COMPUTING THE DENSITY OF TAUTOLOGIES IN PROPOSITIONAL LOGIC BY SOLVING SYSTEM OF QUADRATIC EQUATIONS OF GENERATING FUNCTIONS

TAEHYUN EOM

ABSTRACT. In this paper, we will provide a method to compute the density of tautologies among the set of well-formed formulae consisting of m variables, a negation symbol and an implication symbol, which has a possibility to be applied for other logical systems. This paper contains computational numerical values of the density of tautologies for two, three, and four variable cases. Also, for certain quadratic systems, we will introduce the s-cut concept to make a better approximation when we compute the ratio by brute-force counting, and discover a fundamental relation between generating functions' values on the singularity point and ratios of coefficients, which can be understood as another intepretation of the Szegő lemma for such quadratic systems. With this relation, we will provide an asymptotic lower bound $m^{-1} - (7/4)m^{-3/2} + O(m^{-2})$ of the density of tautologies as m goes to the infinity.

CONTENTS

1. INTRODUCTION

In propositional logic systems, theorems, which are determined by axioms and inference rules, and tautologies, which are determined by valuation, are important objects. The positional logic system on which modern mathematics usually relies is well known to be sound and complete, which means tautologies and theorems are equivalent. Even though tautologies and theorems are equivalent, they are quite different in structure. Tautologies can be checked by the usual buttom-up way, using truth tables recursively, whereas theorems cannot be determined by its subformulae. Hence, we are going to count tautologies.

In fact, just counting tautologies is not interesting, since they are countably infinite. What may be interesting is the probability that a given well-formed formula is a tautology, but this probability should be specified further since the number of total well-formed formulae is infinite. Thus, we will consider the density, which means we will consider the limit of the portion of tautologies in the set of all well-formed formulae with a fixed length. Also, to make this density nonzero, we will consider the case that the number of variables is finite. There are some preceding studies such as [1], which computes the density of tautologies in a logic system with implication and negation on one variable, [2], which computes the asymptotic density, as the number of variables goes to infinity,

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of tautologies in a logic system with implication and negative literals, and [3], which computes the densities in several logic systems based on one variable.

In this paper, we will consider the logic system based on a finite number of variables with implication and negation mainly, but methods in this paper may be used for other logic systems such as 'and' with negation, or 'or' with negation, etc. Basic definitions are in **Section 2**. This section includes a correspondence from a well-formed formula to a subset of the power set of variables which makes logical symbols as a set-theoretical operator. **Section 3** will provide a method to solve the equations and compute the limit ratios completely, by introducing well-organized partitions, which is coset-like for set operators, to reduce the number of equations by merging, which is quotient-like among power sets. We will solve the system of generating functions of well-formed formulae by hierarchical clustering, and apply Szegő lemma to compute the ratio of coefficients. In **Section 4**, we will define the 's-cut', which can be used to estimate the limit ratios for coefficients of generating functions to a fixed generating function by computing first s values when fixed generating function has at-most-quadratic structure and other generating functions consist a quadratic system with a common factor. This generalization can be applied to any finite partitioning of well-formed formulae where logical symbols between partitioned sets are well-defined. With this concept we will prove a relation between ratios of coefficients and values of generating functions on the first singularity, which also can be deduced by Szegő lemma. Moreover, when generating functions are partitioning fixed generating function with specific conditions, we will prove that there is a possibility that s-cut concept will give a valid estimation to the real values. Lastly, **Section 5** will give a computational result, using the result of Section 2, of the density of tautologies for the cases that the number of variables are two, three, or four. Also, it will give a computational evidence that s-cut can be more efficient to estimate for the cases with 1, 2, or 3 variables compare to counting and direct dividing. Lastly, this section will use the relation between ratios and values of generating functions, to give an asymptotic lower bound $m^{-1} - (7/4)m^{-3/2} + (5/4)m^{-2} + O(m^{-5/2})$ for limit ratios as the number of variables, which is denoted as m, goes to the infinity. This result has some reasonable evidences to be conjectured carefully as the largest order term is correct asymptotically.

2. Basic Definitions

Consider the logic system with a (m-element) set of variables X, $\neg$ (negation) and $\to$ (implication). The well-formed formulae of the logic system are defined recursively as follows:

- Every variable is a well-formed formula.
- If ϕ is a well-formed formula, then $\neg\phi$ is a well-formed formula.
- If ϕ and ψ are well-formed formulae, then $[\phi \to \psi]$ is a well-formed formula.

In any well-formed formula, if the left and right ends are parentheses, they can be omitted simultaneously. The length ℓ of a well-formed formula is defined recursively as

$$\begin{aligned} \ell(x) &= 1 \text{ when } x \text{ is a variable,} \\ \ell(\neg\phi) &= \ell(\phi) + 1, \\ \ell([\phi \to \psi]) &= \ell(\phi \to \psi) = \ell(\phi) + \ell(\psi) + 1. \end{aligned}$$

Also, if we have trueness or falseness for each variable, we may extend trueness or falseness to well-formed formulae naturally and uniquely. Hence, we have a trivial bijection between a set of variables $T \subseteq X$ which are assigned to be true, called **an (truth) assignment**, and a valuation v_T which is a function mapping each well-formed formula to true or false. If we use 0 for false and 1 for true, then we have $v_T(\neg\phi) = 1 - v_T(\phi)$ and $v_T(\phi \to \psi) = 1 - v_T(\phi)(1 - v_T(\psi))$. Practically, if we consider m variables $x_0, x_1, \cdots, x_{m-1}$, then we have a natural bijection between $\mathcal{P}(X)$ and $\{n \in \mathbb{Z} \mid 0 \leq n < 2^m\}$ by the binary representation.

Now, for any well-formed formula ϕ, we may define the falsity set F_ϕ of ϕ, which is the set of assignments on variables that make ϕ false. In other words,

$$F_\phi = \{T \subseteq X \mid v_T(\phi) = \text{false}\}.$$

Again, for any possible falsity sets, we may consider its integer representation in $\{n \in \mathbb{Z} \mid 0 \leq n < 2^{2^m}\}$ based on the binary representation, when the logic system has m variables.

A well-formed formula ϕ is a tautology if and only if $F_\phi = \emptyset$; it is an antilogy if and only if $F_\phi = \mathcal{P}(X)$. Let $\mathcal{W}$ be the set of well-formed formulae and from now on, we will fix the number of variables to be m, so $|X| = m$.

Let $W(z)$ be the generating function of all well-formed formulae, i.e.,

$$W(z) = \sum_{\phi \in \mathcal{W}} z^{\ell(\phi)},$$

and $I_A(z)$ be the generating function of all well-formed formulae ϕ such that $F_\phi = A$, i.e.,

$$I_A(z) = \sum_{\phi \in \mathcal{W}, F_\phi = A} z^{\ell(\phi)}.$$

Note that $I_\emptyset(z)$ is the generating function of the set of tautologies.

From the generating rules for well-formed formulae, we get

$$W(z) = mz + zW(z) + zW(z)^2.$$

To find a similar formula for $I_A = I_A(z)$, it is enough to find the condition that $F_{\neg\phi} = A$ and $F_{\phi\to\psi} = A$. From $F_{\neg\phi} = F_\phi^c$ and $F_{\phi\to\psi} = F_\psi \setminus F_\phi$, we obtain the following.

Proposition 2.1. *For any $A \subseteq \mathcal{P}(X)$, if $A = F_x = \{T \subseteq X \mid x \notin T\}$ for some $x \in X$, then we have*

$$I_A = z + zI_{A^c} + \sum_{C \setminus B = A} zI_B I_C,$$

and if there is no such x, then we have

$$I_A = zI_{A^c} + \sum_{C \setminus B = A} zI_B I_C.$$

In practice, if we consder the binary integer intepretation, then F_{x_i} corresponds to the number $\frac{2^{2^m}-1}{2^{2^i}+1} = (2^{2^i} - 1)\prod_{j=i+1}^{m-1}(2^{2^j} + 1) = \prod_{j=0, j\neq i}^{m-1}(2^{2^j} + 1)$ and F_{A^c} corresponds to the number $2^{2^m} - 1 - F_A$, for $0 \leq i < m$.

Note that this construction of a system of equations also can be applied for logic systems with different logical symbols. Also, we may add false variable $\perp$, which is nothing but a variable with $F_\perp = \mathcal{P}(X)$. Of course, these variations of the system will give different densities of tautologies.

Now, we may regard logical symbols as set operators defined on $\mathcal{P}(\mathcal{P}(X))$,

$$\neg F_\phi := F_{\neg\phi} = F_\phi^c$$

and

$$F_\phi \to F_\psi := F_{\phi\to\psi} = F_\psi \setminus F_\phi.$$

Since F_ϕ is defined by falsity, if we extend $\vee$ and $\wedge$ to $\mathcal{P}(\mathcal{P}(X))$, we have $A \vee B = A \cap B$ and $A \wedge B = A \cup B$. Note that this does not match with the usual convention where $A \vee B$ corresponds to $A \cup B$ and $A \wedge B$ to $A \cap B$. We want to find the limit of the portion of tautologies in the set of all well-formed formulae with a fixed length n, i.e.

$$\lim_{n\to\infty} \frac{[z^n]I_\emptyset(z)}{[z^n]W(z)}.$$

This limit is not easy to compute since the recursion for $I_\emptyset(z)$ is difficult to solve. To illustrate how complicated it can be, we first consider the case when $m = 1$, i.e. just one variable. For the one variable case, algebraic analysis on generating functions and limits of ratio between their coefficients are already done in [1] with $\neg$ and $\to$, and in [3] with $\neg$ and $\vee$, etc. But since the equation for $I_\emptyset$ is not written anywhere, we present it without a proof.

Proposition 2.2. *The generating function, $I_\emptyset = I_\emptyset(z)$, for tautologies in the logic system with $\neg$, $\to$ and one variable satisfies the following equation.*

$$\begin{aligned}z^7 I_\emptyset^8 - 8z^6 I_\emptyset^7 + z^5(27 + 2z - 2z^2 + z(1-z)W)I_\emptyset^6 - z^4(50 + 12z - 12z^2 + 6z(1-z)W)I_\emptyset^5 \\ +z^3(55 + 28z - 28z^2 - 4z^3 + z^4 + z(2z+15)(1-z)W)I_\emptyset^4 \\ -z^2(36 + 32z - 32z^2 - 16z^3 + 4z^4 + z(8z+20)(1-z)W)I_\emptyset^3 \\ +z(13 + 18z - 19z^2 - 22z^3 + 4z^4 + z(z^2 + 12z + 15)(1-z)W)I_\emptyset^2 \\ -(2 + 4z - 6z^2 - 12z^3 + z(2z^2 + 8z + 6)(1-z)W)I_\emptyset \\ +(-z - 2z^2) + (z^2 + 2z + 1)(1-z)W = 0\end{aligned}$$

3. The case with more than one variables

Given a graded structure $U = \coprod_{n=0}^{\infty} U_n$, the disjoint union of U_n's, with $|U_n| < \infty$ for all n, we may consider two concepts of density of a subset A of U. The first one is

$$\mu_1(A) = \lim_{n\to\infty} \frac{|A \cap U_n|}{|U_n|}$$

and the second one is

$$\mu_2(A) = \lim_{n\to\infty} \frac{|A \cap \bigcup_{k=0}^n U_k|}{|\bigcup_{k=0}^n U_k|}.$$

Although they are different, if $\{|U_n|\}_{n\geq 0}$ is nondecreasing then μ_1 is stronger than μ_2, by the following proposition.

Proposition 3.1. *If a_n, b_n are nonnegative for all n and satisfy $\sum_{n=0}^{\infty} b_n = \infty$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = r$, then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^n a_k}{\sum_{k=0}^n b_k} = r.$$

This proves that $|U| = \infty$ and $\mu_1(A) = r$ imply $\mu_2(A) = r$, so it is enough to consider only μ_1. Hence, we will try to compute

$$\lim_{n\to\infty} \frac{[z^n]I_A(z)}{[z^n]W(z)},$$

as done in [3] and [1].

Definition 3.2. A partition $\{P_1, \cdots, P_n\}$ of $\mathcal{P}(\mathcal{P}(X))$ is said to be **well-organized for a logical symbol** if the logical symbol is well-defined on the partition in the following sense. For instance, a partition is well-organized for $\to$, if for any i and j, there exists a unique k depending only on i and j such that $F_\phi \in P_i$ and $F_\psi \in P_j$ imply $F_{\phi\to\psi} = F_\phi \to F_\psi \in P_k$, so $P_i \to P_j$ is well-defined as P_k.

This is reminiscent of a quotient group in group theory. Hence, we will give a reminiscent concept for the coset as the following.

Definition 3.3. For any disjoint subsets $A, B \subseteq \mathcal{P}(X)$, the subset $P_{A;B} = \{A \cup Y \mid Y \subseteq B\}$ is called **the standard subclass of $\mathcal{P}(\mathcal{P}(X))$ associated to** (A, B), with $|P_{A;B}| = 2^{|B|}$.

Proposition 3.4. *Given $B \subseteq \mathcal{P}(X)$, the family of subsets of $\mathcal{P}(X)$*

$$P_{;B} = \{P_{A;B} \mid A \cap B = \emptyset\}$$

is a partition of $\mathcal{P}(\mathcal{P}(X))$ such that $|P_{;B}| = 2^{2^m - |B|}$. We call this ***the standard subclass partition associated to*** *B.*

Proof. Since $Y \in P_{A;B}$ if and only if $A = Y \setminus B$, $P_{;B}$ is a partition of $\mathcal{P}(\mathcal{P}(X))$. Clearly, we have $|P_{;B}| = |\{C \subseteq \mathcal{P}(X) \mid C \cap B = \emptyset\}| = 2^{|\mathcal{P}(X)| - |B|} = 2^{2^m - |B|}$. $\square$

It is easy to check that this partition is based on the equivalence relation $A \sim_B C \Leftrightarrow A \setminus B = C \setminus B \Leftrightarrow A \cup B = C \cup B \Leftrightarrow (A \setminus C) \cup (C \setminus A) \subseteq B$. From this, we may extend the definition of the standard subclass to non-disjoint A, B as $P_{A;B} := P_{A \setminus B;B} = \{C \subseteq \mathcal{P}(X) \mid A \setminus B \subseteq C \subseteq A \cup B\}$.

Proposition 3.5. *Standard subclasses satisfy the following equalities, whenever $A \cap B = C \cap B = \emptyset$.*

(a) $\{Y^c \mid Y \in P_{A;B}\} = P_{(A\cup B)^c;B} = P_{A^c \setminus B;B}$,
(b) $\{Y \cap Z \mid Y \in P_{A;B}, Z \in P_{C;B}\} = P_{A\cap C;B}$,
(c) $\{Y \cup Z \mid Y \in P_{A;B}, Z \in P_{C;B}\} = P_{A\cup C;B}$,
(d) $\{Y \setminus Z \mid Y \in P_{A;B}, Z \in P_{C;B}\} = P_{A\setminus C;B}$,
(e) $P_{A;B} \cup P_{A\cup\{y\};B} = P_{A;B\cup\{y\}}$ *if* $y \notin A \cup B$.

Proof. (a) For every $Y \in P_{A;B}$, we have $Y^c \setminus B = Y^c \cap B^c = (Y \cup B)^c = (A \cup B)^c$, and so $Y^c \in P_{(A\cup B)^c;B}$. The first equality is obtained by $|P_{A;B}| = |P_{(A\cup B)^c;B}|$, or by a direct argument that for any Y with $Y \subseteq B \subseteq A^c$, we have $(A\cup B)^c \cup Y = (A^c \cap B^c) \cup Y = A^c \cap (B^c \cup Y) = (A \cup (B \setminus Y))^c$ and $A \cup (B \setminus Y) \in P_{A;B}$.

(b), (c), (d) can be done similarly.

(e) $P_{A;B\cup\{y\}} = \{A \cup Y, A \cup Y \cup \{y\} \mid Y \subseteq B\} = P_{A;B} \cup P_{A\cup\{y\};B}$. $\square$

From (a) to (d) are saying that $P_{;B}$ and $\mathcal{P}(\mathcal{P}(X) \setminus B)$ are isomorphic under set operations that taking the complement(^{c}), intersection($\cap$), union($\cup$), and set difference($\setminus$), by the natural isomorphism $\mathcal{P}(\mathcal{P}(X) \setminus B) \ni A \mapsto P_{A;B}$.

