The Rendering Equation

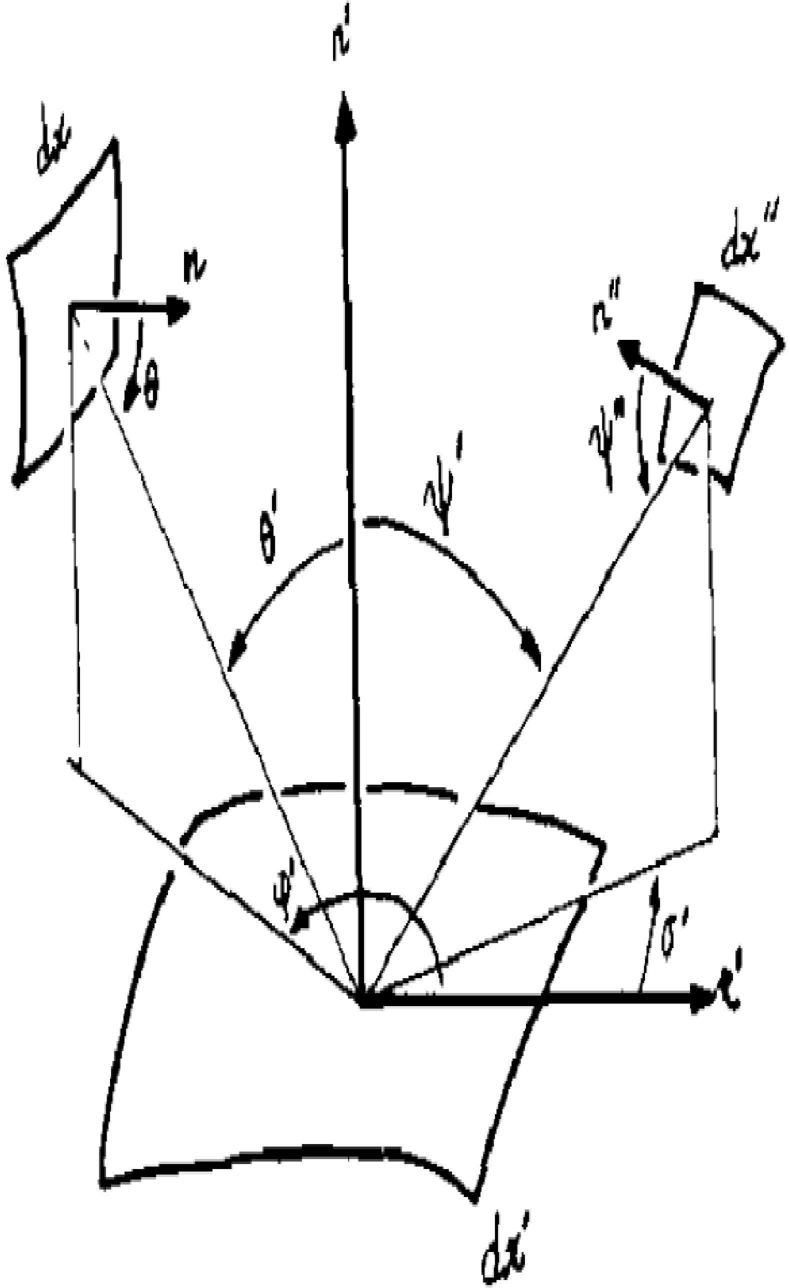
James T. Kajiya

Taekyu Shin

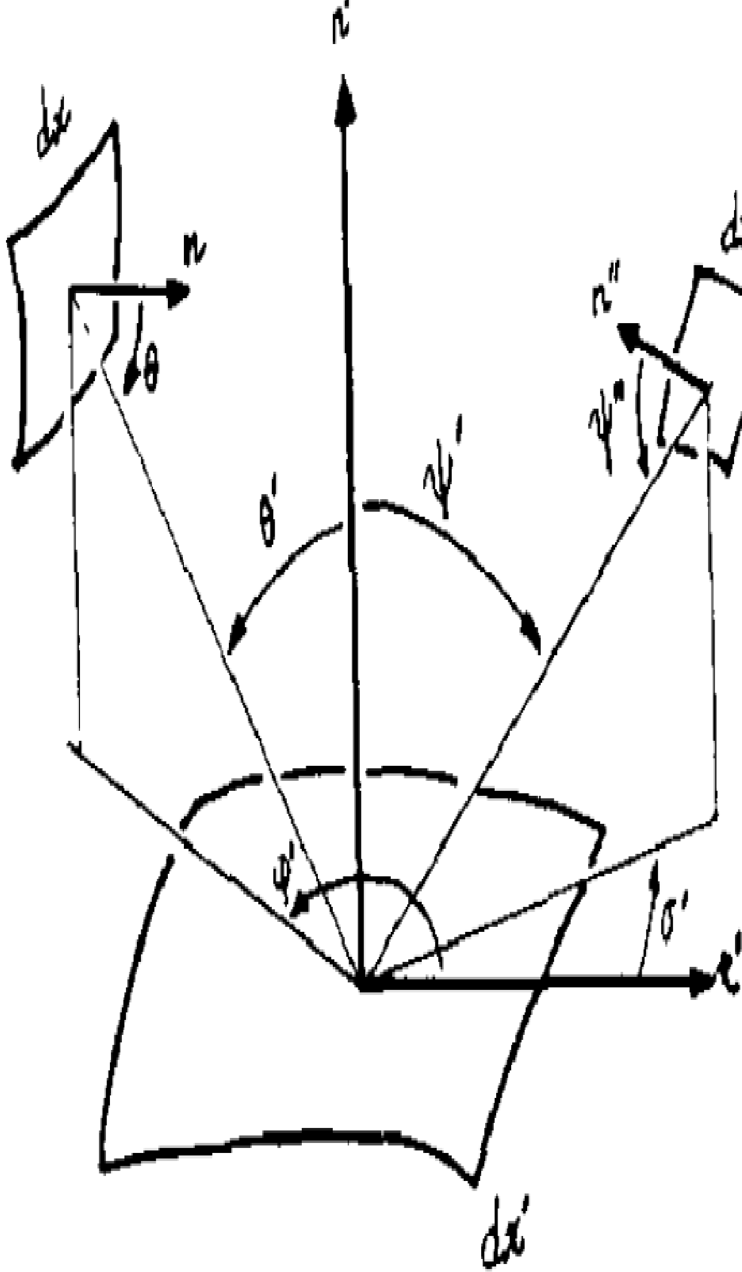
The Rendering Equation -What?

