Exercise

2. Set

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Exercise 1.

1. Let A and B be sets. Show that

   (a) $(A \cap B) \subseteq A$

   (b) $A \cup (B - A) = A \cup B$

   (c) $A \cap B = A$ if and only if $A \cup B = B$

   (d) $A - (A \cap B) = A - B$

   (e) $(A \cup B)^c = A^c \cap B^c$

Answer

(a) $(A \cap B) \subseteq A$

pf. Suppose $x \in (A \cap B)$

by definition of $\cap$, $x \in A \ \wedge \ x \in B$

$\therefore x \in A$

Exercise 1.

(b) $A \cup (B-A) = A \cup B$

Answer

Show $A\cup(B-A) \subseteq A\cup B$ and $A\cup B \subseteq A\cup(B-A)$

① $A\cup(B-A) \subseteq A\cup B$

suppose $x \in A\cup(B-A)$
by definition of $\cup$, $x\in A \lor x\in(B-A)$
case 1) $x\in A$:
by definition of $\cup$, $x\in A\cup B$

case 2) $x\in(B-A)$:
by definition of set difference, $x\in B \land x\notin A$
by definition of $\cup$, $x\in A\cup B$

from case1) & 2), $A\cup(B-A) \subseteq A\cup B$

Exercise 1.

(b) $A \cup (B\text{-}A) = A \cup B$

Answer

② $A \cup B \subseteq A \cup (B\text{-}A)$

suppose $x \in A \cup B$
by definition of $\cup$, $x \in A \vee x \in B$
case 1) $x \in A$:
by definition of $\cup$, $x \in A \cup (B\text{-}A)$

case 2) $x \in B$:
1) $x \in A$: $x \in A \cup (B\text{-}A)$ (by definition of $\cup$)
2) $x \notin A$: $x \in (B\text{-}A)$ (by definition of set difference)
by definition of $\cup$, $x \in A \cup (B\text{-}A)$

from case1) & 2), $A \cup B \subseteq A \cup (B\text{-}A)$

From ① and ②, $A \cup (B\text{-}A) = A \cup B$

Exercise 1.

(c) $A \cap B = A$ if and only if $A \cup B = B$

Answer

① show $A \cap B = A \Rightarrow A \cup B = B$

1) $B \subseteq A \cup B$

suppose $x \in B$
by definition of $\cup$, $x \in A \cup B$
$\therefore B \subseteq A \cup B$ (by definition of sub set)

2) $A \cup B \subseteq B$

suppose $x \in A \cup B$ …. (k)
by definition of $\cup$, $x \in A \vee x \in B$
(i) $x \in A$
since $A = A \cap B$, $x \in A \cap B$
by definition of $\cap$, $x \in A \wedge x \in B$
$\therefore x \in B$ …… (l)
from (k) & (l) , $A \cup B \subseteq B$
from 1) & 2) $A \cup B = B$

Exercise 1.

(c) $A \cap B = A$ if and only if $A \cup B = B$

Answer

② show $A \cup B = B \Rightarrow A \cap B = A$

1) $A \cap B \subseteq A$

suppose $x \in A \cap B$

by definition of $\cap$, $x \in A \wedge x \in B$

$\therefore A \cap B \subseteq A$ (by definition of sub set)

2) $A \subseteq A \cap B$

suppose $x \in A$ …. (k)

by definition of $\cup$, $x \in A \cup B$

since $A \cup B = B$, $x \in B$ …..(l)

from (k) & (l) , $x \in (A \cap B)$ …..(m) (by definition of $\cap$)

from (k) & (m) , $A \subseteq A \cap B$

from 1) & 2) $A \cap B = A$

Exercise 1.

(d) $A - (A \cap B) = A - B$

Answer

Show $A - (A \cap B) \subseteq A - B$ and $A - B \subseteq A - (A \cap B)$

① show $A - (A \cap B) \subseteq A\text{-}B$

Let $x \in A - (A \cap B)$

$x \in A \;\wedge\; x \notin (A \cap B)$ (by definition of set difference)

$x \notin (A \cap B)$: $\neg (x \in A \cap B)$

$\neg (x \in A \;\wedge\; x \in B)$ (by definition of $\cap$)

$\neg (x \in A) \;\vee\; \neg (x \in B)$ (DeMorgan's)

$x \notin A \;\vee\; x \notin B$

$x \in A \wedge (x \notin A \vee x \notin B)$

$\Leftrightarrow (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B)$ (Distributive)

$x \in A \wedge x \notin B$

$x \in A\text{-}B$ (by definition of set difference)

$\therefore A - (A \cap B) \subseteq A\text{-}B$

Exercise 1.

