting of n types of securities is called a vector

$$X = (x_1, x_2, \dots, x_n), \tag{4.2}$$

Where x_i — price share of investments in securities of the type i.

The profitability of the portfolio X is called the value

$$r_X = \frac{p_{X_1} + d_X - p_{X_0}}{p_{X_0}}, \tag{4.3}$$

Where p_{x0} – portfolio value at the beginning of the period;

p_{x1} – portfolio value at the end of the period;

d_x – dividends received on all securities of the portfolio;

The profitability of the portfolio X is expressed by the formula

$$r_X = x_1 r_1 + x_2 r_2 + \ldots + x_n r_n, \tag{4.4}$$

where $r_1, r_2, ..., r_n$ — yields of securities included in the portfolio X.

For each security i from formula (4.1) we have:

$$p_{i1} = p_{i0} + r_i p_{i0} - d_i \tag{4.5}$$

Multiplying equality (4.5) by a multiplier n_i — the number of securities of type i in the portfolio and summing by i, we get

$$\sum_{i=1}^{n} n_i p_{i_1} + \sum_{i=1}^{n} n_i d_i = \sum_{i=1}^{n} n_i p_{i_0} + \sum_{i=1}^{n} n_i p_{i_0} r_i. \tag{4.6}$$

Where:

$\sum_{i=1}^{n} n_i p_{i_1} = p_{X_1}$ – portfolio value at the end of the period;

$n_i p_{i_0}$ – volume of investments into type i securities;

$\sum_{i=1}^{n} n_i p_{i_0} = p_{X_0}$ – portfolio value at the beginning of the period;

$\sum_{i=1}^{n} n_i d_i = d_X$ – dividends received on all securities of the portfolio.

Therefore,

$$p_{X_1} + d_X = p_{X_0} + \sum_{i=1}^{n} n_i p_{i_0} r_i. \qquad (4.7)$$

From here,

$$r_X = \frac{p_{X_1} + d_X - p_{X_0}}{p_{X_0}^0} = \sum_{i=1}^{n} \frac{n_i p_{i_0}}{p_{X_0}^0} r_i = \sum_{i=1}^{n} x_i r_i,$$

which proves the formula (4.4).

The values of securities at the beginning of the period p_{i0} are deterministic values, while at the end of the period they are already random values, therefore both the yields of individual securities and the profitability of the entire portfolio are random values, and we will designate them with capital letters R_i and R_X. The mathematical expectation of the yield of a security is called its efficiency, and the mathematical expectation of the return of a portfolio is called the efficiency of the portfolio.

Let's find expressions for portfolio efficiency $\mu = M(R_X)$ and variance or risk square $\sigma^2 = D(R_X)$ the profitability of the R_X portfolio through the corresponding characteristics of securities.

From formula (4.4) and from the properties of mathematical expectation (the mathematical expectation of the sum is always equal to the sum of mathematical expectations; the constant can be taken as a sign of mathematical expectation) we obtain a formula for the expected return of the portfolio:

$$\mu = x_1\mu_1 + x_2\mu_2 + \ldots + x_n\mu_n, \qquad (4.8)$$

where $\mu_1 = M(R_1)$, $\mu_2 = M(R_2)$, …, $\mu_n = M(R_n)$ — efficiency of securities (mathematical expectations of yields ($R_1, R_2, \ldots, R_n$), components of the portfolio. Denote by $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)^T$ — the vector of efficiencies (expected returns) of the portfolio X, then the formula (4.8) can be written in matrix notation as follows:

$$\mu = \boldsymbol{\mu}^T X. \qquad (4.9)$$

To calculate the risk square, use the formula

$$\sigma^2 = X^T VX, \tag{4.10}$$

Where V — covariance matrix of random variables $R_1, R_2, ..., R_n$.

In the future we will use matrix notation. In this case, all vectors will be thought of as column vectors. Let's denote a vector consisting of some units by $I = (1, 1, ..., 1)^T$.

Thus, two values are associated with each portfolio X: efficiency (expected return) μ and risk σ. The investor would like to have a portfolio that would provide the highest expected return with minimal risk. Such a task, however, is contradictory, since, generally speaking, a large expected return entails an increase in risk. Therefore, the following tasks can be considered:

1) find a portfolio of minimal risk with a given efficiency (with an efficiency of at least a given, with arbitrary efficiency);

2) find a portfolio of maximum efficiency with minimal risk (with a risk not exceeding this level).

In the next paragraph, a portfolio of two securities will be considered as a simpler case and the main properties of such a portfolio will be studied in detail. Their knowledge will greatly facilitate the perception of the general portfolio analysis carried out in the following paragraphs, in which the Markowitz and Tobin portfolios will be considered.

4.2. Portfolio of two securities

4.2.1. Necessary information from probability theory

The variance of a portfolio of two securities is equal to

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2, \tag{4.11}$$

the risk is equal to

$$\sigma = \sqrt{\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2},$$

Where ρ_{12} — correlation coefficient of two securities;

σ_i — risk;

x_i — price share of i–security.

The profitability of the portfolio is equal to

$$\mu = \mu_1 x_1 + \mu_2 x_2, \tag{4.12}$$

Where μ_i —is i-th security profitability.

The normalization condition has the form

$$x_1 + x_2 = 1. \tag{4.13}$$

The yield covariance is defined as

$$\mathrm{Cov}(r_i, r_j) = M(r_i \cdot r_j) - M(r_i)M(r_j); \tag{4.14}$$

$$\mathrm{Cov}(r_i, r_j) = \rho_{ij}\sigma_i\sigma_j; \tag{4.15}$$

$$\rho_{ij} = \frac{\mathrm{Cov}(r_i, r_j)}{\sigma_i\sigma_j}; \tag{4.16}$$

$$|\rho_{ij}| \le 1. \tag{4.17}$$

In the case of independent random variables (yields) R_i, R_j $M(r_i \cdot r_j) = M(r_i)M(r_j)$, so $Cov(r_i, rj) = 0$, that is, covariance is a measure of the dependence of random variables.

Covariance matrix is a matrix whose elements are the corresponding covariances of securities. So, for a portfolio of three securities we have

$$\|\mathrm{cov}(r_1,r_2)\| = \begin{pmatrix} \mathrm{cov}(r_1,r_1) & \mathrm{cov}(r_1,r_2) & \mathrm{cov}(r_1,r_3) \\ \mathrm{cov}(r_2,r_1) & \mathrm{cov}(r_2,r_2) & \mathrm{cov}(r_2,r_3) \\ \mathrm{cov}(r_3,r_1) & \mathrm{cov}(r_3,r_2) & \mathrm{cov}(r_3,r_3) \end{pmatrix}; \qquad (4.18)$$

$$\|\rho(r_1,r_2)\| = \begin{pmatrix} \frac{\mathrm{cov}(r_1,r_1)}{\sigma_1^2} & \frac{\mathrm{cov}(r_1,r_2)}{\sigma_1\sigma_2} & \frac{\mathrm{cov}(r_1,r_3)}{\sigma_1\sigma_3} \\ \frac{\mathrm{cov}(r_2,r_1)}{\sigma_2\sigma_1} & \frac{\mathrm{cov}(r_2,r_2)}{\sigma_2^2} & \frac{\mathrm{cov}(r_2,r_3)}{\sigma_2\sigma_3} \\ \frac{\mathrm{cov}(r_3,r_1)}{\sigma_3\sigma_1} & \frac{\mathrm{cov}(r_3,r_2)}{\sigma_3\sigma_2} & \frac{\mathrm{cov}(r_3,r_3)}{\sigma_3^2} \end{pmatrix}. \qquad (4.19)$$

Considering that

$$\mathrm{Cov}(r_i,r_i) = M(r_i \cdot r_i) - M(r_i)M(r_i) = M(r_i^2) - M^2(r_i) = D(r_i) = \sigma_i^2,$$

we get

$$\|\rho(r_i,r_j)\| = \begin{pmatrix} 1 & \frac{\mathrm{cov}(r_1,r_2)}{\sigma_1\sigma_2} & \frac{\mathrm{cov}(r_1,r_3)}{\sigma_1\sigma_3} \\ \frac{\mathrm{cov}(r_2,r_1)}{\sigma_2\sigma_1} & 1 & \frac{\mathrm{cov}(r_2,r_3)}{\sigma_2\sigma_3} \\ \frac{\mathrm{cov}(r_3,r_1)}{\sigma_3\sigma_1} & \frac{\mathrm{cov}(r_3,r_2)}{\sigma_3\sigma_2} & 1 \end{pmatrix}. \qquad (4.20)$$

Example 4.1. The covariance matrix is given

$$V = \begin{pmatrix} 9 & -8 & 6 \\ -8 & 16 & -11 \\ 6 & -11 & 4 \end{pmatrix}$$

Find the correlation matrix.

There are variances along the diagonal, so for the risks of securities we have:

$$\sigma_1 = \sqrt{9} = 3; \ \sigma_2 = \sqrt{16} = 4; \ \sigma_3 = \sqrt{4} = 2.$$

If according to the formula (4.16) $\rho_{ij} = \frac{Cov(r_i, r_j)}{\sigma_i \sigma_j}$ we calculate the non–diagonal terms (all diagonal terms of the correlation matrix are equal to 1):

$$\rho_{12} = \frac{Cov(r_1, r_2)}{\sigma_1 \sigma_2} = \frac{-8}{3 \cdot 4} = -\frac{2}{3} = \rho_{21},$$

$$\rho_{13} = \frac{Cov(r_1, r_3)}{\sigma_1 \sigma_3} = \frac{6}{3 \cdot 2} = 1 = \rho_{31},$$

$$\rho_{23} = \frac{Cov(r_2, r_3)}{\sigma_2 \sigma_3} = \frac{-11}{4 \cdot 2} = -\frac{11}{8} = \rho_{32}.$$

We obtain the following correlation matrix

$$\left\| \rho(r_i, r_j) \right\| = \begin{pmatrix} 1 & -\frac{2}{3} & 1 \\ -\frac{2}{3} & 1 & -\frac{11}{8} \\ 1 & -\frac{11}{8} & 1 \end{pmatrix}.$$

Remark. The inverse problem of finding the covariance matrix for a given correlation matrix is uncertain: it has no unambiguous solution. This follows from the fact that due to the symmetry of the correlation matrix, only three values are given $\rho(r_1,r_2), \rho(r_1,r_3), \rho(r_2,r_3)$, which allows you to write three equations:

$$\frac{cov(r_1, r_2)}{\sigma_1 \sigma_2} = \rho(r_1, r_2); \quad \frac{cov(r_1, r_3)}{\sigma_1 \sigma_3} = \rho(r_1, r_3); \quad \frac{cov(r_2, r_3)}{\sigma_2 \sigma_3} = \rho(r_2, r_3)$$

for six unknowns

$$\sigma_1, \sigma_2, \sigma_3; cov(r_1, r_2), cov(r_1, r_3), cov(r_2, r_3).$$

4.2.2. The case of complete correlation

In the case of full correlation

$$\rho_{12} = \rho = 1. \quad (4.21)$$

For the risk square (variance) of the portfolio we have:

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\sigma_1\sigma_2 x_1 x_2 = \left(\sigma_1 x_1 + \sigma_2 x_2\right)^2.$$

Extracting the root from both parts, we obtain for the portfolio risk

$$\sigma = \left|\sigma_1 x_1 + \sigma_2 x_2\right| \quad (4.22)$$

Since all variables are non-negative, the module sign can be omitted:

$$\sigma = \sigma_1 x_1 + \sigma_2 x_2 \quad (4.23)$$

Replacing $x_1 \to 1 - t;\ x_2 \to t$, so that $x_1 + x_2 = 1$, we will get:

$$\sigma = \sigma_1 (1 - t) + \sigma_2 t. \quad (4.24)$$

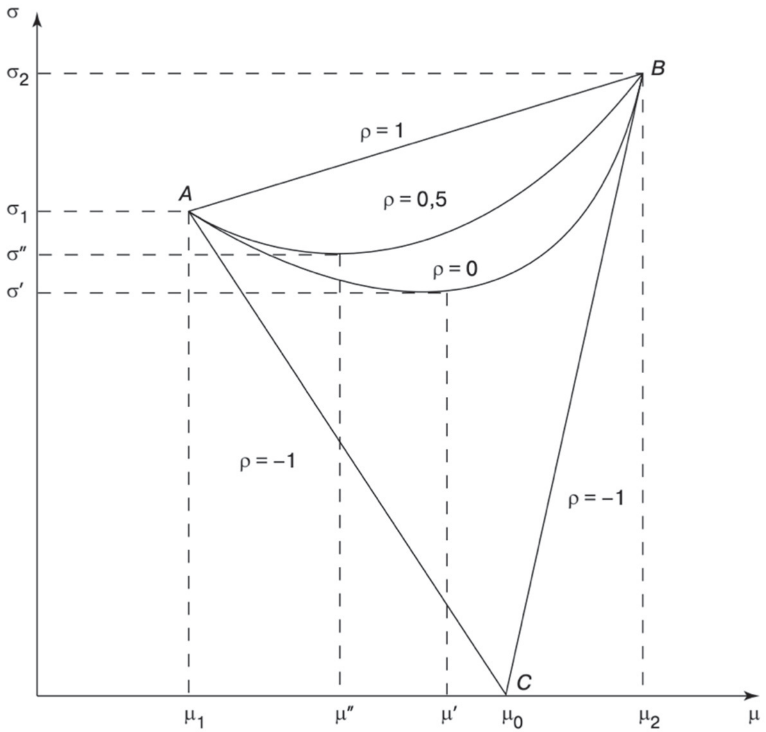

Fig. 4.1. The dependence of the risk of a portfolio of two securities on its effectiveness with fixed parameters of both securities and an increase in the correlation coefficient from –1 to 1.

This is the equation of the segment (AB), where points A and B have the following coordinates: $(\cdot)A=(\mu_1,\sigma_1);(\cdot)B=(\mu_2,\sigma_2)$. runs through values from 0 to 1.

By $t = 0$ the portfolio is at the point A, by $t = 1$ — at the point B. Thus, the permissible set of portfolios in the case of a complete correlation of securities is a segment (AB) (Fig. 4.1).

If an investor forms a portfolio of minimal risk, he must include in it a security of one type that has less risk, in this case security 1, and the portfolio in this case has the form $X = (1, 0)$. Portfolio profitability $\mu = \mu_1$.

When forming a portfolio of maximum profitability, it is necessary to include only security that has a high yield, in this case security 2, and the portfolio in this case has the form $X = (0, 1)$. Portfolio profitability $\mu = \mu_2$.

4.2.3. The case of complete anticorrelation

In case of complete anticorrelation

$$\rho_{12} = \rho = -1. \tag{4.25}$$

For the risk square (variance) of the portfolio we have

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 - 2\sigma_1\sigma_2 x_1 x_2 = (\sigma_1 x_1 - \sigma_2 x_2)^2.$$

Extracting the root from both parts, we obtain for the portfolio risk

$$\sigma = |\sigma_1 x_1 - \sigma_2 x_2|. \tag{4.26}$$

The permissible set of portfolios in the case of complete anticorrelation of securities is two segments (A, C) and (B, C) (Fig. 4.1). With complete anticorrelation, a zero-risk portfolio is possible (point C (μ_0, 0)). Let's find a zero-risk portfolio and its profitability.

From (4.26) we have:

$$\sigma_1 x_1 - \sigma_2 x_2 = 0 \tag{4.27}$$

Substituting in (4.27) $x_2 = 1 - x_1$, we will get

$$\sigma_1 x_1 - \sigma_2 (1 - x_1) = 0$$

$$x_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2} \tag{4.28}$$

$$x_2 = 1 - x_1 = \frac{\sigma_1}{\sigma_1 + \sigma_2} \tag{4.29}$$

Thus, the zero-risk portfolio has the form

$$X = \left(\frac{\sigma_2}{\sigma_1 + \sigma_2}, \frac{\sigma_1}{\sigma_1 + \sigma_2} \right), \tag{4.30}$$

and its profitability is equal to

$$\mu_0 = \frac{\mu_1 \sigma_2 + \mu_2 \sigma_1}{\sigma_1 + \sigma_2}. \tag{4.31}$$

Note that the zero-risk portfolio does not depend on the yields of securities, but is determined only by their risks, and the price share of one security is proportional to the risk of another.

Because $p \leq 1$ then all valid portfolios are inside ($p < 1$) or on the border ($p = 1$) of the triangle ABC.

Example 4.2. For a portfolio of two securities with yield and risk, respectively (0,2; 0,5) and (0,4; 0,7), in the case of complete anticorrelation, find a zero–risk portfolio and its profitability.

First, using the formula (4.30), we will find a zero-risk portfolio

$$X_0 = \left(\frac{\sigma_2}{\sigma_1 + \sigma_2}, \frac{\sigma_1}{\sigma_1 + \sigma_2} \right) = \left(\frac{0,7}{0,5+0,7}, \frac{0,5}{0,5+0,7} \right) = (0,583; 0,417).$$

Then, using the formula (4.31), we will find its profitability

$$\mu_0 = \frac{\mu_1\sigma_2 + \mu_2\sigma_1}{\sigma_1 + \sigma_2} = \frac{0,2 \cdot 0,7 + 0,4 \cdot 0,5}{0,5+0,7} = 0,283.$$

As you can see, the profitability of the portfolio is intermediate between the yields of both securities (but at the same time the risk is zero).

You can check the result for portfolio profitability by calculating it using the formula (4.8):

$\mu = x_1\mu_1 + x_2\mu_2 = 0.583 \cdot 0.2 + 0.417 \cdot 0.4 = 0.283.$

4.2.4. Independent securities

For independent securities

$$\rho_{12} = \rho = 0. \qquad (4.32)$$

For the risk square (variance) of the portfolio we have

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2. \qquad (4.33)$$

We will find a portfolio of minimal risk and its profitability and risk. That is, it is necessary to minimize the objective function

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 \qquad (4.34)$$

On condition

$$x_1 + x_2 = 1. \qquad (4.35)$$

This is a task for a conditional extremum, which could be solved using the Lagrange function.

Let's make up the Lagrange function and find its extremum

$$L = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \lambda\left(x_1 + x_2 - 1\right). \tag{4.36}$$

To find stationary points, we have a system

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 2\sigma_1^2 x_1 + \lambda = 0, \\ \dfrac{\partial L}{\partial x_2} = 2\sigma_2^2 x_2 + \lambda = 0, \\ \dfrac{\partial L}{\partial \lambda} = x_1 + x_2 - 1 = 0. \end{cases} \tag{4.37}$$

Subtracting the second from the first equation, we get

$$\sigma_1^2 x_1 = \sigma_2^2 x_2.$$

Next, using the third equation, we have

$$\sigma_1^2 x_1 = \sigma_2^2 \left(1 - x_1\right).$$

From here

$$x_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2},\ x_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}.$$

Portfolio

$$X = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right), \tag{4.38}$$

and its profitability

$$\mu = \frac{\mu_1 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} + \frac{\mu_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2}. \tag{4.39}$$

Portfolio risk is equal to

$$\sigma = \sqrt{\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2} = \sqrt{\frac{\sigma_1^4 \sigma_2^2 + \sigma_1^2 \sigma_2^4}{\left(\sigma_1^2 + \sigma_2^2\right)^2}} =$$
$$= \sqrt{\frac{\sigma_1^2 \sigma_2^2 \left(\sigma_1^2 + \sigma_2^2\right)}{\left(\sigma_1^2 + \sigma_2^2\right)^2}} = \frac{\sigma_1 \sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} \quad (4.40)$$

In the case of three securities, there is no direct analogy with the formula (4.38) (paragraph 4.2.5).

Example 4.3. Using the formula (4.40), it is easy to demonstrate the impact of diversification on portfolio risk. Let the portfolio consist of two independent securities with risks σ_1=0.1 and σ_2=0.2, respectively. Calculate the portfolio risk using the formula (4.40)

$$\sigma = \frac{\sigma_1 \sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} = \frac{0.1 \cdot 0.2}{\sqrt{0.01 + 0.04}} \approx 0.0894$$

So, the portfolio risk of $\sigma \approx 0.0894$ was lower than the risk of each of the securities (0.1; 0.2). This is an illustration of the principle of diversification: when a portfolio is “smeared” on independent securities, its risk decreases.

4.2.5. Three independent securities

Although this case goes beyond the question of a portfolio of two securities, we consider it here as a generalization of the case of a portfolio of two securities.

For independent securities

$$\rho_{12} = \rho_{13} = \rho_{23} = 0. \quad (4.41)$$

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2. \quad (4.42)$$

For the risk square (variance) of the portfolio we have

We will find a portfolio of minimal risk, its profitability and risk. So it is necessary to minimize the objective function

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2$$

on condition

$$x_1 + x_2 + x_3 = 1. \qquad (4.43)$$

This is a conditional extremum problem, which could be solved using the Lagrange function.

Let's make up the Lagrange function and find its extremum

$$L = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 + \lambda(x_1 + x_2 + x_3 - 1). \qquad (4.44)$$

To find stationary points, we have a system

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 2\sigma_1^2 x_1 + \lambda = 0; \\ \dfrac{\partial L}{\partial x_2} = 2\sigma_2^2 x_2 + \lambda = 0; \\ \dfrac{\partial L}{\partial x_3} = 2\sigma_3^2 x_3 + \lambda = 0; \\ \dfrac{\partial L}{\partial \lambda} = x_1 + x_2 - 1 = 0. \end{cases} \qquad (4.45)$$

Subtracting the second from the first equation, then the third, we get

$$\sigma_1^2 x_1 = \sigma_2^2 x_2, \ \sigma_1^2 x_1 = \sigma_2^2 x_3.$$

From here

$$x_2 = \frac{\sigma_1^2}{\sigma_2^2} x_1, \ x_3 = \frac{\sigma_1^2}{\sigma_3^2} x_1. \qquad (4.46)$$

Substituting (4.46) in the normalization condition

$x_1 + x_2 + x_3 = 1,$ (4.47)

We get

$$x_1 + \frac{\sigma_1^2}{\sigma_2^2}x_1 + \frac{\sigma_1^2}{\sigma_3^2}x_1 = 1. \quad (4.48)$$

From here

$$x_1 = \frac{1}{1+\frac{\sigma_1^2}{\sigma_2^2}+\frac{\sigma_1^2}{\sigma_3^2}} = \frac{\sigma_2^2\sigma_3^2}{\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2}. \quad (4.49)$$

Substituting the resulting value of x_1 in (4.46), we get two more components of the portfolio

The portfolio has the form

$$X = \frac{1}{\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2}\left(\sigma_2^2\sigma_3^2;\sigma_1^2\sigma_3^2;\sigma_1^2\sigma_2^2\right), \quad (4.52)$$

and its profitability is equal to

$$\mu = \frac{\mu_1\sigma_2^2\sigma_3^2+\mu_2\sigma_1^2\sigma_3^2+\mu_3\sigma_1^2\sigma_2^2}{\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2}. \quad (4.53)$$

Portfolio risk is equal to

$$\sigma = \sqrt{\sigma_1^2x_1^2+\sigma_2^2x_2^2+\sigma_3^2x_3^2} = $$
$$= \sqrt{\frac{\left(\sigma_1^2\sigma_2^4\sigma_3^4+\sigma_2^2\sigma_1^4\sigma_3^4+\sigma_3^2\sigma_1^4\sigma_2^4\right)}{\left(\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2\right)^2}} =$$
$$= \frac{\sigma_1\sigma_2\sigma_3}{\sqrt{\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2}}. \quad (4.54)$$

Example 4.4. For a portfolio of three independent securities with profitability and risk, respectively (0,1; 0,4), (0,2; 0,6) and (0,4; 0,8) find the minimum risk portfolio, its risk and profitability.

The minimum risk portfolio has the form (4.52):

$$X=\frac{1}{\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2}\left(\sigma_2^2\sigma_3^2;\sigma_1^2\sigma_3^2;\sigma_1^2\sigma_2^2\right)=$$
$$=\frac{\left(0{,}6^2\cdot 0{,}8^2;\,0{,}4^2\cdot 0{,}8^2;\,0{,}4^2\cdot 0{,}6^2\right)}{0{,}6^2\cdot 0{,}8^2+0{,}4^2\cdot 0{,}8^2+0{,}4^2\cdot 0{,}6^2}=\frac{(0{,}2304;\,0{,}1024;\,0{,}0576)}{0{,}2304+0{,}1024+0{,}0576}=$$
$$=\frac{(0{,}2304;\,0{,}1024;\,0{,}0576)}{0{,}3904}=(0{,}590;\,0{,}263;\,0{,}147).$$

So, $X=(0{,}590;\,0{,}263;\,0{,}147)$.

The risk of the minimum risk portfolio is found by the formula (4.54):

$$\sigma=\frac{\sigma_1\sigma_2\sigma_3}{\sqrt{\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2}}=\frac{0{,}4\cdot 0{,}6\cdot 0{,}8}{\sqrt{0{,}6^2\cdot 0{,}8^2+0{,}4^2\cdot 0{,}8^2+0{,}4^2\cdot 0{,}6^2}}=$$
$$=\frac{0{,}192}{\sqrt{0{,}2304+0{,}1024+0{,}0576}}=\frac{0{,}192}{\sqrt{0{,}3904}}=\frac{0{,}192}{0{,}6348}=0{,}307.$$

$$\mu=\frac{\mu_1\sigma_2^2\sigma_3^2+\mu_2\sigma_1^2\sigma_3^2+\mu_3\sigma_1^2\sigma_2^2}{\sigma_2^2\sigma_3^2+\sigma_1^2\sigma_3^2+\sigma_1^2\sigma_2^2}=$$
$$=\frac{0{,}1\cdot 0{,}6^2\cdot 0{,}8^2+0{,}2\cdot 0{,}4^2\cdot 0{,}8^2+0{,}4\cdot 0{,}4^2\cdot 0{,}6^2}{0{,}6^2\cdot 0{,}8^2+0{,}4^2\cdot 0{,}8^2+0{,}4^2\cdot 0{,}6^2}=$$
$$=\frac{0{,}02304+0{,}02048+0{,}02304}{0{,}2304+0{,}1024+0{,}0576}=\frac{0{,}06656}{0{,}3904}=0{,}1705.$$

Finally, the profitability of the portfolio is calculated by the formula (4.53):

As you can see, the portfolio risk is less than the risk of each individual security, and the portfolio yield is greater than the yield of the first security, slightly less than the yield of the second and less than the yield of the third security.

