#1 Assume the induction statement is not valid so $\exists n > k, \neg P(n)$

Then since $P(k) \wedge P(k+1) \wedge \ldots \wedge P(k+b-1)$, at least one of $P(k), P(k+1)$, $P(k+b-1)$ must be false. But by well ordering prop, there must be a smallest element x_1. If we order the rest of the element $x_2, x_3 \ldots$ then the smallest element a exists in $\{x_k\}$

If $a > b$ then $P(a)$ is false because a is $\neg P(n)$

if $a < b$ then $P(a)$ is false, so we have a contradiction

if $\neg P(a)$ but a is the smallest element, one of $P(a-1), P(a-2) \ldots P(a-b)$ must be false so $a+1 < a$ but contradiction because a has to be the smallest element. Therefore we have a contradiction,

original statement is true.

https://math.stackexchange.com/questions/1403694/ordering-of-natural-numbers didnt neccessarily copy anything but took note in how to prove that n is a power of 2 suffices

#12 This case is only true when for number arrangement $x_1 \dots x_n$ for numbers $1, \dots, n$, when $2x_1, \dots, 2x_n, 2x_1-1, \dots 2x_n-1$ because mean of a $2x_a$ and $2x_b-1$ is not a natural number, $\frac{2x_a + 2x_b - 1}{2} \notin \mathbb{N}$

Therefore, it suffices to prove this fact when n is a power of 2

Basis step when $n=1$ where $n=$ it is possible for arrange $1, 2, 3, 4 \dots 2^n$ without average of two numbers in the arrangement

arrangement $= 1, 2$
mean $= 1.5$
never appears therefore true.

Assumption: Assume you can arrange when $n=k$,

Then to prove you can arrange for $n=k+1$

According to assumption 1, $a_1, a_2, \dots a_{2^k}$ is a right arrangement of $n=k$

then, $2^k+a_1, 2^k+a_2, \dots, 2^k+a_{2^k}$ is the arrangement of $2^k+1, 2^k+2 \dots, 2^{k+1}$

finally $a_1, a_2 \dots, a_{2^k}, 2^k+a_1, 2^k+a_2, \dots, 2^k+a_{2^k}$ is the arrangement for $1, 2 \dots, 2^{k+1}$.
for any 2 int $\leq k$, i and j $\frac{a_i + 2^k + a_j}{2} = \frac{a_i + a_j}{2} + 2^{k-1}$
Therefore, you can arrange for $n=k+1$,
by induction, it is possible for all positive integer n to arrange.

#3 Assuming $f(n)$ is breaks needed for n number of chocolate pieces,

Basis step for $n=1$ $f(n)=0$

for $n=2$ $f(2)=1$

Now for $f(j)=j-1$ for all $i \le j$

But $f(j+1)=j$

for size $j+1$, 1 break produces a piece + piece with $j-1$. Therefore

$f(j+1)=f(j)+1=(j-1)+1=j$

Since now we know $f(j+1)=j$

$f(n)=n-1$

Therefore, by induction, $n-1$ breaks are required for n pieces.

#4 $a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9\ a_{10}$

for consecutive 0s,

After filling $a_1 \sim a_5$ there is 2^5

After filling $a_2 \sim a_6$ there is 2^4

⋮

$a_6 \sim a_{10}$ there is 2^4

for consecutive 1s is also the same because it is symmetrical,

$$2(2^5 + 5 \times 2^4) = 224$$

However, subtract the case for 0000011111 and 1111100000

thus 222 cases.

#5 When $n = pqr$ and p, q, r are prime numbers.

if $A = \{1, 2, \dots, pqr\}$

$P = \{ P \in A \mid$ # of multiple of $p \}$ $|P| = qr$

$Q = \{ Q \in A \mid$ # of multiple of $q \}$ $|Q| = pr$

$R = \{ R \in A \mid$ # of multiple of $r \}$ $|R| = pq$

The set of relative prime $= A - (P + Q + R - ((\#$ divisible by $P) + (\#$ divisible by $Q) + (\#$ divisible by $R)) + pqr)$

$|A| / |P| = p$

$|A| / |Q| = q$

$\vdots$

$=$

$\left| \text{the set of relative prime} \right| = |A| - ((qr + pr + pq) - (p + q + r + 1))$

$pqr - (qr + pr + pq) + (p + q + r) - 1$

#6 Assuming we want to select n members from n math profs and n CS profs such that the chairperson is math,

Selecting the chairperson first, there are n math profs, so n ways to choose the chairperson but $\binom{2n-1}{n-1}$ to choose the rest of the member.

Therefore, total of $n \cdot \binom{2n-1}{n-1}$ to select a committee with math proff as the chairperson.

However, the committee can also be counted where we first count the members, then multiply the ways of finding a chairperson. Counting the members first, since there can be k math profs and $n-k$ CS profs, we can select in $\binom{n}{k}\binom{n}{n-k}$ ways.

but since $\binom{n}{n-k} = \binom{n}{k}$ $= \binom{n}{k}^2$ we can then choose the chairperson from k math profs k ways which is the sum from 1 to n.

Thus two ways of counting are the same $\therefore$

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

#7 when n out of b bins are filled,

the chance this will be the last required throw is

$\frac{b-n}{b}$

To be distributed equally, the probability

is $\frac{b}{b-n}$.

Therefore the expected # of toss is

$$b \sum_{n=1}^{b} \frac{1}{n} = b\left(1+\frac{1}{2} \cdots + \frac{1}{b}\right)$$

#8

a) The probability that Alice sends a 1 but is sent incorrect + probability Alice sends 0 correctly.

$$\frac{1}{3} \times \frac{1}{5} + \frac{2}{3} \times \frac{9}{10}$$

$$= \frac{2}{3}$$

b) Finding probability that 0 was received given that 0 was sent

0T = event 0 was transmitted
0R = event that 0 was received

1T = event 1 was transmitted
1R = event that 1 was received

$$P(0T \mid 0R) = \frac{P(0R \mid 0T) \cdot P(0T)}{P(0R)}$$

$$= \frac{\left(0.9 \times \frac{2}{3}\right) \cdot \frac{2}{3}}{\frac{2}{3} \times 0.9 + \frac{1}{3} \times 0.2} = 0.d$$

#9

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

$\mu = \frac{n}{5}$ $\qquad$ $k = \sqrt{n}$

$$\sigma^2 = \frac{n}{5} \cdot \frac{4}{5} = \frac{4n}{25}$$

$$\leq \frac{4n}{25}\left(\frac{1}{n}\right)$$

upper bound $= \frac{4}{25}$

#10 LHS

$V(X_1+X_2+\ldots+X_n)$

$E((X_1+X_2+\ldots+X_n)^2) = E(X_1^2)+E(X_2^2)+\ldots+E(X_n^2)$

$E[(X_1+X_2+\ldots+X_n)^2]$

$$= \sum_{1\le i\le n} E[X_i^2] + \sum_{\substack{i\ne j,\\ 1\le i,\ j\le n}} E(X_iX_j)$$

However, since pairwise independence means linearity

$E(X_iX_j) = 0\cdot 0 = 0$

$\sum_{1\le i\le n} E(X_i^2)$ $= E(X_1^2)+E(X_2^2)\ldots E(X_n^2)$