$\boxed{\text{(d) } A - (A \cap B) = A - B}$

Answer

Show $A - (A \cap B) \subseteq A - B$ and $A - B \subseteq A - (A \cap B)$

② show $A - B \subseteq A - (A \cap B)$

Let $x \in A - B$

$x \in A$ …(i) $\wedge$ $x \notin B$ (by definition of set difference)

$x \in B^c$ (by definition of complement set)

$x \in A^c \cup B^c$ (by definition of $\cup$)

$x \in (A \cap B)^c$ (DeMorgan's Theorem)

$x \notin (A \cap B)$ …(j)

from i) & j), $x \in A - (A \cap B)$ (by definition of set difference)

$\therefore A\text{-}B \subseteq A - (A \cap B)$

From ① & ②, $A - (A \cap B) = A - B$

Exercise 1.

(e) $(A \cup B)^c = A^c \cap B^c$

Answer

Show $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$

① show $(A \cup B)^c \subseteq A^c \cap B^c$

Let $x \in (A \cup B)^c$

$x \notin A \cup B$ (by definition of complement set)

$\neg((x \in A) \vee (x \in B))$ (by definition of $\cup$)

$\neg(x \in A) \wedge \neg(x \in B)$ (DeMorgan's)

$x \notin A \wedge x \notin B$

$x \in A^c \wedge x \in B^c$

$x \in A^c \cap B^c$

$\therefore (A \cup B)^c \subseteq A^c \cap B^c$

Exercise 1.

(e) $(A \cup B)^c = A^c \cap B^c$

Answer

Show $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$

② show $A^c \cap B^c \subseteq (A \cup B)^c$

Let $x \in A^c \cap B^c$

$x \in A^c \wedge x \in B^c$ (by definition of $\cap$)

$x \notin A \wedge x \notin B$ (by definition of complement)

$\neg(x \in A) \wedge \neg(x \in B)$

$\neg((x \in A) \vee (x \in B))$ (DeMorgan's)

$\neg(x \in A \cup B)$ (by definition of $\cup$)

$x \notin (A \cup B)$

$x \in (A \cup B)^c$

$\therefore A^c \cap B^c \subseteq (A \cup B)^c$

Exercise 2.

2. Let A, B and C be sets.
   Show that $(A\text{-}B)\text{-}C = (A\text{-}C)\text{-}(B\text{-}C)$.

Answer

Show $(A\text{-}B)\text{-}C \subseteq (A\text{-}C)\text{-}(B\text{-}C)$ and $(A\text{-}C)\text{-}(B\text{-}C) \subseteq (A\text{-}B)\text{-}C$

① $(A\text{-}B)\text{-}C \subseteq (A\text{-}C)\text{-}(B\text{-}C)$

suppose $x \in (A\text{-}B)\text{-}C$

$x \in (A\text{-}B) \wedge x \notin C$ (by definition of set difference)

$x \in A \wedge x \notin B \wedge x \notin C$ (by definition of set difference) …… (i)

from (i), $x \in (A\text{-}C)$ (by definition of set difference) ……..(j)

from $x \notin B$, $x \notin (B\text{-}C)$ …….(k)

from (j) & (k), $x \in (A\text{-}C)\text{-}(B\text{-}C)$ (by definition of set difference)

$\therefore (A\text{-}B)\text{-}C \subseteq (A\text{-}C)\text{-}(B\text{-}C)$

Exercise 2.

2. Let A, B and C be sets.
   Show that $(A\text{-}B)\text{-}C = (A\text{-}C)\text{-}(B\text{-}C)$.

Answer

② $(A\text{-}C)\text{-}(B\text{-}C) \subseteq (A\text{-}B)\text{-}C$

suppose $x \in (A\text{-}C)\text{-}(B\text{-}C)$

$x \in (A\text{-}C)$ (by definition of set difference) ------------ (i) and

$x \notin (B\text{-}C)$ (by definition of set difference) ------------ (j)

from (i), $x \in A$ --- (k) $\wedge$ $x \notin C$ --- (l)

from (j) & (l), $x \notin B$ (by def. set difference) --------- (m)

from (k) & (m), $x \in (A\text{-}B)$ (by def. set difference) --------- (n)

from (n) & (l), $x \in (A\text{-}B)\text{-}C$ (by def. set difference)

$\therefore$ $(A\text{-}C)\text{-}(B\text{-}C) \subseteq (A\text{-}B)\text{-}C$

from ① & ②, $(A\text{-}B)\text{-}C = (A\text{-}C)\text{-}(B\text{-}C)$

Exercise 3.