Proposition 3.6. *For any $B \subseteq \mathcal{P}(X)$, $P_{;B}$ is a well-organized partition for $\neg$ and $\rightarrow$.*

Proof. Since $F_{\neg\phi} = F_\phi^c$ and $F_{\phi\rightarrow\psi} = F_\psi \setminus F_\phi$, it directly follows from the above proposition. $\square$

By the same reason, standard subclass partitions are well-organized for $\vee$ and $\wedge$ as well. Hence, we may apply a similar method to the case where $\vee$ or $\wedge$ is a basic logical symbol instead of $\rightarrow$. Also, the converse of this proposition is true. For any given finite set Y, if a partition P of $\mathcal{P}(Y)$ is a well-organized partition for $\setminus$ alone, ^{c} with $\cup$, or ^{c} with $\cap$, then P is a well-organized partition for all of those four operators $\setminus, {}^c, \cap, \cup$, and $P = P_{;B}$ for some $B \subseteq Y$.

Let $I_{A;B}$ be the generating function of $P_{A;B}$, i.e.,

$$I_{A;B}(z) := \sum_{Y \in P_{A;B}} I_Y(z).$$

Since $P_{\emptyset;\mathcal{P}(X)} = \mathcal{P}(\mathcal{P}(X))$, $I_{\emptyset;\mathcal{P}(X)} = W$.

Proposition 3.7. *For any disjoint $A, B \subseteq \mathcal{P}(X)$, $I_{A;B}$ satisfies the following recursion.*

$$I_{A;B}(z) = \#\{x \in X \mid F_x \in P_{A;B}\}z + zI_{(A\cup B)^c;B}(z) + \sum_{\substack{C\setminus D = A \\ C\cap B = D\cap B = \emptyset}} zI_{C;B}(z)I_{D;B}(z).$$

This system of equations satisfies the conditions of Drmota-Lalley-Woods Theorem ([4], p.489), so we have proper analytic functions $h_{A;B}$ such that $I_{A;B}(z) = h_{A;B}(\sqrt{1 - z/r})$ for some $r > 0$. This shows that our generating functions are ready to apply Szegő lemma as [3] and [1], which states that for two generating functions $\sum_{n=0}^{\infty} \sigma_n z^n = \sum_{n=0}^{\infty} \hat{\sigma}_n (1 - z/z_0)^{n/2}$ and $\sum_{n=0}^{\infty} \tau_n z^n =$

$\sum_{n=0}^{\infty} \hat{\tau}_n (1-z/z_0)^{n/2}$ with unique singularity at z_0 in the disk $|z| \leq |z_0|$, we have $\lim_{n\to\infty} \sigma_n/\tau_n = \hat{\sigma}_1/\hat{\tau}_1$ if $\hat{\tau}_1 \neq 0$.

Now, to prove and solve the equation, we begin with the following.

Proposition 3.8. *For any disjoint $A, B \subseteq \mathcal{P}(X)$, we have the following:*

(a) $I_{A;B}(z)$ is a linear combination of elements of the set

$$\{I_{\emptyset;B}(z)\} \cup \{I_{C;B'}(z) \mid C \subsetneq A,\ C \cap B' = \emptyset,\ |B'| = |B| + 1,\ B \subseteq B'\},$$

where the coefficient of $I_{\emptyset;B}$ is ± 1.

(b) $I_{A;B}(z)$ is a linear combination of elements of the set

$$\{I_{B^c;B}(z)\} \cup \{I_{C;B'}(z) \mid A \subseteq C,\ C \cap B' = \emptyset,\ |B'| = |B| + 1,\ B \subseteq B'\},$$

where the coefficient of $I_{B^c;B}$ is ± 1.

Proof. (a) We will induct on $|A|$. It is trivial for $A = \emptyset$. Suppose it is true for every A such that $|A| = n$ and $A \cap B = \emptyset$. Then, if $|A| = n + 1$ and $A \cap B = \emptyset$, choose $y \in A$. Now, from **Proposition 3.5**(e),

$$I_{A;B} = I_{A\setminus\{y\};B\cup\{y\}} - I_{A\setminus\{y\};B},$$

which proves the proposition.

(b) It follows from

$$I_{A;B} = I_{A;B\cup\{y\}} - I_{A\cup\{y\};B}$$

for $y \notin A \cup B$. □

Corollary 3.9. *For any fixed A and A' such that $A \cap B = A' \cap B = \emptyset$, $I_{A';B}$ is a linear combination of elements of the set*

$$\{I_{A;B}\} \cup \{I_{C;B'} \mid C \cap B' = \emptyset,\ |B'| = |B| + 1,\ B \subseteq B'\},$$

where the coefficient of $I_{A;B}$ is ± 1.

Proof. It directly follows from the above proposition, since the coefficient of $I_{\emptyset;B}$ is ± 1. □

Theorem 3.10. *For any disjoint $A, B \subseteq \mathcal{P}(X)$, $I_{A;B}(z)$ is obtained by arithmetic operations and taking quadratic roots. In particular, so is $I_A(z)$.*

Proof. Note that $I_{A;\emptyset} = I_A$ for every $A \subseteq \mathcal{P}(X)$. We will induct on $|B|$ in reverse direction, from the largest to the smallest. If $|B| = |\mathcal{P}(X)|$, then $I_{\emptyset;\mathcal{P}(X)} = W$, so $W(z) = \frac{1-z-\sqrt{(1-z)^2-4mz^2}}{2z}$, which is a composition of arithmetic operations and taking quadratic roots.

Now, assume that it holds for every B with $|B| = n + 1$. Then, for the case $|B| = n$, by **Corollary 3.9**, for any $A \cap B = \emptyset$, the equation for $I_{A;B}$ in **Proposition 3.7** can be written as an at most quadratic equation, where coefficients consist of z, integers and $I_{A';B'}$'s for every A', B' such that $A' \cap B' = \emptyset$ and $|B'| = n + 1$. Moreover, after we simplify the equation for $I_{A;B}$, we find that the coefficient of $I_{A;B}$ is 1 modulo z. In particular, nonzero. Hence, the equation is not trivial, and so, $I_{A;B}$ is a composition of arithmetic operations and taking quadratic roots. Thus, by mathematical induction, it is true for every A, B such that $A \cap B = \emptyset$, so is I_A. □

For fixed disjoint $A, B \subseteq \mathcal{P}(X)$, we will count the number of pairs (C, D) such that $(C \cup D) \cap B = \emptyset$ and $C \setminus D = A$. This is equivalent to counting pairs (C', D') such that $D' \cap (A \cup B) = \emptyset$ and $C' \subseteq D'$ since

$$(C, D) \mapsto (C', D') = (C \setminus A, D)$$

and

$$(C', D') \mapsto (C, D) = (A \cup C', D')$$

are bijections between such pairs (C, D)'s and (C', D')'s. We may easily count the number of (C', D') pairs by choosing D' as a subset of $\mathcal{P}(X) \setminus (A \cup B)$ and C' as a subset of D', as follows.

$$\sum_{k=0}^{2^m-|B|-|A|} \sum_{l=0}^{k} \binom{2^m-|B|-|A|}{k}\binom{k}{l} = 3^{2^m-|B|-|A|}.$$

This is nothing but partitioning elements of $\mathcal{P}(X) \setminus (A \cup B)$ into $C \cap D$, $D \setminus C$ and $(C \cup D)^c$.

Hence, it is easy to write the equation for $I_{A;B}$ when $|A| + |B|$ is big. Thus, we may consider $I_{B^c;B}$ as special, and introduce a simpler notation

$$I_{-;B} := I_{B^c;B}.$$

Proposition 3.11. *For any $A \cap B = \emptyset$, $I_{A,B}(z)$ is a linear combination of*

$$\{I_{-;B'}(z) \mid A \cap B' = \emptyset,\ B \subseteq B'\}.$$

Proof. This follows from **Proposition 3.8**(b). □

The following proposition gives a way to compute exact coefficients when we write $I_{A;B}(z)$ as a linear combination of $I_{-;B'}(z)$'s.

Proposition 3.12. *For any disjoint $A, B \subseteq \mathcal{P}(X)$,*

$$I_{A;B}(z) = (-1)^{|A|} \sum_{B \subseteq B' \subseteq A^c} (-1)^{|B'|} I_{-;B'}(z)$$

Proof. First, $C \in P_{-;B'}$ if and only if ${B'}^c \subseteq C$ which is equivalent to $C^c \subseteq B'$. Hence,

$$P_{-;B'} \cap P_{-;B''} = P_{-;B' \cap B''}$$

is satisfied for any B' and B''. Now, we have

$$P_{A;B} = P_{-;A^c} \setminus \left(\bigcup_{y \in (A \cup B)^c} P_{-;(A \cup \{y\})^c} \right).$$

Thus, by the inclusion-exclusion principle, we get

$$\begin{aligned} I_{A;B}(z) &= \sum_{i=0}^{2^m-|A|-|B|} \sum_{Y \subseteq (A\cup B)^c, |Y|=i} (-1)^i I_{-;(A\cup Y)^c}(z) \\ &= \sum_{Y \subseteq (A \cup B)^c} (-1)^{|Y|} I_{-;(A \cup Y)^c}(z) \\ &= \sum_{B \subseteq B' \subseteq A^c} (-1)^{|{B'}^c \setminus A|} I_{-;B'}(z) \\ &= (-1)^{|A|} \sum_{B \subseteq B' \subseteq A^c} (-1)^{|B'|} I_{-;B'}(z). \end{aligned}$$

□

We consider an equation for $I_{-;B}$, obtained from **Proposition 3.7**,

$$I_{-;B} = \#\{x \mid F_x \in P_{-;B}\}z + zI_{\emptyset;B} + zI_{\emptyset;B}I_{-;B}.$$

Since we have

$$I_{\emptyset;B} = \sum_{B \subseteq B'} (-1)^{|B'|} I_{-;B'} = (-1)^{|B|} I_{-;B} + \sum_{B \subsetneq B'} (-1)^{|B'|} I_{-;B'},$$

adopting short notations $m_B = \#\{x \mid F_x \in P_{-;B}\}$, $\sigma_B = (-1)^{|B|}$ and $I_{\uparrow;B} = \sum_{B \subsetneq B'} (-1)^{|B'|} I_{-;B'}$, we can rewrite the equation as

$$I_{-;B} = z(m_B + I_{\uparrow;B}) + z(\sigma_B + I_{\uparrow;B}) I_{-;B} + z\sigma_B I_{-;B}^2.$$

This gives

$$I_{-;B} = \frac{1 - (\sigma_B + I_{\uparrow;B})z - \sqrt{(1 - (\sigma_B + I_{\uparrow;B})z)^2 - 4\sigma_B(m_B + I_{\uparrow;B})z^2}}{(2\sigma_B)\, z},$$

since $I_{-;B}(0) = 0$.

Naturally, the coefficients of $I_{-;B}$ as a formal power series are always nonnegative and smaller than the corresponding coefficients of W. So the coefficients of $I_{-;B}$ cannot have a larger growth rate than those of W. Since $\frac{1}{2\sqrt{m}+1}$ is the closest singularity of W by **Theorem IV.7** in [4], $I_{-;B}$ cannot have a singularity closer to 0 than $\frac{1}{2\sqrt{m}+1}$. If $I_{-;B}$ has singularity at $s_0 = \frac{1}{2\sqrt{m}+1}$, then we may apply Szegő lemma as [3] and [1], by computing

$$\lim_{z \to s_0^-} \frac{I_{-;B}(z) - I_{-;B}(s_0)}{\sqrt{1 - z/s_0}}$$

or, equivalently,

$$\lim_{z \to s_0^-} -2s_0 I'_{-;B}(z)\sqrt{1 - z/s_0}$$

to get the ratio

$$\lim_{n \to \infty} \frac{[z^n] I_{-;B}(z)}{[z^n] W(z)}.$$

If $I_{-;B}$ has no singularity at s_0, as long as singularities of $I_{-;B}$ are not closer to 0 than s_0, the limit of $I'_{-;B}(z)\sqrt{1 - z/s_0}$ exists and it will give the value 0, where for such a case the growth rate of coefficients of $I_{-;B}$ has strictly slower than that of W, so the ratio will be 0. Hence, this computation also matches for such cases. Now, let

$$\alpha_B = I_{-;B}(s_0),$$
$$\alpha_B^{\uparrow} = \sum_{B \subsetneq B'} (-1)^{|B'|} \alpha_{B'}$$

and

$$\beta_B = \lim_{z \to s_0^-} 2s_0 I'_{-;B}(z)\sqrt{1 - z/s_0},$$
$$\beta_B^{\uparrow} = \sum_{B \subsetneq B'} (-1)^{|B'|} \beta_{B'}.$$

Moreover, to simplify, denote

$$f_B = (1 - (\sigma_B + I_{\uparrow;B})z)^2 - 4\sigma_B z^2 (m_B + I_{\uparrow;B}).$$

Then, we have

$$I'_{-;B}(z) = \frac{-(\sigma_B + I_{\uparrow;B}(z)) - zI'_{\uparrow;B}(z) - \frac{f'_B(z)}{2\sqrt{f_B(z)}}}{(2\sigma_B)\, z} - \frac{I_{-;B}(z)}{z}.$$

Note that the second part becomes 0 when we consider the limit after multiplying $\sqrt{1 - z/s_0}$. By simple computation,

$$f'_B(z) = -2(\sigma_B + I_{\uparrow;B} + I'_{\uparrow;B} z)(1 - (\sigma_B + I_{\uparrow;B})z) - 4\sigma_B z^2 (m_B + I'_{\uparrow;B}) - 8\sigma_B z(m_B + I_{\uparrow;B}).$$

From above, if we define

$$d_B = f_B(s_0)/s_0^2 = (2\sqrt{m} + 1)^2 f_B(s_0) = (2\sqrt{m} + 1 - \sigma_B - \alpha_B^{\uparrow})^2 - 4\sigma_B(m_B + \alpha_B^{\uparrow}),$$

we obtain

$$\alpha_B = \frac{2\sqrt{m} + 1 - \sigma_B - \alpha_B^{\uparrow} - \sqrt{d_B}}{2\sigma_B}$$

and if $d_B \neq 0$, then

$$\beta_B = \beta_B^{\uparrow} \times \frac{-1 + \frac{2\sqrt{m}+1+\sigma_B-\alpha_B^{\uparrow}}{\sqrt{d_B}}}{2\sigma_B}.$$

Moreover, since β_B cannot go to the infinite, so $d_B = 0$ only if $\beta_B^{\uparrow} = 0$. In particular, when $B = \mathcal{P}(X)$, we obtain by simple computation

$$\alpha_{\mathcal{P}(X)} = \sqrt{m},$$
$$\beta_{\mathcal{P}(X)} = \sqrt{2m + \sqrt{m}}.$$

After computing every values of β_B, we can compute the density of well-formed formulae ϕ such that $F_\phi = A$.

Corollary 3.13. *For any $A \subseteq \mathcal{P}(X)$, we have*

$$\lim_{n\to\infty} \frac{[z^n]I_A(z)}{[z^n]W(z)} = \frac{(-1)^{|A|}\sum_{B\subseteq A^c}(-1)^{|B|}\beta_B}{\sqrt{2m+\sqrt{m}}}.$$

In particular,

$$\lim_{n\to\infty} \frac{[z^n]I_\emptyset(z)}{[z^n]W(z)} = \frac{\sum_{B\subseteq\mathcal{P}(X)}(-1)^{|B|}\beta_B}{\sqrt{2m+\sqrt{m}}}.$$

Proof. It directly follows from **Proposition 3.12** and Szegő's lemma. □

Lastly, for different propositional logic systems, there may be some cases in which it is better to consider $I_{\emptyset;B}$ rather than $I_{-;B}$. In such cases, we have a variant of **Proposition 3.12**, which is

$$I_{A;B}(z) = (-1)^{|A|+|B|} \sum_{B\subseteq B'\subseteq A\cup B} (-1)^{|B'|} I_{\emptyset;B'}(z).$$

This is from $P_{\emptyset;B'} \cap P_{\emptyset;B''} = P_{\emptyset;B'\cap B''}$ and

$$P_{A;B} = P_{\emptyset;A\cup B} \setminus \left(\bigcup_{y\in A} P_{\emptyset;(A\setminus\{y\})\cup B} \right)$$

where A, B are disjoint.