$$I(x, x') = g(x, x') \left[\epsilon(x, x') + \int_S \rho(x, x', x'') I(x', x'') dx'' \right]$$

$I(x, x')$	is the related to the intensity of light passing from point x' to point x
$g(x, x')$	is a "geometry" term
$\epsilon(x, x')$	is related to the intensity of emitted light from x' to x
$\rho(x, x'x'')$	is related to the intensity of light scattered from x'' to$x$ by a patch of surface at x'



- Assume that your eye is at x plane
  - How would you measure?
  - You are looking at x’



- Sum up these two
  - light from x’ itself
  - Light reflected from x’ coming from x’’

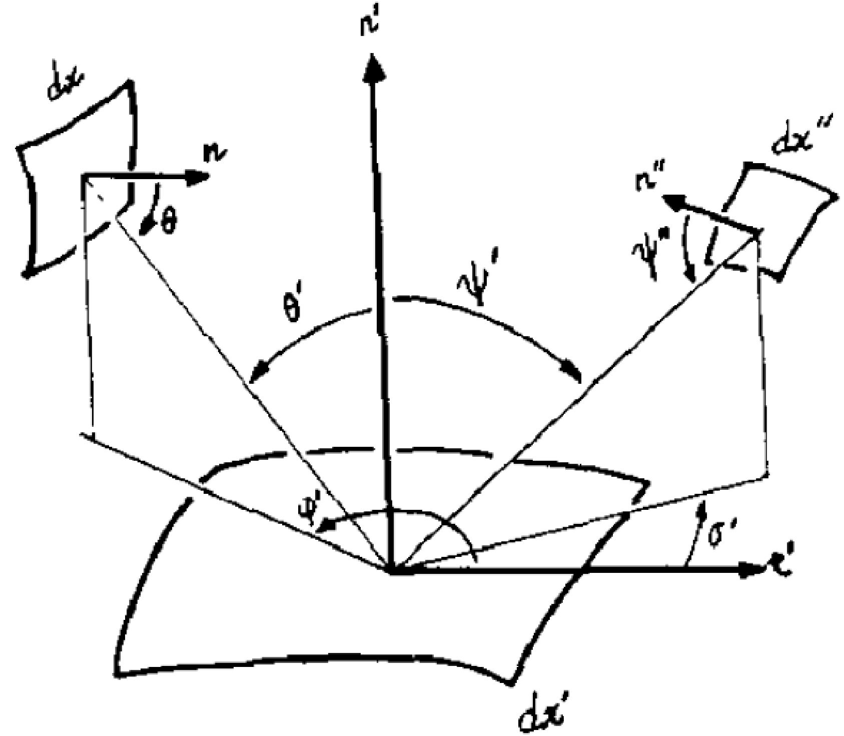
That is this

$$I(x, x') = g(x, x') \left[\epsilon(x, x') + \int_S \rho(x, x', x'') I(x', x'') dx'' \right]$$

- Sum up these two
  - light from x' itself
  - Light reflected from x' coming from x''

$$I(x,x') = g(x,x')\left[\epsilon(x,x') + \int_S \rho(x,x',x'')I(x',x'')dx''\right]$$

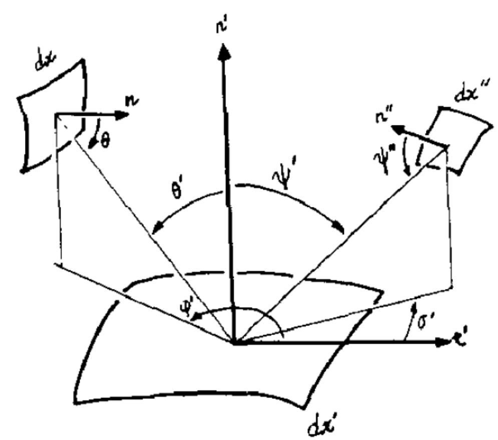
- P(x,x',x'') is BRDF term
- I(x', x'') – light intensity coming from x'' plane.



Energy Transport Function

$$I(x,x') = g(x,x')\left[\epsilon(x,x') + \int_S \rho(x,x',x'')I(x',x'')dx''\right]$$

$$dE = I(x,x')\,dt\,dx\,dx'$$



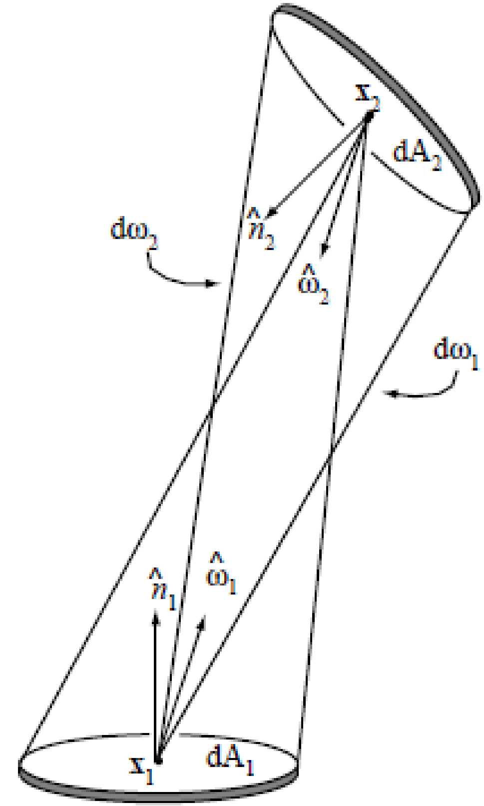
$$dE = \frac{1}{r^2}\epsilon(x,x')dt\,dx\,dx'$$

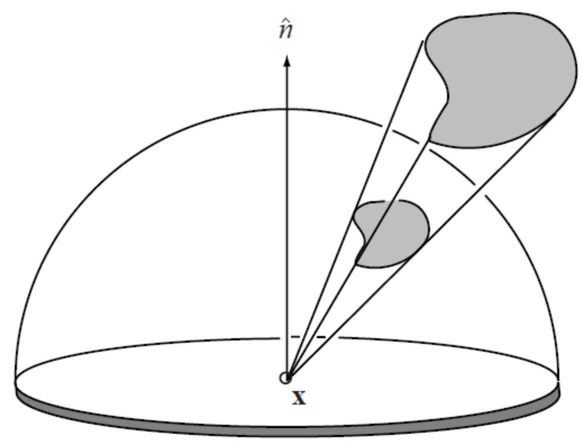
$$dE = \frac{1}{r^2}\rho(x,x',x'')I(x',x'')\,dt\,dx\,dx'\,dx''$$

