Discrete Mathematics

3. Relations

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Binary Relations

Definition:

Let A and B be any two sets. A *binary relation* R from A to B is a subset of $A \times B$.

The notation aRb means $(a,b) \in R$.

– Example:

$$a \leq b \text{ means } (a,b) \in \leq$$

where $\leq$ denotes the relation of *partial ordering*.

Complementary Relations

Definition:

Let $R \subseteq A \times B$ be any binary relation. Then, $\overline{R}$, the *complement* of R, is the binary relation defined by

$$\overline{R} = \{(a,b) \mid (a,b) \notin R\} = (A \times B) - R$$

Note that the complement of $\overline{R}$ is R.

Inverse Relations

Definition:

An inverse (converse) relation of a binary relation $R \subseteq A \times B$, denoted by R^{-1} (R^{c}), is defined to be $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

Theorem:

1. $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$
2. $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$

Relations on a Set

Definition:

1. A (binary) relation from a set A to itself is called a relation *on* the set A.
2. The *identity relation* $\boldsymbol{I}_A$ on a set A is the set,

   $\boldsymbol{I}_A = \{(a,a) \mid a \in A\}$.

Properties of Relations

Definition:

1. A relation R on A is *reflexive* if for every a in A, $(a, a) \in R$.
2. A relation R on A is *irreflexive* if for every a in A, $(a, a) \notin R$.
3. A relation R on A is *symmetric* if for every a and b in A, if $(a,b) \in R$, then $(b,a) \in R$.
4. A relation R on A is *antisymmetric* if for every a and b in A, if $(a,b) \in R$ and $(b,a) \in R$, then $(a=b)$.
5. A relation R on A is *asymmetric* if for every a and b in A, if $(a,b) \in R$, then $(b,a) \notin R$.
6. A relation R on A is *transitive* if for every a, b, and c in A, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

Note "*irreflexive*" ≠ "*not reflexive*"!

Composite Relations

Definition:

Let $R \subseteq A \times B$, and $S \subseteq B \times C$. Then the *composite* of R and S, denoted by $R \circ S$, is defined to be

$R \circ S = \{(a,c) \mid (a,b) \in R \wedge (b,c) \in S$ for some b in $B\}$

Definition:

The n^{th} power R^n of a relation R on a set A can be defined recursively by $R^{n+1} = R^n \circ R$ for all $n \geq 0$ where $R^0 = \boldsymbol{I}_A$.

Theorem:

Let R be a relation. Then

1. $R^n \circ R^m = R^{n+m}$
2. $(R^n)^m = R^{nm}$

Theorem:

Let R_1, R_2, and R_3 be relations on a set A. Then

1. $R_1 \circ (R_2 \cap R_3) \subseteq (R_1 \circ R_2) \cap (R_1 \circ R_3)$
2. $R_1 \circ (R_2 \cup R_3) = (R_1 \circ R_2) \cup (R_1 \circ R_3)$

Proof of $R_1 \circ (R_2 \cap R_3) \subseteq (R_1 \circ R_2) \cap (R_1 \circ R_3)$:

For arbitrary element <x,y>, let $\langle x,y \rangle \in R_1 \circ (R_2 \cap R_3)$ ····· (1)

Then by definition of $\circ$, $\langle x,z \rangle \in R_1$ and $\langle z,y \rangle \in R_2 \cap R_3$ for some z.

By definition of $\cap$, $\langle x,z \rangle \in R_1$ and ($\langle z,y \rangle \in R_2$ and $\langle z,y \rangle \in R_3$)

By *idempotent equivalence*, ($\langle x,z \rangle \in R_1$ and $\langle x,z \rangle \in R_1$) and ($\langle z,y \rangle \in R_2$ and $\langle z,y \rangle \in R_3$)

By *associative and commutative equivalences*,

($\langle x,z \rangle \in R_1$ and $\langle z,y \rangle \in R_2$) and ($\langle x,z \rangle \in R_1$ and $\langle z,y \rangle \in R_3$)

