Translation tips

Assume the domain of all integers

Let $E(x)$ stand for "x is even"

Translate the following into well-formed formulas:

"Every integer is even"

Restate it as : "For every object x in the universe, x is even"

Answer : $(\forall x)(E(x))$

"NOT"

"NO"

"ALL"

"SOME"

"ONLY"

"Not every integer is even"

Restate it as : "It is not the case that every integer is even"

Answer : $\neg (\forall x)(E(x))$

or it can also be interpreted as

: "For some object x in the universe, x is not even"

Answer : $(\exists x) \neg(E(x))$

E_{26} $\neg(\forall x)A(x) \Leftrightarrow (\exists x)\neg A(x)$

"No integer is even"

Restate it as : "For every object x in the universe, x is not even"

Answer : $(\forall x) \neg(E(x))$

or it can also be interpreted as

: "It is not the case that some integer is even"

Answer : $\neg(\exists x) (E(x))$

E_{25} $\neg(\exists x)A(x) \Leftrightarrow (\forall x)\neg A(x)$

Assume the domain of real numbers

Let $I(x)$ stand for "x is an integer"

$E(x)$ stand for "x is even"

Translate the following into well-formed formulas:

"All integers are even"

Restate it as : "For every object x in the universe, ?"

"All integers are even"

Restate it as : "For every object x in the universe, if it is integer, then it is even"

right answer : $(\forall x)\ [I(x) \rightarrow E(x)]$

Restate it as : "For every object x in the universe, it is integer and it is even"

wrong answer : $(\forall x)\ [I(x) \wedge E(x)]$

Description

: We are interested in **not any arbitrary objects** but a **specific type of objects**, that is integers. But $\forall x$ means "for any object in the universe" so we must say "for any object in the universe, if it is integer … " we narrow it down to integers.

Assume the domain of real numbers

Let $I(x)$ stand for "x is an integer"

$E(x)$ stand for "x is even"

Translate the following into well-formed formulas:

"Some integers are even"

Restate it as : "There are objects x in the universe, ?"

"Some integers are even"

Restate it as : "There are objects x in the universe that are integer and even"

right answer : $(\exists x)\ [I(x) \wedge E(x)]$

Restate it as : "For some object in the universe, if it is integer then it is even"

wrong answer : $(\exists x)\ [I(x) \rightarrow E(x)]$

Description

: The former asserts that an even integer exists, while the later does not assert the existence of such an integer. It is a hypothetical statement.

Assume the domain of real numbers

Let $I(x)$ stand for "x is an integer"

$E(x)$ stand for "x is even"

Translate the following into well-formed formulas:

"Only integers are even"

Is equivalent to "If it is even, then it is integer" thus $(\forall x)\ [E(x) \rightarrow I(x)]$

"Integers are even only"

Is equivalent to "If it is integer, then it is even" thus $(\forall x)\ [I(x) \rightarrow E(x)]$

Exercise

1. Logic, Proofs

Exercise 1.

1. Let p, q, and r be the propositions

p : You have the flu.

q : You miss the final examination

r : You pass the course

Express each of these propositions as an English sentence.

(a) $(p \to \neg r) \vee (q \to \neg r)$

(b) $(p \wedge q) \vee (\neg q \wedge r)$

Answer

(a) $(p \to \neg r)$: If you have the flu, then you cannot pass the course.

$(q \to \neg r)$: If you miss the final examination, then you cannot pass the course.

$\therefore (p \to \neg r) \vee (q \to \neg r)$

: It is either the case that if you have the flu then you cannot pass the course or the case that if you miss the final exam then you cannot pass the course.

Exercise 1.

1. Let p, q, and r be the propositions

   p : You have the flu.

   q : You miss the final examination

   r : You pass the course

Express each of these propositions as an English sentence.

(a) $(p \to \neg r) \vee (q \to \neg r)$

(b) $(p \wedge q) \vee (\neg q \wedge r)$

Answer

(b) $(p \wedge q)$: you have the flu and miss the final examination

$(\neg q \wedge r)$: you don't miss the final examination and you pass the course

$\therefore (p \wedge q) \vee (\neg q \wedge r)$

: Either you have the flue and miss the final exam, or you don't miss the final exam and do pass the course.