- We can re-define I
  - in terms of angles
  - dw ➔ picture

$$I(x', x'')$$

$$dE = \boxed{i(\theta', \phi')}\boxed{d\omega}\, dx'_p\, dt$$

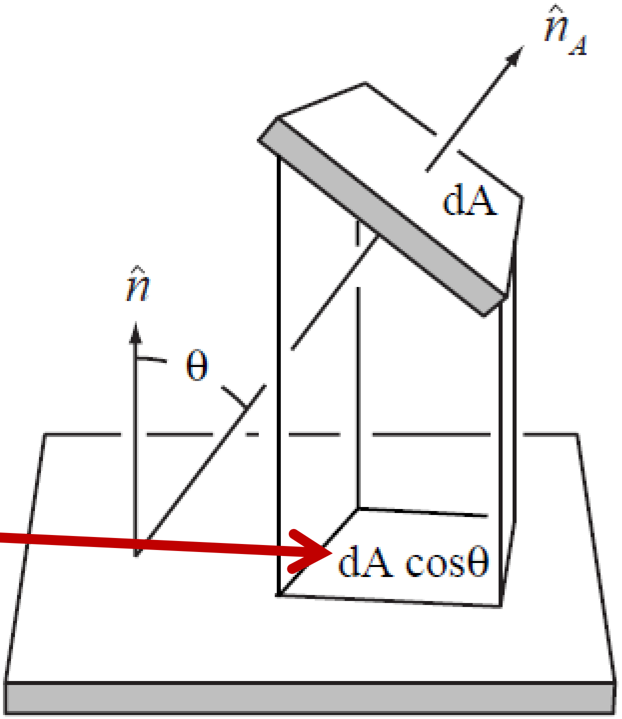


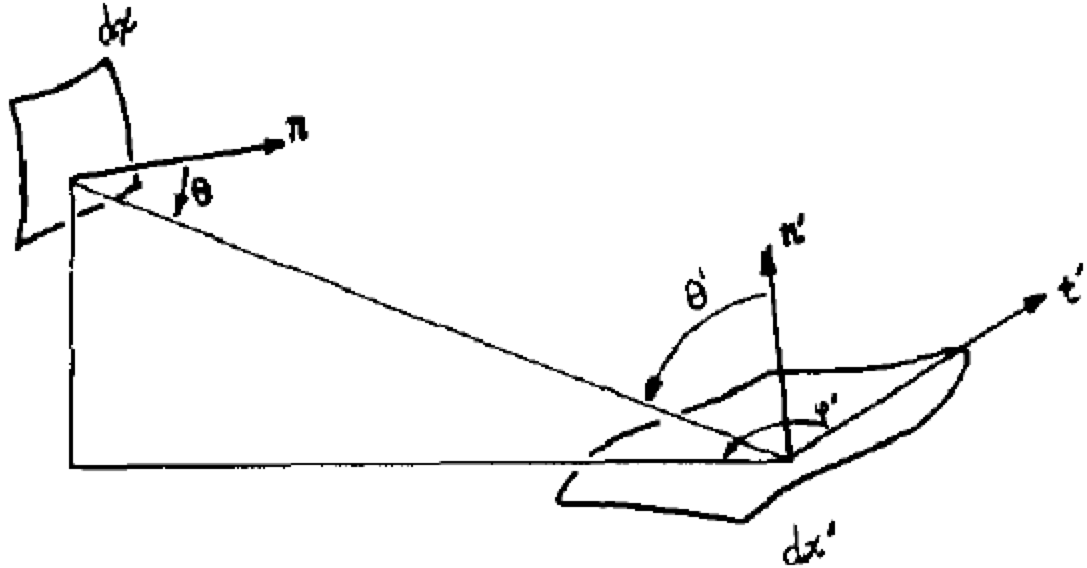


Emitting surface area – projected A

$$dE = i(\theta', \phi')\, d\omega\, \boxed{dx'_p}\, dt$$

dx'_p





$$r = \|x - x'\|$$

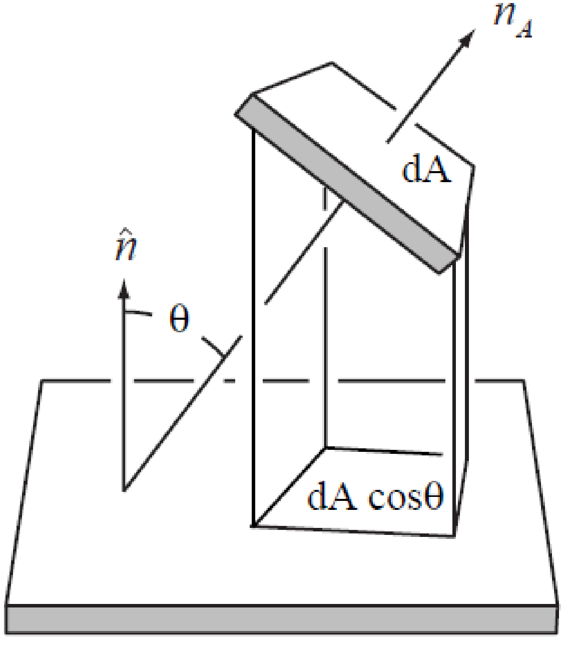
$$dx'_{\mathbf{p}} = dx' \cos\theta$$

$$\cos\theta = \frac{1}{r}\langle \mathbf{n}, x - x' \rangle$$

$$\cos\theta' = \frac{1}{r}\langle \mathbf{n}', x - x' \rangle$$

$$\cos\phi' = \frac{1}{r}\langle \mathbf{t}', x - x' \rangle$$

$$d\omega = \frac{dx_p}{r^2} = \frac{1}{r^2}\cos\theta\, dx$$



Induction

$$dE = i(\theta', \phi') d\omega \, dx'_p \, dt$$

$$d\omega = \frac{dx_p}{r^2} = \frac{1}{r^2} \cos\theta \, dx$$

$$dE = i(\theta', \phi') \frac{1}{r^2} \cos\theta \cos\theta' dt \, dx \, dx'$$

Equating two terms

$$dE = i(\theta', \phi') d\omega \, dx'_p \, dt$$

$$dE = \frac{1}{r^2} \epsilon(x, x') dt \, dx \, dx'$$

- We do that about just $\varepsilon(\theta', \phi')$

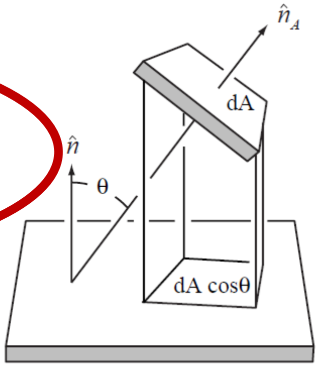
$$I(x, x') = g(x, x') \left[\epsilon(x, x') + \int_S \rho(x, x', x'') I(x', x'') dx'' \right]$$

