

# Domination numbers and homotopy in certain ternary graphs

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joint work with  
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An abstract simplicial complex  $\mathcal{F}$  can be embedded in to the  $\mathbb{R}^{2 \cdot \max\{|F| : F \in \mathcal{F}\} - 1}$

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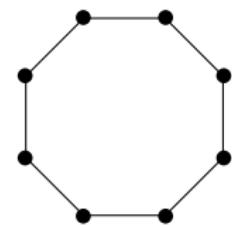
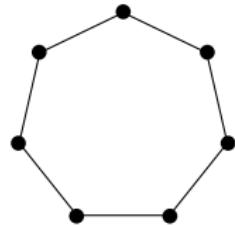
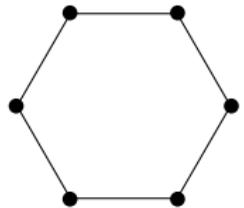
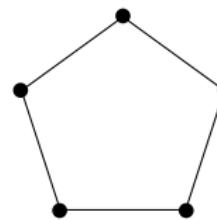
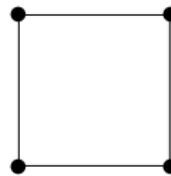
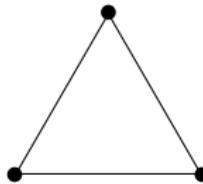
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## Unique Embedding

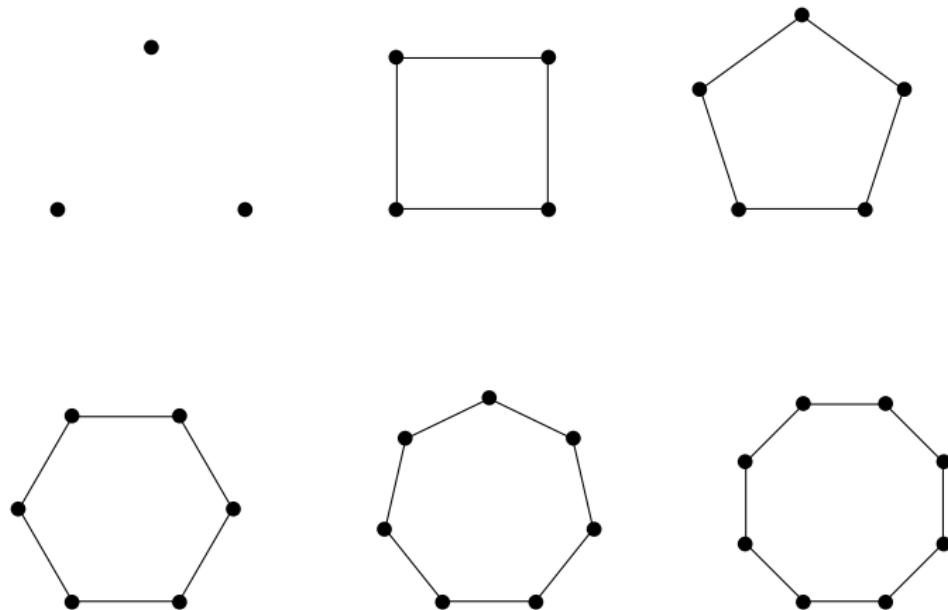
For an abstract simplicial complex, every embeddings are homeomorphic.

Independence complex, Clique complex, Nerve complex, ...

