CSE 4190.101: Discrete Mathematics Fall 2021

Problem Set 3

Instructor: Yongsoo Song **Due on:** Nov 29, 2021

Please submit your answer through eTL. It should be a single PDF file (not jpg), either typed or scanned. **Please include your student ID and name (e.g. 2020-12345 YongsooSong.pdf).** You may discuss with other students the general approach to solve the problems, but the answers should be written in your own words.

- You should cite any reference that you used, and mention what you used it for.
- The reference information should be specific so that TAs are able to find the exact material you used. For example, it is not allowed to simply mention that "I referred a lecture note of Discrete Mathematics class at * university".
- Similarly, if your reference includes a url, type it or submit a separate text file (instead of handwritten address) so that TAs can easily visit the page.
- All references should be publicly accessible. Otherwise, attach the reference to your submission.

Problem 1 (10 points)

Let b be a positive integer. Use the well-ordering property to show that the following form of mathematical induction is a valid method to prove that $P(n)$ is true for all positive integers n.

- **Basis Step:** $P(1), P(2), \ldots, P(b)$ are true.
- **Inductive Step:** For each positive integer k, if $P(k) \wedge P(k+1) \wedge \cdots \wedge P(k+b-1)$ is true, then $P(k+b)$ is true.

Problem 2 (10 points)

Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the average of any two of these numbers never appears between them. [Hint: Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

Problem 3 (10 points)

Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, or any smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

Problem 4 (10 points)

How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Problem 5 (10 points)

Suppose that p, q and r are prime numbers and that $n = pqr$. Use the principle of inclusion–exclusion to find the number of positive integers not exceeding n that are relatively prime to n.

Problem 6 (10 points)

Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

[Hint: Count in two ways the number of ways to select a committee, with n members from a group of n mathematics professors and n computer science professors, such that the chairperson of the committee is a mathematics professor.]

Problem 7 (10 points)

Suppose that balls are tossed into b bins one by one so that each ball is equally likely to fall into any of the bins and that the tosses are independent. What is the expected number of balls tossed until every bin contains a ball?

Problem 8 (10 points)

Alice and Bob communicate using bit strings. Suppose that Alice sends a 1 one-third of the time and a 0 two-thirds of the time. When a 0 is sent, the probability that it is received correctly is 0.9 (so that the probability that it is received incorrectly as a 1 is 0.1). When a 1 is sent, the probability that it is received correctly is 0.8.

1. Find the probability that a 0 is received.
2. Use Bayes' theorem to find the probability that a 0 was transmitted, given that a 0 was received.

Problem 9 (10 points)

Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a biased coin with probability of heads equal to 0.8 is tossed n times deviates from the mean by more than $\sqrt{n}$.

Problem 10 (10 points)

Let $X_1, X_2, \ldots, X_n$ be pairwise independent random variables on a sample space S. Show that

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n).$$