Exercise 2.

2. Let p, q, and r be the propositions

p : You get an A on the final exam.

q : You do every exercise in this book

r : You get an A in this class

Write these propositions using p, q and r and logical connectives.

Answer

(a) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

You get an A on the final : p

You don't do every exercise in this book : $\neg q$

You get an A in this class : r

but, nevertheless : $\wedge$

$\underline{\therefore\ p \wedge \neg q \wedge r}$

Exercise 2.

2. Let p, q, and r be the propositions

p : You get an A on the final exam.

q : You do every exercise in this book

r : You get an A in this class

Write these propositions using p, q and r and logical connectives.

Answer

(b) Getting an A on the final and doing every exercise in this book is sufficient for getting an A in this class.

Getting an A on the final : p

and : $\wedge$

Doing every exercise in this book : q

sufficient for : $\rightarrow$

Getting an A in this class : r

$\underline{\boldsymbol{\therefore (p \wedge q) \rightarrow r}}$

Exercise 3.

3. Assume the domain of all people.
Let $J(x)$ stand for "x is a junior",
$S(x)$ stand for "x is a senior",
and $L(x, y)$ stand for "x likes y".
Translate the following into well-formed formulas:

Answer

(a) All people like some juniors.

$(\forall x)(\exists y)$?

$(\exists y)(J(y) \wedge L(x,y))$

Recall; "Some integers are even" : $(\exists x)\,[I(x) \wedge E(x)]$

$(\forall x)(\exists y)(J(y) \wedge L(x,y))$

Exercise 3.

3. Assume the domain of all people.
Let $J(x)$ stand for "x is a junior",
$S(x)$ stand for "x is a senior",
and $L(x, y)$ stand for "x likes y".
Translate the following into well-formed formulas:

Answer

(b) Some people like all juniors.

$(\exists x)(\forall y)$?

$(\forall y)(J(y)\rightarrow L(x,y))$

Recall; "All integers are even" : $(\forall x)\ [I(x) \rightarrow E(x)]$

$(\exists x)(\forall y)(J(y)\rightarrow L(x,y))$

Exercise 3.

3. Assume the domain of all people.
Let $J(x)$ stand for "x is a junior",
$S(x)$ stand for "x is a senior",
and $L(x, y)$ stand for "x likes y".
Translate the following into well-formed formulas:

Answer

(c) Only seniors like juniors.

$(\forall x)(\forall y)$?

Recall; "Only integers are even" : $(\forall x)\ [E(x) \rightarrow I(x)]$

$(\forall x)(\forall y)((J(y) \wedge L(x,y)) \rightarrow S(x))$

Exercise 4.

4. Let $B(x)$ stand for "x is a boy", $G(x)$ stand for "x is a girl", and $T(x,y)$ stand for "x is taller than y".
Complete the well-formed formula representing the given statement by filling out ? part.

Answer

(a) Only girls are taller than boys: $(?)(\forall y)((? \wedge T(x,y)) \rightarrow ?)$

$(\forall x)(\forall y)((\mathbf{B}(y) \wedge T(x,y)) \rightarrow \mathbf{G}(x))$

Recall; " Only seniors like juniors."
:$(\forall x)(\forall y)((\mathrm{J}(y) \wedge \mathrm{L}(x,y)) \rightarrow \mathrm{S}(x))$

Exercise 4.

4. Let $B(x)$ stand for "x is a boy", $G(x)$ stand for "x is a girl", and $T(x,y)$ stand for "x is taller than y". Complete the well-formed formula representing the given statement by filling out ? part.

