ANSWERS

In addition to the answer, we have provided a difficulty rating for each problem. Our scale is 1-7, with 7 being the most difficult. These are only approximations, and how difficult a problem is for a particular student will vary. Below is a general guide to the ratings:

Difficulty 1/2/3 - One concept; one- to two-step solution; appropriate for students just starting the middle school curriculum.
4/5 - One or two concepts; multistep solution; knowledge of some middle school topics is necessary.
6/7 - Multiple and/or advanced concepts; multistep solution; knowledge of advanced middle school topics and/or problem-solving strategies is necessary.

Probability Stretch

Answer		Difficulty			
1.	15.38	(2)	6.	50	(4)
2.	1/4	(3)	7.	20	(2)
3.	0.2	(4)	8.	9/16	(4)
4.	2/5	(3)	9.	1/5	(3)
5.	3/8	(3)	10.	0.17	(4)

Patterns Stretch

Answer		Difficulty			
11.	1480	(4)	16.	2601	(3)
12.	423	(3)	17.	2	(4)
13.	1457	(3)	18.	4/3	(4)
14.	36	(4)	19.	125	(4)
15.	1365	(3)	20.	243	(5)

Travel Stretch

Answer		Difficulty			
21.	$2\frac{2}{3}$	(5)	26.	3	(3)
22.	2:35	(4)	27.	$7\frac{1}{2}$	(5)
23.	880	(3)	28.	$1\frac{1}{4}$	(3)
24.	3	(4)	29.	5	(4)
25.	2	(4)	30.	5/8	(3)

Warm-Up 1

Answer		Difficulty			
31.	−19	(1)	36.	18	(2)
32.	14	(2)	37.	1/2	(2)
33.	2112	(2)	38.	0	(3)
34.	1/221	(3)	39.	452	(4)
35.	$60\sqrt{(14)}$	(3)	40.	47	(4)

Warm-Up 2

Answer		Difficulty			
41.	5	(2)	46.	5/18	(3)
42.	32	(4)	47.	9/2	(4)
43.	81	(4)	48.	25	(3)
44.	600	(2)	49.	16	(3)
45.	4	(3)	50.	4π	(4)

Warm-Up 3

Answer		Difficulty			
51.	−1009	(3)	56.	13/18	(3)
52.	12	(3)	57.	25	(4)
53.	6/25	(5)	58.	348	(3)
54.	10,227	(3)	59.	1,000,000	(3)
55.	$2\sqrt{5}$	(4)	60.	$4\frac{1}{20}$	(3)

** The plural form of the units is always provided in the answer blank, even if the answer appears to require the singular form of the units.*

Warm-Up 4

Answer	Difficulty				
61.	1/2	(3)	66.	3π	(4)
62.	900	(4)	67.	432	(3)
63.	50	(3)	68.	4	(4)
64.	21	(4)	69.	27	(3)
65.	191	(4)	70.	48	(5)

Warm-Up 5

Answer	Difficulty				
71.	17	(3)	76.	1/3	(2)
72.	6	(2)	77.	15	(4)
73.	120	(5)	78.	63	(3)
74.	9	(3)	79.	4	(3)
75.	4	(3)	80.	1	(5)

Warm-Up 6

Answer	Difficulty				
81.	1320	(2)	86.	3/8	(4)
82.	128	(4)	87.	100	(2)
83.	5	(3)	88.	80	(2)
84.	3.81	(2)	89.	315	(4)
85.	126	(5)	90.	1/4	(3)

Warm-Up 7

Answer	Difficulty				
91.	12	(2)	96.	$\sqrt{3}/4$	(5)
92.	160	(4)	97.	9	(4)
93.	$5\sqrt{2}$	(5)	98.	11	(4)
94.	161	(3)	99.	3	(4)
95.	7/36	(3)	100.	1000	(3)

Warm-Up 8

Answer	Difficulty				
101.	68	(3)	106.	11/18	(4)
102.	40	(3)	107.	$\sqrt{2} - 1$	(5)
103.	144	(4)	108.	4	(5)
104.	1/7	(3)	109.	173	(5)
105.	81	(3)	110.	1/4	(5)

Warm-Up 9

Answer	Difficulty				
111.	79/111	(2)	116.	180	(3)
112.	8/11	(5)	117.	6	(4)
113.	17/24	(5)	118.	36	(7)
114.	$100\sqrt{10}$	(4)	119.	85	(2)
115.	305	(4)	120.	148	(4)

Warm-Up 10

Answer	Difficulty		
121. 23	(3)	126. $6\sqrt{7}$	(6)
122. 100	(3)	127. 9	(4)
123. 3/8	(5)	128. 105	(5)
124. 5	(3)	129. $6/\pi$	(6)
125. 6720	(5)	130. 30	(4)

Warm-Up 11

Answer	Difficulty		
131. 20	(4)	136. 132	(6)
132. 76	(4)	137. 350	(4)
133. 3/8	(5)	138. 28	(4)
134. 25	(4)	139. 18	(3)
135. 175/256	(5)	140. 8/27	(5)

Warm-Up 12

Answer	Difficulty		
141. 867	(3)	146. 81	(5)
142. 58	(5)	147. 58	(4)
143. 241	(5)	148. 12	(3)
144. 300	(3)	149. $10+2\sqrt{35}$ or $2\sqrt{35}+10$	(6)
145. 0	(3)	150. 8	(5)

Warm-Up 13

Answer	Difficulty		
151. 5	(4)	156. 9490	(5)
152. $18\sqrt{3}$	(3)	157. 13	(4)
153. 53	(3)	158. 105	(5)
154. 91	(4)	159. 17	(5)
155. 3	(4)	160. 25	(4)

Warm-Up 14

Answer	Difficulty		
161. 30	(4)	166. $(11\sqrt{10})/8$	(5)
162. 52	(2)	167. 45	(6)
163. 1/9	(5)	168. 140	(4)
164. 1632	(4)	169. 7	(5)
165. 3/5	(5)	170. 1/27	(5)

Workout 1

Answer	Difficulty		
171. 3/8	(2)	176. 33	(3)
172. 2.8	(4)	177. 14	(3)
173. 263	(5)	178. 144	(4)
174. 135	(3)	179. 5000 or 5000.00	(3)
175. 30.4	(2)	180. 87,672	(3)

Workout 2

Answer	Difficulty		
181. 100	(3)	186. 383	(2)
182. 2.5	(5)	187. 250	(5)
183. 62.5	(5)	188. 0.162	(2)
184. 55.71	(2)	189. 3	(4)
185. 33	(4)	190. 67.5	(4)

Workout 3

Answer	Difficulty		
191. 1530	(3)	196. 70	(3)
192. 45	(3)	197. 150π	(3)
193. 34	(3)	198. 127,750,000	(2)
194. 44	(4)	199. 26	(4)
195. 14	(3)	200. 70	(3)

Workout 4

Answer	Difficulty		
201. 75	(2)	206. 1/3	(4)
202. 1.01	(3)	207. 32/45	(3)
203. 28	(3)	208. 30	(2)
204. 0.924	(5)	209. 1300	(3)
205. 35.7	(3)	210. 2	(4)

Workout 5

Answer	Difficulty		
211. 3375	(5)	216. 3	(3)
212. 57	(4)	217. 20 or 20.00	(3)
213. 1,036,800	(5)	218. 4	(4)
214. 11	(5)	219. 50	(2)
215. $64 + 128\sqrt{2}$ or $128\sqrt{2} + 64$	(5)	220. 4/7	(4)

Workout 6

Answer	Difficulty		
221. 37.9	(5)	226. 52	(5)
222. 25	(2)	227. 2996	(3)
223. 7.6	(4)	228. 6	(3)
224. 28	(3)	229. 11	(4)
225. 1.47	(5)	230. 360	(4)

Workout 7

Answer	Difficulty		
231. 45	(4)	236. 78	(5)
232. 4	(6)	237. 128	(5)
233. Thursday	(4)	238. −1516.75	(4)
234. 8/51	(3)	239. 225 or 225.00	(3)
235. 72	(5)	240. 24π	(5)

Workout 8

Answer	Difficulty		
241. 144	(4)	246. 11	(5)
242. 816	(5)	247. 26	(5)
243. 1729	(6)	248. 195	(6)
244. 5	(4)	249. $3\sqrt{7}$	(6)
245. 4.7	(6)	250. 87	(6)

MATHCOUNTS Problems Mapped to Common Core State Standards (CCSS)

Forty-two states, the District of Columbia, four territories and the Department of Defense Education Activity (DoDEA) have voluntarily adopted the Common Core State Standards (CCSS). As such, MATHCOUNTS considers it beneficial for teachers to see the connections between the *2017-2018 MATHCOUNTS School Handbook* problems and the CCSS. MATHCOUNTS not only has identified a general topic and assigned a difficulty level for each problem but also has provided a CCSS code in the Problem Index (pages 54-55). A complete list of the Common Core State Standards can be found at www.corestandards.org.

The CCSS for mathematics cover K-8 and high school courses. MATHCOUNTS problems are written to align with the NCTM Standards for Grades 6-8. As one would expect, there is great overlap between the two sets of standards. MATHCOUNTS also recognizes that in many school districts, algebra and geometry are taught in middle school, so some MATHCOUNTS problems also require skills taught in those courses.

In referring to the CCSS, the Problem Index code for each of the Standards for Mathematical Content for grades K-8 begins with the grade level. For the Standards for Mathematical Content for high school courses (such as algebra or geometry), each code begins with a letter to indicate the course name. The second part of each code indicates the domain within the grade level or course. Finally, the number of the individual standard within that domain follows. Here are two examples:

- *6.RP.3 → Standard #3 in the Ratios and Proportional Relationships domain of grade 6*
- *G-SRT.6 → Standard #6 in the Similarity, Right Triangles and Trigonometry domain of Geometry*

Some math concepts utilized in MATHCOUNTS problems are not specifically mentioned in the CCSS. Two examples are the Fundamental Counting Principle (FCP) and special right triangles. In cases like these, if a related standard could be identified, a code for that standard was used. For example, problems using the FCP were coded 7.SP.8, S-CP.8 or S-CP.9 depending on the context of the problem; SP → Statistics and Probability (the domain), S → Statistics and Probability (the course) and CP → Conditional Probability and the Rules of Probability. Problems based on special right triangles were given the code G-SRT.5 or G-SRT.6, explained above.

There are some MATHCOUNTS problems that either are based on math concepts outside the scope of the CCSS or based on concepts in the standards for grades K-5 but are obviously more difficult than a grade K-5 problem. When appropriate, these problems were given the code SMP for Standards for Mathematical Practice. The CCSS include the Standards for Mathematical Practice along with the Standards for Mathematical Content. The SMPs are (1) Make sense of problems and persevere in solving them; (2) Reason abstractly and quantitatively; (3) Construct viable arguments and critique the reasoning of others; (4) Model with mathematics; (5) Use appropriate tools strategically; (6) Attend to precision; (7) Look for and make use of structure and (8) Look for and express regularity in repeated reasoning.

PROBLEM INDEX

It is difficult to categorize many of the problems in the *MATHCOUNTS School Handbook*. It is very common for a MATHCOUNTS problem to straddle multiple categories and cover several concepts. This index is intended to be a helpful resource, but since each problem has been placed in exactly one category and mapped to exactly one Common Core State Standard (CCSS), the index is not perfect. In this index, the code **9 (3) 7.SP.3** refers to problem 9 with difficulty rating 3 mapped to CCSS 7.SP.3. For an explanation of the difficulty ratings refer to page 49. For an explanation of the CCSS codes refer to page 53.

Number Theory

45	(3)	4.OA.4
56	(3)	4.OA.4
59	(3)	SMP
68	(4)	N-RN.1
69	(3)	S-CP.9
78	(3)	SMP
84	(2)	SMP
109	(5)	SMP
110	(5)	8.EE.2
111	(2)	7.NS.2
119	(2)	4.OA.4
121	(3)	6.NS.4
130	(4)	SMP
134	(4)	6.NS.4
139	(3)	SMP
148	(3)	7.NS.3
151	(4)	SMP
153	(3)	6.EE.2
156	(5)	6.NS.4
161	(4)	F-BF.2
193	(3)	SMP
194	(4)	SMP
201	(2)	7.NS.3
227	(3)	8.EE.2
229	(4)	6.NS.4
231	(4)	6.NS.4
233	(4)	SMP
237	(5)	6.NS.4
241	(4)	N-RN.2
246	(5)	SMP

Logic

83	(3)	S-CP.9
92	(4)	SMP
131	(4)	SMP
145	(3)	SMP
167	(6)	SMP
216	(3)	S-CP.9
235	(5)	SMP

Plane Geometry

32	(2)	4.G.2
43	(4)	7.G.6
53	(5)	7.G.4
61	(3)	7.G.6
66	(4)	G-C.2
82	(4)	7.G.6
106	(4)	7.G.4
107	(5)	7.G.6
113	(5)	8.G.8
118	(7)	G-SRT.4
123	(5)	G-SRT.6
133	(5)	7.G.6
136	(6)	G-SRT.5
149	(6)	A-REI.4
152	(3)	G-SRT.6
158	(5)	G-SRT.6
160	(4)	G-C.2
163	(5)	G-SRT.5
170	(5)	G-SRT.5
178	(4)	8.G.5
187	(5)	G-SRT.6
192	(3)	8.G.5
197	(3)	G-C.2
204	(5)	7.G.1
211	(5)	G-SRT.5
215	(5)	G-SRT.6
221	(5)	G-SRT.6
225	(5)	G-SRT.6
232	(6)	G-C.2
236	(5)	G-C.2
240	(5)	7.G.6
245	(6)	G-SRT.6
250	(6)	G-SRT.6

Measurement

40	(4)	8.G.5
49	(3)	5.MD.1
50	(4)	G-SRT.5
55	(4)	8.G.8
67	(3)	6.G.1
96	(5)	7.G.6
126	(6)	8.G.8
132	(4)	8.G.5
143	(5)	8.G.7
155	(4)	7.RP.3
172	(4)	6.RP.3
181	(3)	6.EE.7
183	(5)	8.G.7
198	(2)	6.RP.3

Solid Geometry

60	(3)	7.G.6
74	(3)	G-GMD.3
129	(6)	G-GMD.3
177	(3)	8.G.9
190	(4)	8.G.9
208	(2)	8.G.9
219	(2)	7.G.6
230	(4)	G-GMD.3

Coordinate Geometry

79	(3)	G-C.2
93	(5)	8.G.8
112	(5)	8.F.3
166	(5)	8.F.3
203	(3)	6.G.1
238	(4)	8.G.1
249	(6)	8.G.8

Algebraic Expressions & Equations

38	(3)	6.EE.2
42	(4)	6.EE.9
47	(4)	8.F.3
51	(3)	A-REI.4
52	(3)	6.EE.9
71	(3)	6.EE.7
73	(5)	A-REI.4
80	(5)	8.EE.8
81	(2)	6.EE.3
89	(4)	N-RN.1
91	(2)	7.EE.4
99	(4)	N-RN.1
101	(3)	A-CED.1
120	(4)	N-RN.1
142	(5)	N-RN.1
150	(5)	8.EE.8
176	(3)	6.EE.2
185	(4)	6.EE.7
199	(4)	A-SSE.3
202	(3)	A-REI.4
209	(3)	A-CED.1
210	(4)	A-SSE.3
217	(3)	7.EE.4
218	(4)	A-REI.4
224	(3)	A-CED.1

Statistics

57	(4)	6.SP.5
65	(4)	6.SP.5
87	(2)	6.SP.5
98	(4)	6.SP.5
105	(3)	6.SP.5
138	(4)	SMP
147	(4)	7.EE.4
175	(2)	6.SP.2
186	(2)	6.SP.2
223	(4)	6.SP.5

Problem Solving (Misc.)

54	(3)	SMP
58	(3)	4.OA.4
70	(5)	S-CP.9
72	(2)	SMP
128	(5)	SMP
157	(4)	G-CO.3
162	(2)	A-CED.1
180	(3)	7.NS.3
212	(4)	SMP

General Math

31	(1)	3.OA.8
33	(2)	SMP
35	(3)	6.EE.2
36	(2)	4.OA.2
37	(2)	3.OA.8
39	(4)	SMP
44	(2)	5.OA.1
62	(4)	SMP
88	(2)	7.NS.3
100	(3)	N-RN.2
114	(4)	N-RN.2
124	(3)	6.NS.4
174	(3)	6.NS.3

Probability, Counting & Combinatorics

Probability Stretch[1]

34	(3)	7.SP.8
46	(3)	7.SP.8
64	(4)	S-CP.9
75	(3)	S-CP.9
76	(2)	7.SP.8
85	(5)	S-CP.9
95	(3)	7.SP.8
103	(4)	8.EE.1
104	(3)	7.SP.7
116	(3)	S-CP.9
125	(5)	S-CP.9
135	(5)	7.SP.8
140	(5)	7.SP.7
144	(3)	S-CP.9
146	(5)	S-CP.9
154	(4)	S-CP.9
159	(5)	SMP
168	(4)	S-CP.9
171	(2)	7.SP.7
189	(4)	S-CP.9
196	(3)	7.SP.8
206	(4)	7.SP.8
207	(3)	7.SP.7
213	(5)	S-CP.9
226	(5)	SMP
234	(3)	7.SP.8
242	(5)	S-CP.9
247	(5)	S-CP.9
248	(6)	8.G.7

[1] CCSS 7.SP.8 & S-CP.9

Percents & Fractions

48	(3)	6.RP.3
77	(4)	7.RP.3
86	(4)	7.NS.3
137	(4)	A-CED.1
179	(3)	7.RP.3
184	(2)	6.RP.3
188	(2)	6.NS.1
195	(3)	6.NS.1
200	(3)	A-CED.1
205	(3)	7.RP.3
222	(2)	7.RP.3
239	(3)	7.NS.3

Proportional Reasoning

21	(5)	6.RP.3
22	(4)	6.EE.7
23	(3)	6.RP.3
24	(4)	6.EE.7
25	(4)	6.RP.3
26	(3)	6.RP.3
27	(5)	6.RP.3
28	(3)	6.RP.3
29	(4)	6.RP.3
30	(3)	6.RP.3
41	(2)	5.MD.1
63	(3)	4.OA.2
90	(3)	7.RP.2
102	(3)	6.RP.3
108	(5)	6.RP.3
117	(4)	6.RP.3
122	(3)	7.RP.3
127	(4)	7.RP.3
169	(5)	SMP
173	(5)	6.RP.3
182	(5)	7.RP.2
220	(4)	7.EE.4
228	(3)	7.RP.2
244	(4)	SMP

Sequences, Series & Patterns

Patterns Stretch[2]

94	(3)	F-BF.2
97	(4)	F-BF.2
115	(4)	F-BF.2
141	(3)	F-LE.1
164	(4)	F-LE.2
165	(5)	F-LE.2
191	(3)	F-LE.2
214	(5)	F-LE.2
243	(6)	F-BF.2

[2] CCSS F-BF.2

SOLUTIONS

The solutions provided here are only *possible* solutions. It is very likely that you or your students will come up with additional—and perhaps more elegant—solutions. Happy solving!

Probability Stretch

1. In a standard deck of 52 playing cards, the red number cards greater than 6 are the 7, 8, 9 and 10 in the suits of diamonds and hearts. That's a total of 8 cards. The percent probability that Perta randomly selects one of these 8 cards, then, is $8/52 \approx$ **15.38**%.