By definition of $\circ$, ($\langle x,y \rangle \in R_1 \circ R_2$) and ($\langle x,y \rangle \in R_1 \circ R_3$)

By definition of $\cap$, $\langle x,y \rangle \in R_1 \circ R_2 \cap R_1 \circ R_3$ ····· (2)

From (1) and (2), by definition of $\subseteq$, $R_1 \circ (R_2 \cap R_3) \subseteq (R_1 \circ R_2) \cap (R_1 \circ R_3)$

Proof of $R_1 \circ (R_2 \cup R_3) = (R_1 \circ R_2) \cup (R_1 \circ R_3)$:

For arbitrary element $<x,y>$, let $<x,y> \in R_1 \circ (R_2 \cup R_3)$ ····· (1)

Then by definition of $\circ$, $<x,z> \in R_1$ and $<z,y> \in R_2 \cup R_3$ for some z.

By definition of $\cup$, $<x,z> \in R_1$ and ($<z,y> \in R_2$ or $<z,y> \in R_3$)

By *distributive equivalence*, ($<x,z> \in R_1$ and $<z,y> \in R_2$) or

($<x,z> \in R_1$ and $<z,y> \in R_3$)

By definition of $\circ$, ($<x,y> \in R_1 \circ R_2$) or ($<x,y> \in R_1 \circ R_3$)

By definition of $\cup$, $<x,y> \in (R_1 \circ R_2) \cup (R_1 \circ R_3)$····· (2)

From (1) and (2), by definition of $\subseteq$,

$R_1 \circ (R_2 \cup R_3) \subseteq (R_1 \circ R_2) \cup (R_1 \circ R_3)$ ····· (3)

For arbitrary element $<x,y>$, let $<x,y> \in (R_1 \circ R_2) \cup (R_1 \circ R_3)$ ····· (4)

By definition of $\cup$, $<x,y> \in R_1 \circ R_2$ or $<x,y> \in R_1 \circ R_3$

By definition of $\circ$, ($<x,z> \in R_1$ and $<z,y> \in R_2$ for some z) or

($<x,w> \in R_1$ and $<w,y> \in R_3$ for some w)

By definition of $\cup$, ($<x,z> \in R_1$ and $<z,y> \in R_2 \cup R_3$ for the set R_3) or

($<x,w> \in R_1$ and $<w,y> \in R_2 \cup R_3$ for the set R_2)

By definition of $\circ$, $(<x,y> \in R_1 \circ (R_2 \cup R_3))$ or $(<x,y> \in R_1 \circ (R_2 \cup R_3))$

By *idempotent equivalence*, $<x,y> \in R_1 \circ (R_2 \cup R_3) \cdots\cdots$ (5)

From (4) and (5), by definition of $\subseteq$,

$$(R_1 \circ R_2) \cup (R_1 \circ R_3) \subseteq R_1 \circ (R_2 \cup R_3) \cdots\cdots \quad (6)$$

From (3) and (6), by definition of =, $R_1 \circ (R_2 \cup R_3) = (R_1 \circ R_2) \cup (R_1 \circ R_3)$

Theorem:

Let R be a relation on a set A, i.e. $R \subseteq A \times A$, and I_A be a identity relation on a set A, ($I_A=\{<x,x> | x \in A\}$).

Then the following holds:

1. R is *reflexive* iff $I_A \subseteq R$
2. R is *irreflexive* iff $I_A \cap R = \varnothing$
3. R is *symmetric* iff $R = R^{-1}$
4. R is *asymmetric* iff $R \cap R^{-1} = \varnothing$
5. R is *antisymmetric* iff $R \cap R^{-1} \subseteq I_A$
6. R is *transitive* iff $R \circ R \subseteq R$

Walk, path, cycle, loop, sling

Definition:

Given a directed graph G=<N, V> where N is a set of nodes and V is a set of edges,

1. A *walk* is a sequence $x_0, x_1, \ldots, x_n$ of the vertices of a directed graph such that $x_i x_{i+1}$, $0 \le i \le n-1$, is an edge.
2. The *length of a walk* is the number of edges in the walk.
3. If a walk holds $x_i \neq x_j$ ($i \neq j$) $i, j = 0, \ldots, n$, (i.e., no edge is repeated), the walk is called a *path*.
4. If a walk holds $x_i \neq x_j$ ($i \neq j$) $i, j = 0, \ldots, n$, except $x_0 = x_n$, the walk is called a *cycle*.
5. A *loop* is a cycle of length one.
6. A *sling* is a cycle of length two.