4.2.6. Risk-free security

Let one of the two portfolio securities be risk-free. A portfolio of n–securities, including a risk-free one, bears the name of Tobin, who first investigated it, and has properties significantly different from those of a portfolio consisting only of risky securities (paragraph 4.4). Here we will consider how the inclusion of a risk-free security in a portfolio of two securities affects the effective portfolio set.

So, we have two securities: $1(\mu_1, 0)$ and $2(\mu_2, \sigma_2)$, while $\mu_1 < \mu_2$ (otherwise it would be necessary to form a portfolio (1,0) consisting only of risk-free security, and we would have a risk–free portfolio of maximum profitability).

We have the following equations:

$$\begin{aligned} &\mu = \mu_1 x_1 + \mu_2 x_2, \\ &\sigma = \sigma_2 x_2, \\ &x_1 + x_2 = 1. \end{aligned} \tag{4.55}$$

It is easy to get an acceptable set of portfolios from them

$$\begin{aligned} &\mu = \mu_1 \cdot (1 - x_2) + \mu_2 \cdot x_2 = \\ &= \mu_1 + (\mu_2 - \mu_1) \cdot x_2 = \mu_1 + (\mu_2 - \mu_1) \cdot \frac{\sigma}{\sigma_2}, \end{aligned}$$

which is a segment

$$\mu = \mu_1 + (\mu_2 - \mu_1) \cdot \frac{\sigma}{\sigma_2}. \tag{4.56}$$

When σ=0, the portfolio is at the point $1(\mu_1, 0)$, and at $\sigma = \sigma_2$ —at the point $2(\mu_2, \sigma_2)$ (Fig. 4.2).

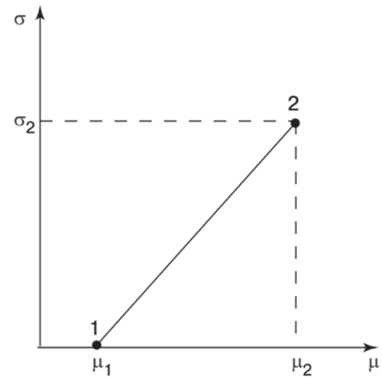

Fig. 4.2. Allowable set of portfolios consisting of two securities, one of which is risk–free

Although this case is very simple, two conclusions can be drawn from it:

1) the allowable set of portfolios does not depend on the correlation coefficient (although usually a risk-free security is considered uncorrelated with other (risky) securities;

2) the allowable set of portfolios has narrowed from a triangle to a segment.

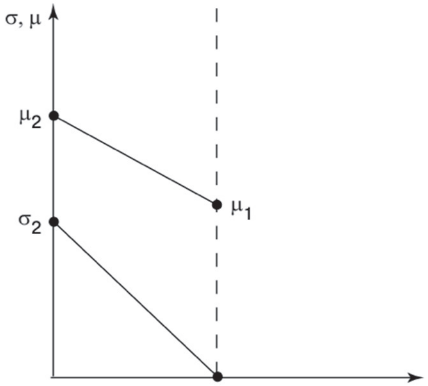

Fig. 4.3. Dependence of the profitability and risk of the portfolio on the share of risk-free securities x_1

A similar effect occurs in the case of Tobin's portfolio.

In conclusion, we give the dependence of the profitability and risk of the portfolio on the share of risk–free securities (Fig. 4.3).

Portfolio risk decreases linearly from σ_2 by $x_1 = 0$ to zero by $x_1 = 1$, while the yield also linearly decreases from μ_2 by $x_1 = 0$ to μ_1 by $x_1 = 1$.

4.2.7. Fixed Efficiency Portfolio

In the case of a portfolio of two securities, the task of portfolio efficiency or its risk uniquely determines the portfolio (except in the case of $\mu_1 = \mu_2$, when only the portfolio risk assignment uniquely determines the portfolio itself – see below for more details).

When setting the efficiency of a portfolio, it is definitely found as a solution to the system

$$\begin{cases} \mu = \mu_1 x_1 + \mu_2 x_2, \\ x_1 + x_2 = 1, \end{cases} \tag{4.57}$$

and when setting the portfolio risk, as a solution to the system

$$\begin{cases} \sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2, \\ x_1 + x_2 = 1. \end{cases} \tag{4.58}$$

Therefore, in the case of a portfolio of two securities, it is not necessary to talk about the minimum limit (the minimum risk of the portfolio at a given efficiency). The reasoning about this is erroneous.

Let's consider the first case when the portfolio efficiency is set. Suppose, $\mu_1 \neq \mu_2$. The portfolio is definitely located as a solution to the system (4.57):

$$\begin{cases} \mu = \mu_1 x_1 + \mu_2 x_2, \\ x_1 + x_2 = 1. \end{cases}$$

Expressing x_2 from the second equation and substituting it into the first, we get:

$$\mu = x_1\mu_1 + x_2\mu_2 = x_1\mu_1 + (1-x_1)\mu_2 = x_1(\mu_1-\mu_2) + \mu_2.$$

From here we find

$$x_1 = \frac{\mu-\mu_2}{\mu_1-\mu_2},\; x_2 = \frac{\mu_1-\mu}{\mu_1-\mu_2}. \qquad (4.59)$$

Substituting these expressions into the expression for the risk square of the portfolio, we get:

$$\sigma^2 = \frac{\sigma_1^2(\mu-\mu_2)^2 + \sigma_2^2(\mu-\mu_1)^2 - 2\sigma_1\sigma_2\rho_{12}(\mu-\mu_1)(\mu-\mu_2)}{(\mu_2-\mu_1)^2}. \qquad (4.60)$$

This equation is mistakenly called the minimum boundary equation. In fact, it is an equation of the (unambiguous) relationship of portfolio risk with its effectiveness.

Only in the case of $\mu_1 = \mu_2$, when for all values x_1 and x_2 the equality is fulfilled $\mu = \mu_1 = \mu_2$ and the permissible set of portfolios from the triangle narrows to a (vertical) segment, we can talk about the minimum boundary, which in this case consists of a single point (μ, σ_1) (by $\sigma_1 < \sigma_2$) or (μ, σ_2) (by $\sigma_1 > \sigma_2$).

Consider the various limiting cases discussed above.

1. Cases of complete correlation** ($\rho_{12} = 1$) **and complete anticorrelation

($\rho_{12} = -1$).

Due to the fact that the correlation coefficient ρ does not exceed in absolute value 1, let's start exploring the equation (4.60) for extreme values $\rho = \pm 1$.

First, let's give general considerations. It is known that for ρ= ±1 the random variables R_1 and R_2 are linearly dependent.

Without limitation of generality, it can be considered that $R_2 = aR_1 + b$. Then the profitability of the portfolio will be written as follows.

$$R_X = x_1 R_1 + (1-x_1)R_2 = (x_1 + a(1-x_1))R_1 + (1-x_1)b.$$

so

$$\sigma^2 = (x_1 + a(1-x_1))^2 \sigma_1^2,\ \mu = (x_1 + a(1-x_1))\,\mu_1 + (1-x_1)\,b.$$

After excluding the parameter x_1 we get a relation of the form:

$$\sigma^2 = (c\mu + d)^2,$$

that is, the risk as a function of efficiency will have the form of a segment or angle (Fig. 4.1).

Now we study equation (4.60) in the cases ρ= ±1.

complete correlation case ($\rho_{12} = 1$):

$$\sigma = \left| \frac{\sigma_1(\mu-\mu_2) - \sigma_2(\mu-\mu_1)}{(\mu_2-\mu_1)} \right|. \qquad (4.61)$$

The case of complete anticorrelation ($\rho_{12} = -1$):

$$\sigma = \left| \frac{\sigma_1(\mu-\mu_2) + \sigma_2(\mu-\mu_1)}{(\mu_2-\mu_1)} \right|. \qquad (4.62)$$

2. ***Independent securities*** ($\rho_{12} = 0$). Equation (4.60) takes the form:

$$\sigma^2 = \frac{\sigma_1^2(\mu-\mu_2)^2 + \sigma_2^2(\mu-\mu_1)^2}{(\mu_2-\mu_1)^2}. \qquad (4.63)$$

Next, it will be shown that for intermediate values of the correlation coefficient, the portfolio ρ risk as a function of its effectiveness has the form (4.71):

$$\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}. \qquad (4.64)$$

If we find the type of dependence of portfolio risk on its effectiveness for a fixed portfolio (μ_1, σ_1), (μ_2, σ_2), but with different values of the correlation coefficient ρ, it is possible to come to the following conclusion: with an increase in the correlation coefficient from –1 to 1, a decrease occurs μ_M. At the same time, the graph of the portfolio's risk dependence on its effectiveness becomes more and more elongated along the abscissa axis, that is, with a fixed change in the expected return, the increase μ in risk σ becomes less and less (Fig. 4.1).

If we imagine that $x_1 \in [0,1]$, so, and $x_2 \in [0,1]$, that's from the first formula (4.57), it follows that $\mu \in [\mu_1, \mu_2]$ under the assumption $\mu_1 < \mu_2$, since is their convex combination. Portfolios form part of the border *AMB*, and namely, its part connecting the points, it is its part connecting the points (μ_1, σ_1) and (μ_2, σ_2) (Fig. 4.1). Thus, in the case of $n = 2$ and with an additional assumption $x_1 \geq 0$, $x_2 \geq 0$ a lot of portfolios are pieces of hyperbolas or broken lines connecting points (μ_1, σ_1) and (μ_2, σ_2).

4.2.8. Portfolio of a given risk

Let's say the portfolio risk is set.

The portfolio is now found as a (single–digit or two–digit) solution to the system.

$$\begin{cases} \sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + 2\rho_{12}\sigma_1\sigma_2 x_1 x_2, \\ x_1 + x_2 = 1. \end{cases} \qquad (4.65)$$

Expressing x_2 from the second equation and substituting it into the first, we get:

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2\left(1 - x_1^2\right) + 2\rho_{12}\sigma_1\sigma_2 x_1\left(1 - x_1\right). \qquad (4.66)$$

After elementary transformations, we obtain a quadratic equation for x_1:

$$x_1^2\left(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2\right) + 2\sigma_2\left(\rho_{12}\sigma_1 - \sigma_2\right)x_1 + \left(\sigma_2^2 - \sigma^2\right) = 0.$$

Solving this equation, we find x_1– portfolio component

$$x_1 = \frac{-\sigma_2\left(\rho_{12}\sigma_1 - \sigma_2\right) \pm \sqrt{D}}{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}, \qquad (4.67)$$

where

$$D = \sigma_2^2\left(\rho_{12}\sigma_1 - \sigma_2\right)^2 - \left(\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2\right)\left(\sigma_2^2 - \sigma^2\right) =$$
$$= \sigma_1^2\left(\sigma^2 - \sigma_2^2 + \rho_{12}^2\sigma_2^2\right) + \sigma_2^2\sigma^2 - 2\rho_{12}\sigma_1\sigma_2\sigma^2.$$

For portfolio component x_2 we have:

$$x_2 = 1 - \frac{-\sigma_2\left(\rho_{12}\sigma_1 - \sigma_2\right) \pm \sqrt{D}}{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2} = \frac{\sigma_1\left(\sigma_1 - \rho_{12}\sigma_2\right) \mp \sqrt{D}}{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}. \qquad (4.68)$$

Thus, a portfolio with a given portfolio risk σ has the form:

$$X = \left(\frac{-\sigma_2\left(\rho_{12}\sigma_1 - \sigma_2\right) \pm \sqrt{D}}{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}; \frac{\sigma_1\left(\sigma_1 - \rho_{12}\sigma_2\right) \mp \sqrt{D}}{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}\right). \qquad (4.69)$$

Consider the various limiting cases discussed above.

$$D = \sigma_1^2\left(\sigma^2 - \sigma_2^2\right) + \sigma_2^2\sigma^2; \qquad (4.70)$$

$$X = \left(\frac{\sigma_2^2 \pm \sqrt{D}}{\sigma_1^2 + \sigma_2^2}; \frac{\sigma_1^2 \mp \sqrt{D}}{\sigma_1^2 + \sigma_2^2}\right). \qquad (4.71)$$

1. ***Independent securities*** ($\rho_{12} = 0$):

This is a minimum risk portfolio not found above, having the form a portfolio with risk σ

$$X=\left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2};\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right),$$

2. ***The case of complete correlation*** ($\rho_{12} = 1$):

$$D=\sigma_1^2\sigma^2+\sigma_2^2\sigma^2-2\sigma_1\sigma_2\sigma^2=\sigma^2\left(\sigma_1-\sigma_2\right)^2; \tag{4.72}$$

$$X=\left(\frac{-\sigma_2\pm\sigma}{\sigma_1-\sigma_2};\frac{\sigma_1\mp\sigma}{\sigma_1-\sigma_2}\right). \tag{4.73}$$

3. ***The case of complete anticorrelation*** ($\rho_{12} = -1$):

$$D=\sigma_1^2\sigma^2+\sigma_2^2\sigma^2+2\sigma_1\sigma_2\sigma^2=\sigma^2\left(\sigma_1+\sigma_2\right)^2; \tag{4.74}$$

$$X=\left(\frac{\sigma_2\pm\sigma}{\sigma_1+\sigma_2};\frac{\sigma_1\mp\sigma}{\sigma_1+\sigma_2}\right). \tag{4.75}$$

$$X=\left(\frac{\sigma_2}{\sigma_1+\sigma_2};\frac{\sigma_1}{\sigma_1+\sigma_2}\right), \tag{4.76}$$

From here it is easy to get a zero-risk portfolio by putting $\sigma = 0$: naturally, coinciding with the above obtained in another way.

4.3. Portfolios from n-papers. Markowitz Portfolios

4.3.1. Minimal risk portfolio for a given efficiency

The first of these tasks was set and solved by Markowitz. So, we consider the following task: we need to find a portfolio $X =$

$= (x_1, x_2, ..., x_n)^T$, which would minimize the risk σ and provided a given amount of expected profitability μ.

In the mathematical formulation, the problem looks like this:

find the minimum of the objective function

$$\frac{1}{2}\sigma^2 = \frac{1}{2}X^T VX \to \min \tag{4.77}$$

under the conditions

$$\vec{\mu}^T X = \mu; \tag{4.78}$$

$$I^T X = 1. \tag{4.79}$$

Note that the numerical multiplier in the objective function is introduced for convenience. We are looking for a minimum of the risk square – this is also due to technical considerations. Condition (4.78) provides this level of efficiency. The condition (4.79) follows from the definition of the vector ***X***.

If we additionally assume that the vector ***X*** consists of non-negative numbers

$$X \geq 0, \tag{4.80}$$

then the components ***X*** can be interpreted as the shares of investments invested in the corresponding asset. In general, among the numbers x_1, x_2, ..., x_n negative ones may occur, which means a debt obligation.

In the following, we assume that the covariance matrix V is positively defined, and the vector of efficiencies is $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)^T$ vector ***I***; in other words, not all efficiencies are equal.

The first assumption, in particular, means that the covariance matrix is non-degenerate and is fulfilled in practice for risky assets (stocks). In particular, there is an inverse matrix V^{-1}, which is also positively defined. The case of a risk-free asset will be considered later. If the second assumption is violated, the task has a simpler solution, which will be indicated below.

Let's turn to the following constants:

$$\alpha = I^T V^{-1} I,\ \beta = I^T V^{-1} \vec{\mu} = \vec{\mu}^T V^{-1} I,\ \gamma = \vec{\mu}^T V^{-1} \vec{\mu},\ \delta = \alpha\gamma - \beta^2. \quad (4.81)$$

Equality for β follows from the symmetry of the matrix V^{-1}.

We prove that the constants α, γ, δ are positive numbers. Positivity of numbers α, γ follows from the fact that for any positive definite matrix M and any non-zero vector W number $W^T M W$ will be positive.

To prove the positivity of the number δ as a vector W, consider the vector $\alpha\mu - \beta I$. It is nonzero, since, by assumption, the vector of efficiencies μ is not collinear to vector I. We have

$$(\alpha\vec{\mu} - \beta I)^T V^{-1} (\alpha\vec{\mu} - \beta I) = \alpha^2\gamma - 2\alpha\beta^2 + \alpha\beta^2 = \alpha^2\gamma - \alpha\beta^2 = \alpha\delta > 0.$$

Therefore $\delta > 0$.

$$X = V^{-1}(\lambda I + \nu\vec{\mu}),\ \lambda = (\gamma - \beta\mu)/\delta,\ \nu = (\alpha\mu - \beta)/\delta. \quad (4.82)$$

We prove that the problem of finding the optimal portfolio with the objective function (4.77) under conditions (4.78)—(4.79) has a unique solution:

The objective function (4.77) is a quadratic form with a positive definite matrix V. Any such form is reduced by some non–degenerate linear transformation to the form $y_1^2 + y_2^2 + \ldots + y_n^2$. In this case, the constraints (4.78)—(4.79) turn into a system of two linear equations with respect to the new variables $y_1, y_2, \ldots, y_n$.. Let P be a plane in space R^n (dimension n–1 or n–2) defined by these constraints. The initial problem is reduced to finding the point closest to the origin in the plane P. As you know, such a problem has a unique solution.

To find explicit formulas (4.82), we use the Lagrange multiplier method. Consider the Lagrange function for the optimization problem (4.77)—(4.79):

$$L(X,\lambda,\nu)=\frac{1}{2}X^TVX+\lambda\left(1-I^TX\right)+\nu\left(\mu-\vec{\mu}^TX\right).$$

Equating the derivatives with respect to X, λ, ν, to zero, we obtain a system of three equations

$$\begin{cases} VX=\lambda I+\nu\vec{\mu}, \\ I^TX=1, \\ \vec{\mu}^TX=\mu. \end{cases} \tag{4.83}$$

Let's express the unknown $\boldsymbol{X}$ from the first equation

$$X=V^{-1}\left(\lambda I+\nu\vec{\mu}\right) \tag{4.84}$$

and we substitute the second and third equations of the system

$$\begin{cases} \alpha\lambda+\beta\nu=1, \\ \beta\lambda+\gamma\nu=\mu. \end{cases} \tag{4.85}$$

System (4.85) determinant $\delta \neq 0$ (we proved above that $\delta > 0$), so it has the only solution

$$\lambda=\frac{\gamma-\beta\mu}{\delta},\ \nu=\frac{\alpha\mu-\beta}{\delta}. \tag{4.86}$$

These formulas together with equality (4.84) give a solution to the optimization problem (4.77)—(4.79):

$$X=V^{-1}\left(\frac{\gamma-\beta\mu}{\delta}I+\frac{\alpha\mu-\beta}{\delta}\vec{\mu}\right). \tag{4.87}$$

So, for each value of the expected return μ there is a single portfolio X that provides a minimum value of risk $\sigma = \sigma_{\min}$, that is, the function is defined

$$\sigma=\sigma(\mu). \tag{4.88}$$

The graph of the function (4.88) is called the minimum boundary.

4.3.2. Minimum boundary and its properties

Next, we will consider the solution of two more problems about the Markowitz portfolio: a portfolio of minimal risk with an efficiency of at least a given one and a portfolio of minimal risk with arbitrary efficiency. To solve them, we will use the already obtained solution of the problem of the Markovitz portfolio of minimal risk for a given efficiency, as well as the idea of the minimum boundary, to the detailed description of which we proceed.

As already mentioned, the graph of the dependence of the minimum risk of the portfolio on its effectiveness, i.e. the graph of the function (4.88), is called the minimum boundary. We show that such is a branch of a hyperbola whose equation has the form:

$$\sigma = \sqrt{\frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}}. \qquad (4.89)$$

To find the desired equation, it is enough to substitute the found solution $\boldsymbol{X}$ in the expression for σ:

$$\begin{aligned}\sigma^2 &= X^T V X = \left(V^{-1}(\lambda I + \nu\vec{\mu})\right)^T V \cdot V^{-1}(\lambda I + \nu\vec{\mu}) = \\ &= \left(\lambda I^T + \nu\vec{\mu}^T\right)V^{-1}(\lambda I + \nu\vec{\mu}) = \lambda^2\alpha + 2\lambda\nu\beta + \nu^2\gamma = \\ &= \lambda(\lambda\alpha + \nu\beta) + \nu(\lambda\beta + \nu\gamma).\end{aligned} \qquad (4.90)$$

Substituting values for expressions in parentheses from the system (4.85), we have:

$$\sigma^2 = \lambda + \nu\mu = \frac{\gamma - \beta\mu}{\delta} + \frac{\alpha\mu - \beta}{\delta}\mu = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}, \qquad (4.91)$$

from where we get the equation of the minimum boundary. Let's bring it into the canonical form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

or in our variables

$$\frac{\sigma^2}{a^2}-\frac{\mu^2}{b^2}=1.$$

To do this, select the full square on the right side of the equation

$$\sigma^2=\frac{\alpha}{\delta}\left[\left(\mu-\frac{\beta}{\alpha}\right)^2+\frac{\gamma}{\alpha}-\frac{\beta^2}{\alpha^2}\right]=\frac{\alpha}{\delta}\left[\left(\mu-\frac{\beta}{\alpha}\right)^2+\frac{\delta}{\alpha^2}\right]=\frac{\alpha}{\delta}\left(\mu-\frac{\beta}{\alpha}\right)^2+\frac{1}{\alpha}.$$

The canonical equation of the minimum boundary has the form:

$$\alpha\sigma^2-\frac{\alpha^2}{\delta}\left(\mu-\frac{\beta}{\alpha}\right)^2=1,$$

or

$$\frac{\sigma^2}{a^2}-\frac{\tilde{\mu}^2}{b^2}=1, \qquad (4.92)$$

where

$$a^2=1/\alpha,\ b^2=\delta/\alpha^2,\ \tilde{\mu}=\mu-\frac{\beta}{\alpha}. \qquad (4.93)$$

The minimum bound is a branch of a hyperbola with asymptotes $\sigma=\sqrt{\frac{\alpha}{\delta}}\left|\mu-\frac{\beta}{\alpha}\right|$ and the absolute minimum $M\left(\frac{\beta}{\alpha},\frac{1}{\sqrt{\alpha}}\right)$. We obtain an asymptote equation that has the form: $y=\pm\frac{b}{a}x$ or in our variables

$$\sigma=\pm\frac{a}{b}\tilde{\mu}=\pm\frac{a}{b}\left|\mu-\frac{\beta}{\alpha}\right|=\frac{\alpha}{\alpha\sqrt{\delta\alpha}}\left|\mu-\frac{\beta}{\alpha}\right|=\sqrt{\frac{\alpha}{\delta}}\left|\mu-\frac{\beta}{\alpha}\right|. \qquad (4.94)$$

It is not difficult to see that in the degenerate case, when all expected returns coincide and are equal μ, the minimum boundary is reduced to a single point, $M\left(\mu, \frac{1}{\sqrt{\alpha}}\right)$ and $X = \frac{1}{\alpha} \cdot V^{-1} \cdot I$.

The graph of the minimum boundary is shown in Fig. 4.4. On it *AMB* is the minimum boundary, $M\left(\frac{\beta}{\alpha}, \frac{1}{\sqrt{\alpha}}\right)$ is the point of the absolute minimum, the dotted line indicates the asymptotes.

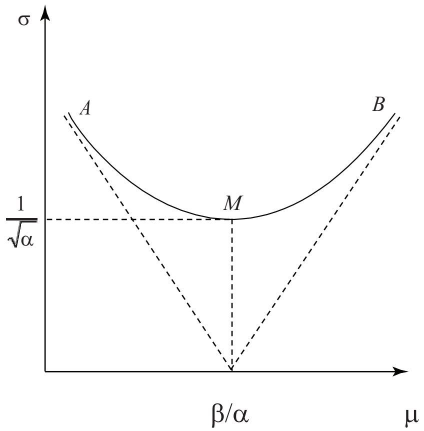

Fig. 4.4. View of the minimum boundary for the Markowitz portfolio

Since the investor is interested in increasing the efficiency of μ, it is clear that he will choose a point on the more profitable part of the minimum boundary, namely on the *MB* curve, which is called the effective boundary.

$$V = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 9 & -2 \\ 0 & -2 & 4 \end{pmatrix}.\ V^{-1} = \frac{1}{14} \cdot \begin{pmatrix} 16 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

Example 4.5. Given a portfolio of three securities with yields $\mu_1 = 10\%$; $\mu_2 = 20\%$; $\mu_3 = 30\%$ and the covariance matrix

Find a portfolio of minimal risk with profitability μ = 25% and his risk. It is required to write the equation of the minimum boundary. Note that V is a positive definite. Find the inverse matrix V^{-1}:

Find the constants α, β, γ, σ:

$$\alpha = I^T V^{-1} I,\ \beta = I^T V^{-1}\vec{\mu} = \vec{\mu}^T V^{-1} I,\ \gamma = \vec{\mu}^T V^{-1}\vec{\mu},\ \delta = \alpha\gamma - \beta^2,$$

where $I = (1, 1, 1)^T$, $\mu = (10; 20; 30)^T$.

$$\alpha = I^T V^{-1} I = \frac{1}{14}(1,1,1)\cdot\begin{pmatrix}16 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{14}(19,5,6)\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{15}{7},$$

$$\beta = I^T V^{-1}\vec{\mu} = \frac{1}{14}(1,1,1)\cdot\begin{pmatrix}16 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 4\end{pmatrix}\cdot\begin{pmatrix}10\\20\\30\end{pmatrix} = \frac{1}{14}(19,5,6)\cdot\begin{pmatrix}10\\20\\30\end{pmatrix} = \frac{235}{7},$$

$$\gamma = \vec{\mu}^T V^{-1}\vec{\mu} = \frac{1}{14}(10,20,30)\cdot\begin{pmatrix}16 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 4\end{pmatrix}\cdot\begin{pmatrix}10\\20\\30\end{pmatrix} =$$

$$= \frac{1}{14}(230,90,150)\cdot\begin{pmatrix}10\\20\\30\end{pmatrix} = \frac{4300}{7},$$

$$\delta = \alpha\gamma - \beta^2 = \frac{4300}{7}\cdot\frac{15}{7} - \left(\frac{235}{7}\right)^2 = \frac{9275}{49} = 189{,}3.$$

We will also find constants λ and v:

$$\lambda = (\gamma - \beta\mu)/\delta \text{ и } \nu = (\alpha\mu - \beta)/\delta,$$

$$\lambda = (\gamma - \beta\mu)/\delta = \left(\frac{4300}{7} - \frac{235}{7}\cdot 25\right)\Big/189{,}3 = -225/189{,}3 = -1{,}19,$$

$$\nu = (\alpha\mu - \beta)/\delta = \left(\frac{15}{7}\cdot 25 - \frac{235}{7}\right)\Big/189{,}3 = 20/189{,}3 = 0{,}106.$$

Now let's define a portfolio of minimal risk with profitability μ = 25%:

$$X = V^{-1}(\lambda I + \nu\vec{\mu}) = \frac{1}{14}\cdot\begin{pmatrix}16 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 4\end{pmatrix}\cdot\begin{pmatrix}-1{,}19 + 0{,}106\cdot 10\\ -1{,}19 + 0{,}106\cdot 20\\ -1{,}19 + 0{,}106\cdot 30\end{pmatrix} =$$

$$= \frac{1}{14}\cdot\begin{pmatrix}16 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 4\end{pmatrix}\cdot\begin{pmatrix}-0{,}13\\ 0{,}93\\ 1{,}99\end{pmatrix} = \frac{1}{14}\cdot\begin{pmatrix}1{,}77\\ 3{,}59\\ 8{,}76\end{pmatrix} = (0{,}12;\ 0{,}26;\ 0{,}62)^T.$$

Thus, a portfolio of minimal risk with profitability μ = 25% equal to $X = (0{,}12;\ 0{,}26;\ 0{,}62)^T$: it is necessary to take 12% of the first type of security, 26% of the second and 62% of the third type.

