Parameter Estimation of Polya's Distribution

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1 Introduction

A topic model is a type of statistical model for discovering the abstract "topics" that occur in a collection of documents. Intuitively, given that a document is about a particular topic, one would expect particular words to appear in the document more or less frequently.The idea behind Topic modelling (LDA) is to model documents as arising from multiple topics, where a topic is defined to be a distribution over a fixed vocabulary words. Specifically, the assumption is k topics are associated with a collection and that each document exhibits these topics with different proportions.

In this project our goal is to estimate parameters of Poly's distribuiotn using different methods like maximum likelihood and method of moments, and compare their performance with the Cramer Rao Lower bound. In Polya distribution we have k parameters $p_1, p_2, ..., p_k$ for multinomial distribution representing probabilities for k categories. These k parameters are random and coming from Dirichlet distribution with parameters $\alpha_1, \alpha_2, ..., \alpha_k$. In this project we explore a simple case where number of categories are just two. Hence, the multinomial reduces to Binomial and Dirichlet reduces to Beta distribution.

2 Cramer Rao Lower Bound

The probability mass function for Polya distributin is:

$$p(x \mid \alpha) = \frac{n!}{\prod\limits_k n_{k!}} \frac{\Gamma(\sum\limits_k \alpha_k)}{\Gamma(n + \sum\limits_k \alpha_k)} \prod_k \frac{\Gamma(n_k + \alpha_k)}{\Gamma(\alpha_k)} \tag{1}$$

where,

the parameter vector is: $\theta = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_k \end{bmatrix}$

$n = n(x) =$ length of a data sample x

$n_k = n_k(x) =$ number of k category elements in x

Now, for m observations $x_1, x_2, ..., x_m$

$$p(x_1, x_2, ..., x_m \mid \alpha) = \prod_{i=1}^{m} \left(\frac{n_i!}{\prod_k n_{ik!}} \frac{\Gamma(\sum_k \alpha_k)}{\Gamma(n_i + \sum_k \alpha_k)} \prod_k \frac{\Gamma(n_{ik} + \alpha_k)}{\Gamma(\alpha_k)} \right) \tag{2}$$

The log likelihood is:

$$log\ p(x_1, x_2, ..., x_m \mid \alpha) \tag{3}$$

$$= \sum_{i=1}^{m} log\left(\frac{n_i!}{\prod_k n_{ik}!} \frac{\Gamma(\sum_k \alpha_k)}{\Gamma(n_i + \sum_k \alpha_k)} \prod_k \frac{\Gamma(n_{ik} + \alpha_k)}{\Gamma(\alpha_k)} \right) \tag{4}$$

$$= \sum_{i=1}^{m} \left(log(n_i!) - \sum_k log(n_{ik}!)! + log(\Gamma(\sum_k \alpha_k)) - log(\Gamma(n_i + \sum_k \alpha_k)) + \sum_k log(\Gamma(n_{ik} + \alpha_k)) - \sum_k log(\Gamma(\alpha_k)) \right) \tag{5}$$

Differentiating in terms of α_k:

$$\frac{d\ log\ p(D \mid \alpha)}{d\ \alpha_k} = \sum_{i=1}^{m} \left(\psi(\sum_k \alpha_k) - \psi(n_i + \sum_k \alpha_k) + \psi(n_{ik} + \alpha_k) - \psi(\alpha_k) \right) \tag{6}$$

$$\frac{d^2\ log\ p(D \mid \alpha)}{d\ \alpha_k^2} = \sum_{i=1}^{m} \left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) + \psi'(n_{ik} + \alpha_k) - \psi'(\alpha_k) \right) \tag{7}$$

$$\frac{d^2\ log\ p(D \mid \alpha)}{d\ \alpha_k\ d\ \alpha_j} = \sum_{i=1}^{m} \left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) \right) \tag{8}$$