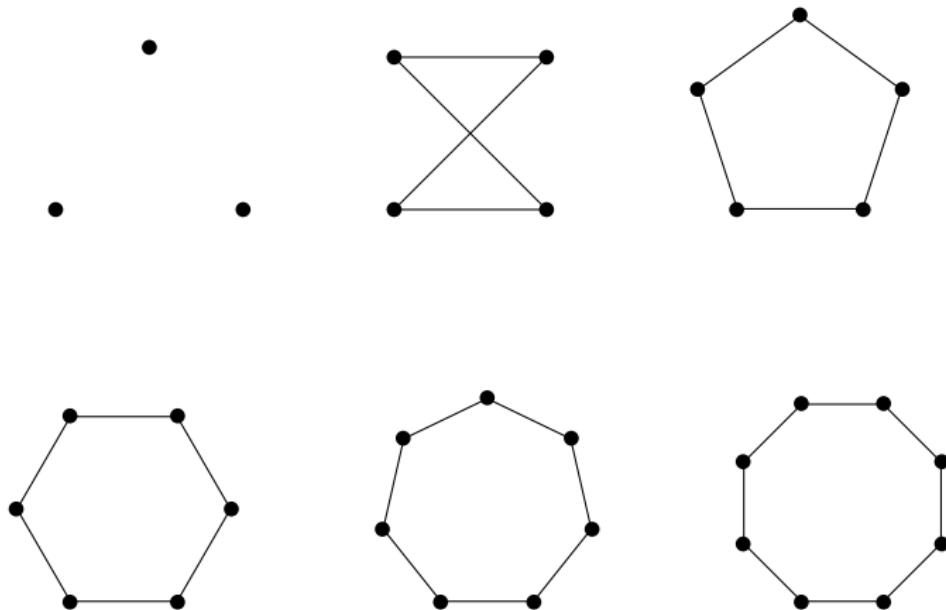
# Independence complex of a cycle



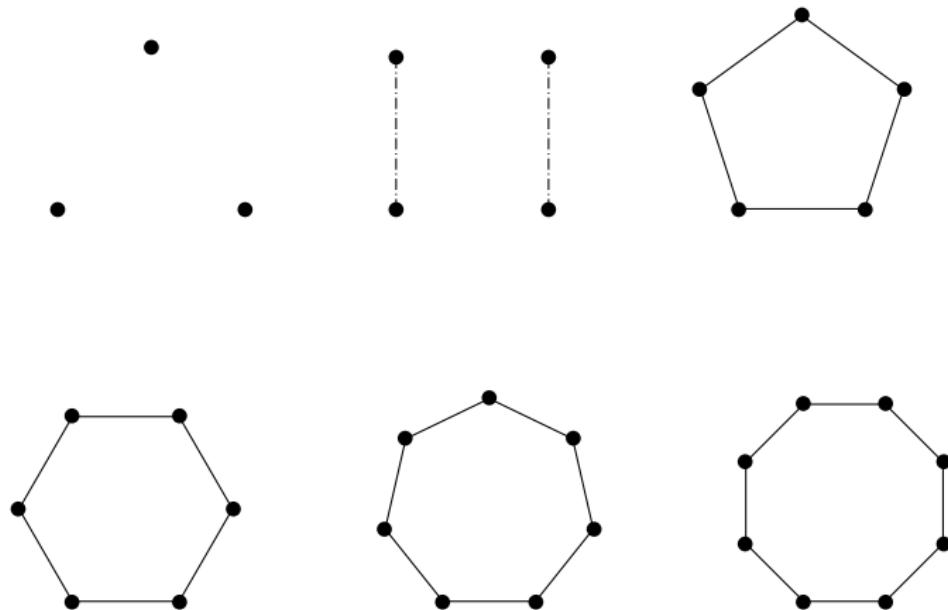
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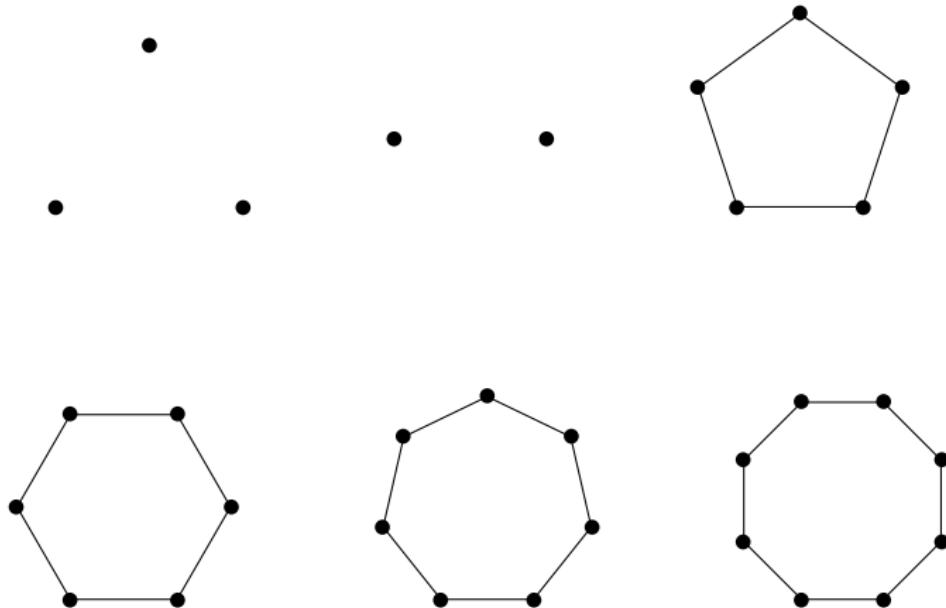
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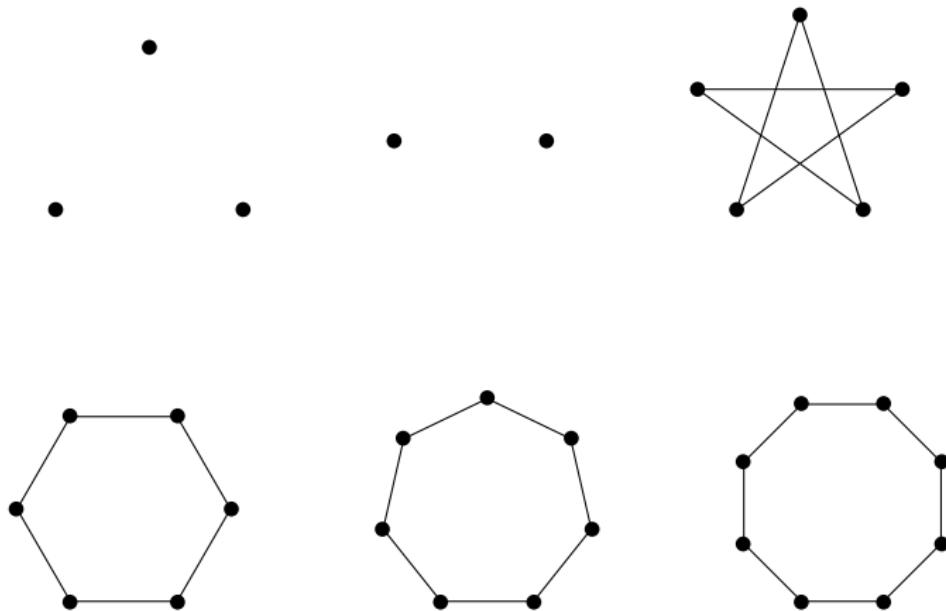
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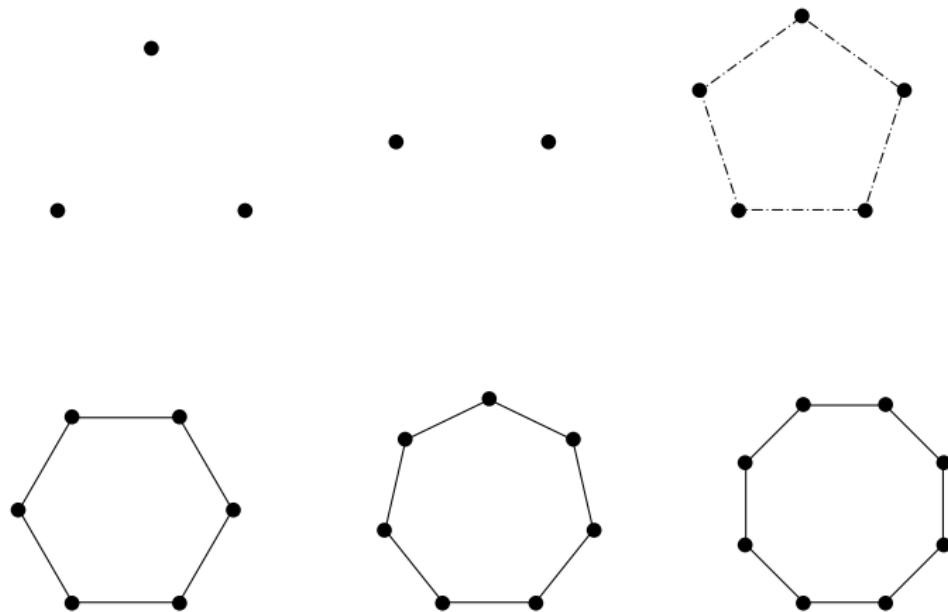
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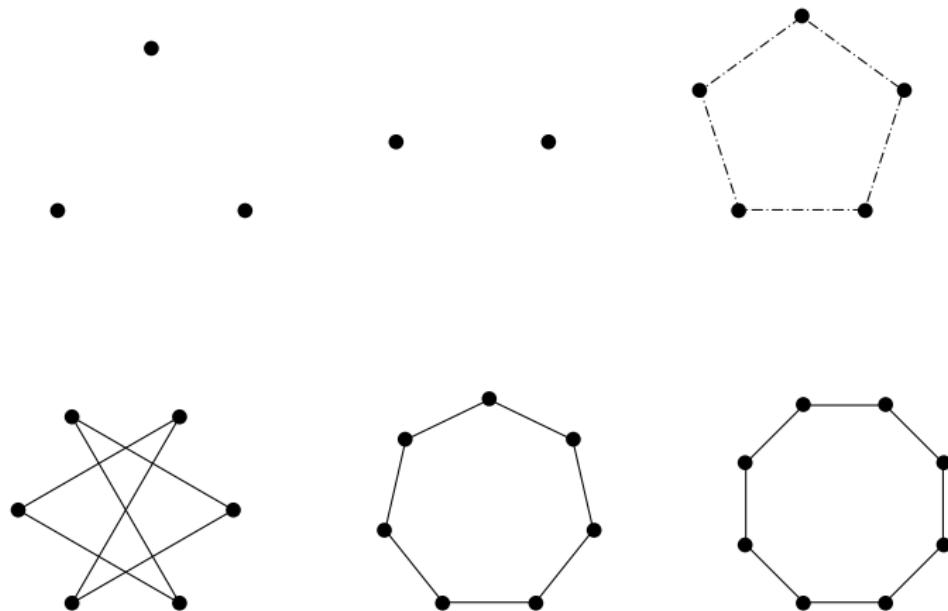