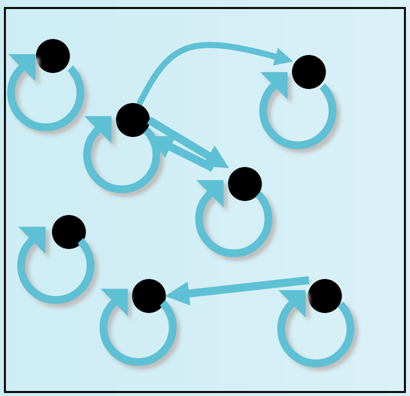
Theorem:

Given a directed graph G=<N, R> where N is a set of nodes and R is a set of edges,

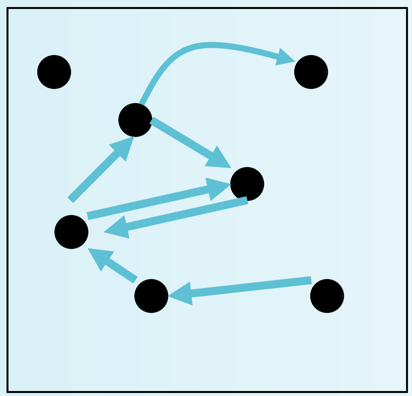
1. R is *reflexive* iff G has a loop at every node.
2. R is *irreflexive* iff G has no loop at any node.
3. R is *symmetric* iff if G has a walk of length one between two distinct nodes, then it has a sling between them.
4. R is *asymmetric* iff if G has a walk of length one between two distinct nodes, then it has no sling between them and no loop at any node.
5. R is *antisymmetric* iff if G has a walk of length one between two distinct nodes, then it has no sling between them.
6. R is *transitive* iff if G has a walk of length two between two nodes, then it has a walk of length one between them.

Digraph Reflexive, Symmetric

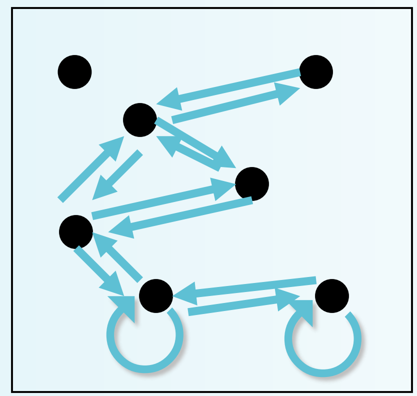
It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.



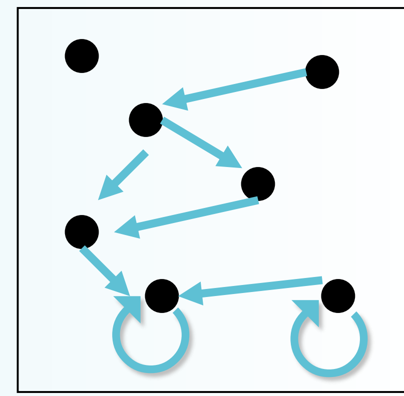
Reflexive: Every node has a loop



Irreflexive: No node has a loop



Symmetric: Every link is bidirectional



Antisymmetric: No link is bidirectional

Closures of Relations

Definition:

For any property X, the "X closure" of a set R is defined as the "smallest" superset of R that has the given property.