4. Numerical Theory

In this section, we will focus on systems of at-most-quadratic equations of generating functions which is more general. Consider a power series $Z(x) = \sum_{n=0}^{\infty} Z_n x^n$ with limit ratio

$$\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = \frac{1}{r} > 1,$$

satisfying at-most-quadratic relation of the form

$$Z(x) = f(x) + g(x)Z(x) + h(x)(Z(x))^2,$$

where g, h are polynomials, and f is a power series with a ratio condition

$$\lim_{n\to\infty} \frac{[x^n]f(x)}{[x^n]Z(x)} = \lim_{n\to\infty} \frac{f_n}{Z_n} = \gamma.$$

With this generating function Z as a base, we will consider a system of quadratic equations of generating functions related to Z. We will consider power series $A_i(x) = \sum_{n=0}^{\infty} A_{i\,n} x^n$ for $i = 1, \cdots, N$. First, these generating functions are comparable with $Z(x)$ by

$$\lim_{n\to\infty} \frac{[x^n]A_i(x)}{[x^n]Z(x)} = \lim_{n\to\infty} \frac{A_{i\,n}}{Z_n} = \beta_i,$$

the limit ratios of coefficients. Second, these generating functions satisfy the following system of at-most-quadratic relations

$$A_i(x) = f_i(x) + \sum_{j=1}^{N} g_{ij}(x)A_j(x) + \sum_{j,k=1}^{N} h(x)h_{ijk}(x)A_j(x)A_k(x)$$

where g_{ij}, h_{ijk}'s are some polynomials, and $f_i(x) = \sum_{n=0}^{\infty} f_{i\,n}x^n$ are power series. Here, h is the polynomial in the equation of $Z(x)$. Lastly, f_i's have ratio conditions

$$\lim_{n\to\infty} \frac{[x^n]f_i(x)}{[x^n]Z(x)} = \lim_{n\to\infty} \frac{f_{i\,n}}{Z_n} = \gamma_i,$$

for $i = 1, \cdots, N$. Also, for any formal power series $F(x) = \sum_{n=0}^{\infty} F_n x^n$, define

$$F^{\leq s}(x) = \sum_{n=0}^{s} F_n x^n.$$

Definition 4.1. With the above conditions on $r, \gamma, \gamma_i, g, h, g_{ij}$, and h_{ijk}, the **s-cut solution** $(\beta_1^{(s)}, \cdots, \beta_N^{(s)})$ for the system $(Z, A_1, \cdots, A_N)$ is defined as an N-tuple of numbers satisfying

$$\beta_i^{(s)} = \gamma_i + \sum_{j=1}^{N} g_{ij}(r)\beta_j^{(s)} + \sum_{j,k=1}^{N} h(r)h_{ijk}(r)\left(A_j^{\leq s}(r)\beta_k^{(s)} + A_k^{\leq s}(r)\beta_j^{(s)}\right) + \sum_{j,k=1}^{N} \zeta_s h_{ijk}(r)\beta_j^{(s)}\beta_k^{(s)},$$

where $\zeta_s = 1 - \gamma - g(r) - 2h(r)Z^{\leq s}(r)$, and the **$s$-cut operator** is the map $C_s : (x_1, \cdots, x_N) \mapsto (c_1, \cdots, c_N)$ where for $i = 1, 2, \ldots, N$,

$$c_i = \gamma_i + \sum_{j=1}^{N} g_{ij}(r)x_j + \sum_{j,k=1}^{N} h(r)h_{ijk}(r)\left(A_j^{\leq s}(r)x_k + A_k^{\leq s}(r)x_j\right) + \sum_{j,k=1}^{N} \zeta_s h_{ijk}(r)x_j x_k.$$

In other words, an s-cut solution is a fixed point of the s-cut operator. Also, note that we may add Z as A_0 to the system of equations for $A_1, \cdots, A_N$ freely. So, we may consider Z as one of A_i's if we need. Moreover, we may deal with cubic terms of form $h(x)^2A_i(x)A_j(x)A_k(x)$ by introducing $B_{ij}(x) = h(x)A_j(x)A_j(x)$, and it can be generalized to any degree.

The following is a motivation to consider the s-cut concept.

Theorem 4.2. *Suppose formal power series $Z, A_1, \cdots, A_N$ satisfy*

$$Z(x) = f(x) + g(x)Z(x) + h(x)Z(x)^2,$$

$$A_i(x) = f_i(x) + \sum_{j=1}^{N} g_{ij}(x)A_j(x) + \sum_{j,k=1}^{N} h(x)h_{ijk}(x)A_j(x)A_k(x)$$

where g, h, g_{ij}, h_{ijk} are polynomials for $i, j, k = 1, \cdots, N$. Also, assume that there exist r, $\beta_1, \ldots, \beta_N$, γ, $\gamma_1, \cdots, \gamma_N$ such that

$$\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = \frac{1}{r} > 1, \quad \lim_{n\to\infty} \frac{A_{in}}{Z_n} = \beta_i, \quad \lim_{n\to\infty} \frac{f_n}{Z_n} = \gamma, \quad \lim_{n\to\infty} \frac{f_{in}}{Z_n} = \gamma_i,$$

and let β be N-tuple $(\beta_1, \cdots, \beta_N)$. Lastly, assume that Z, h have only non-negative coefficients and $Z(r)$ is bounded. Then

$$\beta = \lim_{s\to\infty} C_s(\beta).$$

Proof. Since

$$1 = \frac{f_n}{Z_n} + \sum_{u=0}^{\deg g} g_u \frac{Z_{n-u}}{Z_n} + \sum_{u=0}^{\deg h} h_u \sum_{v=0}^{n-u} \frac{Z_v Z_{n-u-v}}{Z_n},$$

$\lim_{n\to\infty} \frac{Z_{n-m}}{Z_n} = r^m$, and $\lim_{n\to\infty} \frac{f_n}{Z_n} = \gamma$, we have

$$\lim_{n\to\infty} \sum_{u=0}^{\deg h} h_u \sum_{v=s+1}^{n-u-s-1} \frac{Z_v Z_{n-u-v}}{Z_n} = 1 - \gamma - g(r) - 2h(r)Z^{\leq s}(r) = \zeta_s.$$

Then, using the notation $|g|(x)$ to denote $\sum_n |g_n| x^n$, we have

$$|\zeta_s| \leq 1 + |\gamma| + |g|(r) + 2h(r)Z(r),$$

which implies that ζ_s is bounded. If we let $C_s(\beta) = (c_1^{(s)}, \ldots, c_N^{(s)})$, then by a similar argument, if we apply $\lim_{n\to\infty}$ to the formula of A_{in} after dividing by Z_n, we get

$$\beta_i - c_i^{(s)} = \lim_{n\to\infty} \sum_{j,k=1}^{N} \sum_{t=0}^{\deg h_{ijk}} (h_{ijk})_t \sum_{u=0}^{\deg h} h_u \sum_{v=s+1}^{n-t-u-s-1} \Delta_{njktuv}$$

where

$$\Delta_{njktuv} = \frac{A_{jv} A_{k,n-t-u-v}}{Z_n} - \frac{\beta_j \beta_k Z_v Z_{n-t-u-v}}{Z_n}.$$

Then, for any $\epsilon > 0$, choose s so that $s \leq \min\{u, v\}$ implies $|\beta_j \beta_k - \frac{A_{jv} A_{ku}}{Z_u Z_v}| < \epsilon$ for any j, k; and choose n so that $n > 2s + \deg h_{ijk} + \deg h$ for every i, j, k. Then, for any v such that $s < v < n - t - u - s$,

$$\begin{aligned} |\Delta_{njktuv}| &= \left| \frac{A_{jv} A_{k,n-t-u-v}}{Z_n} - \frac{\beta_j \beta_k Z_v Z_{n-t-u-v}}{Z_n} \right| \\ &\leq \epsilon \times \frac{Z_v Z_{n-t-u-v}}{Z_{n-t}} \frac{Z_{n-t}}{Z_n}. \end{aligned}$$

Hence, if we apply $\lim_{n\to\infty}$, we get

$$|\beta_i - c_i^{(s)}| \leq \epsilon \sum_{j,k=1}^{N} \sum_{t=0}^{\deg h_{ijk}} |(h_{ijk})_t| \zeta_s r^t.$$

Thus, $|\beta_i - c_i^{(s)}| \to 0$ as $s \to \infty$. Hence $\beta = \lim_{s\to\infty} C_s(\beta)$. □

From now on, we will assume that Z, f, g, h, f_i satisfy the following three conditions: (1) Z, f, f_i are formal power series, (2) g, h are polynomials and (3) r, γ, γ_i are well-defined.

Proposition 4.3. *Suppose that all the coefficients of Z, g, h are nonnegative, h is nonzero or γ is nonnegative, and $f(r), Z(r)$ converge. Then we have*

$$0 \leq \sqrt{(1 - g(r))^2 - 4f(r)h(r)} - \gamma \leq \zeta_s \leq 1 - \gamma - g(r)$$

and

$$\lim_{s\to\infty} \zeta_s = \sqrt{(1 - g(r))^2 - 4f(r)h(r)} - \gamma = 1 - \gamma - g(r) - 2h(r)Z(r).$$

In this case, we set $\zeta_\infty = \lim_{s\to\infty} \zeta_s$ and call it ***the impurity of the equation*** *$Z = f + gZ + hZ^2$. Moreover, if f has no singularity in $\{z \in \mathbb{C} \mid |z| < r + \epsilon\}$ for some $\epsilon > 0$, then both γ and the impurity, ζ_∞, are zero.*

Proof. First, we have $\zeta_s = 1-\gamma-g(r)-2h(r)Z^{\le s}(r) \le 1-\gamma-g(r)$. Moreover,

$$\zeta_s = \lim_{n\to\infty}\sum_{u=0}^{\deg h} h_u \sum_{v=s+1}^{n-u-s-1} \frac{Z_v Z_{n-u-v}}{Z_n}$$

gives $\zeta_s \ge 0$ always. If $h = 0$, then we have

$$\zeta_s = 1-\gamma-g(r) = \sqrt{(1-g(r))^2-4f(r)h(r)}-\gamma = 0$$

where $1-g(r)\ge 0$ is from $\gamma \ge 0$. Now, if h is nonzero, then we have

$$Z(x) = \frac{1-g(x)-\sqrt{(1-g(x))^2-4f(x)h(x)}}{2h(x)}.$$

Since $f(r), Z(r)$ converge, it gives

$$Z(r) = \frac{1-g(r)-\sqrt{(1-g(r))^2-4f(r)h(r)}}{2h(r)}.$$

Hence,

$$\zeta_s = 1-g(r)-2h(r)Z^{\le s}(r)-\gamma \ge 1-g(r)-2h(r)Z(r)-\gamma$$

and

$$0 \le \lim_{s\to\infty}\zeta_s = 1-g(r)-2h(r)Z(r)-\gamma = \sqrt{(1-g(r))^2-4f(r)h(r)}-\gamma.$$

Now, consider the case that f has no singularity in $\{z\in\mathbb{C} \mid |z| < r+\epsilon\}$. By **Theorem IV.7** in [4], r is the closest singularity to zero of Z. If h is zero, then

$$Z(x) = \frac{f(x)}{1-g(x)}.$$

Since f has no singularity in $\{z\in\mathbb{C} \mid |z| < r+\epsilon\}$, it means $g(r) = 1$. Hence, $\zeta_s = 1-\gamma-g(r) = -\gamma \le 0$, so the impurity and γ are zero. For the case that h is nonzero, f has no singularity in $\{z\in\mathbb{C} \mid |z| < r+\epsilon\}$ and g, h are polynomials, so

$$(1-g(r))^2-4f(r)h(r) = 0.$$

Then $1-g(r)-2h(r)Z(r) = 0$ and $\lim_{s\to\infty}\zeta_s = -\gamma$. Hence, it is enough to prove that $\gamma = 0$. This can be induced from again **Theorem IV.7** in [4], which gives $\limsup (f_n)^{1/n} \le \frac{1}{r+\epsilon}$. $\square$

Combining **Theorem 4.2** and **Proposition 4.3**, we directly obtain the following.

Theorem 4.4. *Suppose that our system of equations on formal power series $Z(x), A_1(x), \cdots, A_N(x)$ satisfies that the coefficients of Z, g, h are nonnegative, h is nonzero, $f(r)$, $Z(r)$, $A_1(r)$, $\cdots$, $A_N(r)$ converge and the impurity is zero. Then*

$$\beta_i = \gamma_i + \sum_{j=1}^{N} g_{ij}(r)\beta_j + \sum_{j,k=1}^{N} h(r)h_{ijk}(r)(A_j(r)\beta_k + A_k(r)\beta_j).$$

whenever $\beta_i = \lim_{n\to\infty}\frac{[x^n]A_i(x)}{[x^n]Z(x)}$ exists for every i.

Note that this result can be understood as an application of Szegő's lemma, just differentiate and multiply $\sqrt{1-x/r}$ and take limit. Moreover, this is linear on β_j's when $A_j(r)$'s are given, and linear on $A_j(r)$'s when β_j's are given. Note that if $A_j(r)$ are given and γ_i's are zero, then it is a homogeneous linear system on β_j's, in which case we need more conditions to solve completely. This theorem gives an alternative practical method to compute

$$\lim_{n\to\infty}\frac{[z^n]I_A(z)}{[z^n]W(z)}$$

which uses $\alpha_B = I_{\setminus;B}(s_0)$, since we have natural additional condition

$$\sum_{A\subseteq\mathcal{P}(X)} \lim_{n\to\infty} \frac{[z^n]I_A(z)}{[z^n]W(z)} = 1.$$

From the equation $Z = f + gZ + hZ^2$, g and h show recursive structures of the object counted by Z, and f counts basic elements. Hence it is natural that basic elements do not form so large a portion among objects to focus on its recursive structures, which means γ, the limit portion of the basic elements among the whole objects, is natural to be 0.

Since for any polynomial δ, δZ is also a formal power series which satisfies the ratio condition, we may define new f as $f + \delta Z$ to modify the value of γ. Also, from equation $Z = f + gZ + hZ^2$, we can make a different equation by multiplying the constant c and rewrite as $Z = (cf + (1-c)Z) + cgZ + chZ^2$. We will give a name to these conversions, and show that even when we convert γ by these conversions, the impurity is a kind of an invariant, so the zeroness of the impurity is preserved, and hence we can change γ safely.

Definition 4.5.

(a) If $\gamma \neq 1$, $\gamma - \widehat{\gamma}$ conversion of equation $Z = f + gZ + hZ^2$ is defined as $Z = \widehat{f} + \widehat{g}Z + \widehat{h}Z^2$ where

$$\widehat{f} = \frac{1-\widehat{\gamma}}{1-\gamma} f + \frac{\widehat{\gamma}-\gamma}{1-\gamma} Z,$$
$$\widehat{g} = \frac{1-\widehat{\gamma}}{1-\gamma} g,$$
$$\widehat{h} = \frac{1-\widehat{\gamma}}{1-\gamma} h.$$

(b) If $\delta(x)$ is a polynomial, δ conversion is defined as $Z = \widetilde{f} + \widetilde{g}Z + \widetilde{h}Z^2$ where

$$\widetilde{f} = f + \delta Z,$$
$$\widetilde{g} = g - \delta,$$
$$\widetilde{h} = h.$$

Proposition 4.6.

(a) For $\gamma - \widehat{\gamma}$ conversion, we have

$$\lim_{n\to\infty} \frac{\widehat{f}_n}{Z_n} = \widehat{\gamma}.$$

(b) For $\gamma - \widehat{\gamma}$ conversion, $\frac{\zeta_s}{1-\gamma}$ is invariant and, moreover, if $Z(r)$ converges, then $\frac{\zeta_\infty}{1-\gamma}$ is invariant.

(c) For δ conversion, we have

$$\widetilde{\gamma} = \gamma + \delta(r).$$

(d) For δ conversion, ζ_s is invariant and if $Z(r)$ converges, then ζ_∞ is invariant.

Proof. (a),(c) are simple computation.

(b) We have

$$\begin{aligned}\widehat{\zeta}_s &= 1 - \widehat{\gamma} - \widehat{g}(r) - 2\widehat{h}(r)Z^{\leq s}(r)\\ &= \frac{1-\widehat{\gamma}}{1-\gamma}\left(1 - \gamma - g(r) - 2h(r)Z^{\leq s}(r)\right)\\ &= \frac{1-\widehat{\gamma}}{1-\gamma}\zeta_s.\end{aligned}$$

(d) We have

$$\begin{aligned}\widetilde{\zeta}_s &= 1-\widetilde{\gamma}-\widetilde{g}(r)-2\widetilde{h}(r)Z^{\le s}(r)\\ &= 1-\gamma-\delta(r)-g(r)+\delta(r)-2h(r)Z^{\le s}(r)\\ &= \zeta_s.\end{aligned}$$

□

Now, we are going to compute the numerical estimations of the ratio β_i's by computing s-cut solutions, which means we expect that

$$\lim_{s\to\infty}\beta_i^{(s)} = \lim_{n\to\infty}\frac{A_{in}}{Z_n} = \beta_i$$

is satisfied. Since the equation for s-cut solution is quadratic, existence and uniqueness are not guaranteed. Hence, we will provide some condition for existence, uniqueness and above convergence of s-cut solution.