Find the portfolio risk

$$\sigma = \sqrt{X^T V X} = (0{,}12;\ 0{,}26;\ 0{,}62)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 9 & -2 \\ 0 & -2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0{,}12 \\ 0{,}26 \\ 0{,}62 \end{pmatrix} =$$

$$= \sqrt{(-0{,}14;\ 0{,}98;\ 1{,}96)\begin{pmatrix} 0{,}12 \\ 0{,}26 \\ 0{,}62 \end{pmatrix}} = \sqrt{1{,}4532} = 1{,}205.$$

The risk of the portfolio turned out to be slightly greater than the risk of the first security (σ_1 = 1), but there is less risk of the second (σ_2 = 3) and the third (σ_3 = 2) of securities. At the same time, its yield (25%) is 15% higher than the yield of the first security, 5% higher than the yield of the second and only 5% less than the yield of the third security.

Note an interesting fact: the share (price) of the second security in the portfolio of minimal risk turned out to be more than 2 times higher than the share of the first, despite the fact that the risk of the second security is 3 times higher than the risk of the first. This means that portfolio risk largely depends on the correlation of securities, and not only on their individual risks.

Let's write down the form of the minimum boundary. According to the formula (4.89)

$$\sigma = \sqrt{\frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}}.$$

Substituting the values of constants we found here, α, β, γ, δ, we will get

$$\sigma = \sqrt{\left(\frac{15}{7} \cdot \mu^2 - 2 \cdot \frac{235}{7} \cdot \mu + \frac{4300}{7}\right) \Big/ \left(\frac{9275}{49}\right)} =$$

$$= \sqrt{0.011 \cdot \mu^2 - 0.355 \cdot \mu + 3.245}$$

So, the minimum boundary has the form:

$$\sigma = \sqrt{0.011 \cdot \mu^2 - 0.355\mu + 3.245}$$

or

$$31.6 \cdot \sigma = \sqrt{11 \cdot \mu^2 - 355\mu + 3245}$$

4.3.3. Markovitz portfolio of minimal risk with an efficiency not less than the specified

Along with the task (1) ((4.77)—(4.79)) let's find a portfolio of minimal risk from all portfolios of efficiency at least given (task (1')). We will call such a portfolio the optimal Markowitz portfolio.

To do this, consider the optimization problem: find the minimum of the objective function

$$\frac{1}{2}\sigma^2 = \frac{1}{2} X^T V X \to \min \tag{4.95}$$

under the conditions

$$\mu^T X \geq \mu \tag{4.96}$$

$$I^T X = 1. \tag{4.97}$$

From the structure of the quadratic function (4.91), which sets the equation of the minimum boundary, it can be seen that problems (1) and (1') have the same solution for any $\mu \geq \mu_0 = \frac{\beta}{\alpha}$, exactly

$$\sigma_{\min} = \sqrt{\frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}}$$, and the portfolio

$$X = V^{-1}\left(\frac{\gamma - \beta\mu}{\delta} \cdot I + \frac{\alpha\mu - \beta}{\delta} \cdot \overline{\mu}\right)$$

At $\mu \leq \beta/\alpha$ the problems under consideration have different solutions: for example, the solution of problem (1') for all $\mu \leq \beta/\alpha$ is a single solution of problem (1) for $\mu \leq \beta/\alpha$ (Fig. 4.4). Exactly

$$\sigma_{\min} = \sqrt{\frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}} = \frac{1}{\sqrt{\alpha}} \qquad (4.98)$$

and the portfolio

$$X = V^{-1}\left(\frac{\gamma - \beta\mu_0}{\delta} I + \frac{\alpha\mu_0 - \beta}{\delta} \vec{\mu}\right) =$$
$$= V^{-1}\left(\frac{\gamma - \beta^2/\alpha}{\delta} I + \frac{\beta - \beta}{\delta} \vec{\mu}\right) = \frac{1}{\alpha} V^{-1} I. \qquad (4.99)$$

It is clear that at $\mu \leq \beta/\alpha$ it makes no sense to solve problem (1) — it is necessary to solve problem (1') and the solution of this problem will be better than the solution of problem (1), because the efficiency of the portfolio, which is the solution of problem (1'), is equal to $\mu_0 = \beta/\alpha$, that is, even more than required, and the variance is equal to $\sigma^2 = 1/\alpha$, that is, even less than that of the portfolio, which is the solution to problem (1) given μ.

4.3.4. Minimum Risk Portfolio

Let's solve another optimization problem in which we need to find a portfolio of minimal risk from all possible portfolios, that is, portfolios of any efficiency.

To do this, you need to find the minimum of the objective function

$$\frac{1}{2}\sigma^2 = \frac{1}{2}X^T VX \to \min \tag{4.100}$$

on condition

$$I^T X = 1. \tag{4.101}$$

The Lagrange function in this case has the form:

$$L(X,\lambda) = \frac{1}{2}X^T VX + \lambda(1 - I^T X) \tag{4.102}$$

Equating the derivatives with respect to X, λ to zero, we obtain a system of two equations

$$\begin{cases} VX = \lambda I \\ I^T X = 1 \end{cases} \tag{4.103}$$

Let's express the unknown X from the first equation

$$X = V^{-1}\lambda I \tag{4.104}$$

and substitute the second equation of the system. Receive:

$$\lambda = \frac{1}{I^T V^{-1} I} = \frac{1}{\alpha} \tag{4.105}$$

For X we have

$$X = \frac{V^{-1} I}{I^T V^{-1} I}. \tag{4.106}$$

So, there is a minimal risk portfolio

$$X = \frac{V^{-1} I}{I^T V^{-1} I} = \frac{V^{-1} I}{\alpha} \tag{4.107}$$

The minimum dispersion itself is equal to

$$X^T VT = \left(\frac{I^T V^{-1}}{\alpha}\right) V \left(\frac{V^{-1} I}{\alpha}\right) = \frac{I^T V^{-1} I}{\alpha^2} = \frac{1}{\alpha} \tag{4.108}$$

So, the inverse value of the parameter α is numerically equal to the minimum variance of all portfolios.

It is interesting to note that the minimum dispersion and the minimum risk portfolio itself are determined exclusively by the matrix V (or, more precisely, V^{-1}). However, the effectiveness of such a portfolio also depends on the vector $\vec{\mu}$, and it is equal to

$$\mu = \vec{\mu}^T X = \frac{\vec{\mu}^T V^{-1} I}{\alpha} = \frac{\beta}{\alpha} \quad (4.109)$$

Thus, the efficiency of the minimum risk portfolio is equal to β/α. Note that the assumption $\delta > 0$ is not used here.

The qualitatively obtained result can be found in the graph of the minimum boundary.

Let's write the dispersion (covariance) of portfolio profitability in the form:

$$V = \sum_{i,j=1}^{n} x_i V_{ij} x_j = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} V_{ij} x_j \quad (4.110)$$

Let's call the value $V_i = \sum_{j=1}^{n} V_{ij} x_j$ portfolio covariance of the yield of the i–th security. A column vector with components from portfolio covariances, i.e. the vector VX, is called the vector of portfolio covariances.

A characteristic property of a minimal risk portfolio: a portfolio has minimal risk if and only if all portfolio covariances in it are the same.

Really, there is an optimal portfolio of minimal risk $X = \frac{V^{-1} I}{\alpha}$ and there is a vector of portfolio covariances in it $R = VX = \frac{VV^{-1} I}{\alpha} = \frac{1}{\alpha}$. Conversely: suppose that for some portfolio X all portfolio covariances are the same, i.e. $VX = cI$. Then $X = cV^{-1} I$. Because $I^T X = 1$, that $c = 1/\alpha$. Dispersion of such a portfolio is equal to $X^T V X = c^2 I^T V^{-1} V V^{-1} I = 1/\alpha$, which coincides with the lowest value of portfolio variance.

Note that the characteristic property of the minimal risk portfolio follows from the general property of the conditional extremum of the problems of the type considered in this paragraph: at the extremum point, the gradient of the objective function is proportional to the normal vector of the hyperplane that defines a linear constraint in the form of equality.

4.3.5. Portfolio of maximum efficiency out of all risk portfolios no more than the specified

Along with portfolios of minimal risk, it also makes sense to look for portfolios of maximum efficiency from a certain set of portfolios.

This task is reduced to solving the following optimization problem:

$$\vec{\mu}^T X \to \max \tag{4.111}$$

find the maximum of the objective function under the conditions

$$\frac{1}{2} X^T V X = \frac{1}{2}\sigma^2 \tag{4.112}$$

$$I^T X = 1 \tag{4.113}$$

The direct approach — the compilation of the Lagrange function, etc. — does not lead to the solution of the problem. Therefore, the following approach is proposed. Earlier we obtained that for a portfolio that is a solution to problem (1),

$$\frac{1}{2}\sigma^2 = \frac{1}{2} X^T V X \to \min$$

under the conditions

$$\vec{\mu}^T X = \mu,$$

$$I^T X = 1.$$

dispersion and efficiency are related by the formula (4.71) $\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}$

On the plane (μ, V) we draw the curve (4.71). In Figure 4.5, the set of portfolios is shaded, β/α, $1/\alpha$ — the efficiency and dispersion of the portfolio of minimal risk.

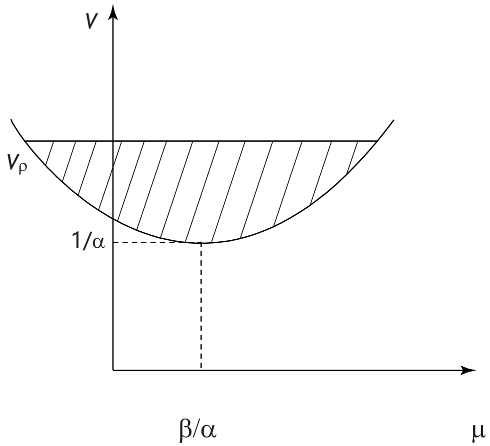

Figure. 4.5. Finding the Markovitz portfolio of maximum efficiency from all risk portfolios of no more than a given

So, if we fix the efficiency of the portfolio μ, then the lowest point of the shaded set lying on the corresponding vertical is the Markowitz portfolio —the solution to problem (1).

If we fix the variance of the portfolio V, then the rightmost point of the shaded set, lying on the corresponding horizontal, will obviously give a solution to the problem (4.111)–(4.113), i.e. a portfolio of maximum efficiency and a given risk.

Thus, the solution of the problem (4.111)—(4.113) can be obtained as follows:

1) calculate constants α, β, γ, δ according to the formulas (4.61);

2) for a given value of the dispersion of portfolio V, solve the quadratic equation (4.71)

$$\alpha\mu^2 - 2\beta\mu + \gamma = V\delta \qquad (4.114)$$

3) find the largest root μ_0 of this equation:

$$\mu_0 = \frac{\beta}{\alpha} + \sqrt{\left(V - \frac{1}{\alpha}\right)\frac{\delta}{\alpha}} \tag{4.115}$$

Its abscissa is spaced from β/α to the right by an amount $\sqrt{\left(V - \frac{1}{\alpha}\right)\frac{\delta}{\alpha}}$

At the same time $\left(V - \frac{1}{\alpha}\right)$ there is an excess of the specified portfolio dispersion over its minimum value. It is clear that the solution of the problem (4.111)—(4.113) exists only when $V \geq 1/\alpha$;

4) find the vector X by formulas (4.62).

Now it can be seen that the considered problem (4.111)—(4.113) is formally equivalent to the more general problem of finding the maximum of the objective function under the conditions

$$\vec{\mu}^T X \to \max \tag{4.116}$$

$$\frac{1}{2} X^T V X = \frac{1}{2}\sigma^2 \leq \frac{1}{2} V \tag{4.117}$$

$$I^T X = 1 \tag{4.118}$$

4.4. Tobin's Portfolio

The situation changes dramatically if there is a risk-free security on the market. It is assumed that the yield of a risk-free security is a random variable that is not correlated with the yield of other risky securities, therefore, if there is a risk-free security in the matrix, there are zero rows and columns of covariances, which is why the reasoning used when considering Markowitz portfolios turns out to be incorrect. The effectiveness of a risk-free security is denoted by μ_f and we will consider it positive.

4.4.1 Tobin's portfolio of minimal risk from all portfolios of a given efficiency

So, let's assume that together with n–risky assets, the investor's portfolio includes a risk-free security with a deterministic return $\mu_f = R_f$ and a share in the portfolio of x_f. In this case, the task (4.57)—(4.59) will look like this:

$$\frac{1}{2}\sigma^2 = \frac{1}{2}X^T V X \to \min \tag{4.119}$$

under the conditions

$$\mu_f x_f + \vec{\mu}^T X = \mu \tag{4.120}$$

$$x_f + I^T X = 1 \tag{4.121}$$

The expression for the risk square has not changed due to the risk-free nature of the added asset. In this case, first considered by Tobin, the form of the minimum boundary will change.

First of all, we reformulate the problem (4.119)—(4.121). To do this, we exclude the variable xf from the relations by multiplying (4.121) by x_f and deducting from (4.121):

$$\left(\vec{\mu} - \mu_f I\right)^T X = \mu - \mu_f \tag{4.122}$$

To solve the problem (4.119), (4.122), we compose a Lagrange function and write down the necessary extremum conditions for it:

$$L = \frac{1}{2}X^T V X - \lambda\left((\vec{\mu} - \mu_f I)^T X - \mu + \mu_f\right)$$

$$\begin{cases} VX = \left(\vec{\mu} - \mu_f I\right)\lambda \\ (\vec{\mu} - \mu_f I)^T X = \mu - \mu_f \end{cases} \tag{4.123}$$

We express X from the first equation of the system (4.103) and substitute it into the second

$$\left(\vec{\mu} - \mu_f I\right)^T V^{-1}(\vec{\mu} - \mu_f I)\lambda = \mu - \mu_f \tag{4.124}$$

Denote

$$d = \sqrt{\left(\vec{\mu} - \mu_f I\right)^T V^{-1}\left(\vec{\mu} - \mu_f I\right)} = \sqrt{\alpha\mu_f^2 - 2\beta\mu_f + \gamma} \qquad (4.125)$$

This definition is correct because the vectors $\vec{\mu}$ and I are not collinear, and the matrix V^{-1} is positively defined. Therefore, from (4.124)

$$\lambda = \frac{\mu - \mu_f}{d^2}$$

$$X = \frac{\mu-\mu_f}{d^2} V^{-1}(\vec{\mu} - \mu_f I) \qquad (4.126)$$

— the desired vector of risk shares, the risk-free share is found as follows from the ratio (4.121):

$$x_f = 1 - I^T X = 1 - \frac{\mu-\mu_f}{d^2} I^T V^{-1}\left(\vec{\mu} - \mu_f I\right) \qquad (4.127)$$

Now it is not difficult to find the equation of the minimum boundary. To do this, it is enough to substitute the found X in the expression for the risk square

$$\begin{aligned}\sigma^2 &= \left(\frac{\mu - \mu_f}{d^2}\right)^2 \left(V^{-1}\left(\vec{\mu} - \mu_f I\right)\right)^T V V^{-1}\left(\vec{\mu} - \mu_f I\right) \\ &= \left(\frac{\mu - \mu_f}{d^2}\right)^2 \left(\vec{\mu} - \mu_f I\right)^T V^{-1}\left(\vec{\mu} - \mu_f I\right) = \left(\frac{\mu - \mu_f}{d}\right)^2\end{aligned}$$

Thus, the equation of the minimum boundary

$$\sigma^2 = \left(\frac{\mu-\mu_f}{d}\right)^2 \qquad (4.128)$$

It is usually assumed that the expected return on the portfolio should not be less than the return on a risk-free asset, i.e. $\mu \geq \mu_f$. In the opposite case, it would be necessary to form a portfolio of only one of them. Therefore, equation (4.128) turns into a linear:

$$\sigma = \frac{\mu-\mu_f}{d} \qquad (4.129)$$

We prove that the line (4.129) is tangent to the graph of the minimum boundary (4.69). To prove, we find the intersection points of the hyperbola (4.69) and the line (4.129), solving their equations together, and make sure that there is one such point. Equating the right parts of (4.128) and (4.69), we get $\left(\frac{\mu-\mu_f}{d}\right)^2=\frac{\alpha\mu^2-2\beta\mu+\gamma}{\delta}$

Next, we get a quadratic equation with respect to μ and find its root:

$$\mu^2\left(\frac{\delta}{d^2}-\alpha\right)+2\mu\left(\beta-\frac{\delta}{d^2}\mu_f\right)+\left(\frac{\delta}{d^2}\mu_f^2-\gamma\right)=0 \qquad (4.131)$$

The discriminant of this equation is zero:

$$4\left(\beta-\frac{\delta}{d^2}\mu_f\right)^2-4\left(\frac{\delta}{d^2}-\alpha\right)\left(\frac{\delta}{d^2}\mu_f^2-\gamma\right)=0 \qquad (4.132)$$

This proves that the line (4.129) is tangent to the graph of the minimum boundary (4.69).

Now let's find the coordinates of the tangent point (coordinates of the tangent portfolio):

$$\mu=\frac{-2\left(\beta-\frac{\delta}{d^2}\mu_f\right)}{2\left(\frac{\delta}{d^2}-\alpha\right)}=-\frac{\beta d^2-\delta\mu_f}{\delta-ad^2}=$$

$$=-\frac{a(\gamma-\beta\mu_f)\left(\mu_f-\frac{\beta}{a}\right)}{a^2\left(\mu_f-\frac{\beta}{a}\right)^2}=-\frac{(\gamma-\beta\mu_f)}{a\left(\mu_f-\frac{\beta}{a}\right)}$$

So, the efficiency of the tangent portfolio μ_T is equal to:

$$\mu_T=\frac{\gamma-\beta\mu_f}{\beta-a\mu_f} \qquad (4.133)$$

Substituting the found efficiency value μ_T into the tangent equation, we find the risk of the tangent portfolio σ_T:

$$\sigma = \frac{\mu - \mu_f}{d} = \frac{\dfrac{\gamma - \beta\mu_f}{\beta - a\mu_f} - \mu_f}{d} =$$
$$= \frac{\gamma - 2\beta\mu + a\mu_f^2}{d(\beta - a\mu_f)} = \frac{d^2}{d(\beta - a\mu_f)} = \frac{d}{d(\beta - a\mu_f)}$$

So, for the coordinates of the tangent portfolio we have

$$\mu_T = \frac{\gamma - \beta\mu_f}{\beta - a\mu_f}, \quad \sigma_T = \frac{d}{\beta - a\mu_f} \qquad (4.134)$$

In this case, the tangent portfolio T itself is from (4.126) by substitution $\mu = \mu_T$:

$$T = \frac{\mu_T - \mu_f}{d^2} V^{-1}(\vec{\mu} - \mu_f I) \qquad (4.135)$$

It is also possible to show that the line (4.129) is tangent to the graph of the minimum boundary (4.69) geometrically (Fig. 4.6). Every minimal portfolio is a linear combination of a risk-free asset and a risky part lying on the minimum boundary. Therefore, any such point lies on the ray *FA*, where the point *F* corresponds to a risk-free asset. From point *A*, you can move along the horizontal axis to point B, lying on the tangent *FT*, which has the same risk and higher profitability. Therefore, the tangent *FT* is the desired minimum boundary.

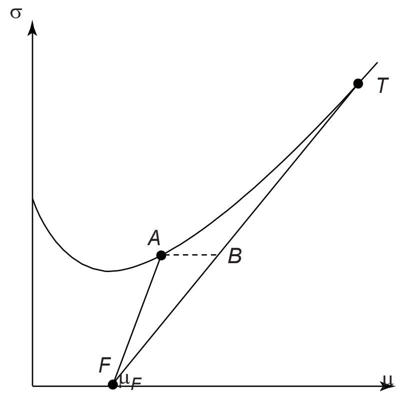

Figure. 4.6. The minimum boundary of the Tobin portfolio, tangent portfolio

The points of the minimum boundary are represented as a linear combination

$M = \lambda F + (1 - \lambda)T$

moreover, when the point moves from *F* to *T*, the parameter λ changes from 1 to 0.

Example 4.6. The portfolio consists of three securities: a risk-free one with an efficiency (expected return) of 5% and two risky ones with an efficiency of 10 and 15%, respectively, and a covariance matrix $\begin{pmatrix} 9 & 5 \\ 5 & 36 \end{pmatrix}$.

Find Tobin's portfolios of expected returns of 10, 11, 12% and minimum risk and their risks.

From (4.125) and (4.126) we have the following expressions for the parameter *d* and the desired portfolio *X*:

$$d = \sqrt{\left(\vec{\mu} - \mu_f I\right)^T V^{-1}\left(\vec{\mu} - \mu_f I\right)}, \quad X = \frac{\mu - \mu_f}{d^2} V^{-1}\left(\vec{\mu} - \mu_f I\right)$$

1) μ = 10%

Here $\vec{\mu} = (10; 15)^T$, $\mu = 10, \mu_f = 5$ Calculate the parameter *d*:

$$d^2 = \left(\vec{\mu} - \mu_f I\right)^T V^{-1}\left(\vec{\mu} - \mu_f I\right) = \frac{1}{299}(5; 10)\begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} =$$
$$= \frac{1}{299}(130; 65)\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \frac{1{,}300}{299} = 4.35$$

Now you can find a portfolio *X*:

$$X = \frac{\mu - \mu_f}{d^2} V^{-1}\left(\vec{\mu} - \mu_f I\right) = \frac{10 - 5}{4.35} * \frac{1}{299}\begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} =$$

$$= 0.00384 * \begin{pmatrix} 130 \\ 65 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix}$$

$x_f = 1 - x_1 - x_2 = 0.25$

Thus, the portfolio of expected return of 10% and minimal risk has the form:

$$X = (0.5; 0.25;\ x_f = 0.25)$$

Portfolio risk is equal to:

$$\sigma = \sqrt{X^T V X} = \sqrt{(0.5; 0.25)\begin{pmatrix} 9 & 5 \\ 5 & 36 \end{pmatrix}\begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix}} =$$

$$\sqrt{(5.75; 11.5)\begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix}} = 2.4.$$

The risk of the portfolio is less than the risk of each of the risky securities, which are equal to $\sqrt{9} = 3$ and $\sqrt{36} = 6$ respectively for the first and second securities;

2) μ = 11%

Here $\vec{\mu} = (10; 15)^T,\ \mu = 11, \mu_f = 5$

Parameter *d* is still equal to 4.35.

Find portfolio *X*:

$$X = \frac{\mu - \mu_f}{d^2} V^{-1}(\vec{\mu} - \mu_f I) = \frac{11 - 5}{4.35} * \frac{1}{299}\begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} =$$

$$= 0.00461 * \begin{pmatrix} 130 \\ 65 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.3 \end{pmatrix}$$

$x_f = 1 - x_1 - x_2 = 0.1$

Thus, the portfolio of expected return of 11% and minimal risk has the form:

$X = (0.6; 0.3;\ x_f = 0.1)$

Portfolio risk is equal to:

$$\sigma = \sqrt{X^T V X} = \sqrt{(0.6; 0.3)\begin{pmatrix} 9 & 5 \\ 5 & 36 \end{pmatrix}\begin{pmatrix} 0.6 \\ 0.3 \end{pmatrix}} = \sqrt{(6.9; 13.8)\begin{pmatrix} 0.6 \\ 0.3 \end{pmatrix}} = 2.88$$

Despite the increase in the required yield by 1%, the portfolio risk remains less than the risk of each of the risky securities;

3) Here $\vec{\mu} = (10; 15)^T$, $\mu = 12, \mu_f = 5$ Calculate the parameter *d*:

$$d^2 = (\vec{\mu} - \mu_f I)^T V^{-1}(\vec{\mu} - \mu_f I) = \frac{1}{299}(5; 10)\begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} =$$

$$= \frac{1}{299}(130; 65)\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \frac{1{,}300}{299} = 4.35$$

Now you can find a portfolio *X*:

$$X = \frac{\mu - \mu_f}{d^2} V^{-1}(\vec{\mu} - \mu_f I) = \frac{12 - 5}{4.35} * \frac{1}{299}\begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} =$$

$$= 0.00538 * \begin{pmatrix} 130 \\ 65 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.35 \end{pmatrix}$$

The sum of the shares of already risky assets exceeds one, so it is not possible to form a Tobin portfolio with an expected return of 12% (or higher) and minimal risk.

