Discrete Mathematics

2. Sets

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Introduction to Set Theory

- A *set* is a new type of structure, representing an *unordered* collection of zero or more *distinct* (different) objects.

- Set theory deals with operations between, relations among, and statements about sets.

Naive Set Theory

- A set is any collection of objects (*elements*) that we can describe. (Basic premise)

- The naive set theory, however, leads to logical inconsistencies, known as *paradoxes*:

  Russell's paradox:

  1. A set being a member of itself: Possible from the case that the set of concepts is itself a concept, and hence this set is apparently a member of itself. The assertions ($x \notin x$) and ($x \in x$) are therefore predicates which can be used to define sets:

  2. Define S to be $S=\{x \mid x \notin x\}$.

  3. Is S a member of itself?

- Set theory is formulated to avoid *Russell's paradox*: Restrictions on the ways in which sets can be related, which imply that *no set is permitted to be a member of itself*. (Other *paradoxes* exist?)

Basic notations for Sets

- For sets, we'll use variables S, T, U, …
- We can denote a set S in writing by listing all of its elements in curly braces:
  - $\{a, b, c\}$ is the set of 3 objects denoted by a, b, and c.
- *Set builder notation*: For any predicate symbol P, $\{x | P(x)\}$ is *the set of all x such that P(x).* (or *the set of all x holding the property P.)*

Basic properties of Sets

- Sets are inherently *unordered*: No matter what objects a, b, and c denote, $\{a, b, c\} = \{a, c, b\} = \{b, a, c\} = \{b, c, a\} = \{c, a, b\} = \{c, b, a\}$.

- All elements are *distinct* (unequal): Multiple listings make no difference!

  If $a=b$, then $\{a, b, c\} = \{a, c\} = \{b, c\} = \{a, a, b, a, b, c, c, c, c\}$.

  This set contains at most 2 elements!

Infinite Sets

- Conceptually, sets may be *infinite* (*i.e.*, not *finite*, without end, unending).

- Symbols for some special infinite sets:
  $\boldsymbol{N} = \{1, 2, \ldots\}$, The **N**atural numbers.
  $\boldsymbol{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, The Integers.
  $\boldsymbol{R}$ = The "**R**eal" numbers, such as 374.1828471929498181917281943125…

- Infinite sets come in different sizes!

Empty Set

Definition:

A set which does not contain any elements is an empty set, denoted by $\varnothing$ or {} or $\{x \mid \text{False}\}$

Example:

$x \notin \varnothing$ for any x

Subset and Superset

Definition:

Let S and T be any two sets. S is a subset of T (*T is a superset of S*), denoted by $S \subseteq T$, *if and only if* every element of S is an element of T, *i.e.*,

$(\forall x)((x \in S) \rightarrow (x \in T))$.

Example:

$\varnothing \subseteq S$, $S \subseteq S$.

Set Equality

Definition:

Let A and B be any two sets. A and B are said to be equal *if and only if* they contain exactly the same elements, i.e., $A=B$ *if and only if* $(A \subseteq B) \land (B \subseteq A)$.

Note that it does not matter *how the set is defined or denoted.*

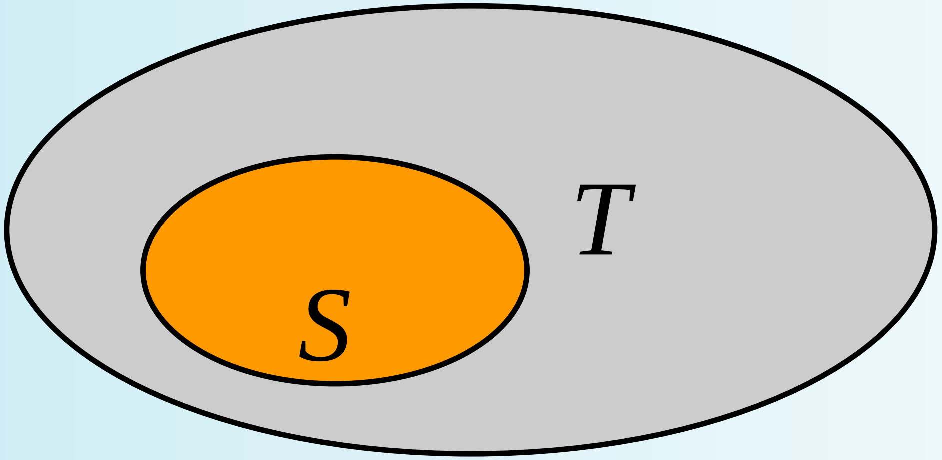
Example:

$\{1, 2, 3, 4\} = \{x \mid x$ is an integer where $x>0$ and $x < 5\} =$
$\{x \mid x$ is a positive integer whose square is > 0 and $< 25\}$

Proper Subset and Superset

Definition:

Let S and T be any two sets. S is a proper subset of T (T is a proper superset of S), denoted by $S \subset T$, *if and only if* $S \subseteq T$ and $S \neq T$.



Example:
$\{1,2\} \subset \{1,2,3\}$

Venn Diagram equivalent of $S \subset T$

Sets are objects, too!

The objects that are elements of a set may *themselves* be sets.

Example:

Let $S=\{x \mid x \subseteq \{1,2,3\}\}$. Then

$S=\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Note that $1 \neq \{1\} \neq \{\{1\}\}$.

Element of (Member of)

Definition:

1. $x \in S$ ("x is in S") is the proposition that object x is an *element* or *member* of set S.

   Example:

   $3 \in \boldsymbol{N}$, "a" $\in \{x \mid x$ is a letter of the alphabet$\}$

2. $x \notin S = \neg(x \in S)$ "x is not in S"

Cardinality and Finiteness

The *cardinality* of S, denoted by $|S|$, is a measure of how many different elements S has.

Example:

$|\varnothing|=0$, $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$, $|\{\{1,2,3\},\{5\}\}| = 2$.

If $|S| \in \boldsymbol{N}$, then S is said to be *finite*. Otherwise, S is said to be *infinite*.

Power Set

Definition:

Let S be a set. The *power set* $\wp(S)$ of S is the set of all subsets of S, i.e., $\wp(S) = \{x \mid x \subseteq S\}$.

Example: $\wp(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.

Sometimes $\wp(S)$ is written $\mathbf{2}^S$.

Note that for finite S, $|\wp(S)| = 2^{|S|}$.

It turns out that $|\wp(\boldsymbol{N})| > |\boldsymbol{N}|$.

There are different sizes of infinite sets where ***N*** *is a set of all natural numbers.*

Ordered *n*-tuples

Definition:

For $n \in \boldsymbol{N}$, an *ordered n-tuple* or a *sequence of length n* is defined to be $(a_1, a_2, \ldots, a_n)$. The *first* element is a_1, *etc.*

These are like sets, except that duplicates matter and the order makes a difference.

Note $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.

Empty sequence, singlets, pairs, triples, quadruples, quin<u>tuples</u>, …, *n*-tuples.

Cartesian Products of Sets

Definition:

Let A and B be any two sets. The *Cartesian product* $A \times B$ is defined to be

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

Example:

$$\{a, b\} \times \{1,2\} = \{(a,1), (a, 2), (b,1), (b, 2)\}$$

Note that for two finite sets, A and B,

1. $|A \times B| = |A||B|$.
2. $A \times B \neq B \times A$.

Union Operator

Definition:

Let A and B be any two sets. The *union* $A \cup B$ of A and B is the set containing all elements that are either in A, or in B (or, of course, in both), i.e.,

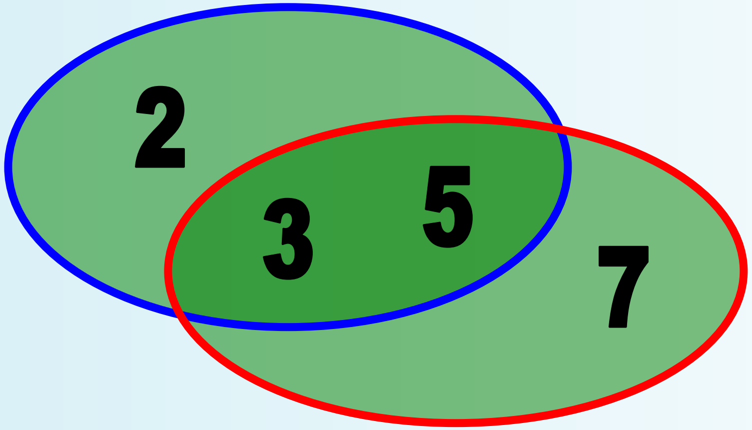
$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