Definition 4.7. Suppose that a formal power series Z satisfies

$$Z(x) = f(x)+g(x)Z(x)+h(x)Z(x)^2$$

with a formal power series f and polynomials g, h. Then, $A_1,\cdots,A_N$ satisfying

$$A_i(x) = f_i(x)+\sum_{j=1}^{N} g_{ij}(x)A_j(x)+\sum_{j,k=1}^{N} h(x)h_{ijk}(x)A_j(x)A_k(x)$$

are a **natural partition of** Z if

$$\begin{aligned}Z(x) &= \sum_{i=1}^{N} A_i(x),\\ f(x) &= \sum_{i=1}^{N} f_i(x),\\ g(x) &= \sum_{i=1}^{N} g_{ij}(x),\\ 2 &= \sum_{i=1}^{N}(h_{ijk}(x)+h_{ikj}(x)).\end{aligned}$$

Also, a natural partition system $(Z, A_1,\cdots,A_N)$ is **nonnegative** if all the coefficients of Z, A_i, g, h, g_{ij}, h_{ijk}, and γ,γ_i are nonnegative. From nonnegativity, we have $[x^n]Z(x)\ge[x^n]f(x)$ and $[x^n]A_i(x)\ge[x^n]f_i(x)$ which imply $\gamma,\gamma_i\le 1$, and since sum of $h_{ijk}(x)$'s is a constant polynomial, every $h_{ijk}(x)$ is also a constant polynomial.

Proposition 4.8. *Let $(c_1,\cdots,c_N)$ be a fixed point of the s-cut operator C_s for a natural partition system $(Z, A_1,\cdots,A_n)$ with nonzero ζ_s. Then $(c_1,\cdots,c_N)$ is on the hyperplane $x_1+\cdots+x_N=\frac{\gamma}{\zeta_s}$ or $x_1+\cdots+x_N=1$ in $\mathbb{R}^N$.*

Proof.

$$\begin{aligned}\sum_{i=1}^N c_i =& \sum_{i=1}^N \gamma_i + \sum_{i=1}^N \sum_{j=1}^N g_{ij}(r)c_j + \sum_{i=1}^N \sum_{j,k=1}^N h(r)h_{ijk}(r)(A_j^{\leq s}(r)c_k + A_k^{\leq s}(r)c_j) \\ &+ \sum_{i=1}^N \sum_{j,k=1}^N \zeta_s h_{ijk}(r)c_jc_k \\ =&\gamma + \sum_{j=1}^N g(r)c_j + \frac{1}{2}\sum_{j,k=1}^N \sum_{i=1}^N h(r)(h_{ijk}(r)+h_{ikj}(r))(A_j^{\leq s}(r)c_k + A_k^{\leq s}(r)c_j) \\ &+ \frac{1}{2}\sum_{j,k=1}^N \sum_{i=1}^N \zeta_s(h_{ijk}(r)+h_{ikj}(r))c_jc_k \\ =&\gamma + g(r)\sum_{j=1}^N c_j + \sum_{j,k=1}^N h(r)(A_j^{\leq s}(r)c_k + A_k^{\leq s}(r)c_j) + \sum_{j,k=1}^N \zeta_s c_jc_k \\ =&\gamma + g(r)\sum_{j=1}^N c_j + 2h(r)Z^{\leq s}(r)\sum_{j=1}^N c_j + \zeta_s\left(\sum_{j=1}^N c_j\right)^2.\end{aligned}$$

Hence,

$$(\zeta_s+\gamma)\sum_{j=1}^N c_j = \gamma + \zeta_s\left(\sum_{j=1}^N c_j\right)^2,$$

which proves the proposition. □

Proposition 4.9. *The s-cut operator C_s of a nonnegative natural partition system $(Z, A_1, \cdots, A_N)$ has a fixed point in*

$$H := \{(x_1,\cdots,x_N)\in\mathbb{R}^N : 0\leq x_i\leq 1, \sum_{i=1}^N x_i = 1\}.$$

Proof. Let $C_s(x_1,\cdots,x_N) = (c_1,\cdots,c_N)$. If $(x_1,\cdots,x_N)\in H$, then as in the proof of **Proposition 4.8**, we have

$$\sum_{i=1}^N c_i = \gamma + g(r)\cdot 1 + 2h(r)Z^{\leq s}(r)\cdot 1 + \zeta_s\cdot 1^2 = 1.$$

In the proof of **Proposition 4.3** we obtain $\zeta_s \geq 0$ from the fact that coefficients of Z and h are nonnegative. Hence, we have $c_i \geq 0$ for every i. Then, $\sum_{i=1}^N c_i = 1$ implies $c_i \leq 1$, so $(c_1,\cdots,c_N)\in H$.

Now, H is a convex compact set in $\mathbb{R}^N$, so by Brouwer fixed point theorem, C_s has a fixed point in H. □

By simple computation, we have

$$\frac{\partial c_i}{\partial x_j} = g_{ij}(r) + \sum_{k=1}^N h(r)(h_{ijk}(r)+h_{ikj}(r))A_k^{\leq s}(r) + \zeta_s\sum_{k=1}^N (h_{ijk}(r)+h_{ikj}(r))x_k.$$

From this, we have the following result.

Proposition 4.10. *For the Jacobian J of the s-cut operator C_s of a nonnegative natural partition system $(Z, A_1, \cdots, A_n)$,*

$$\|J(x_1, \cdots, x_N)\|_1 = 1 - \zeta_s - \gamma + 2\zeta_s \left(\sum_{i=1}^{N} x_i \right)$$

on $[0, \infty)^N$, where $\|\cdot\|_1$ denotes the 1-norm of a matrix. In particular, $\|J\|_1 = 1 - \gamma + \zeta_s$ on H.

Proof. Note that $\|B\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |b_{ij}|$ for any $m \times n$ matrix B. Since the system is nonnegative, $\frac{\partial c_i}{\partial x_j} \ge 0$ on $[0, \infty)^N$. Then,

$$\begin{aligned}
\sum_{i=1}^{N} \frac{\partial c_i}{\partial x_j} =& \sum_{i=1}^{N} g_{ij}(r) + \sum_{k=1}^{N} h(r) \sum_{i=1}^{N} (h_{ijk}(r) + h_{ikj}(r)) A_k^{\le s}(r) \\
&+ \zeta_s \sum_{k=1}^{N} \sum_{i=1}^{N} (h_{ijk}(r) + h_{ikj}(r)) x_k \\
=& g(r) + \sum_{k=1}^{N} h(r) 2 A_k^{\le s}(r) + \zeta_s \sum_{k=1}^{N} 2 x_k \\
=& g(r) + 2h(r) Z^{\le s}(r) + 2\zeta_s \sum_{k=1}^{N} x_k \\
=& 1 - \zeta_s - \gamma + 2\zeta_s \sum_{k=1}^{N} x_k,
\end{aligned}$$

which proves the result. □

This result is also true when nonnegative condition is weakened: For instance γ and γ_i may not be nonnegative. Moreover, we have the following result for general p-norms.

Proposition 4.11. *For the Jacobian J of the s-cut operator C_s of a natural partition system,*

$$\|J(x_1, \cdots, x_N)\|_p \ge |1 - \gamma + \zeta_s|$$

on H. Note that $|1 - \gamma + \zeta_s| = 1 - \gamma + \zeta_s$ when the given system is nonnegative, since we have $\gamma \le 1$ and $\zeta_s \ge 0$.

Proof. Let J^T denote the transpose of the Jacobian. We have

$$J^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = (1 - \gamma + \zeta_s) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

on H, from $\sum_{i=1}^{N} \frac{\partial c_i}{\partial x_j} = 1 - \zeta_s - \gamma + 2\zeta_s \sum_{k=1}^{N} x_k = 1 - \gamma + \zeta_s$. Hence, $1 - \gamma + \zeta_s$ is an eigenvalue of J^T, so is an eigenvalue of J. Thus, we get $\|J\|_p \ge |1 - \gamma + \zeta_s|$. □

Since the norm of the Jacobian of the s-cut operator C_s can be larger than 1, especially when $\gamma = 0$, this fact may induce some convergence problem when we try to find an s-cut solution by applying fixed point iteration method on C_s. Hence, we may consider modification.

Definition 4.12. The σ-shifted s-cut operator $\widetilde{C_s^\sigma}$ is defined as

$$\widetilde{C_s^\sigma}(x) = C_s(x) - \sigma \left(\sum_{i=1}^{N} x_i - 1 \right) \cdot (1, 1, \cdots, 1).$$

Since $\widetilde{C_s^\sigma}(x) = C_s(x)$ for all $x \in H$, fixed points of C_s on H are fixed points of $\widetilde{C_s^\sigma}$. Moreover,

$$\widetilde{J} = J - \begin{bmatrix} \sigma & \sigma & \cdots & \sigma \\ \sigma & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \sigma & \cdots & \cdots & \sigma \end{bmatrix} = J - \sigma\mathbf{1},$$

where $\widetilde{J}$ is the Jacobian for $\widetilde{C_s^\sigma}$.

From the Banach contraction principle, we deduce the following.

Proposition 4.13. *If the Jacobian $\widetilde{J}$ of $\widetilde{C_s^\sigma}$ satisfies $\|\widetilde{J}\| < 1$ for a matrix norm $\|\cdot\|$ on H, then $\widetilde{C_s^\sigma}$ is a contraction on H, and C_s has the unique fixed point on H.*

Note that since H is compact, $\|\widetilde{J}\| < 1$ is enough to apply the Banach contraction principle rather than the condition that there exists $K < 1$ such that $\|\widetilde{J}\| \leq K$. From $\|A - B\| \geq |\|A\| - \|B\||$, it would be best to choose σ satisfying $\|J\| = \|\sigma\mathbf{1}\|$, and one of such choice is $\sigma = \frac{1-\gamma+\zeta_s}{N}$, which is from the 1-norm. Hence, we will call the s-cut operator shifted by this value as the **standard shifted s-cut operator**.

Corollary 4.14. *The Jacobian $\tilde{J}$ of the standard shifted s-cut opertor $\widetilde{C_s}$ of a nonnegative natural partition system satisfies $\|\tilde{J}\|_1 < 1$ on H if $1 - 2\gamma + 2\zeta_s > 0$ and*

$$\frac{\partial c_i}{\partial x_j} < \frac{1-\gamma+\zeta_s}{N} + \max\left\{\frac{1-\gamma+\zeta_s}{N(1-2\gamma+2\zeta_s)}, \frac{1}{2(N-1)}\right\}.$$

Note that $1 - 2\gamma + 2\zeta_s \leq 0$ implies $1 - \gamma + \zeta_s \leq \frac{1}{2}$, which means $\|J\|_1 < 1$ is already satisfied without shifting.

Proof. Since $\|\widetilde{J}\|_1 = \max\left\{\sum_{i=1}^N \left|\frac{\partial c_i}{\partial x_j} - \frac{1-\gamma+\zeta_s}{N}\right| \mid j = 1, \cdots, N\right\}$ and we have $\sum_{i=1}^N \frac{\partial c_i}{\partial x_j} = 1 - \gamma + \zeta_s$ already, it is enought to prove that a_is arranged as $\max\{\frac{1-\gamma+\zeta_s}{N(1-2\gamma+2\zeta_s)}, \frac{1}{2(N-1)}\} + \frac{1-\gamma+\zeta_s}{N} > a_1 \geq a_2 \geq \cdots \geq a_m \geq \frac{1-\gamma+\zeta_s}{N} \geq a_{m+1} \geq \cdots \geq a_N \geq 0$ satisfying $\sum_{i=1}^N a_i = 1 - \gamma + \zeta_s$ satisfies $\sum_{i=1}^N |a_i - \frac{1-\gamma+\zeta_s}{N}| < 1$. Since $\frac{1-\gamma+\zeta_s}{N}$ is the mean of a_is, we may assume $m < N$. Easily,

$$\begin{aligned}
\sum_{i=1}^N \left|a_i - \frac{1-\gamma+\zeta_s}{N}\right| &= \sum_{i=1}^m (a_i - \frac{1-\gamma+\zeta_s}{N}) + \sum_{i=m+1}^N (\frac{1-\gamma+\zeta_s}{N} - a_i) \\
&= \frac{1-\gamma+\zeta_s}{N}(N-2m) + \sum_{i=1}^m a_i - \sum_{i=m+1}^N a_i \\
&= \frac{1-\gamma+\zeta_s}{N}(N-2m) + 2\sum_{i=1}^m a_i - (1-\gamma+\zeta_s) \\
&= 2\sum_{i=1}^m a_i - \frac{2m}{N}(1-\gamma+\zeta_s).
\end{aligned}$$

If $\frac{1}{2(N-1)} \geq \frac{1-\gamma+\zeta_s}{N(1-2\gamma+2\zeta_s)}$, we have $a_i < \frac{1}{2(N-1)} + \frac{1-\gamma+\zeta_s}{N}$, so

$$\begin{aligned}
&2\sum_{i=1}^m a_i - \frac{2m}{N}(1-\gamma+\zeta_s) \\
&\qquad < 2m(\frac{1}{2(N-1)} + \frac{1-\gamma+\zeta_s}{N}) - \frac{2m}{N}(1-\gamma+\zeta_s) = \frac{m}{N-1} \leq 1.
\end{aligned}$$

For the other case, if $m > N\left(1 - \frac{1}{2(1-\gamma+\zeta_s)}\right)$, we have

$$\begin{aligned}2\sum_{i=1}^{m} a_i - \frac{2m}{N}(1-\gamma+\zeta_s) &\leq 2\sum_{i=1}^{N} a_i - \frac{2m}{N}(1-\gamma+\zeta_s)\\ &\leq 2(1-\gamma+\zeta_s) - \frac{2m}{N}(1-\gamma+\zeta_s)\\ &= 2(1-\gamma+\zeta_s)(1-\frac{m}{N}) < 1,\end{aligned}$$

and if $m \leq N(1 - \frac{1}{2(1-\gamma+\zeta_s)})$, we have $a_i < \frac{1-\gamma+\zeta_s}{N(1-2\gamma+2\zeta_s)} + \frac{1-\gamma+\zeta_s}{N}$, so

$$\begin{aligned}&2\sum_{i=1}^{m} a_i - \frac{2m}{N}(1-\gamma+\zeta_s)\\ &< 2m\left(\frac{1-\gamma+\zeta_s}{N} + \frac{1-\gamma+\zeta_s}{N(1-2\gamma+2\zeta_s)}\right) - \frac{2m}{N}(1-\gamma+\zeta_s)\\ &= \frac{m}{N}\frac{2(1-\gamma+\zeta_s)}{1-2\gamma+2\zeta_s}\\ &\leq \left(1 - \frac{1}{2(1-\gamma+\zeta_s)}\right)\frac{2(1-\gamma+\zeta_s)}{1-2\gamma+2\zeta_s} = 1.\end{aligned}$$

□

This corollary gives a condition to have the unique s-cut solution by computing the 1-norm of the shifted s-cut operator. Finding the best choice to shift based on the matrix 1-norm of the Jacobian is equivalent to find σ from given nonnegative sequences $a^{(1)}, \cdots, a^{(N)}$ satisfying $\sum_i a_i^{(1)} = \cdots = \sum_i a_i^{(N)}$ such that minimizes the $\max\left\{\sum_i \left|a_i^{(j)} - \sigma\right| \mid j = 1, \cdots, N\right\}$. For each j, it is well-known that the median minimizes $\sum_i \left|a_i^{(j)} - \sigma\right|$, compare with that mean minimizes $\sum_i (a_i^{(j)} - \sigma)^2$, where the standard shift operator is defined as to choose σ as the mean, which is easier to compute than the median. Hence, it may possible to refine the condition to have unique s-cut solution by considering the median rather than the mean. In such case, we may have to use some variant of the iteration method, which uses different iteration function for each iteration.

Lastly, we will prove the following.

Theorem 4.15. *Suppose that a nonnegative natural partition system $(Z, \{A_i\})$ satisfies the following:*

- $\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = \frac{1}{r} > 1$,
- $\lim_{n\to\infty} \frac{A_{in}}{Z_n} = \beta_i$,
- *there exist a common contraction factor $K < 1$ and a sequence of proper shifting factor $\{\sigma_s\}$ of the s-cut operator C_s satisfying $\left|\widetilde{C_s^{\sigma_s}}(x) - \widetilde{C_s^{\sigma_s}}(y)\right| \leq K|x-y|$ for any $x, y \in H$ except for finitely many s.*

Then, there exists a sequence of s-cut solutions on H, $\beta^{(s)} = (\beta_1^{(s)}, \cdots, \beta_N^{(s)})$, converging to $\beta = (\beta_1, \cdots, \beta_N)$ as $s \to \infty$.