Answer

(b) Some girls are taller than boys: $(\exists x)(?)(G(x) \wedge (? \to ?))$

Some people are taller than boys
: $(\exists x)(\forall y)((\mathrm{B}(y) \to T(x,y))$

Recall; "some people like all juniors"
: $(\exists x)(\forall y)(\mathrm{J}(y) \to \mathrm{L}(x,y))$

$(\exists x)(\forall \boldsymbol{y})(G(x) \wedge ((\mathbf{B}(\boldsymbol{y}) \to \boldsymbol{T(x,y)}))$

Exercise 4.

4. Let $B(x)$ stand for "x is a boy", $G(x)$ stand for "x is a girl", and $T(x,y)$ stand for "x is taller than y". Complete the well-formed formula representing the given statement by filling out ? part.

Answer

(c) Girls are taller than boys only: (?) $(\forall y)((G(x) \wedge ?) \rightarrow ?)$

Recall; " Only girls are taller than boys."

$(\forall x)(\forall y)((B(y) \wedge T(x,y)) \rightarrow G(x))$

$(\forall \boldsymbol{x})(\forall y)((G(x) \wedge \boldsymbol{T(x,y)}) \rightarrow \mathbf{B(y)})$

Exercise 4.

4. Let $B(x)$ stand for "x is a boy", $G(x)$ stand for "x is a girl", and $T(x,y)$ stand for "x is taller than y".
Complete the well-formed formula representing the given statement by filling out ? part.

Answer

(d) Some girls are not taller than any boy: $(\exists x)(?)(G(x) \wedge (? \to ?))$

some people are taller than any boy : $(\exists x)(\forall y)\ (B(y) \to T(x,y))$
some people are not taller than any boy : $(\exists x)(\forall y)\ (B(y) \to \neg T(x,y))$

$(\exists x)(\boldsymbol{\forall y})(G(x) \wedge (\mathbf{B(y)} \to \boldsymbol{\neg T(x,y)}))$

Exercise 4.

4. Let $B(x)$ stand for "x is a boy", $G(x)$ stand for "x is a girl", and $T(x,y)$ stand for "x is taller than y".
Complete the well-formed formula representing the given statement by filling out ? part.

Answer

(e) No girl is taller than any boy: $(?)(\forall y)((B(y) \wedge ?) \rightarrow ?)$

all people are not taller than any boy: $(\forall x)(\forall y)((B(y)) \rightarrow \neg T(x,y))$
all girls are not taller than any boy: $(\forall x)(\forall y)((G(x) \wedge B(y)) \rightarrow \neg T(x,y))$
or $(\forall x)(\forall y)(G(x)\rightarrow(B(y) \rightarrow \neg T(x,y))$

$(\forall x)(\forall y)((B(y) \wedge \mathbf{G(x)}) \rightarrow \neg T(x,y))$

Exercise 5.

5. Prove formally using inference rules that $R \wedge (P \vee Q)$ logically follows from $(P \vee Q)$, $(Q \to R)$, $(P \to M)$, and $\neg M$.

Answer

(1)	$P \to M$	P	
(2)	$\neg M$	P	
(3)	$\neg P$	T , (1), (2) and I_{12}	I_{12} : $\neg Q, P \to Q \Rightarrow \neg P$
(4)	$P \vee Q$	P	
(5)	Q	T , (3), (4) and I_{10}	I_{10} : $\neg P , P \vee Q \Rightarrow Q$
(6)	$Q \to R$	P	
(7)	R	T , (5), (6) and I_{11}	I_{11} : $P , P \to Q \Rightarrow Q$
(8)	$R \wedge (P \vee Q)$	T , (4), (7) and I_9	I_9 : $P , Q \Rightarrow P \wedge Q$

Exercise 6.

6. Prove that if n is a positive integer, then n is a even if and only if $7n+4$ is even

Answer

① Prove if n is even, then $7n+4$ is even

If n is even, then $n = 2k$ where $k \in \{1, 2, 3, \ldots\}$

$7n + 4 = 7 * 2k + 4 = 2(7k+2)$

$\therefore$ n is even

② Prove if $7n+4$ is even, then n is even

If n is odd, $7n+4$ is odd

If $n \neq$ even (odd), then

$$\begin{aligned} 7n+4 &= 7(2k+1)+4 \\ &= 14k + 11 \\ &= 2(7k+5)+1 \end{aligned}$$

$\therefore$ $7n+4$ is odd

From ① and ②, *n is even if and only if 7n+4 is even*

Exercise 7.