2. The table shows all the ways to make 45 cents from nickels, dimes and quarters. Only two of the cups contain three or more dimes. Therefore, the probability that Max randomly selects one of these cups is 2/8 = **1/4**.

quarters	0					1		
dimes	**4**	**3**	2	1	0	2	1	0
nickels	1	3	5	7	9	0	2	4

3. Danya can get a total of 10 with two or three chips if the first two chips drawn are 4 and 6, which can occur in 2 ways, or if the first three chips drawn are 2, 3 and 5, which can occur in 6 ways. There are $5 \times 4 = 20$ ways to randomly select two chips, and there are $5 \times 4 \times 3 = 60$ ways to randomly select three chips. The probability that Danya's total will equal 10 at some point is 2/20 + 6/60 = 1/10 + 1/10 = 2/10 = **0.2**.

4. The probability of pulling out two green socks is $2/5 \times 1/4 = 1/10$. The probability of pulling out two blue socks is $3/5 \times 2/4 = 3/10$. The probability, therefore, of randomly pulling out a matching pair of socks is 1/10 + 3/10 = 4/10 = **2/5**.

5. The probability that the nickel comes up heads is 1/2. The probability that one or more of the other two coins comes up heads is 3/4. The probability that at least two heads come up, with one of them being the nickel, is $1/2 \times 3/4$ = **3/8**.

6. After that circuit is turned on, lights A and B will blink together every 55 seconds. Lindsey sees light A blink alone. Because light B blinks once every 11 seconds, and it did not blink this time, it will blink in one of the 10 following seconds, with equal likelihood. In 5 of those cases, it will blink before or at the same time as light A; and in the other 5 cases, light A will blink alone first. Therefore the probability that the next light to blink will be light A blinking alone is 5/10 = 0.50 = **50**%. *Alternative solution:* Whether or not light B blinks with A, A will always blink alone twice before the next time light B blinks. When she sees light A blink alone, it could be either the first or the second time, so the probability that it's the first time is **50**%.

7. Multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, … . It appears that every fifth multiple of 3 is also a multiple of 5. The percent probability that a randomly selected multiple of 3 is also a multiple of 5 is 1/5 = **20**%.

8. Starting at the top of the figure and at each subsequent junction, there is a 1/2 probability of choosing a path to the right and a 1/2 probability of choosing a path to the left. The number of junctions leading to each of the eight endpoints varies. There are only two junctions in the path ending at ❸, for a probability of $1/2 \times 1/2 = 1/4$. There are three junctions in each of the paths ending at ❶ and ❼, so there is a $1/2 \times 1/2 \times 1/2 = 1/8$ probability of ending at either. There are four junctions in the path ending at ❺, for a probability of $1/2 \times 1/2 \times 1/2 \times 1/2 = 1/16$. Therefore, the probability of randomly choosing a path that ends at an odd number is 1/8 + 1/4 + 1/16 + 1/8 = **9/16**.

9. Recall that a number is divisible by 4 if the two-digit number formed by the tens and ones digits is divisible by 4. Using the digits 1, 2, 3, 4 and 5, the following two-digit multiples of four can be formed: 12, 24, 32 and 52. There are $5 \times 4 = 20$ permutations of two digits chosen from the digits 1, 2, 3, 4 and 5. Therefore, the probability that the five-digit number is divisible by 4 is 4/20 = **1/5**.

10. The inner circle of radius 1 has an area of $\pi \times 1^2 = \pi$ units^2. The combined area of the four numbered sectors is 4/6 = 2/3 the area of the inner circle, or $2\pi/3$ units^2. The area of the dartboard is $\pi \times 2^2 = 4\pi$ units^2. The probability of a dart landing in one of the numbered sectors is $(2\pi/3)/(4\pi)$ = $1/6 \approx$ **0.17**.

Patterns Stretch

stage	1	2	3	4
dots	1	4	9	16

11. The table shows the number of dots at each of the first four stages of the dot pattern. We can see that the number of dots at Stage n is n^2. If we let $n = 27$, then the difference between the numbers of dots in the figures at Stage 27 and Stage 47 is $(n + 20)^2 - n^2 \rightarrow n^2 + 40n + 400 - n^2 \rightarrow 40n + 400$ dots, or $40(27) + 400 = 1080 + 400 =$ **1480** dots. *Alternative solution:* If we recognize that the difference between the numbers of dots in the figures at Stage 27 and Stage 47 is the difference of two squares, then we get a difference of $47^2 - 27^2 = (47 + 27)(47 - 27) = (74)(20) =$ **1480** dots.

12. The terms in the sequence are 1, 2, 3, 6, 11, 20, 37, 68, 125, 230, 423, …. The 11th term is **423**.

13. The series 2 + 5 + 8 + 11 + 14 + … + 89 + 92 has 31 terms, and the average value of a term is (2 + 92)/2 = 47. The sum of the terms is then 31×47 = **1457**.

14. The difference between the first and second terms is $2x + 11 - x = x + 11$. The difference between the second and third terms is $4x - 3 - (2x + 11) = 4x - 3 - 2x - 11 = 2x - 14$. Since the difference between consecutive terms is constant, we set these two differences equal to each other and get $x + 11 = 2x - 14 \rightarrow x = 25$. Substituting, we see that the difference between consecutive terms is $x + 11 = 25 + 11 =$ **36**.

15. Filling in the missing terms, we get the series 1 + 4 + 16 + 64 + 256 + 1024 = **1365**.

16. The nth consecutive odd positive integer has a value of $2n - 1$. For example, we know that 31 is the 16th consecutive odd positive integer since when $n = 16$, $2n - 1 = 31$. The sum of the 16 terms is then $1 + 3 + 5 + 7 + ... + 29 + 31 = 32/2 \times 16 = 16^2 = 256$. In general, the sum of the first n consecutive odd positive integers is n^2. Therefore, the sum of the first 51 consecutive odd positive integers is $51^2 =$ **2601**.

17. Let $x = 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots$. So, $x/2 = 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + \ldots$. Substituting this back into the first equation yields $x = 1 + x/2$. Solving for x, we get $2x = 2 + x$ and $x =$ **2**.

18. Let $x = 1 + 1/4 + 1/16 + 1/64 + 1/256 + \ldots$. So, $x/4 = 1/4 + 1/16 + 1/64 + 1/256 + 1/1024 + \ldots$. Substituting this back into the first equation yields $x = 1 + x/4$. Solving for x, we get $4x = 4 + x \rightarrow 3x = 4$ and $x =$ **4/3**.

19. Based on the information provided, we can find $f^5(x)$ in a number of ways. For example, $f^5(x) = f^2(f^3(x)) = f^3(f^2(x)) = f^4(f(x)) = f(f^4(x)) = ax + b$. We can easily derive either $f^3(x)$ or $f^4(x)$ using the information provided in the problem. We have $f^3(x) = f^2(f(x)) = 4(2x + 3) + 9 = 8x + 12 + 9 = 8x + 21$. So, $f^5(x) = f^2(f^3(x)) = 4(8x + 21) + 9 = 32x + 84 + 9 = 32x + 93$. We see that $a = 32$, $b = 93$ and $a + b = 32 + 93 =$ **125**.

20. Let x, rx, r^2x, and r^3x represent the degree measures of the angles of this quadrilateral. If $r = 2$, our angle measures are x, $2x$, $4x$ and $8x$. Then $x + 2x + 4x + 8x = 360 \rightarrow 15x = 360 \rightarrow x = 24$. The largest angle has degree measure $8 \times 24 = 192$ degrees. If $r = 3$, our angle measures are x, $3x$, $9x$ and $27x$. Then $x + 3x + 9x + 27x = 360 \rightarrow 40x = 360 \rightarrow x = 9$. The largest angle has degree measure $27 \times 9 = 243$ degrees. Letting $r = 4$, or any higher value, yields non-integer angle measures. So, the largest possible degree measure of an angle in this quadrilateral is **243** degrees.

Travel Stretch

21. To determine the average speed for the entire round-trip, we need the total distance and the total time. If Jack and Jill travel d miles uphill, then they travel another d miles downhill, for a total distance of $2d$ miles. Since time = distance/speed, it follows that the time to travel uphill is $d/2$ hours and the time to travel downhill is $d/4$ hours, for a total time of $d/2 + d/4 = (2d + d)/4 = 3d/4$ hours. Now we can use these values in the formula speed = distance/time to get $2d/(3d/4) = 2d \times 4/(3d) = 8/3 =$ $\mathbf{2\frac{2}{3}}$ mi/h.

22. Since distance = speed × time, we know that when they meet in t hours, Jack will have traveled $4t$ miles and Jill will have traveled $2t$ miles. We also know that the total distance traveled is 1.5 miles. Therefore, $4t + 2t = 1.5 \rightarrow 6t = 1.5 \rightarrow t = 1/4$ hour, or 15 minutes. They will meet at **2:35** p.m.

23. We need to determine how far Jill has traveled up the hill when she meets Jack. Jill traveled at a speed of 2 mi/h for 1/4 hour. That's a total distance of $2 \times 1/4 = 1/2$ mile $= 1/2 \times 5280 \times 1/3 = 1760/2 =$ **880** yards.

24. Let t represent the time it takes Alysha to drive from home to the market. We know that her walking speed is 5 mi/h and it takes her $t + 21$ minutes to walk to the market from home. Converting her speed to miles per minute, we get $5 \div 60 = 1/12$ mile per minute. We are told that she drives eight times her walking speed, so her driving speed is $8 \times 1/12 = 8/12 = 2/3$ mile per minute. Since Alysha takes the same route to the store whether she walks or drives, we have the equation $(2/3)t = (1/12)(t + 21)$. Solving for t, we have $8t = t + 21 \rightarrow 7t = 21 \rightarrow t = 3$. So, the time it takes Alysha to drive to the market is **3** minutes.

25. We know that Alysha drives to the market at a speed of 2/3 mile per minute and it takes her 3 minutes. That means the distance from her home to the market is $2/3 \times 3 =$ **2** miles.

26. Running at a speed of 6 mi/h, Jana's total time to complete the 2-mile path is $2 \div 6 = 1/3$ hour, or 20 minutes. Riding his bicycle at a speed of 10 mi/h, Zhao's total time to complete the path is $2/10 = 1/5$ hour, or 12 minutes. That's a difference of $20 - 12 = 8$ minutes. Since Jana starts running 5 minutes before Zhao starts riding, she will reach the end of the path $8 - 5 =$ **3** minutes after Zhao does.

27. Jana runs for 5 minutes, or 1/12 hour, before Zhao begins riding. During that time she travels a distance of $6 \times 1/12 = 1/2$ mile. Let t represent the number of hours it will take Zhao to catch up to Jana. When Zhao catches up to her, Jana will have traveled a total of $6t + 1/2$ miles, while Zhao will have traveled $10t$ miles. They will have traveled the same distance, so we can set these two quantities equal to each other to get the equation $6t + 1/2 = 10t$. Solving for t, we get $1/2 = 4t \rightarrow t = 1/8$ hour, or $\mathbf{7\frac{1}{2}}$ minutes. *Alternative solution:* Let t be the number of minutes it takes Zhao to catch up with Jana. He is traveling at 10 mi/h, or 1/6 mile per minute, so he will travel $t \times 1/6$ miles before he catches up with her. She is traveling at 6 mi/h, or 1/10 mile per minute, and will have traveled $t + 5$ minutes before he overtakes her. Therefore $t \times 1/6 = (t + 5) \times 1/10 \rightarrow 5t = 3t + 15 \rightarrow 2t = 15 \rightarrow t =$ $\mathbf{7\frac{1}{2}}$ minutes.

28. Recall from the previous problem that Jana will have traveled $6t + 1/2$ miles when Zhao catches up with her. That's 6(1/8) + 1/2 = 3/4 + 1/2 = (3 + 2)/4 = 5/4 = $\mathbf{1\frac{1}{4}}$ miles. *Alternative solution:* Recall from the previous problem that Jana will have jogged for 5 + 7 1/2 = 12 1/2 minutes before Zhao catches up with her. Since she is traveling at 1/10 mile per minute, the distance she jogged will be 25/2 × 1/10 = 5/4 = $\mathbf{1\frac{1}{4}}$ miles.

29. Let c represent the speed of the river's current. Then Ansel's speed was $20 + c$ downstream and $20 - c$ upstream. The time to travel the 10 miles downstream and then back upstream was $10/(20 + c) + 10/(20 - c)$. Since we know that the entire round-trip took 64 minutes, or 16/15 hours, we have $10/(20 + c) + 10/(20 - c) = 16/15$. Solving for c, we find that $(200 + 10c + 200 - 10c)/(400 - c^2) = 16/15 \rightarrow 400/(400 - c^2) = 16/15 \rightarrow 15(400) = 16(400 - c^2) \rightarrow 6000 = 6400 - 16c^2 \rightarrow 16c^2 = 400 \rightarrow c^2 = 25 \rightarrow c = 5$. Thus, the speed of the river's current is **5** mi/h.

30. The time to travel upstream for 10 miles was 10/(20 − 5) = 10/15 = 2/3 hour, or 40 minutes. So, of the entire 64-minute round-trip, the fraction that was spent traveling upstream was 40/64 = **5/8**.

Warm-Up 1

31. We evaluate the expression according to the Order of Operations as follows: 5 − 5 × 5 + 5 ÷ 5 = 5 − 25 + 1 = −20 + 1 = **−19**.

32. First we note that a convex heptagon is a figure with 7 sides and 7 vertices. Each vertex has an interior angle measure less than 180 degrees. Also, a diagonal is any line segment that connects two vertices that are not connected by a side. A diagonal can be drawn from each vertex of the heptagon to 4 other vertices, excluding itself and its two neighbors. If we multiply 7 × 4 = 28, we have counted each diagonal at both ends, so a convex heptagon actually has a total of 28 ÷ 2 = **14** diagonals.

33. A palindrome is a number (or word) that reads the same forward and backward. Since we are looking for the first palindrome year after 2018, we'll start by considering years that begin and end with 2 and have two middle digits that are the same. We have 2002, 2112, 2222, 2332, etc. The first year after 2018 that is a palindrome, then, is **2112**.

34. There are four aces in a deck of 52 cards, so the probability that the first card selected at random is an ace is 4/52 = 1/13. If an ace is indeed chosen, then there are three aces left in the remaining 51 cards. The probability that the second card selected at random is an ace is 3/51 = 1/17. The probability that these two events both occur is 1/13 × 1/17 = **1/221**. *Alternative solution:* There are "4 choose 2" or ${}_4C_2$ = 4!/(2! × 2!) = (4 × 3)/2 = 6 ways to pick two aces out of "52 choose 2" or ${}_{52}C_2$ = 52!/(50! × 2!) = (52 × 51)/2 = 1326 ways to pick any two cards. That's a probability of 6/1326 = **1/221**.

35. To simplify this radical expression, we will rewrite the product in terms of prime factors. Then we will take the square root of all the perfect square factors. We have $\sqrt{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 10} = \sqrt{2 \times 3 \times (2 \times 2) \times 5 \times (2 \times 3) \times 7 \times (2 \times 5)} = \sqrt{2^5 \times 3^2 \times 5^2 \times 7} = 2^2 \times 3 \times 5 \times \sqrt{2 \times 7} = \mathbf{60\sqrt{14}}$.

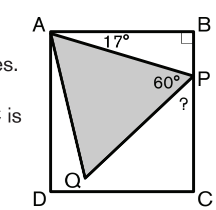
36. The temperature went from 13 °F to −5 °F, which is a drop of 13 − (−5) = 13 + 5 = **18** °F.

37. Using the Order of Operations, we get 1 × 2 + 3 ÷ 6 × 5 − 4 = 2 + (1/2) × 5 − 4 = 2 + 5/2 − 4 = 5/2 − 4/2 = **1/2.**

38. First, we evaluate 2 ® 1 inside the parentheses to get $2^2 - 1^2 = 4 - 1 = 3$. Using the result to evaluate 3 ® 3, we get $3^2 - 3^2 =$ **0**.

39. The three-digit number and the two-digit number differ by 288, so let's start by filling in the blanks of the subtraction problem __ __ __ − __ __ = 288 with the digits 7, 8, 2, 3, and 0. In the ones place, 0 − 2 leaves an 8 (after borrowing). In the tens place of the three-digit number we had to reduce the digit by one to give 10 to the ones place, so the 7 could become a 6 and 16 − 8 = 8. That leaves the 3 for the hundreds place of the three-digit number. This works since 370 − 82 = 288. So, the three-digit number is 370 and the two-digit number is 82, and the desired sum is 370 + 82 = **452**.

40. The figure shows equilateral triangle APQ inside rectangle ABCD. In right triangle PAB, the measure of angle PAB is 17 degrees. Since the two acute angles in a right triangle are complementary, we know that the measure of angle APB must be 90 − 17 = 73 degrees. We also know that the measure of angle APQ is 60 degrees. Angle BPC is a straight angle, so the measure of angle QPC is 180 − 73 − 60 = **47** degrees.



Warm-Up 2

41. Kim will have to buy a whole number of balls of yarn, so we need to round 750 ÷ 180 = 25 ÷ 6 = 4 1/6 balls to **5** balls of yarn.

42. Let's say C is Chris' age in 1992 and J is Joseph's age in 1992. The first sentence translates to $C = (1/2)J$ or $2C = J$, and the second sentence translates to $C + 6 = (2/3)(J + 6)$. Substituting $2C$ for J in the second equation, we get $C + 6 = (2/3)(2C + 6) \rightarrow C + 6 = (4/3)C + 4 \rightarrow 2 = (1/3)C \rightarrow C = 6$. So, Chris must have been 6 in 1992. In 2018, Chris will be 2018 − 1992 = 26 years older, which will make him **32** years old.

43. At 5:42, the time elapsed is 42 of the 60 minutes in the 5 o'clock hour. Since the minute hand will make the complete revolution during that hour, at 5:42 it has traveled 42/60 = 7/10 of the full 360 degrees, or (7/10) × 360 = 252 degrees. The hour hand makes 1/12 of a complete revolution every hour. So, from 12:00 to 5:00, it travels 5/12 of the full 360 degrees, or (5/12) × 360 = 150 degrees. By 5:42, it has traveled another 7/10 of 1/12 of the full 360 degrees, or (7/10) × (1/12) × 360 = 21 degrees, for a total of 150 + 21 = 171 degrees. The measure of the angle between the hands at 5:42 is 252 − 171 = **81** degrees.

44. The expression 12 × 37 + 12 × 7 + 12 × 6 can be rewritten as 12 × (37 + 7 + 6), which equals 12 × 50 = **600**.

45. The prime factorization of 2018 is 2 × 1009, since 1009 is prime. Thus, 1, 2, 1009 and 2018 are the **4** positive factors of 2018.

46. The possible values for the sum of the two top faces that are *at least* 9 are 9, 10, 11 and 12. There are 4 ways to roll a 9, 3 ways to roll a 10, 2 ways to roll an 11 and only 1 way to roll a 12. That's 4 + 3 + 2 + 1 = 10 rolls out of the 6 × 6 = 36 possible rolls, for a probability of 10/36 = **5/18**.

47. The letter m in the equation $y = mx + b$ represents the slope of the line. We need to calculate the ratio of the change in the y-values to the change in the x-values. Using the points (6, 13) and (10, 31), we have a slope of $m = (31 - 13)/(10 - 6) = 18/4 =$ **9/2**.

48. The least common multiple (LCM) of 12 and 20 is 60. So let the number of 12-ounce cans of soda that Dewey buys be 5, and let the number of 20-ounce bottles of soda that Peppar buys be 3. They each get a total of 60 ounces, but Dewey spends 5 × 1.00 = \$5.00, and Peppar spends 3 ×1.25 = \$3.75. Peppar spends 5.00 − 3.75 = \$1.25 less than Dewey, which is 1.25 ÷ 5.00 × 100 = 25**,** so P = **25**.