Note that $A \cup B$ contains all the elements of A and it contains all the elements of B:

$$(A \cup B \supseteq A) \wedge (A \cup B \supseteq B)$$

Example of Union

- $\{a,b,c\} \cup \{2,3\} = \{a,b,c,2,3\}$
- $\{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}$



Intersection Operator

Definition:

Let A and B be any two sets. The *intersection* $A \cap B$ of A and B is the set containing all elements that are simultaneously in A and in B, i.e.,

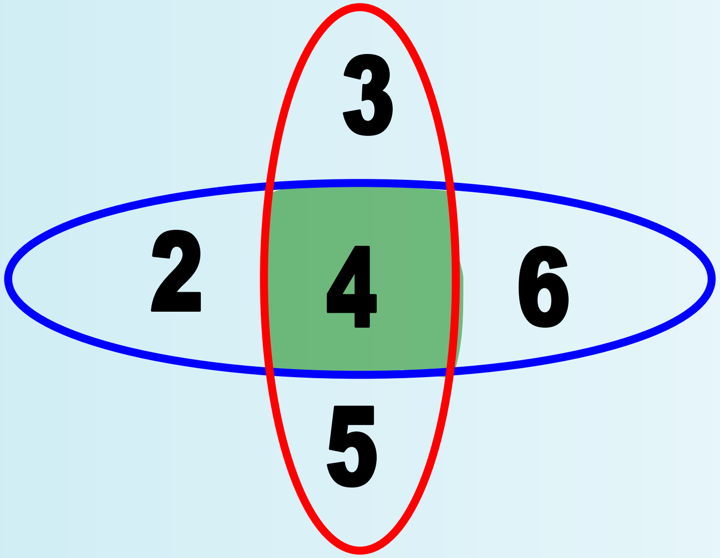
$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

Note that $A \cap B$ is a subset of A and it is a subset of B:

$$(A \cap B \subseteq A) \wedge (A \cap B \subseteq B)$$

Example of Intersection

- $\{a,b,c\} \cap \{2,3\} = \varnothing$
- $\{2,4,6\} \cap \{3,4,5\} = \{4\}$



Disjointedness

Definition:

Let A and B be any two sets. A and B are called *disjoint if and only if* their intersection is empty ($A \cap B = \emptyset$).

Example:

The set of even integers is disjoint with the set of odd integers.

Inclusion-Exclusion Principle

How many elements are in $A \cup B$?

$|A \cup B| = |A| + |B| - |A \cap B|$.

Example:

How many students are on our class email list?

Consider a set $E = I \cup M$ where

$I = \{s \mid s \text{ turned in an information sheet}\}$ and

$M = \{s \mid s \text{ sent the TAs their email address}\}$.

Since some students did both,

$|E| = |I \cup M| = |I| + |M| - |I \cap M|$

Set Difference

Definition:

Let A and B be any two sets.

1. The set *difference*, $A–B$, *of* A *and* B is the set of all elements that are in A but not in B.
2. $A–B$ is also called the *complement of* B *with respect to* A.

Example

1. $\{1,2,3,4,5,6\} - \{2,3,5,7,9,11\} =$ **{1,4,6}**
2. $\boldsymbol{Z} - \boldsymbol{N} = \{\ldots, -1, 0, 1, 2, \ldots\} - \{1, \ldots\}$

   $= \{x \mid x \text{ is an integer but not a nat. number}\}$

   $= \{x \mid x \text{ is a negative integer or } x = 0\}$

   $= \{\ldots, -3, -2, -1, 0\}$

Universal Set & Complement of a Set

Definition (Universal Set):

A set is a universal set or a universe of discourse, denoted by U, if it includes every set under discussion.