Example 4.7. For the condition of the previous example, it is necessary to find a tangent portfolio, its expected profitability and risk. So, the portfolio consists of three securities: a risk-free one with an efficiency (expected return) of 5% and two risky ones with an efficiency of 10 and 15%, respectively, and a covariance matrix $\begin{pmatrix} 9 & 5 \\ 5 & 36 \end{pmatrix}$

The desired tangent portfolio *T* has the form (4.135):

$$T = \frac{\mu_T - \mu_f}{d^2} V^{-1}(\vec{\mu} - \mu_f I)$$

and for its coordinates from (4.134) we have

$$\mu_T = \frac{\gamma - \beta\mu_f}{\beta - a\mu_f}, \quad \sigma_T = \frac{d}{\beta - a\mu_f}$$

From (4.125) we have the following expression for the parameter *d*:

$$d = \sqrt{\left(\vec{\mu} - \mu_f I\right)^T V^{-1} \left(\vec{\mu} - \mu_f I\right)},$$

$$d^2 = \left(\vec{\mu} - \mu_f I\right)^T V^{-1} \left(\vec{\mu} - \mu_f I\right) = \frac{1}{299}(5; 10)\begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} =$$
$$= \frac{1}{299}(130; 65)\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \frac{1300}{299} = 4.35$$

Find the constants α, β, γ:

$\alpha = I^T V^{-1} I,\ \beta = I^T V^{-1} \vec{\mu} = \vec{\mu}^T V^{-1} I,\ \gamma = \vec{\mu}^T V^{-1} \vec{\mu}$

Here $I = (1, 1)^T$, $\mu = (10, 15)^T$

$$\alpha = I^T V^{-1} I = (1, 1) * \frac{1}{299}\begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{299}(31.4) * \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{35}{299}$$
$$= 0.117$$

$$\beta = I^T V^{-1} \vec{\mu} = (1, 1) * \frac{1}{299}\begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 10 \\ 15 \end{pmatrix} = \frac{1}{299}(31.4) * \begin{pmatrix} 10 \\ 15 \end{pmatrix}$$
$$= \frac{370}{299} = 1.24$$

$$\gamma = \vec{\mu}^T V^{-1} \vec{\mu} = (10, 15) * \frac{1}{299}\begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix}\begin{pmatrix} 10 \\ 15 \end{pmatrix} =$$

$$= \frac{1}{299}(285, 85) * \begin{pmatrix} 10 \\ 15 \end{pmatrix} = \frac{4125}{299} = 13.80$$

Now let's find the coordinates of the tangent portfolio *T*:

$$\mu_T = \frac{\gamma - \beta\mu_f}{\beta - a\mu_f} = \frac{13.8 - 1.24 * 5}{1.24 - 0.117 * 5} = \frac{7.6}{0.655} = 11.6$$

$$\sigma_T = \frac{d}{\beta - a\mu_f} = \frac{\sqrt{4.35}}{1.24 - 0.117 * 5} = \frac{2{,}086}{0{,}655} = 3.18$$

And finally, we find the tangent portfolio *T*:

$$T = \frac{\mu_T - \mu_f}{d^2} V^{-1}(\vec{\mu} - \mu_f I) =$$
$$= \frac{11{,}6-5}{4{,}35} * \frac{1}{299} \begin{pmatrix} 36 & -5 \\ -5 & 9 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} =$$
$$= 0{,}00507 \begin{pmatrix} 130 \\ 65 \end{pmatrix} = \begin{pmatrix} 0{,}6597 \\ 0{,}3298 \end{pmatrix}$$

So, the tangent portfolio is equal to $T = (0.66; 0.33)$, i.e. it includes 66% of the first security and 33% of the second security and practically does not include risk-free security.

The risk of the tangent portfolio ($\sigma_T = 3.18$) is slightly higher than the risk of the first security ($\sigma_1 = 3$) and almost half the risk of the second security ($\sigma_2 = 6$).

The yield of the tangent portfolio $\mu_T = 11.6\%$ is the maximum yield at which it is possible to form a portfolio of minimal risk.

Taking into account the results of the previous example in addition, it can be concluded that with an increase in profitability from 10 to 12%, the portfolio risk increases from 2.4% (at $\mu = 10\%$) to 2.88% (at $\mu = 11\%$) and further to 3.18% (at $\mu = 11.6\%$ for the tangent portfolio). And at $\mu > 11.6\%$, it is no longer possible to form a portfolio of minimal risk.

4.4.2. Portfolio of maximum efficiency from all risk portfolios no more than the specified

Along with the Tobin problem (4.119)—(4.121), we consider the optimization problem:

$$\mu_f x_f + \vec{\mu}^T X \to \max \quad (4.136)$$

under the conditions

$$\frac{1}{2}\sigma^2 = \frac{1}{2} X^T V X < \frac{1}{2} V \quad (4.137)$$

$$x_f + I^T X = 1 \quad (4.138)$$

To solve the problem, consider the plane (μ, σ) (in the variables efficiency-risk) (Fig. 4.7). On this plane we will draw a polyline

$\sigma = |\mu - \mu_f|/d$

where

$$d = \sqrt{(\vec{\mu} - \mu_f I)^T V^{-1} (\vec{\mu} - \mu_f I)} = \sqrt{a\mu_f^2 - 2\beta\mu_f + \gamma}$$

In Figure 4.7, the set of portfolios for the situation under consideration is shaded.

So, if we fix the efficiency of the portfolio, then the lowest point of the shaded set lying on the corresponding vertical is Tobin's portfolio – the solution of the problem (4.119)–(4.121). If we fix the risk of the portfolio σ, then the rightmost point of the shaded set, lying no higher than the corresponding horizontal, i.e. exactly on it, will obviously give a solution to the problem (4.136)–(4.138), i.e. a portfolio of maximum efficiency and limited risk. From (4.129) we find

$\mu = \mu_f + d * \sigma$ (4.139)

After that, for the found μ by the formula (4.126), we determine the original portfolio *X*.

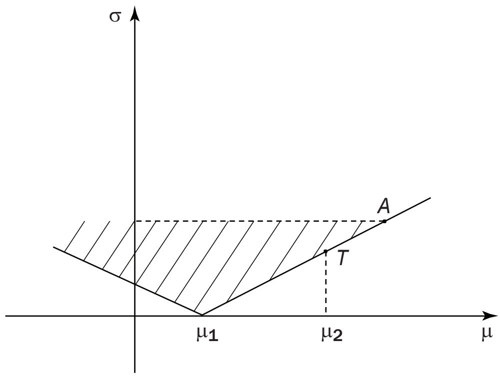

Figure. 4.7. To find a portfolio of maximum efficiency from all risk portfolios of no more than a given

4.5. Optimal non-negative portfolio

4.5.1. Kuhn-Tucker Theorem

In this section we will consider only non-negative portfolios $X \geq 0$. The condition (4.60) should be added to the conditions of the optimal problem (4.57) – (4.59). Non-negativity of portfolio components means that they can now be interpreted as price (cost) shares of investments in a particular portfolio security. At the same time, however, both the algorithm for solving the problem and the solution itself change. Conditions (4.58)–(4.60) define a convex polyhedron, i.e. a bounded closed set. The solution of optimal problems under such conditions (the presence of restrictions not only in the form of equalities, but also in the form of inequalities) has its own specifics. These tasks are the tasks of finding extremes of convex (concave) functions defined on convex sets. The solution of optimal problems of this type requires the modification of a method based on the search for extremes of the La Grange function, which makes up an algorithm for finding conditional extremes (if there are restrictions only in the form of equalities). Such a modification can be made using the Kuhn-Tucker theorem and related theorems, the meaning of which boils down to the fact that the point the local extremum (and even stationary) of a convex (concave) function given on a convex set is a point on it global extremum (fig. 4.8)

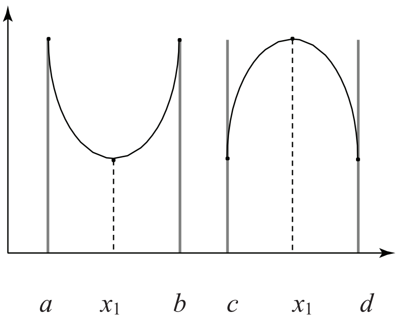

Figure. 4.8. An illustration of the fact that the point of the local extremum (and even stationary) of a convex (concave) function given on a convex set is the point of the global extremum on it

We formulate several such theorems.

1. Let X_0 be the point of the local (conditional) extremum of the function $f(X)$ on the convex set M. Then:

a) if X_0 — is the point of the local minimum, and $f(X)$ — is convex on the set M, then X_0 — is a point of global minimum on the set M;

b) if X_0 — is the point of the local maximum, and $f(X)$ — is concave on the set M, then X_0 — is the point of the global maximum on the set M.

2. If a function $f(X)$ is strictly convex (strictly concave) on a concave set M, then it can have at most one extremum on this set.

3. The Kuhn-Tucker Theorem.

Let a convex set M be given by a system of constraints

$$\begin{cases} g_i(x_1, x_2, \ldots, x_n) \geq 0, \ \ i = 1, 2, \ldots, l, \\ g_i(x_1, x_2, \ldots, x_n) = 0, j = l+1, l+2, \ldots, m; \end{cases} \tag{4.140}$$

$f, g_1, \ldots, g_m$ — differentiable and concave functions on the set M.

Then the criterion (necessary and sufficient condition) of the global maximum on the set M at the point $X_0 \in M$ is the existence of an m–dimensional vector of Lagrange multipliers λ, satisfying the conditions:

$$\begin{aligned} &L'_{xi}(X, \vec{\lambda}) = 0, && i = 1, \ldots, n; \\ &\lambda_j g_j(X) = 0, && j = 1, \ldots l; \\ &\lambda_k \geq 0, && k = 1, \ldots, l; \end{aligned} \tag{4.141}$$

4.5.2. Profitability of a non–negative portfolio

We prove that in a non-negative portfolio, the yield μ lies on the segment $[\mu_{min}, \mu_{max}]$, where μ_{min} and μ_{max} — minimum and maximum values of yields of individual securities included in the portfolio

$$\mu_{\min} = \min(\mu_1, \mu_2, \ldots, \mu_n) \tag{4.142}$$

$$\mu_{\max} = \max(\mu_1, \mu_2, \ldots, \mu_n) \tag{4.143}$$

that is, that the yield μ satisfies the inequalities

$$\mu_{\min} \leq \mu \leq \mu_{\max} \tag{4.144}$$

In the formula

$$\mu = x_1\mu_1 + x_2\mu_2 + \cdots + x_n\mu_n$$

all μ_i, i = 1, 2, ..., n replace first with $\mu_{\min}$, then on $\mu_{\max}$ and, using the non-negativity condition of all x_i, receive

$$\mu_{\min}(x_1 + x_2 + \cdots + x_n) = \mu_{\min} \leq \mu \leq \mu_{\max}(x_1 + x_2 + \cdots + x_n) = \mu_{\max}$$

From here

$$\mu_{\min} \leq \mu \leq \mu_{\max}$$

So, we proved that the yield of a non-negative portfolio cannot be less than the minimum and greater than the maximum yield of individual securities included in the portfolio.

Taking into account the obtained restrictions on the profitability of a non-negative portfolio, the problem of finding a portfolio of minimal risk with a given profitability (efficiency) has the same form as for an arbitrary portfolio. In this case, however, it is necessary to impose the requirement of non-negativity on the variables. So, the Markowitz problem (4.57) – (4.60) looks like this.

Find the minimum of the objective function

$$\frac{1}{2}\sigma^2 = \frac{1}{2}X^TVX \to \min \tag{4.145}$$

under the conditions

$$\vec{\mu}^TX = \mu, \qquad I^TX = 1, \qquad X \geq 1 \ . \tag{4.146}$$

This is the classical formulation of the Markowitz problem. Since the allowable set is compact, the desired portfolio exists. Suppose the matrix V is positively defined, then, taking into account the strict convexity of the objective function, the linearity of the boundary and the differentiability of the functions under consideration, we conclude that the Kuhn-Tucker conditions are necessary and sufficient conditions for a conditional minimum:

$$X^T V - \lambda I^T - v\vec{\mu}^T \geq 0, \qquad (X^T V - \lambda I^T - \vec{\mu}^T)X = 0,$$
$$I^T X = 1, \qquad \vec{\mu}^T X = \mu, \quad X \geq 0 \tag{4.147}$$

Solving this system of equations and inequalities is, in general, extremely difficult. With a small number of n securities, it is possible to solve the system (4.147) by iterating over the cases.

We prove that the problem (4.145)–(4.146) has a solution for any

$\mu \epsilon [\mu_{\min}, \mu_{\max}]$

We also prove that the minimal boundary is a convex curve consisting of a finite number of hyperbolic pieces.

Since the objective function is strictly convex, and the admissible solutions form a convex polyhedron, the solution to this problem exists if the admissible set is nonempty. Denote by $X_{\min}$ (accordingly $X_{\max}$) portfolios corresponding to $\mu_{\min}$ ($\mu_{\max}$). It is clear that for any $\mu \epsilon [\mu_{\min}, \mu_{\max}]$ there is a single convex combination $X_{\min}$ and $X_{\max}$, for which the value of the expected return is taken μ.

Thus, taking into account the received restrictions on profitability μ the problem of finding a portfolio of minimal risk with a given profitability (efficiency) has a single solution.

We now prove the convexity of the minimal boundary. Let's choose two parameter values $\mu: \mu_1, \mu_2 \, \epsilon [\mu_{\min}, \mu_{\max}]$, for which there are non-negative portfolios X_1, X_2. Portfolio $X = tX_1 + (1-t)X_2, \; t\epsilon[0,1]$ will be valid for the value $\mu = t\mu_1 + (1-t)\mu_2$. Now the convexity of the minimum boundary follows from the convexity of the objective function

$$\sigma^2(\mu) \leq \sigma^2(X) \leq t\, \sigma^2(X_1) + (1-t)\, \sigma^2(X_2) = t\sigma^2(\mu_1) + (1-t)\, \sigma^2(\mu_2) \tag{4.148}$$

Note that when finding the optimal portfolio, as well as when solving optimal linear programming problems, an important role is played by the so-called corner points – the minimum boundary in which its analytical task changes, or points belonging to segments, planes for which there is no neighborhood entirely belonging to this segment, plane.

There is an algorithm for finding corner points, similar to the algorithm of the simplex method. If all the corner points are known, then the initial problem of finding the minimum boundary equation reduces to several problems of finding the minimum boundary equation for two points.

4.5.3. Non–negative portfolio of two securities

As above, in the case of an arbitrary portfolio, consideration of a non-negative portfolio of n–securities will begin with the simplest case of a portfolio of two securities.

Let's first consider a non-negative portfolio of two independent securities:

$$\rho_{12}=\rho=0 \tag{4.149}$$

For the risk square (variance) of the portfolio we have:

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 \tag{4.150}$$

We will find a non-negative portfolio of minimal risk and its profitability and risk. That is, it is necessary to minimize the objective function

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 \tag{4.151}$$

Under the conditions

$$x_1 + x_2 = 1$$
$$x_1 \geq 0;\ x_2 \geq 0 \tag{4.152}$$

This is a conditional extremum problem, which is solved using Kuhn-Tucker conditions.

Let's make up the Lagrange function and find its extremum

$$L = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \lambda(x_1 + x_2 - 1) + \lambda_1 x_1 + \lambda_2 x_2 \quad (4.153)$$

The presence of the last two terms is associated with the non-negativity conditions of the portfolio components.

To find stationary points, we have a system

$$\begin{cases} \frac{\partial L}{\partial x_1} = 2\sigma_1^2 x_1 + \lambda + \lambda_1 = 0 \\ \frac{\partial L}{\partial x_2} = 2\sigma_2^2 x_2 + \lambda + \lambda_2 = 0 \\ \frac{\partial L}{\partial \lambda} = x_1 + x_2 - 1 = 0 \\ \lambda_1 x_1 = 0; \ \lambda_2 x_2 = 0 \end{cases} \quad (4.154)$$

Consider the various possibilities following from the last two conditions (non-negativity conditions):

1) $\lambda_1 = \lambda_2 = 0$. In this case, we get the same portfolio as in the absence of non-negativity conditions

$$X = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right)$$

2) $\lambda_1 = x_2 = 0$. The portfolio has the form $X = (1, 0)$.

3) $\lambda_2 = x_1 = 0$. The portfolio has the form $X = (0, 1)$.

If we depict the received portfolios on a plane (x_1, x_2), it is easy to see (Fig. 4.9) that point 1 $\left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}, \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} \right)$ lies on the segment connecting points 2 and 3.

Thus, the set of effective portfolios is a segment connecting the points (0,1) and (1,0).

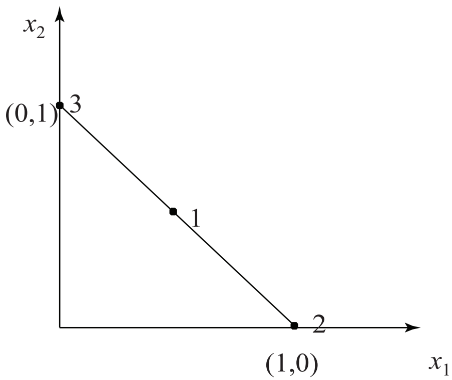

Figure. 4.9. A non-negative portfolio of two securities on a plane (x_1, x_2)

4.5.4. Examples of non-negative portfolios of three independent securities

Let the risks of securities of the three types be equal

$\sigma_1 = 1$, $\sigma_2 = 2$, $\sigma_3 = 3$, and their expected returns

$\mu_1 = 10\%$, $\mu_2 = 20\%$, $\mu_3 = 30\%$.

To find the effective boundary, it is necessary to find the point of the global minimum of the convex function $\sigma^2(x_1, x_2, x_3)$ using the Kuhn-Tucker conditions.

Since the covariance matrix of securities returns is non-degenerate, the risk square of the portfolio

$$\sigma^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 \qquad (4.155)$$

is a strictly convex function. Effective portfolio with expected return μ we will look for as the minimum point of the function σ^2 on a set of constraints.

Let's make up the Lagrange function:

$$L = x_1^2 + 4x_2^2 + 9x_3^2 + \lambda(1 - x_1 - x_2 - x_3) +$$
$$+\kappa(\mu - 0.1x_1 - 0.2x_2 - 0.3x_3) + v_1x_1 + v_2x_2 + v_3x_3 \qquad (4.156)$$

where $\lambda, \kappa, v_1, v_2, v_3$ –parameters. Back to the original constraints

$$\begin{aligned} &x_1 + x_2 + x_3 = 1 \\ &0.1x_1 + 0.2x_2 + 0.3x_3 = \mu \\ &x_i \geq 0, \quad i = 1, 2, 3 \end{aligned} \qquad (4.157)$$

Kuhn-Tucker conditions are added:

$$\begin{aligned} &\frac{\partial L}{\partial x_1} = 2x_1 - \lambda - 0.1\kappa + v_1 = 0 \\ &\frac{\partial L}{\partial x_2} = 8x_2 - \lambda - 0.2\kappa + v_2 = 0 \\ &\frac{\partial L}{\partial x_3} = 18x_3 - \lambda - 0.3\kappa + v_3 = 0 \\ &v_ix_i = 0, \quad i = 1, 2, 3 \\ &v_i \geq 0, \quad i = 1, 2, 3 \end{aligned} \qquad (4.158)$$

This system of conditions seems to be quite complex. However, consideration of special cases makes the task easier.

1) $\nu_1 = \nu_2 = \nu_3 = 0$.

From the Kuhn-Tucker conditions we find

$$X = \left(\frac{1}{2}(\lambda + 0.1\kappa); \frac{1}{8}(\lambda + 0.2\kappa); \frac{1}{18}(\lambda + 0.3\kappa)\right) \qquad (4.159)$$

Then, from the initial constraints, we obtain the system

$$\begin{cases} \frac{49}{72}\lambda + \frac{11}{120}\kappa = 1 \\ \frac{11}{120}\lambda + \frac{3}{200}\kappa = \mu \end{cases} \qquad (4.160)$$

we find

$$\lambda = \frac{108 - 660\mu}{13}; \qquad \kappa = \frac{620 + 12100\mu}{39} \qquad (4.161)$$

From here

$$X^*(\mu) = \left(\frac{9}{2} - \frac{55\mu}{2} + \frac{31}{39} + \frac{605\mu}{39};\ \frac{9}{8} - \frac{55\mu}{8} + \frac{62}{39} + \frac{1210\mu}{39}; 1 - \frac{55\mu}{72} + \frac{31}{117} + \frac{605\mu}{117}\right) == \left(\frac{413}{78} - \frac{935\mu}{78}; \frac{847}{312} - \frac{7535\mu}{312}; \frac{148}{117} - \frac{37125\mu}{8424}\right) \quad (4.162)$$

Since $X \geq 0$, then $\mu \leq 0.44$. In fact, as follows from what we have proved, the portfolio yield properties are $\mu \leq \mu max = 0.3$;

2) $\nu_1 = \nu_2 = x_3 = 0$.

From the Kuhn-Tucker conditions we find:

$$X = \left(\frac{1}{2}(\lambda + 0.1\kappa); \frac{1}{8}(\lambda + 0.2\kappa); 0\right), \quad v_3 = \lambda + 0.3\kappa \quad (4.163)$$

Then, from the initial constraints, we obtain the system

$$\begin{cases} \frac{1}{2}(\lambda + 0.1\kappa) + \frac{1}{8}(\lambda + 0.2\kappa) = 1 \\ \frac{1}{20}(\lambda + 0.1\kappa) + \frac{1}{40}(\lambda + 0.2\kappa) = \mu \end{cases} \quad (4.164)$$

we find

$$\lambda = -120\mu + 16; \qquad \kappa = 1000\mu - 120 \quad (4.165)$$

From here

$$X^*(\mu) = (-10\mu + 2;\ 80\mu - 8;\ 0), \ \ v_3 = 180\mu - 20 \quad (4.166)$$

Since $X \geq 0$, $v_3 \geq 0$, then $\mu \epsilon [10\%; 20\%]$

Considering further all other special cases:

3) $\nu_1 = x_2 = \nu_3 = 0$;

4) $x_1 = \nu_2 = \nu_3 = 0$;

5) $x_1 = x_2 = \nu_3 = 0$, portfolio $X = (0, 0, 1)$;

6) $x_1 = \nu_2 = x_3 = 0$, portfolio $X = (0, 1, 0)$;

7) $\nu_1 = x_2 = x_3 = 0$, portfolio $X = (1, 0, 0)$,

we'll find the other *corner portfolios*.

Further, since the covariance matrix of securities yields is non-degenerate, it follows from the Kuhn-Tucker conditions that the effective set of portfolios is a finite-link polyline in R^n, the vertices of which are corner portfolios. So every effective portfolio is a linear combination of adjacent corner portfolios.

Here is another simpler example. The covariance matrix of yields of securities of three types is given by the formula

$$V = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \quad (4.167)$$

and their expected returns — equalities:

$$\mu_1 = 3\%,\ \mu_2 = 6\%,\ \mu_3 = 9\%. \quad (4.168)$$

Find an effective boundary.

We find the point of the global minimum of the convex function $\sigma^2(x)$ using the Kuhn-Tucker conditions. Since the variances of the yields of securities of all three types are the same, the least risky portfolio— vector

$$\left(\frac{1}{3}; \frac{1}{3}; \frac{1}{3}\right) \quad (4.169)$$

The expected profitability of such a portfolio will be

$$\mu_0 = \frac{3+6+9}{3} = 6\% \quad (4.170)$$

Since the covariance matrix of securities returns is non-degenerate, the function (the square of portfolio risk)

$$\sigma_X^2 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \sigma_3^2 x_3^2 \quad (4.171)$$

strictly convex. We will look for an effective portfolio with an expected return μ as the minimum point of the function σ_X^2 on a set of constraints. The corresponding Lagrange function has the form:

$$L = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda(1 - x_1 - x_2 - x_3) +$$
$$+ \kappa(\mu - 3x_1 - 6x_2 - 9x_3) + v_1 x_1 + v_2 x_2 + v_3 x_3 \quad (4.172)$$

Effective portfolios are found by examining the following conditions.

Initial constraints:

$$\begin{gathered} x_1 + x_2 + x_3 = 1 \\ 3x_1 + 6x_2 + 9x_3 = \mu \\ x_i \geq 0, \;\; i = 1, 2, 3 \end{gathered} \quad (4.173)$$

Kuhn-Tucker Conditions:

$$\begin{aligned} &\frac{\partial L}{\partial x_1} = x_1 - \lambda - 3\kappa + v_1 = 0 \\ &\frac{\partial L}{\partial x_2} = x_2 - \lambda - 6\kappa + v_2 = 0 \\ &\frac{\partial L}{\partial x_3} = x_3 - \lambda - 9\kappa + v_3 = 0 \\ &v_i x_i = 0, \;\; i = 1, 2, 3 \\ &v_i \geq 0, \;\; i = 1, 2, 3 \end{aligned} \quad (4.174)$$

Consider special cases:

1) $\nu_1 = \nu_2 = \nu_3 = 0$.