49. A diameter of 9 inches is 3/4 of one foot. We want to know how many 3/4 of a foot there are in 12 feet. Dividing, we get 12 ÷ (3/4) = 12 × 4/3 = 16. So, to create the 12-foot wall, Gerald will have to stack **16** logs.

50. The area of a square of side length s is s^2. So, a square of area 8 units^2 has side length $s = \sqrt{8} = 2\sqrt{2}$ units. By properties of 45-45-90 right triangles, we know that a square of side length s has diagonal length $s\sqrt{2}$. So, a square of side length $2\sqrt{2}$ units, has diagonal length $2\sqrt{2} \times \sqrt{2} =$ 4 units. Since the inscribed square's diagonal is a diameter of the circle, it follows that the circle has radius $r = 2$ units and area $\pi r^2 = \pi \times 2^2 =$ **4π** units^2.

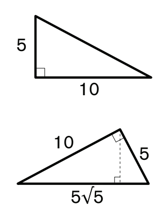
Warm-Up 3

51. If we take the square root of each side we see that $y = \pm(y + 2018)$. So $y = y + 2018$ or $y = -y - 2018$. Solving the first equation leads to the false statement 0 = 2018. Solving the second equation yields $2y = -2018 \rightarrow y =$ **−1009**, which is the only valid answer.

52. Let M and C represent Maura's and Cara's current ages, respectively. The first sentence translates to $M + 5 = C$, and the second sentence translates to $C - 7 = 2(M - 7)$. Substituting $M + 5$ for C in the second equation, we get $M + 5 - 7 = 2(M - 7) \rightarrow M - 2 = 2M - 14 \rightarrow M = 12$. So, Maura is **12** years old.

53. The area of the entire dartboard is the area of the circle of radius 10 inches, or $\pi \times 10^2 = 100\pi$ in^2. The area of the yellow region of the dart board is the difference between the areas of the circle of radius 5 inches and the circle of radius 1 inch, or $\pi \times 5^2 - \pi \times 1^2 = 25\pi - \pi = 24\pi$ in^2. Thus, $24\pi/(100\pi) =$ **6/25** of the dartboard's total area is colored yellow, and that fraction is the desired probability.

54. In the 28 years preceding 2018, there are 7 leap years, which each contain 1 more day than a non-leap year. Therefore, the total number of days in those 28 years is 365 × 28 + 7 = 10,220 + 7 = **10,227** days.



55. If the triangle is positioned as shown in the top figure, it has base length 10 cm and height 5 cm, and it has area $(1/2) \times b \times h = (1/2) \times 5 \times 10 = 25$ cm^2. Using the Pythagorean Theorem, we can determine that a right triangle with legs of length 5 cm and 10 cm has a hypotenuse of length $\sqrt{(5^2 + 10^2)} = \sqrt{(25 + 100)} = \sqrt{125} = 5\sqrt{5}$ cm. Now, if the triangle is positioned as shown in the bottom figure, it has base length $5\sqrt{5}$ cm. We can determine the height of the triangle in this position since we know that $(1/2) \times b \times h = (1/2) \times 5\sqrt{5} \times h = 25$. Solving for h, we get $h = 50/(5\sqrt{5}) = 50\sqrt{5}/25 =$ **2√5** cm.

56. Of the 18 two-digit multiples of 5 from 10 to 95, only 13 have exactly two distinct prime factors, as shown. (Note that some of these numbers have more than one copy of their distinct prime factors.) The probability of choosing one of these 13 at random, then, is **13/18**.

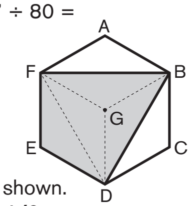
$10 = 2 \times 5$	$25 = 5^2$	$40 = 2^3 \times 5$	$55 = 5 \times 11$	$70 = 2 \times 5 \times 7$	$85 = 5 \times 17$
$15 = 3 \times 5$	$30 = 2 \times 3 \times 5$	$45 = 3^2 \times 5$	$60 = 2^2 \times 3 \times 5$	$75 = 3 \times 5^2$	$90 = 2 \times 3^2 \times 5$
$20 = 2^2 \times 5$	$35 = 5 \times 7$	$50 = 2 \times 5^2$	$65 = 5 \times 13$	$80 = 2^4 \times 5$	$95 = 5 \times 19$

57. In a stem-and-leaf plot, commonly the numbers to the left of the vertical line are the tens digits and the numbers to the right of the vertical line are the corresponding ones digits of the numbers in the data set. There are 24 numbers in this set. Recall that in a set of data with an even number of data points, the median is the mean of the two middle numbers when the data points have been arranged in ascending or descending order. In this case, the median is (20 + 24) ÷ 2 = 22, and there are 12 numbers less than 22 and 12 greater than 22. Next, to determine the lower quartile, we find the median of the 12 numbers less than 22. In this case, the lower quartile is (11 + 11) ÷ 2 = 11. To determine the upper quartile, we find the median of the 12 numbers greater than 22. In this case, the upper quartile is (35 + 37) ÷ 2 = 36. The interquartile range, then, is 36 − 11 = **25**.

58. Locating the primes in each row (starting from 36 and working backward), numbers are crossed off in the following order: 31, 26, 21, 16, 11, 6, 29, 24, 23, 18, 19, 14, 9, 4, 17, 12, 13, 8, 3, 7, 2, 5. The sum of the 14 remaining numbers is 1 + 10 + 15 + 20 + 22 + 25 + 27 + 28 + 30 + 32 + 33 + 34 + 35 + 36 = **348**.

59. In mathematics, the exclamation point indicates the "factorial" of a number, which is the product of the positive integers less than or equal to the number. We are asked to evaluate 1,000,000! ÷ 999,999! = (1,000,000 × 999,999 × 999,998 × ⋯ × 2 × 1)/(999,999 × 999,998 × ⋯ × 2 × 1). Once we get rid of the common factors in the numerator and denominator, the only number that remains is 1,000,000 in the numerator. Therefore, the answer is **1,000,000**.

60. One cubic yard of topsoil is 3 × 3 × 3 = 27 ft^3 of topsoil. The garden is 10 × 8, which is 80 ft^2, so the depth of the soil will be 27 ÷ 80 = 27/80 feet, which is equivalent to 27/80 × 12 = 81/20 = $\mathbf{4\frac{1}{20}}$ inches.



Warm-Up 4

61. We will draw a few more line segments, between vertices D and F and between each of the vertices B, D and F and center G, as shown. We now have six congruent obtuse triangles, four of which are in the interior of quadrilateral BDEF. The desired ratio of areas is 2/4 = **1/2**.

62. The expression can be evaluated as follows: $\frac{11!-(9+1)(9!)}{8(7!)} = \frac{11!-(10)(9!)}{8!} = \frac{11\times10!-10!}{8!} = \frac{10!(11-1)}{8!} = \frac{10!(10)}{8!} = \frac{10\times9\times8!\times10}{8!} = 10^2 \times 9 =$ **900**.

63. There are 30 × 20 = 600 drops of eyeglass cleaner in the bottle. David needs 3 × 4 = 12 drops to clean both sides of both lenses, so he can clean his glasses 600 ÷ 12 = **50** times.

64. There are ${}_3C_2$ = 3!/(1! × 2!) = 3 ways to choose 2 meats. There are ${}_4C_2$ = 4!/(2! × 2!) = (4 × 3)/(2 × 1) = 6 ways to choose vegetables. There are 3 × 4 = 12 ways to choose 1 meat and 1 vegetable. That's 3 + 6 + 12 = **21** possible combinations. *Alternative solution:* The rules really amount simply to choosing 2 of the 7 toppings, and this can be done in ${}_7C_2$ = 7!/(5! × 2!) = (7 × 6)/(2 × 1) = **21** ways.

65. If we add the 150 pounds of the first weighing and the 168 pounds of the last weighing, we have the sum of the average of sheep A and B and the average of sheep C and D. We can now subtract the 127 pounds of the second weighing, which is the average of sheep B and C, to get the average of sheep A and D. That's 150 + 168 − 127 = **191** pounds.

66. The measure of an angle inscribed in a circle is half the angle measure of the arc it intercepts. Since $m\angle ABC$ = 90 degrees, it follows that the measure of arc ABC is 180 degrees, and it has length equal to half the circumference of the circle. The circle has radius 3 meters, so the length of arc ABC is **3π** meters.

67. The wall has an area of 6 × 8 = 48 ft^2. Each square foot will require 3 × 3 = 9 of the 4-inch square tiles. Thus, 9 × 48 = **432** tiles are needed.

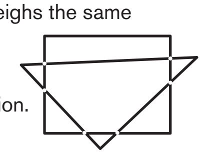
68. To find the units digit of a product, multiply the units digits of the factors and take the units digit of the result. Each of the factors in the problem is quite large, but there is a cycle to the units digits of both the powers of 2 and the powers of 7. In the powers of 2, we get units digits 2, 4, 8, 6 and then the pattern repeats. Since 2017 is one more than a multiple of 4 (the number of digits in the pattern), the 2017th power of 2 will have a units digit of 2. Similarly, in the powers of 7, we get units digits 7, 9, 3, 1 and then the pattern repeats. The units digit of the 2017th power of 7 is 7. Finally, the units digit of the product of these powers is just the units digit of 2 × 7 = 14, which is **4**.

69. Each of the hundreds, tens and units digits of these three-digit numbers can be a 3, a 4 or a 5. That's a total of 3 × 3 × 3 = **27** integers.

70. The 10 paths that start at cell A and move only to adjacent cells and include all five cells are ABCDE, ABCED, ABECD, ABEDC, ADCBE, ADCEB, ADEBC, ADECB, AEBCD and AEDCB. By symmetry, there are also 10 paths that start at cells B, C and D. The 8 paths that start at E are EABCD, EADCB, EBADC, EBCDA, ECBAD, ECDAB, EDABC and EDCBA. In all, there are 10 × 4 + 8 = **48** paths.

Warm-Up 5

71. Let p, c and e represent the weights of a pencil, paper clip and eraser, respectively. From the information given, we have $p + 5c = 2e$ and $p = 29c$. Substituting the $29c$ for p in the first equation, we get $29c + 5c = 2e$. This simplifies to $34c = 2e \to e = 17c$. So, an eraser weighs the same as **17** paper clips.



72. Each of the triangle's three sides can intersect twice with a side of the square, as shown, for a maximum of **6** points of intersection.

73. Substituting $x = 0$ into the function, we get $p(0) = a(0)^2 + b(0) + c = 4 \to c = 4$. Substituting $x = 1$ and $c = 4$ into the function, we get $p(1) = a(1)^2 + b(1) + 4 = 15 \to a + b = 11$. Substituting $x = 2$ and $c = 4$ into the function, we get $p(2) = a(2)^2 + b(2) + 4 = 36 \to 4a + 2b = 32 \to 2a + b = 16$. Subtracting the second and third equations yields $(2a + b) - (a + b) = 16 - 11 \to 2a + b - a - b = 5 \to a = 5$. Substituting back into the second equation, we see that $5 + b = 11 \to b = 6$. So, $a = 5$, $b = 6$, $c = 4$ and $abc = 5 \times 6 \times 4 =$ **120**.

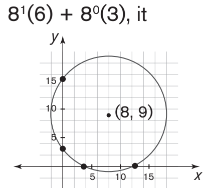
74. A sphere of radius r has volume $(4/3) \times \pi \times r^3$ and surface area $4 \times \pi \times r^2$. Because the given sphere's volume is numerically three times its surface area, we have $(4/3) \times \pi \times r^3 = 3 \times 4 \times \pi \times r^2$. Solving for r, we get $r^3/r^2 = (3 \times 4 \times \pi)/[(4/3) \times \pi] \rightarrow r = \mathbf{9}$ units.

75. We'll use B, G, R, O and Y to represent the colors Blue, Green, Red, Orange and Yellow, respectively. We know that B must be the middle layer. If subsequent candles are made with no color next to a color it touched in the original candle, then G must be the top or bottom layer, since it touched B in the original candle. If G is the top layer, then the two possible color arrangements are: GOBRY and GOBYR. If G is the bottom layer we have the reverse of those two arrangements: YRBOG and RYBOG. That makes **4** different candles.

76. No matter what color the pointer lands on with the first spin, the probability is **1/3** that he will match that color on his second spin.

77. Let m represent the number of shots Kevin has made and a represent the number of shots he has attempted. Currently, his ratio of made shots to attempted shots is $m/a = 1/3 \rightarrow 3m = a$. If he makes the next 5 shots, the new ratio will be $(m + 5)/(a + 5) = 1/2 \rightarrow 2(m + 5) = a + 5 \rightarrow 2m + 10 = a + 5 \rightarrow 2m + 5 = a$. Setting these two expressions for a equal to each other, we get $3m = 2m + 5 \rightarrow m = 5$. This accounts for 1/3 of the initial $3 \times 5 =$ **15** shots Kevin has attempted up to now.

78. The binary number 110011_2 is $2^5(1) + 2^4(1) + 2^3(0) + 2^2(0) + 2^1(1) + 2^0(1) = 32 + 16 + 2 + 1 = 51$ in base ten. Since $51 = 8^1(6) + 8^0(3)$, it follows that 110011_2 is **63** base eight.

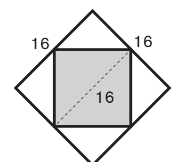


79. Since the circle's radius of 10 units is greater than both 8 and 9, the circle intersects each axis at two points, as shown, for a total of **4** points.

80. If we multiply the equations, we get $(x + (1/y)) \times (y + (1/x)) = (1/5) \times 20 \rightarrow xy + 2 + 1/(xy) = 4 \rightarrow xy + 1/(xy) = 2 \rightarrow (xy)^2 + 1 = 2xy \rightarrow (xy)^2 - 2xy + 1 = 0 \rightarrow (xy - 1)^2 = 0 \rightarrow xy - 1 = 0 \rightarrow xy = \mathbf{1}$. This is confirmed by substituting $1/y$ for x in the equation $x + (1/y) = 1/5$ to get $(1/y) + (1/y) = 1/5 \rightarrow 2/y = 1/5 \rightarrow y = 10$, and $x = 1/y = 1/10$.

Warm-Up 6

81. We know that $3x + 5 = 13$. Since $3x + 4$ is one less than $3x + 5$, we have $3x + 4 = 12$. Similarly, $3x + 3 = 11$ and $3x + 2 = 10$. Thus, $(3x + 2)(3x + 3)(3x + 4) = 10 \times 11 \times 12 =$ **1320**.

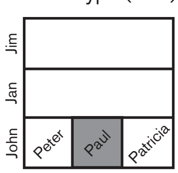


82. We will get the greatest possible area when the rectangle is a square. Since the diagonal of our square is 16 units, the area of the square is half the area of a 16 by 16 square, as shown. The maximum area, then, is $(1/2) \times 16^2 = (1/2) \times 256 =$ **128** units^2.

83. In order to satisfy rules I *and* II, at least one of the options in each rule must be true. The **5** pairs of numbers that work are (0, 0), (1, 1), (−1, 1), (1, −1), (−1, −1).

84. Since we are looking for the least possible sum, let's consider using the digits 1 through 6. We'll use the 1 and 2 as the units digits, so we have 1.BC + 2. EF. Now we need to arrange the digits 3, 4, 5 and 6 to get the smallest sum. Assigning the digits in any of the following ways gives us the smallest sum: 1.35 + 2.46 = 1.46 + 2.35 = 1.36 + 2.45 = 1.45 + 2.36 = **3.81**.

85. The problem requires that Matt choose one of each kind of fastener. So let's imagine that Matt first selects one of each kind and sets them aside. The question now is how many ways he can choose the other 5 fasteners. To make an organized list of all possibilities is a challenging and worthwhile exercise. We will take a shortcut known to some as "stars and bars." Five stars will represent the 5 fasteners, and 4 bars will separate the fasteners into the 5 different categories in the given order: wood screws, sheet metal screws, hex bolts, carriage bolts and lag bolts. The arrangement *****|||| represents 5 wood screws and none of the other types. The arrangement *|*|*|*|* represents one of each type of fastener. All possible combinations can be represented with this notation. The number of ways to arrange 9 items, with 5 being of one type (stars) and 4 being of another type (bars), is $9!/(5! \times 4!) = (9 \times 8 \times 7 \times 6)/(4 \times 3 \times 2 \times 1) =$ **126** ways.



86. The figure shows the share of the farm each heir is given. Jim gets a third of the farm in the first division. When John's third is then divided among his heirs, Peter, Paul and Patricia, each gets a third of his third, which is a ninth of the farm. After Paul sells his ninth of the farm, 8/9 of the farm remains. Jim, who still owns 1/3 = 3/9 of the original farm, owns 3/8 of the remaining 8/9 of the farm. Therefore, from the most recent sale, Jim should receive **3/8** of the proceeds.

87. For Olivia to achieve an average score of 90 on the four tests, the sum of all four scores would need to be $4 \times 90 = 360$. The sum of her first three test scores is $82 + 86 + 92 = 260$. The score Olivia needs to get on the fourth test is $360 - 260 =$ **100**. *Alternative solution:* Olivia's first test was 8 points below her desired average, the second was 4 points below, and the third was 2 points above. Thus, she has a deficit of $8 + 4 - 2 = 10$ points to make up and therefore needs to score 10 points above 90, or **100**.

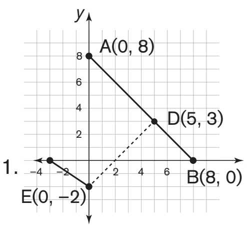
88. After the preschool fee of \$330, Cody paid an additional 770 − 330 = \$440 for after-school care. Dividing \$440 by the hourly rate of \$5.50, we find that Cody's son must have spent **80** hours in after-school care.

89. Let $n - 1$, n and $n + 1$ be positive consecutive integers such that $(n - 1) \times (n) \times (n + 1) = 16 \times [(n - 1) + (n) + (n + 1)]$. Simplifying and solving for n, we have $n(n^2 - 1) = 16 \times 3n \rightarrow n^3 - n = 48n \rightarrow n^3 - 49n = 0 \rightarrow n(n^2 - 49) = 0$. So, $n = 0$ or $n^2 - 49 = 0 \rightarrow (n + 7)(n - 7) = 0 \rightarrow n = -7$ or $n = 7$. Since n is a positive integer, it follows that the three consecutive numbers are 6, 7 and 8. The difference between their product and sum is $6 \times 7 \times 8 - (6 + 7 + 8) = 336 - 21 =$ **315**.

90. The ratio of the surface area of Wilbur's plane to the surface area of Orville's plane is the square of the scale factor of Wilbur's mini replica of Orville's plane. Since Wilbur's mini replica has linear dimensions that are 1/2 the size of Orville's model airplane, it follows that the ratio between the lift forces on Wilbur's and Orville's planes is $(1/2) \times (1/2) =$ **1/4**.

Warm-Up 7

91. Simplifying, we have $6/78 < 1/n < 5/55 \rightarrow 1/13 < 1/n < 1/11$. This compound inequality is true when $n =$ **12**.

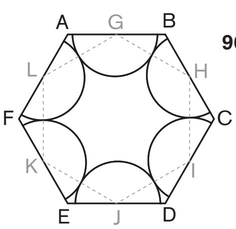


92. If we add the wins of both Flying Turtles and Dolphins, we have to then subtract the 19 wins for the games they played against each other regardless of who won those games. The value of $F + D$, then, is $95 + 84 - 19 =$ **160**.

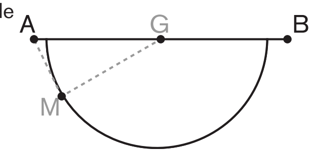
93. The shortest distance between the two lines is the segment from E(0, −2) drawn perpendicular to the segment with endpoints A(0, 8) and B(8, 0). Since segment AB has slope $(0 - 8)/(8 - 0) = -1$, the desired segment must have a slope of 1. It intersects segment AB at D(5, 3). As the figure shows, segment DE has length $\sqrt{(5^2 + 5^2)} = \sqrt{50} =$ **$5\sqrt{2}$** units.