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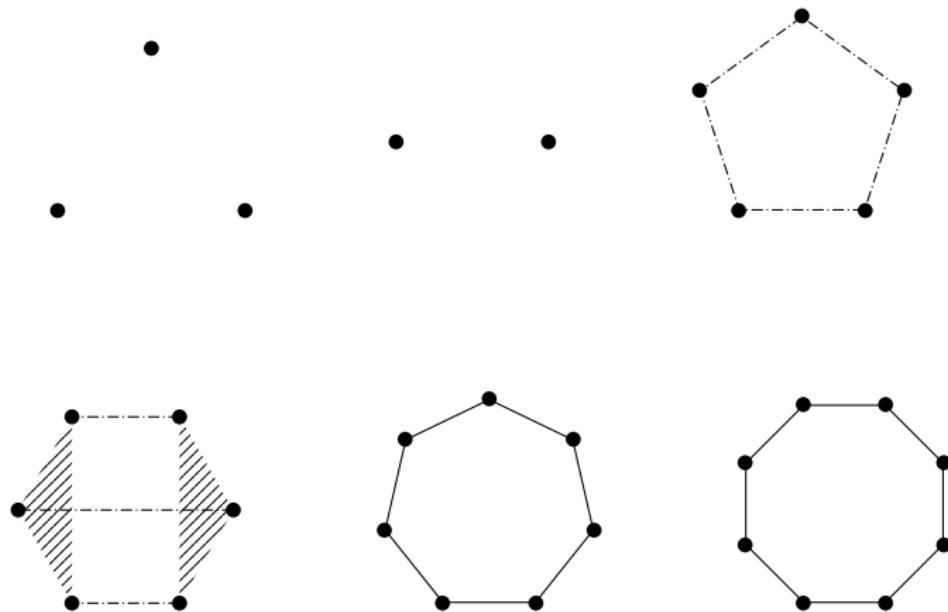
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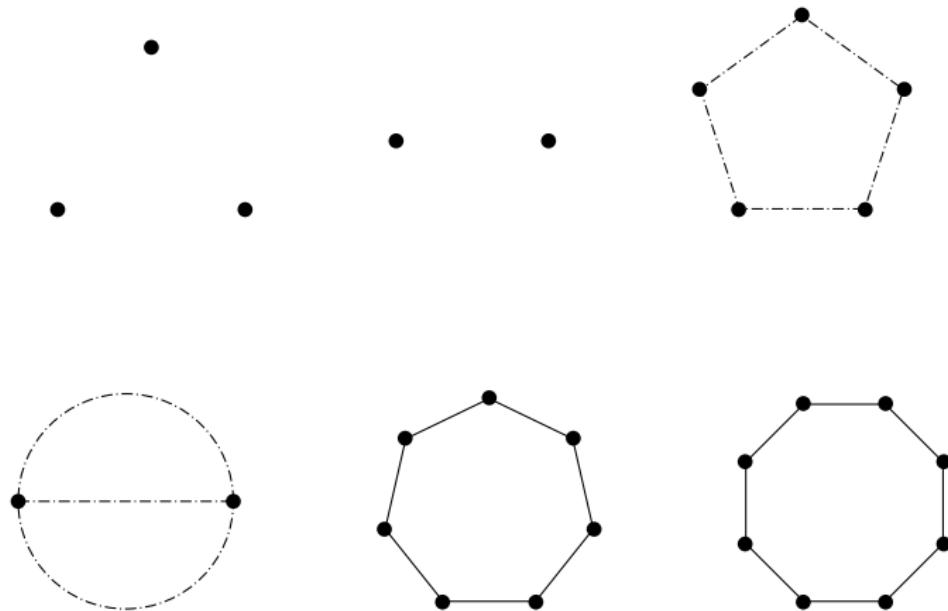
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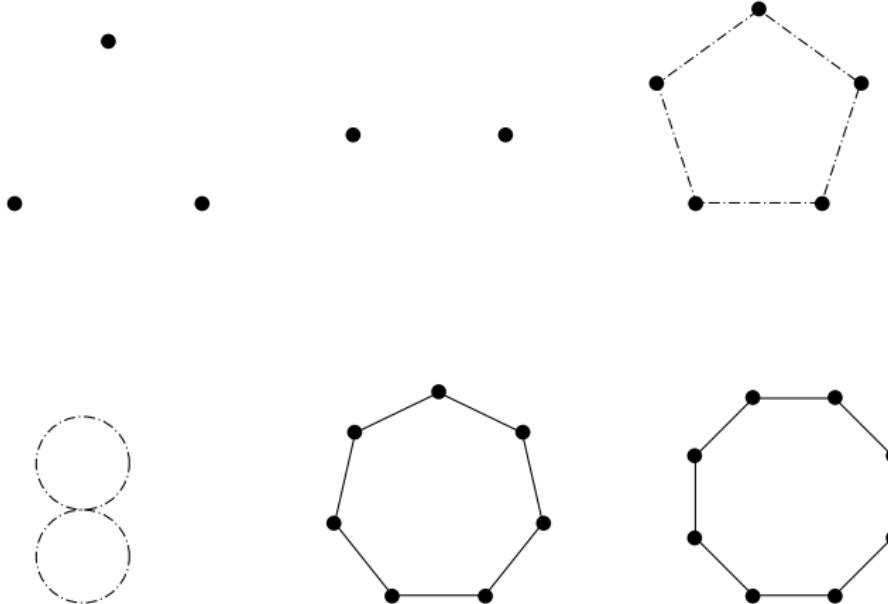
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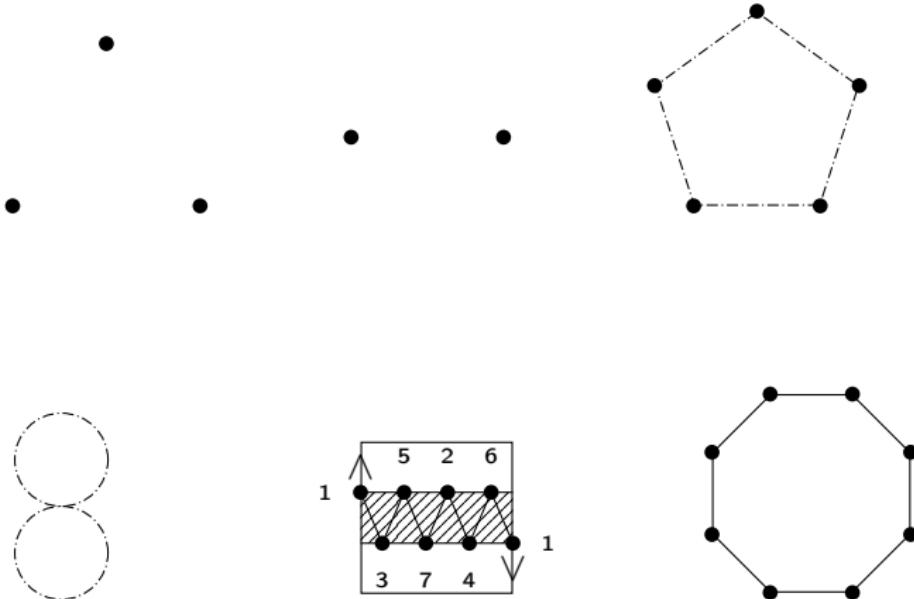
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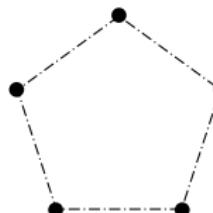