Definition (Complement of a Set):

Let A be a set. The *complement* of A in U, denoted by $\overline{A}$, is the set of all elements of U which are not elements of A, i.e.,

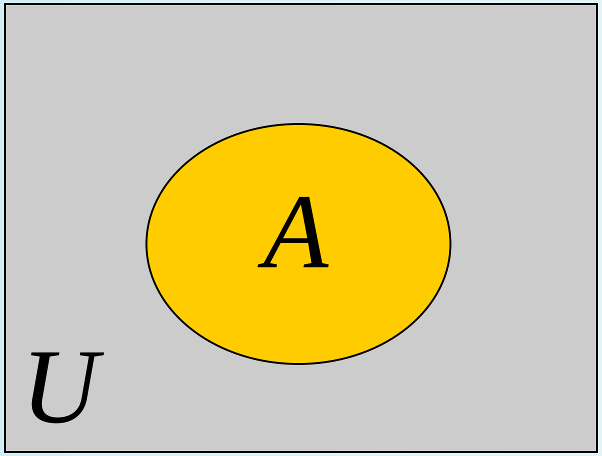
$$\overline{A} = U - A.$$

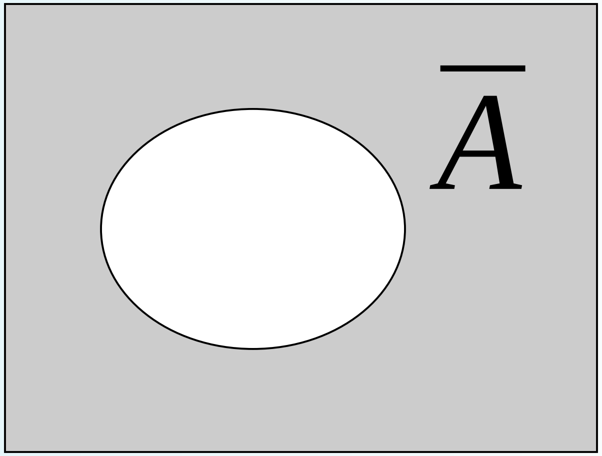
Example:

If U=$\boldsymbol{N}$, $\overline{\{3,5\}} = \{1, 2, 4, 6, 7, \ldots\}$

An equivalent definition, when U is clear:

$$\overline{A} = \{x \mid x \notin A\}$$





Set Identity Theorems

For any sets, A, B, and C, the following holds:

1. *Identity*: $A \cup \varnothing = A, \quad A \cap U = A$
2. *Domination*: $A \cup U = U, \quad A \cap \varnothing = \varnothing$
3. *Idempotent*: $A \cup A = A = A \cap A$
4. *Double complement*: $\overline{(\overline{A})} = A$
5. Commutative: $A \cup B = B \cup A, \quad A \cap B = B \cap A$
6. *Associative*: $A \cup (B \cup C) = (A \cup B) \cup C$

   $A \cap (B \cap C) = (A \cap B) \cap C$

DeMorgan’s Theorem for Sets

Theorem:

Let A and B be sets. Then the following holds:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Example:

Let A, B, and C be sets.

Prove (formally) that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof:

// $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$//

	(Rule)	(Tautology)	(Justification)
1. $x \in A \cap (B \cup C)$	AP		
2. $(x \in A) \wedge (x \in B \cup C)$	Def		definition of $\cap$, $1 \rightarrow 2$
3. $(x \in A) \wedge ((x \in B) \vee (x \in C))$	Def		definition of $\cup$, $2 \rightarrow 3$
4. $[(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)]$	T	E_6	$3 \rightarrow 4$
5. $(x \in A \cap B) \vee (x \in A \cap C)$	Def		definition of $\cap$, $4 \rightarrow 5$
6. $x \in (A \cap B) \cup (A \cap C)$	Def		definition of $\cup$, $5 \rightarrow 6$
7. $x \in A \cap (B \cup C) \rightarrow x \in (A \cap B) \cup (A \cap C)$	CP		$1,6 \rightarrow 7$
8. $\forall x\, [x \in A \cap (B \cup C) \rightarrow x \in (A \cap B) \cup (A \cap C)]$	UG		$7 \rightarrow 8$
9. $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$	Def		definition of $\subseteq$, $8 \rightarrow 9$

// $(A\cap B)\cup(A\cap C) \subseteq A\cap(B\cup C)$//

	(Rule)	(Tautology)	(Justification)
10. $x\in(A\cap B)\cup(A\cap C)$	AP		
11. $x\in(A\cap B) \vee x\in(A\cap C)$	Def		definition of $\cup$, 10→11
12. $[(x\in A)\wedge(x\in B)]\vee[(x\in A)\wedge (x\in C)]$	Def		definition of $\cap$, 11→12
13. $(x\in A) \wedge[(x\in B))\vee (x\in C)]$	T	E_6	12→13
14. $(x\in A) \wedge(x\in B \cup C)$	Def		definition of $\cup$, 13→14
15. $x\in A \cap(B \cup C)$	Def		definition of $\cap$, 14→15
16. $x\in(A\cap B)\cup(A\cap C) \rightarrow x\in A \cap(B \cup C)$	CP		10,15→16
17. $\forall x\, [x\in(A\cap B)\cup(A\cap C) \rightarrow x\in A \cap(B \cup C)]$	UG		16→17
18. $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$	Def		definition of $\subseteq$, 17→18
19. $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$	Def		definition of =, 9,18→19

Theorem:

If A and B are two sets, the following statements are equivalent.