Proof. From **Theorem 4.2**, $\lim_{s\to\infty} C_s(\beta) = \beta$ is satisfied. We may assume s is large enough to have a common contraction constant K. Then, we have

$$|\beta - \beta^{(s)}| \leq |\beta - C_s(\beta)| + |C_s(\beta) - \beta^{(s)}| = |\beta - C_s(\beta)| + |C_s(\beta) - C_s(\beta^{(s)})|.$$

Since $C_s = \widetilde{C_s^{\sigma}}$ on H, C_s is also a contraction on H with same contraction constant. Hence,

$$|\beta - \beta^{(s)}| \leq |\beta - C_s(\beta)| + |C_s(\beta) - C_s(\beta^{(s)})| \leq |\beta - C_s(\beta)| + K|\beta - \beta^{(s)}|.$$

Thus,

$$|\beta - \beta^{(s)}| \leq \frac{1}{1-K}|\beta - C_s(\beta)| \to 0$$

as $s \to \infty$. $\square$

Note that except for finitely many s's, each $\beta^{(s)}$ is uniquely determined.

5. Estimated results for the multivariable cases

We now go back to the original problem, computing the density of tautologies. Even if our logic system has more than 1 variable, we have a method to get an exact formula of the density of tautologies and of antilogies. But the formulae will include nearly 2^m nested quadratic roots, which makes visualization difficult. Hence, in the following table we only provide numerical results for densities of tautologies and antilogies, when the number of variables is two, three and four computed by Sage.

density	tautologies	antilogies
$m=2$	0.33213..	0.09710..
$m=3$	0.27003..	0.06625..
$m=4$	0.22561..	0.04868..

It is somewhat surprising that if we partition the set of well-formed formulae into classes according to F_ϕ, the largest class is the class of tautologies and the second largest is the class of antilogies when the number of variables is two, three or four, which is false when there is only one variable. But it is easy to prove that the density of tautologies is $\Omega(\frac{1}{m})$ and the density of antilogies is $\Omega(\frac{1}{m\sqrt{m}})$, where the number of classes is 2^{2^m}. The next paragraph proves it.

Note that if ψ is a tautology, then $\neg\neg\psi$ is a tautology and that for any well-formed formula ϕ, $p \to [\phi \to p]$ is a tautology for any variable $p \in X_m$. Since these two types have no common elements, we have

$$[z^n]I_\emptyset(z) \geq [z^{n-2}]I_\emptyset(z) + m \cdot ([z^{n-4}]W(z)).$$

Now we deduce

$$[z^n]W(z) \simeq \sqrt{\frac{2m+\sqrt{m}}{4\pi n^3}}(2\sqrt{m}+1)^n$$

from the expression

$$\begin{aligned} W(z) &= \frac{1-z-\sqrt{(1-(2\sqrt{m}+1)z)(1+(2\sqrt{m}-1)z)}}{2z} \\ &= \sqrt{m} - \sqrt{2m+\sqrt{m}}\sqrt{1-(2\sqrt{m}+1)z} + O(1-(2\sqrt{m}+1)z). \end{aligned}$$

Hence, $\lim_{n\to\infty} \frac{[z^{n-2}]W(z)}{[z^n]W(z)} = \frac{1}{(2\sqrt{m}+1)^2}$ and $\lim_{n\to\infty} \frac{[z^{n-4}]W(z)}{[z^n]W(z)} = \frac{1}{(2\sqrt{m}+1)^4}$. Thus,

$$\lim_{n\to\infty} \frac{[z^n]I_\emptyset(z)}{[z^n]W(z)} \geq \left(\frac{1}{(2\sqrt{m}+1)^2} \lim_{n\to\infty} \frac{[z^{n-2}]I_\emptyset(z)}{[z^{n-2}]W(z)}\right) + \frac{m}{(2\sqrt{m}+1)^4}$$

and so,

$$\lim_{n\to\infty} \frac{[z^n]I_\emptyset(z)}{[z^n]W(z)} \geq \frac{\sqrt{m}}{4(\sqrt{m}+1)(2\sqrt{m}+1)^2},$$

where we have $\frac{\sqrt{m}}{4(\sqrt{m}+1)(2\sqrt{m}+1)^2} = \Theta(\frac{1}{m})$. Thus, the density of tautologies is $\Omega(\frac{1}{m})$. For antilogies, we can get $\Omega(\frac{1}{m\sqrt{m}})$ from

$$[z^n]I_{\mathcal{P}(X)}(z) \geq [z^{n-1}]I_\emptyset(z),$$

since every $\neg\phi$ is an antilogy for any tautology ϕ.

Now, we will show that using s-cut solution is efficient to compute the approximation of the limit value. The following table compares exact values, the ratios at s, which are $\frac{[z^s]I_A}{[z^s]W}$, and s-cut solutions for the density of tautologies and of antilogies when the number of variables and s change.

			$s=10$		$s=50$		$s=200$	
		value	ratio	cut-sol	ratio	cut-sol	ratio	cut-sol
$m=1$	taut	0.4232	0.3102	0.4243	0.4142	0.4233	0.4210	0.4233
	anti	0.1632	0.1868	0.1642	0.1612	0.1634	0.1628	0.1633
$m=2$	taut	0.3321	0.2374	0.3345	0.3206	0.3323	0.3293	0.3322
	anti	0.0971	0.0996	0.0982	0.0947	0.0972	0.0965	0.0971
$m=3$	taut	0.2700	0.1913	0.2732	0.2581	0.2703	0.2670	0.2701
	anti	0.0663	0.0637	0.0673	0.0641	0.0663	0.0657	0.0663

Values for $m=1$ is from **Theorem 22** of [1]. The s-cut solution is computed by Sage using fixed point iteration, starting with $(1,0,\cdots,0)$. This shows that s-cut solution converges faster than just computing ratio, so we can compute more accurate values with less exact numbers of well-formed formulae in each class. Since for quadratic generating functional equations, we need every $a_1,\cdots,a_{n-1}$ values to compute a_n and computing a_n itself is also time-consuming, so even though computing s-cut solution takes more time than just dividing, computing s-cut solution gives advantages in memory, also possible in time, for fixed accuracy.

The above deduction of the density of tautologies, $\Omega(\frac{1}{m})$, is similar to results in [2] and [5], which give that in the logic system with $\to$ and negative literals the density of tautologies is asymptotically same as the density of simple tautologies, i.e., $\frac{7}{8m}+O(\frac{1}{m^2})$. A simple tautology, which is defiend in [2], is a tautology of the form

$$\phi_1 \to [\phi_2 \to [\cdots \to [\phi_n \to p]\cdots]],$$

which can be simplified with the canonical form of an expression, defined in [2], as

$$\phi_1,\cdots,\phi_n \mapsto p$$

where each ϕ_i is a well-formed formula and p is a variable, with condition $\phi_i = p$ for some i, or for some distinct pair i and j, ϕ_i is a variable and $\phi_j = \bar{\phi}_i$. Here, $\bar{x}$ means negative literal of x. The former is called a simple tautology of the first kind, and the latter is called a simple tautology of the second kind. But there are some differences between our case and the given cases. Firstly, for the case of implication with negative literals, there are no antilogies. Secondly, we have to negate, rather than using negative literals, which increases the length of the formula. It introduces the factor $\sqrt{m}$ in asymptotic ratio, which changes the order.

With these facts in mind, we will try to compute the asymptotic density of tautologies as the number of variables goes to the infinity. In the following, our m-element variable set X is considered as the set of variables $\{x_0,x_1,\cdots,x_{m-1}\}$, so it will generate a chain sturcture as the number of variable changes.

Definition 5.1. Let $\overline{X} = \{x_0,x_1,\cdots\}$ be a countably infinite set of variables, and $\overline{\mathcal{W}}$ be the set of well-formed formulae of $\overline{X}$. For any $\sigma \in S_{\{0,1,2,\ldots\}} =: S_\infty$, the set of all permutations of $\{0,1,2\ldots\}$ with a finite support, we have a natural action on $\overline{\mathcal{W}}$ defined as

$$\begin{aligned} \sigma x_i &= x_{\sigma(i)}, \\ \sigma\neg\phi &= \neg\sigma\phi, \\ \sigma[\phi\to\psi] &= \sigma\phi\to\sigma\psi. \end{aligned}$$

A formula $\phi \in \overline{\mathcal{W}}$ is a **type formula** if for every occrurence of x_i, there must exist occurrences of $x_0,\cdots,x_{i-1}$ before it. The **type of a well-formed formula** ψ is the type formula ϕ such that there exists $\sigma \in S_\infty$ satisfying $\psi = \sigma\phi$. It is easy to prove that the type of a well-formed formula exists uniquely. For any well-formed formula ψ, $[\psi]$ is the set of well-formed formulae with the same type as ψ, and $[\psi]_m$ be the elements in $[\psi]$ consisting of $x_0, \cdots, x_{m-1}$. Note that $[\psi]$ is just the S_∞-orbit in $\mathcal{W}$, and for any ψ consisting of $x_0, \cdots, x_{m-1}$, the set $[\psi]_m$ is nothing but the S_m-orbit.

For any formula $\phi \in \overline{\mathcal{W}}$, $\|\phi\|$ is the number of distinct variables in ϕ. In other words, this is the minimum m such that the type of ϕ consists of $x_0, \cdots, x_{m-1}$. Lastly, $|\phi|$ is defined as

$$|\phi| = \|\phi\| - \frac{1}{2}\ell(\phi).$$

From the definition of the action, we obtain the following.

Proposition 5.2. *For any $\sigma \in S_\infty$, we have $F_{\sigma\phi} = \{\sigma T \mid T \in F_\phi\}$ where $\sigma T = \{x_{\sigma i} \mid x_i \in T\}$. In particular, ϕ is a tautology or an antilogy if and only if its type is a tautology or an antilogy, respectively.*

The following is a motivation for $|\cdot|$.

Proposition 5.3. *For any type formula ϕ and $m \geq \|\phi\|$, we have*

$$\left.\left(\sum_{\psi \in [\phi]_m} z^{\ell(\psi)}\right)\right|_{z=\frac{1}{2\sqrt{m}+1}} = \frac{m^{\underline{\|\phi\|}}}{(2\sqrt{m}+1)^{\ell(\phi)}} = \Theta(m^{|\phi|}).$$

where $m^{\underline{k}}$ is the falling factorial $m(m-1)\cdots(m-k+1)$.

From **Theorem 4.4**, we have relation between the generating function value at the singularity point and the limit ratio of coefficients, so it can be expected that tautologies with large $|\cdot|$ values dominate the density of tautologies.

Now, we will prove basic properties of $|\cdot|$. We begin with the following lemma.

Lemma 5.4. *(a) If ϕ has no $\neg$'s, then ϕ is true when the rightmost variable is true.*

(b) If ϕ has no repeated variables, then there is an assignment that makes ϕ true, and there is an assignment that makes ϕ false.

(c) Suppose that ϕ has no repeated variables, no $\neg$'s and that p is a variable in ϕ with $\phi \neq p$. Then, there are an assignment that p is true and an assignment that p is false, which make ϕ true.

(d) Suppose that ϕ has no repeated variables, no $\neg$'s and that p is not the rightmost variable of ϕ. Then, there is an assignment that makes ϕ false and p is true.

Proof. For any assignment set $T \subseteq \overline{X}$ and a well-formed formula ϕ, let T_ϕ be the set of variables in T which appear in ϕ. By definition of the valuation,

$$v_T(\phi) = v_{T_\phi}(\phi)$$

is natural.

(a) We will induct on the length of ϕ. If ϕ is a variable, then done. Otherwise, since $\phi = \psi \to \eta$ and η is true by induction hypothesis, so is ϕ.

(b) By induction, if ϕ is a variable, then done. Otherwise, it is trivial when $\phi = \neg\psi$ for some ψ, since $\neg$ reverses trueness and falseness. Now, if $\phi = \psi \to \eta$, then there are an assignment T on ψ that makes ψ true and an assignment S on η that makes η false, by induction hypothesis. Since ϕ has no repeated variables, we have $(T_\psi \cup S_\eta)_\psi = T_\psi$ and $(T_\psi \cup S_\eta)_\eta = S_\eta$ It gives

$$v_{T_\psi \cup S_\eta}(\phi) = 1 - v_{T_\psi}(\psi)(1 - v_{S_\eta}(\eta)) = 0.$$

Hence, $T_\psi \cup S_\eta$ is an assignment that makes ϕ false. Similarly, there is an assignment that makes ϕ true.

(c) Since ϕ has no $\neg$'s and ϕ is not p, $\phi = \psi \to \eta$ for some ψ, η. If ψ has p, then there exists an assignment T that makes η true by (b). Then, T_η is an assignment that makes ϕ true and does not contain p, so p is false, and $T_\eta \cup \{p\}$ is an assignment that makes ϕ true and contains p, so p is true. Similarly, when η has p, it is done by using an assignment that makes ψ false.

(d) Since ϕ has no $\neg$'s and p is not the rightmost variable, clearly we have $\phi \neq p$. If p is not a variable of ϕ, then the case (b) is applicable. So we consider the case that $\phi = \psi \to \eta$ for some ψ and η. First, suppose ψ has p. Then, by (b), there is an assignment T that makes η false. Now, if $\psi \neq p$, then by (c), there is an assignment S that makes ψ true and contains p. Then, $T_\eta \cup S_\psi$ contains p and makes ϕ false, since ϕ has no repeated variables. If $\psi = p$, then for any assignment T that makes η false, $T \cup \{p\}$ is an assignment that makes ϕ false.

Now, if η has p, then by the induction hypothesis, there is an assignment T that contains p and makes η false. By (b), there is an assignment S that makes ψ true, so there is an assignment $T_\eta \cup S_\psi$ that makes ϕ false. ☐

Then, we have the following.

Proposition 5.5.

(a) For any well-formed formula ϕ, $|\phi| \leq \frac{1}{2}$.

(b) For any tautology ϕ, $|\phi| \leq -\frac{1}{2}$. Moreover, $|\phi| = -\frac{1}{2}$ if and only if ϕ does not contains $\neg$ symbol, ϕ has unique variable appears twice, and every other variable in ϕ appears only once.

(c) For any antilogy ϕ, $|\phi| \leq -1$.

Proof. (a) Induction on the length. At first, $|x_i| = 1 - \frac{1}{2} = \frac{1}{2}$, and $|\neg\phi| = |\phi| - \frac{1}{2} \leq 0$. Finally, we have

$$|\phi \to \psi| \leq \|\phi\| + \|\psi\| - \frac{1}{2}(\ell(\phi) + \ell(\psi) + 1) = |\phi| + |\psi| - \frac{1}{2} \leq \frac{1}{2}.$$

(b) For a well-formed formula ϕ, the number of occurrences of variables is exactly the number of occurrences of $\to$'s plus 1. Let R be the number of variables in ϕ that do not appear first time in ϕ, Y be the number of occurrences of $\to$'s, and N be the number of occurrences of $\neg$'s. Then, we have

$$\begin{aligned}|\phi| &= \|\phi\| - \frac{1}{2}\ell(\phi)\\ &= Y + 1 - R - \frac{1}{2}(Y + (Y+1) + N)\\ &= \frac{1}{2} - R - \frac{1}{2}N.\end{aligned}$$

Hence, $|\phi| \geq 0$ implies $R = 0$, so ϕ has no repeated variables. Then, by **Lemma 5.4**(b), ϕ is not a tautology. The remaining part follows from the fact that $R \geq 1$ and $|\phi| = -\frac{1}{2}$ imply $N = 0$.

(c) By **Lemma 5.4**(a) and (b), any antilogy ϕ needs at least one $\neg$ and repeated variables. Hence, $R \geq 1$ and $N \geq 1$, so $|\phi| \leq -1$. ☐

It is remarkable that every simple tautology ϕ of the first kind with exactly one repetition and without $\neg$'s has $|\phi| = -\frac{1}{2}$, and every simple tautology ϕ of the second kind with exactly one repetition and one $\neg$ has $|\phi| = -1$. Actually, the converse holds, for the first kind.

Proposition 5.6. *Suppose ϕ is a tautology and $|\phi| = -\frac{1}{2}$. Then, there are well-formed formulae $\psi_1, \cdots, \psi_k, \eta$ without $\neg$'s, pairwise common variables, and repeated variables such that ϕ is $\psi_1, \cdots, \psi_k, p \mapsto \eta$ where p is the rightmost variable of η. Here, $k = 0$ is possible.*

Proof. First, ϕ has no $\neg$'s and has the unique repeated variable p which appears twice, by above proposition. Hence, $\phi = \psi \to \eta$ for some ψ, η.

Suppose ψ and η have no common variables. Then, by **Lemma 5.4**(a), there is an assignment T that makes ψ true. Hence, if there is an assignment S that makes η false, $T_\psi \cup S_\eta$ makes $\phi = \psi \to \eta$ false. Thus, there is no assignment that makes η false, so η is again a tautology. This implies that p must in η, since every tautology has at least one repeated variable. Hence, η is again a tautology with

$|\eta| = -\frac{1}{2}$. Then, by induction on length, η is $\psi_2, \cdots, \psi_k, p \mapsto \eta'$ and so, ϕ is $\psi, \psi_2, \cdots, \psi_k, p \mapsto \eta'$. Thus, done.