$$\frac{1}{r^2}\varepsilon(x,x')dt\,dx\,dx' = \varepsilon(\theta',\phi')\,dw\,dx'_p\,dt$$

$$d\omega = \frac{dx_p}{r^2} = \frac{1}{r^2}\cos\theta\,dx$$

$$\frac{1}{r^2}\varepsilon(x,x')dt\,dx\,dx' = \varepsilon(\theta',\phi')\frac{1}{r^2}\cos\theta\,dx\,dx'_p\,dt$$

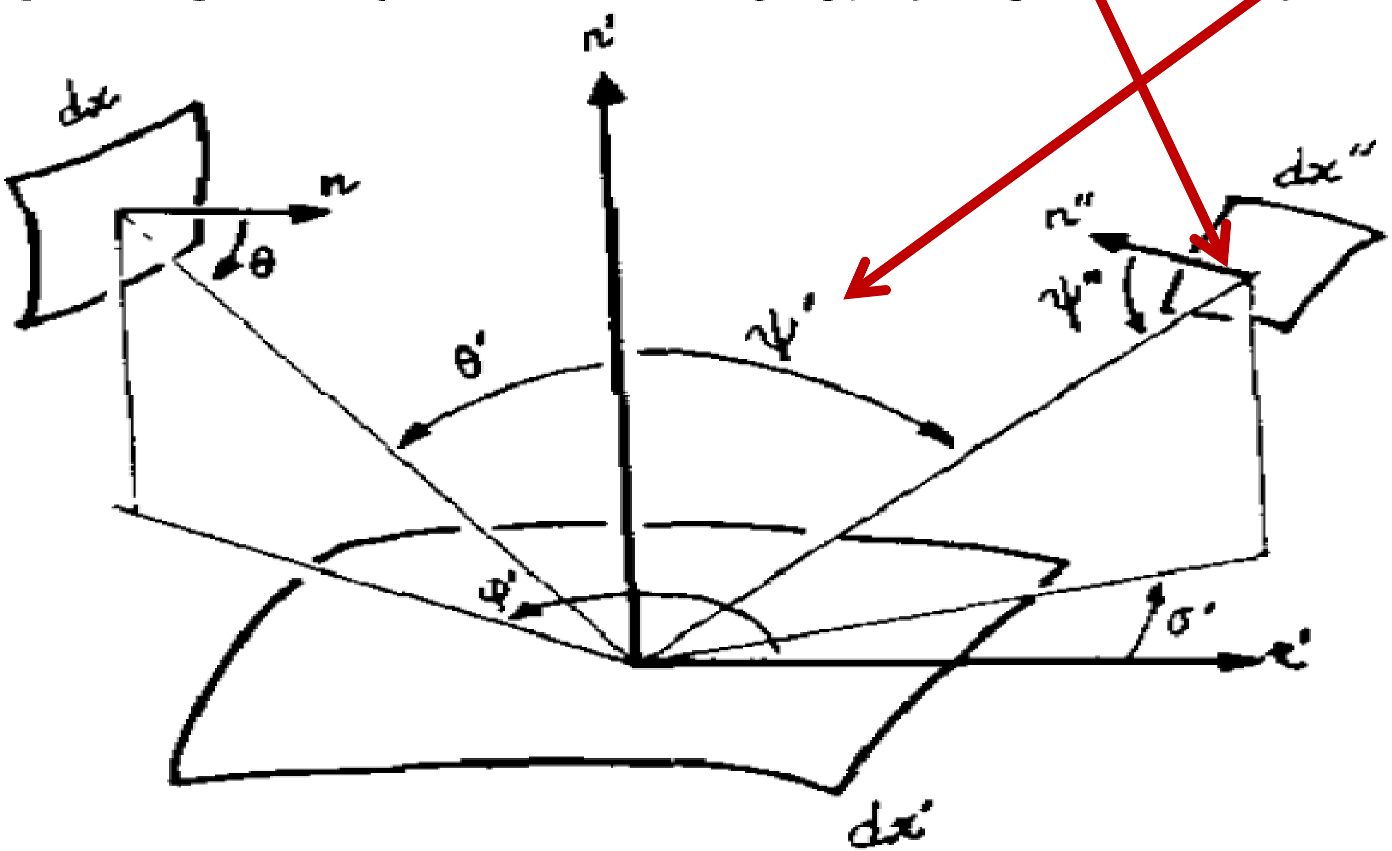
$$\frac{1}{r^2}\varepsilon(x,x')\;dx' = \varepsilon(\theta',\phi')\frac{1}{r^2}\cos\theta\;dx'_p$$

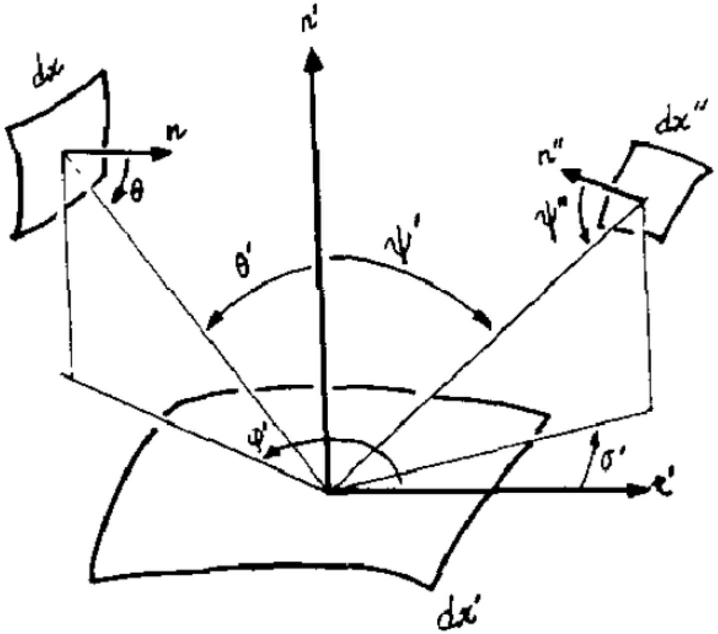


$$\varepsilon(x,x') = \varepsilon(\theta',\phi')\cos\theta\;\cos\theta'$$

- How about reflectance terms?