1. $A \subseteq B$
2. $A \cap B = A$
3. $A \cup B = B$

Proof of $A \subseteq B$ if and only if $A \cap B = A$

Formal proof:

// $A \subseteq B$ if $A \cap B = A$ //

	(Rule)	(Tautology)	(Justification)
1. $x \in A$	AP		
2. $A \cap B = A$	P		
3. $x \in A \cap B$	Def		definition of =, 1,2→3
4. $x \in B$	Def		definition of ∩ , 3→4
5. $(x \in A) \rightarrow (x \in B)$	CP		1,4→5
6. $\forall x[(x \in A) \rightarrow (x \in B)]$	UG		5→6
7. $A \subseteq B$	Def		definition of ⊆, 6 →7

// $A \subseteq B$ only if $A \cap B = A$ //

	(Rule)	(Tautology)	(Justification)
1. $x \in A \cap B$	AP		
2. $x \in A$	Def		definition of $\cap$, $1 \rightarrow 2$
3. $(x \in A \cap B) \rightarrow (x \in A)$	CP		$1,2 \rightarrow 3$
4. $\forall x[(x \in A \cap B) \rightarrow (x \in A)]$	UG		$3 \rightarrow 4$
5. $A \cap B \subseteq A$	Def		definition of $\subseteq$, $4 \rightarrow 5$
6. $x \in A$	AP		
7. $A \subseteq B$	P		
8. $x \in B$	Def		definition of $\subseteq$, $6,7 \rightarrow 8$
9. $x \in A \cap B$	Def		definition of $\cap$, $6,8 \rightarrow 9$
10. $(x \in A) \rightarrow (x \in A \cap B)$	CP		$6,9 \rightarrow 10$
11. $\forall x[(x \in A) \rightarrow (x \in A \cap B)]$	UG		$10 \rightarrow 11$
12. $A \subseteq A \cap B$	Def		definition of $\subseteq$, $11 \rightarrow 12$
13. $A \cap B = A$	Def		definition of $=$, $5,12 \rightarrow 13$

Informal proof:

// $A \subseteq B$ if $A \cap B = A$ //

For arbitrary element x, let $x \in A$ (assume $x \in A$) ····· (1)

From the given condition that $A \cap B = A$, $x \in A \cap B$ by definition of =.

Then by definition of $\cap$, $x \in B$ ····· (2)

From (1)&(2), by definition of $\subseteq$, $A \subseteq B$

// $A \subseteq B$ only if $A \cap B = A$ //

For arbitrary element x, let $x \in A \cap B$ ····· (3)

Then $x \in A$ by definition of $\cap$. ····· (4)

From (3)&(4), by definition of $\subseteq$, $A \cap B \subseteq A$ ····· (5)

For arbitrary element x, let $x \in A$ ····· (6)

Then from the given condition that $A \subseteq B$, $x \in B$ by definition of $\subseteq$. ··· (7)

From (6)&(7), by definition of $\cap$, $x \in A \cap B$ ····· (8)

From (6)&(8), by definition of $\subseteq$, $A \subseteq A \cap B$ ····· (9)

From (5)&(9), by definition of =, $A \cap B = A$

Example:

Let A, B, and C be three nonempty sets.

Prove that $A-(B \cup C) = (A-B) \cap (A-C)$.

Proof:

//Show that $A-(B \cup C) \subseteq (A-B) \cap (A-C)$//

For arbitrary element x, let $x \in A-(B \cup C)$. ····· (1)

Then by definition of $-$, $(x \in A) \wedge \neg(x \in B \cup C)$.

By definition of $\cup$, $(x \in A) \wedge \neg((x \in B) \vee (x \in C))$.

By DeMorgan's equivalence, $(x \in A) \wedge (\neg(x \in B) \wedge \neg (x \in C))$.