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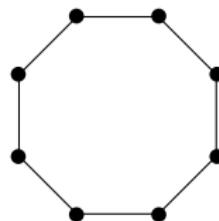
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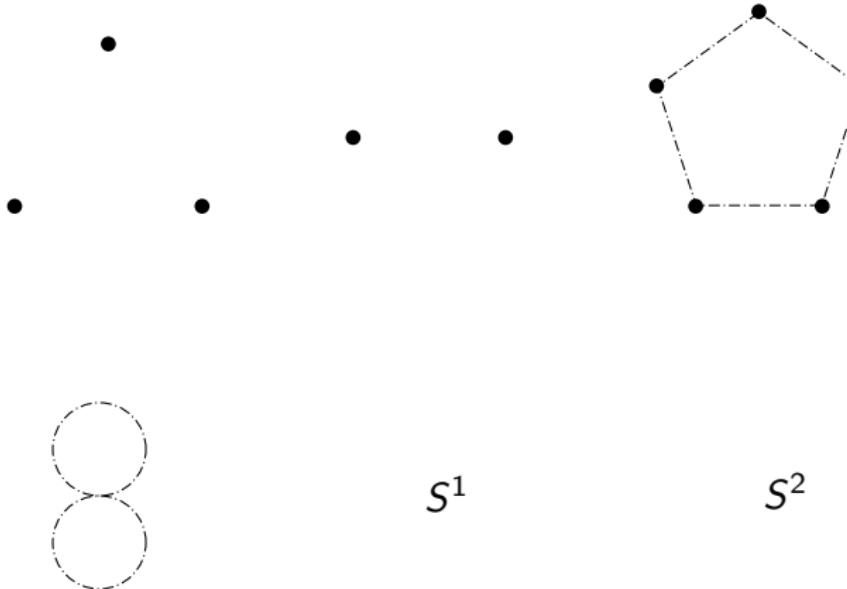
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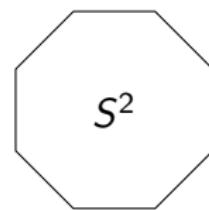
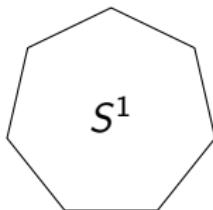
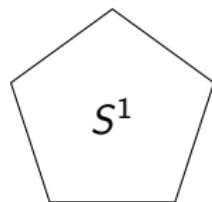
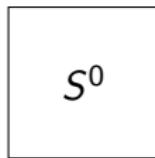
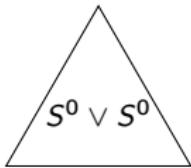
$$S^1$$