Now, for beta binomial case (k = 2), we have two parameters α_1 and α_2. Hence the FIM is:

$$FIM = -E \begin{bmatrix} \frac{d^2\ log\ p(D \mid \alpha)}{d\ \alpha_1^2} & \frac{d^2\ log\ p(D \mid \alpha)}{d\ \alpha_1\ d\ \alpha_2} \\ \frac{d^2\ log\ p(D \mid \alpha)}{d\ \alpha_2\ d\ \alpha_1} & \frac{d^2\ log\ p(D \mid \alpha)}{d\ \alpha_2^2} \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} \sum_{i=1}^{m} \left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) + \psi'(n_{i1} + \alpha_1) - \psi'(\alpha_1) \right) & \sum_{i=1}^{m} \left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) \right) \\ \sum_{i=1}^{m} \left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) \right) & \sum_{i=1}^{m} \left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) + \psi'(n_{i2} + \alpha_2) - \psi'(\alpha_2) \right) \end{bmatrix} \tag{10}$$

$$FIM_{11} = -E\left[\sum_{i=1}^{m}\left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) + \psi'(n_{i1} + \alpha_1) - \psi'(\alpha_1)\right)\right] \tag{11}$$

$$= -\sum_{i=1}^{m}\left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) + E\left[\psi'(n_{i1} + \alpha_1)\right] - \psi'(\alpha_1)\right) \tag{12}$$

$$= -m * \left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) + E\left[\psi'(n_{i1} + \alpha_1)\right] - \psi'(\alpha_1)\right) \tag{13}$$

$$FIM_{12} = FIM_{21} = -E\left[\sum_{i=1}^{m}\left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k)\right)\right] \tag{14}$$

$$= -\sum_{i=1}^{m} E\left[\left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k)\right)\right] \tag{15}$$

$$= -m * \left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k)\right) \tag{16}$$

$$FIM_{22} = -E\left[\sum_{i=1}^{m}\left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) + \psi'(n_{i2} + \alpha_2) - \psi'(\alpha_2)\right)\right] \tag{17}$$

$$= -\sum_{i=1}^{m}\left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) + E\left[\psi'(n_{i2} + \alpha_2)\right] - \psi'(\alpha_2)\right) \tag{18}$$

$$= -m * \left(\psi'(\sum_k \alpha_k) - \psi'(n_i + \sum_k \alpha_k) + E\left[\psi'(n_{i2} + \alpha_2)\right] - \psi'(\alpha_2)\right) \tag{19}$$

After inverting FIM we get the $CRLB$ for α_1 and α_2 from FIM^{-1}:

$$CRLB_{\alpha_1} = (FIM^{-1})_{11} \tag{20}$$

$$CRLB_{\alpha_2} = (FIM^{-1})_{22} \tag{21}$$

3 Maximum Likelihood Estimation

To find the MLE of the parameters we start by taking log likelihood of the equation 1.

$$log\ p(x_1, x_2, ..., x_m \mid \alpha)$$

$$= \sum_{i=1}^{m}\left(log(n_i!) - \sum_k log(n_{ik}!)! + log(\Gamma(\sum_k \alpha_k)) - log(\Gamma(n_i + \sum_k \alpha_k)) + \sum_k log(\Gamma(n_{ik} + \alpha_k)) - \sum_k log(\Gamma(\alpha_k))\right) \tag{22}$$

Then from this α_k can be found using iterative method. One method suggested in [1] is using fixed point iteration. The idea is to guess an initial α_k, find a function that bounds F from below which is tight at α_k , then optimize this function to arrive at α_k^{new} which converges the function.
In [1], Minka come up with the final fixed point iteration using the following bounds. First equation 22 can be bounded using the following bounds [2]:

$$log\Gamma(z) - log\Gamma(z+n) \geq log\Gamma(\hat{z}) - log\Gamma(\hat{z}+n) + [\Psi(\hat{z}+n) - \Psi(\hat{z})](\hat{z} - z) \tag{23}$$

$$log\Gamma(z+n) - log\Gamma(z) \geq log\Gamma(\hat{z}+n) - log\Gamma(\hat{z}) + \hat{z}[\Psi(\hat{z}+n) - \Psi(\hat{z})](logz - log\hat{z}) \tag{24}$$

Then substituting equation 23 and 24 in equation 22 simplified and differentiating with α_k gives.

$$\frac{d\ log\ p(D \mid \alpha)}{d\ \alpha_k} = \sum_{i=1}^{m} \left(\frac{\alpha_k \psi(\sum_k \alpha_k) - \psi(n_i + \sum_k \alpha_k)}{\alpha_k^{new}} + \Psi(n_{ik} + \alpha_k) - \psi(\alpha_k) \right) \tag{25}$$