$$i(\theta', \phi') = \rho(\theta', \phi', \psi', \sigma')\, i(\psi', \sigma')\, d\omega'' \cos\psi'$$





$$r'' = \|x' - x''\|$$

$$dx''_p = dx'' \cos\psi''$$

$$\cos\psi' = \frac{1}{r''}\langle \mathbf{n}', x' - x''\rangle$$

$$\cos\psi'' = \frac{1}{r''}\langle \mathbf{n}'', x' - x''\rangle$$

$$\cos\sigma' = \frac{1}{r''}\langle \mathbf{t}', x' - x''\rangle$$

$$d\omega'' = \frac{dx''_p}{r''^2} = \frac{1}{r''^2}\cos\psi'' dx''$$

- We already derived :

$$\varepsilon(x, x^{'}) = \varepsilon(\theta^{'}, \phi^{'}) \cos\theta \ \cos\theta^{'}$$

- You see the pattern?

$$\rho(x, x', x'') = \rho(\theta', \phi', \psi', \sigma') \cos\theta \cos\theta'$$

$$I(x,x') = g(x,x')\left[\epsilon(x,x') + \int_S \rho(x,x',x'')I(x',x'')dx''\right]$$

$$I = g\epsilon + gMI$$

$$(1 - gM)I = g\epsilon$$

- What is Newmann Seris? $(\mathrm{Id} - T)^{-1} = \sum_{n=0}^{\infty} T^n$
- Convergence=invertible == This case

$$I = (1 - gM)^{-1} g\epsilon$$
$$= g\epsilon + gMg\epsilon + gMgMg\epsilon + g(Mg)^3\epsilon \cdots$$

- We can solve this! ➔ Wait a minute. Is it?

Utah approximation

$$
\begin{aligned}
I &= (1 - gM)^{-1} g\epsilon \\
&= g\epsilon + gMg\epsilon + gMgMg\epsilon + g(Mg)^3\epsilon \cdots
\end{aligned}
$$

$$I = g\epsilon + gM\epsilon_0$$

Ray tracing approximation

- Turner Whitted proposed

$$I = g\epsilon + gM_0 g\epsilon_0 + gM_0 gM_0 g\epsilon_0 + \cdots$$

The reflection and refraction of his lighting model

first model very similar to blinn phong

second model more complicated

Distributed Ray Tracing Approximation

Distribution Ray Tracing

$L_o(\theta_o,\phi_o) =$

$(1/\Omega_{cone})\int k_s L_{sp}(\theta_{sp},\phi_{sp})\ d\omega +$

$k_d \int \cos\theta_{so} \cos\theta_{fs} L_{e,so}/r^2\ dA + k_a L_a$

The Radiosity Approximation

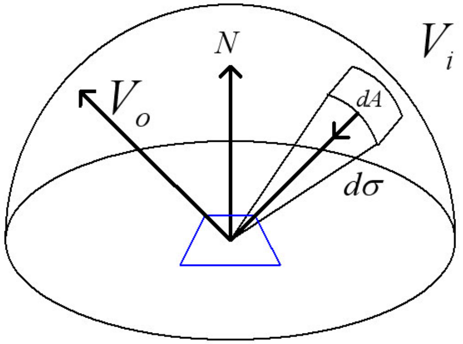
$$\rho(\theta', \phi', \psi', \sigma') = \rho_0$$

$$\begin{aligned} dB(x') &= dx' \int_{hemi} i(\theta', \phi') \cos\theta' \, d\omega \\ &= dx' \int_{hemi} \frac{I(x, x') r^2}{\cos\theta} d\omega \\ &= dx' \int_S I(x, x') \, dx \end{aligned}$$

$$dB(x') = dx' \int_{hemi} i(\theta', \phi') \cos\theta' \, d\omega$$

$$= dx' \int_{hemi} \frac{I(x, x') r^2}{\cos\theta} d\omega$$

$$= dx' \int_S I(x, x') dx$$



$$d\omega = \frac{dx_p}{r^2} = \frac{1}{r^2} \cos\theta \, dx$$

$$I(x, x') = i(\theta', \phi') \frac{1}{r^2} \cos\theta \cos\theta'$$

$$I(x,x') = g(x,x')\left[\epsilon(x,x') + \int_S \rho(x,x',x'')I(x',x'')dx''\right]$$

$$dx' \int_S I(x,x')dx$$

$$dB(x') = dx' \int \Big\{ g(x,x')\epsilon(x,x')$$

$$+ g(x,x') \int \rho(x,x',x'')I(x',x'')dx'' \Big\} dx$$

- g(x,x') = 1

No occlusion

$$
\begin{aligned}
dB_e(x') &= dx' \int \frac{\epsilon(x, x')}{r^2} dx \\
&= dx' \int \epsilon(\theta', \phi') \cos\theta' \frac{\cos\theta \, dx}{r^2} \\
&= dx' \int \epsilon(\theta', \phi') \cos\theta' d\omega \\
&= dx' \pi \epsilon_0
\end{aligned}
$$

$$d\omega = \frac{dx_p}{r^2} = \frac{1}{r^2} \cos\theta \, dx$$

$$\varepsilon(x, x') = \varepsilon(\theta', \phi') \cos\theta \ \cos\theta'$$

ϵ_0 is a constant representing hermispherical emittance

$$
\begin{aligned}
dB_r(x') &= dx' \int \frac{1}{r^2} \int \rho(x, x', x'') I(x', x'') dx'' dx \\
&= dx' \int \frac{1}{r^2} \rho(\theta', \phi', \psi', \sigma') \cos\theta \cos\theta' dx \\
&\quad \times \int I(x', x'') dx'' \\
&= dx' \rho_0 \int \cos\theta \, d\omega \int I(x', x'') dx'' \\
&= dx' \rho_0 \pi H(x')
\end{aligned}
$$

Sum up emittance and reflectance

$$I(x,x') = g(x,x')\left[\epsilon(x,x') + \int_S \rho(x,x',x'')I(x',x'')dx''\right]$$

$dx'\pi\epsilon_0$ $\qquad$ $dx'\rho_0\pi H(x')$

$$dB(x') = \pi[\epsilon_0 + \rho_0 H(x')]dx'$$

- The point of this equation.
  - We need to get H(x’), then we can get radiosity.