94. We are just counting the numbers determined by the sequence 317, 319, ..., 637. We do so as follows: $(317 - 315)/2 = 2/2 = 1$ term, $(319 - 315)/2 = 4/2 = 2$ terms, and so forth. So, $(637 - 315)/2 = 322/2 =$ **161** terms.

95. Only perfect square numbers have an odd number of factors. Though there are $6 \times 6 = 36$ possible outcomes when two standard dice are rolled, there are only 11 possible sums: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,12. Of these, only 4 and 9 are perfect squares. There are 3 ways to roll a sum of 4: (1, 3), (2, 2) and (3, 1). There are 4 ways to roll a sum of 9: (3, 6), (4, 5), (5, 4) and (6, 3). That's a total of $3 + 4 = 7$ sums with an odd number of factors, for a probability of **7/36**.



96. If we draw a second regular hexagon, with vertices at the midpoints of the sides of the original hexagon, then the midpoints of the sides of this smaller hexagon are the points of tangency between adjacent semicircles. If we zoom in on one part of this figure, we see a 30-60-90 triangle, labeled AGM in the figure shown. Side AG of triangle AGM is half of side AB of the original hexagon and has length $s/2$. Side GM of triangle AGM is a radius of a semicircle and the long leg of the 30-60-90 triangle. The ratio of GM = r to AG = $s/2$, which is the ratio of the long leg to the hypotenuse of a 30-60-90 triangle, is $r/(s/2) = \sqrt{3}/2$. So, the ratio of r to s is **$\sqrt{3}/4$**.



97. Gaylon uses 8 digits for every date he writes down. If we divide 2018 by 8, we get 252 with a remainder of 2. The 253rd day of the year occurs in the month of September, so Gaylon writes 09 for the month of September. The 2018th digit he writes down is **9**.

98. April has 30 days, so Zeus needs to throw a total of $30 \times 12 = 360$ lightning bolts in all of April. So far he has thrown $7 \times 15 = 105$ lightning bolts. In the next $30 - 7 = 23$ days, he needs to throw $360 - 105 = 255$ lightning bolts. That's an average of about $255 \div 23 \approx$ **11** lightning bolts per day.

99. Both 9 and 27 are powers of 3, so we can rewrite the equation as $(3^2)^{2x^2-6} = (3^3)^{x^2-1}$. Using the laws of exponents, we can rewrite this as $3^{2(2x^2-6)} = 3^{3(x^2-1)} \rightarrow 3^{4x^2-12} = 3^{3x^2-3}$. Since the base is 3 on both sides of the equation, the expressions will be equal when their exponents are equal. To determine what positive value of x makes this equation true, we need only solve the equation $4x^2 - 12 = 3x^2 - 3$ to see that $x^2 = 9$, so $x =$ **3**.

100. The increase from 60 to 120 decibels is 60 decibels, which is three increases of 20 decibels. Since every increase of 20 decibels corresponds to sound becoming 10 times as loud, the rock concert must be $10 \times 10 \times 10 =$ **1000** times as loud. as a conversation

Warm-Up 8

101. The sum of any three consecutive integers is divisible by 3. Since 206 is not, we know it is either the sum of the first, second and fourth terms or of the first, third and fourth terms. If we call the first term in Pamela's sequence x, then the four consecutive terms are x, $x + 1$, $x + 2$ and $x + 3$. Based on the two options above, we can have either $x + x + 1 + x + 3 = 206 \rightarrow 3x = 206 - 4 \rightarrow 3x = 202$ or $x + x + 2 + x + 3 = 206 \rightarrow 3x = 206 - 5 \rightarrow 3x = 201$. The latter is the only one that yields an integer value for x. Solving, we find $x = 67$. The other integer is $x + 1 =$ **68**.

102. Benjamin is walking up at a rate of 1 flight per 10 seconds, and the escalator is moving down at a rate of 1 flight per 20 seconds. So, his overall rate of progress is $1/10 - 1/20 = 1/20$ flight per second. It takes him 20 seconds to walk up one flight and **40** seconds to walk up two flights on the escalator.

103. The prime factorization of K is equal to the product of the prime factorizations of factors 168 and 900, which is $K = 168 \times 900 = (2^3 \times 3 \times 7) \times (2^2 \times 3^2 \times 5^2) = 2^5 \times 3^3 \times 5^2 \times 7^1$. If we list the available powers of each prime, *including the zero power*, as shown, we can create all the positive divisors of K by multiplying one number from each column in all possible ways. This means there are $6 \times 4 \times 3 \times 2 =$ **144** divisors of K.

1	1	1	1
2	3	5	7
4	9	25	
8	27		
16			
32			

104. The sum of the 7 integers from 1 to 7 is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. Since $28 - 4 = 24$, to get a sum of 24, Emma needs to choose every integer except the 4. By symmetry, the probability that she leaves out the 4 is the same as the probability that she leaves out any one of the 7 numbers, which is **1/7**.

105. Each data set is only three numbers, the profits for three weeks. Since the medians and largest numbers of the two sets are the same, the difference in the means of the sets must depend entirely on the difference in their lowest numbers. Cart A's mean for the three weeks is \$27 more, so the difference in the lowest week's profit values is 3 × \$27 = **\$81**.

106. Since this question is about a ratio, we have the freedom to assign convenient values to the radii. Let's choose radii of 1, 2 and 3 units for the inner circles. This choice makes the radius of the largest circle 6 units and gives it an area of $\pi \times 6^2 = 36\pi$ units^2. The total area of the inner circles is $\pi \times 1^2 + \pi \times 2^2 + \pi \times 3^2 = \pi + 4\pi + 9\pi = 14\pi$ units^2. The shaded area is the difference $36\pi - 14\pi = 22\pi$ units^2. Finally, the fraction that is shaded is $22\pi/36\pi =$ **11/18**.

107. As in Problem 106, since the question is about a ratio, we have the freedom to assign convenient values to the lengths. If we say that the side length of the regular octagon is 1 unit, then four of the shaded isosceles right triangles can be rearranged to form a unit square. Since there are eight of those triangles, the total area of the shaded regions is 2 units^2. The side length of the congruent squares is $1/\sqrt{2} + 1 + 1/\sqrt{2} = \sqrt{2}/2 + 1 + \sqrt{2}/2 = 1 + \sqrt{2}$ units, so the area of one of them is $(1+\sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$ units^2. The area of the octagon is 1 unit^2 less than the area of the square, so it's $2+2\sqrt{2}$ units^2. The ratio of the shaded area to the area of the octagon is $2/(2 + 2\sqrt{2}) = 1/(1 + \sqrt{2})$, but this answer is not in simplest radical form. To "rationalize the denominator," we multiply this fraction by a well-chosen form of the number 1, known as the conjugate as follows: $1/(1 + \sqrt{2}) \times (1 - \sqrt{2})/(1 - \sqrt{2}) = (1 - \sqrt{2})/(-1) =$ $\boldsymbol{\sqrt{2} - 1}$.

108. If we let M be the number of hours it takes Markus to make a basket, then it takes Avi $M + 1/2$ hour to make a basket. After 28 hours, Markus has made $28/M$ baskets and Avi has made $28/(M + 1/2)$ baskets, which is one fewer basket. Therefore, we can write the following equation: $28/M = 28/(M + 1/2) + 1$. We now multiply both sides of this equation by $M(M + 1/2)$, which gives us $28(M + 1/2) = 28M + M(M + 1/2)$. Distributing on both sides, we get $28M + 14 = 28M + M^2 + (1/2)M$, which we simplify to $14 = M^2 + (1/2)M$, which we further simplify to $2M^2 + M - 28 = 0$. The trinomial expression on the left can be rewritten as the following product of two binomials: $(2M - 7)(M + 4) = 0$. This product will equal zero if $2M - 7 = 0$ or $M + 4 = 0$. The solution to the first possibility is $M = 7/2 = 3\ 1/2$ hours, which makes sense. The solution to the second possibility is $M = -4$ hours, which does not make sense. Since it takes Markus 3 1/2 hours to make a basket, it must take Avi **4** hours to make a basket.

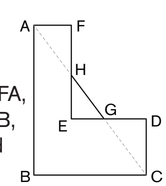
109. We will restate the given properties of N using modular arithmetic. The statement "$N \equiv 1 \pmod 2$" means that N leaves a remainder of 1 when divided by 2. We know the following: $N \equiv 1 \pmod 2$, $N \equiv 2 \pmod 3$, $N \equiv 3 \pmod 5$ and $N \equiv 5 \pmod 7$. The first two statements can be rewritten as: $N \equiv -1 \pmod 2$ and $N \equiv -1 \pmod 3$. Since 2 and 3 are relatively prime and the remainder is now the same, we can say that $N \equiv -1 \pmod 6$, which means that N is 1 less than a multiple of 6. Similarly, the last two original statements can be rewritten as $N = -2 \pmod 5$ and $N = -2 \pmod 7$. Now since 5 and 7 are relatively prime, we can write $N \equiv -2 \pmod{35}$. We can now just consider a list of numbers that are 2 less than a multiple of 35: 33, 68, 103, 138 and finally 173. Only this last number, **173**, is also 1 less than a multiple of 6, so this is our least possible value of N. (Incidentally, the next value of N would be $173 + 6 \times 35 = 173 + 210 = 383$.)

110. The prime factorization of 1000^3, or 1,000,000,000, is $2^9 \cdot 5^9$. This number has $10 \times 10 = 100$ positive factors. To combine these prime factors in ways that produce perfect square factors, we calculate that each of 5 perfect square powers of 2 (namely 2^0, 2^2, 2^4, 2^6 and 2^8) can be multiplied by each of 5 perfect square powers of 5 (namely 5^0, 5^2, 5^4, 5^6 and 5^8). That's $5 \times 5 = 25$ perfect square factors, which is $25/100 =$ **1/4** of the factors of 1000^3.

Warm-Up 9

111. Since Sola's lucky common fraction is $f = 0.\overline{711}$, $1000 \times f = 711.\overline{711}$. Subtracting the first equation from the second, we get $999 \times f = 711 \rightarrow f = 711/999 =$ **79/111**.

112. The equation for line m can be rewritten in slope-intercept form as $y = 2x - 7/3$. Since line n is perpendicular to m, its slope is the negative reciprocal of 2, which is $-1/2$. Given this slope and the point (6, 2), we can write the equation for line n in point-slope form as $y - 2 = (-1/2)(x - 6) \rightarrow y - 2 = (-1/2)x + 3 \rightarrow y = (-1/2)x + 5$. Using the information given, we can write the equation for line k as $y = 5x + 1$. We want to find the x-coordinate of the point where lines n and k intersect, so we set those two expressions for y equal to each other to get $(-1/2)x + 5 = 5x + 1 \rightarrow -x + 10 = 10x + 2 \rightarrow 8 = 11x \rightarrow x =$ **8/11**.



113. Triangle ABC has side lengths that form the Pythagorean Triple 6-8-10. So AC = 10. If we label the intersection of segment AC with side FE as H and the intersection of segment AC with side DE as G, we see that three triangles all similar to ABC are formed – HFA, HEG and CDG. We can find the lengths of segments AH and GC by setting up ratios using the known lengths. Since HA/AF = AC/CB, we have $HA/2 = 10/6 \rightarrow HA = 20/6 = 10/3$. And since CG/CD = AC/AB, we have $CG/3 = 10/8 \rightarrow CG = 30/8 = 15/4$. The desired fraction is $(10/3 + 15/4) \div 10 = 1/3 + 3/8 =$ **17/24** of the segment AC.

114. We can interpret the reading of the Richter scale as the exponent to which the base 10 is raised. Since 7.5 − 5 = 2.5, a reading of 7.5 is $10^{2.5}$ times stronger than a reading of 5 on the Richter scale. In simplest radical form, that's $10^{2.5} = 10^{5/2} = \sqrt{(10^5)}$ = $\mathbf{100\sqrt{10}}$ times stronger.

115. In a geometric sequence with first term a and common ratio between consecutive terms r, the third term can be expressed as ar^2, and the seventh term can expressed as ar^6. Since we know that $ar^2 = 45$ and $ar^6 = 3645$, we have $ar^6/(ar^2) = 3645/45 \rightarrow r^4 = 81 \rightarrow r = 3$ or $r = -3$. Substituting either value for r yields $a = 5$. The two possible sequences are 5, 15, 45, 135, 405, 1215, 3645 and 5, −15, 45, −135, 405, −1215, 3645. The least possible sum of the first five terms, then, is 5 − 15 + 45 − 135 + 405 = **305**.

116. There are 6 consonant tiles available for the first letter of the word. For each consonant that may be chosen for the first letter, there are 3 vowel tiles available for the second letter, 2 vowel tiles available for the third letter and 5 consonant tiles available for the last letter. The number of four-letter words that can be formed, then, is 6 × 3 × 2 × 5 = **180** words.

117. Let s represent the constant speed at which the girls run. Rebecca runs for 13 minutes, or 13/60 hour at speed s, and Susan runs for 7 minutes, or 7/60 hour at speed s. Rebecca covers $(13/60)s$ miles, and Susan covers $(7/60)s$ miles, so they run a total distance of $(20/60)s$ = $(1/3)s$ miles. When they meet, they will have covered a total of 2 miles. Therefore, $(1/3)s = 2 \rightarrow s =$ **6** mi/h.

118. Some Mathletes may find the 8-15-17 right triangle that has an area of (1/2) × 8 × 15 = 60 units². But there is a Heronian triangle with a smaller area. Heron's formula for the area of a triangle with side lengths a, b and c and semiperimeter s is $A = \sqrt{[s(s-a)(s-b)(s-c)]}$. We can let $a = 17$ and try positive integer values for b and c that are less than 17 and have a sum greater than 17. The first sum greater than 17 is 18, but this would give us a semiperimeter of 35/2. This won't work since we are looking for an integer area. The next sum greater than 17 is 19. This gives us a semiperimeter of 36/2 = 18. This might work. What integer values could we use for the two sides that sum to 19? There is only one combination that will give us an integer area. The triangle with sides 17, 9 and 10 has an area of $\sqrt{[18(18-17)(18-9)(18-10)]} = \sqrt{(18 \times 1 \times 9 \times 8)} = \sqrt{(1296)}$ = **36** units²

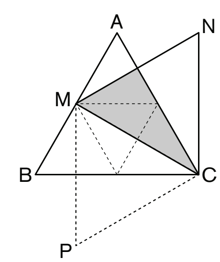
119. The first eight primes are 2, 3, 5, 7, 11, 13, 17 and 19. Each of these primes has two positive factors, 1 and itself. So, the sum of all the numbers Brian writes down is 2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + (1 × 8) = 77 + 8 = **85**.

120. If we add the two equations, we get $2\sqrt{x} = 24 \rightarrow \sqrt{x} = 12 \rightarrow x = 144$. Substituting into the first equation, we get $12 - \sqrt{y} = 10 \rightarrow -\sqrt{y} = -2 \rightarrow y = 4$. The sum is $x + y = 144 + 4 =$ **148**.

Warm-Up 10

121. Since (1!)! × (2!)! × (3!)! × (4!)! = 1! × 2! × 6! × 24!, there is no need to expand this product further. The greatest prime factor of the product is a factor of 24! = 24 × 23 × 22 × ⋯ × 2 × 1. It is **23**.

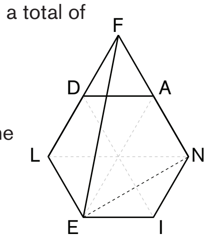
122. In the half hour with the wall charger, Anita's cell phone battery charged 0.5/1.5 = 1/3 of a complete charge. In the 1 hour with her computer, the phone battery charged an additional 1/3 of a complete charge, bringing the total to 1/3 + 1/3 = 2/3 of a complete charge. To charge the remaining 1/3, it will take 1/3 of the 5 hours required to fully charge her phone battery using the portable charger. That's 5/3 hours, which is 5/3 × 60 = **100** minutes.



123. We can solve this problem geometrically if we draw the additional equilateral triangle CMP and introduce a few more lines. Triangle ABC is now subdivided into 8 congruent 30-60-90 triangles, 3 of which are inside of △CMN, so the fraction is **3/8**. For an algebraic solution, we could assign a side length of 2 units to △ABC. Then segment AM would be 1 unit, segment MC would be $\sqrt{3}$ units, and the area of △ABC would be $1/2 \times 2 \times \sqrt{3} = \sqrt{3}$ units². The side length of △CMN would be $\sqrt{3}$ units, its altitude would be $\sqrt{3} \times \sqrt{3}/2 = 3/2$, and its area would be $1/2 \times \sqrt{3} \times 3/2 = (3\sqrt{3})/4$ units². Half of triangle CMN is in △ABC, which is an area of $(3\sqrt{3})/8$ units². The fraction of the area of △ABC inside △CMN, then, is $(3\sqrt{3})/8 \div \sqrt{3}$ = **3/8**.

124. Factoring, we get $3^7 - 27 = 3^7 - 3^3 = 3^3(3^4 - 1) = 27(81 - 1) = 27 \times 80 = 2^4 \times 3^3 \times 5$. The greatest prime factor is **5**.

125. There are 8! = 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 40,320 ways to arrange 8 differently covered balls with no restrictions. Since we only want one of the 3 × 2 × 1 = 6 ways that the red, green and yellow balls can be ordered, we divide this 40,320 by 6 to get **6720** ways. *Alternative solution:* We could, instead, choose the 3 spots for the red, green and yellow balls as a group, out of the 8 positions. This can be done in ${}_8C_3$ = 8!/(5! × 3!) = (8 × 7 × 6)/(3 × 2 × 1) = 8 × 7 = 56 ways. Since the remaining 5 balls can be ordered in 5! ways, we have a total of 56 × 5! = **6720** ways.



126. This regular hexagon can be subdivided into six equilateral triangles, each of side length 6 units, as shown. Triangle DAF is congruent to one of these six triangles. So, triangle DAF has side length 6 units and FN = 2 × 6 = 12 units. Segment EN is twice the altitude of one of the equilateral triangles. So, EN = $2 \times (6\sqrt{3})/2 = 6\sqrt{3}$ units. We can calculate the length of segment FE using the Pythagorean Theorem to get FE = $\sqrt{[12^2 + (6\sqrt{3})^2]} = \sqrt{(144 + 108)} = \sqrt{(252)}$ = $\mathbf{6\sqrt{7}}$ units.

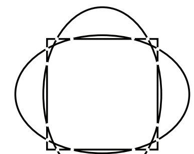
127. Since Annette can pick 4 baskets in one hour, she must have picked 2 baskets in the half hour the three girls worked together. Similarly, Mary must have picked 2.5 baskets in the half hour. That means Lynn picked the remaining 6 − 2 − 2.5 = 1.5 baskets in the half hour. By herself, Lynn must be able to pick 2 × 1.5 = 3 baskets per hour, so she can pick 3 × 3 = **9** baskets in 3 hours.

128. No matter how Kayla divides the stones, it will take 14 divisions to end up with 15 piles of one stone, and the sum of all 14 products will always be the same. (This is worth trying to prove!) We will systematically separate 1 stone at a time to get the following sum of products: $14 + 13 + 12 + \cdots + 2 + 1 = (14 \times 15) \div 2 = 7 \times 15 =$ **105**.