By idempotent equivalence, $((x \in A) \wedge (x \in A)) \wedge (\neg(x \in B) \wedge \neg (x \in C))$.

By associative and commutative equivalences,

$((x \in A) \wedge \neg (x \in B)) \wedge ((x \in A) \wedge \neg (x \in C))$.

By definition of $-$, $(x \in (A-B)) \wedge (x \in (A-C))$.

By definition of $\cap$, $(x \in (A-B) \cap (A-C))$. ····· (2)

From (1) & (2), by definition of $\subseteq$, $A-(B \cup C) \subseteq (A-B) \cap (A-C)$. ····· (3)

//Show that $(A-B) \cap (A-C) \subseteq A-(B \cup C)$ //

For arbitrary element x, let $x \in (A-B) \cap (A-C)$. ····· (4)

Then by definition of $\cap$, $(x \in (A-B)) \wedge (x \in (A-C))$.

By definition of $-$, $((x \in A) \wedge \neg (x \in B)) \wedge ((x \in A) \wedge \neg (x \in C))$.

By associative and commutative equivalences,

$((x \in A) \wedge (x \in A)) \wedge (\neg(x \in B) \wedge \neg (x \in C))$.

By idempotent equivalence, $(x \in A) \wedge (\neg(x \in B) \wedge \neg (x \in C))$.

By DeMorgan's equivalence, $(x \in A) \wedge \neg((x \in B) \vee (x \in C))$.

By definition of $\cup$, $(x \in A) \wedge \neg(x \in B \cup C)$.

By definition of $-$, $x \in A-(B \cup C)$. ····· (5)

From (4) & (5), by definition of $\subseteq$, $(A-B) \cap (A-C) \subseteq A-(B \cup C)$. ····· (6)

From (3) & (6), by definition of $=$, $(A-B) \cap (A-C) = A-(B \cup C)$.

Generalized Unions & Intersections

- Since union & intersection are commutative and associative, we can extend them from operating on *ordered pairs* of sets (A, B) to operating on sequences of sets $(A_1, \ldots, A_n)$, or even unordered *sets* of sets.

Generalized Union

1. Binary union operator: $A \cup B$
2. n-ary union:
   $A_1 \cup A_2 \cup \ldots \cup A_n = ((\ldots((A_1 \cup A_2) \cup \ldots) \cup A_n)$
   (grouping & order is irrelevant)
3. "Big $\cup$" notation: $\bigcup_{i=1}^{n} A_i$
4. For infinite sets of sets: $\bigcup_{A \in X} A$

Generalized Intersection

1. Binary intersection operator: $A \cap B$
2. n-ary intersection:
   $A_1 \cap A_2 \cap \ldots \cap A_n \equiv ((\ldots((A_1 \cap A_2) \cap \ldots) \cap A_n)$
   (grouping & order is irrelevant)
3. "Big ∩" notation: $\bigcap_{i=1}^{n} A_i$
4. For infinite sets of sets: $\bigcap_{A \in X} A$

Exercise

1. Let A and B be sets. Show that

   (a) $(A \cap B) \subseteq A$

   (b) $A \cup (B\text{-}A) = A \cup B$

   (c) $A \cap B = A$ if and only if $A \cup B = B$

   (d) $A\text{-} (A \cap B) = A\text{-}B$

   (e) $\neg(A \cup B) = \neg A \cap \neg B$

2. Let A, B and C be sets. Show that

   $(A\text{-}B)\text{-}C = (A\text{-}C)\text{-}(B\text{-}C)$.

3. Let A and B be two sets. Prove or disprove each of the followings:

(a) $\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$ where $\wp(A)$ is the power set of the set A.

(b) $\wp(A \cup B) \subseteq \wp(A) \cup \wp(B)$

4. Which of the following are true for all sets, A, B, and C ? Give a counter example if the answer is false (No proof is necessary if the answer is true).

(a) If $A \cap B = \emptyset$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

(b) If $A \in B$ and $\neg(B \subseteq C)$, then $\neg(A \in C)$.

(c) If $A \in B$ and $B \in C$, then $\neg(A \in C)$.

(d) $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

(e) $\emptyset \in A$.

(f) If $A \subseteq B$ and $B \in C$, then $A \subseteq C$

(g) If $A \in B$, then $\{A\} \subseteq B$