From the Kuhn-Tucker conditions we find

$$X = (\lambda + 3\kappa;\ \lambda + 6\kappa;\ \lambda + 9\kappa) \quad (4.175)$$

Then, from the initial constraints, we obtain the system

$$\begin{cases} 3\lambda + 18\kappa = 1 \\ 18\lambda + 126\kappa = \mu \end{cases} \quad (4.176)$$

We find

$$\lambda = \frac{7-\mu}{3}; \qquad \kappa = \frac{\mu}{18} - \frac{1}{3} \tag{4.177}$$

From here

$$X^*(\mu) = \left(\frac{4}{3} - \frac{\mu}{6};\ \frac{1}{3}; \frac{\mu}{6} - \frac{2}{3}\right) \tag{4.178}$$

Since $X \geq 0$, then $\mu \in [4; 8]$. In this case, $X^*(\mu)$ — an effective portfolio if $\mu \geq \mu_0 = 6$

2) $x1 = v2 = v3 = 0$.

From the Kuhn-Tucker conditions we find

$$X = (0;\ \lambda + 6\kappa;\ \lambda + 9\kappa), \qquad v_1 = -\lambda - \ 3\kappa \tag{4.179}$$

Then, from the initial constraints, we obtain the system

$$\begin{cases} 2\lambda + 15\kappa = 1 \\ 15\lambda + 117\kappa = \mu \end{cases} \tag{4.180}$$

We find

$$\lambda = \frac{117-15\mu}{9}; \qquad \kappa = \frac{2\mu-15}{9} \tag{4.181}$$

From here

$$X^*(\mu) = \left(0;\ 3 - \frac{\mu}{3}; \frac{\mu}{3} - 2\right), \qquad v_1 = \mu - 8 \tag{4.182}$$

Since $X \geq 0,\ v_1 \geq 0$, then $\mu \in [8; 9]$Having considered the other special cases, we will find all corner portfolios:

3) $x_1 = x_2 = v_3 = 0$, portfolio $X = (0, 0, 1)$;

4) $x_1 = v_2 = x_3 = 0$, portfolio $X = (0, 1, 0)$; 5) $v_1 = x_2 = x_3 = 0$, portfolio $X = (1, 0, 0)$.

It is clear from the formulas obtained that effective portfolios form a polyline in R^3 with vertices:

$$X^*(6) = \left(\frac{1}{3};\frac{1}{3};\frac{1}{3}\right),\ X^*(8) = \left(0;\frac{1}{3};\frac{2}{3}\right), X^*(6) = (0;0;1),$$

$$X^*(3) = (1;0;0), \quad X^*(6) = (0;1;0) \tag{4.183}$$

The effective set of portfolios is a polyline in R^3 drawn through the resulting points (corner portfolios).

4.5.5. Maximum risk portfolio with non-negative components

At first glance, it seems an absurd task to look for a portfolio of maximum risk, since all investors seek to minimize the risk of the portfolio. However, considering such a task may make sense. Firstly, increasing the profitability of a portfolio, as a rule, entails an increase in its risk, therefore, the search for highly effective portfolios is closely related to the consideration of high-risk portfolios. Secondly, venture capital investment always implies high risks, so investing in the development of new technologies is also associated with considering portfolios with high risks. And finally, in practice, there may be a situation when the task of a portfolio of minimal risk (as opposed to a portfolio of maximum risk) cannot be solved or for some reason such a portfolio cannot be formed. In this situation, knowledge of the structure of portfolios with high risks will allow the investor to avoid them.

So, let's solve the optimization problem:

$$\frac{1}{2}\sigma^2 = \frac{1}{2}X^TVX \to \max \tag{4.184}$$

under the conditions

$$I^TX = 1, \qquad X \geq 1 \tag{4.185}$$

Since the set of portfolios under consideration is compact (recall that the number of different types of securities is finite), then there is a maximum point. Let Z be some point of maximum, $c = z_1 + z_2$, $T = \{(t_1,\ t_2,\ z_3, \dots, z_n\ )/\ t_1 +\ t_2 = c, t_1, t_2 \geq\ 0\}$. Since the admissible set of the problem is convex, then *T* lies in a valid set. Convex function $\frac{1}{2}X^TVX$ on the set *T* takes the largest value at one of the two ends *T,* that is, one of the variables z_1, z_2 must be equal to 0. From this reasoning it follows that only one of the components of the point Z may be different from 0.

Therefore, the maximum risk portfolio with non-negative components consists simply of maximum risk securities. And the maximum value of the risk of such a portfolio coincides with the risk of maximum risk securities.

The conclusions obtained remain valid even if there is a risk-free security on the market.

4.5.6. Maximum efficiency portfolio with non–negative components

To find a portfolio of maximum efficiency with non-negative components, it is necessary to solve an optimization problem:

$$\vec{\mu}^T X \to \max \tag{4.186}$$

under the conditions

$$I^T X = 1,\ X \geq 1 \tag{4.187}$$

The problem under consideration is a linear programming problem. It is known from the theory of linear programming that in the optimal solution of the problem (4.185)–(4.187) only one variable can be different from zero. Therefore, the desired portfolio consists only of the most efficient security.

4.5.7. Minimum risk portfolio with non–negative components

If there is a risk-free security, then a portfolio made up only of it is the one you are looking for. If there is no risk-free security, then the matrix

V it can be considered positive definite. In this case, we will solve the optimization problem

$$\frac{1}{2}\sigma^2 = \frac{1}{2}X^T V X \to \min \tag{4.188}$$

under the conditions

$$I^T X = 1, X \geq 1 \tag{4.189}$$

Since the allowable set is compact, the desired portfolio exists. Taking into account the strict convexity of the objective function, the linearity of the constraint and the differentiability of the functions under consideration, we conclude that the Kuhn-Tucker conditions give the necessary and sufficient conditions for a conditional minimum:

$$X^T V - \lambda I^T \geq 0, (X^T V - \lambda I^T)X = 0,\ I^T X = 1, X \geq 1 \tag{4.190}$$

Solving this system of equations and inequalities is, in general, extremely difficult. With a small number of n–securities, it is possible to solve the system (4.190) by iterating over the cases.

4.5.8. Portfolio diversification

Diversification (from latin *diversus* – different and *facere* – to do, eng. *diversification*) in the field of finance – this is the distribution of investments across different financial instruments.

Diversification of the investment portfolio — this is the distribution of funds between different investment objects in order to avoid serious losses in the event of a drop in the prices of one or more assets of the investment portfolio.

In paragraph 3.7.1, it was said that the diversification method (in relation to uncorrelated financial transactions) is based on the following statement (proved in the same paragraph): the risk of a (composite) financial transaction consisting of *n* uncorrelated financial transactions, inversely proportional to its average income $\sqrt{n}$ and, consequently, with the growth of n, the relative risk of a composite financial transaction decreases.

Thus, the relative risk of a composite financial transaction with growth *n* decreases. When proving the claim, it was assumed that the income of the financial transactions that make up the operation *X*, they are values of the same order, as well as their risks.

In the same paragraph, it is proved that with an increase in the number of uncorrelated operations, their arithmetic mean has an efficiency of the order of efficiency of each of these operations, and the risk of $\left(\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}\,/n \infty 1/\sqrt{n}\right)$it turns out to be inversely proportional $\sqrt{n}$ and, consequently, with the growth of n decreases.

This effect is called **the effect of diversification** and it means that it is necessary to carry out various, unrelated or negatively correlated operations. (It's also known as the "don't put all your eggs in one basket" principle.) With such a strategy, the effectiveness of a financial transaction or portfolio is averaged, and the risk is reduced.

Let us consider the question of changing the minimum boundary when replenishing the Markowitz portfolio with a new security x_{n+1}. The answer to the question is given by the following statement.

Let a security be added to the portfolio $X = (x_1, x_2, \ldots., x_n)$ so that we get a portfolio $\tilde{X} = (X, x_{n+1})$. Then for the equations of minimum bounds $\sigma_X(\mu)$ and $\sigma_{\tilde{X}}(\mu)$ for all μ, the inequality holds.

$$\sigma_{\tilde{X}}(\mu) \le \sigma_X(\mu) \tag{4.191}$$

Indeed, the task (4.77) — (4.79) for the portfolio X is a special case of a similar task for a portfolio $\tilde{X}$, namely, it is necessary to put $x_{n+1} = 0$. Hence follows the inequality (4.191).

Thus, the replenishment of the portfolio with a new asset at least does not worsen the situation for the investor, since the minimum risk at the same yield does not increase. In practice, however, it is not always possible to form an optimal portfolio (a portfolio of minimal risk), in this case, the risk of the portfolio may increase when adding a new asset.

In general, the "smearing" of the portfolio by a larger number of uncorrelated or negatively correlated securities reduces the risk of the portfolio.

Control questions and tasks

1. Define the profitability of the security and portfolio.

2. Derive the formula for the profitability of a portfolio of n–securities through the profitability of individual securities.

3. The covariance matrix is given $V = \begin{pmatrix} 10 & -5 & 6 \\ -5 & 11 & -7 \\ 6 & -7 & 12 \end{pmatrix}$. Find the correlation matrix.

4. Data on the distribution of yields of two securities A and B are given in the table. Find the covariance and correlation coefficient of these securities.

A	–8	–1/2	7	29/2	22
B	28	19	10	1	–8
P (probability)	0.2	0.1	0.3	0.05	0.35

5. A portfolio of two securities. The case of complete correlation.

6. A portfolio of two securities. The case of complete anticorrelation.

7. A portfolio of two independent securities is given. How do you determine the minimum risk portfolio and its profitability?

8. How to find a minimum risk portfolio of two independent securities whose variances are 10 and 15?

9. 9. A portfolio of three independent securities. Find the minimum risk portfolio and its profitability.

10. Find a minimum risk portfolio of three independent securities whose variances are 9, 16 and 25.

11. How to find a portfolio of two independent securities, one of which is risk-free?

12. Describe Markowitz's portfolios.

13. How to determine the minimum risk portfolio for a given efficiency?

14. What is the minimum boundary and its properties?

15. Prove that the equation of the minimum boundary $\sigma = \sqrt{\frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}}$ is a branch of the hyperbola. Find its asymptotes.

16. The portfolio consists of two securities *A* and *B*. Expected returns are equal 0.2 and 0.4, and the risks 0.3 and 0.5. The correlation coefficient is equal to 0.2. Find a minimal risk portfolio, its risk and profitability.

17. The portfolio consists of two securities *A* and *B*. Expected returns are equal 0.6 and 0.4, and the risks 0.1 and 0.5. The correlation coefficient is equal to –0.3. Find the minimum risk portfolio and its profitability.

18. The portfolio consists of two securities *A* and *B*. Expected returns are equal 0.5 and 0.8, and the risks 0.2 and 0.6. The correlation coefficient is equal to $^1/_2$. Find the minimum risk portfolio and its profitability.

19. For a portfolio of two securities with yield and risk, respectively (0.3; 0.6) and (0.5; 0.9) in the case of complete anticorrelation, find a zero-risk portfolio and its profitability.

20. A portfolio of minimal risk with an efficiency not less than the specified.

21. Describe minimum risk portfolio.

22. Portfolio of maximum risk efficiency no more than a given.

23. Given portfolio of three securities with yields μ = 5%; 1 μ = 0%; 1 μ = 5% and the covariance matrix $V = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 16 & -3 \\ 0 & -3 & 25 \end{pmatrix}$. Find a minimal risk portfolio with a return of μ = 12% and its risk. Write the equation of the minimum boundary.

24. Describe Tobin's portfolio.

25. Find a tangent portfolio, its expected return and risk if the portfolio consists of three securities: risk-free with efficiency (expected return) of 7% and two risky ones with an efficiency of 12 and 20% respectively and a covariance matrix $\begin{pmatrix} 4 & -2 \\ -2 & 25 \end{pmatrix}$

26. The portfolio consists of three securities: a risk-free one with an expected return of 4% and two risky ones with an efficiency of 8 and 12%, respectively, and a covariance matrix $\begin{pmatrix} 3 & -4 \\ -4 & 6 \end{pmatrix}$. Find Tobin's portfolios of expected returns of 7%, 8%, 10% and 11% and minimum risk and their risks.

27. How do you determine the minimum risk portfolio from all portfolios for a given performance?

28. How do you determine the portfolio of maximum efficiency from all risk portfolios of no more than a given?

29. Describe the Kuhn-Tucker theorem.

30. Find the profitability of a non-negative portfolio.

31. How is the maximum risk portfolio with non-negative components determined?

32. How do you find a portfolio of maximum efficiency with non-negative components?

33. How do you find a minimal risk portfolio with non-negative components?

34. How do you find a portfolio of minimal risk, given efficiency with non-negative components?

Chapter 5

Bonds

According to the sources of financing, the company's financial resources are divided into its own, borrowed, attracted and state. In addition to loans, a bond loan or bonds issued by the issuer for borrowing money can act as borrowed funds. The state, municipality, corporations, financial or commercial institutions can act as an issuer.

5.1. Basic concepts

Bond — this is a security that indicates that its holder has granted a loan to the issuer for a fixed, usually long term, and provides its owner with a specified income. This income is usually lower than from other securities, at the same time it is more reliable and stable than, for example, dividends on shares, since it does not depend on fluctuations in the conjuncture. In this regard, pension funds, insurance companies, mutual funds, invest free resources in bonds and etc.

The bond is characterized by the following parameters:

— **redemption date** ($t = T$, where T – the time of circulation of the bond from the date of issue);

— **maturity date** ($n = T - \tau$, where τ – current date);

— **nominal value** (N) – the amount of money paid to the owner of the bond on the maturity date. The nominal value is usually indicated on the bond itself;

— **redemption value** (if it differs from the nominal);

— **coupon income** (C) – permanent payments that are paid to the owner annually by **coupon rate** (rate of income) $c = C/N$. If coupon payments are not provided, then such a bond is called **coupon – free**. The income on it is formed due to the exchange rate difference in the value of the bond.

5.2. Current value of the bond

Each bond has a payment flow associated with it, consisting of the annual payment of coupon income and the payment of the nominal value at the maturity date. Therefore, at the moment of time *t* you can talk about *the current value P* bonds. Let *r* — the refinancing rate (interest rate), and there is exactly one remaining until the bond is repaid *n* years. Then we have:

$$P = \sum_{k=1}^{n} \frac{C}{(1+r)^k} + \frac{N}{(1+r)^n} \qquad (5.1)$$

Coupon payments $C = cN$ form a simple rent, so that formula (5.1) can be rewritten in closed form

$$P = cN\frac{1-(1+r)^{-n}}{r} + N(1+r)^{-n} \qquad (5.2)$$

Example 5.1. It is necessary to find the current value of a bond with a nominal value of 1000 days units, with a maturity of five years and annual payments at a coupon rate of 15% at an annual interest rate of 20%. We have $N = 1000$, $n = 5$, $c = 0.15$, $r = 0.2$. Substituting these values into the formula (5.2), we get:

$$P = cN\frac{1-(1+r)^{-n}}{r} + N(1+r)^{-n} = 150\frac{1-1{,}2^{-5}}{0.2} + 1000 * 1.2^{-5}$$
$$= 448.59 + 401.88 = 850.47$$

Thus, the current value of the bond $P = 850.47$, which is less than the nominal value.

5.3. Current yield and yield to maturity

A potential investor investing in bonds must make a choice between many bonds available on the market. To this end, he should compare the parameters of various bonds, which can be different indicators of profitability, average term, duration, modified duration, convexity, etc. Let's consider the main parameters by which the

the choice of bonds, indicators of profitability, as which we consider the current yield and yield to maturity.

The coupon interest rate is a measure of yield only when the bond is sold at par, so let's pay attention to the current yield and the yield to maturity. After the bond is issued, it enters the market, where it is freely sold and bought at the market price V, which does not coincide with the current value calculated according to formulas (5.1), (5.2). In this case, the ratio of the market price of the bond V to the nominal value N is called **bond exchange rate** (K):

$$K = \frac{V}{N} * 100 \tag{5.3}$$

5.3.1. Current yield of the bond

The current yield of the i–bond is equal to the ratio of coupon payments

$$cN = C \tag{5.4}$$

to the market price of the bond V:

$$i = \frac{cN}{V} = \frac{C}{V} \tag{5.5}$$

Example 5.2. Let the bond rate be 105, the coupon yield is 15%. It is required to find the current yield of the bond. Substituting in (5.5) $V = KN/100$, we get:

$i = 100c/K = 15/105 = 0.14285 = 14.285\%$

Thus, the current yield of the bond is equal to 14.285%.

Note that if coupon payments are made p–times per year at the rate of c/p, then, in this case, the current yield of the bond is calculated using the formula (5.5). It follows from it that if the bond is bought at a discount ($V < N$), then the current yield of the bond is greater than the coupon rate ($i > c$), if the bond is bought with a premium $V > N$, then the current yield of the bond is less than the coupon rate ($i < c$).

If coupon payments (coupon income) change over time and their values are known, then it is possible to find the average current yield of the bond:

$$\bar{\iota} = \frac{\sum_{j=1}^{n} c_j}{n} \frac{N}{V} \tag{5.6}$$

If, under the terms of the bond issue, a constant relative increase in coupon payments (coupon income) is provided for q, then coupon payments form a geometric progression with a denominator q: c, cq, cq^2, ...

Calculating the sum of the n–terms of the geometric progression by the formula

$$S_n = c\frac{1 - q^n}{1 - q} \tag{5.7}$$

we get an expression for the average coupon rate

$$\bar{c} = c\frac{1 - q^n}{1 - q} * \frac{1}{n} \tag{5.8}$$

At the same time, the current yield of such bonds is equal to

$$\bar{\iota} = \bar{c}\frac{N}{V} \tag{5.9}$$

5.3.2. Yield to maturity

The current yield from the point of view of assessing the effectiveness of investing in bonds has a significant drawback, since it does not take into account the second part of the income on bonds – the change in the value of the bond by the end of its term. Therefore, it is enough to say that bonds with zero coupon income, the current yield of which is zero, often turn out to be very profitable for the investor when taking into account their entire life.

A more important indicator is the yield to maturity. This value serves as a substitute for the interest rate r in a situation where the current value *P* the bond does not match its market value *V*.

If the market price of the bond is known *V*, its nominal value *N*, maturity date *n* and coupon rate *c*, then the yield to maturity is defined as the solution of the equation:

$$V = \sum_{k=1}^{n} \frac{cN}{(1+p)^k} + \frac{N}{(1+p)^n} \tag{5.10}$$

Summing up the geometric progression in the first term with $a_1 = \frac{1}{(1+p)}$, $q = \frac{1}{(1+p)}$, we obtain an equivalent equation for p:

$$V = cN\frac{1-(1+p)^{-n}}{p} + N(1+p)^{-n} \tag{5.11}$$

The problem of the existence and uniqueness of such a solution is solved in the same way as in the case of determining the internal rate of return of a special type of payment flow. Namely, from the formula (5.10) it can be seen that the right part is a decreasing function of the argument $\rho > -1$, taking any positive values, so that for any $V > 0$ there is a unique solution to equation (5.10). For large values of n, approximate formulas are used to find the yield to extinction. One of the following formulas is given in:

$$\rho \approx \frac{2(cn+1-K)}{K-1+n(1+K)}. \tag{5.12}$$

Dividing both parts of equation (5.11) by *N*, we get

$$K = c\frac{1-(1+\rho)^{-n}}{\rho} + (1+\rho)^{-n}. \tag{5.13}$$

To derive the formula (5.12), we use the expansion in a series

$$\frac{n\rho}{1-(1+\rho)^{-n}} = 1 + \frac{n+1}{2}\rho + \frac{n^2-1}{12}\rho^2 + \ldots \,. \qquad (5.14)$$

We will rewrite the formula (5.13) in the form:

$$\frac{K-1}{n} \cdot \frac{n\rho}{1-(1+\rho)^{-n}} = c - \rho. \qquad (5.15)$$

We substitute the formula (5.14) in (5.15), limiting ourselves only to the linear part of the decomposition. We will get

$$\frac{K-1}{n}\left(1+\frac{n+1}{2}\rho\right) = c - \rho,$$

where

$$\rho = \frac{2(cn+1-K)}{K-1+(K+1)n}. \qquad (5.16)$$

Standard methods, such as the Newton tangent method, are used to refine approximate formulas. It can be shown that if the market price is greater than the nominal value, then the yield to maturity is less than the coupon rate. It turns out that, in general, the following statements are fulfilled:

1) the market price of a bond is equal to its face value if and only if the yield to maturity is equal to the coupon rate;

2) the market price of a bond is greater than its face value if and only if the yield to maturity is less than the coupon rate;

3) the market price of a bond is less than its face value if and only if the yield to maturity is greater than the coupon rate.

The proof follows from the fact that the function $V = V(r)$ is decreasing in its domain of definition, and all three statements follow from the equality $V(c) = N$, which follows directly from formula (5.11).

Example 5.3. How do you determine the yield to maturity of a bond with a maturity of eight years, a face value of $3,000 and a coupon rate of 8% if: 1) it sells for $3,000; 2) its market price will increase by 10%; 3) decrease by 5%?

In the first case, the bond is sold at face value, so the yield to maturity is equal to the coupon rate $r = 8\%$.

In the second case, the bond is sold at a premium of $3,300, so we should expect the yield to maturity to fall below 10%. Indeed, in this case $K = 1{,}1$, so that calculations using formula (5.12) give the following approximate value:

$$\rho \approx \frac{2(cn+1-K)}{K-1+n(1+K)} = \frac{2(0{,}08\cdot 8+1-1{,}1)}{1{,}1-1+8(1+1{,}1)} = \frac{1{,}08}{16{,}9} = 0{,}0639 = 6{,}39\%.$$

In the third case, the bond is sold at a discount for $2,850, therefore, according to the theorem, the yield should be more than 8%. In that case $K = 0.95$.

Calculations give in this case $\rho = 8.87\%$.

$$\rho \approx \frac{2(cn+1-K)}{K-1+n(1+K)} = \frac{2(0{,}08\cdot 8+1-0{,}95)}{0{,}95-1+8(1+0{,}95)} = \frac{1{,}38}{15{,}55} = 0{,}0887 = 8{,}87\%.$$

Note that in the considered cases, changes in yield are proportional to changes in the bond $\frac{10\%}{5\%} \approx \frac{1{,}61\%}{0{,}87\%}$ $(2 \approx 1.85)$.

We will discuss this fact in the following paragraphs.

5.4. Dependence of yield to maturity of a bond on parameters

Let us study the dependence of the yield to maturity of a bond on the parameters included in formula (5.13):

$$K = c\frac{1-(1+\rho)^{-n}}{\rho} + (1+\rho)^{-n}, \qquad (5.17)$$

where $K = \frac{V}{N}$ – bond price.

Note that $K = 1$ if the bond is sold at face value, $K < 1$ if the bond is sold at a discount, and $K > 1$ if the bond is sold at a premium.

Let us prove that if the bond yield ρ does not change during the time of its circulation, then the size of the discount or premium decreases with a decrease in the period of its circulation n. Formula (5.17) can be rewritten as:

$$K = \frac{c + (\rho - c)(1+\rho)^{-n}}{\rho}. \qquad (5.18)$$

Then at $r > c$, the function $K = K(n)$ decreases, so that as n decreases, the value of K increases, and the discount value equal to $N - V = N(1 - K)$ decreases. Similarly, when $r < c$, the function $K = K(n)$ increases, so that as n decreases, the value of K decreases, and the value of the premium, equal to $V - N = N(K - 1)$, decreases. Under these conditions, the price of a bond with a decrease in the term of its circulation approaches the nominal value.

Figure 5.1 shows the dependency graphs $K = K(n)$ for two n–year bonds with the same face value and the same coupon rate. At the same time, the yield of one bond (with a discount, see the bottom chart) is greater than the yield of the second one (with a premium, see the top chart). The abscissa shows the time scale.

Example 5.4. It is required to find the change in the discount of a bond with maturity n_1=7 years with a nominal value of N = 5,000, coupon rate c = 8% and yield to maturity r = 10% when selling it now and in a year.

The discount I (the difference between the face value N and the current market value of the bond V) when selling it at the moment is equal to $I_1 = N - V_1$.

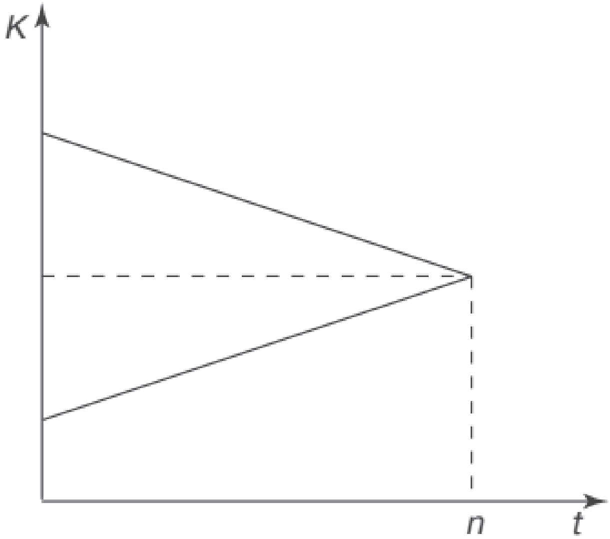

Fig. 5.1. Dependency graphs $K = K(n)$ for two n–year bonds with the same face value and coupon rate

Calculate the current market value of the bond at the moment V_1 according to the formula (5.11):

$$V_1 = cN \cdot \frac{1-(1+\rho)^{-n}}{\rho} + N(1+\rho)^{-n} =$$

$$= 0.08 \cdot 5{,}000 \cdot \frac{1-(1+0.1)^{-7}}{0.1} + 5{,}000 \cdot (1+0.1)^{-7} =$$

$$= 1{,}947.37 + 2{,}565.79 = 4{,}513.16$$

The discount I when selling a bond in a year is equal to $I_2 = N - V_2$. Calculate the current market value of the bond V_2 in a year by the same formula (5.11):

$$V_2 = 0.08 \cdot 5{,}000 \cdot \frac{1-(1+0.1)^{-6}}{0.1} + 5{,}000 \cdot (1+0.1)^{-6} =$$
$$= 1{,}742.10 + 2{,}822.37 = 4{,}564.47$$

Now let's find the discounts:

$$I_1 = N - V_1 = 5{,}000 - 4{,}513.16 = 486.84$$
$$I_2 = N - V_2 = 5{,}000 - 4{,}564.47 = 435.53$$

Thus, the discount decreased from 486.84 to 435.53. Let us prove that if the yield to maturity of the bond does not change during the time of its circulation, the value of the discount or premium will decrease the faster, the shorter the circulation period.