3. Let A and B be two sets. Prove or disprove each of the followings
   (a) $P(A) \cup P(B) \subseteq P(A \cup B)$ where $P(A)$ is the power set of the set A.

Answer

Let $x \in (P(A) \cup P(B))$.

by definition of $\cup$, $(x \in P(A)) \vee (x \in P(B))$
by definition of power set, $(x \subseteq A) \vee (x \subseteq B)$

case 1) $x \subseteq A$:
by definition of $\cup$, $x \subseteq A \cup B$
by definition of power set, $x \in P(A \cup B)$

case 2) $x \subseteq B$:
by definition of $\cup$, $x \subseteq A \cup B$
by definition of power set, $x \in P(A \cup B)$

from case 1) & 2), $P(A) \cup P(B) \subseteq P(A \cup B)$

Exercise 3.

3. Let A and B be two sets. Prove or disprove each of the followings

(b) $P(A \cup B) \subseteq P(A) \cup P(B)$

Answer

(False)

$A = \{1\}, \quad B = \{2\}$

$P(A) = \{\varnothing, \{1\}\}$

$P(B) = \{\varnothing, \{2\}\}$

$P(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$

$P(A) \cup P(B) = \{\varnothing, \{1\}, \{2\}\}$

Exercise 4.

4. Which of the following are true for all sets, A, B, and C ? Give a counter example if the answer is false (No proof is necessary if the answer is true).

(a) If $A \cap B = \varnothing$ and $B \cap C = \varnothing$, then $A \cap C = \varnothing$.

Answer

(a) If $A \cap B = \varnothing$ and $B \cap C = \varnothing$, then $A \cap C = \varnothing$.

$\Rightarrow$ false

i) $A = \{1,2\}$, $B=\{3,4\}$, $C=\{1,5,6\}$

$A \cap B = \varnothing$, $B \cap C = \varnothing$, but $A \cap C = \{1\} \neq \varnothing$

ii) $A = C$

$A \cap C \neq \varnothing$

Exercise 4.

(b) If $A \in B$ and $\neg(B \subseteq C)$, then $\neg(A \in C)$.

(c) If $A \in B$ and $B \in C$, then $\neg(A \in C)$.

Answer

(b) If $A \in B$ and $\neg(B \subseteq C)$, then $\neg(A \in C)$.

$\Rightarrow$ false

i) if $B \cap C = \varnothing$: $\neg(A \in C)$

ii) if $A \in B \cap C$: $A \in C$

iii) $B = \{A, 1, 2\}$, $C=\{A, 2, 3\}$

$A \in C$

(c) If $A \in B$ and $B \in C$, then $\neg(A \in C)$.

$\Rightarrow$ false

$B = \{A\}$, $C=\{\{A\}, A\}$ 일 때 $A \in C$

Exercise 4.

(d) $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

(e) $\emptyset \in A$.

(f) If $A \subseteq B$ and $B \in C$, then $A \subseteq C$

(g) If $A \in B$, then $\{A\} \subseteq B$

Answer

(d) $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

$\Rightarrow$ true

(e) $\emptyset \in A$.

$\Rightarrow$ false

A = {1}

(f) If $A \subseteq B$ and $B \in C$, then $A \subseteq C$

$\Rightarrow$ false

A={1} B={1,2} C={{1,2}}

(g) If $A \in B$, then $\{A\} \subseteq B$

$\Rightarrow$ true

Discrete Mathematics Exam1 10/7/2014

1. (3) Define each of the following in propositional logic:
(a)Well-formed formula
(b)Tautology
(c)Valid consequence

2. (8) Given that L(x,y), M(x) and W(x) are statements for "x loves y", "x is a man", and "x is a woman", respectively, translate the following into the well-formed formulas.
(a)There exists a man who loves every woman.
(b)A woman who is loved by a man does not love him.
(c)Every man loves a woman
(d)Some men are loved by every woman.

3.(10) Let B(x) stand for "x is a boy", G(x) stand for "x is a girl", and T(x,y) stand for "x is taller than y", Complete the well-formed formula representing the given statement By filling out ? Part.
(a)Only girls are taller than boys: $(?)(\forall y)((? \wedge T(x,y)) \rightarrow ?)$
(b) Some girls are taller than boys: $(\exists x)(?)(G(x) \wedge (? \rightarrow ?))$
(c) Girls are taller than boys only: $(?)\ (\forall y)((\mathrm{G}(x) \wedge ?) \rightarrow ?)$
(d) Some girls are not taller than any boy: $(\exists x)(?)(G(x) \wedge (? \rightarrow ?))$
(e) No girl is taller than any boy: $(?)(\forall y)((B(y) \wedge ?) \rightarrow ?)$