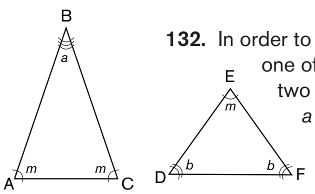
129. The diameter of the sphere is equal to the side length s of the cube. The volume of the cube is s^3, and the volume of the sphere is $4/3 \times \pi \times (s/2)^3 = \pi s^3/6$. The ratio of the volume of the cube to the volume of the sphere is $s^3/(\pi s^3/6) =$ **6/π**. Note that because the sphere is inside the cube, the ratio has to be greater than 1, of course. It is interesting that the ratio is almost 2 (because $\pi \approx 3.1$), meaning the sphere takes up only a little more than half the space inside the cube.

130. To get the maximum possible value of #(5x) – #(4x), we will use the greatest possible value of x, which is something slightly less than 30. If x were equal to 30, then we would have #(5x) = #(5 × 30) = #150. The greatest even integer less than 150 is 148. If x is some real number slightly less than 30, we will still get 148. Similarly, for #(4x), we get #(4 × 30) = #120 = 118. The difference is 148 – 118 = **30**.

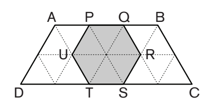
Warm-Up 11



131. Each ellipse can intersect the square in 8 different places, and the two ellipses can intersect each other in 4 different places, so there are a maximum of 2 × 8 + 4 = **20** points of intersection.



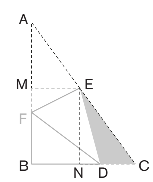
132. In order to have only three different angle measures in the two triangles, there must be two angles of degree measure m in one of the triangles and one angle of degree measure m in the other triangle, as shown. Let a and b represent the other two angle measures. Since the sum of the three different angle measures is 156, we can write the following equations: $a + 2m = 180$, $m + 2b = 180$, $a + m + b = 156$. If we subtract the third equation from the first, we get $m - b = 24 \rightarrow m = b + 24$. Substituting this value for m in the second equation yields $3b + 24 = 180 \rightarrow 3b = 156 \rightarrow b = 52$. We can now substitute this value of b back into the second equation to get $m = 180 - 2 \times 52 =$ **76** degrees. We should also note that $a + 2(76) = 180 \rightarrow a + 152 = 180 \rightarrow a = 28$.



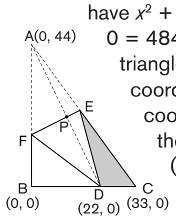
133. As the figure shows, trapezoid ABCD can be subdivided into 16 congruent equilateral triangles. The hexagon PQRSTU accounts for 6/16 = **3/8** of the area of the trapezoid.

134. Since 2, 3 and 5 are relatively prime, the least common multiple is 2 × 3 × 5 = 30. From 30 up to 990 there are 960 ÷ 30 = 32 multiples of 30, 33 multiples if we include 990. These are the only integers that are multiples of 2, 3 and 5. From this list, we need to eliminate the multiples of 4 × 30 = 120 that are divisible by 8. There are 8 of them. That leaves 33 – 8 = **25** integers.

135. Since the individual digits are small, we don't have to worry about the sum in any place value "carrying over" to the next place value. For any given value of A, there is exactly 1 out of the 4 values for E that will make a sum of 5. The same reasoning applies to each of the four place values in the sum ABCD + EFGH. Since the probability in each place value is 1/4 that we get a sum of 5, the probability is 3/4 that we won't get a sum of 5. The probability that we will not get any sums of 5 is 3/4 × 3/4 × 3/4 × 3/4 = 81/256. So, the probability that we will get a sum of 5 must be 1 – 81/256 = **175/256**.



136. Right triangle ABC has sides in the ratio 3:4:5. We can draw two similar triangles by adding two additional segments to our figure. From point E, extend a line parallel to side BC to intersect with AB at point M. Then again from point E, extend a line parallel to side AB to intersect with side BC at point N. Triangle AME is similar to triangle ABC. If we label the length ME as x, then AM is $(4/3)x$. The length EN is $44 - (4/3)x$ and ND is $22 - x$. We know that AE and ED are equal since segment ED was created from folding triangle AEF down. We can use the Pythagorean Theorem for triangles AME and END and solve for x. We have $x^2 + ((4/3)x)^2 = (22 - x)^2 + (44 - (4/3)x)^2 \rightarrow x^2 + (16/9)x^2 = 484 - 44x + x^2 + 1936 - (352/3)x + (16/9)x^2 \rightarrow 0 = 484 - 44x + 1936 - (352/3)x \rightarrow (484/3)x = 2420 \rightarrow x = 15$. Now that we have x, we know that EN = 44 – 4/3 × 15 = 24. So triangle CDE, with base 11 cm, has altitude 24 cm and area 1/2 × 11 × 24 = **132** cm². *Alternative solution:* If we place the figure on a coordinate plane, the coordinates of the various points are A(0,44), B(0,0), C(33,0), D(22,0). Let P be the midpoint of AD; then P has coordinates that are the average of the coordinates of A and the coordinates of D, so P = (11, 22). The slope of line AD is –2, so the slope of line FE is 1/2. Because line FE passes through P, we can write its equation using point-slope form as $y - 22 = (1/2)(x - 11)$. We can also write the equation of side AC as $x/33 + y/44 = 1$. This system can be solved to get the intersection point E as (15, 24). So the area of triangle CDE is (1/2) × 11 × 24 = **132** cm².



137. Using p for the number of passes after the first eight games, we have $(0.7p + 49)/(p + 50) = 0.74 \rightarrow 0.7p + 49 = 0.74p + 37 \rightarrow 0.04p = 12 \rightarrow p = 300$. So, in the first eight games Jason threw 300 passes, and we are told that, in the ninth game, he threw 50 passes. In total, Jason threw 300 + 50 = **350** passes.

138. The seven numbers must be distinct positive integers with a mean of 20. That means the sum of the seven integers must be 140. If the set of integers is arranged in ascending order, the median is the fourth number. The set {1, 2, 3, 4, 14, 16, 100} works, with the least possible value of the fourth number being 4, which is the least possible median. Now to maximize the fourth number, we make the first three numbers as small as possible, giving them values of 1, 2 and 3. That leaves a sum of 140 – (1 + 2 + 3) = 140 – 6 = 134 to be split among the remaining four integers in the set of seven integers. Since the integers all must be distinct, let's consider four consecutive integers with a sum of 134. We have $n + (n + 1) + (n + 2) + (n + 3) = 134 \rightarrow 4n + 6 = 134 \rightarrow 4n = 128 \rightarrow n = 32$. The set {1, 2, 3, 32, 33, 34, 35} works, and the greatest possible median, then, is 32. That's a difference of 32 – 4 = **28**.

139. The product of the digits will equal the units digit if and only if the tens digit is a 1 or the units digit is a zero. The **18** integers that meet this criterion are 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 30, 40, 50, 60, 70, 80, and 90.

140. Only perfect squares have an odd number of positive integer divisors. The only perfect squares she could roll are 1 and 4. So, on any roll of the die, the probability is 2/6 = 1/3 that she will step 1 meter to the right and 2/3 that she will step 1 meter to the left. There are six ways that Colleen could end up right where she started after four rolls: RRLL, RLRL, RLLR, LRRL, LRLR, LLRR. In each case the probability is 1/3 × 1/3 × 2/3 × 2/3 = 4/81, for a total probability of 6 × 4/81 = **8/27**.

Warm-Up 12

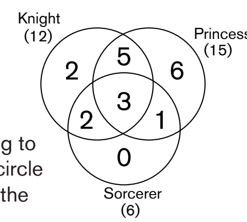
141. There are 50 multiples of 2 from 1 to 100, inclusive. The sum of these numbers is 2 + 4 + 6 + ⋯ + 98 + 100 = (2 + 100) × 50 ÷ 2 = 102 × 25 = 2550. There are 33 multiples of 3 from 1 to 100, inclusive. The sum of these numbers is 3 + 6 + 9 + ⋯ + 96 + 99 = (3 + 99) × 33 ÷ 2 = 51 × 33 = 1683. The absolute difference is 2550 – 1683 = **867**.

142. The general form of a cubic polynomial is $p(x) = ax^3 + bx^2 + cx + d$. We have four specific values for $p(x)$, so we can set up a system of four equations and solve for a, b, c and d. First, we have $p(0) = 4$, so $a(0^3) + b(0^2) + c(0) + d = 4 \rightarrow d = 4$. Then, we have $p(1) = 10$, so $a(1^3) + b(1^2) + c(1) + 4 = 10 \rightarrow a + b + c = 6$. Next, we have $p(-1) = 2$, so $a(-1)^3 + b(-1)^2 + c(-1) + 4 = 2 \rightarrow -a + b - c = -2$. Finally, we have $p(2) = 26$, so $a(2^3) + b(2^2) + c(2) + 4 = 26 \rightarrow 8a + 4b + 2c = 22$. Our system of equations includes $a + b + c = 6$, $-a + b - c = -2$ and $8a + 4b + 2c = 22$. If we add the first and the second equations, we get $2b = 4 \rightarrow b = 2$. Substituting 2 into the first and third equations, we now have $a + 2 + c = 6 \rightarrow a + c = 4$ and $8a + 4(2) + 2c = 22 \rightarrow 8a + 8 + 2c = 22 \rightarrow 8a + 2c = 14 \rightarrow 4a + c = 7$. Subtracting these two resulting equations yields $4a + c - (a + c) = 7 - 4 \rightarrow 3a = 3 \rightarrow a = 1$. Substituting, we see that $1 + c = 4 \rightarrow c = 3$. We can now substitute these values for a, b, c and d in the general cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ to get $p(x) = x^3 + 2x^2 + 3x + 4$. So, $p(3) = (3^3) + 2(3^2) + 3(3) + 4 = 27 + 2(9) + 9 + 4 = 27 + 18 + 13 =$ **58**.

143. The Pythagorean Theorem tells us that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. Since we are dealing with an obtuse triangle, the sum of the squares of the shortest two sides will be less than the square of the longest side. In other words, we want $a^2 + b^2 < c^2$. To achieve the greatest possible perimeter for our obtuse triangle, we will make $c = 100$ and find the greatest integer values of a and b that satisfy this inequality. The square of 100 is 10,000. We want a and b to be nearly equal, so we are looking for a perfect square near 5000. Since $70^2 = 4900$, we'll let $a = 70$ and $b = 71$. Then $70^2 + 71^2 = 4900 + 5041 = 9941$, which is less than 10,000. Since $71^2 + 72^2 = 5041 + 5184 = 10{,}225$, which is greater than 10,000, we see that the greatest possible perimeter is 70 + 71 + 100 = **241** inches.

144. Each of the 25 staff members gives a fist bump to 24 coworkers, but this counts each fist bump twice, so the total is 25 × 24 ÷ 2 = **300** fist bumps.

145. A Venn diagram can help us solve this problem. We draw three overlapping circles, one labeled for each of the three roles knight, princess and sorcerer. Let's start with the 3 students who were willing to play any of the three roles. This number goes in the center, where all three circles overlap. Since there are 8 students willing to play either the knight or the princess, there must be 8 – 3 = 5 who were willing to play the knight or the princess but not the sorcerer. By similar reasoning, we find that there are 5 – 3 = 2 who were willing to play the knight or the sorcerer but not the princess, and 4 – 3 = 1 who was willing to play the princess or the sorcerer but not the knight. Finally, we can fill in the remaining numbers in the regions that are in one circle only. That's 12 – (3 + 5 + 2) = 12 – 10 = 2 willing to play only the knight, 15 – (3 + 5 + 1) = 15 – 9 = 6 willing to play only the princess, and 6 – (3 + 2 + 1) = 6 – 6 = **0** willing to play only the sorcerer.



146. The table shown demonstrates a method of counting all the combinations and sequences of 1, 2 or 3 hops Frankie can use to jump 8 units forward. This is calculated using the formula for the number of permutations of n objects with n_1 identical objects of type 1, n_2 identical objects of type 2, …, and n_k identical objects of type k, which is $n!/[(n_1)! \times (n_2)! \times \cdots \times (n_k)!]$. In total, he can hop from 0 to 8 in 1 + 7 + 6 + 15 + 20 + 10 + 6 + 12 + 1 + 3 = **81** ways.

hop combinations	hop sequences	# ways
1-1-1-1-1-1-1-1	7!/(7!)	1
1-1-1-1-1-1-2	7!/(6! × 1!)	7
1-1-1-1-1-3	6!/(5! × 1!)	6
1-1-1-1-2-2	6!/(4! × 2!)	15
1-1-1-2-3	5!/(3! × 1! × 1!)	20
1-1-2-2-2	5!/(2! × 3!)	10
1-1-3-3	4!/(2! × 2!)	6
1-2-2-3	4!/(1! × 2! × 1!)	12
2-2-2-2	4!/(4!)	1
2-3-3	3!/(1! × 2!)	3

147. The sum of the five integers is $10 + 12 + 26 + x + x = 48 + 2x$. If the mean is 10, then we have $(48 + 2x)/5 = 10 \rightarrow 48 + 2x = 50 \rightarrow x = 1$. The five numbers, in ascending order, are 1, 1, 10, 12 and 26, which satisfies the condition that the median also be 10. If the mean is 12, then we have $(48 + 2x)/5 = 12 \rightarrow 48 + 2x = 60 \rightarrow x = 6$. The five numbers, in ascending order, are 6, 6, 10, 12 and 26, which does not give us a median of 12. So, x cannot be 6. If the mean is 26, then we have $(48 + 2x)/5 = 26 \rightarrow 48 + 2x = 130 \rightarrow x = 41$. The five numbers, in ascending order, are 10, 12, 26, 41 and 41, which satisfies the condition that the median also be 26. Finally, if the mean is x, then we have $(48 + 2x)/5 = x \rightarrow 48 + 2x = 5x \rightarrow 48 = 3x \rightarrow x = 16$. The five numbers, in ascending order, are 10, 12, 16, 16 and 26, which satisfies the condition that the median also be 16. The sum of the possible values of x is 1 + 41 + 16 = **58**.

(3, 97)	(17, 83)	(41, 59)
(11, 89)	(29, 71)	(47, 53)

148. Each of the 6 ordered pairs shown can be reversed for a total of **12** ordered pairs of primes with a sum of 100.

149. Using the given information, we can set up two equations. Because we know the area is 10 cm^2, we can write $1/2 \times b \times h = 10 \rightarrow bh = 20$. We also know this is a right triangle with hypotenuse length 10 cm. This gives us the equation $b^2 + h^2 = 10^2$. If we double the first equation and add it to the second, we have $b^2 + 2bh + h^2 = 140 \rightarrow (b + h)^2 = 140 \rightarrow b + h = \sqrt{140} = 2\sqrt{35}$. The perimeter of the triangle is $\mathbf{10 + 2\sqrt{35}}$ cm.

150. If we multiply the second equation by 4 and add it to the first equation, we eliminate the y term completely and get the much simpler equation $18/(x + 1) = 18/3$. Since the numerators are equal, the denominators must also be equal. So, $x + 1 = 3 \rightarrow x = 2$. Going back to the original equations, we can double the first equation and subtract it from the second equation to eliminate x completely and get $(-18)/(y - 3) = (-18)/3$. Again, since the numerators are equal, the denominators must also be equal. So, $y - 3 = 3 \rightarrow y = 6$. Therefore, $x + y = 2 + 6 =$ **8**. *Alternative solution:* Let $u = 3/(x + 1)$ and $v = 3/(y - 3)$. That yields the much simpler system of equations $2u + 8v = 10$ and $4u - 2v = 2$. Solving this system leads to $u = v = 1$ and then $x = 2$ and $y = 6$. Again, the result is $x + y = 2 + 6 =$ **8**.

Warm-Up 13

151. The numbers could all be 1, resulting in the ordered triple (1, 1, 1). The numbers could all be 0, resulting in the ordered triple (0, 0, 0). Since if one of the numbers is 0, they all must be 0 and otherwise they are all nonzero, the first two equations imply that $n^2 = 1$ and $n = 1$ or $n = -1$. Two of the numbers could be -1 and the third could be 1, resulting in the ordered triples $(-1, -1, 1)$, $(-1, 1, -1)$ and $(1, -1, -1)$. These are the **5** possible ordered triples.

152. By the properties of 30-60-90 right triangles, we know that an equilateral triangle with side length 6 cm has altitude $3\sqrt{3}$ cm. The rectangle with the least area that can enclose this equilateral triangle would have dimensions $a = 6$ cm and $b = 3\sqrt{3}$ cm. The product of the side lengths, then, is $ab = 6 \times 3\sqrt{3} = \mathbf{18\sqrt{3}}$ cm^2.

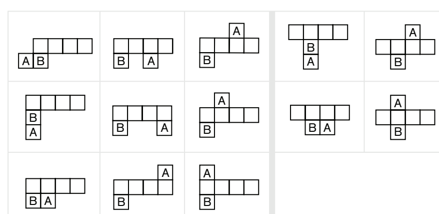
153. Since $44^2 = 1936 < 2018 < 2025 = 45^2$, we know that there are 44 positive integers less than 2018 that are perfect squares. Since $12^3 = 1728 < 2018 < 2197 = 13^3$, we know that there are 12 positive integers less than 2018 that are perfect cubes. If we were to list all these numbers, those that are perfect sixth powers would appear on both lists. Since $3^6 = 729 < 2018 < 4096 = 4^6$, we know that there are 3 positive integers less than 2018 that are perfect sixth powers. Therefore, the total number of positive integers less than 2018 that are perfect squares or perfect cubes is $44 + 12 - 3 =$ **53** integers.

154. Let's try to count all the possibilities in an organized manner. Alexander can get 12 of one type of cookie in 3 ways. He can get 11 of one type and 1 of another type in 3 ways. If there are 12 glazed, then there is only 1 way to complete the order with chocolate and cherry. If there are 11 glazed, then there are 2 ways (1 chocolate or 0 chocolate). If there are 10 glazed, then there are 3 ways. And so on, until if there are 0 glazed, then there are 13 ways (anywhere from 12 to 0 chocolate). So the answer is $1 + 2 + \ldots + 13 = 13 * 14 / 2 =$ **91** assortments. *Alternative solution:* Let's, instead, use the counting technique known as "stars and bars." The idea is that we arrange 12 stars to represent the cookies and two bars to separate the cookies into the three different categories. For example, the arrangement ****|**|****** represents the possibility that Alexander buys 4 of the first type of cookie, 2 of the second type, and 6 of the third type. Our question now becomes: How many ways can we arrange the 12 stars and 2 bars? Thus, the number of assortments of a dozen cookies he can buy is ${}_{14}C_2 = 14!/(12! \times 2!) = (14 \times 13)/(2 \times 1) = 7 \times 13 =$ **91** assortments.

155. That Gabriel and Isabel each start with 20 coins is not important. After the exchange, Gabriel has 23 coins and Isabel has 17 coins, and Gabriel has twice as much money as Isabel. Since we are looking for the greatest possible combined value, let's suppose that Gabriel has 23 quarters. This would be \$5.75 which is not an even number, so we can swap one quarter for a dime, which gives Gabriel \$5.60. If this works, Isabel would have to have \$2.80. She could have 8 quarters, 7 dimes and 2 nickels, which is 17 coins, but we want the fewest dimes Isabel could have. If she has 9 quarters, 3 dimes and 5 nickels, that's 17 coins worth \$2.80. We can't do any better than that, so **3** dimes is the fewest dimes Isabel could have.

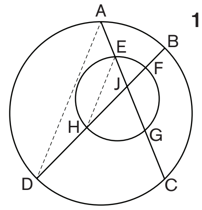
156. We are looking for the least common multiple of 130 and 365. The prime factorization of 365 is 5×73. We know 130 has 5 as a factor, but not 73. The LCM is $130 \times 73 =$ **9490**.

157. The **13** distinct hexominoes that have exactly four squares in a row are shown. To find them all, we systematically "moved" square A around the figure, leaving square B in place for the first 9 hexominoes. Then we shifted square B to the right one unit and "moved" square A around the figure, keeping only the new arrangements that were not duplicates.