Now, assume that ψ and η have a common variable. Then, from the uniqueness of the repeated variable of ϕ, it must be p. If $\psi \neq p$, then by **Lemma 5.4**(b), there is an assignment T on η that makes η false. If $p \in T$, then by **Lemma 5.4**(c), there is an assignment S on ψ that makes ψ true and $p \in S$. Also, if $p \notin T$, then we have an assignment S on ψ that makes ψ true and $p \notin S$. Then, $T_\eta \cup S_\psi$ makes $\phi = \psi \to \eta$ false, which is a contradiction. So $\psi = p$.

Then, we have $\phi = p \to \eta$. If p is not the rightmost variable of η, then by **Lemma 5.4**(d), there is an assignment T on η that makes η false and $p \in T$. Hence, T makes ϕ false, which is a contradiction. Thus, p is the rightmost variable of η. $\square$

From these propositions and **Proposition 5.3** we can guess that the density of tautologies is of $\frac{1}{m}$ order: since the maximum $|\cdot|$ of well-formed formulae is $\frac{1}{2}$ and the maximum $|\cdot|$ of tautologies is $-\frac{1}{2}$, we may expect $\frac{m^{-\frac{1}{2}}}{m^{\frac{1}{2}}} = \frac{1}{m}$ order. Similarily, for antilogies, we may expect $\frac{1}{m\sqrt{m}}$ order.

Definition 5.7. In the following, $k \geq 1$.

(a) A well-formed formula ϕ is a **simple tautology of the first kind**, if there exist well-formed formulae $\psi_1, \cdots, \psi_k$ and a variable p such that ϕ is

$$\psi_1, \cdots, \psi_k \mapsto p$$

with $\psi_i = p$ for some i. Let $\mathcal{S}_1$ be the set of simple tautologies of the first kind.

(b) A well-formed formula $\psi_1, \cdots, \psi_k \mapsto p$ is a **strict simple tautology of the first kind**, if $\psi_1 = p$ and $\psi_2, \cdots, \psi_k \neq p$. Let $\mathcal{S}_c$ be the set of strict simple tautologies of the first kind.

(c) A well-formed formula ϕ is a **simple tautology of the second kind**, if there exist well-formed formulae $\psi_1, \cdots, \psi_{k+2}$ and a variable p so ϕ is

$$\psi_1, \cdots, \psi_{k+1} \mapsto \psi_{k+2}$$

where ψ_{k+2} is not $\eta_1 \to \eta_2$ form well-formed formula, and there exists distinct $i, j \leq k+1$ such that $\psi_i = p$ and $\psi_j = \neg p$. Here, if ψ_{k+2} is $\eta_1 \to \eta_2$, then $\psi_1, \cdots, \psi_{k+1} \mapsto \psi_{k+2}$ is same as $\psi_1, \cdots, \psi_{k+1}, \eta_1 \mapsto \eta_2$. So actually, the restriction for ψ_{k+2} is only for determining $\psi_1, \cdots, \psi_{k+2}$ uniquely. Let $\mathcal{S}_2$ be the set of simple tautologies of the second kind.

Proposition 5.8.

(a) The generating function S_1 of $\mathcal{S}_1$ is

$$\frac{mz^3}{(1+z^2-zW(z))(1-zW(z))}$$

and

$$\lim_{n\to\infty} \frac{[z^n]S_1(z)}{[z^n]W(z)} = \frac{m(4m+6\sqrt{m}+3)}{(\sqrt{m}+1)^2(2m+3\sqrt{m}+2)^2} = \frac{1}{m} - \frac{7}{2m\sqrt{m}} + \frac{7}{m^2} + O(\frac{1}{m^2\sqrt{m}}).$$

(b) The generating function S_c of $\mathcal{S}_c$ is

$$\frac{mz^3}{1+z^2-zW(z)}$$

and

$$\lim_{n\to\infty} \frac{[z^n]S_c(z)}{[z^n]W(z)} = \frac{m}{(2m+3\sqrt{m}+2)^2} = \frac{1}{4m} - \frac{3}{4m\sqrt{m}} + \frac{19}{16m^2} + O(\frac{1}{m^2\sqrt{m}}).$$

Proof. (a) The generating function of well-formed formulae of the form $\psi_1, \cdots, \psi_k \mapsto p$ is

$$mz(zW(z)) + mz(zW(z))^2 + mz(zW(z))^3 + \cdots = \frac{mz^2W(z)}{1-zW(z)}.$$

Here, mz term is for the variable p, and $zW(z)$ term is for the ψ_i with $\to$ symbol. Now we select those of the first kind by using the fact that a given well-formed formula is not a simple tautology of the first kind if and only if every ψ_i is not p. We induce that the generating function of such well-formed formulae of the first kind is

$$mz(z(W(z)-z)) + mz(z(W(z)-z))^2 + mz(z(W(z)-z))^3 + \cdots = \frac{mz^2(W(z)-z)}{1+z^2-zW(z)}.$$

Hence, we have

$$S_1(z) = \frac{mz^2W(z)}{1-zW(z)} - \frac{mz^2(W(z)-z)}{1+z^2-zW(z)} = \frac{mz^3}{(1+z^2-zW(z))(1-zW(z))}.$$

Then, by Szegő's lemma, when we take $s_0 = \frac{1}{2\sqrt{m+1}}$, we have

$$\lim_{n\to\infty}\frac{[z^n]S_1(z)}{[z^n]W(z)} = \frac{\lim_{z\to s_0^-} S_1'(z)\sqrt{1-z/s_0}}{\lim_{z\to s_0^-} W'(z)\sqrt{1-z/s_0}}.$$

Now, we have

$$\begin{aligned} S_1'(z) &= -\frac{mz^3((1+z^2-zW(z))(-zW'(z)) + (1-zW(z))(-zW'(z))}{(1+z^2-zW(z))^2(1-zW(z))^2} + R(z) \\ &= \frac{mz^4(2+z^2-2zW(z))W'(z)}{(1+z^2-zW(z))^2(1-zW(z))^2} + R(z) \end{aligned}$$

where $\lim_{z\to s_0^-} R(z)\sqrt{1-z/s_0} = 0$. Thus, we have

$$\lim_{n\to\infty}\frac{[z^n]S_1(z)}{[z^n]W(z)} = \frac{ms_0^4(2+s_0^2-2s_0W(s_0))}{(1+s_0^2-s_0W(s_0))^2(1-s_0W(s_0))^2} = \frac{m(4m+6\sqrt{m}+3)}{(\sqrt{m}+1)^2(2m+3\sqrt{m}+2)^2}.$$

Also, from $S_1(z)(1+z^2-zW(z))(1-zW(z)) = mz^3$ and $z(W(z))^2 = W(z) - mz - zW(z)$, we have an equation

$$S_1(z) = mz^3 + (m-1)z^2S_1(z) + z(1+z+z^2)W(z)S_1(z),$$

and if we use **Theorem 4.4**, we get

$$\lim_{n\to\infty}\frac{[z^n]S_1(z)}{[z^n]W(z)} = \frac{s_0(1+s_0+s_0^2)S_1(s_0)}{1-(m-1)s_0^2 - s_0(1+s_0+s_0)^2W(s_0)} = \frac{m(4m+6\sqrt{m}+3)}{(\sqrt{m}+1)^2(2m+3\sqrt{m}+2)^2}.$$

which corresponds to the result from Szegő's lemma.

(b) This can be done similarly as (a). ☐

This lower bound of the density of tautologies from (a) of the above proposition is quite improved from the first result

$$\frac{\sqrt{m}}{4(\sqrt{m}+1)(2\sqrt{m}+1)^2}.$$

To improve more, we will consider the following.

Definition 5.9. Let $\mathcal{B}$ be a set of tautologies.

(a) The strong $\mathcal{B}$-category is a partition of well-formed formulae consisting of strong $\mathcal{B}$-tautologies ($\mathcal{T}_*$), $\mathcal{B}$-unknowns ($\mathcal{U}_*$), and $\mathcal{B}$-antilogies ($\mathcal{A}_*$) determined by $\mathcal{B}$ such that
- $\phi \in \mathcal{T}_*$ if and only if $\phi \in \mathcal{B}$; ϕ is $\neg\psi$ form where $\psi \in \mathcal{A}_*$; or ϕ is $\psi \to \eta$ form where $\eta \in \mathcal{T}_*$.
- $\phi \in \mathcal{A}_*$ if and only if ϕ is $\neg\psi$ form where $\psi \in \mathcal{T}_*$; or ϕ is $\psi \to \eta$ form where $\psi \in \mathcal{T}_*$ and $\eta \in \mathcal{A}_*$.
- $\phi \in \mathcal{U}_*$ if and only if $\phi \notin \mathcal{T}_* \cup \mathcal{A}_*$.

The following table shows this recursive classification.

	$\mathcal{T}_*$	$\mathcal{U}_*$	$\mathcal{A}_*$
$\neg$	$\mathcal{A}_*$	$\mathcal{U}_*$	$\mathcal{T}_*$
$\mathcal{T}_* \to$	$\mathcal{T}_*$	$\mathcal{U}_*$	$\mathcal{A}_*$
$\mathcal{U}_* \to$	$\mathcal{T}_*$	$\mathcal{U}_*$	$\mathcal{U}_*$
$\mathcal{A}_* \to$	$\mathcal{T}_*$	$\mathcal{U}_*$	$\mathcal{U}_*$

(b) The weak $\mathcal{B}$-category is a partition of well-formed formulae consisting of strong $\mathcal{B}$-tautologies ($\mathcal{T}^*$), $\mathcal{B}$-unknowns ($\mathcal{U}^*$), and $\mathcal{B}$-antilogies ($\mathcal{A}^*$) determined by $\mathcal{B}$ such that
- $\phi \in \mathcal{T}^*$ if and only if $\phi \in \mathcal{B}$; ϕ is $\neg\psi$ form where $\psi \in \mathcal{A}^*$; or ϕ is $\psi \to \eta$ form where $\eta \in \mathcal{T}^*$ or $\psi \in \mathcal{A}^*$.
- $\phi \in \mathcal{A}^*$ if and only if ϕ is $\neg\psi$ form where $\psi \in \mathcal{T}^*$; or ϕ is $\psi \to \eta$ form where $\psi \in \mathcal{T}^*$ and $\eta \in \mathcal{A}^*$.
- $\phi \in \mathcal{U}^*$ if and only if $\phi \notin \mathcal{T}^* \cup \mathcal{A}^*$.

The following table shows this recursive classification.

	$\mathcal{T}^*$	$\mathcal{U}^*$	$\mathcal{A}^*$
$\neg$	$\mathcal{A}^*$	$\mathcal{U}^*$	$\mathcal{T}^*$
$\mathcal{T}^* \to$	$\mathcal{T}^*$	$\mathcal{U}^*$	$\mathcal{A}^*$
$\mathcal{U}^* \to$	$\mathcal{T}^*$	$\mathcal{U}^*$	$\mathcal{U}^*$
$\mathcal{A}^* \to$	$\mathcal{T}^*$	$\mathcal{T}^*$	$\mathcal{T}^*$

(c) A well-formed formula ϕ is weak (resp. strong) $\mathcal{B}$-basic if ϕ is a weak (resp. strong) $\mathcal{B}$-tautology and ϕ is not a weak (resp. strong) $(\mathcal{B} \setminus \{\phi\})$-tautology.
(d) The set $\mathcal{B}$ is weak (resp. strong) basic if every $\phi \in \mathcal{B}$ is weak (resp. strong) $\mathcal{B}$-basic.
(e) A set of $\mathcal{B}$-tautologies $\mathcal{B}'$ is a weak (resp. strong) basis of $\mathcal{B}$ if $\mathcal{B}'$ is weak (resp. strong) basic and every $\phi \in \mathcal{B}$ is a weak (resp. strong) $\mathcal{B}'$-tautology.

This proposition is true for both weak and strong category.

Proposition 5.10. *Suppose $\mathcal{B}$ is a set of tautologies.*

(a) For a basis $\widetilde{\mathcal{B}}$ of $\mathcal{B}$, $\widetilde{\mathcal{B}}$-category is same as $\mathcal{B}$-category.
(b) Every well-formed formula ϕ of a basis $\widetilde{\mathcal{B}}$ of $\mathcal{B}$ is $\mathcal{B}$-basic.
(c) Every $\mathcal{B}$ has a basis and a well-formed formula ϕ is in a basis $\widetilde{\mathcal{B}}$ if and only if ϕ is $\mathcal{B}$-basic. In particular, there is a unique basis $\widetilde{\mathcal{B}}$ of $\mathcal{B}$, which is a subset of $\mathcal{B}$.

Proof. (a) With induction on the length of well-formed formulae, it comes from the recursive structure of categories.

(b) If ϕ is not $\mathcal{B}$-basic, then ϕ is $(\mathcal{B} \setminus \{\phi\})$-tautology. Now, for every well-formed formula ψ with $\ell(\psi) < \ell(\phi)$, $\mathcal{B}$-category, $(\mathcal{B} \setminus \{\phi\})$-category, $\widetilde{\mathcal{B}}$-cateogory and $(\widetilde{\mathcal{B}} \setminus \{\phi\})$-category are all same. Hence, ϕ is a $(\widetilde{\mathcal{B}} \setminus \{\phi\})$-tautology, contradicting that $\widetilde{\mathcal{B}}$ is basic.

(c) It is enough to show that every $\mathcal{B}$-basic ϕ is in $\widetilde{\mathcal{B}}$ and the set of $\mathcal{B}$-basic well-formed formulae is a basis. If ϕ is $\mathcal{B}$-basic, then ϕ is not a $(\mathcal{B} \setminus \{\phi\})$-tautology and so not a $(\widetilde{\mathcal{B}} \setminus \{\phi\})$-tautology. Since $\widetilde{\mathcal{B}}$ is a basis, ϕ is a $\widetilde{\mathcal{B}}$-tautology, and so $\phi \in \widetilde{\mathcal{B}}$.

Let $\widehat{\mathcal{B}}$ be the set of $\mathcal{B}$-basic well-formed formulae. Then, by the definition of basic well-formed formula, $\widehat{\mathcal{B}} \subseteq \mathcal{B}$. Since $\phi \in \widehat{\mathcal{B}}$ is not a $(\mathcal{B} \setminus \{\phi\})$-tautology, it is not a $(\widehat{\mathcal{B}} \setminus \{\phi\})$-tautology, and so $\widehat{\mathcal{B}}$ is basic. Let ψ be a shortest $\mathcal{B}$-tautology that is not a $\widehat{\mathcal{B}}$-tautology. Then, for every shorter well-formed formula ψ' than ψ, $(\mathcal{B} \setminus \{\psi\})$-category is same as $\widehat{\mathcal{B}}$-category. Now, ψ is not $\mathcal{B}$-basic, so is a $(\mathcal{B} \setminus \{\psi\})$-tautology, and hence ψ is a $\widehat{\mathcal{B}}$-tautology, which is a contradition. So every $\mathcal{B}$-tautology is a $\widehat{\mathcal{B}}$-tautology, and so, $\widehat{\mathcal{B}}$ is a basis of $\mathcal{B}$. $\square$

Proposition 5.11. *For a set $\mathcal{B}$ of tautologies, every strong $\mathcal{B}$-tautology is a weak $\mathcal{B}$-tautology, and every weak $\mathcal{B}$-tautology is a tautology. Hence, every weak $\mathcal{B}$-basic well-formed formula is strong $\mathcal{B}$-basic.*

The following system of equations naturally follows from the structure of $\mathcal{B}$-categories.