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## Ternary graph

A graph is **ternary** if it has no induced cycles of length 0 modulo 3.

# From Kalai, Meshulam and Engström

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$\chi(I(G))$  for ternary graphs (Chudnovsky, Scott, Seymour and Spirkl, 2020)

For a ternary graph  $G$ ,  $\left| \sum_{A:\text{indep}} (-1)^{|A|} \right| \leq 1$

Betti numbers of ternary graphs (Zhang and Wu, 2025, Arxiv 2020)

For a ternary graph  $G$ , the sum of reduced Betti number of  $I(G)$  is at most 1.

Homotopy type of the ternary graph (J. Kim, 2022)

A graph  $G$  is ternary iff  $I(H)$  is either contractible or homotopy equivalent to a sphere for every induced subgraph  $H$  of  $G$ .

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For a forest  $F$ , if  $I(F)$  is not contractible, then

$$d(F) = \gamma(F) - 1 = i(F) - 1$$

where  $\gamma(F)$  is the domination number and  $i(F)$  is the independent domination number of  $F$ .

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This result is also true for  $C_5, C_8, \dots$ , but not for  $C_4, C_7, \dots$

## (0, 1)-ternary graph

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- For a  $(0,1)$ -ternary graph  $G$ , if  $I(G)$  is not contractible, then

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- For a  $(0,1)$ -ternary hypergraph  $H$ , if  $I(H)$  is not contractible, then  $d(H) = \gamma(H) - 1$ .

## Sketch of the proof

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Graph point of view, Hà and Woodroffe, 2014

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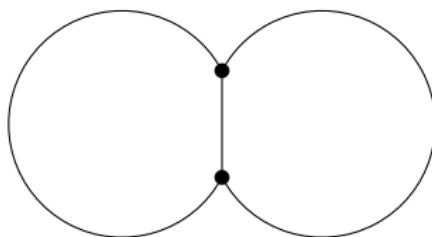
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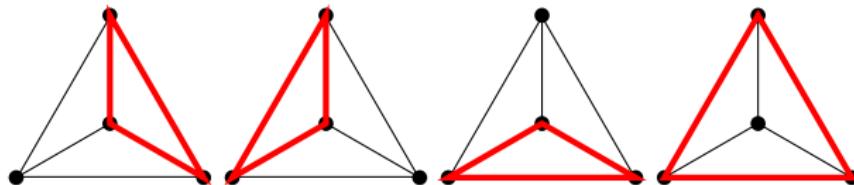
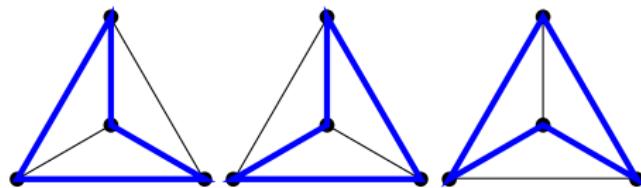
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# Structure of $(0, 1)$ -ternary graphs. Part 1

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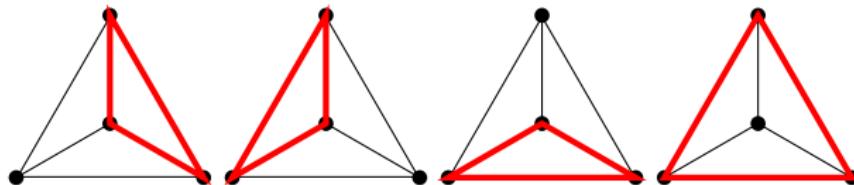
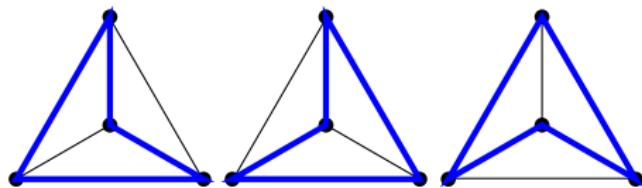
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This is generally true for any graph  $G$  such that every cycle of length  $k (\neq 0)$  modulo  $m$

## 2-connected and ear-decomposition

For a 2-connected  $(0, 1)$ -ternary graph  $G$  and its ear-decomposition sequence  $G_0 \leq G_1 \leq \dots \leq G_k = G$ , let  $u, v$  be two endpoints of the last ear  $E$ .

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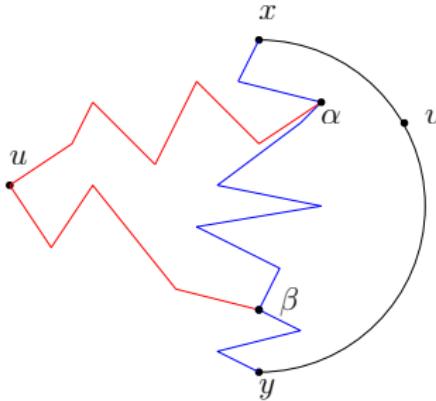
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Since  $G$  is 2-connected, there exists two paths  $P, P'$  connecting  $u$  and  $v$  in  $G_{k-1}$ . Since  $P \cup E$ ,  $P' \cup E$  and  $P \cup P'$  have length 2 modulo 3,  $P$ ,  $P'$  and  $E$  have length 1 modulo 3.