Finally, equation 25 can be set to zero to solve α_k^{new}

$$\alpha_k^{new} = \alpha_k \frac{\sum\limits_{i=1}^{m} \Psi(n_{ik} + \alpha_k) - \Psi(\alpha_k)}{\sum\limits_{i} \Psi(n_i + \sum\limits_{k} \alpha_k) - \Psi(\sum\limits_{k} \alpha_k)} \tag{26}$$

We can also simplify equation 26 using the following gamma simplifications.

$$\Psi(n + x) - \Psi(x) = \frac{d}{dx}(log \frac{\Gamma(n + x)}{\Gamma(x)}) \tag{27}$$

$$= \frac{d}{dx}(\sum_{i=0}^{n-1}(log(x + i)) \tag{28}$$

$$= \sum_{i=0}^{n-1} \frac{1}{(x + i)} \tag{29}$$

Then using the above simplification equation 26 can be reduced to:

$$\alpha_k^{new} = \alpha_k \frac{\sum\limits_{i=1}^{m} \sum\limits_{j=0}^{(n_{ik}-1)} \frac{1}{\alpha_k + j}}{\sum\limits_{i=1}^{m} \sum\limits_{j=0}^{(n_i-1)} \frac{1}{\sum\limits_{k} \alpha_k + j}} \tag{30}$$

$$\tag{31}$$

4 Method of Moments

In method of moment parameter estimation we usually relate population moments with sample moments and then solve for unknown parameters. We estimate the Beta Binomial parameters in the same way. We know for Beta-Binomial:

$$E[X] = \frac{n\alpha}{\alpha + \beta} \tag{32}$$

$$E[X^2] = \frac{n\alpha(n + n\alpha + \beta)}{(\alpha + \beta)(1 + \alpha + \beta)} \tag{33}$$

Now, the first and second order moments from the data:

$$m_1 = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{34}$$

$$m_2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \tag{35}$$

Equating first and second order moments with sample moments:

$$m_1 = \frac{n\alpha}{\alpha + \beta} \tag{36}$$

$$m_2 = \frac{n\alpha(n + n\alpha + \beta)}{(\alpha + \beta)(1 + \alpha + \beta)} \tag{37}$$

From 36 we have:

$$\beta = \frac{\alpha(n - m_1)}{m_1} \tag{38}$$

Dividing 37 by 36:

$$\frac{m_2}{m_1} = \frac{n + n\alpha + \beta}{(1 + \alpha + \beta)} \tag{39}$$

Replacing β in 39 from 38 we have:

$$\frac{m_2}{m_1} = \frac{nm_1 + nm_1\alpha + n\alpha - m_1\alpha}{m_1 + m_1\alpha + n\alpha - \alpha m_1} \tag{40}$$

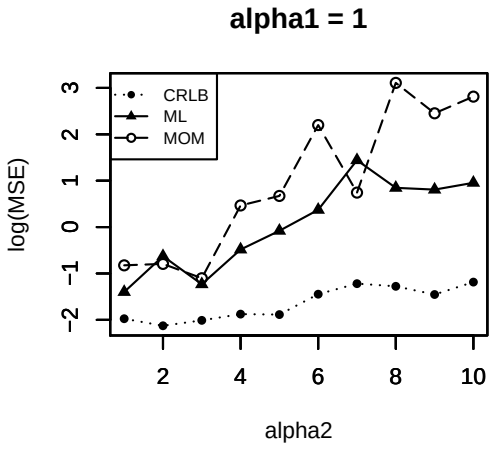
Solving for α:

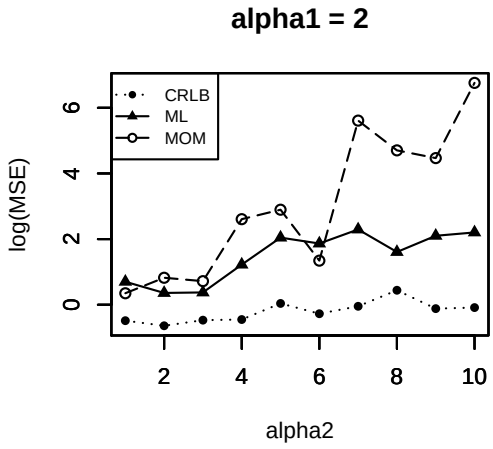
$$\alpha = \alpha_1 = \frac{nm_1 - m_2}{n(\frac{m_2}{m_1} - m_1 - 1) + m_1} \tag{41}$$