Proposition 5.12. *Let $\mathcal{B}$ be a set of tautologies.*

(a) Let B_*, T_*, U_*, A_* be the generating functions of the strong basis of $\mathcal{B}$, strong $\mathcal{B}$-tautologies, strong $\mathcal{B}$-unknowns, and strong $\mathcal{B}$-antilogies, respectively. Then the following system of equations is satisfied.

$$\begin{aligned}T_*(z) &= B_*(z) + zA_*(z) + zT_*(z)W(z),\\ U_*(z) &= mz - B_*(z) + zU_*(z) + z[U_*(z)W(z) + A_*(z)W(z) - A_*(z)T_*(z)],\\ A_*(z) &= zT_*(z) + zA_*(z)T_*(z).\end{aligned}$$

(b) Let B^*, T^*, U^*, A^* be the generating functions of the weak basis of $\mathcal{B}$, weak $\mathcal{B}$-tautologies, weak $\mathcal{B}$-unknowns, and weak $\mathcal{B}$-antilogies, respectively. Then the following system of equations is satisfied.

$$\begin{aligned}T^*(z) &= B^*(z) + zA^*(z) + z[T^*(z)W(z) + A^*(z)W(z) - A^*(z)T^*(z)],\\ U^*(z) &= mz - B^*(z) + zU^*(z) + zU^*(z)W(z),\\ A^*(z) &= zT^*(z) + zA^*(z)T^*(z).\end{aligned}$$

Note that these systems of equations have fixed number of equations whenever m, the number of variables of the propositional logic system, changes, so it makes easy to analyze an asymptotic behavior as $m \to \infty$. Also, for fixed $\mathcal{B}$, we have

$$\lim_{n\to\infty} \frac{[z^n]I_\emptyset(z)}{[z^n]W(z)} \geq \lim_{n\to\infty} \frac{[z^n]T^*(z)}{[z^n]W(z)} \geq \lim_{n\to\infty} \frac{[z^n]T_*(z)}{[z^n]W(z)}$$

so computing $\lim_{n\to\infty} \frac{[z^n]T^*(z)}{[z^n]W(z)}$ or $\lim_{n\to\infty} \frac{[z^n]T_*(z)}{[z^n]W(z)}$ will give a lower bound for the asymptotic density of tautologies.

Proposition 5.13. *(a) $\mathcal{S}_c$ is the strong basis of $\mathcal{S}_1$.*

(b) The weak basis of $\mathcal{S}_1$ is the set of well-formed formulae of the form $\psi_1 \to [\cdots \to [\psi_k \to p]\cdots]$ where $\psi_1 = p$, and $\psi_2, \cdots, \psi_k$ are not p nor $\mathcal{S}_1$-antilogy. Its generating function satisfies

$$B^*(z) = \frac{mz^3}{1 + z^2 - zW(z) + zA^*(z)}.$$

which naturally satisfies

$$B^*(z) = mz^3 - z^2B^*(z) + z[B^*(z)W(z) - B^*(z)A^*(z)].$$

Now, we may solve the equation for $\mathcal{S}_1$-strong case algebraically, by using the identity $A_*(z) = \frac{zT_*(z)}{1-zT_*(z)}$, to obtain

$$\begin{aligned}T_*(z) &= \frac{1 - z^2 + zS_c(z) - zW(z) - \sqrt{(1 - z^2 + zS_c(z) - zW(z))^2 - 4zS_c(z)(1 - zW(z))}}{2z(1 - zW(z))}\\ A_*(z) &= \frac{zT_*(z)}{1 - zT_*(z)},\\ U_*(z) &= \frac{mz - S_c(z) + zA_*(z)^2}{1 - z - z(W(z) + A_*(z))} = \frac{mz - S_c(z) + zA_*(z)(W(z) - T_*(z))}{1 - z - zW(z)}.\end{aligned}$$

For $s_0 = \frac{1}{2\sqrt{m}+1}$, we have

$$T_*(s_0) = \frac{\sqrt{m}(2m+4\sqrt{m}+3)}{2m+3\sqrt{m}+2} - \frac{(2\sqrt{m}+1)^2}{\sqrt{m}+1}\sqrt{\frac{m(4m^3+24m^2\sqrt{m}+60m^2+84m\sqrt{m}+70m+33\sqrt{m}+7)}{(2\sqrt{m}+1)^4(2m+3\sqrt{m}+2)^2}}.$$

and it is also possible to compute $A_*(s_0)$ and $U_*(s_0)$. Note that if we substitute $1/y$ for $\sqrt{m}$, then $yT_*(s_0)$, $yU_*(s_0)$ and $yA_*(s_0)$ are analytic about y near 0. So we have series expansions

$$\begin{aligned} T_*(s_0) &= \frac{1}{2\sqrt{m}} - \frac{5}{4m} + \frac{17}{8m\sqrt{m}} + O(\frac{1}{m^2}), \\ A_*(s_0) &= \frac{1}{4m} - \frac{3}{4m\sqrt{m}} + O(\frac{1}{m^2}), \\ U_*(s_0) &= \sqrt{m} - \frac{1}{2\sqrt{m}} + \frac{1}{m} - \frac{11}{8m\sqrt{m}} + O(\frac{1}{m^2}). \end{aligned}$$

Then, by **Theorem 4.4**, if we let $\gamma = \lim_{n\to\infty} \frac{[z^n]S_c(z)}{[z^n]W(z)}$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{[z^n]T_*(z)}{[z^n]W(z)} &= \frac{(T_*(s_0) - 1/s_0)(T_*(s_0) + \gamma/s_0)}{T_*(s_0)(1/s_0 - \sqrt{m}) + A_*(s_0) + \sqrt{m}/s_0 - 1/s_0^2 + 1} \\ &= \frac{(T_*(s_0) - 1/s_0)(T_*(s_0) + \gamma/s_0)}{T_*(s_0)(\sqrt{m}+1) + A_*(s_0) - \sqrt{m}(2\sqrt{m}+3)} \\ &= \frac{1}{m} - \frac{7}{2m\sqrt{m}} + \frac{31}{4m^2} + O(\frac{1}{m^2\sqrt{m}}), \end{aligned}$$

which gives a slight improvement from $\lim_{n\to\infty} \frac{[z^n]S_1(z)}{[z^n]W(z)}$.

To use this method of undetermined coefficients of power series for weak class case, we need to prove that $yT^*(s_0)$, $yU^*(s_0)$, $yA^*(s_0)$ and $yB^*(s_0)$ are also analytic about $y = \frac{1}{\sqrt{m}}$ near 0. We will prove that our equations have analytic solutions near $y = 0$, and there are unique solutions for B, T, U in a bounded region for fixed small y, so our analytic solutions match with real solutions that we want.

We will consider the general case, i.e., the case with arbitrary $B^*(z)$. First, the equation $U^*(z) = mz - B^*(z) + zU^*(z) + zU^*(z)W(z)$ is actually equivalent to

$$mz - B^*(z) = U^*(z)(1 - z - zW(z)) = \frac{mzU^*(z)}{W(z)} = mz\left(1 - \frac{T^*(z) + A^*(z)}{W(z)}\right).$$

Moreover, it is easy to check that a system of equations

$$\begin{aligned} T^*(z) &= B^*(z) + zA^*(z) + z[T^*(z)W(z) + A^*(z)W(z) - A^*(z)T^*(z)], \\ A^*(z) &= zT^*(z) + zA^*(z)T^*(z), \end{aligned}$$

is actually equivalent to

$$\begin{aligned} W(z)B^*(z) &= mz(T^*(z) + A^*(z)), \\ A^*(z) &= zT^*(z) + zA^*(z)T^*(z). \end{aligned}$$

Then, with

$$
\begin{aligned}
s_0 &= \frac{1}{2\sqrt{m}+1} = \frac{y}{2+y} = \frac{y}{2} - \frac{y^2}{4} + \frac{y^3}{8} - \frac{y^4}{16} + \cdots , \\
m &= \frac{1}{y^2}, \\
W(s_0) &= \sqrt{m} = \frac{1}{y},
\end{aligned}
$$

we have the system of equations

$$
\begin{aligned}
T^*(s_0) &= B^*(s_0) + \frac{y}{y+2}A^*(s_0) + \frac{y}{y+2}\left[\frac{T^*(s_0)}{y} + \frac{A^*(s_0)}{y} - A^*(s_0)T^*(s_0)\right], \\
A^*(s_0) &= \frac{y}{y+2}T^*(s_0) + \frac{y}{y+2}A^*(s_0)T^*(s_0),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
(y+2)B^*(s_0) &= T^*(s_0) + A^*(s_0), \\
A^*(s_0) &= \frac{y}{y+2}T^*(s_0) + \frac{y}{y+2}A^*(s_0)T^*(s_0).
\end{aligned}
$$

Note that since B^*, T^*, A^* are generating functions, which are bounded by W, the values of $T^*(s_0)$, $B^*(s_0)$, $A^*(s_0)$ satisfy $yB^*(s_0)$, $yT^*(s_0)$, $yA^*(s_0) \leq 1$ for each $y = m^{-1/2}$ where m is a positive integer. Then, we need to solve

$$
\begin{aligned}
(y+2)[yB^*(s_0)] &= [yT^*(s_0)] + [yA^*(s_0)], \\
[yA^*(s_0)] &= \frac{y + [yA^*(s_0)]}{y+2}[yT^*(s_0)].
\end{aligned}
\tag{1}
$$

in $[0,1]^3$. Now, assume that we have an equation $B^*(z) = \Theta(B^*(z), T^*(z), A^*(z); m, z, W(z))$, and define $\theta(b,t,a;w) = w\Theta(b/w, t/w, a/w; \frac{1}{w^2}, \frac{w}{w+2}, \frac{1}{w})$. We define

$$
\begin{aligned}
\lambda(b,t,a;w) &= \left(\theta(b,t,a;w), (w+2)b - a, \frac{w+a}{w+2}t\right), \\
\tilde{\lambda}(b,t,a;w) &= \left(\theta(b,t,a;w), \frac{b}{2} + \frac{(w+1)a + (w+3)t - at}{2(w+2)}, \frac{w+a}{w+2}t\right).
\end{aligned}
$$

As we said, the set of fixed points of λ and $\tilde{\lambda}$ are same. Now, solving our original system of equations (1) for $yB^*(s_0)$, $yT^*(s_0)$, $yA^*(s_0)$ is equivalent to finding a fixed point of λ when w is fixed as y. Assume that we have a unique solution b_0, t_0, a_0 in $\{(b,t,a) \in \mathbb{C}^3 \mid |b|, |t|, |a| \leq 1\}$ satisfying $(b_0, t_0, a_0) = \lambda(b_0, t_0, a_0; 0)$, in other words, a fixed point at $w = 0$. Since we have $a_0 = \frac{a_0 t_0}{2}$ and $t_0 = 2b_0 - a_0$, this gives $t_0 = 2b_0$ and $a_0 = 0$. Then, for $\epsilon > 0$, we say $D \subseteq \mathbb{C}^3$ is a **proper ϵ-region** if it satisfies following:

- D is closed and bounded, i.e. compact.
- D contains an open neighborhood of (b_0, t_0, a_0),
- θ is analytic about w, b, t, a when $|w| < \epsilon$ and $(b,t,a) \in D$,
- $\tilde{\lambda}(D; w) \subseteq D$ when $|w| < \epsilon$,
- if $y < \epsilon$, then every solution $(yB^*(s_0), yT^*(s_0), yA^*(s_0))$ in $[0,1]^3$ of (1) is in D.

For the last condition, it is sufficient to show that if $(b,t,a) = \lambda(b,t,a;y)$ and $|b|, |t|, |a| \leq 1$, then $(b,t,a) \in D$. Hence, by the analytic implicit function theorem, we will get the existence of analytic solution when the determinant of the Jacobian

$$
\det J_1 = \det \frac{\partial(\mathrm{id} - \lambda)}{\partial(b,t,a)}
$$

is nonzero at (b_0, t_0, a_0) where $w = 0$, and by the Banach contraction principle, we will get the uniqueness of solution for fixed $w = y = m^{-1/2}$ when the Jacobian

$$J_2 = \frac{\partial \tilde{\lambda}}{\partial(b,t,a)}$$

has norm value less than 1 whenever $|w| < \epsilon$ and $(b,t,a) \in D$ for some fixed norm. Here, we are using $\tilde{\lambda}$ since the Jacobian of λ contains $w+2$ entry, which makes hard to get small norm. By simple computation, we have

$$\det J_1(b,t,a;w) = \frac{2+2w+a-t}{2+w} - \frac{2+2w+a-t}{2+w}\frac{\partial\theta}{\partial b} - (2+w-t)\frac{\partial\theta}{\partial t} - (a+w)\frac{\partial\theta}{\partial a}.$$

and

$$J_2(b,t,a;w) = \begin{bmatrix} \frac{\partial\theta}{\partial b} & \frac{\partial\theta}{\partial t} & \frac{\partial\theta}{\partial a} \\ \frac{1}{2} & \frac{w+3-a}{2(w+2)} & \frac{w+1-t}{2(w+2)} \\ 0 & \frac{w+a}{w+2} & \frac{t}{w+2} \end{bmatrix}.$$

Moreover, if Θ is a function of A^* only, then we may reduce the number of variables by considering

$$\widehat{\lambda}(a;w) = \frac{w+a}{w+2}((w+2)\theta(a;w) - a) = (w+a)\theta(a;w) - \frac{a(w+a)}{w+2},$$

which gives

$$\widehat{J_1}(a;w) = \frac{2w+2a+2}{w+2} - a\theta'(a;y) - \theta(a;y) = 1 - \widehat{J_2}(a;w),$$
$$\widehat{J_2}(a;w) = a\theta'(a;w) + \theta(a;w) - \frac{w+2a}{w+2}.$$

We have free to choose J_1 or $\widehat{J_1}$ to check the existence of analytic solution, and J_2 or $\widehat{J_2}$ to check the uniqueness of solution. Of course, we need to variate the definition of proper region and choose properly to use $\widehat{J_2}$. Lastly, for the proper ϵ-region with $\epsilon < 1$, suppose (b,t,a) is a solution of $(b,t,a) = \lambda(b,t,a;w)$ satisfying $|b|, |t|, |a| \le 1$ where $|w| < \epsilon$. A proper ϵ-region must contain every such (b,t,a), and we want to find ϵ-region as narrow as possible to get uniqueness easily. Note that we have $a = \frac{w+a}{w+2}t$, which gives

$$|a| = \left|\frac{wt}{2+w-t}\right| \le \frac{|w||t|}{|2+w|-|t|} \le \frac{\epsilon|t|}{2-\epsilon-|t|} \le \frac{\epsilon|t|}{1-\epsilon},$$

so it is reasonable to try to take proper ϵ-region as a subset of $\{(b,t,a) \mid |b|, |t|, |a| \le 1, |a| \le \frac{\epsilon}{1-\epsilon}|t|\}$.

Now, consider $\mathcal{S}_1$-weak case. We have two choices of $\Theta(B^*(z), T^*(z), A^*(z); z, m, W(z))$. One is

$$\Theta(B^*(z), T^*(z), A^*(z); z, m, W(z)) = mz^3 - z^2B^*(z) + z[B^*(z)W(z) - B^*(z)A^*(z)],$$

and the other is

$$\Theta(B^*(z), T^*(z), A^*(z); z, m, W(z)) = \frac{mz^3}{1+z^2-zW(z)+zA^*(z)}.$$

Note that the latter is a function of A^* only. If we take the latter as our Θ, then we have

$$\theta(b,t,a;w) = \frac{w^2}{w+2} \cdot \frac{1}{2w^2+3w+2+(w+2)a}.$$

Since $\theta(b,t,0;0) = 0$ always, so $b_0 = t_0 = a_0 = 0$ is a unique solution. Now, if $\epsilon \le \frac{1}{8}$, $|w| < \epsilon$ and $|a| \le \frac{1}{7}$, then we have

$$|\theta(b,t,a;w)| \le \frac{\epsilon^2}{2-|w|} \cdot \frac{1}{2-3|w|-2|w|^2-|2+w||a|} \le \frac{\frac{1}{64}}{2-\frac{1}{8}} \cdot \frac{1}{2-\frac{3}{8}-\frac{2}{64}-(2+\frac{1}{8})\frac{1}{7}} = \frac{28}{4335}.$$

Hence, if we define

$$D = \{(b,t,a) \mid |b| \le \frac{28}{4335}, |t| \le \frac{33}{35}, |a| \le \frac{1}{7}\},$$

then D is closed, bounded region containing an open neighborhood of $(b_0, t_0, a_0) = (0,0,0)$. Moreover, if $(b,t,a) \in D$, then

$$|\theta(b,t,a;w)| \le \frac{28}{4335},$$
$$\left|\frac{b}{2} + \frac{(w+1)a + (w+3)t - at}{2(w+2)}\right| \le \frac{14}{4335} + \frac{(1+\frac{1}{8})\frac{1}{7} + (3+\frac{1}{8}) + \frac{1}{7}}{2(2-\frac{1}{8})} = \frac{14}{4335} + \frac{32}{35} \le \frac{33}{35} < 1$$
$$\left|\frac{w+a}{w+2}t\right| \le \frac{\frac{1}{8}+\frac{1}{7}}{2-\frac{1}{8}} = \frac{1}{7},$$

and so $\tilde{\lambda}(D;w) \subseteq D$. Now, if $(b,t,a) = \lambda(b,t,a;w)$ and $|b|, |t|, |a| \le 1$, then we have

$$|a| \le \frac{\epsilon}{1-\epsilon}|t| \le \frac{1}{7},$$
$$|b| = |\theta(b,t,a;w)| \le \frac{28}{4335},$$
$$|t| = \left|\frac{b}{2} + \frac{(w+1)a + (w+3)t - at}{2(w+2)}\right| \le \frac{33}{35},$$

and so $(b,t,a) \in D$. Thus, D is a proper ϵ-region. Note that we may choose smaller D. For example, from $|t| \le \frac{33}{35}$, we may get $|a| \le \frac{33}{7\cdot 35}$ and from this bound of a, we can get smaller bounds for t and $\theta(b,t,a;w)$. Hence we may repeat this bootstrap process to make D smaller and smaller. Lastly, we will consider Jacobians. We will choose J_1, J_2 rather than $\widehat{J}_1$ and $\widehat{J}_2$. By direct computation, we have

$$\det J_1(b,t,a;w) = \frac{2+2w+a-t}{2+w} + (w+a)\frac{w^2}{w+2} \cdot \frac{w+2}{(2w^2+3w+2+(w+2)a)^2},$$
$$J_2(b,t,a;w) = \begin{bmatrix} 0 & 0 & -\frac{w^2}{(2w^2+3w+2+(w+2)a)^2} \\ \frac{1}{2} & \frac{w+3-a}{2(w+2)} & \frac{w+1-t}{2(w+2)} \\ 0 & \frac{w+a}{w+2} & \frac{t}{w+2} \end{bmatrix}.$$