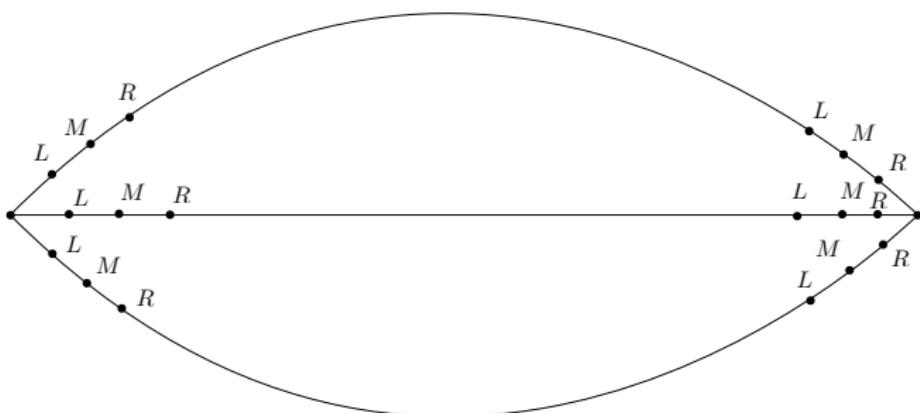
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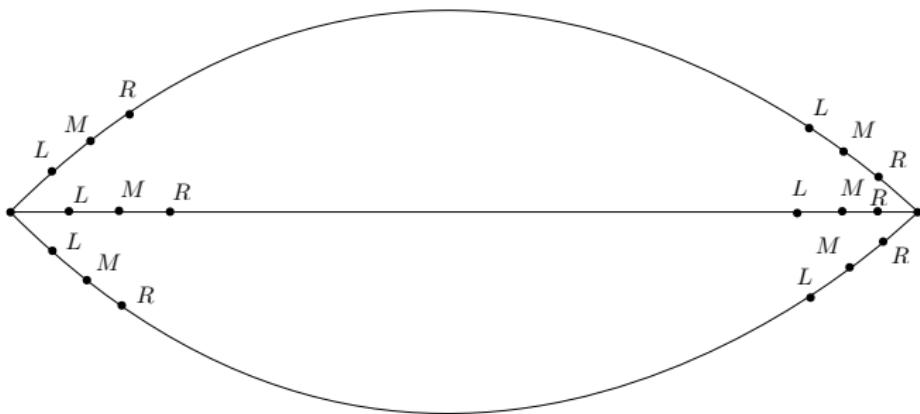
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# Structure of minimum independent dominating sets

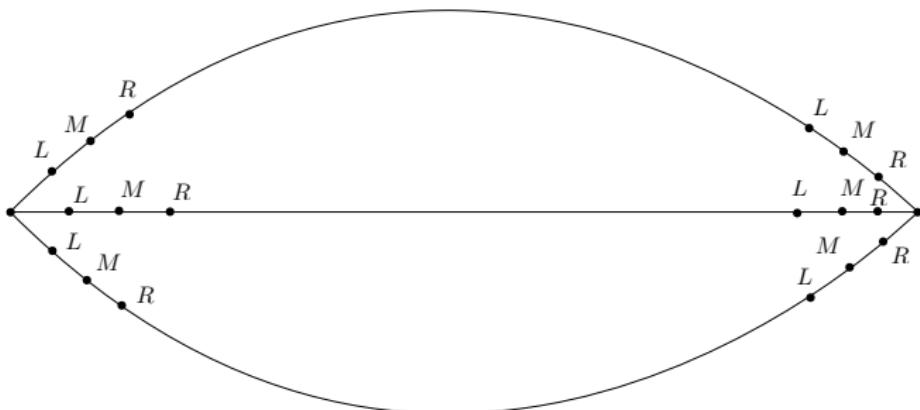


# Structure of minimum independent dominating sets



The number of vertices is  $3k + 2$ , and its independent domination number is  $k + 1$ .

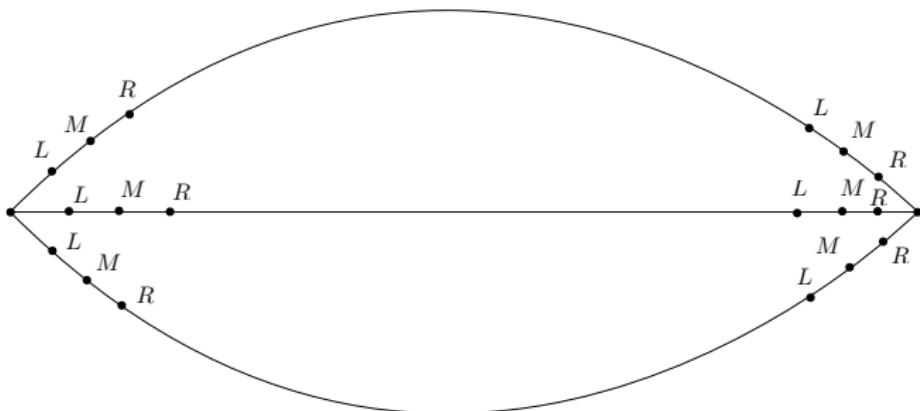
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- If both endpoints are not in a minimum size independent dominating set, then one path has pattern  $L \cdots L(LR)R \cdots R$  and others have  $M \cdots M$ .