4. (6) Prove formally the following using the inference rules and tautologies:

(a) $(P \rightarrow Q), (Q \rightarrow \neg R), R, (P \vee (J \wedge S)) \Rightarrow (J \wedge S))$

(b) $(\exists x)(\forall y)\ (S(x) \wedge (F(y) \rightarrow L(x,y))) \Rightarrow (\forall y)(\exists x)(F(y) \rightarrow (L(x,y) \wedge S(x))))$

5. (3) Prove informally that

For every three sets, A,B, and C, $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

Discrete Mathematics Exam1 10/7/2014

1. (3) Define each of the following in propositional logic:
(a)Well-formed formula
(b)Tautology
(c)Valid consequence

Answer
(a) Well-formed formula

Definition:

1. Any proposition variable is a wff.
2. For any wff P, $\neg P$ is a wff.
3. If P and Q are wffs, then $(P \wedge Q)$, $(P \vee Q)$, $(P \rightarrow Q)$ and $(P \leftrightarrow Q)$ are wffs.
4. A finite string of symbols is a wff only when it is constructed by steps 1, 2, and 3.

2. (8) Given that L(x,y), M(x) and W(x) are statements for “x loves y”, “x is a man”, and “x is a woman”, respectively, translate the following into the well-formed formulas.
(a)There exists a man who loves every woman.
(b)A woman who is loved by a man does not love him.
(c)Every man loves a woman
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(a)There exists a man who loves every woman.
(b)A woman who is loved by a man does not love him.
(c)Every man loves a woman
(d)Some men are loved by every woman.

Answer

(a) There exists a man who loves every woman.

$(\exists x)(\forall y)(\ )$
$(\exists x)(\forall y)(\ M(x) \wedge (W(y) \rightarrow L(x,y))\ )$

(b) A woman who is loved by a man does not love him.

$(\forall x)(\forall y)(\ )$
$(\forall x)(\forall y)(\ (W(x) \wedge L(y,x) \wedge M(y)) \rightarrow \neg L(x,y)\ )$
or $(\forall x)(\forall y)(\ (W(x) \wedge L(y,x)) \rightarrow (M(y) \rightarrow \neg L(x,y))\ )$
or $(\forall x)(\forall y)(\ W(x) \rightarrow (L(y,x) \rightarrow (M(y) \rightarrow \neg L(x,y))\ )\ )$

2. (8) Given that L(x,y), M(x) and W(x) are statements for "x loves y", "x is a man", and "x is a woman", respectively, translate the following into the well-formed formulas.

(a)There exists a man who loves every woman.

(b)A woman who is loved by a man does not love him.

(c)Every man loves a woman

(d)Some men are loved by every woman.

Answer

(c) Every man loves a woman

$(\forall x)(\exists y)(\ )$

$(\forall x)(\exists y)(M(x) \rightarrow (L(x,y) \wedge W(y)))$

(d) Some men are loved by every woman.

$(\exists x)(\forall y)(\ )$

$(\exists x)(\forall y)(M(x) \wedge (W(y) \rightarrow L(y,x)))$

4. (6) Prove formally the following using the inference rules and tautologies:

(a) $(P \rightarrow Q), (Q \rightarrow \neg R), R, (P \vee (J \wedge S)) \Rightarrow (J \wedge S))$

(b) $(\exists x)(\forall y)\ (S(x) \wedge (F(y) \rightarrow L(x,y))) \Rightarrow\ (\forall y)(\exists x)(F(y) \rightarrow (L(x,y) \wedge S(x))))$

Answer

(a) $(P \rightarrow Q), (Q \rightarrow \neg R), R, (P \vee (J \wedge S)) \Rightarrow (J \wedge S))$

1. R	P
2. $(Q \rightarrow \neg R)$	P
3. $R \rightarrow \neg Q$	T 2 and E18, E1
4. $\neg Q$	T 1,3 and I11
5. $(P \rightarrow Q)$	P
6. $\neg P$	T 4,5 and I12
7. $(P \vee (J \wedge S))$	P
8. $(J \wedge S)$	T 6.7 and I10

4. (6) Prove formally the following using the inference rules and tautologies:

(a) $(P \rightarrow Q), (Q \rightarrow \neg R), R, (P \vee (J \wedge S)) \Rightarrow (J \wedge S))$

(b) $(\exists x)(\forall y)\ (S(x) \wedge (F(y) \rightarrow L(x,y))) \Rightarrow (\forall y)(\exists x)(F(y) \rightarrow (L(x,y) \wedge S(x))))$