158. Since the side length of the hexagon is 6, we know that the height of the square is 6. By properties of 30-60-90 right triangles, we know that the heights of the hexagon and small equilateral triangle are $6\sqrt{3}$ and $3\sqrt{3}$, respectively. The height of the large equilateral triangle, then, is $6\sqrt{3} + 6 + 3\sqrt{3} = 6 + 9\sqrt{3}$. To find the base length of the large equilateral triangle, we divide its height by $\sqrt{3}/2$ to get $(6 + 9\sqrt{3}) \div (\sqrt{3}/2) = (6 + 9\sqrt{3}) \times 2/\sqrt{3} = 12/\sqrt{3} + (18\sqrt{3})/\sqrt{3} = 4\sqrt{3} + 18$. The area of the small equilateral triangle is $1/2 \times 6 \times 3\sqrt{3} = 9\sqrt{3}$. The area of the square is $6 \times 6 = 36$. The area of the hexagon is six times the area of the small equilateral triangle, or $6 \times 9\sqrt{3} = 54\sqrt{3}$. The area of the large equilateral triangle is $1/2 \times (4\sqrt{3} + 18) \times (6 + 9\sqrt{3}) = 1/2 \times (24\sqrt{3} + 108 + 108 + 162\sqrt{3}) = 1/2 \times (216 + 186\sqrt{3}) = 108 + 93\sqrt{3}$. Now we subtract from this the sum of the areas of the three small shapes to get $(108 + 93\sqrt{3}) - (9\sqrt{3} + 36 + 54\sqrt{3}) = 72 + 30\sqrt{3} = a + b\sqrt{c}$. Thus, $a + b + c = 72 + 30 + 3 =$ **105**.

159. We will write our ordered pairs in alphabetical order as (A, B, M, N). Abhi can have any of the prime numbers 2, 3, 5 or 7, and Noreen can have any of the perfect squares 1, 4 or 9. If Bryan has half of Abhi's number, then Abhi has to have 2 and Bryan has 1. That gives us (2, 1, 3, 4), (2, 1, 5, 4), (2, 1, 6, 4), (2, 1, 3, 9) and (2, 1, 10, 9), where Meghna gets the sum of two other values, as long as the sum does not exceed 10. If Bryan has half of Noreen's number, then Noreen has to have 4 and Bryan has 2. That gives us (3, 2, 5, 4), (3, 2, 6, 4), (3, 2, 7, 4), (5, 2, 6, 4), (5, 2, 7, 4), (5, 2, 9, 4), (7, 2, 6, 4) and (7, 2, 9, 4). Finally, if Bryan has half of Meghna's value, then Meghna's value has to be the sum of Abhi's and Noreen's values and this sum must be even. The possibilities are (2, 3, 6, 4), (3, 2, 4, 1), (5, 3, 6, 1) and (7, 4, 8, 1). That's $5 + 8 + 4 =$ **17** assignments.

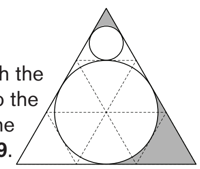


160. The measure of an inscribed angle is half the measure of the arc that it intercepts. Since arc EF has measure 10 degrees, the measure of angle EHF is 10 ÷ 2 = 5 degrees. By the same reasoning, the measure of angle GEH is 98 ÷ 2 = 49 degrees. These two angles are two of the three angles in triangle EJH, so the measure of the third angle, EJH, must be 180 – (5 + 49) = 180 – 54 = 126 degrees. This is the same angle that is in triangle AJD, so the measure of angle AJD is also 126 degrees. Angle JAD is the same as angle CAD, which has measure 83 ÷ 2 = 41.5 degrees. The remaining angle in triangle AJD must have measure 180 – (126 + 41.5) = 180 – 167.5 = 12.5 degrees. This angle, ADJ, is half the measure of the arc AB, so the measure of arc AB is 2 × 12.5 = **25** degrees.

Warm-Up 14

161. The prime factors of n must include the prime factors of 22 and 14, which include 11 and 7. These two primes are not close enough together to be among three consecutive integers. Since we know that n is a multiple of 22, let's consider 21 and 22, which are consecutive and have factors of 7 and 11. We are looking for the least possible value of k, a factor of n, so let the three consecutive positive integers be 20, 21 and 22. Then their product is 20 × 21 × 22 = 9240, and k would have to be 9240 ÷ (22 × 14) = **30**.

162. In the middle column and middle row, 1 + 7 + 13 + 19 + 25 = 10 + 24 + 13 + 2 + 16 = 65. Since 65 must also be the sum of the numbers in every other row and column, we can determine that the missing numbers are 20, 15, 11 and 6. Their sum is **52**. *Alternative solution:* The sum of the numbers in the third row of the magic square is 10 + 24 + 13 + 2 + 16 = 65. The sums of the numbers in the first, second, fourth and fifth rows, respectively, are 45, 50, 54, 59. Since each of these rows should sum to 65, the sum of the missing numbers is 4 × 65 – (45 + 50 + 54 + 59) = **52**.



163. The larger equilateral triangle can be subdivided into 9 equilateral triangles that are all congruent to the equilateral triangle with the smaller circle inscribed in it, as shown in the figure. The centroid of the larger equilateral triangle is 1/3 of the way up the triangle, so the top of the larger circle is 2/3 of the way up. That means the horizontal tangent at the top of the circle is 1/3 of the way down from the top. The ratio of the areas of shaded regions K and P is the same as the ratio of the area of the smaller circle to the larger circle, **1/9**.

164. We will start with the sum of all the numbers, subtract the sum of the multiples of 2 and 3 and then add back the multiples of 6, which will have been subtracted twice. The sum of all the numbers from 2 to 99 is (2 + 99) × 98 ÷ 2 = 101 × 49 = 4949. The sum of the multiples of 2 strictly between 1 and 100 is 2 + 4 + 6 + ⋯ + 96 + 98 = (2 + 98) × 49 ÷ 2 = 50 × 49 = 2450. The sum of the multiples of 3 between 1 and 100 is 3 + 6 + 9 + ⋯ + 96 + 99 = (3 + 99) × 33 ÷ 2 = 51 × 33 =1683. The sum of the multiples of 6 between 1 and 100 is 6 + 12 + 18 + ⋯ + 90 + 96 = (6 + 96) × 16 ÷ 2 = 51 × 16 = 816. The sum we want is 4949 – 2450 – 1683 + 816 = **1632**.

165. There is a 1/3 probability that James will win on his first spin. For him to win on his second spin, James would have to get a PASS, then John would have to get a PASS and then James would have to get WIN, so the probability is 2/3 × 2/3 × 1/3 = 4/27. If we continue this reasoning, we get an infinite sum that looks like this: 1/3 + [$(2/3)^2$ × 1/3] + [$(2/3)^4$ × 1/3] + [$(2/3)^6$ ×1/3] + ⋯. This is an infinite geometric series, and there is a formula that many Mathletes know and can use to calculate the answer. Let's look for a simpler way to think about it. In any given round, there is a 1/3 = 3/9 chance that James wins with his spin and a 2/3 × 1/3 = 2/9 chance that John wins. There is a 2/3 × 2/3 = 4/9 chance that neither of them wins and they play again. The insight is to realize that we can completely ignore the double-pass situations and just look at the odds of James winning to John winning in any round, which is 3/9 to 2/9. Since the odds are 3 to 2, then the probability that James wins is **3/5** and the probability that John wins is 2/5.

166. Rewriting the equation of the line in slope-intercept form, we get $y_1 = (1/3)x + 7/3$. Side AB, with slope (6 – 0)/(2 – 0) = 3 and y-intercept 0 is given by $y_2 = 3x$. We can set the expressions for y_1 and y_2 equal to each other to determine the x-coordinate of the point of intersection N. We get $(1/3)x + 7/3 = 3x \rightarrow x + 7 = 9x \rightarrow 7 = 8x \rightarrow x = 7/8$. Substituting back into the equation for y_2, we see that $y_2 = 3(7/8) = 21/8$. So the line intersects side AB at N(7/8, 21/8). Side BC, with slope (2 – 6)/(8 – 2) = −4/6 = −2/3 and endpoint (2, 6), is given by the equation $y_3 - 6 = (-2/3)(x - 2) \rightarrow y_3 - 6 = (-2/3)x + 4/3 \rightarrow y_3 = (-2/3)x + 22/3$. We can set the expressions for y_1 and y_3 equal to each other to determine the x-coordinate of the point of intersection M. We get $(1/3)x + 7/3 = (-2/3)x + 22/3 \rightarrow x = 15/3 = 5$. Substituting back into the equation for y_3, we see that $y_3 = (-2/3)(5) + 22/3 = -10/3 + 22/3 = 12/3 = 4$. So the line intersects side BC at M(5, 4). The length of segment MN is $\sqrt{[(5 - 7/8)^2 + (4 - 21/8)^2]} = \sqrt{[(33/8)^2 + (11/8)^2]} = \sqrt{[(1089 + 121)/64]} = \sqrt{(1210)}/8 =$ **(11√10)/8** units.

167. If Edna starts with the first bottle on the left and samples every bottle, she will eventually taste a bitter poison and know that the previous bottle contains the magic potion. If Edna samples every even-numbered bottle, she will have to check the odd-numbered bottle right before the first bitter poison she gets. If the odd bottle is tasteless, then it contains the magic potion; if it's bitter, then she will become sick and unable to drink anything else, but she will have succeeded in being able to identify the bottle that contains the magic potion. Likewise, if Edna samples every third bottle, she can go back after she gets the first bitter bottle, but she has to go back to check the bottle after the last tasteless bottle. If we continue with this line of reasoning, we realize that Edna can skip-count by any number she wants. As soon as she gets to a bitter bottle, she has to go back to the bottle that comes right after the last tasteless bottle and check every bottle until she gets to another bitter one. At this point, some students will arrive at the idea of using the square root of 1000, which is a little more than 31. Edna could set out to check multiples of 31 until she comes to a bitter bottle. Edna would check bottles 31, 62, 93, …, up until 992, which is 32 × 31. In the worst-case scenario, bottle 991 would contain the magic potion. Edna would find bottle 992 to be the first bitter one she samples and she would have to go back to check bottles 962 through 991, which would all be tasteless. She would then know that bottle 991 contains the magic potion, but she would have sampled 32 + 30 = 62 bottles. (The total is the same if she counts by 32s instead of 31s.) This is pretty good, but it turns out that Edna can do better. She can take bigger steps at the beginning and reduce the size of her jump as she goes. If she reduces the jump by 1 each time, then what we want is a triangular number that is close to 1000. The 45th triangular number is 45 × 6 ÷ 2 = 1035, so Edna should first sample bottle 45, then bottle 45 + 44 = 89, then bottle 89 + 43 = 132, etc. At any point, when she samples a bitter bottle, she will have to go back and check the bottles in between, and the total number of bottles sampled is always **45** bottles.

short side	long side	qty
1	$\sqrt{2}$	36
2	$2\sqrt{2}$	25
3	$3\sqrt{2}$	16
4	$4\sqrt{2}$	9
5	$5\sqrt{2}$	4
6	$6\sqrt{2}$	1
$\sqrt{2}$	2	30
$2\sqrt{2}$	4	15
$3\sqrt{2}$	6	4

168. It's fairly easy to see the silver rectangles in the figure that have the following dimensions: $1\times\sqrt{2}$, $2\times2\sqrt{2}$, $3\times3\sqrt{2}$, $4\times4\sqrt{2}$, $5\times5\sqrt{2}$ and $6\times6\sqrt{2}$. Less obvious, however, are the silver rectangles in the figure that are $\sqrt{2}\times2$, $2\sqrt{2}\times4$ and $3\sqrt{2}\times6$. For all these rectangles, the ratio of the length of the short side to the length of the long side is exactly $1:\sqrt{2}$. You might miss the last three sizes if you fail to realize that $\sqrt{2}/2 \times \sqrt{2}/\sqrt{2} = 2/(2\sqrt{2}) = 1/\sqrt{2}$, $(2\sqrt{2})/4 \times \sqrt{2}/\sqrt{2} = 4/(4\sqrt{2}) = 1/\sqrt{2}$ and $(3\sqrt{2})/6 \times \sqrt{2}/\sqrt{2} = 6/(6\sqrt{2}) = 1/\sqrt{2}$. The table shows the number of silver rectangles in each of these nine sizes. There are a total of 36 + 25 + 16 + 9 + 4 + 1 + 30 + 15 + 4 = **140** silver rectangles.

169. Since Du passes Priya just as he completes his 4th lap, Priya must be finishing her 3rd lap. At the same moment, Amanda must be finishing 3 1/2 laps, which is the average of 3 and 4 laps. If we double the time, then Du will complete his 8th lap, Amanda will complete her 7th lap, and Priya will complete her 6th lap at the same moment. Since the difference between Amanda and Priya is 1 lap, it must be that the first time she passes Priya, Amanda has completed **7** laps.

170. We saw in problem 163 that the ratio of the areas of the circles inside the equilateral triangle is 1 to 9. The radius of the middle circle in the current problem is 1/3 of the altitude of the equilateral triangle, so if we call the radius of the middle circle 3 units, then the radius of the smallest circle will be 1 unit and the radius of the largest circle will be $3\sqrt{3}$ units. The ratio of the areas of these circles is equal to the square of the ratio of their radii. So the ratio we want is $1^2/(3\sqrt{3})^2 = 1/(9 \times 3) =$ **1/27**.

Workout 1

171. The 8 possible equally-likely outcomes when flipping three coins are HHH, HHT, HTH, HTT, THH, THT, TTH and TTT. Three of the 8 outcomes have exactly two heads, so the probability is **3/8**.

172. Since 300 km/h = 300 ÷ 60 × 1000 ÷ 60 = 250/3 m/s, it follows that Gary's average rate of acceleration is 250/3 ÷ 30 ≈ **2.8** m/s^2.

173. Gordon's weight on Jupiter would be 100 × 318 ÷ 11^2 ≈ **263** pounds.

174. Half of the sum of the domestic gross and international gross was 550 ÷ 2 = 275, so the greatest advertising budget for financial success was 275 − 140 = **135** million dollars.

175. The mean number of days per month in 2018 is 365 ÷ 12 ≈ **30.4** days.

176. The value of A is $(-2)^2 - 2(-2) + 6 = 4 + 4 + 6 = 14$, and the value of B is $(5 \times (-2)^2 - 1)/(-2 + 3) = (20 - 1)/1 = 19$. Therefore, $A + B = 14 + 19 =$ **33**.

177. With a base diameter of 18 cm, the radius is 18 ÷ 2 = 9 cm, and we are told that the volume of the cone is 1187.5 cm^3. The formula for the volume of a cone is $V = 1/3 \times \pi \times r^2 \times h$. So, we have $1/3 \times \pi \times (9)^2 \times h = 1187.5 \rightarrow 27\pi h = 1187.5 \rightarrow h = 1187.5/(27\pi) \approx$ **14** cm.

178. A regular decagon has exterior angles of 360 ÷ 10 = 36 degrees. So, the interior angles must be 180 – 36 = **144** degrees.

179. A decrease of 20% means the portfolio's value decreased to 80%, or 4/5, of its original value. An increase of 25% means the portfolio's value increased to 125%, or 5/4, of its previous value. The value of Elliott's portfolio at the end of February, then, was 5000 × 4/5 × 5/4 = $**5000**.

180. The decade from January 1, 2011, through December 31, 2020, includes 3 leap days, for the years 2012, 2016 and 2020. The number of days is 10 × 365 + 3 = 3653 days. The total number of hours in the decade is 3653 × 24 = **87,672** hours.

Workout 2

181. If we call the length $16x$ and the height $9x$, then the equation for the area of the screen is $16x \times 9x = 576 \rightarrow 144x^2 = 576 \rightarrow x^2 = 576/144 \rightarrow x^2 = 4 \rightarrow x = 2$. The length of the screen must be 16 × 2 = 32 inches, and its height must be 9 × 2 = 18 inches. The perimeter is 2(32 + 18) = 2(50) = **100** inches.

182. Using Alex's biking pace of 7 minutes per mile and the total round-trip time of 20 + 15 = 35 minutes, we can set up the proportion $7/1 = 35/x$ and solve to see that $7x = 35 \rightarrow x = 5$. So, the round-trip distance is 5 miles, and the trip from home to school is 5 ÷ 2 = **2.5** miles.

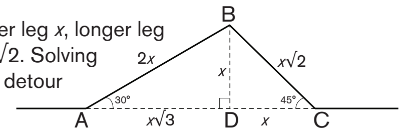
183. Bruce, Lawson and the goal are located at the vertices of a 3-4-5 right triangle. The distance from Lawson to the goal is 50 meters. The puck will reach the goal in 40 ÷ 50 = 0.8 second. To reach the goal at the same time as the puck, Lawson must skate at 50 ÷ 0.8 = **62.5** m/s.

184. To calculate a tax of 9% on a bill of $619, we multiply to get 0.09 × 619 = $**55.71**.

185. If we say Sara gets x dollars per cake, then Linda gets $x - 11$ dollars per cake. For them to earn the same amount each hour, the following must be true: $4x = 6(x - 11)$. Solving for x, we get $4x = 6x - 66 \rightarrow 2x = 66 \rightarrow x = 33$. Sara must be paid $**33** per cake.

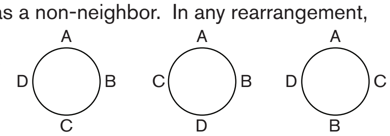
186. If Jason wants the mean of his six drives to be 400 yards, the sum of his drives must be 6 × 400 = 2400 yards. So far the sum of his drives is 394 + 401 + 387 + 414 + 421 = 2017 yards. The last drive must be at least 2400 − 2017 = **383** yards. *Alternative solution:* If we consider each distance relative to 400 yards, then the result of the first five drives is −6 + 1 − 13 + 14 + 21 = 17 yards. To ensure a mean of at least 400 yards for all six drives, the sixth drive must be at least −17, or 400 − 17 = **383** yards.

187. The distance from A to C along the original route is 1000 km. The 30-60-90 right triangle ABD has shorter leg x, longer leg $x\sqrt{3}$ and hypotenuse $2x$. The 45-45-90 right triangle BCD has two legs of length x and hypotenuse of length $x\sqrt{2}$. Solving the equation $x\sqrt{3} + x = 1000$, we get $x(\sqrt{3} + 1) = 1000 \rightarrow x = 1000/(\sqrt{3} + 1)$. So the difference between the detour and the intended route is $2(1000/(\sqrt{3} + 1)) + (1000/(\sqrt{3} + 1))\sqrt{2} - 1000 \approx$ **250** km.



188. This is calculated as follows: 0.12 × 3/4 × 1.8 = **0.162**.

189. When four people sit around a circular table, each person has two neighbors, leaving only one person as a non-neighbor. In any rearrangement, there must be at least one repeat neighbor. As the figure shows, from the perspective of person A, we basically give each of persons B, C and D a turn at being the non-neighbor across from A. Thus, four people can be seated in **3** ways.



190. The ratio of the larger radius to the smaller radius is 4.5/3 = 3/2. This is the scale factor between the two similar containers. The volume of the larger container will be greater by a factor that is the cube of this scale factor. We are told that the volume of the smaller container is 20 fluid ounces, so the volume of the larger container will be $20 \times (3/2)^3 = 20 \times 27/8 = 135/2 =$ **67.5** fluid ounces.