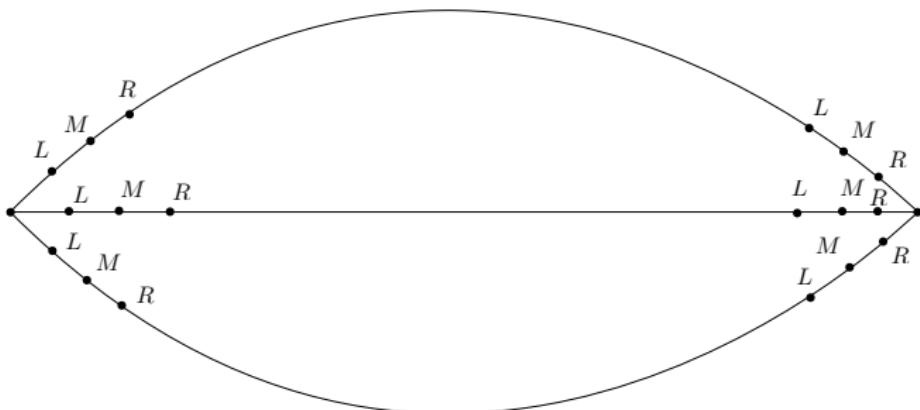
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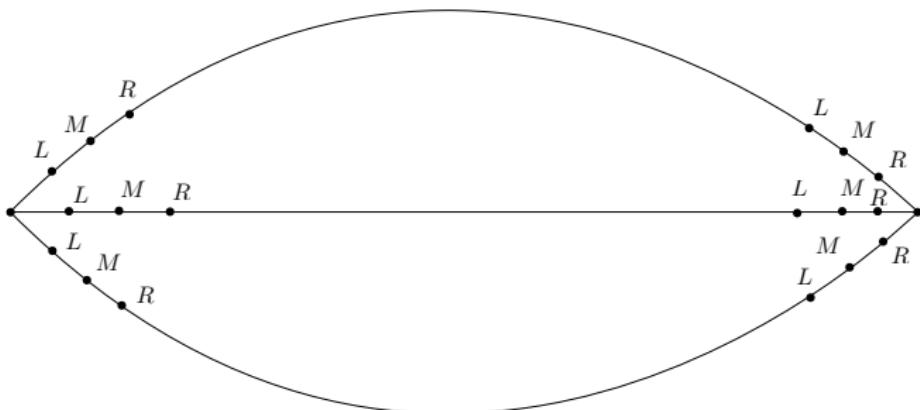
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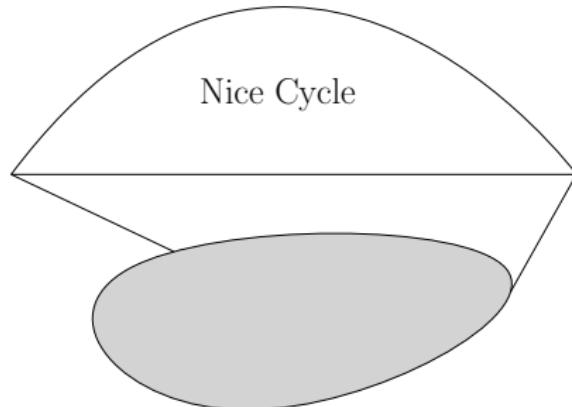
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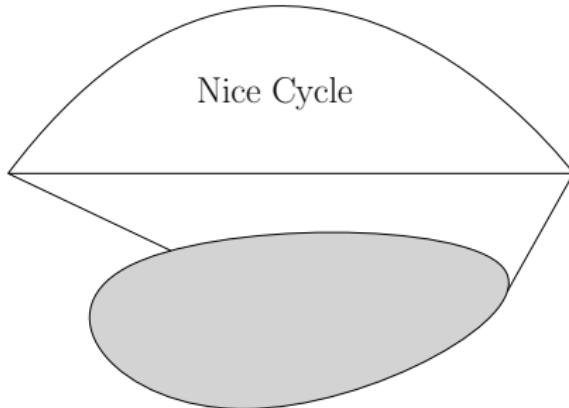
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- Both endpoints cannot be included in a minimum size independent dominating set simultaneously.

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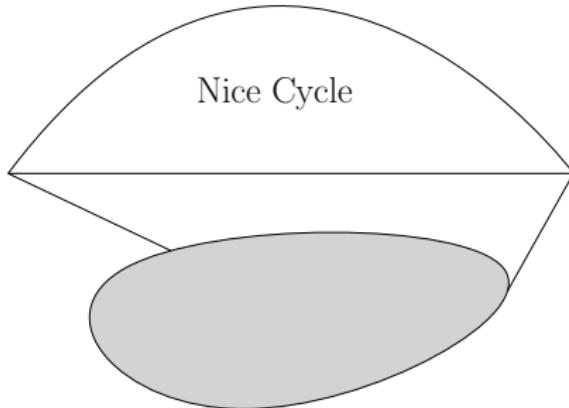
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Independent domination of 2-connected  $(0,1)$ -ternary graphs

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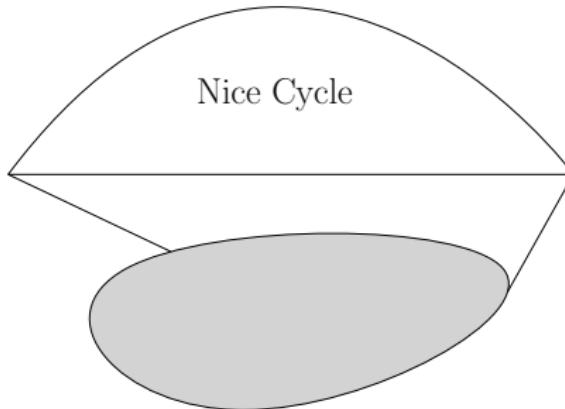
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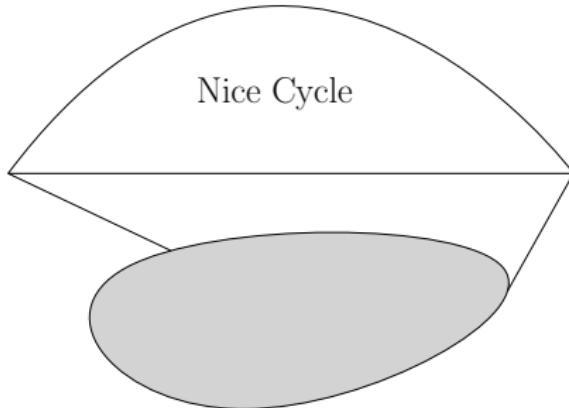


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- For any  $v \in V(G)$ ,  $i(G - v) \geq i(G)$ .