Theorem:

1. The *reflexive closure* of a relation R on A is obtained by adding (a,a) to R for each $a \in A$, *i.e.*, $r(R) = R \cup I_A$.
2. The *symmetric closure* of R on A is obtained by adding (b,a) to R for each (a,b) in R, *i.e.*, $s(R) = R \cup R^{-1}$.
3. The *transitive closure* or *connectivity relation* of R on A is obtained by repeatedly adding (a,c) to R for each $(a,b),(b,c)$ in R, *i.e.*, $t(R) = \bigcup_{n \in \mathbf{Z}^+} R^n$

Proof of $t(R) = \bigcup_{i=1}^{\infty} R^i$

(1) Show that $\bigcup_{i=1}^{\infty} R^i \subseteq t(R)$.

By induction it is shown that $R^n \subseteq t(R)$ for every $n \geq 1$.

1. (Basis: n=1)

   By definition of transitive closure $t(R)$, $R \subseteq t(R)$.

2. (Induction: Assuming $R^n \subseteq t(R)$ holds for $n=k$, show that it holds for $n=k+1$)

   Let $<a,b> \in R^{k+1}$. Since $R^{k+1}=R^k \circ R$, there exists some $c \in A$ such that $<a,c> \in R^k$ and $<c,b> \in R$. By induction hypothesis that $R^k \subseteq t(R)$ and basis that $R \subseteq t(R)$, $<a,c> \in t(R)$ and $<c,b> \in t(R)$. Since $t(R)$ is transitive, $<a,b> \in t(R)$. Thus $R^{k+1} \subseteq t(R)$.

Since $R^n \subseteq t(R)$ for every $n \geq 1$, $\bigcup_{i=1}^{\infty} R^i \subseteq t(R)$.

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(2) Show that $t(R) \subseteq \bigcup_{i=1}^{\infty} R^i$.

Claim that $\bigcup_{i=1}^{\infty} R^i$ is transitive:

Let $<a,b>$ and $<b,c>$ be arbitrary elements of $\bigcup_{i=1}^{\infty} R^i$.

Then for some integers $n \geq 1$ and $m \geq 1$, $<a,b> \in R^n$ and $<b,c> \in R^m$. By definition of $\circ$, $<a,c> \in R^n \circ R^m$. Since $R^n \circ R^m = R^{n+m}$ by the previous theorem, $<a,c> \in \bigcup_{i=1}^{\infty} R^i$. Thus $\bigcup_{i=1}^{\infty} R^i$ is transitive.

Since $\bigcup_{t=1}^{\infty} R^i$ is a transitive relation which contains R and $t(R)$ is the smallest transitive relation containing R, it follows that $t(R) \subseteq \bigcup_{i=1}^{\infty} R^i$.

From (1) & (2), by definition of =, $t(R) = \bigcup_{i=1}^{\infty} R^i$.

Equivalence Relations

Definition:

A relation R on a set A is called an *equivalence relation* if it is *reflexive*, *symmetric*, and *transitive*.

Equivalence Classes

Definition:

Let R be any equivalence relation on a set A. For each a in A, the *equivalence class* of a with respect to R, denoted by $[a]_R$, is

$$[a]_R = \{ b \mid \langle a,b \rangle \in R \}$$

Examples:

1. "Strings a and b are the same length."
   - $[a]$ = the set of all strings of the same length as a.
2. "Integers a and b have the same absolute value."
   - $[a]$ = the set $\{a, -a\}$
3. "Real numbers a and b have the same fractional part (*i.e.*, $a - b \in \mathbf{Z}$)."
   - $[a]$ = the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$
4. "Integers a and b have the same residue modulo m." (for a given $m>1$)
   - $[a]$ = the set $\{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\}$

Theorem:

Let R be an equivalence relation on a set A.

1. For every x in A, $x \in [x]_R$.
2. If $<x, y> \in R$, then $[x]_R=[y]_R$.

Theorem:

Let R be an equivalence relation on a set A.

If $<x, y> \notin R$, then $[x]_R \cap [y]_R = \varnothing$.

Partition and Covering of a Set

Definition:

Let S be a given set and $\pi = \{A_1, A_2, \ldots, A_m\}$ where each A_i , i=1, … m, is a non-empty subset of S and $\bigcup_{i=1}^{m} A_i = S$.