7. Let P, Q, R and S be statement variables.
   Prove formally the following.
   (a) $\neg P \vee Q, \neg Q \vee R, R \rightarrow S \Rightarrow P \rightarrow S$
   (b) $\neg P \wedge (P \vee Q) \Rightarrow Q$

Answer

(a) $\neg P \wedge Q, \neg Q \vee R, R \rightarrow S \Rightarrow P \rightarrow S$

(1)	$\neg P \vee Q$	P	
(2)	$\neg Q \vee R$	P	
(3)	$R \rightarrow S$	P	
(4)	$P \rightarrow Q$	T , (1) and E_{16}	$E_{16} : P \rightarrow Q \Leftrightarrow \neg P \vee Q$
(5)	$Q \rightarrow R$	T , (2) and E_{16}	
(6)	$P \rightarrow R$	T , (4), (5) and I_{13}	$I_{13} : P \rightarrow Q , Q \rightarrow R \Rightarrow R$
(7)	$P \rightarrow S$	T , (3), (6) and I_{13}	

Exercise 7.

7. Let P, Q, R and S be statement variables.
   Prove formally the following.
   (a) $\neg P \wedge Q, \neg Q \vee R, R \rightarrow S \Rightarrow P \rightarrow S$
   (b) $\neg P \wedge (P \vee Q) \Rightarrow Q$

Answer

(b) $\neg P \wedge (P \vee Q) \Rightarrow Q$

(1)	$\neg P \wedge (P \vee Q)$	P	
(2)	$\neg P$	T , (1) I_1	$I_1 : P \wedge Q \Rightarrow P$
(3)	$P \vee Q$	T , (1) I_2	$I_2 : P \wedge Q \Rightarrow Q$
(4)	Q	T , (2), (3) and I_{10}	$I_{10} : \neg P , P \vee Q \Rightarrow Q$

Exercise 8.

8. Show the following implication.

(a) $(\forall x)(P(x) \vee Q(x)),\ (\forall x)\neg P(x) \Rightarrow (\exists x)Q(x)$

(b) $\neg((\exists x)P(x) \wedge Q(a)) \Rightarrow (\exists x)P(x) \rightarrow \neg Q(a)$

Answer

(a) $(\forall x)(P(x) \vee Q(x)),\ (\forall x)\neg P(x) \Rightarrow (\exists x)Q(x)$

(1)	$(\forall x)(P(x) \vee Q(x))$	P
(2)	$(\forall x)\neg P(x)$	P
(3)	$\neg P(a)$	T, (2) U.S.
(4)	$P(a) \vee Q(a)$	T, (1) U.S.
(5)	$Q(a)$	T, (3),(4) I_{10}
(6)	$(\exists x)Q(x)$	T, (5) E.G.

$I_{10} : \neg P,\ P \vee Q \Rightarrow Q$

Exercise 8.

8. Show the following implication.

(a) $(\forall x)(P(x) \vee Q(x)), (\forall x)\neg P(x) \Rightarrow (\exists x)Q(x)$

(b) $\neg((\exists x)P(x) \wedge Q(a)) \Rightarrow (\exists x)P(x) \rightarrow \neg Q(a)$

Answer

(b) $\neg((\exists x)P(x) \wedge Q(a)) \Rightarrow (\exists x)P(x) \rightarrow \neg Q(a)$

(1)	$\neg((\exists x)P(x) \wedge Q(a))$	P	
(2)	$\neg(\exists x)P(x) \vee \neg Q(a)$	T, (1) E_8	$E_8 : \neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
(3)	$(\exists x)P(x) \rightarrow \neg Q(a)$	T, (2) E_{16}	$E_{16} : P \rightarrow Q \Leftrightarrow \neg P \vee Q$