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## Independent domination number vs Sphere dimension

For a  $(0,1)$ -ternary graph  $G$ , if  $I(G)$  is not contractible, then

$$i(G) - 1 \leq d(G).$$

This uses a Mayer-Vietoris argument based on  $G - v$  and  $G - N(v)$  to prove  $\tilde{H}_s(I(G)) = 0$  for  $s \leq i(G) - 2$ .

## Domination number vs Edge domination number

For a  $(0,1)$ -ternary graph  $G$ ,

$$\gamma(L(G)) \leq \gamma(G).$$

Sketch of the proof) Generally, for a  $(0,1)$ -ternary graph  $G$  and a minimum size dominating set  $W$ , there exists  $W' \subseteq W$  and edges  $e_1, \dots, e_{|W'|}$  where each  $e_i$  attaches to a vertex in  $W'$  such that  $W \setminus W'$  with  $e_1, \dots, e_{|W'|}$  dominates edges.

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- $\gamma(L(G)) \leq \gamma(L(G) - \{e_1, \dots, e_{|W'|}\}) + |W'|$ .
- $\gamma(L(G) - \{e_1, \dots, e_{|W'|}\}) \leq \gamma(G') \leq |W| - |W'|$ .

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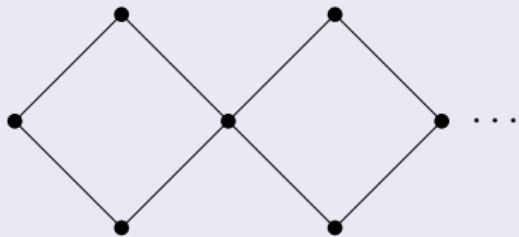
$$\gamma(L(G)) \leq \gamma(G).$$

Sketch of the proof) Generally, for a  $(0,1)$ -ternary graph  $G$  and a minimum size dominating set  $W$ , there exists  $W' \subseteq W$  and edges  $e_1, \dots, e_{|W'|}$  where each  $e_i$  attaches to a vertex in  $W'$  such that  $W \setminus W'$  with  $e_1, \dots, e_{|W'|}$  dominates edges.

- $\gamma(L(G)) \leq \gamma(L(G) - \{e_1, \dots, e_{|W'|}\}) + |W'|$ .
- $\gamma(L(G) - \{e_1, \dots, e_{|W'|}\}) \leq \gamma(G') \leq |W| - |W'|$ .
- $\gamma(L(G)) \leq |W| - |W'| + |W'| = |W| = \gamma(G)$ .

# With cycles of length 1 modulo 3

Connected  $C_4$

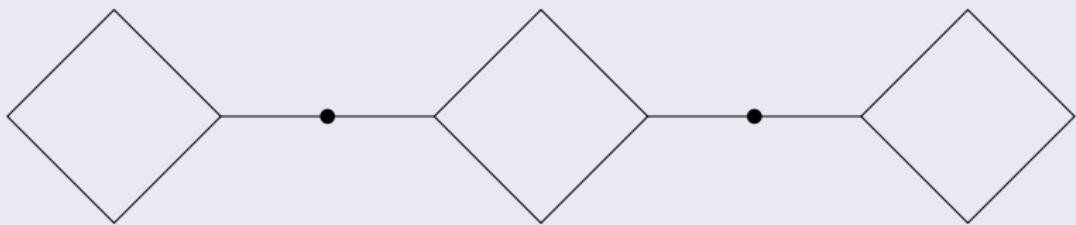


$3k - 1$  consecutive induced  $C_4$ .

- $I(\bullet) \simeq S^{2k-1}$
- $\gamma(\bullet) = i(\bullet) = 3k$

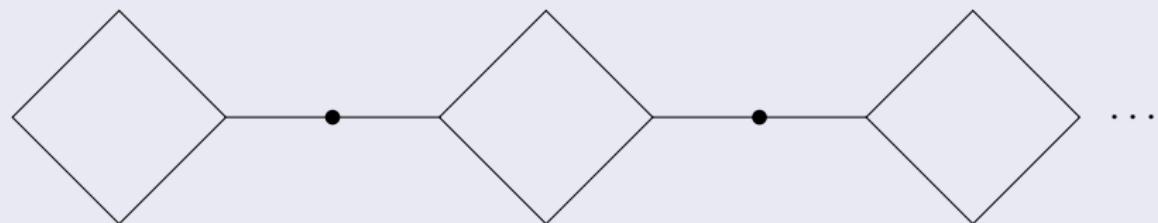
# With cycles of length 1 modulo 3

Connected  $C_4$  by path of length 2



- $I(\bullet) \simeq S^3$
- $\gamma(\bullet) = i(\bullet) = 5$
- $\gamma(L(\bullet)) = 6$

Connected  $C_4$  by path of length 2



For the case with  $3k$  induced  $C_4$

- $I(\bullet) \simeq S^{4k-1}$
  - $\gamma(\bullet) = i(\bullet) = \lceil \frac{9k}{2} \rceil$
  - $\gamma(L(\bullet)) = 6k$

## Remaining problems

- Is there any other parameter to represent the  $d(G)$ , which explains cases with length 1 modulo 3 cycles?
- $\frac{2}{3}\gamma(G) \leq d(G) + 1 \leq \gamma(G)$ ?
- If  $\gamma(G) \neq i(G)$ , then  $I(G)$  is contractible?

The end.