1. Then the set π is called a *covering* of S, and the sets $A_1, A_2, \ldots, A_m$ are said to *cover* S.
2. If the elements of π, which are subsets of S, are mutually disjoint, then π is called a *partition* of S, and the sets $A_1, A_2, \ldots, A_m$ are called the *blocks* of the partition.

Quotient Set

Definition:

Let R be an equivalence relation on a set A. Then $A/R = \{[x]_R | x \in A\}$ is called a *quotient set of A modulo R.*

Theorem:

Let R be an equivalence relation on a set A. Then the quotient set of A modulo R, $A/R = \{[x]_R | x \in A\}$, is a partition of A.

Relation induced by the Partition

Definition:

Let A be a set. Let $\pi = \{A_1, A_2, \ldots, A_n\}$ be a partition of A. R_π is a *relation induced by the partition* π and defined as follows.

$R_\pi = \{<x, y> | (x \in A_i) \wedge (y \in A_i) \text{ for some } i\}$

Theorem:

Let A be a set. Let $\pi = \{A_1, A_2, \ldots, A_n\}$ be a partition of A and R_π be the relation induced by the partition π. Then, R_π is an equivalence relation on A.

Refinement

Definition:

Let π_1 and π_2 be two partitions of a set A. π_2 is a *refinement* of π_1, (π_2 refines π_1), if for every block B_i in π_2, there exists some block A_j in π_1 such that $B_i \subseteq A_j$.

Theorem:

Let π and π' be two partitions of a nonempty set A and let R_π and $R_{\pi'}$ be the equivalence relations induced by π and π' respectively. Then π' refines π if and only if $R_{\pi'} \subseteq R_\pi$.

Proof

// π' refines π if $R_{\pi'} \subseteq R_{\pi}$ //

Let $B_i \in \pi'$. ····· (1)

Then for every x and y in B_i , $<x,y> \in R_{\pi'}$ by definition of $R_{\pi'}$. From the given condition that $R_{\pi'} \subseteq R_{\pi}$, $<x,y> \in R_{\pi}$ by definition of $\subseteq$. By definition of R_{π} , then there exists some A_j in π such that x and y in A_j , and thus $B_i \subseteq A_j$ by definition of $\subseteq$. ····· (2)

From (1) & (2), by definition of refinement, π' refines π.

// π' refines π only if $R_{\pi'} \subseteq R_{\pi}$ //

Let $<x,y> \in R_{\pi'}$. ····· (3)

Then by definition of $R_{\pi'}$, there exists some B_i in π' such that x and y in B_i . From the given condition that π' refines π , there exists some A_j in π such that $B_i \subseteq A_j$ and thus x and y in A_j by definition of $\subseteq$. By definition of R_{π} , then $<x,y> \in R_{\pi}$. ····· (4)

From (3) & (4), by definition of $\subseteq$, $R_{\pi'} \subseteq R_{\pi}$.

Partial Orderings

Definition:

1. A relation *R* on a set *S* is called a *partial ordering* or *partial order* iff it is *reflexive*, *antisymmetric*, and *transitive*.
2. A set *S* together with a *partial ordering R* is called a *partially ordered set*, or *poset*, denoted by (*S*, *R*).

Example:

Consider the "greater than or equal to" relation $\geq$ (defined by $\{(a, b) \mid a \geq b\}$). Is $\geq$ a partial ordering on the set of integers?

Proof:

1. $\geq$ is reflexive, because $a \geq a$ for every integer a.
2. $\geq$ is antisymmetric, because if $a \geq b \wedge b \geq a$, then $a=b$.
3. $\geq$ is transitive, because if $a \geq b$ and $b \geq c$, then $a \geq c$.

Consequently, $(\mathbf{Z}, \geq)$ is a partially ordered set.

Example:

Is the "inclusion relation" $\subseteq$ on the power set of a set S a partial ordering ?