Markov chains for solving integral equations

$$x = a + Mx$$

- (I-M)x = a
- X=(I-M)^(-1) * a

$$x = \sum_{k=0}^{\infty} M^k a$$

$$x = a + \sum_{k=1}^{\infty} M^k a$$

- Previously

$$\begin{aligned} I &= (1 - gM)^{-1} g\epsilon \\ &= g\epsilon + gMg\epsilon + gMgMg\epsilon + g(Mg)^3\epsilon \cdots \end{aligned}$$

Finite Dimensional Distribution

- Non-deterministic process

$$x = a + \sum_{k=1}^{\infty} M^k a$$

$$\hat{x}_1 = \left(\prod_{i=0}^{l(\omega)} m_{n_{i-1} n_i}\right) a_{n_{l(\omega)}} \frac{1}{p(\omega)}$$

$$\mathbb{P}^X_{i_1 \dots i_k}(A_1 \times \cdots \times A_k) := \mathbb{P}\left\{\omega \in \Omega \,\middle|\, X_{i_j}(\omega) \in A_j \text{ for } 1 \leq j \leq k\right\}.$$

using Markov Chain

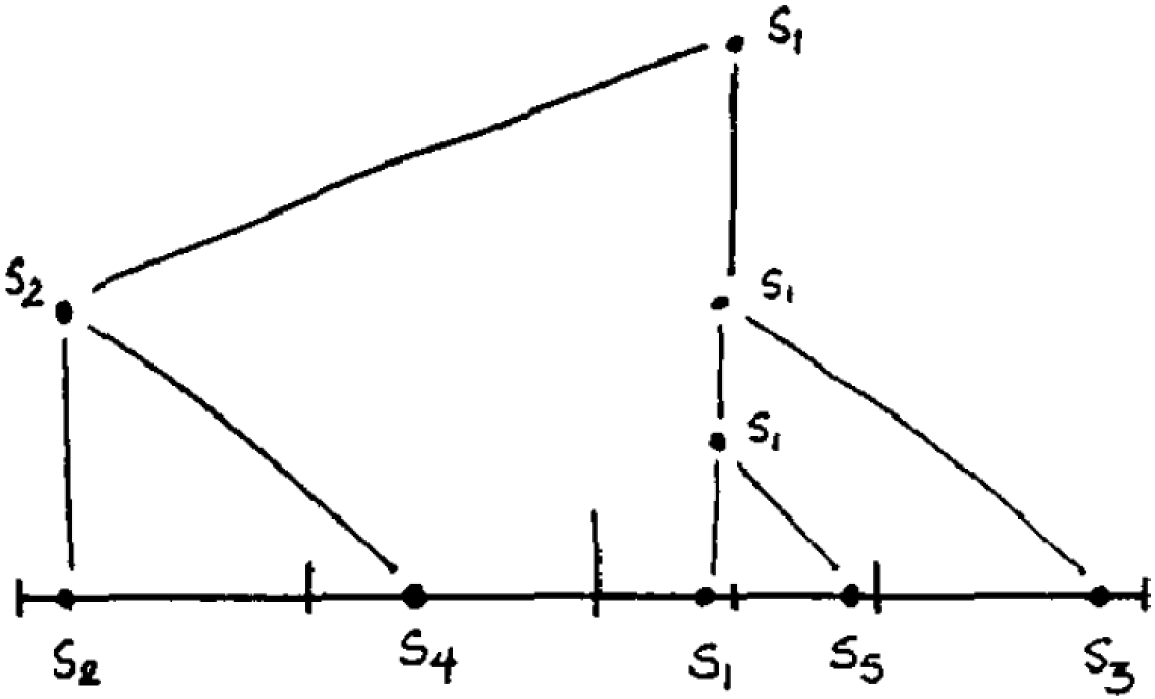
- We can get I from Markov chains

$$\hat{x}_1 = \left(\prod_{i=0}^{l(\omega)} m_{n_{i-1} n_i}\right) a_{n_{l(\omega)}} \frac{1}{p(\omega)}$$

$$p(\omega) = p(x_{l(\omega)}, x_{l(\omega)-1}) \cdots p(x_2, x_1) \cdot p(x_1, x_0) \cdot p(x_0)$$

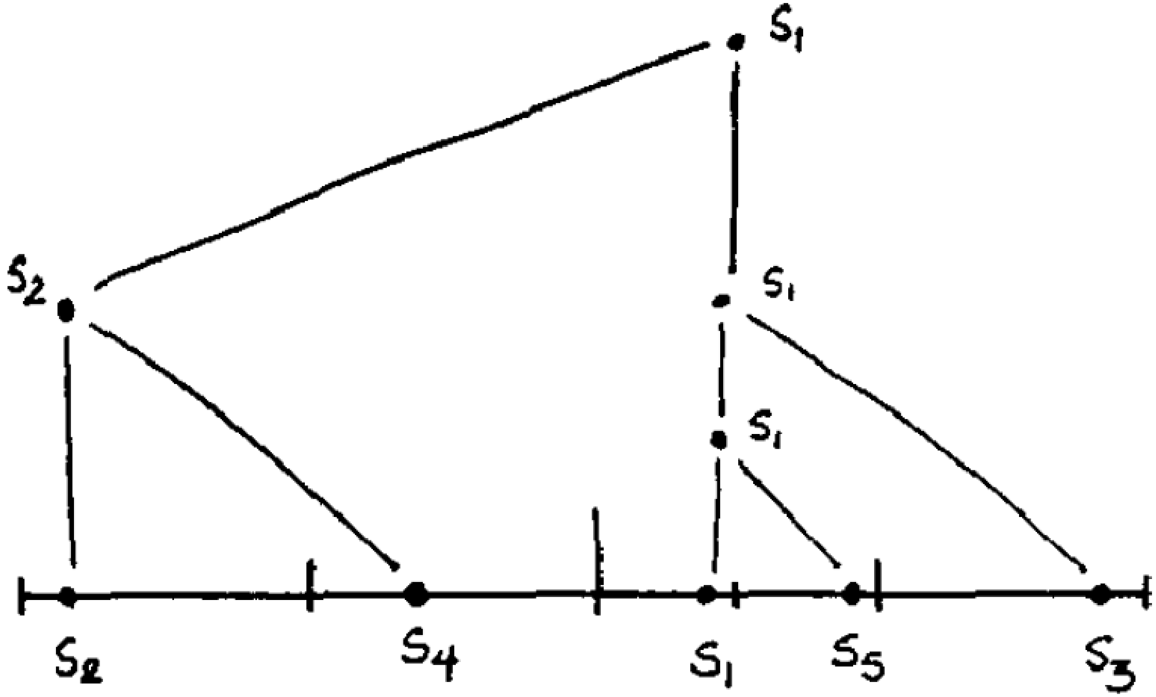
$$I(x, x') = g(x, x') \left[\epsilon(x, x') + \int_S \rho(x, x', x'') I(x', x'') dx''\right]$$