Workout 3

191. The number pattern is 15, 17, 20, 22, 25, 27, …. We can separate this into the number pattern of the odd rows, which is 15, 20, 25, …, and the pattern of the even rows, which is 17, 22, 27, …. In both cases, the terms increase by 5 each time, so the last term will be 14 × 5 = 70 more than the first term. The number of seats in the 15 odd rows is 15 × (15 + 85) ÷ 2 = 15 × 100 ÷ 2 = 15 × 50 = 750 seats, and the number of seats in the 15 even rows is 15 × (17 + 87) ÷ 2 = 15 × 104 ÷ 2 = 15 × 52 = 780 seats. The total number of seats in all 30 rows is 750 + 780 = **1530** seats.

192. A pentagon can be subdivided into three triangles, so the interior angles' sum must be 3 × 180 = 540 degrees. The three known angles have a sum of 110 + 120 + 130 = 360, so the remaining two angles must have a sum of 540 − 360 = 180. Since one of the remaining angles is three times the measure of the other, we can divide the 180 degrees by 4 to get the smallest angle of **45** degrees. The other angle is 180 − 45 = 135 degrees.

193. The possible prime values for a are 2, 3, 5 and 7. The possible composite values for b are 4, 6, 8, 9 and 10. The possible perfect square values for c are 1, 4 and 9. The possible perfect cube values for d are 1 and 8. If we choose the greatest from each of these lists, we get the distinct numbers $a = 7$, $b = 10$, $c = 9$ and $d = 8$. The greatest possible sum of four numbers chosen as described is 7 + 10 + 9 + 8 = **34**.

194. If p and q are both odd primes, then the product pq would also be odd, and r would have to be even. Since the only even prime is 2, the product pq would have to be 73 − 2 = 71, but 71 is prime. This means that either p or q will have to be 2 so that we get an even product pq. We'll let $p = 2$. Now, in order to minimize the sum $p + q + r$, we want q, which gets doubled, to be a large prime so that r can be small. If $q = 31$, we get 2 × 31 + 11 = 73, and $p + q + r = 2 + 31 + 11 =$ **44**.

195. If we let B be Bella's speed, T be Thomas's speed and M be Tam's speed, then we have the two equations $B = 0.4T$ and $B = 0.35M$, which means that $0.4T = 0.35M \rightarrow M = (0.4/0.35)T \rightarrow M = (8/7)T$. Tam is 1/7 ≈ 0.14 = **14**% faster than Thomas.

196. There are three possibilities that qualify as rain on at least one of the two days. First, the probability that there is rain on both days is 0.5 × 0.4 = 0.20. Second, the probability that it rains on the first day but not on the second day is 0.5 × 0.6 = 0.30. Third, the probability that it does not rain on the first day but does rain on the second day is 0.5 × 0.4 = 0.20. The combined probability for rain on at least one day, then, is 0.2 + 0.3 + 0.2 = 0.7, which is **70%**. *Alternative solution:* We could have calculated the probability that it does not rain on either day, which is 0.5 × 0.6 = 0.3, and subtracted this from 1 to get 1 − 0.3 = 0.7, or **70**%.

197. A 60 degree sector of a circle accounts for 60/360 = 1/6 of the circle's area. So, the area of the 60 degree sector of a circle of radius 30 feet is $(1/6) \times \pi \times 30^2 =$ **150π** ft^2.

198. Half of $700 million is $350 million, and this amount was made 365 times. The weight of that many $1 bills was 350,000,000 × 365 = 127,750,000,000 grams. Since 1000 grams is equivalent to a kilogram, that's 127,750,000,000 ÷ 1000 = **127,750,000** kg.

n	$B(n)$
22	33,902
23	34,408
24	34,776
25	35,000
26	35,074
27	34,992

199. The graph of the given cubic function is an "S" shape rotated 90 degrees. If we factor $B(n)$, we get $B(n) = n(45 - n)(45 + n)$. The zeros of this function will be −45, 0 and 45. So the maximum we are looking for occurs somewhere between $x = 0$ and $x = 45$. We would expect the maximum to occur close to the middle of the interval. Since 22.5 is the exact middle, we can start by looking at 22 and 23. We see that the function value increases from 22 to 23, so let's continue until we see a decrease. The function begins to decrease between 26 and 27. The greatest harvest value is at **26** trees per acre as shown in the table.

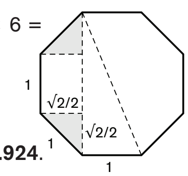
200. There were p cupcakes three days ago. Two days ago, 20% of the cupcakes were eaten, so $(4/5) \times p$ cupcakes remained. We don't know how many cupcakes there were yesterday, but let's say that the number from two days ago got multiplied by some factor x, so that there were $(4/5) \times xp$ cupcakes yesterday. Today, there are 30% fewer cupcakes than yesterday, or $(7/10) \times (4/5) \times xp = (14/25) \times xp$ cupcakes. We are told that this quantity is equal to $(1/2) \times p$. We can solve the equation $(14/25) \times xp = (1/2) \times p$ to get $(14/25)x = 1/2 \rightarrow x = 25/28$. Thus, we need to find a number that is a multiple of 5 (so that there were a whole number of cupcakes three days ago) and 4/5 of which is a multiple of 28, which means that it is a multiple of $28 \div 4 = 7$. The smallest number that is a multiple of both 5 and 28 is $5 \times 28 = 70$. If $p = 70$, then the numbers of cupcakes on the last few days were: 70, 56, 50, 35. We get half of p for today, so it works. The least possible value of p is **70**.

Workout 4

201. Multiplying 15 times the units digits 1, 2, 3, 4, 5 or 6 will result in a two-digit number. Here are the first six multiples of 15: 15, 30, 45, 60, 75 and 90. Since $75 = 15 \times 5$ and $7 + 5 = 12$, we conclude that the two-digit integer in question is **75**.

202. The trinomial $x^2 + 8x + 15$ can be factored into the product of two binomials $(x + 3)(x + 5)$. So, $(x^2 + 8x + 15)/(x + 5) = 4.01 \rightarrow (x + 3)(x + 5)/(x + 5) = 4.01 \rightarrow x + 3 = 4.01 \rightarrow x =$ **1.01**.

203. Since the first and second ordered pairs both have 2 as the x-coordinate, we can use the side that connects these two points as a base. It has length $2 - (-6) = 8$ units. The altitude from the base to the third point has length $2 - (-5) = 7$ units. The area of the triangle is $(1/2) \times 8 \times 7 =$ **28** units^2.



204. The sum of the interior angles of a regular n-gon is $180 \times (n - 2)$. So the sum of the interior angles of a regular octagon is $180 \times 6 = 1080$. Each interior angle has degree measure $1080 \div 8 = 135$. In the figure, the two shaded triangles are 45-45-90 right triangles with hypotenuse length 1 unit. Using the properties of 45-45-90 right triangles, we can determine that the length of each leg is $(1/\sqrt{2}) \times (\sqrt{2}/\sqrt{2}) = \sqrt{2}/2$ units. As the figure shows, the medium diagonal has length $\sqrt{2}/2 + 1 + \sqrt{2}/2 = \sqrt{2} + 1$ units. To find the length of the long diagonal, we use the Pythagorean Theorem as follows: $\sqrt{(1^2 + (\sqrt{2} + 1)^2)} = \sqrt{(1 + (2 + 2\sqrt{2} + 1))} = \sqrt{(4 + 2\sqrt{2})}$ units. The ratio of the length of the medium diagonal to the length of the long diagonal, then, is $(\sqrt{2} + 1)/(\sqrt{(4 + 2\sqrt{2})}) \approx$ **0.924**.

205. If the cost of the meal was x dollars, then the meal plus the first 18% tip was $1.18x$. The cost after the second tip was added was $1.15 \times 1.18x = 1.357x$. Emalee actually paid a **35.7%** tip. Note that the answer is not 18 + 15 = 33%, because the second tip was 15% of the larger amount $1.18x$.

206. From the first "no," Penner learns that the card is neither red nor a multiple of 2. That eliminates all 10 red cards and the 5 even cards from each of the other colors. The 15 remaining cards are odd numbers that are blue, green or yellow. From the next answer, "yes," Penner learns that the card is either blue or a multiple of 3. The 9 remaining cards are odd blues, the green 3 and 9 and the yellow 3 and 9. From the next answer, "no," Penner learns that the card is neither green nor a multiple of 5. That eliminates all the green cards and the blue 5, leaving the 1, 3, 7 and 9 that are blue and the 3 and 9 that are yellow. Finally, from the last answer, "yes," Penner learns that the card is either yellow or a multiple of 7. That leaves the 7 that is blue and the 3 and 9 that are yellow. With these 3 cards to choose from, the probability that Penner guesses Tell's secret card is **1/3**.

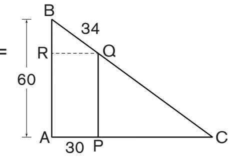
207. There are 90 two-digit numbers, so the denominator of our probability is 90. There are 9 two-digit numbers with digits that have an absolute difference of 0, namely 11, 22, 33, 44, 55, 66, 77, 88 and 99. Now we just need to count the numbers with digits that have an absolute difference of 1, such as 10 and 12. There are 2 of these in every "decade" except the 90s, so there must be $2 \times 9 - 1 = 17$ of them. Since there are $9 + 17 = 26$ numbers that have digits with an absolute difference of 0 or 1, there must be $90 - 26 = 64$ numbers that have digits with an absolute difference greater than 1. The probability is, thus, 64/90 = **32/45**.

208. Ngorongoro Crater is 10 miles across, so its radius is 5 miles. Its depth of 2000 feet is 2000/5280 = 25/66 mile. The formula for the volume of a cylinder of radius r and height h is $\pi \times r^2 \times h$. This particular cylindrical crater has a volume of $\pi \times 5^2 \times (25/66)$. So, it will take $625\pi/66 \approx$ **30** mi^3 of water to fill the crater.

209. If there are x candies in each of the 25 bags on the second day, that would be $25x$ candies in all, which equals the original w candies. When Mady redistributes the candies into 26 bags on the third day, there are $x - 2$ candies in each bag, so we have the equation $26(x - 2) = 25x \rightarrow 26x - 52 = 25x \rightarrow x = 52$. The value of w must be $25 \times 52 =$ **1300**.

210. If $f(x) = ax^2 + bx + c$ and $f(0) = 4$, then $a(0)^2 + b(0) + c = 4 \rightarrow c = 4$. We are told that $f(2) = 2$. Substituting 4 for c, we get $a(2)^2 + b(2) + 4 = 2 \rightarrow 4a + 2b + 4 = 2 \rightarrow 4a + 2b = -2 \rightarrow 2a + b = -1$. Similarly, since $f(4) - f(3) = 4$, we have $(a(4)^2 + b(4) + 4) - (a(3)^2 + b(3) + 4) = 4 \rightarrow 16a + 4b + 4 - 9a - 3b - 4 = 4 \rightarrow 7a + b = 4$. Subtracting the two equations yields $(7a + b) - (2a + b) = 4 - (-1) \rightarrow 7a + b - 2a - b = 4 + 1 \rightarrow 5a = 5 \rightarrow a = 1$. Substituting this back into $2a + b = -1$ yields $2(1) + b = -1 \rightarrow b = -3$. So, $f(x) = x^2 - 3x + 4$ and $f(1) = (1)^2 - 3(1) + 4 = 1 - 3 + 4 =$ **2**.

Workout 5

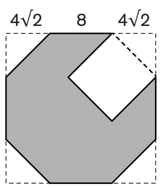


211. If we draw segment RQ parallel to AC, as shown, then we create triangle RBQ, which is similar to triangle ABC. We know that BQ = 34 and RQ = AP = 30. Using the Pythagorean Theorem, we find that BR = $\sqrt{(34^2 - 30^2)} = \sqrt{(1156 - 900)} = \sqrt{256}$ = 16. Since AB = 60, it follows that AR = PQ = 60 − 16 = 44. Triangles ABC and PQC are similar, so the ratios of corresponding sides are equal. In particular, AB/PQ = AC/PC → 60/44 = (30 + PC)/PC → 15/11 = (30 + PC)/PC → 15 × PC = 11 × (30 + PC) → 15PC = 330 + 11PC → 4PC = 330 → PC = 82.5. So, AC = 30 + 82.5 = 112.5, and triangle ABC has area 1/2 × 112.5 × 60 = **3375** units^2.

212. Since all ten digits must be used in the five two-digit numbers, the greatest range between the first and the last number will occur in a list such as 98, 76, 54, 32, 10, and the range is 98 − 10 = 88. The least positive range will occur in a list such as 50, 46, 37, 28, 19, and the range is 50 − 19 = 31. The difference between the greatest and the least possible ranges is 88 − 31 = **57**.

213. There are 6! = 720 ways to arrange the donuts and 6! = 720 ways to arrange the cookies. Mackenzie also has to decide whether to start with a donut or a cookie, so there are 2 × 720 × 720 = **1,036,800** ways to create an arrangement with alternating donuts and cookies.

214. Since the *n*th worker takes *n* hours to complete the job alone, that worker completes exactly 1/*n* of the job each hour. Collectively, *k* people must complete 1 + 1/2 + 1/3 + ⋯ + 1/*k* of the job each hour. The reciprocal of this sum is the fraction of an hour it would take the group to complete 1 job. We want to find the value of *k* for which the previous sum is first greater than 3, since its reciprocal would be less than 1/3 of an hour, or 20 minutes. That occurs when *k* = 11 and 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10 + 1/11 ≈ 3.02. This means 11 people could do more than 3 of the jobs in an hour, so each job would take less than 20 minutes. The fewest workers needed, then, to complete the job in under 20 minutes, working together, is **11** workers.



215. Since the perimeter of the octagon is 64 cm, each side of the octagon has length 64 ÷ 8 = 8 cm. If we draw a square around the octagon, as shown, the 45-45-90 triangles in the four corners have hypotenuse length 8 cm and side length $8/\sqrt{2} \times \sqrt{2}/\sqrt{2} = 8\sqrt{2}/2 = 4\sqrt{2}$ cm. The enclosing square has side length $4\sqrt{2} + 8 + 4\sqrt{2} = 8 + 8\sqrt{2}$ cm and area $(8 + 8\sqrt{2})^2 = 64 + 128\sqrt{2} + 128 = 192 + 128\sqrt{2}$ cm^2. The four triangles in the corners can be combined to make a single 8-by-8 square, so we need to subtract 64 cm^2 from the area of the square to get the area of the octagon, which is $192 + 128\sqrt{2} - 64 = 128 + 128\sqrt{2}$ cm^2. Finally, we need to subtract the area of the 8-by-8 cutout square to get a total area of $128 + 128\sqrt{2} - 64 =$ **$64 + 128\sqrt{2}$** cm^2.

216. Of the 16 distinct sets of inputs that can be applied on the far left of the function machine, (1, 1, 1, 1), (1, 1, 1, 0) and (1, 1, 0, 1) are the only **3** inputs that result in an output of 1.

217. If the amount Bob paid was *x*, then Joe and Randell each paid 2*x*. Twenty-five percent, or 1/4, of $80 is 80 ÷ 4 = $20. So, the group paid a total of 80 + 20 = $100. We have 2*x* + 2*x* + *x* = 100 → 5*x* = 100 → *x* = 20. Bob must have paid $**20**, and Joe and Randell each paid $40.

a − *b*	*a* + *b*	*a*	*b*
1	144	72.5	71.5
2	72	37	35
3	48	25.5	22.5
4	36	20	16
6	24	15	9
8	18	13	5
9	16	12.5	3.5
12	12	12	0

218. The left side of the equation $a^2 - b^2$ is called a difference of two squares and can be factored so that we get the equivalent equation $(a - b)(a + b) = 144$. We now consider the following factor pairs of 144: 1 × 144, 2 × 72, 3 × 48, 4 × 36, 6 × 24, 8 × 18, 9 × 16, 12 × 12. We are looking for the pairs of factors that are the difference and sum of two positive integers *a* and *b*. Consider *a* − *b* = 1 and *a* + *b* = 144. If we add these equations, we get 2*a* = 145 → *a* = 72.5, and *b* = 144 − 72.5 = 71.5. The table shows the values of *a* and *b* for each factor pair of 144. There are **4** positive integer pairs whose squares have a difference of 144

219. The formula for the volume of a rectangular prism is $V = l \times w \times h$. Since 1 m = 100 cm and 10 mm = 1 cm, the volume of this box is 0.5 × 1 × 100 = **50** cm^3.

220. Let *c* represent the number of cars there used to be on campus, and *b* represent the number of bicycles there used to be. Then the current number of cars is 0.7*c*, and the current number of bicycles is 1.2*b*. We are told that there is a 1:3 ratio between the numbers of cars and bicycles. So, 1/3 = 0.7*c*/1.2*b* → *c*/*b* = 1.2/2.1 = 12/21 = **4/7**.

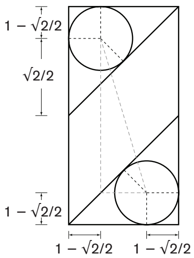
Workout 6

221. If we let 2*x* inches equal the side length of the square, then the distance from the center of the square to the midpoint of a side is *x* inches. Using the properties of 45-45-90 right triangles, we can determine that the distance from the center to a corner of the square is $x\sqrt{2}$ inches. A radius of the circle passes through each corner of the square and has length $x\sqrt{2} + 1$ inches. A radius that passes through the midpoint of a side of the square is *x* + 2 inches. Since all radii are equal, it follows that $x\sqrt{2} + 1 = x + 2 \to x\sqrt{2} - x = 1 \to x(\sqrt{2} - 1) = 1 \to x = 1/(\sqrt{2} - 1) \times (\sqrt{2} + 1)/(\sqrt{2} + 1) = \sqrt{2} + 1$. The circle has radius $x + 2 = \sqrt{2} + 1 + 2 = \sqrt{2} + 3$ inches and area $\pi \times (\sqrt{2} + 3)^2 = (11 + 6\sqrt{2})\pi$ in^2. The square has side length $2x = 2 \times (\sqrt{2} + 1) = 2\sqrt{2} + 2$ inches and area $(2\sqrt{2} + 2)^2 = 12 + 8\sqrt{2}$ in^2. The area of the gasket is the difference $((11 + 6\sqrt{2})\pi) - (12 + 8\sqrt{2}) \approx$ **37.9** in^2.

222. Getting half off the regular price on half the socks is the same as getting one quarter off the regular price on all the socks. Jeffrey paid 3/4, or 75% of the regular price for all the socks. That's a savings of **25**%.

223. The median of 10 numbers is the average of the 5th and 6th numbers when they are listed in order. The lowest possible median of a list of positive integers is 1. We can get a median of 1 if we include six 1s. We then want to raise the mean as much as possible by including four 20s. Our data set is {1, 1, 1, 1, 1, 1, 20, 20, 20, 20}, with a median of 1 and a mean of 86 ÷ 10 = 8.6. The absolute difference between the median and the mean is 8.6 − 1 = **7.6**. We get the same answer with the set {1, 1, 1, 1, 20, 20, 20, 20, 20, 20}, which has a median of 20 and a mean of 124 ÷ 10 = 12.4. The absolute difference is again 20 − 12.4 = **7.6**.

224. Let x represent the number of tourists beyond the first 15 tourists. From the information provided, we have $(x + 15)(520 - 5x) = 12{,}740 \rightarrow 520x - 5x^2 + 7800 - 75x = 12{,}740 \rightarrow 5x^2 - 445x + 4940 = 0 \rightarrow x^2 - 89x + 988 = 0$. Factoring, we see that $(x - 76)(x - 13) = 0 \rightarrow x - 76 = 0 \rightarrow x = 76$ or $x - 13 = 0 \rightarrow x = 13$. Since the maximum number of tourists is 36, our solution is $x = 13$, meaning there were 15 + 13 = **28** tourists.