First, $\det J_1(0,0,0;0) = 1 \ne 0$, so we have local analytic solution about w from the analytic implicit function theorem. Then, for the (1,3)-entry of J_2, we have

$$|(J_2)_{13}| \le \frac{\epsilon^2}{(2-3|w|-2|w|^2-|2+\epsilon||a|)^2} \le \frac{\frac{1}{8^2}}{\left(2-\frac{3}{8}-\frac{2}{64}-\left(2+\frac{1}{8}\right)\frac{1}{7}\right)^2} = \left(\frac{28}{289}\right)^2,$$

when $|w| < \epsilon \le \frac{1}{8}$ and $(b,t,a) \in D$. Now, sum of the absolute values of the second column is bounded by

$$\frac{\epsilon+3+|a|}{2(2-\epsilon)} + \frac{\epsilon+|a|}{2-\epsilon} = \frac{3(\epsilon+|a|+1)}{2(2-\epsilon)}$$

and of the third column is bounded by

$$\left(\frac{28}{289}\right)^2 + \frac{\epsilon+1+|t|}{2(2-\epsilon)} + \frac{|t|}{2-\epsilon} = \left(\frac{28}{289}\right)^2 + \frac{\epsilon+3|t|+1}{2(2-\epsilon)}.$$

Here, both of them become less than 1 as $\epsilon \to 0$, so there is $\epsilon_0 \le \frac{1}{8}$ such that $w < \epsilon_0$ implies $\|J_2\|_\infty < 1$. Hence, by the Banach contraction principle, we have uniqueness of solutions for each such w, so the values of local analytic solution must match to true values of $yB^*(s_0)$, $yT^*(s_0)$ and

$yA^*(s_0)$. Then, $yW(s_0) = 1$ and $yU^*(s_0) = yW(s_0) - yT^*(s_0) - yA^*(s_0)$, so it is also true for $yU^*(s_0)$.

From this result, we may assume

$$\begin{aligned}
B^*(s_0) &= \frac{b_{-1}}{y} + b_0 + b_1 y + b_2 y^2 + \cdots , \\
T^*(s_0) &= \frac{t_{-1}}{y} + t_0 + t_1 y + t_2 y^2 + \cdots , \\
U^*(s_0) &= \frac{u_{-1}}{y} + u_0 + u_1 y + u_2 y^2 + \cdots , \\
A^*(s_0) &= \frac{a_{-1}}{y} + a_0 + a_1 y + a_2 y^2 + \cdots ,
\end{aligned}$$

where $b_{-1}, t_{-1}, u_{-1}, a_{-1} \geq 0$, since we are considering generating functions. Then, we have a series of quadratic equations

$$\begin{aligned}
B^*(z) &= mz^3 - z^2 B^*(z) + z[B^*(z)W(z) - B^*(z)A^*(z)], \\
T^*(z) &= B^*(z) + zA^*(z) + z[T^*(z)W(z) + A^*(z)W(z) - A^*(z)T^*(z)], \\
U^*(z) &= mz - B^*(z) + zU^*(z) + zU^*(z)W(z), \\
A^*(z) &= zT^*(z) + zA^*(z)T^*(z),
\end{aligned}$$

and if we write this equation in terms of y, we will get

$$\begin{aligned}
B^*(s_0) &= \frac{y}{(2+y)^3} - \frac{y^2}{(2+y)^2} B^*(s_0) + \frac{1}{2+y} B^*(s_0) - \frac{y}{2+y} B^*(s_0)A^*(s_0), \\
T^*(s_0) &= B^*(s_0) + \frac{y}{2+y} A^*(x) + \frac{1}{2+y} T^*(s_0) + \frac{1}{2+y} A^*(s_0) - \frac{y}{2+y} A^*(s_0)T^*(s_0), \\
U^*(s_0) &= \frac{1}{y(2+y)} - B^*(s_0) + \frac{y}{2+y} U^*(s_0) + \frac{1}{2+y} U^*(s_0), \\
A^*(s_0) &= \frac{y}{2+y} T^*(s_0) + \frac{y}{2+y} A^*(s_0)T^*(s_0).
\end{aligned}$$

Then, the method of undetermined coefficients gives

$$\begin{aligned}
B^*(s_0) &= \frac{1}{4\sqrt{m}} - \frac{1}{2m} + \frac{9}{16m\sqrt{m}} + O(\frac{1}{m^2}), \\
T^*(s_0) &= \frac{1}{2\sqrt{m}} - \frac{1}{m} + \frac{5}{4m\sqrt{m}} + O(\frac{1}{m^2}), \\
U^*(s_0) &= \sqrt{m} - \frac{1}{2\sqrt{m}} + \frac{3}{4m} - \frac{5}{8m\sqrt{m}} + O(\frac{1}{m^2}), \\
A^*(s_0) &= \frac{1}{4m} - \frac{5}{8m\sqrt{m}} + O(\frac{1}{m^2}).
\end{aligned}$$

Now, by **Theorem 4.4** again, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{[z^n]B^*(z)}{[z^n]W(z)} &= \frac{1}{4m} - \frac{3}{4m\sqrt{m}} + \frac{9}{8m^2} + O(\frac{1}{m^2\sqrt{m}}), \\
\lim_{n\to\infty} \frac{[z^n]T^*(z)}{[z^n]W(z)} &= \frac{1}{m} - \frac{5}{2m\sqrt{m}} + \frac{29}{8m^2} + O(\frac{1}{m^2\sqrt{m}}).
\end{aligned}$$

This is a lower bound of the asymptotic density of weak tautologies from simple tautologies of the first kind, so is of tautologies. Finally, we are going to consider both the first and second kind of simple tautologies. From **Proposition 5.3** and **Proposition 5.5**, since all simple tautologies of the second kind have $\neg$ symbol in it, we expect that this does not change the $\frac{1}{\sqrt{m}}$ order term of $T^*(s_0)$, but it will give an improvement on $\frac{1}{m}$ order term. Hence, it will not change the $\frac{1}{m}$ order term of ratio, but it will give an improvement on $\frac{1}{m\sqrt{m}}$ order term of it.

We have to start from finding the basis of $\mathcal{S}_1 \cup \mathcal{S}_2$. Let us consider weak sense partition, and use simple notations B, T, U, A for generating functions of basis, tautologies, unknowns, and antilogies, respectively. A well-formed formula $\psi_1, \cdots, \psi_{k-1} \mapsto \psi_k$ such that ψ_k is a variable or $\neg\eta$ for a well-formed formula η is $(\mathcal{S}_1 \cup \mathcal{S}_2)$-basic if and only if one of the following is true.

- First, $k \geq 2$, and there is a variable p such that ψ_1, ψ_k are p, $\psi_2, \cdots, \psi_{k-1}$ are not p, and $\psi_2, \cdots, \psi_{k-1}$ are not $(\mathcal{S}_1 \cup \mathcal{S}_2)$-antilogies.
- There is a variable p and $i < k$ such that ψ_1 is p, ψ_i is $\neg p$, ψ_k is not p, ψ_k is not $\neg\eta$ for an $(\mathcal{S}_1 \cup \mathcal{S}_2)$-antilogy η, and for any $1 < j < k$, ψ_j is not an $(\mathcal{S}_1 \cup \mathcal{S}_2)$-antilogy nor p.
- There is a variable p and $i < k$ such that ψ_1 is $\neg p$, ψ_i is p, ψ_k is not p, ψ_k is not $\neg\eta$ for an $(\mathcal{S}_1 \cup \mathcal{S}_2)$-antilogy η, and for any $1 < j < k$, ψ_j is not an $(\mathcal{S}_1 \cup \mathcal{S}_2)$-antilogy nor $\neg p$.

Also, these three conditions are pairwise disjoint. The generating function for the first case is

$$\frac{mz^3}{1 - z[W(z) - z - A(z)]},$$

for the second case is

$$mz^2\left(\frac{(m-1)z + z[W(z) - A(z)]}{1 - z[W(z) - z - A(z)]} - \frac{(m-1)z + z[W(z) - A(z)]}{1 - z[W(z) - z - z^2 - A(z)]}\right),$$

and for the third case is

$$mz^3\left(\frac{(m-1)z + z[W(z) - A(z)]}{1 - z[W(z) - z^2 - A(z)]} - \frac{(m-1)z + z[W(z) - A(z)]}{1 - z[W(z) - z - z^2 - A(z)]}\right).$$

Deducing these formulae is similar to the proof of **Proposition 5.8**.(a). To apply the method to computing the density of weak tautologies from $\mathcal{S}_1$ case, we have to consider the existence of proper region D. If Θ is a function of only $A^*(z)$ and $\theta(b, t, 0; 0) = 0$, then to prove the existence of proper region D, it is enough to choose $\epsilon > 0$ such that there exists $\delta > 0$ satisfies

- if $|w| < \epsilon$ and $|a| \leq \frac{\epsilon}{1-\epsilon}$, then $|\theta(b, t, a; w)| \leq \delta$, and
- $\frac{\delta}{2} + \frac{3}{2(2-\epsilon)(1-\epsilon)} \leq 1$.

If these conditions are satisfied, then $D = \{(b, t, a) \mid |b| \leq \delta, |t| \leq 1,\ |a| \leq \frac{\epsilon}{1-\epsilon}\}$ will be a proper ϵ-region. Then, we may compute Jacobians and check $\det J_1(0, 0, 0; 0)$ is nonzero and a norm of J_2 is less than 1, where we may reduce D by bootstrap argument and ϵ freely, if is needed. By direct computation, we can show $\theta(b, t, 0; 0) = 0$ is really true for this case either, and hence, other process to prove analyticity is almost automatic.

After we get the analyticity, we have to consider a system of quadratic equations including the generating function of the basis. We have following system of equation.

$$
\begin{aligned}
B_1(z) =& mz^3 - z^2 B_1(z) + z(W(z)B_1(z) - A(z)B_1(z)), \\
B_2(z) =& m(m-1)z^3 + mz^3[W(z) - A(z)] - z^2 B_2(z) \\
& + z[W(z)B_2(z) - A(z)B_2(z)], \\
B_3(z) =& m(m-1)z^3 + mz^3[W(z) - A(z)] - (z^2 + z^3)B_3(z) \\
& + z[W(z)B_3(z) - A(z)B_3(z)], \\
B_4(z) =& m(m-1)z^4 + mz^4[W(z) - A(z)] - z^3 B_4(z) \\
& + z[W(z)B_4(z) - A(z)B_4(z)], \\
B_5(z) =& m(m-1)z^4 + mz^4[W(z) - A(z)] - (z^2 + z^3)B_5(z) \\
& + z[W(z)B_5(z) - A(z)B_5(z)], \\
B(z) =& B_1(z) + B_2(z) - B_3(z) + B_4(z) - B_5(z), \\
T(z) =& B(z) + zA(z) + z(T(z)W(z) + A(z)W(z) - A(z)T(z)), \\
U(z) =& mz - B(z) + zU(z) + zU(z)W(z), \\
A(z) =& zT(z) + zA(z)T(z).
\end{aligned}
$$

From this system of equations, we have a series solution

$$
\begin{aligned}
B_1(s_0) =& \frac{1}{4\sqrt{m}} - \frac{1}{2m} + \frac{9}{16m\sqrt{m}} + O(\frac{1}{m^2}), \\
B_2(s_0) =& \frac{\sqrt{m}}{4} - \frac{1}{4} - \frac{3}{16\sqrt{m}} + \frac{5}{8m} - \frac{47}{64m\sqrt{m}} + O(\frac{1}{m^2}), \\
B_3(s_0) =& \frac{\sqrt{m}}{4} - \frac{1}{4} - \frac{3}{16\sqrt{m}} + \frac{9}{16m} - \frac{35}{64m\sqrt{m}} + O(\frac{1}{m^2}), \\
B_4(s_0) =& \frac{1}{8} - \frac{3}{16\sqrt{m}} + \frac{1}{16m} + \frac{3}{32m\sqrt{m}} + O(\frac{1}{m^2}), \\
B_5(s_0) =& \frac{1}{8} - \frac{3}{16\sqrt{m}} + \frac{9}{32m\sqrt{m}} + O(\frac{1}{m^2}), \\
B(s_0) =& \frac{1}{4\sqrt{m}} - \frac{3}{8m} + \frac{3}{16m\sqrt{m}} + O(\frac{1}{m^2}), \\
T(s_0) =& \frac{1}{2\sqrt{m}} - \frac{3}{4m} + \frac{1}{2m\sqrt{m}} + O(\frac{1}{m^2}), \\
U(s_0) =& \sqrt{m} - \frac{1}{2\sqrt{m}} + \frac{1}{2m} + O(\frac{1}{m^2}), \\
A(s_0) =& \frac{1}{4m} - \frac{1}{2m\sqrt{m}} + O(\frac{1}{m^2}),
\end{aligned}
$$

and by **Theorem 4.4**, we get

$$\lim_{n\to\infty}\frac{[z^n]B(z)}{[z^n]W(z)} = \frac{1}{4m} - \frac{1}{2m\sqrt{m}} + \frac{5}{16m^2} + O(\frac{1}{m^2\sqrt{m}}),$$
$$\lim_{n\to\infty}\frac{[z^n]T(z)}{[z^n]W(z)} = \frac{1}{m} - \frac{7}{4m\sqrt{m}} + \frac{5}{4m^2} + O(\frac{1}{m^2\sqrt{m}}),$$
$$\lim_{n\to\infty}\frac{[z^n]U(z)}{[z^n]W(z)} = 1 - \frac{1}{m} + \frac{5}{4m\sqrt{m}} - \frac{1}{8m^2} + O(\frac{1}{m^2\sqrt{m}}),$$
$$\lim_{n\to\infty}\frac{[z^n]A(z)}{[z^n]W(z)} = \frac{1}{2m\sqrt{m}} - \frac{9}{8m^2} + O(\frac{1}{m^2\sqrt{m}}),$$

and this result shows only improvement in $\frac{1}{m\sqrt{m}}$ order term, as we expected.

For the upper bound of the density, we have an upper bound

$$1 - \lim_{n\to\infty}\frac{[z^n]A(z)}{[z^n]W(z)}$$

so we finally conclude

$$\frac{1}{m} - \frac{7}{4m\sqrt{m}} + \frac{5}{4m^2} + O(\frac{1}{m^2\sqrt{m}}) \le \lim_{n\to\infty}\frac{[z^n]I_\emptyset(z)}{[z^n]W(z)} \le 1 - \frac{1}{2m\sqrt{m}} + \frac{9}{8m^2} + O(\frac{1}{m^2\sqrt{m}}).$$

We may improve the upper bound slightly by dividing the class unknowns into unknowns and not tautologies nor antilogies. In such partitioning, $\mathcal{B}$-tautologies and $\mathcal{B}$-antilogies are not changed, and by same argument, we may compute, with proper analyticity assumption, the density of not tautologies nor antilogies has lower bound

$$\frac{1}{4m} - \frac{5}{16m\sqrt{m}} + \frac{5}{32m^2} + O(\frac{1}{m^2\sqrt{m}}),$$

and this gives an upper bound

$$1 - \frac{1}{4m} - \frac{3}{16m\sqrt{m}} + \frac{31}{32m^2} + O(\frac{1}{m^2\sqrt{m}}).$$

But this upper bound is still too far from the lower bound. Moreover, we have reasonable conjecture with **Proposition 5.3** that we cannot improve the first term $1/m$ for the limit density and indeed, this result is asymptotically correct. In other words, we may expect that m times the density of tautologies will converge to 1.

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E-mail address, T. Eom: taeheom@kaist.ac.kr

(T. Eom) DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, DAEJEON, 34141, REPUBLIC OF KOREA