Proof:

1. $\subseteq$ is reflexive, because $A \subseteq A$ for every set A.
2. $\subseteq$ is antisymmetric, because if $A \subseteq B \wedge B \subseteq A$, then $A = B$.
3. $\subseteq$ is transitive, because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Consequently, $(\wp(S), \subseteq)$ is a partially ordered set

Partially Ordered Sets

In a poset, the notation $a \leq b$ denotes that $(a, b) \in \leq$.

Note that the symbol $\leq$ is used to denote the relation in any poset, not just the "less than or equal" relation. The notation $a < b$ denotes that $a \leq b$, but $a \neq b$. If $a < b$, we say "a is less than b" or "b is greater than a".

For two elements a and b of a poset $(S, \leq)$, it is possible that neither $a \leq b$ nor $b \leq a$. For instance, in $(\wp(\mathbf{Z}), \subseteq)$, $\{1, 2\}$ is not related to $\{1, 3\}$, and vice versa, since neither is contained within the other.

Definition:

1. The elements a and b of a poset $(S, \leq)$ are called comparable if either $a \leq b$ or $b \leq a$.

2. The elements a and b of a poset $(S, \leq)$ are called incomparable if neither $a \leq b$ nor $b \leq a$.

Definition :

If (S, ≤) is a poset and every two elements of S are comparable, (S, ≤) is called a *totally ordered* or *linearly ordered set*, and ≤ is called a *total order* or *linear order*. A totally ordered set is also called a *chain*.

Example 1**:** Is (**Z**, ≤) a totally ordered poset?

Yes, because a ≤ b or b ≤ a for all integers a and b.

Example 2**:** Is ($\mathbf{Z}^+$, |) a totally ordered poset?

No, because it contains incomparable elements such as 5 and 7.