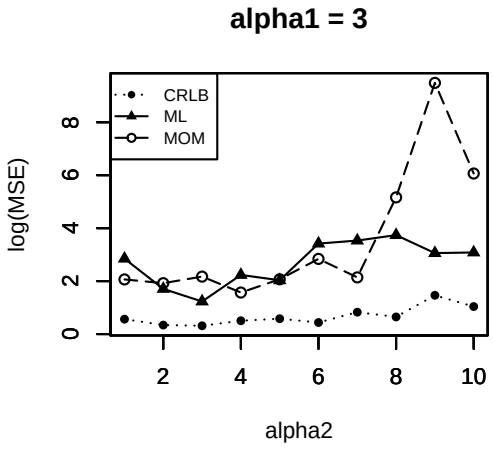
Replacing the value of α in 38 we get:

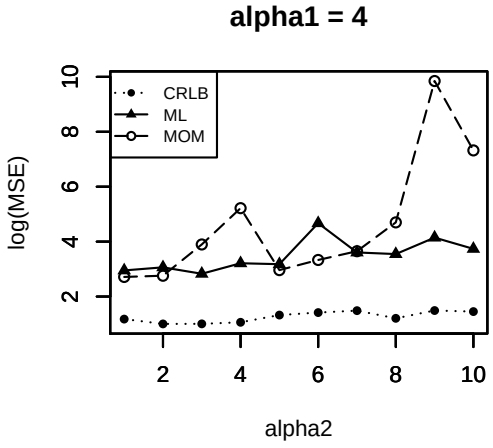
$$\beta = \alpha_2 = \frac{(n - m_1)(n - \frac{m_2}{m_1})}{n(\frac{m_2}{m_1} - m_1 - 1) + m_1} \tag{42}$$