It is known from mathematical analysis that the statement of the theorem is equivalent to the concavity of the function $|V–N|$ or the inequality

$|V–N|_n^n < 0.$

In other words, the graph *V(n)* has the form shown in Figure 5.1. To prove it, we write the formula (5.11) in the form

$$V - N = N\left(\frac{c}{\rho} - 1 + \left(1 - \frac{c}{\rho}\right)(1+\rho)^{-n}\right). \tag{5.19}$$

Then

$$(V - N)_n'' = N\left(1 - \frac{c}{\rho}\right)(1+\rho)^{-n}\ln^2(1+\rho). \tag{5.20}$$

If the bond is sold at a premium, i.e. $r < c$, then $V > N$ and $|V - N| = = V - N$.

Then the formula (5.20) shows that $(V - N)_n^n < 0$. If about the bond is sold at a discount, i.e. $r > c$, then $V < N$ and $|V - N| = N - V$. In this case, also $(N - V)_n^n < 0$.

Let us prove that a decrease in the yield of a bond will lead to an increase in its market price by an amount greater than the corresponding decrease in the market price with an increase in the yield by the same amount.

We are talking about the nature of the dependence of the decreasing function *V = V(r).* We fix the yield to maturity r and deviate from the selected value by the value $\Delta r > 0$ left and right. When the yield changes to maturity, the price of the bond will also change. Suppose

$$\Delta V_1 = V(\rho - \Delta\rho) - V(\rho) > 0,$$

$$\Delta V_2 = V(\rho) - V(\rho + \Delta\rho) > 0.$$

The theorem states that the inequality holds

$$\Delta V_1 > \Delta V_2. \tag{5.21}$$

The assertion of the theorem follows from the convexity of the function *V = V(r)*, and this property follows directly from the form of the function (5.10).

Example 5.5. Find the change in the current market value of a bond with maturity n_1= 7 years with a face value of N = \$5000, coupon rate c = 8% and yield to maturity r = 10% with an increase and decrease in yield to maturity by 2%.1. Let's calculate the current market value of the bond at the moment V_0 by the formula (5.11) (see example 5.4):

$$V_0 = cN \cdot \frac{1-(1+\rho)^{-n}}{\rho} + N(1+\rho)^{-n} =$$

$$= 0.08 \cdot 5000 \cdot \frac{1-(1+0.1)^{-7}}{0.1} + 5000 \cdot (1+0.1)^{-7} =$$

$$= 1947.37 + 2565.79 = 4513.16$$

2. Determine the current market value of the bond at the present moment V_1 with an increase in yield to maturity by 2% according to the formula

(5.11):

$$V_1 = cN \cdot \frac{1-(1+\rho)^{-n}}{\rho} + N(1+\rho)^{-n} =$$

$$= 0.08 \cdot 5000 \cdot \frac{1-(1+0.12)^{-7}}{0.12} + 5000 \cdot (1+0.12)^{-7} =$$

$$= 1825.50 + 2261.75 = 4087.25$$

Find the change in the current market value of the bond

$\Delta V_1 = V_1 - V_0 = 4087.25 - 4513.16 = -\$425.91$

So, with an increase in yield to maturity by 2%, the market value of the bond decreased by \$425.91.

3. Let us now calculate the current market value of the bond at the present moment V_2 with a decrease in yield to maturity by 2% according to the same formula (5.11):

$$V_2 = cN \cdot \frac{1-(1+\rho)^{-n}}{\rho} + N(1+\rho)^{-n} =$$

$$= 0.08 \cdot 5{,}000 \cdot \frac{1-(1+0.08)^{-7}}{0.08} + 5{,}000 \cdot (1+0.08)^{-7} =$$

$$= 2{,}082.55 + 2{,}917.45 = 5{,}000$$

This result could have been predicted, since with a decrease in yield to maturity by 2%, it became equal to the coupon rate, while the market value of the bond becomes equal to the face value.

Find the change in the current market value of the bond:

$\Delta V_2 = V_0 - V_1 = \$5{,}000 - \$4{,}513.16 = \$486.84$

So, with a decrease in yield to maturity by 2%, the market value of the bond decreased by \$486.84.

Note that, in accordance with the statement proved above, a decrease in the yield of a bond led to an increase in its market price by (\$486.84) greater than the corresponding decrease in the market price with an increase in yield by the same amount (\$425.91).

The following statement shows the dependence of the relative change in the price of a bond on the coupon rate. Let us prove that the relative change in the price of a bond (in %) as a result of a change in yield to maturity will be the smaller, the higher the coupon rate. We write formula (5.11) for the price of a bond in the form:

$$P(c) = N(ac + b), \tag{5.22}$$

$a = \dfrac{1-(1+\rho)^{-n}}{\rho} > 0$, $b = (1 + \rho)^{-n} > 0$. $c = (1 + \rho)^{-n} > 0$

Differentiating (5.22) with respect to c, we obtain

$$dP(c) = Nadc. \tag{5.23}$$

Replacing the increment of the function with a differential and dividing (5.23) by (5.22), we write the relative increment of the bond price

$$\frac{\Delta P}{P} \approx \frac{Na\Delta c}{N(ac+b)} = \frac{a\Delta c}{ac+b}. \tag{5.24}$$

Now the assertion follows from the fact that $\frac{a}{ac+b}$ − decreasing function of the argument c for positive values of *a, b*.

5.5. Additional characteristics of the bond

5.5.1. Average time of receipt of income

In addition to the yield of a bond, it is also necessary to be able to assess its risk associated with the bond term: the longer the term to maturity, the higher the risk. In addition to the timing itself, it is necessary to take into account the distribution of income over time. For this kind of valuation of bonds, the average period of receipt of income from a bond is introduced, which is studied in this paragraph.

The average term of receipt of income is the weighted average value of all types of receipts (income) from the bond. The sums of receipts (revenues) are taken as weights. It should be noted that the average term of receipt of income from a bond differs from the average life of a bond, which averages only the terms of payment of the face value of bonds (allowing early redemption), but does not take into account the terms of coupon income payments. The average bond life *(t)* is determined by the formula

$$\bar{t} = \sum_{i=1}^{k} x_i t_i, \tag{5.25}$$

where k – number of episodes*;*
x_i – consecutive shares of bond repayment, $\sum_{i=1}^{k} x_i = 1$.

The average period of receipt of income from the bond (T) is found as follows:

$$T = \frac{\sum_{i=1}^{n} t_i S_i}{\sum_{i=1}^{n} S_i}, \tag{5.27}$$

where S_i −amount of income*;*
n −the term of the bond*;*
t_i −timing of receipt coupon income.

The calculation by formula (5.27) can be simplified.

The amount of coupon income and face value, which is in the denominator, is equal to:

$$\sum_{i=1}^{n} S_i = cNn + N. \tag{5.28}$$

The sum of the terms, weighted by the amount of income, standing in the numeral body, is equal to

$$\sum_{i=1}^{n} t_i S_i = cN\sum_{i=1}^{n} t_i + nN. \tag{5.29}$$

$$T = \frac{cN\sum_{i=1}^{n} t_i + nN}{cNn + N}. \tag{5.30}$$

In the case $ti = 1,2,\ldots,n \sum_{i=1}^{n} ti = \frac{n(n+1)}{2}$.

Dividing the numerator and denominator (5.30) by nN, we obtain

$$T = \frac{\frac{c(n+1)}{2} + 1}{c + \frac{1}{n}}. \tag{5.31}$$

Note that the average period of income from bonds does not depend on the nominal value. The higher the coupon yield, the shorter the average term. Let us prove the last assertion. Let $c_2 > c_1$, the bond term is n, let us show that $T_1 > T_2$ for $n > 1$.

From (5.31) we have

$$T_1 = \frac{\frac{c_1(n+1)}{2} + 1}{c_1 + \frac{1}{n}} > \frac{\frac{c_2(n+1)}{2} + 1}{c_2 + \frac{1}{n}} = T_2.$$

$$\left[\frac{c_1(n+1)}{2} + 1\right]\left[c_2 + \frac{1}{n}\right] > \left[\frac{c_2(n+1)}{2} + 1\right]\left[c_1 + \frac{1}{n}\right].$$

Expanding the brackets, we get

$$\frac{c_1c_2(n+1)}{2}+c_2+\frac{c_1(n+1)}{2n}+\frac{1}{n}>\frac{c_1c_2(n+1)}{2}+c_1+\frac{c_2(n+1)}{2n}+\frac{1}{n},$$

or

$$c_2+\frac{c_1(n+1)}{2n}>c_1+\frac{c_2(n+1)}{2n}.$$

Next we have

$$c_2-c_1>\frac{(c_2-c_1)(n+1)}{2n},$$

or abbreviating to $(c_2 - c_1) > 0$,

$$1>\frac{n+1}{2n},$$

whence we have

$2n > n + 1, n > 1$.

So, it has been proved that with a bond life of more than a year, the average period of receipt of income decreases with an increase of coupon income of bonds. In addition, for bonds with coupon income $T < n$, and for zero-coupon bonds or interest payments at the end of the term, $T < n$. Bonds with consecutive redemption of face values (for example, serial ones) have a shorter average term than bonds with redemption at the end of the term.

Now let the coupons be paid p–times a year. Then for the sum of payment terms we have:

$$\sum_{i=1}^{np} t_i=\frac{np(n+1/p)}{2}, \qquad (5.32)$$

where n – is the term of the bond in the years;

ti=1/p, 2/p,…,n.

The average period of receipt of income from a bond instead of formula (5.31) is determined by the expression

$$T = \frac{\frac{c\left(n+1/p\right)}{p}+1}{c+\frac{1}{n}}. \tag{5.33}$$

Increasing the rate of payment of interest on the bond reduces the average period of receipt of income from the bond.

In conclusion, let us explain the meaning of the introduced concept "average period of receipt of income from a bond". It gives that moment in time of the total bond term at which the amount of credit services (credit service – the product of the loan amount by its term) are equal to each other up to this moment and after. The shorter the average term, the sooner the owner of the bond receives a return on it and, consequently, the lower the risk.

5.5.2. Bond duration

The concept of the average term of receipt of income from a bond introduced above has the obvious drawback that it ignores the time value of money. This drawback is absent in another value that takes into account not the size of income, but their discounted values. This value is called duration.

Consider first the general situation. Let there be a stream of payments

$$\{(t_1, R_1), (t_2, R_2), \ldots, (t_n, R_n)\}, \tag{5.34}$$

so we can talk about the current value P of the flow (5.34) relative to the interest rate y:

$$P(y)=\sum_{k=1}^{n} R_k (1+y)^{-t_k}. \tag{5.35}$$

We differentiate the function (5.35) with respect to the argument y:

$$P'(y)=-\sum_{k=1}^{n} t_k R_k (1+y)^{-t_k-1}=-\frac{1}{1+y}\sum_{k=1}^{n} t_k R_k (1+y)^{-t_k} \tag{5.36}$$

and divide both sides of equality (5.36) by *P(y)*. We get the ratio

$$\frac{P'(y)}{P(y)}=-\frac{1}{1+y}\frac{\sum_{k=1}^{n} t_k R_k (1+y)^{-t_k}}{\sum_{k=1}^{n} R_k (1+y)^{-t_k}}=-\frac{1}{1+y}\sum_{k=1}^{n} w_k t_k; \tag{5.37}$$

$$w_k=\frac{R_k (1+y)^{-t_k}}{\sum_{k=1}^{n} R_k (1+y)^{-t_k}}=\frac{R_k (1+y)^{-t_k}}{P} \tag{5.38}$$

— weight coefficients that determine the weight of each payment R_k in the present value of the entire flow (5.34). The sum of all weight coefficients is equal to one

$$\sum_{k=1}^{n} w_k = 1. \tag{5.39}$$

Macaulay introduced a new concept – “duration (Macaulay duration) of the flow of payments.” The duration of the flow of payments (5.39) is called the value

$$D=\sum_{k=1}^{n} w_k t_k. \tag{5.40}$$

We will consider positive cash flows in what follows. In this case, all weight coefficients w_k– positive numbers, the sum of which is equal to one. Therefore, duration is the center of gravity of payments on the time scale (Fig. 5.2).

Fig. 5.2. Duration as the center of gravity of payments on the timeline

Example 5.6. It is necessary to find the duration of the payment flow {(100, 1), (200.2), (300, 3), (400, 4)} at the interest rate y = 12%.

Let's bring the flow to the initial moment of time:

P=100·1.12^{-1}+200·1.12^{-2}+300·1.12^{-3}+400·1.12^{-4}
=89.29+159.44+213.53+254.21=716.47.

Next, we find the weight coefficients using the formula (5.38)

$$w_k = \frac{R_k\left(1+y\right)^{-t_k}}{P},$$
$$w_1 = \frac{100\cdot 1{,}12^{-1}}{716{,}47} = 0{,}125,\ w_2 = \frac{200\cdot 1{,}12^{-2}}{716{,}47} = 0{,}223,$$
$$w_3 = \frac{300\cdot 1{,}12^{-3}}{716{,}47} = 0{,}298,\ w_4 = \frac{400\cdot 1{,}12^{-4}}{716{,}47} = 0{,}355.$$

It is easy to check that the sum of all weights is equal to one: $w_1+w_2+w_3+w_4= 1$.

Now by formula (5.40) we find

$$D = \sum_{k=1}^{4} w_k t_k = 0{,}125\cdot 1 + 0{,}223\cdot 2 + 0{,}298\cdot 3 + 0{,}355\cdot 4 = 2{,}885.$$

5.5.3. Duration properties

Let's consider some properties of duration for positive streams of payments

1. If n = 1, then $D = t_n$. If n > 1, then $D < t_n$.

2. The relation is fulfilled

$$\frac{P'(y)}{P(y)} = -\frac{D}{1+y} = -MD. \tag{5.41}$$

This formula is one of the main formulas related to duration. It shows that the duration, more precisely, the modified duration determines the sensitivity of the bond price to changes in the interest rate level in the market. This is the main value of this indicator. The modified duration for a bond with coupon income payments *p*–once a year is equal to

$\mathrm{MD} = \frac{D}{1+y/p}$.

3. *D* = *D(y)*−decreasing function of the interest rate *y*.

The first property is obvious, the second is a direct consequence equality (5.37), to prove the third property, we find the derivative of the duration from the formula

$$D(y) = \frac{\sum_{k=1}^{n} t_k R_k (1+y)^{-t_k}}{\sum_{k=1}^{n} R_k (1+y)^{-t_k}};$$

$$D'(y) =$$

$$= \frac{-\sum_{k=1}^{n} t_k^2 R_k (1+y)^{-t_k-1} \cdot \sum_{k=1}^{n} R_k (1+y)^{-t_k} + \sum_{k=1}^{n} t_k R_k (1+y)^{-t_k} \cdot \sum_{k=1}^{n} t_k R_k (1+y)^{-t_k-1}}{P^2} =$$

$$= \frac{\left(\sum_{k=1}^{n} t_k R_k (1+y)^{-t_k}\right)^2 - \sum_{k=1}^{n} R_k (1+y)^{-t_k} \cdot \sum_{k=1}^{n} t_k^2 R_k (1+y)^{-t_k}}{(1+y)P^2}.$$

The necessary result is obtained from the following statement: for any sequence 0< t_1< t_2< ...< t_n and positive numbers $a_1, a_2,$ ...,a_n the inequality is fulfilled

$$\left(\sum_{k=1}^{n} t_k a_k\right)^2 < \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} t_k^2 a_k. \quad (5.42)$$

To prove it, it suffices to apply the Cauchy-Bunyakovsky inequality:

$$\left(\vec{a}, \vec{b}\right)^2 \le \left(\vec{a}, \vec{a}\right) \cdot \left(\vec{b}, \vec{b}\right),$$

if put $\bar{a} = (\sqrt{a_1}, \ldots, \sqrt{a_n})$, $\bar{b} = (t_1\sqrt{a_1}, \ldots, t_n\sqrt{a_n})$. The inequality is strict since the vectors $\bar{a}$ and $\bar{b}$ are not collinear. Let $a_k = R_k(1+y)^{-k}$ and apply (5.42). In the expression for the derivative $D'(y)$ the numerator has a negative sign, and this proves the third property.

Let us apply the above material to the case of bonds. Then the flow of payments relative to the interest rate (yield to maturity) y has the form:

$$\{(1, cN); (2, cN); \ldots (n, cN + N)\}$$

where c–coupon rate;
N–nominal cost;
n–maturity.

Properties 1–3 in the case of bonds are formulated as follows.

1. For a zero–coupon bond (c=0), the duration is the same as the maturity repayment $D = n$.

2. For a relative change in the price of a bond $\frac{\Delta V}{V}$ when changing profitability ΔV the approximate formula

$$\frac{\Delta V}{V} \approx -\frac{D}{1+y}\Delta y. \quad (5.43)$$

3. $D = D(y)$ is a decreasing function with respect to return to maturity y.

Let's derive a formula for finding the duration. Let us show that the duration of a bond does not depend on the face value and is given by the formula

$$D=\frac{1+y}{y}-\frac{n(c-y)+1+y}{c\left((1+y)^{n}-1\right)+y}. \tag{5.44}$$

From formula (5.41) it follows that

$$D=-\frac{P'_y}{P}(1+y). \tag{5.45}$$

For the current value of the bond, we have

$$P=cN\frac{1-(1+y)^{-n}}{y}+N(1+y)^{-n}, \tag{5.46}$$

Where

$$P'_y=N\left(c\frac{n(1+y)^{-n-1}y-1+(1+y)^{-n}}{y^2}-n(1+y)^{-n-1}\right),$$

that's why

$$-P'_y(1+y)=N\left(n(1+y)^{-n}-c\frac{ny(1+y)^{-n}+(1+y)\left((1+y)^{-n}-1\right)}{y^2}\right). \tag{5.47}$$

Substituting (5.46) and (5.47) into the duration formula (5.45), we obtain

$$D = \frac{n(1+y)^{-n} - c\dfrac{ny(1+y)^{-n} + (1+y)\left((1+y)^{-n} - 1\right)}{y^2}}{c\dfrac{1-(1+y)^{-n}}{y} + (1+y)^{-n}} =$$

$$= \frac{ny^2(1+y)^{-n} - cny(1+y)^{-n} + c(1+y)\left(1-(1+y)^{-n}\right)}{y\left(c\left(1-(1+y)^{-n}\right) + y(1+y)^{-n}\right)} =$$

$$= \frac{ny^2 - cny + c(1+y)\left((1+y)^n - 1\right)}{y\left(c\left((1+y)^n - 1\right) + y\right)} =$$

$$= \frac{1+y}{y} - \frac{n(c-y) + 1 + y}{c\left((1+y)^n - 1\right) + y}.$$

This proves formula (5.44).

4. If the bond is sold at face value, i.e. $c = y$, then

$$D = \frac{1+y}{y} \cdot \left(1-(1+y)^{-n}\right) \qquad (5.48)$$

which directly follows from formula (5.44).

5. $D = D(c)$ is a decreasing function of the coupon rate c. Formula (5.44) can be rewritten in the following form:

$$D = a - \frac{nc+b}{kc+y},$$

where $a, b, k > 0$ are constants

then

$$D = a - \frac{n}{k} + \frac{\frac{ny}{k} - b}{kc + y}.$$

Therefore, it suffices to show that the inequality $ny - bk > 0$ holds.

Let's go back to the original notation:

$$ny - (1 + y - ny)\left((1 + y)^n - 1\right) =$$
$$= ny(1 + y)^n - (1 + y)\left((1 + y)^n - 1\right) > 0.$$

Let's change the variable $t = 1 + y,\ t \geq 1$. Then

$n(t-1)t^n - t(t^n-1) = (t-1)(nt^n - (t^n + \ldots + t)) \geq 0,$

which is what was required.

6. For perpetual bonds ($n \to \infty$)

$$D_\infty = \frac{1 + y}{y}. \tag{5.49}$$

Indeed, passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\left(\frac{1+y}{y} - \frac{n(c-y)+1+y}{c\left((1+y)^n - 1\right) + y}\right) = \frac{1+y}{y}.$$

The most difficult task is to determine the dependence D from argument n. It is true that if the coupon rate greater than or equal to yield to maturity ($c \geq y$), then $D = D(n)$ is an increasing function of n.

If $c < y$, then the function $D(n)$ has a unique maximum whose approximate estimate is

$$n_{\max} \approx \frac{1}{\ln(1+y)} + \frac{1+y}{y-c}.$$

Therefore *D(n)* increases as $n < n_{max}$ and decreases as $n > n_{max}$.

The derivative with respect to *n* has the form:

$$D'(n) = \frac{c(1+y)^n \left(n(c-y)\ln(1+y) + (1+y)\ln(1+y) - (c-y)\right) + (c-y)^2}{d^2},$$

where $d = c((1+y)^n - 1) + y$.

The function *D(n)* has a horizontal asymptote $D = D_\infty$, and graph *D(n)* (Figure 5.3) that intersects the asymptote if and only if equality $n(c-y)+1+y=0$.

So if $c \geq y$, then this equation has no solution for positive values of *n*, so the inequality $D(n) < D_\infty$. Since equation $D'(n) = 0$ has at most one root, then the graph *D(n)* looks like this (Figure 5.3, I).

Let now $c < y$. Then the graph intersects the asymptote at $n_0 = \frac{1+y}{y-c}$.

Therefore, there is a single maximum of the function D(n), the approximate value of which can be obtained from the equation

$n\,(c-y)\ln(1+y) + (1+y)\ln(1+y) - (c-y) = 0,$

so

$$n_{\max} \approx \frac{1+y}{y-c} + \frac{1}{\ln(1+y)}.$$

A typical graph looks like (Fig. 5.3, curve II).

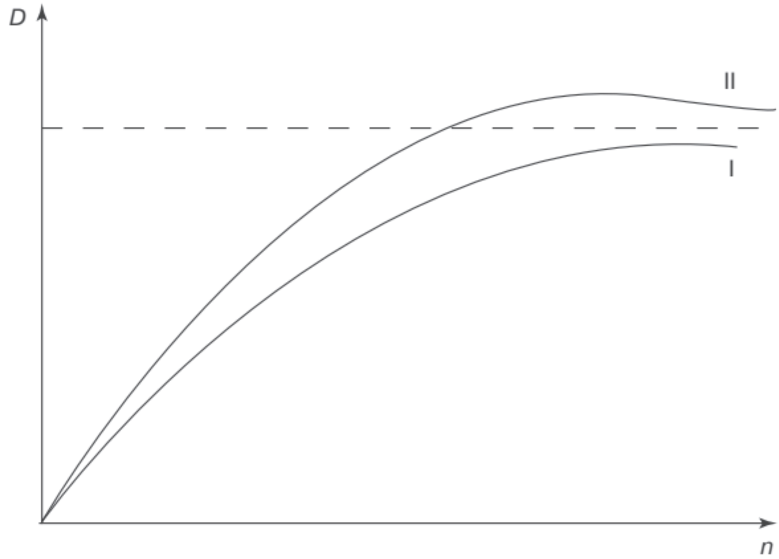

Fig. 5.3. Dependence of duration D of a bond on its maturity n at $c \geq y$ (I) and $c < y$ (II)

Example 5.7. Find the duration of a bond sold at face value with a maturity of n = 10 years and a coupon rate c = 8% (with an annual payment). Since the bond is sold at face value, then $y = c = 0.08$ and you can use the formula (5.48):

$$D = \frac{1+y}{y}\left(1-(1+y)^{-n}\right) = \frac{1{,}08}{0{,}08}\left(1-1{,}08^{-10}\right) = 7{,}25.$$

Example 5.8. A bond with a maturity of n = 15 years and a coupon rate of c = 10% (paid annually) has a yield to maturity of y = 8%. Find its duration. We use the formula (5.44).

Example 5.9. The duration of the bond is D = 10. It is known that its yield to maturity increased from 12 to 13.5%. Determine the percentage change in the price of the bond.

We use the formula (5.43)

$$\frac{\Delta V}{V} \approx -\frac{D}{1+y}\Delta y = -\frac{10}{1+0{,}12}\cdot 1{,}5\% = -13{,}39\%.$$

Thus, the price of the bond decreased by 13.39%.

To refine the approximate formula (5.48), the concept of convexity is introduced.

5.5.4. Bond Bulge

The convexity of a bond *W(y)* for a given yield y is the value

$$W(y) = \frac{V''(y)}{V(y)}(1+y)^2. \tag{5.50}$$

To find the convexity, use the formula that is obtained is obtained by direct differentiation of formula (5.10) with the change of variable *p* to *y*:

$$W(y) = \frac{c}{K}\sum_{k=1}^{n} k(k+1)(1+y)^{-k} + \frac{n(n+1)}{K}(1+y)^{-n}. \tag{5.51}$$

Recall that *c* is the coupon rate, *K* = *V/N* is the rate of the bond, *n* is the maturity, and *y* is the yield of the bond. In principle, it is possible to obtain a closed formula for convexity, similar to formula (5.44) for duration, but we will not do this. The main application of convexity is a refinement of the approximate formula (5.43). Namely, the assertion is true that for a relative change in the price of a bond $\Delta V/V$ when yield changes by Δy approximation formula

$$\frac{\Delta V}{V} \approx -\frac{D}{1+y}\Delta y + \frac{1}{2}W\cdot(\Delta y)^2. \tag{5.52}$$

To prove formula (5.45), it is necessary to expand the function *V(y)* into a Taylor series up to a term of the second order of smallness $(\Delta y)^2$.

Figure 5.4 shows the graphs of dependencies *V* = *V(y)* for two bonds in which, at $y = y_0$ yields and durations coincide, however, the convexity of one (shown by a dash) is greater than the other (solid line).