Hasse Diagram

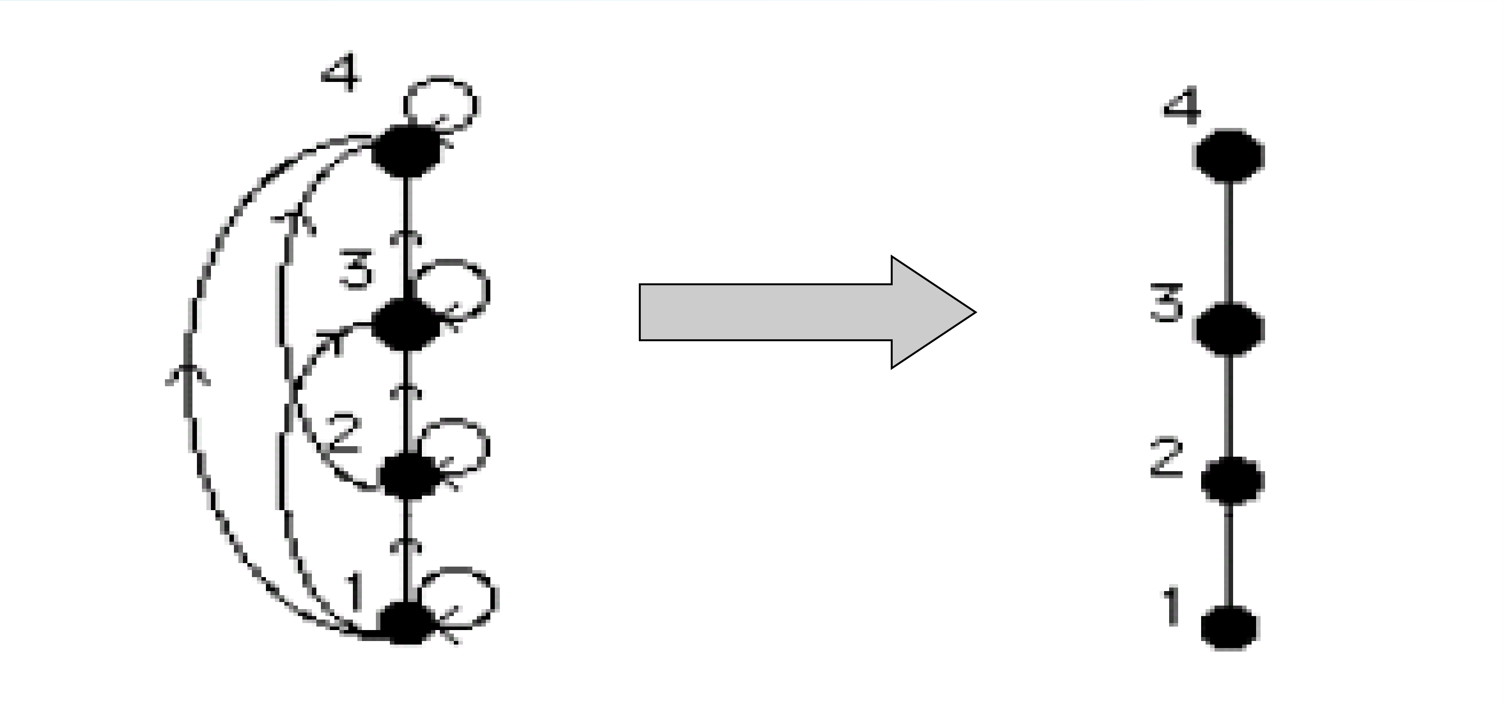
Definition **:**

Let G be a digraph representing a poset, $(A, \leq)$.
The Hasse diagram of $(A, \leq)$ is constructed from G by

1. All loops are omitted.
2. An arc is not present in a Hasse diagram if it is implied by the transitivity of the relation.
3. All arcs point upward and arrow heads are not used.

Example of Hasse Diagram

{<*a*, *b*> | *a*≤*b* } on {1, 2, 3, 4}



Greatest Elements and Least Elements

Definition:

Let $(A, \leq)$ be a poset and B be a subset of A.

1. An element $a \in B$ is *a greatest element* of B iff for every element $a' \in B$, $a' \leq a$.
2. An element $a \in B$ is *a least element* of B iff for every element $a' \in B$, $a \leq a'$.

Theorem:

Let $(A, \leq)$ be a poset and $B \subseteq A$. if a and b are greatest (least) elements of B, then $a=b$

Least Upper Bound (lub)

Definition:

Let $(A, \leq)$ be a poset and B be a subset of A.

1. An element $a \in A$ is an *upper bound* for B iff for every element $a' \in B$, $a' \leq a$.
2. An element $a \in A$ is a *least upper bound* (*lub*) for B iff a is an upper bound for B and for every upper bound a' for B, $a \leq a'$.

Greatest Lower Bound (glb)

Definition:

Let $(A, \leq)$ be a poset and B be a subset of A.

1. An element $a \in A$ is a *lower bound* for B iff for every element $a' \in B$, $a \leq a'$.
2. An element $a \in A$ is a *greatest lower bound* (*glb*) for B iff a is a lower bound for B and for every lower bound a' for B, $a' \leq a$.

lub and glb

***Theorem*:**

Let $(A, \leq)$ be a poset and $B \subseteq A$.

1. If b is a greatest element of B, then b is a lub of B.
2. If b is an upper bound of B and $b \in B$, then b is a greatest element of B.

***Theorem*:**

Let $(A, \leq)$ be a poset and $B \subseteq A$.

If a least upper bound (or a greatest lower bound) for B exists, then it is unique.

Lattices

Definition:

A poset, $(A, \leq)$, is a *lattice* if every pair of elements in A has a *lub* and a *glb*.

Theorem:

Let $<L, \leq>$ be a lattice. If $x*y$ $(x+y)$ denotes the *glb* (*lub*) for $\{x, y\}$, then the following holds: for any a, b, and c in L,

(i) $a*a=a$	(i') $a+a=a$	(*idempotent*)
(ii) $a*b=b*a$	(ii') $a+b=b+a$	(*commutative*)
(iii) $(a*b)*c= a*(b*c)$	(iii') $(a+b)+c= a+(b+c)$	(*associative*)
(iv) $a*(a+b)=a$	(iv') $a+(a*b)=a$	(*absorption*)