Answer

(b) $(\exists x)(\forall y)\ (S(x) \wedge (F(y) \rightarrow L(x,y))) \Rightarrow (\forall y)(\exists x)(F(y) \rightarrow (L(x,y) \wedge S(x))))$

1. $(\exists x)(\forall y)\ (S(x) \wedge (F(y) \rightarrow L(x,y)))$	P
2. $(\forall y)\ (S(a) \wedge (F(y) \rightarrow L(a,y)))$	ES 1
3. $(S(a) \wedge (F(y) \rightarrow L(a,y)))$	US 2
4. $F(y) \rightarrow L(a,y)$	T 3 and I2
5. $F(y)$	AP
6. $L(a,y)$	T 4,5 and I11
7. $S(a)$	T 3 and I1
8. $L(a,y) \wedge S(a)$	T 6,7 and I9
9. $F(y) \rightarrow (L(a,y) \wedge S(a))$	CP 5,8
10. $(\exists x)\ F(y) \rightarrow (L(x,y) \wedge S(x))$	EG 9
11. $(\forall y)(\exists x)\ F(y) \rightarrow (L(x,y) \wedge S(y))$	UG 10

5. (3) Prove informally that
For every three sets, A,B, and C, $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

Answer

1) $C \subseteq A$ if $(A \cap B) \cup C = A \cap (B \cup C)$

For arbitrary element x, Assume $x \in C$ …. (1)
From $x \in C$, by def. of $\cup$
$x \in (A \cap B) \cup C$
From $(A \cap B) \cup C = A \cap (B \cup C)$, by def. of =
$x \in A \cap (B \cup C)$
From $x \in A \cap (B \cup C)$, by def. of $\cap$
$x \in A$ …..(2)
From (1),(2), by def. of $\subseteq$
$C \subseteq A$

5. (3) Prove informally that

For every three sets, A,B, and C, $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

Answer

2) $C \subseteq A$ only if $(A \cap B) \cup C = A \cap (B \cup C)$

① **$(A \cap B) \cup C \subseteq A \cap (B \cup C)$ if** $C \subseteq A$

For arbitrary element x, Assume $x \in (A \cap B) \cup C$ …. (1)

From $x \in (A \cap B) \cup C$, by def. of $\cup$

$x \in (A \cap B) \vee x \in C$

From $x \in (A \cap B) \vee x \in C$, by def. of $\cap$

$((x \in A) \wedge (x \in B)) \vee x \in C$

From $((x \in A) \wedge (x \in B)) \vee x \in C$, by distributive

$((x \in A) \vee (x \in C)) \wedge ((x \in B) \vee (x \in C))$

Then from the given condition that $C \subseteq A$, $((x \in A) \vee (x \in A)) \wedge ((x \in B) \vee (x \in C))$

by def. of $\subseteq$

From $((x \in A) \vee (x \in A)) \wedge ((x \in B) \vee (x \in C))$, by idempotent, by def. of $\cup$

$(x \in A) \wedge (x \in B \cup C)$

From $(x \in A) \wedge (x \in B \cup C)$, be def. of $\cap$

$x \in A \cap (B \cup C)$ …. (2)

From (1),(2) be def. of $\subseteq$

$(A \cap B) \cup C \subseteq A \cap (B \cup C)$ …(3)

5. (3) Prove informally that
For every three sets, A,B, and C, $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

Answer

2) $C \subseteq A$ only if $(A \cap B) \cup C = A \cap (B \cup C)$

② **$A \cap (B \cup C) \subseteq (A \cap B) \cup C$ if** $C \subseteq A$
For arbitrary element x, Assume $x \in A \cap (B \cup C)$ …. (4)
From $x \in A \cap (B \cup C)$, by def. of $\cap$
$(x \in A) \wedge (x \in (B \cup C))$
From $(x \in A) \wedge (x \in (B \cup C))$, by def. of $\cup$
$(x \in A) \wedge ((x \in B) \vee (x \in C))$,
From $(x \in A) \wedge ((x \in B) \vee (x \in C))$, by distributive
$((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$
From $((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$,
$((x \in A) \wedge (x \in B)) \vee ((x \in C))$
From $((x \in A) \wedge (x \in B)) \vee ((x \in C))$, by idempotent, by def. of $\cup$ and $\cap$
$x \in (A \cap B) \cup C)$ …(5)
From (4),(5) by def. of $\subseteq$
$A \cap (B \cup C) \subseteq (A \cap B) \cup C)$ …(6)
From (3),(6) be def. of =
$(A \cap B) \cup C = A \cap (B \cup C)$