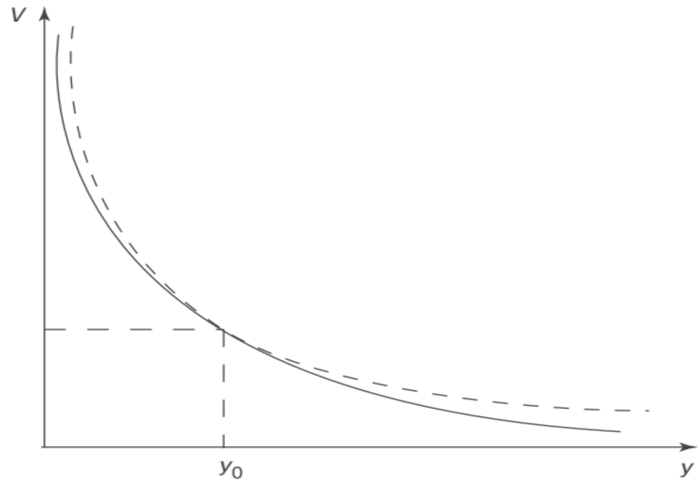

Fig. 5.4. Dependences of bond price *V* on bond yield y *V* = *V(y)* for two bonds, for which, at *y* = *y0* yields and durations coincide

We will consider this situation in more detail in the next paragraph.

5.6. Bond Portfolio Immunization

It is understood that under the immunization of a portfolio of bonds is a portfolio management, which allows you to maintain the level of its profitability for a certain period, despite jumps in the market interest rate. An example of an immunized portfolio would be a bond portfolio held by, say, a pension fund, if the duration of the bond portfolio is equal to the duration of the liabilities of that fund.

Consider the immunization theorem following.

Let's assume that the debt *R* needs to be repaid in exactly *n* years. The duration of such payment is equal to *n*. One way to repay the debt is to buy a zero-coupon *n*–year bond with a face value of *N* = *R* at an annual interest rate of *r*. Then buying a bond will cost $P = N/(1+r)^2$. Samuelson pointed out the possibility of replacing one bond with two so that at a given interest rate, the current value does not change, but only increases when the interest rate changes. This is because the current value is a decreasing function of the interest rate, and bonds with different maturities react differently to changes in the interest rate.

So, let it be required to pay off a debt in the amount of R at time t. Buying a bond with a face value of $N = R$ and a maturity of t will pay off the debt. Let's call it bond I, the current value of which is equal to

$$P_1 = N/(1 + r)^2. \tag{5.53}$$

Consider two zero-coupon bonds with par values $N_1$and N_2 and maturities respectively $t_1$and t_2, and the inequality

$t_1 < t < t_2$.

The portfolio consisting of these bonds will be called bond II. Bond II has a present value

$$P_2 = \frac{N_1}{(1+r)^{t_1}} + \frac{N_2}{(1+r)^{t_2}}. \tag{5.54}$$

Require that for $r = r_0$ the conditions

$$\begin{cases} P_1(r_0) = P_2(r_0), \\ D_1(r_0) = D_2(r_0). \end{cases} \tag{5.55}$$

These conditions ensure the equivalence of two cash flows associated with bonds I and II, and the equality of their durations at $r = r_0$. In this case, the graphs of the functions P_1 *(r)* and P_2 *(r)* look the same as in Fig. 6, i.e. they touch at some point $r = r_0$. We want to show that for the remaining values of r, the inequality

$$P_1(r) < P_2(r). \tag{5.56}$$

To do this, it suffices to make sure that the same inequality is true for the second derivatives for $r = r_0$. We have

$$P_1' = -Nt(1+r)^{-t-1},$$

$$P_1'' = Nt(t+1)(1+r)^{-t-2} = \frac{N}{(1+r)^2}\left(\frac{t^2}{(1+r)^t} + \frac{t}{(1+r)^t}\right) =$$

$$= \frac{P}{(1+r)^2}\left(\frac{t^2 N}{P(1+r)^t} + D_1\right). \tag{5.57}$$

Similarly

$$P_2' = -N_1 t_1 (1+r)^{-t_1-1} - N_2 t_2 (1+r)^{-t_2-1},$$

$$P_2'' = \frac{P_2}{(1+r)^2}\left(\frac{t_1^2 N_1}{P_2(1+r)^{t_1}} + \frac{t_2^2 N_2}{P_2(1+r)^{t_2}} + D_2\right). \tag{5.58}$$

Because $P_1 = P_2 = P$ and $D_1 = D_2 = D$, then it remains to verify that

$$\frac{t_1^2 N_1}{P(1+r)^{t_1}} + \frac{t_2^2 N_2}{P(1+r)^{t_2}} > \frac{t^2 N}{P(1+r)^t}. \tag{5.59}$$

Because the inequality (5.59) can be rewritten as follows:

$$w_1 = \frac{N_1}{P(1+r)^{t_1}}, w_2 = \frac{N_2}{P(1+r)^{t_2}}, w_1 + w_2 = 1,$$

$$w_1 t_1^2 + w_2 t_2^2 > t^2, \tag{5.60}$$

moreover, it follows from the equality of durations that

$$t = w_1 t_1 + w_2 t_2. \tag{5.61}$$

Substituting the expression for t in (5.60) and opening the brackets, we easily get

$$t^2 = w_1^2t_1^2 + w_2^2t_2^2 + 2w_1w_2t_1t_2$$
$$= w_1(1 - w_2)t_1^2 + w_2(1 - w_1)t_2^2 + 2w_1w_2t_1t_2 =$$
$$= w_1t_1^2 + w_2t_2^2 - w_1w_2(t_1 - t_2)^2 < w_1t_1^2 + w_2t_2^2 \quad (5.62)$$

In order to ensure the fulfillment of the conditions (5.55), it is sufficient to require that the weights of payments w_1, w_2 satisfy the system of equations

$$\begin{cases} w_1 + w_2 \\ t_1w_1 + t_2w_2 \end{cases} \quad (5.63)$$

In the considered situation, it is said that bond II immunizes bond I.

Example 5.10. Build a portfolio of three- and five-year bonds, immunizing a four-year bond with a nominal value of $3000 for an interest rate of 15%.

Let's write down the system of equations (5.63)

$$\begin{cases} w_1 + w_2 = 1 \\ 3w_1 + 5w_2 = 4 \end{cases}$$

from where we find

$$w_1 = \frac{1}{2}, w_2 = \frac{1}{2}$$

Now let's determine the current values of all bonds:

$$P = 3{,}000 \times 1.15^{-4} = 1{,}715.26;\ P_1 = \frac{1}{2}P = 857.63.$$

Next, we will find the nominal values of the bonds included in the portfolio

$$N_1 = P_1 \times 1.15^3 = 1{,}304.35;\ N_2 = P_2 \times 1.15^5 = 1{,}725.$$

Thus, the immunizing portfolio consists of a three-year bond with nominal value 1304.35 and five-year with nominal 1725. It is easy to check that, if the interest rate changes, the present value of this portfolio will be higher.

5.7. Bond portfolio

A bond portfolio consisting of bonds of various types, maturity dates, coupon yield sizes and other characteristics has its own yield, average term of receipts, duration, modified duration, convexity and other parameters that characterize the portfolio as a whole. Consider the calculation of these portfolio characteristics in accordance with

5.7.1. Bond portfolio yield

Recall how the yield of a separate bond is determined (paragraph 5.3.2). If the market price of the bond is known, its nominal value N, maturity date n and coupon rate c, then the yield to maturity r is defined as the solution of the equation

$$V = \sum_{k=1}^{n} \frac{cN}{(1+p)^k} + \frac{N}{(1+p)^n}$$

or equivalently, as the solution of the equation

$$V = cN\frac{1-(1+p)^{-n}}{p} + N(1+p)^{-n} \qquad (5.65)$$

When determining the yield of a bond portfolio, it is also found as a solution to an equation in which the sum of the given values of the total income stream is equated to the total market value of the bonds that make up the portfolio

$$\sum_{i=1}^{n} \frac{S_i}{(1+p)^i} = \sum_{k=1}^{m} q_k P_k, \qquad (5.66)$$

where *Si* — total income from bonds at a time $t = i$;
q_k —quantity of bonds of type *k*;
P_k —the price of the bond of type *k*;
M — кnumber of types of bonds in the portfolio;
n —the maximum period of payment of income.

In general, the solution of equation (5.66) with respect to the interest rate (portfolio yield) ρ. It is found by iterative methods, for example, the Newton-Raphson method, or based on linear interpolation. In the latter case, you can use the formula

$$\rho = \rho' + \frac{P' - P}{P' - P''}(\rho' - \rho''), \tag{5.67}$$

where ρ, ρ″ — the minimum and maximum values of the yield of the bond portfolio, limiting the interval, within which an unknown value of portfolio r profitability is expected,

P — market value of the portfolio;

P, P'' — estimated portfolio values when applying bids ρ, ρ″.

It is more convenient to determine the rate value not by the formula (5.67), but as the average weighted value of the profitability of the entire set of investments. Now under ρ we will mean the average weighted value of profitability. The market value of the corresponding number of bonds is taken as the weight (q_kP_k or q_kK_k):

$$\rho = \frac{\sum_k \rho_k q_k P_k}{\sum_k q_k P_k} = \frac{\sum_k \rho_k q_k K_k}{\sum_k q_k K_k}, \tag{5.68}$$

where K_k — the rate of bonds of the type k.

It is more adequate, however, to use as weights the product of the duration of each type of bond (D_k) on the value of the corresponding number of bonds:

$$\rho = \frac{\sum_k \rho_k D_k q_k P_k}{\sum_k D_k q_k P_k} = \frac{\sum_k \rho_k D_k q_k K_k}{\sum_k D_k q_k K_k}$$

5.7.2. Average term of receipt of income from the bond portfolio

The average term of receipt of the income of the bond portfolio (*T0*) as a whole is found as a weighted average. The values of the bonds are taken as weights:

$$T_0 = \frac{\sum_k T_k q_k P_k}{\sum_k q_k P_k} \tag{5.70}$$

where T_k — the average term of receipt of income of bonds of the type k.

A portfolio with a shorter average term of income, all other things being equal, has less risk than a portfolio with a longer term.

5.7.3. Duration of bond portfolio and its convexity

Duration is the average weighted duration of payments of proceeds from a bond with weights equal to discounted values of incomes.

Let's establish a relationship between the duration of the bond portfolio and the durations of individual bonds of this portfolio.

Let the portfolio consist of two bonds with income streams R_k and S_k and durations.

$$D_1 = \frac{\sum_{k=1}^{n} t_k R_k (1+y)^{-t_k}}{\sum_k D_k q_k P_k} = \frac{\sum_{k=1}^{n} t_k R_k (1+y)^{-t_k}}{P_1} \text{ and}$$

$$D_2 = \frac{\sum_{k=1}^{n} t_k R_k (1+y)^{-t_k}}{P_2}$$

For combined income flow from two bonds, we have

$$D_0 = \frac{\sum_{k=1}^{n} t_k R_k (1+y)^{-t_k} + \sum_{k=1}^{n} t_k S_k (1+y)^{-t_k}}{P_1+P_2} \tag{5.72}$$

If in the portfolio q1 and q2 bonds, then income streams increase in proportion to these values. We have

$$D_0 = \frac{q_1 \sum_{k=1}^{n} t_k R_k (1+y)^{-t_k} + q_2 \sum_{k=1}^{n} t_k S_k (1+y)^{-t_k}}{q_1 P_1 + q_2 P_2}$$

From (5.71) follows

$$D_1 P_1 = \sum_{k=1}^{n} t_k R_k (1+y)^{-t_k} \text{ and } D_2 P_2 = \sum_{k=1}^{n} t_k S_k (1+y)^{-t_k} \tag{5.73}$$

Therefore

$$D_0 = \frac{D_1 q_1 P_1 + D_2 q_2 P_2}{q_1 P_1 + q_2 P_2} \tag{5.74}$$

Thus, we conclude that the duration of the bond portfolio is equal to the average weighted duration of individual bonds of this portfolio with weights equal to the bond values.

Generalizing (5.74) to the case *m*-kinds of bonds, we will get

$$D_0 = \frac{\sum_{k=1}^{m} D_k q_k P_k}{\sum_{k=1}^{m} q_k P_k} = \sum_{k=1}^{m} D_k h_k$$

where h_k — value share of bonds of the type *k*.

The value share of bonds can be obtained based not only on bond prices, but also on their exchange rates. In this case $h_k = \frac{q_k P_k}{\sum_{k=1}^{m} q_k P_k} = \frac{q_k K_k}{\sum_{k=1}^{m} q_k K_k}$

while

$$\sum_{k=1}^{m} h_k = 1$$

Bulge of the bond portfolio (C_0), like the duration, there is an average weighted convexity of individual bonds of a given portfolio with weights equal to the bond values, and is determined by the formula.

$$C_0 = \frac{\sum_{k=1}^{m} C_k q_k P_k}{\sum_{k=1}^{m} q_k P_k} = \sum_{k=1}^{m} C_k h_k$$

Control questions and tasks

1. List and define the parameters that characterize the bond.

2. Define and give a formula for the current value of the bond.

3. Define the exchange rate (exchange rate value) of the bond. Give an example.

4. Define and give a formula for the current yield of the bond. Illustrate with an example.

5. Define and give a formula for the yield of the bond to maturity. Give an example.

6. What is the relationship between the market price of a bond and its nominal value at different ratios of yield to maturity and coupon rate?

7. Analyze the dependence of the yield to maturity of the bond on the parameters.

8. Print an approximate formula for the relative change in the price of a bond when its yield changes.

9. Prove that a decrease in the yield of a bond will lead to an increase in its market price by an amount greater than the corresponding decrease in the market price with an increase in yield by the same amount.

10. Prove that the relative change in the bond price (as a percentage) as a result of the change in the yield to maturity will be the smaller the higher the coupon rate. Define and give a formula for the average term of receipt of the bond income. Confirm this with an example.

12. Prove that if the life of the bond is more than a year, the average period of income receipt decreases with the growth of the coupon yield of the bond. Give an example.

13. Prove that increasing the multiplicity of interest payments on a bond reduces the average term of receipt of income from the bond.

14. Explain the meaning of the concept of "the average period of receipt of income from the investment".

15. Define the formula for the duration of the bond. Give an example.

16. List the duration properties for positive payment flows.

17. Prove that for a coupon–free bond (c = 0), the duration coincides with the maturity of D = n.

18. Prove that for a relative change in the bond price DV V when the yield changes by when the yield changes by Dy the approximate formula is valid

$$\frac{\Delta V}{V} = -\frac{D}{1+y}\Delta y$$

19. Prove that the duration of the bond does not depend on the nominal value and is given by the formula

$$D = \frac{1+y}{y} = -\frac{n(c-y)+1+y}{c((1+y)^n-1)+y}$$

20. Prove that the duration of the bond is a decreasing function of the yield to maturity y.

21. Prove that the duration of the bond is a decreasing function of the k–full rate c.

22. Prove that for perpetual bonds $(n \to \infty)$

$$D_\infty = \frac{1+y}{y}$$

23. Define and give a formula for the modified duration of the bond. Give an example.

24. Define the convexity of the bond and give a formula for calculating it.

25. Prove, that for relative change of the bond's price for the relative change in the bond price when the yield changes by *Dy* the approximate formula is valid (5.57).

$$\frac{\Delta V}{V} \approx \frac{D}{1+y}\Delta y + \frac{1}{2} W \times (\Delta y)^2$$

26. Define the immunization of the bond portfolio.

27. Formulate and prove the immunization theorem of the bond portfolio.

28. Define the yield of the bond portfolio.

29. What values are taken as weights when calculating the weighted average yield of a bond portfolio?

30. Define the average term of receipt of income of the bond portfolio.

31. What values are taken as weights when calculating the average term of income receipt of a bond portfolio as a weighted average value?

32. How can I deduce the relationship between the duration of a bond portfolio and the durations of individual bonds in this portfolio?

33. How to express the value share of bonds in terms of their exchange value (bond rates)?

34. Define and give a formula for the convexity of the bond portfolio.

35. What values are taken as weights when calculating the bulge of a bond portfolio as a weighted average?

36. Find the change in the price of a bond with a maturity of n = 7 years, yield y = 8, coupon rate c = 7 with a decrease in yield to 5%.

CHAPTER 6

CAPITAL STRUCTURE: MODIGLIANI – MILLER THEORY (MM THEORY)

Under the capital structure understand the ratio between own and borrowed funds of the company. Does the capital structure affect the main parameters of the company, such as the cost of capital, profit, company value and others, and if so, how? The choice of the optimal capital structure, i.e. such a capital structure that minimizes the weighted average cost of capital and maximizes the value of the company is one of the most important tasks solved by the financial manager and company management. The first serious study of the influence of the company's capital structure on its performance was the work of Modigliani and Miller (MM) [9]. Prior to the advent of this study, there was an approach (let's call it traditional), which was based on the analysis of empirical data [4-5].

Modigliani and Miller (MM) in their work [9] came to a conclusion that is fundamentally different from the conclusions of the traditional approach. They showed that, under their assumptions, the choice of the ratio between equity and debt does not affect either the value of the company or the cost of capital.

6.1. Modigliani-Miller theory without taxes

Modigliani and Miller (MM) in their first paper [6] come to conclusions which are fundamentally different from the conclusions of a traditional approach. Under the assumptions (see Sect. 6.3 for details), that there are no taxes, no transaction costs, no bankruptcy costs, perfect market exists with symmetry information, equivalence in borrowing costs for both companies and investors etc. they showed that choosing of the ratio between the debt and equity capital does not affect company value as well as capital costs (Fig. 6.1).

Under the above assumptions, Modigliani and Miller have analyzed the impact of financial leverage, supposing the absence of any taxes (on corporate profit as well as individual one). They have formulated and proven two following statements.

Without taxes the total cost of any company is determined by the value of its EBIT – Earnings Before Interest and Taxes, discounted with fix rate k_0, corresponding to group of business–risk of this company:

$$V_L = V_U = \frac{EBIT}{k_0} \,. \tag{6.1}$$

Index L means financially dependent company (using debt financing), while index U means a financially independent company.

Authors suppose that both companies belong to the same group of business-risk and k_0 corresponds to required profitability of a financially independent company, having the same business-risk.

Because, as it follows from the formula (Eq. 6.1), the value of the company does not depend on the value of debt, then according to the Modigliani-Miller theorem (Modigliani et al. 1958), in the absence of taxes, the value of the company is independent of the method of its funding. This means as well that weighted average cost of capital *WACC* of this company does not depend on its capital structure and is equal to the capital cost, which this company will have under the founding by equity capital only.

$$V_0 = V_L ; CF / k_0 = CF / WACC \;,$$

and thus $WACC = k_0$.

Note, that first Modigliani-Miller theorem is based on the suggestion about the independence of weighted average cost of capital and debt cost on leverage level.

From the first Modigliani-Miller theorem [6] it is easy to derive an expression for the equity cost

$$WACC = k_0 = k_e w_e + k_d w_d \,. \tag{6.2}$$

Finding from here k_e, one gets

$$k_e = \frac{k_0}{w_e} - k_d \frac{w_d}{w_e} = \frac{k_0(S+D)}{S} - k_d \frac{D}{S} = k_0 + (k_0 - k_d)\frac{D}{S} = k_0 + (k_0 - k_d)L \tag{6.3}$$

Here D – value of debt capital of the company;

S – value of equity capital of the company;

$k_d, w_d = \dfrac{D}{D+S}$ – cost and fraction of debt capital of the company;

$k_e, w_e = \dfrac{S}{D+S}$ – cost and fraction of equity capital of the company;

$L = D/S$ – financial leverage.

Thus we come to second statement (theorem) of Modigliani-Miller theory about the equity cost of financially dependent (leverage) company [6].

Equity cost of leverage company k_e could be found as equity cost of financially independent company k_0 of the same group of risk, plus premium for risk, which value is equal to production of difference $(k_0 - k_d)$ on leverage level L:

$$k_e = k_0 + (k_0 - k_d)L \quad (6.4)$$

Formula (Eq. 6.4) shows that equity cost of the company increases linearly with leverage level (Fig. 6.1).

The combination of these two Modigliani-Miller statements implies that the increasing of level of debt in the capital structure of the company does not lead to increased value of firms, because the benefits gained from the use of more low-cost debt capital markets will be exactly offset by an increase in risk (we are speaking about the financial risk, the risk of bankruptcy), and, therefore, by an increase in cost of equity capital firms: investors will increase the required level of profitability under increase the risk, by which a higher level of debt in the capital structure is accompanied.

In this way, the Modigliani-Miller theorem [6] argues that, in the absence of the taxes the capital structure of the company does not affect on the value of the company and on its weighted average cost of capital, WACC, and equity cost increasing linearly with the increasing financial leverage.

Explanations, given by Modigliani-Miller under receiving of their conclusions, are the following (Modigliani et al. 1958). Value of the company depends on profitability and risk only, and does not depend on the capital structure. Based on the principle of preservation of the value, they postulate that the value of the company, which is equal to the sum of its own and borrowed funds, is not changed when the ratio between its parts is changed. An important role in the justification of Modigliani-Miller statements plays an existence of an arbitral awards opportunities for the committed markets. Two identical companies, differing only by the leverage level, must have the same value. If this is not the case, the arbitration aligns business cost: investors of less cost company can invest capital in a company of more value. Selling of shares of the first company and buying of stock of the second company will continue until the value of both companies is not equalized.

Most of Modigliani and Miller's assumptions (Modigliani et al. 1958), of course, are unrealistic. Some assumptions can be removed without changing the conclusions of the model. However, assuming no costs of bankruptcy and the absence of taxes (or the presence of only corporate taxes) are crucial – the change of these assumptions alters conclusions. The last two assumptions rule out the possibility of signaling and agency costs and, thus, also constitute a critical prerequisite.

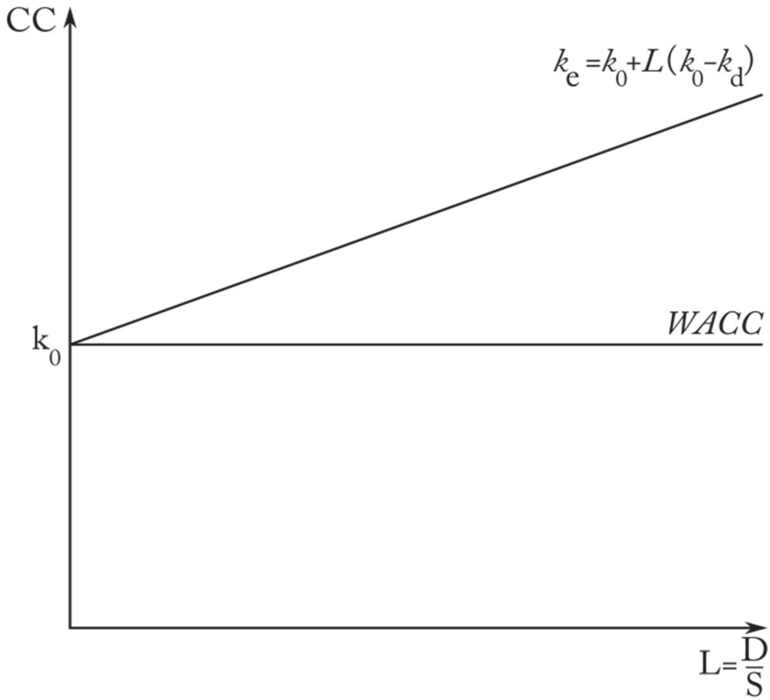

Fig. 6.2 Dependence of equity cost $\boldsymbol{k_e}$ and WACC on leverage level L within Modigliani-Miller theory without taxes (CC means capital cost)

6.2 Modigliani-Miller theory with taxes

In real situations, taxes on the profit of companies always exist. Since the interest paid on debt is excluded from the tax base, it leads to the so-called effect of "tax shield": the value of the company that used the borrowed capital (leverage company) is higher than the value of the company that is financed entirely by the equity (non–leverage company). The value of the "tax shield" for one year is equal to $k_d Dt$, where D is the value of debt, t – the income tax rate, k_d – the interest on the debt (or debt capital cost) [7]. The value of the "tax shield" for perpetuity company for all time of its existence is equal to (we used the formula for the sum of terms of an infinitely decreasing geometric progression),

$$(PV)_{TS} = k_d Dt \sum_{t=1}^{\infty} (1 + k_d)^{-t} = Dt \qquad (6.5)$$

and the cost of leverage company is equal to

$$V = V_0 + Dt, \qquad (6.6)$$

where V_0 is the value of financially independent company.

Thus, we obtain the following result obtained by Modigliani and Miller in 1963 [7].

The value of a financially dependent company is equal to the value of the company of the same risk group used no leverage, increased by the value of tax shield arising from financial leverage and equal to the product of rate of corporate income tax T and the value of debt D.

Now let us obtain the expression for the equity capital cost of the company under the existence of corporate taxes.

Accounting, that $V_0 = CF/k_0$, and that the ratio of debt capital $w_d = D/V$, one gets

$$V = CF/k_0 + w_d VT. \qquad (6.7)$$

Because the value of leverage company $V = CF/WACC$, for weighted average cost of capital, WACC, we get

$$WACC = k_0(1 - w_d T). \tag{6.8}$$

From here the dependence of $WACC$ on leverage $L = D/S$ becomes the following

$$WACC = k_0(1 - LT/(1 + L)) \tag{6.9}$$

On the other hand, on definition of the weighted average cost of capital with "tax shield" accounting we have

$$WACC = k_0 w_e + k_d w_d (1 - T). \tag{6.10}$$

Equating (Eq. 6.9) and (Eq. 6.11), one gets

$$k_0(1 - w_d T) = k_0 w_e + k_d w_d (1 - T), \tag{6.11}$$

and from here for equity cost we get the following expression

$$k_e = k_0 \frac{(1 - w_d T)}{w_e} - k_d \frac{w_d}{w_e}(1 - T) = k_0 \frac{1}{w_e} - k_0 \frac{w_d}{w_e} T - k_d \frac{D}{S}(1 - T) =$$
$$= k_0 \frac{D + S}{S} - k_0 \frac{D}{S} T - k_d \frac{D}{S}(1 - T) = k_0 + L(1 - T)(k_0 - k_d). \tag{6.12}$$

So, we get the following statement, obtained by Modigliani and Miller [7].