Proof of $a*(a+b)=a$:

//Show $a*(a+b) \le a$ //

$a*(a+b)$ is a *glb* of $\{a, a+b\}$

By definition of *glb*, $a*(a+b)$ is a *lb* of $\{a, a+b\}$

By definition of *lb*, $a*(a+b) \le a$ and $a*(a+b) \le a+b$ ····· (1)

//Show $a \le a*(a+b)$ //

$(a+b)$ is a *lub* of $\{a, b\}$

By definition of *lub*, $(a+b)$ is a *ub* of $\{a, b\}$

By definition of *ub*, $a \le (a+b)$ and $b \le (a+b)$ ····· (2)

Since $\le$ is *reflexive*, $a \le a$ ····· (3)

From (2) & (3), a is a *lb* of $\{a, a+b\}$ ····· (4)

However, $a*(a+b)$ is a *glb* of $\{a, a+b\}$ ····· (5)

By definition of *glb*, from (4) & (5), $a \le a*(a+b)$ ····· (6)

Since $\le$ is *antisymmetric*, from (1) & (6), $a*(a+b)=a$

Exercise

1. For each of the following relation R on set A, state whether or not R is *reflexive*, *irreflexive*, *symmetric*, *asymmetric*, *antisymmetric*, *and transitive.*

   (a) $A = \{1, 2, \ldots., 9\}$

   $R = \{<x,y> \mid x+y=10\}$

   (b) A = a set of real numbers

   $R = \{<x,y> \mid |x| \le |y| \}$

   (c) A = a set of natural numbers

   $\mathrm{R} = \{<x,y> \mid x\text{-}y=2k,\ \ k \in A \}$

2. Suppose that R and S are reflexive relations on a set A.

   Prove or disprove each of theses statements

   (a) $R \cup S$ is reflexive

   (b) $R \cap S$ is reflexive

3. Show that the relation R on a set A is symmetric if and only if $R=R^{-1}$, where R^{-1} is the inverse relation.

4. Let R_1 and R_2 be arbitrary relations on a set A.
Prove or disprove the following assertions.
(a) If R_1 and R_2 are reflexive, then $R_1 \circ R_2$ is reflexive.
(b) If R_1 and R_2 are transitive, then $R_1 \circ R_2$ is transitive.
(c) If R_1 and R_2 are symmetric, then $R_1 \circ R_2$ is symmetric.

5. Show that the relation R on a set A is symmetric if and only if $R=R^{-1}$, where R^{-1} is the inverse relation.

6. Let A be a set of ordered pairs of positive integers and R be a relation on A such that $\langle(x,y),(u,v)\rangle \in R$ if and only if $x+v = y+u$. Determine whether or not R is an equivalence relation.

7. Let R_1 and R_2 be two equivalence relations on a nonempty set A. Prove or disprove the following :

   (a) $R_1 \cup R_2$ an equivalence relation.

   (b) $R_1 \cap R_2$ an equivalence relation.

8. If R is a partial ordering relation on a set X and $A \subseteq X$, show that $R \cap (A \times A)$ is a partial ordering on A.

9. Let S be a set of all partitions defined on a nonempty set A. The relation R on a set S is defined to be $<\pi_1, \pi_2> \in R$ if and only if π_1 refines π_2 (π_1 is the refinement of π_2).

(a) Show that R is a partial ordering.

(b) Is a p.o. set $<S, R>$ a lattice? If yes, prove it. Otherwise, explain why.

10. Let $<A, \leq>$ be a lattice. Prove that for every x, y, and z in A,

(a) $x*(y*z) = (x*y)*z$

(b) $x+(x*y) = x$

where $x*y$ is glb(x,y) and $x+y$ is lub(x,y).

11. Let $<E(A), \subseteq>$ be a p.o.set where $E(A)$ is a set of all equivalence relations defined on a set A.

(a) For every x and y in $E(A)$, is $x \cap y$ the glb of $\{x,y\}$?

(b) For every x and y in $E(A)$, is $x \cup y$ the lub of $\{x,y\}$?