Hierachical sampling for one pixel



- Stratified scheme(one jittered sample per sub-grid)
- After taking s1 sample, now we know that we will take 5 samples
  - Sequential Uniform Sampling

Hierachical Integration (log instead of polynomials)

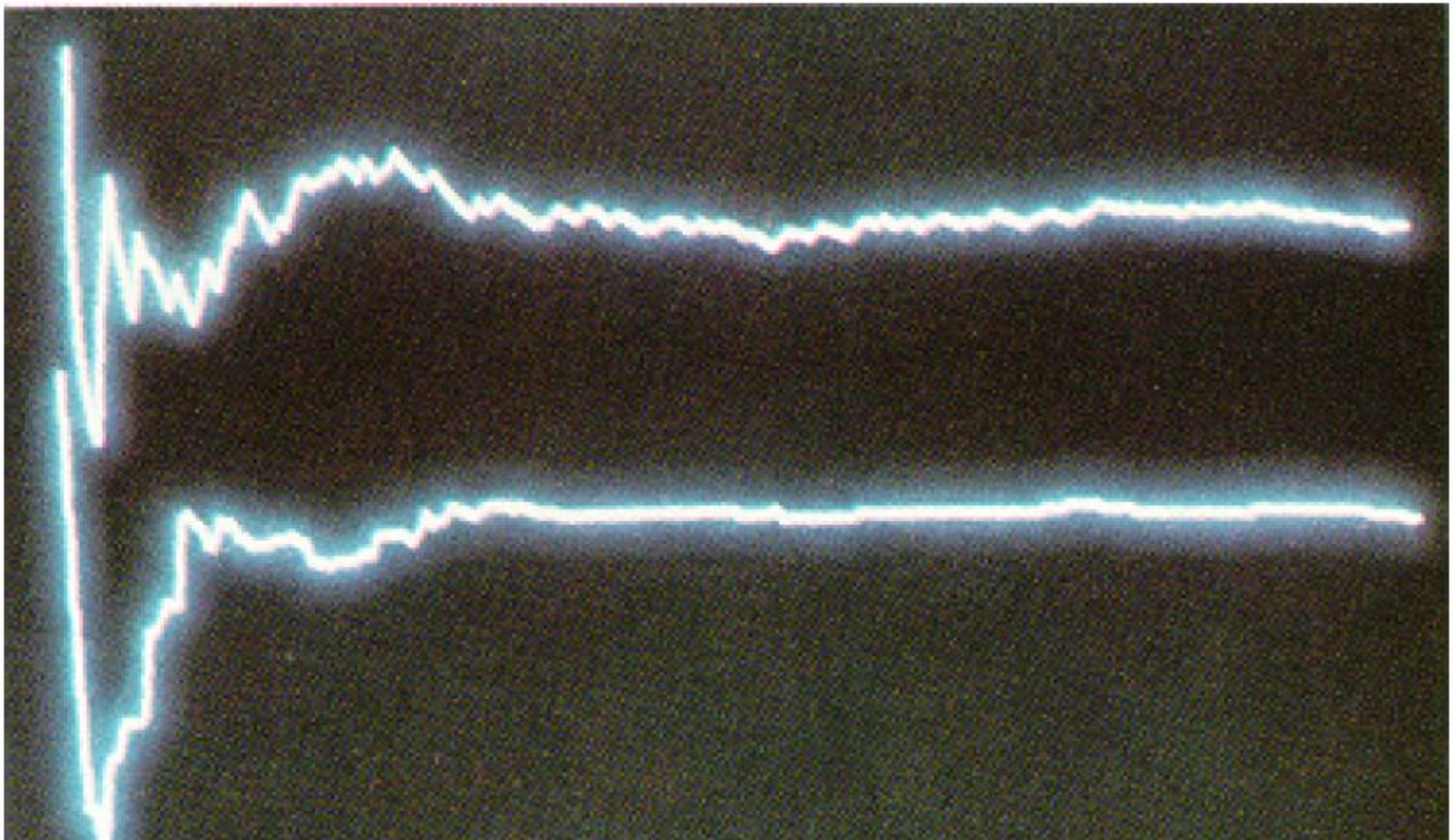


- S1 → S2 → (S1 + S2) * 0.5 → S3 →

  ((S1 * 0.5 + S2 * 0.5) + S3)* 0.5 +

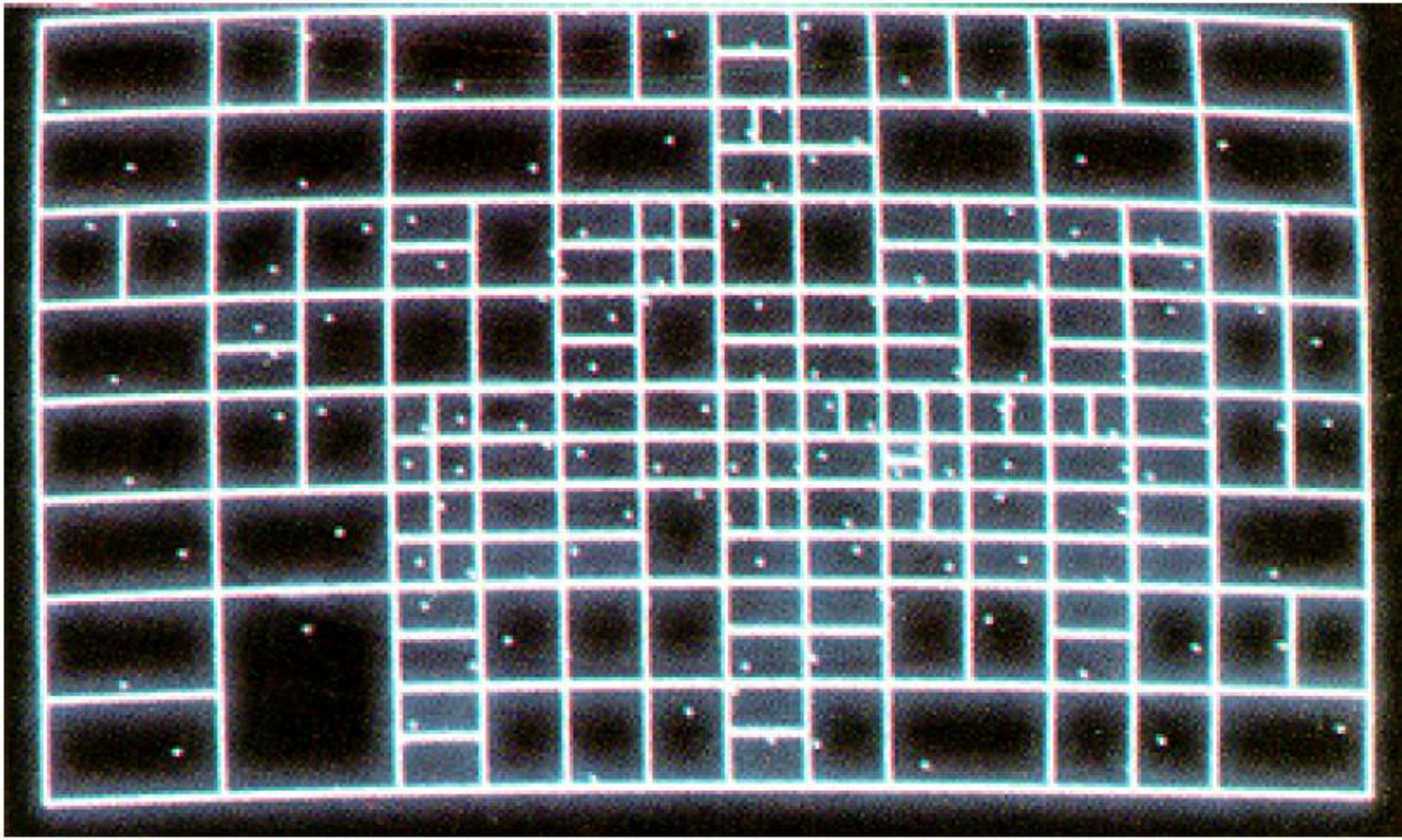
  --- Why better than monte carlo integration?

Hierachical Integration