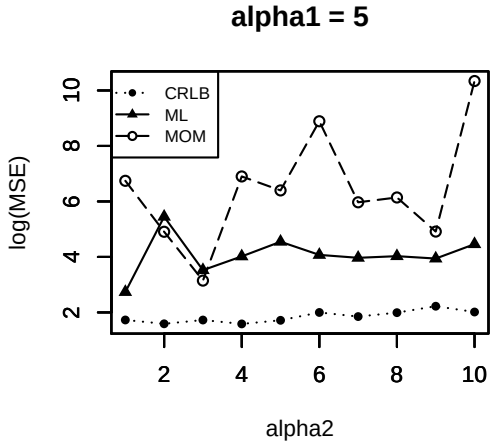
5 Experimental Result

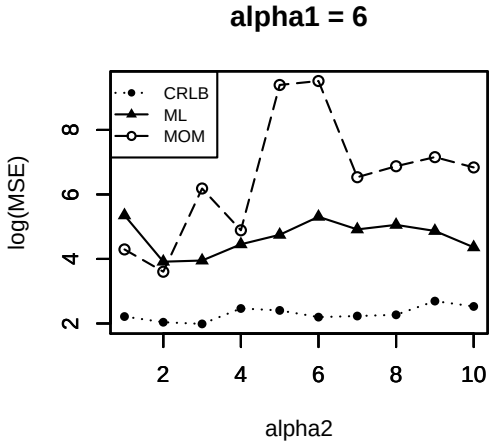

alpha1 = 1
CRLB
ML
MOM
log(MSE)
alpha2

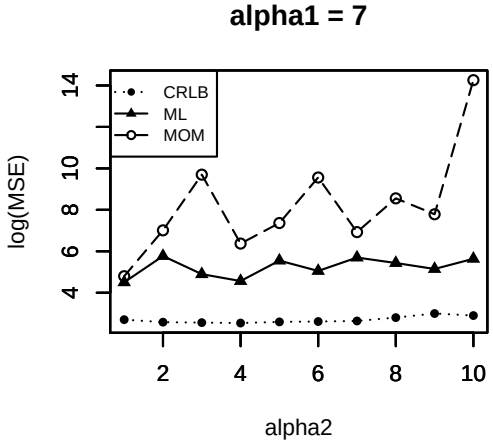

alpha1 = 2
CRLB
ML
MOM
log(MSE)
alpha2

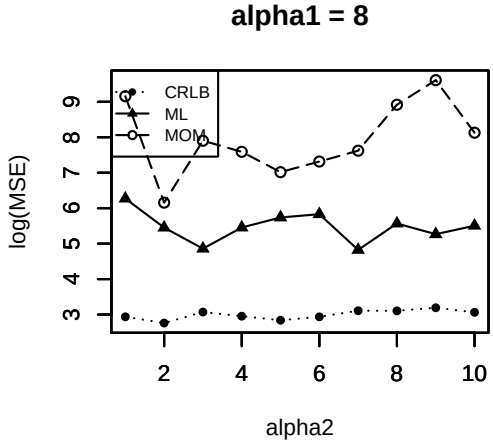

alpha1 = 3
CRLB
ML
MOM
log(MSE)
alpha2

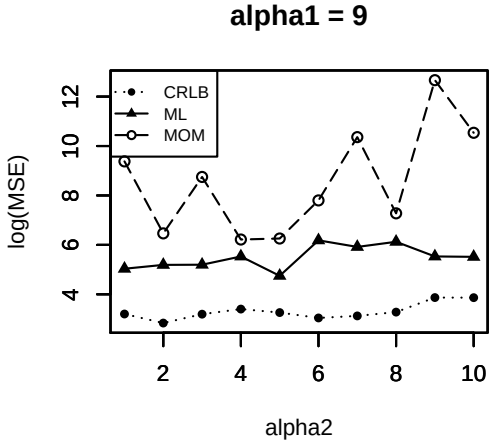

alpha1 = 4
CRLB
ML
MOM
log(MSE)
alpha2

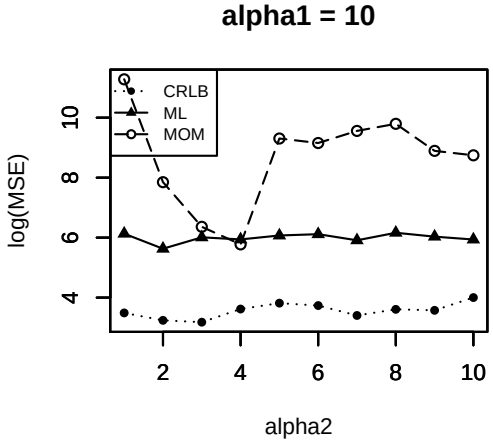

alpha1 = 5
CRLB
ML
MOM
log(MSE)
alpha2


alpha1 = 6
CRLB
ML
MOM
log(MSE)
alpha2


alpha1 = 7
CRLB
ML
MOM
log(MSE)
alpha2


alpha1 = 8
CRLB
ML
MOM
log(MSE)
alpha2


alpha1 = 9
CRLB
ML
MOM
log(MSE)
alpha2


alpha1 = 10
CRLB
ML
MOM
log(MSE)
alpha2

The graphs compare CRLB and the MSE of the two methods discussed above in logarithmic scale. In our experiment, we took 10 by 10 grid to generate the data sample from Beta binomial distribution. For each generated data we estimate the parameters for the beta binomial. These plots compare CRLB and MSE of ML and MOM for a given α_1 to the range of α_2. The x-axis shows the range of α_2 corresponding to the MSE on the y-axis.

The result shows CRLB has the smallest value compared to all estimation methods for all experiments. Our empirical result agrees to the analytical explanation of CRLB, which claims CRLB is the lowest bound for MSE of any estimator. Furthermore, the lower bound variance for the examples generated from larger α_k parameters tends to have bigger value compared to data generated from smaller parameters.

In the other hand, MLE achieves lower MSE than MOM for most examples generated. However, in few experiments there exist a situation where MOM beats MLE. For example, we can see some points in the plots of α_1 less than 6 where MOM has smaller MSE value than MLE. Analytically, we expect MLE to outperform MOM for large sample size as MLE is asymptotically efficient. However, in the smaller sample size dataset there could be a situation where MOM could beat MLE . The result of some plot in our experiment shows MOM could also achieve better estimation than MLE. This is because our experiments is based on small sample size (i.e. m=20 document size).

References

[1] Thomas P. Minka. Estimating a Dirichlet distribution. 2012.

[2] B.N. Guo and F. Qi, Inequalities and Monotonicity for the Ratio of Gamma Functions, Taiwanese Journal of Mathematics, Vol 19, No. 7. pp. 407-409. (1976).