Equity cost of the leverage company k_e paying tax on profit could be found as equity cost of financially independent company $k_0$of the same group of risk, plus premium for risk, which value is equal to production of difference $(k_0 - k_d)$ on leverage level *L* and on tax shield (*1-T*).

It should be noted that the formula (Eq. 6.12) is different from the formula (Eq. 6.4) without tax only by the multiplier (1-T) in term, indicating a premium for risk. As the multiplier is less than unit, the corporate tax on profits leads to the fact that capital is growing with the increasing of financial leverage slower than it would have been without them.

Analysis of formulas (Eq. 6.4), (Eq. 6.9) and (Eq. 6.12) leads to following conclusions. With leverage grows:

1) value of company increases,

2) Weighted average cost of capital WACC decreases from k_0 (at $L = 0$) up to $k_0(1-T)$ (at $L = \infty$) (when the company is funded solely by borrowed funds).

3) equity cost increasing linearly from k_0 (at $L = 0$) up to ∞ (at $L = \infty$).

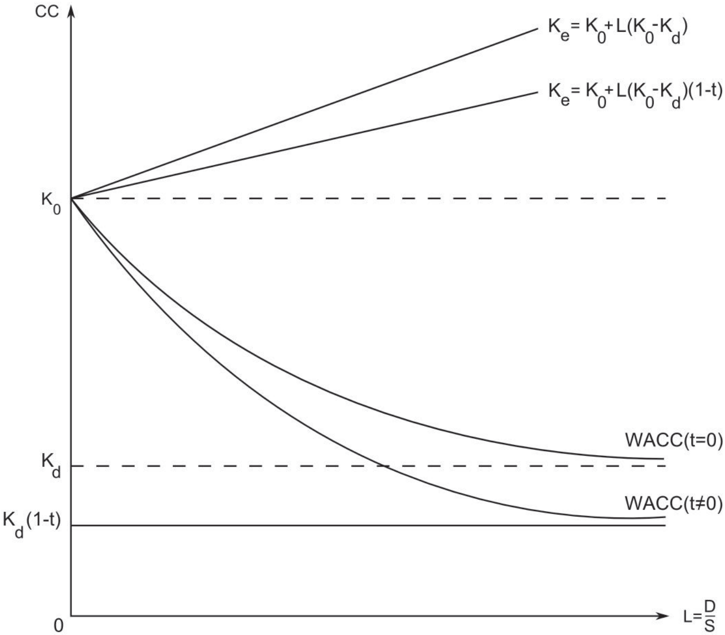

Fig. 6.3 Dependence of equity capital cost, debt cost and *WACC* on leverage in Modigliani-Miller theory without taxes ($t = 0$) and with taxes ($t \neq 0$)

Within their theory Modigliani and Miller (Modigliani et al. 1963) had come to the following conclusions. With the growth of financial leverage (Fig. 6.3):

1) the company value increases,

2) the weighted average cost of capital decreases from k_0 (for $L = 0$) up to $k_0(1 - T)$ (for $L = \infty$, when the company is financed entirely with borrowed funds).

3) the cost of equity capital increases linearly from k_0 (for $L = 0$) up to ∞ (for $L = \infty$).

6.3 Main assumptions of Modigliani-Miller theory

The most important assumptions of the Modigliani-Miller theory are as following.

1. Investors behave rationally and instantaneously see profit opportunity and inadequate investment risk. Therefore, the possibility of a stable situation of the arbitration, i.e. obtaining the risk-free profit on the difference in prices for the same asset cannot be kept any length of time – reasonable investors quickly take advantage of it for their own purposes and equalize conditions in the market. This means that in a developed financial market, the same risk should be rewarded by the same rate of return.

2. Investment and financial market opportunities should be equally accessible to all categories of investors – whether institutional or individual investors, large or small, rapidly growing or stable, experienced or relatively inexperienced.

3. Transaction costs associated with funding are very small. In practice, the magnitude of transaction costs is inversely proportional to the amount of finance involved, so this assumption is the more consistent with reality than the large sums involved: i.e. in attracting small amounts the transaction costs can be high, while, as in attracting large loans, as well as

during placement of shares at a significant amount the transaction costs can be ignored.

4. Investors get money and provide funds to borrowers at risk-free rates. In all probability, this assumption is due to the fact that the lender seeks to protect himself by using one or other guarantees, pledge of assets, the right to pay claims on third parties, the treaty provisions restricting the freedom of the borrower to act to the detriment of the creditor. Lender's risk is really small, but its position can be considered risk-free with respect to the position of the borrower and, accordingly, should be rewarded by a risk-free rate of return.

5. Companies have only two types of assets: risk-free lending and risk their own.

6. There is no possibility of bankruptcy, in other words, no matter to what level borrowing companies increase their financial leverage, they are not threatened with bankruptcy. There are no bankruptcy costs.

7. There are no corporate taxes and taxes on personal income of investors. If the personal income tax can indeed be neglected because the assets of the company are separated from the assets of shareholders, the corporate income taxes should be considered in the development of more realistic theories (which was done by Modigliani and Miller in the second paper devoted to the capital structure).

8. Companies are in the same class of risky companies.

9. All financial flows are perpetuity.

10. Companies have the same information.

11. Management of the company maximizes the equity of the company.

Example 6.1. The capitalization of the company is 15,000, the company's income for the period is 1500. Determine the cost of borrowed capital, the cost of equity at a zero level of leverage, if at a leverage level L=3 the cost of equity is 25%.

For the capitalization of the company, we have the following expressions

$$V = CF / WACC = CF / k_0$$

Find the cost of equity at zero level of leverage k0:

$$k_0 = CF / V = 1500 / 15000 = 10\% .$$

Further from the formula for the cost of equity

$$k_e = k_0 + (k_0 - k_d) L$$

find the cost of borrowed capital kd

$$25\% = 10\% + (10\% - k_d) \cdot 3 = 40\% - 3 k_d$$

Hence kd = 5%.

Example 6.2. The company's income for the period is 2500. Determine the capitalization of the company, the cost of equity at zero leverage level, if at the leverage level L=2 the cost of equity is 30%, and the cost of borrowed capital is 8%.

From the formula for the cost of equity

$$k_e = k_0 + (k_0 - k_d) L$$

find the cost of equity at zero level of leverage k0

$$30\% = k_0 + (k_0 - 8\%)\, 2 = 3 k_0 - 16\%$$

Hence k_0 =15.(3)%.

For the capitalization of the company, we have the following expressions

$$V = CF / k_0$$

We find the capitalization of the company

$$V = 2500 / 0.1533 = 16304.38$$

Tasks

6.1. The capitalization of the company is 10,000, the company's income for the period is 2000. Determine the cost of borrowed capital, the cost of equity at zero leverage level, if at the leverage level L=2 the cost of equity is 30%.

6.2. The capitalization of the company is 20,000, the company's income for the period is 4000. Determine the cost of borrowed capital, the cost of equity at zero leverage level, if at the leverage level L=3 the cost of equity is 35%.%.

6.3. The capitalization of the company is 25,000, the company's income for the period is 5,000. Determine the cost of borrowed capital, the cost of equity at a zero-leverage level, if at a leverage level L=0.5, the cost of equity is 25%.

6.4. The company's income for the period is 4000. Determine the capitalization of the company, the cost of equity at zero leverage level, if at the level of leverage L=3 the cost of equity is 35%, and the cost of borrowed capital is 10%.

6.5. The capitalization of the company is 25,000. Determine the company's income for the period, the cost of equity at zero leverage level, if at the level of leverage L=2.5 the cost of equity is 40%, and the cost of borrowed capital is 12%.

6.6. The capitalization of the company is 42,000. Determine the company's income for the period, the cost of equity at zero level of leverage, if at the level of leverage L=1.5 the cost of equity is 30%, and the cost of borrowed capital is 15%.

6.7. The company's income for the period is 3500. Determine the company's capitalization, the cost of equity at zero leverage level, if at the leverage level L=1.2 the cost of equity is 40%, and the cost of borrowed capital is 12%.

6.8. At what level of leverage will the cost of equity ke be 40% if the cost of equity at zero leverage is 20% and the cost of debt is 10%.

6.9. At what level of leverage will the cost of equity ke be 35% if the cost of equity at zero leverage is 18% and the cost of debt is 12%.

6.10. The capitalization of the company is 10000, the company’s income for the period is 2000. Determine the cost of equity at zero level of leverage, the dependence of the cost of equity on the level of leverage (L=0; 1;2;3;....10), if the cost of borrowed capital equals 11%. Provide a graph of the dependence of the cost of equity on the level of leverage.

6.11. The capitalization of the company is 20,000, the company’s income for the period is 3000. Determine the cost of equity at zero level of leverage, the dependence of the cost of equity on the level of leverage (L=0; 1;2;3;....10), if the cost of borrowed capital equals 10%. Provide a graph of the dependence of the cost of equity on the level of leverage.

CHAPTER 7

CAPITAL STRUCTURE THEORY: BRUSOV-FILATOVA-OREKHOVA THEORY (BFO THEORY)

One of the serious limitations of the Modigliani-Miller theory is the suggestion about the perpetuity of the companies. In 2008, Brusov, Filatova and Orekhova (Filatova et al. 2008) have lifted up this limitation and shown that the accounting of the finite lifetime of the company leads to significant changes of all Modigliani-Miller results (Modigliani et al. 1958, 1963, 1966): capitalization of the company is changed, as well as the equity cost, k_e, and the weighted average cost of capital, WACC, in the presence of corporative taxes. Besides a number of qualitatively new effects in corporate finance, obtained in the Brusov-Filatova-Orekhova theory [5], is absent in Modigliani-Miller theory.

Only in the absence of corporative taxes we give a rigorous proof of the Brusov-Filatova-Orekhova theory, that the cost of company equity, k_e, as well as its weighted average cost, WACC, do not depend on the lifetime or age of the company, so the Modigliani-Miller theory could be generalized for arbitrary lifetime companies.

Until recently (before 2008, when the first paper by Brusov, Filatova and Orekhova [5] appeared) the basic theory (and the first quantitative one) of the cost of capital and capital structure of companies was the theory of Nobel Prize winners Modigliani and Miller. One of the serious limitations of the Modigliani-Miller theory is the suggestion about the perpetuity of the companies. We lift up this limitation and show that the accounting of the finite lifetime of the company leads to change of the equity cost, k_e, as well as of the weighted average cost of capital WACC in the presence of corporative taxes. The effect of leverage on the cost of equity capital of the company, k_e, with an arbitrary lifetime, and its weighted average cost of WACC is investigated. We give a rigorous proof of the Brusov-Filatova-Orekhova theory, that in the absence of corporative taxes cost of company equity, k_e, as well as its weighted average cost, WACC, do not depend on the lifetime of the company.

7.1 Companies with arbitrary age: the Brusov-Filatova-Orekhova equation

Let us consider the situation with finite age companies. First of all, we will find the value of tax shields, TS , of the company for n years

$$TS = k_d DT \sum_{t=1}^{n} (1+k_d)^{-t} = DT\left[1-(1+k_d)^{-n}\right]. \tag{7.1}$$

(We used the formula for the sum of n terms of a geometric progression).

Here, D is the value of debt capital; k_d – the cost of debt capital, T – income tax rate.

Next, we use the Modigliani-Miller theory:

The value of a financially dependent company is equal to the value of the company of the same risk group using no leverage, increased by the value of tax shield arising from financial leverage and equal to the product of rate of corporate income tax T and the value of debt D.

$$V = V_0 + DT \tag{7.2}$$

This theory was formulated by Modigliani and Miller for perpetuity companies, but we modify it for a company with a finite lifetime.

$$V = V_0 + TS = V_0 + k_d DT \sum_{t=1}^{\infty} (1+k_d)^{-t} =$$
$$= V_0 + w_d VT\left[1-(1+k_d)^{-n}\right] \tag{7.3}$$

$$V\left(1-w_d VT\left[1-(1+k_d)^{-n}\right]\right)=V_0 \quad (7.4)$$

There is a common use of the following two formulas for the cost of the financially independent and financially dependent companies:

$$V_0 = CF/k_0 \text{ and } V = CF/WACC. \quad (7.5)$$

However, these almost always used formulas were derived for perpetuity company and in case of a company with a finite lifetime they must be modified in the same manner as the value of tax shields [5]

$$V_0 = CF\left[1-(1+k_0)^{-n}\right]/k_0;\ V = CF\left[1-(1+WACC)^{-n}\right]/WACC. \quad (7.6)$$

From formula (Eq. 7.4) we get the Brusov-Filatova-Orekhova equation for *WACC*

$$\frac{1-(1+WACC)^{-n}}{WACC}=\frac{1-(1+k_0)^{-n}}{k_0\left[1-\omega_d T\left(1-(1+k_d)^{-n}\right)\right]}. \quad (7.7)$$

Here, S – the value of own (equity) capital of the company, $w_d=\frac{D}{D+S}$ – the share of debt capital; $k_e, w_e=\frac{S}{D+S}$ – the cost and the share of the equity of the company,

$L = D/S$ – financial leverage.

At n=1 we get the Myers formula for a one-year company

$$WACC = k_0 - \frac{(1+k_0)k_d}{1+k_d}w_d T \quad (7.8)$$

For $n = 2$ one has

$$\frac{1-(1+WACC)^{-2}}{WACC}=\frac{1-(1+k_0)^{-2}}{k_0\left[1-\omega_d T\left(1-(1+k_d)^{-2}\right)\right]}. \tag{7.9}$$

This equation can be solved for *WACC* analytically:

$$WACC=\frac{1-2\alpha\pm\sqrt{4\alpha+1}}{2\alpha}, \tag{7.10}$$

where

$$\alpha=\frac{2+k_0}{(1+k_0)^2\left[1-\omega_d T\frac{2k_d+k_d^2}{(1+k_d)^2}\right]}. \tag{7.11}$$

For $n = 3$ and $n = 4$ equation for the WACC becomes more complicate, but it still can be solved analytically, while for n> 4 it can be solved only numerically.

We would like to make **an important methodological notice**: taking into account the finite life–time of the company, all formulas, without exception, should be received with use formulas (Eq. 7.6) instead of their perpetuity limits (Eq. 7.5).

Below, we will describe the algorithm for the numerical solution of the equation (Eq. 7.7).

Algorithm for finding of WACC in case of arbitrary company age

Let us return back to *n*-year company. We have the following equation for WACC in n–year case

$$\frac{1-(1+WACC)^{-n}}{WACC}-A(n)=0, \tag{7.12}$$

where

$$A(n) = \frac{1-(1+k_0)^{-n}}{k_0\left[1-\omega_d T\left(1-(1+k_d)^{-n}\right)\right]}. \tag{7.13}$$

The algorithm of the solving of the equation (7.12) should be as following:

1) Putting the values of parameters k_0, ω_d, T and given n, we calculate $A(n)$;

2) We determine two WACC values, for which the left part of the equation (7.12) has opposite signs. It is obviously that as these two values we can use $WACC_1$ and $WACC_\infty$, because $WACC_1 > WACC_n > WACC_\infty$ for finite $n \geq 2$.

3) Using, for example, the bisection method, we can solve the equation (7.12) numerically.

Tasks

7.1. Using Microsoft Excel, find the dependence of the weighted average cost of capital, WACC, on the level of leverage L at t=0.2, n=2, k0=0.28, kd=0.2.

7.2. Using Microsoft Excel, find the dependence of the weighted average cost of capital, WACC, on the level of leverage L at t=0.2, n=2 and 5, k0=0.28, kd=0.2. Show the resulting dependencies on one graph and analyze them.

7.3. Using Microsoft Excel, find the dependence of the weighted average cost of capital, WACC, on the level of leverage L at t=0.15 and 0.2, n=2, k0=0.26, kd=0.22. Show the resulting dependencies on one graph and analyze them.

7.4. Using Microsoft Excel, find the dependence of the weighted average cost of capital, WACC, on the level of leverage L at t=0.2, n=3, k0=0.28, kd=0.26; 0.24; 0.22; 0.2. Show the resulting dependencies on one graph and analyze them.

7.5. Using Microsoft Excel, find the dependence of the weighted average cost of capital, WACC, on the share of debt capital wd at t=0.2, n=2, k0=0.28, kd=0.2.

7.6. Using Microsoft Excel, find the dependence of the weighted average cost of capital, WACC, on the share of debt capital wd at t=0.2, n=2, k0=0.2, kd=0.14.

7.7. Find the dependence of the weighted average cost of capital, WACC, of a three-year company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 30% , the cost of borrowed capital 26%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of tables and graphs.

7.8. Find the dependence of the weighted average cost of capital, WACC, of a two-year company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 30% , the cost of borrowed capital 22%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of tables and graphs.

7.9. Find the dependence of the weighted average cost of capital, WACC, of a five-year company on the leverage level (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 18%, the cost of equity at a zero leverage level of 24%, the cost of borrowed capital 20%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of tables and graphs.

7.10. Find the dependence of the weighted average cost of capital, WACC, of a one-year company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 20% , the cost of borrowed capital 16%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of tables and graphs.

7.11. Find the dependence of the weighted average cost of capital, WACC, of a five-year company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at income tax rates of 16% and 20%, the cost of equity at zero level leverage 26%, cost of borrowed capital 22%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.12. Find the dependence of the weighted average cost of capital, WACC, of a three-year company on the leverage level (L=0; 1; 2; 3; .... 10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 26%, cost of borrowed capital 24%, 22%, 20%, 18%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.13. Find the dependence of the weighted average cost of capital, WACC, of a four-year-old company on the leverage level (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 14% , the cost of borrowed capital 12%, 10%, 8%, 6%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.14. Find the dependence of the weighted average cost of capital, WACC, of a four-year-old company on the income tax rate (t=0; 0.1; 0.2;....1) in the BFO theory at the level of leverage L=2, the cost of equity at zero level leverage 16%, cost of borrowed capital 12%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.15. Find the dependence of the weighted average cost of capital, WACC, of a four-year company on the income tax rate (t=0; 0.1; 0.2;....1) in the BFO theory at leverage levels L=2, L=4, cost of equity capital at a zero level of leverage 16%, the cost of borrowed capital 12%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of a table and a summary graph.

7.16. Find the dependence of the weighted average cost of capital, WACC, of a three-year company on the income tax rate (t=0; 0.1; 0.2;....1) in the BFO theory with leverage L=1.5, the cost of equity with zero level of leverage 16%, cost of borrowed capital 12%, 10%, 8%, 6%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of a table and a summary graph.

7.17. Find the dependence of the weighted average cost of capital, WACC, of a company on its age (n=1; 3; 5; 10; 20; 30; 50) in the BFO theory at a leverage level of L=2; loan capital 12%, income tax rate 20%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of a table and a summary graph.

7.18. Find the dependence of the weighted average cost of capital, WACC, of the company on its age (n=1; 3; 5; 10; 20; 30; 50) in the BFO theory at leverage levels L=1, 2, 3, the cost of equity at a zero-leverage level of 30%, cost of borrowed capital 22%, income tax rate 20%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of a table and a summary graph.

7.19. Find the dependence of the weighted average cost of capital, WACC, of the company on its age (n=1; 3; 5; 10; 20; 30; 50) in the BFO theory at a leverage level of L=2, the cost of equity at a zero leverage level of 30%, costs debt capital 28%, 26%, 22%, income tax rate 20%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of a table and a summary graph.

7.20. Find the dependence of the cost of equity, ke, of a three-year-old company on the leverage level (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 30% , the cost of borrowed capital 26%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of tables and graphs.

7.21. Find the dependence of the cost of equity, ke, of a two-year-old company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 28% , the cost of borrowed capital 22%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of tables and graphs.

7.22. Find the dependence of the cost of equity, ke, of a five-year-old company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 18%, the cost of equity at a zero leverage level of 24% , the cost of borrowed capital 20%.

Note: Calculations are to be carried out in Microsoft Excel using the "“Parameter Selection” option. Present the results in the form of tables and graphs.

7.23. Find the dependence of the cost of equity, ke, of a one-year-old company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 20% , the cost of borrowed capital 16%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of tables and graphs.

7.24. Find the dependence of the cost of equity, ke, of a five-year company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at income tax rates of 16% and 20%, the cost of equity at a zero leverage level of 26%, cost of borrowed capital 22%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of a table and a summary graph.

7.25. Find the dependence of the cost of equity, ke, of a three-year-old company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 26% , cost of borrowed capital 24%, 22%, 20%, 18%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of a table and a summary graph.

7.26. Find the dependence of the cost of equity, ke, of a four-year-old company on the level of leverage (L=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 14% , the cost of borrowed capital 12%, 10%, 8%, 6%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of a table and a summary graph.

7.27. Find the dependence of the cost of equity, ke, of a four-year-old company on the income tax rate (t=0; 0.1; 0.2;....1) in the BFO theory at the level of leverage L=2, the cost of equity at a zero leverage level of 16%, cost of borrowed capital 12%.

Note: Calculations are to be carried out in Microsoft Excel using the “Parameter Selection” option. Present the results in the form of a table and a summary graph.

7.28. Find the dependence of the cost of equity, ke, of a four-year-old company on the income tax rate (t=0; 0.1; 0.2;....1) in the BFO theory at leverage levels L=2, L=4, the cost of equity capital at a zero leverage level of 16%, the cost of borrowed capital 12%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.29. Find the dependence of the cost of equity, ke, of a three-year-old company on the income tax rate (t=0; 0.1; 0.2;....1) in the BFO theory with leverage L=1.5, the cost of equity with a zero leverage level of 16%, cost of borrowed capital 12%, 10%, 8%, 6%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.30. Find the dependence of the cost of equity, ke, of the company on its age (n=1; 3; 5; 10; 20; 30; 50) in the BFO theory at the level of leverage L=2; loan capital 12%, income tax rate 20%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.31. Find the dependence of the cost of equity, ke, of the company on its age (n=1; 3; 5; 10; 20; 30; 50) in the BFO theory at leverage levels L=1, 2, 3, the cost of equity at a zero leverage level of 30%, cost of borrowed capital 22%, income tax rate 20%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.32. Find the dependence of the cost of equity, ke, of the company on its age (n=1; 3; 5; 10; 20; 30; 50) in the BFO theory at the level of leverage L=2; debt capital 28%, 26%, 22%, income tax rate 20%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.33. Find the dependence of the weighted average cost of capital, WACC, of a three-year company on the share of borrowed funds (wd=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 30 %, the cost of borrowed capital 26%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of tables and graphs.

7.34. Find the dependence of the weighted average cost of capital, WACC, of a two-year company on the share of borrowed funds (wd=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at zero leverage 28%, the cost of borrowed capital 22%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of tables and graphs.

7.35. Find the dependence of the weighted average cost of capital, WACC, of a five-year company on the share of borrowed funds (wd=0; 1;2;3;....10) in the BFO theory at an income tax rate of 18%, the cost of equity at a zero leverage level of 24 %, the cost of borrowed capital 20%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of tables and graphs.

7.36. Find the dependence of the weighted average cost of capital, WACC, of a one-year company on the share of borrowed funds (wd=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 20%, the cost of borrowed capital 16%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of tables and graphs.

7.37. Find the dependence of the weighted average cost of capital, WACC, of a one-year company on the share of borrowed funds (wd=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 24%, the cost of borrowed capital 18%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of tables and graphs.

7.38. Find the dependence of the weighted average cost of capital, WACC, of a five-year company on the share of borrowed funds (wd=0; 1;2;3;....10) in the BFO theory at income tax rates of 16% and 20%, the cost of equity at a zero leverage level of 26%, cost of borrowed capital 22%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.39. Find the dependence of the weighted average cost of capital, WACC, of a three-year company on the share of borrowed funds (wd=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 26 %, cost of borrowed capital 24%, 22%, 20%, 18%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.40. Find the dependence of the weighted average cost of capital, WACC, of a four-year-old company on the share of borrowed funds (wd=0; 1;2;3;....10) in the BFO theory at an income tax rate of 20%, the cost of equity at a zero leverage level of 14 %, cost of borrowed capital 12%, 10%, 8%, 6%.

Note: Calculations are to be carried out in Microsoft Excel using the "Parameter Selection" option. Present the results in the form of a table and a summary graph.

7.41 The capitalization of the company is 20,000. Find the weighted average cost of capital, WACC, dependence of the company's capitalization, V, equity cost, ke, debt capital, kd, on the share of borrowed funds, wd.

LITERATURE FOR ADDITIONAL READING

1. *Kellison S.G.* The theory of interest. Irwin/McGraw-Hill, 1991.

2. *Alexander Solla*, Financial Mathematics: A Study Guide for Exam FM, 2015; Create Space Independent Publishing Platform;

3. *Richard G. Schroeder, Myrtle W. Clark, et al.*; Financial Accounting Theory and Analysis: Text and Cases Wiley; 14th edition 2022

4. *Marek Capiński, Tomasz Zastawniak* Mathematics for Finance: An Introduction to Financial Engineering (Springer Undergraduate Mathematics Series) 2nd ed. 2011

5. *Brusov, P.; Filatova, T.; Orehova, N.; Eskindarov, M. Modern Corporate Finance, Investments, Taxation and Ratings*, 2nd ed.; Springer Nature Publishing: Switzerland, 2018; pp. 1–571.

6. *Brusov, P.; Filatova, T.; Orehova, N.; The Brusov-Filatova-Orekhova theory of capital structure: Applications in Corporate Finance, Investments, Taxation and Ratings*, Springer Nature Publishing: Switzerland, 2023; pp. 1–723.

7. *Brusov, P.; Filatova, T.; Orehova, N.; Generalized Modigliani–Miller Theory: Applications in Corporate Finance, Investments, Taxation and Ratings*: Switzerland, 2021; pp. 1–362.

8. Modigliani, F.; Miller, M.H. The cost of capital, corporation finance and the theory of investment. *Am. Econ. Rev.* 1958, *48*, 261–296.