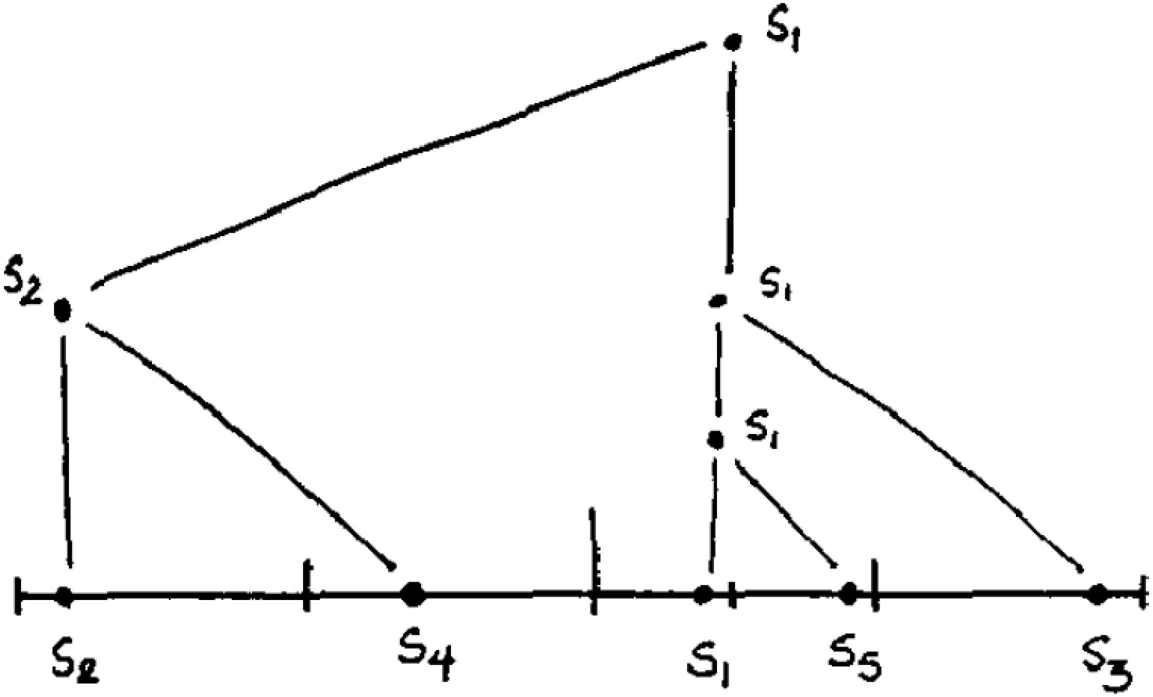
- Monte Carlo Integration VS Hierachical Integration
  - Hierachical Integration converges faster

Hierachical Integration



- Monte Carlo Integration VS Hierachical Integration
  - Hierachical Integration converges faster

Adaptive Hierachical Integration

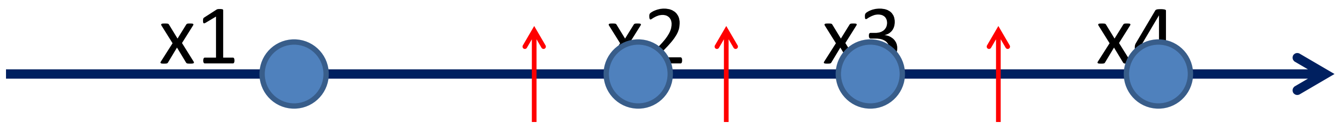


- Weighted average

Non-uniform sampling

- Assume some probability density function p(x)

p(x1) = p(x2)= p(x3) = p(x4) = 0.25



Raytracing VS Rendering Equation