225. The two lines drawn to the midpoint of each side of the rectangle create two 45-45-90 right triangles. As the figure shows, radii drawn to each circle's points of tangency with the sides of the right triangle form a square and two congruent kites. Each isosceles right triangle has leg length 1 inch and hypotenuse length $\sqrt{2}$ inches. The long side of each kite has length $\sqrt{2}/2$ inch, making the radius of each circle $1 - \sqrt{2}/2$ inch. The vertical distance between the centers of the circles is $2 - 2(1 - \sqrt{2}/2) = 2 - 2 + \sqrt{2} = \sqrt{2}$ inches. The horizontal distance between the centers of the circles is $1 - 2(1 - \sqrt{2}/2) = 1 - 2 + \sqrt{2} = \sqrt{2} - 1$ inch. Using the Pythagorean Theorem, we see that the distance between the centers of the circles is $\sqrt{((\sqrt{2})^2 + (\sqrt{2} - 1)^2)} = \sqrt{(2 + 2 - 2\sqrt{2} + 1)} = \sqrt{(5 - 2\sqrt{2})} \approx$ **1.47** inches.

226. If we consider the smallest rectangle to be a 1×1 square, we can make an organized list to count the number of rectangles in the figure. For each size, the number of rectangles is as follows: 1×1 - <u>12</u>, 1×2 - <u>6</u>, 2×1 - <u>6</u>, 1×3 - <u>4</u>, 3×1 - <u>4</u>, 1×4 - <u>2</u>, 4×1 - <u>2</u>, 2×2 - <u>1</u>, 2×3 - <u>2</u>, 3×2 - <u>2</u>, 2×4 - <u>1</u>, 4×2 - <u>1</u>, 3×3 - <u>4</u>, 3×4 - <u>2</u>, 4×3 - <u>2</u>, 4×4 - <u>1</u>. That's a total of 12 + 6 + 6 + 4 + 4 + 2 + 2 + 1 + 2 + 2+ 1 + 1 + 4 + 2 + 2 + 1 = **52** rectangles.

227. The set of all three-digit integers that are perfect squares and whose digits, when reversed, also form a perfect square is {121, 144, 169, 441, 484, 676, 961}. The sum of these integers is 121 + 144 + 169 + 441 + 484 + 676 + 961 = **2996**.

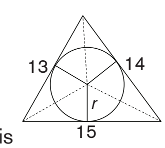
228. The ratio of Shandra's weight to her little sister's weight is 96/72 = 4/3. For the seesaw to be perfectly balanced, the fulcrum should be positioned so that the 14-foot beam is split in a 3 to 4 ratio. Thus the fulcrum should be placed at a distance of 3/7 ×14 = **6** feet from Shandra, which results in a distance of 4/7 × 14 = 8 feet from her sister. (Notice that 96 × 6 = 72 × 8 = 576.)

229. Since we know that the quantities are distinct and pairwise relatively prime, the least we can hope for is 1 + 2 + 3 + 5 = 11. Let's see if we can find a scenario that works with those quantities and totals \$1.08. We can start by letting $p = 3$. That leaves 1.08 − 0.03 = \$1.05 left to make with only nickels, dimes and quarters. If we then let $n = 1$, $d = 5$ and $q = 2$, our total is \$1.08. Thus, the least possible value of $p + n + d + q$ is **11**.

230. The 12-foot altitude of this right square pyramid meets the square base at its center, which is 5 feet away from the midpoint of a side of the square base. This forms a 5-12-13 triangle, which is a Pythagorean Triple. The 13-foot side is the slant height of each of the four triangular faces of the pyramid. The combined area of the four triangular faces is 4 × 1/2 × 10 × 13 = 260 ft^2. The area of the 10-foot by 10-foot square base is 10 × 10 = 100 ft^2. The total surface area of the pyramid, then, is 260 + 100 = **360** ft^2.

Workout 7

231. The least common multiple (LCM) of 10, 12 and 18 is 180. The monetary value of 180 quarters is 180 ÷ 4 = **\$45**.



232. We can use Heron's formula to find the area of the triangle given its three side lengths. A triangle with side lengths a, b and c and with semiperimeter $s = (a + b + c)/2$ has area $A = \sqrt{(s(s - a)(s - b)(s - c))}$. This triangle has side lengths 13, 14 and 15 units. So, $s = (13 + 14 + 15)/2 = 42/2 = 21$ units. The area of the triangle, then, is $\sqrt{(21(21 - 13)(21 - 14)(21 - 15))} = \sqrt{(21 \times 8 \times 7 \times 6)} = \sqrt{(7 \times 3 \times 2 \times 2 \times 2 \times 7 \times 3 \times 2)} = \sqrt{(7^2 \times 3^2 \times 4^2)} = \sqrt{(7 \times 3 \times 4)^2} = 7 \times 3 \times 4 = 84$ units^2. As the figure shows, the total area of this triangle is the sum of the areas of the three triangles with bases of lengths 13, 14 and 15 units, each with altitude equal to the radius of the circle. So, we have $1/2 \times 13 \times r + 1/2 \times 14 \times r + 1/2 \times 15 \times r = 84 \rightarrow (1/2)r \times 42 = 84 \rightarrow (1/2)r = 2 \rightarrow r =$ **4** units.

233. There are 24 hours in a day, so 2^{18} hours from any day is $2^{18} \div 24 = 32{,}768/3 = 10{,}922\ 2/3$ days later. Since Francisco is born at 1:00 a.m., the extra 2/3 of a day is still the same day, 2/3 × 24 = 16 hours later, at 5:00 p.m. We just need to determine the value of 10,922 (mod 7), which is the remainder when 10,922 is divided by 7. Since 10,920 is a multiple of 7, it follows that 10,922 (mod 7) ≡ 2. Since Francisco is born on Tuesday, he gets married on the day of the week that is 2 days after Tuesday, which is **Thursday**.

234. Erica will have two consecutive numbers if and only if she draws a 3 or a 5 of any suit. There are four suits, so Erica can draw any of 8 cards out of the remaining 51 cards in the deck. The probability is **8/51**.

235. There are 5 numbers from 3 to 7, namely 3, 4, 5, 6 and 7. A set of 5 elements has $2^5 = 32$ subsets. Excluding the null set and the complete set, there are at most 30 possibilities to test. If we consider the clues carefully, we should not have to try all 30 subsets. We might notice that statements 5 and 7 cannot both be true. The only composite numbers available are even, and any even factor will make the product even. This means we cannot have both 5 and 7 in the set. Furthermore, if statement 3 is correct, then we can't have either 5 or 7 since two odds would make the sum even. If we left 3 out of the set and just included 5 or 7, then statement 3 would be correct, in which case 3 should be included. If we try to exclude all the odd numbers, then statement 6 would be false and the set would contain only the number 4. But that would make statement 5 true, which is a contradiction. All this means that 3 must be the only odd and only prime number in the set. If 3 is the only prime, then statement 6 is correct and we include 6. Now we have to include 4 also, so that statement 5 will be false. The set must be $S = \{3, 4, 6\}$, the statements are true, true, false, true, false in that order, and the product of the numbers in S is $3 \times 4 \times 6 =$ **72**.

236. We can divide the circle into 6 congruent sectors, each composed of an equilateral triangle and a shaded segment. The area of the shaded segment is the difference between the area of the sector, which is 1/6 the area of the circle, and the area of the equilateral triangle. Since the regular hexagon is composed of 6 congruent equilateral triangles of side length 12 meters, it follows that the radius of the circle also is 12 meters. Thus, the area of each sector is $1/6 \times 12^2 \times \pi = 24\pi$ m². Using properties of 30-60-90 right triangles, we see that the area of one equilateral triangle is $1/2 \times 12 \times 6\sqrt{3} = 36\sqrt{3}$ m². If follows, then, that the area of one segment is $24\pi - 36\sqrt{3}$ m². The total area of the shaded regions is $6 \times (24\pi - 36\sqrt{3}) = 144\pi - 216\sqrt{3} \approx$ **78** m².

237. To get the greatest possible value of $k - m$, we want to make m as small as possible. Working backward, we can subtract 6 from 200 a total of 33 times and get 2 (because $200 - 33 \times 6 = 2$). If we continue to subtract 6, the result is a negative number. But Mason must take an even number of steps, since Kendra takes only half as many steps. Let's have Mason start at 8 and count up by 6 a total of 32 times. Then Kendra would have to start at $200 - 16 \times 4 = 136$. This gives us the greatest possible difference $k - m = 136 - 8 =$ **128**.

238. If we rewrite the equation $3x - 4y = 13$ in slope-intercept form, we get $y = 0.75x - 3.25$. The y-intercept is -3.25. To translate the line 2018 units to the right, we substitute $x - 2018$ for x and get $y = 0.75(x - 2018) - 3.25 \rightarrow y = 0.75x - 1513.5 - 3.25 \rightarrow y = 0.75x - 1516.75$. So, the y-intercept of the translated line is **−1516.75**.

239. If we subtract Ron's base salary from this week's earnings, we get 383.75 − 215 = $168.75, which must have been 15% of his sales this week. Now we can set up the percent proportion $168.75/x = 15/100 \rightarrow 15x = 168.75 \times 100 \rightarrow 15x = 16{,}875 \rightarrow x = 1125$. This means that Ron sold $1125 worth of wallets during his five days of work this week. His average daily sales must have been 1125 ÷ 5 = **$225**.

240. A regular hexagon is made up of six equilateral triangles, so the area of each triangle must be $216\sqrt{3} \div 6 = 36\sqrt{3}$ in². The area of an equilateral triangle of side length s is $\sqrt{3}/4 \times s^2$. We have $36\sqrt{3} = \sqrt{3}/4 \times s^2 \rightarrow s^2 = 144 \rightarrow s = 12$ inches. This is also the radius of the circle that circumscribes the hexagon. A circle with a radius of 12 inches has a diameter of 24 inches and a circumference of **24π** inches.

Workout 8

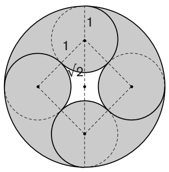
241. Comparing 5 & 3 = 18 and 10 & 3 = 72, we see that the result of (2 × 5) & 3 is 72 ÷ 18 = 4 times the result of 5 & 3. This suggests that the first number might get squared, which means that $m = 2$. Next, comparing 5 & 3 and 5 & 6, we see that the result of 5 & (2 × 3) is 36 ÷ 18 = 2 times as great as 5 & 3. This suggests that the second number might be raised to the first power, which means that $n = 1$. Substituting 2 for m and 1 for n in the expression $k \times A^m \times B^n$ for $A = 5$ and $B = 3$ yields $k \times 5^2 \times 3^1 = 18 \rightarrow 75k = 18 \rightarrow k = 18/75 = 6/25$. Let's confirm this with 10 & 3 and 5 & 6. We have $k \times 10^2 \times 3^1 = 72 \rightarrow 300k = 72 \rightarrow k = 72/300 = 6/25$ and $k \times 5^2 \times 6^1 = 36 \rightarrow 150k = 36 \rightarrow k = 36/150 = 6/25$. Now that we've confirmed that $k = 6/25$, we can calculate the value of 10 & 6 = $(6/25) \times 10^2 \times 6^1 = 3600/25 =$ **144**.

242. This is another problem that can be solved using the "stars and bars" technique. We imagine a line of 15 stars that represent the coins and 3 bars that represent partitions that separate the coins into the 4 groups of coins in the order stated: pennies, nickels, dimes, quarters. We now determine the number of permutations of 18 objects with 15 identical stars and 3 identical bars. There are $18!/(15! \times 3!) = (18 \times 17 \times 16)/(3 \times 2 \times 1) = 3 \times 17 \times 16 =$ **816** possible combinations.

a	r	ar	ar^2	$a + ar + ar^2$
1	2	2	4	7
1	3	3	9	13
1	4	4	16	21
1	5	5	25	31
1	6	6	36	43
2	2	4	8	14
2	3	6	18	26
2	4	8	32	42
3	2	6	12	21
3	3	**9**	**27**	**39**
4	3/2	6	9	19
4	2	8	16	28
4	3	12	36	52
4	5/2	**10**	**25**	**39**
5	3	15	45	65

243. There are 12 edges on a rectangular prism, but there are only three different lengths, so the sum of these three different lengths must be $3 \times 13 = 39$. We need to find three distinct positive integers that are in geometric progression and have a sum of 39. If we let a represent the least integer, then the next two integers are ar and ar^2. We know that $a + ar + ar^2 = 39$. The table shows the results of our systematic guess-and-check strategy. It turns out that, although the three dimensions have to be integers, the value of r does not have to be an integer. In particular, when a is a perfect square, we can divide by a factor twice, so r can be a rational number. In any case, the two possible sets of dimensions are (3, 9, 27) and (4, 10, 25). The prism has volume $a \times ar \times ar^2 = (ar)^3$. So, in the first case, with $a = 3$ and $r = 3$, the prism has volume $(3 \times 3)^3 = 9^3 = 729$ in³. In the second case, with $a = 4$ and $r = 5/2$, the prism has volume $(4 \times 5/2)^3 = 10^3 = 1000$ in³. The desired sum is **1729** in³.

244. The length of the lower left beam is divided so that the right and left sides are in the ratio 2:3, with a total of 16 pounds hanging from the left side. To maintain balance, the weight associated with each side is inversely proportional to the length of its beam. It follows, then, that the total weight on the right side must be 3/2 the total weight on the left, or $3/2 \times 16 = 24$ pounds. This can be accomplished using the fewest weights with a 16-pound weight and an 8-pound weight. Now let's consider the overall mobile, whose top beam is divided so that the right and left sides are in the ratio 5:2, with a total of $16 + 24 = 40$ pounds hanging from the left side. To maintain balance, the total weight on the right side must be 2/5 the total weight on the left side, or $2/5 \times 40 = 16$ pounds. We now need to consider the lower right beam, whose length is divided so that the right and left sides are in the ratio 3:1, with a combined 16 pounds hanging from the pair of sides. To maintain balance, $1/4 \times 16 = 4$ pounds needs to hang from the right side and $3/4 \times 16 = 12$ pounds needs to hang from the left. This can be accomplished using the fewest weights by hanging an 8-pound weight and a 4-pound weight on the left and hanging a 4-pound weight on the right. With the weights as described, the mobile is in balance by adding the fewest weights possible, which is **5** weights.



245. Suppose the radius of each of the four non-overlapping circles is 1 unit. Then the square created by connecting the centers of these four circles has side length $1 + 1 = 2$ units and area $2^2 = 4$ units^2. By properties of 45-45-90 right triangles, we know that the diagonal of the square has length $2\sqrt{2}$ units, making the larger circle's radius $1 + \sqrt{2}$ units and its area $\pi \times (1 + \sqrt{2})^2 = (3 + 2\sqrt{2})\pi$ units^2. The area of the region that is not shaded is the difference between the area of the square and the combined area of four quarter circles of radius 1 unit, which is $\pi \times 1^2 = \pi$ units^2. The unshaded region has area $4 - \pi$ units^2. So, $(4 - \pi)/[(3 + 2\sqrt{2})\pi] \approx 0.047$ of the figure is not shaded. That accounts for about **4.7**%.

246. The best strategy for Martin to follow can be found by looking at the problem backward and asking "How wide a range of numbers can Martin handle with g guesses?" Suppose he has just 1 guess. Then clearly he cannot handle any range with more than 1 number. If he has 2 guesses, then he can handle the range from 1 to 3, inclusive, because he can first guess the middle number, 2, and if that is not correct, then by knowing whether his guess is too high or too low, he will be able to guess correctly the second time. If he has 3 guesses, then he can handle the range from 1 to 7 by initially guessing 4, again the middle number. If that is wrong, then he has narrowed the range to 3 numbers and can apply his strategy on that range with his remaining 2 guesses. The range pattern is 1, 3, 7, 15, ... With g guesses, Martin can handle a range of $2^g - 1$ numbers. Because $2^{10} - 1 = 1023$ and $2^{11} - 1 = 2047$, 10 guesses will not be enough to handle the range from 1 to 2018, but 11 guesses will be enough. The strategy is to make a guess right in the middle of the narrowed range at each stage until the range has been narrowed to one number. The answer is n = **11**.

247. Suppose the book list contains a novels by Twain, b novels by Hemingway and c novels by Steinbeck. If there are exactly 100 ways for Austen to select two of an author's novels, then we have ${}_aC_2 + {}_bC_2 + {}_cC_2 = 100$. If we were to calculate the values of ${}_nC_2$ for n = 2, 3, 4, 5, 6, …, we would get the list of triangular numbers, namely 1, 3, 6, 10, 15, …. The table shows the values of n for which ${}_nC_2$ is a triangular number less than 100. We are looking for groups of three triangular numbers with a sum of 100. Since we want to maximize $a + b + c$, we should look for three numbers that are close in value. The group of three that works is 36, 36 and 28, which corresponds to 9, 9 and 8 novels. The greatest possible number of novels on the list is **26** novels.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
${}_nC_2$	1	3	6	10	15	21	28	36	45	55	66	78	91

m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$
2	1	3	4	**5**
3	2	5	12	**13**
4	1	15	8	**17**
4	3	7	24	25
5	2	21	20	**29**
5	4	9	40	**41**
6	1	35	12	**37**
6	5	11	60	61
7	2	45	28	**53**

248. The table shows a way to generate Pythagorean Triples by choosing positive integer values m and n such that $m > n$, GCF(m, n) = 1, and one of them is even. Then we compute $m^2 - n^2$, $2mn$ and $m^2 + n^2$, which is a primitive Pythagorean Triple. If we systematically assign values of m and n, we can catch all primitive Pythagorean Triples. There are 7 primes less than 60 that can be the length of the hypotenuse of a right triangle. The sum of these primes is $5 + 13 + 17 + 29 + 41 + 37 + 53 =$ **195**.

249. We are given the point A(a, 0). So, let's say that the coordinates of point B are (0, b). Since the area of the whole triangle is 54 units^2, the product ab must be twice as much, so we have $ab = 108$. Suppose, then, that the coordinates of the intersection of the line $y = 0.6x$ and segment AB are (x, $0.6x$). This allows us to write two more equations like the previous one: $a \times 0.6x = 28$ and $bx = 80$. We now have a system of three equations with three unknowns. We'll rewrite the second equation as $ax = 140/3$. We only need the value of a, so we don't necessarily need to find b and x. If we multiply the first and second equations, we get $ab \times ax = 108 \times 140/3 \rightarrow a^2bx = 5040$. Now we can divide by the third equation to eliminate the bx on the left. The result is $a^2 = 5040/80 = 63 \rightarrow a = \sqrt{63} =$ $\mathbf{3\sqrt{7}}$.

250. The hexagon can be subdivided into 6 equilateral triangles. The base of each triangle is 10 meters. By the properties of 30-60-90 right triangles, we know that the altitude of each equilateral triangle is $5\sqrt{3}$ meters. The area of the whole hexagon, then, is $6 \times 1/2 \times 10 \times 5\sqrt{3} = 150\sqrt{3}$ m^2. Although there are seven solar discs across a long diagonal of the hexagon, the diameter of those discs is not 20/7 since the discs on the ends do not intersect the vertices of the hexagon. To find the radius of the solar discs, we draw a perpendicular segment from the center of the center disc to the midpoint of one side of the hexagon. The length of this altitude is $5\sqrt{3}$ meters, and it represents $3\sqrt{3} + 1$ radii of the discs. The radius of each disc is $(5\sqrt{3})/(3\sqrt{3} + 1)$, so the area of the 37 discs is $37 \times \pi \times [(5\sqrt{3})/(3\sqrt{3} + 1)]^2$. This accounts for $[37 \times \pi \times [(5\sqrt{3})/(3\sqrt{3} + 1)]^2]/[150\sqrt{3}] \approx 0.87$ of the hexagon, which is **87**%.