1loop Results (Work in Progress)

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Contents

1 Useful Formulae

1.1 Product of differentials on the torus and the sphere

For a product of differentials $\prod_{i=2}^{n} dz_i$ on the torus, we have in the sphere degenertaion limit

$$\prod_{i=2}^{n} dz_i = \prod_{i=2}^{n}\left(d\sigma_i \frac{\sigma_{+-}}{\sigma_{i+}\sigma_{i-}}\right) \tag{1}$$

$$= \left(\prod_{i=2}^{n} d\sigma_i\right)\frac{\sigma_{1+}\sigma_{1-}}{\sigma_{+-}}\left(\prod_{i=1}^{n}\frac{\sigma_{+-}}{\sigma_{i+}\sigma_{i-}}\right) \tag{2}$$

where σ_i, σ_+ and σ_- are points on the sphere. We also used the notation $\sigma_{ij} = \sigma_i - \sigma_j$.

Using the following identity (check $(-1)^n$ factor?, probably wrong)

$$\prod_{i=1}^{n}\frac{\sigma_{+-}}{\sigma_{i+}\sigma_{i-}} = (-1)^n \sum_{\rho\in S_n}\frac{\sigma_{+-}}{\sigma_{+\rho(1)}\sigma_{\rho(1)\rho(2)}\cdots\sigma_{\rho(n-1)\rho(n)}\sigma_{\rho(n)-}} \tag{3}$$

where S_n is all permutations of the set $\{12...n\}$, equation (2) becomes

$$\prod_{i=2}^{n} dz_i = (-1)^n\left(\prod_{i=2}^{n} d\sigma_i\right)\frac{\sigma_{1+}\sigma_{1-}}{\sigma_{+-}}\sigma_{-+}\sum_{\rho\in S_n}\frac{\sigma_{+-}}{\sigma_{+\rho(1)}\sigma_{\rho(1)\rho(2)}\cdots\sigma_{\rho(n-1)\rho(n)}\sigma_{\rho(n)-}\sigma_{-+}} \tag{4}$$

$$= (-1)^n\left(\prod_{i=2}^{n} d\sigma_i\right)\sigma_{+-}\sigma_{-1}\sigma_{1+}\sum_{\rho\in S_n}\frac{1}{\sigma_{+\rho(1)}\sigma_{\rho(1)\rho(2)}\cdots\sigma_{\rho(n-1)\rho(n)}\sigma_{\rho(n)-}\sigma_{-+}} \tag{5}$$

Note that the factors preceding the sum in equation (5) are related to $\mathrm{vol}\,\mathrm{SL}(2,C)$ and the marked points σ_+ and σ_- (? check this) by

$$\left(\prod_{i=2}^{n} d\sigma_i\right)\sigma_{+-}\sigma_{-1}\sigma_{1+} = \frac{\prod_{A=\{1,\cdots,n,+,-\}} d\sigma_A}{\mathrm{vol}\,\mathrm{SL}(2,C)} \tag{6}$$

Thus in general the product of differentials $\prod_{i=2}^{n} dz_i$ in the sphere degerenation limit is given by

$$\prod_{i=2}^{n} dz_i = \frac{\prod_A d\sigma_A}{\mathrm{vol}\,\mathrm{SL}(2,C)}(-1)^n\sum_{\rho\in S_n}\frac{1}{\sigma_{+\rho(1)}\sigma_{\rho(1)\rho(2)}\cdots\sigma_{\rho(n-1)\rho(n)}\sigma_{\rho(n)-}\sigma_{-+}} \tag{7}$$

$$= \frac{\prod_A d\sigma_A}{\mathrm{vol}\,\mathrm{SL}(2,C)}(-1)^n\sum_{\rho\in S_n}\frac{1}{(+\rho(1)\cdots\rho(n)-)} \tag{8}$$

where we used the notation

$$(+\rho(1)\cdots\rho(n)-) \coloneqq \sigma_{+\rho(1)}\cdots\sigma_{\rho(n)-}\sigma_{-+} \tag{9}$$

1.2 General formula for the half integrand I_n

Schematically, the amplitude from the CHY formula is written as

$$A_{\mathrm{CHY}} = \int d\mathcal{M}_{\mathrm{CHY}}\mathcal{I}_{\mathrm{L}}\mathcal{I}_{\mathrm{R}} \tag{10}$$

and the string amplitude is similarly written as

$$A_{\mathrm{String}} = \int d\mathcal{M}_{\mathrm{String}}\hat{\mathcal{I}}_{\mathrm{L}}\hat{\mathcal{I}}_{\mathrm{R}} \tag{11}$$

The half integrand $\mathcal{I}_n$ for an n-point diagram at one-loop is the function

$$\mathcal{I}_n(k_i, \epsilon_i, z_i, l, \tau) \tag{12}$$

where k_i are the incoming/outgoing momenta, ϵ_i are the polarisations, z_i are points on the torus, l is the loop momentum and τ is the modular parameter on the torus (? check).

The basic building blocks for the half integrand $\mathcal{I}_n$ are the coefficients $C_n(k_i, \epsilon_i)$, the functions $g^{(n)}(z_i - z_j, \tau)$ and the loop momentum l.

$\mathcal{I}_n$ is also constrained by monodromy and modular (? check) properties. Each $g^{(n)}(z_i - z_j, \tau)$ function contributes a weight of n and each factor of loop momentum l contributes a weight of 1. In general $\mathcal{I}_n$ has a weight of $n-4$. For example, $\mathcal{I}_4$ has weight 0 and $\mathcal{I}_5$ has weight 1.

1.3 Meromorphic Functions $g^{(n)}$

The objects $g^{(n)}(z, \tau)$ are generated from the Kronecker-Eisentein series,

$$F(z, \alpha, \tau) \equiv \frac{\theta'(0,\tau)\theta(z+\alpha,\tau)}{\theta(\alpha,\tau)\theta(z,\tau)} \equiv \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z,\tau) \tag{13}$$

where $\theta(z, \tau)$ is the odd Jacobi theta function given by

$$\theta(z,\tau) \equiv \sin(\pi z) \prod_{n=1}^{\infty} (1 - e^{2\pi i(n\tau + z)})(1 - e^{2\pi i(n\tau - z)}) \tag{14}$$

and $\theta'(z, \tau) \equiv \partial_z \theta(z, \tau)$.

Note that $\theta(-z, \tau) = -\theta(z, \tau)$, i.e. it is an odd in the argument z. There are in total 4 Jacobi theta functions relevant for string theory (the other 3 functions are even).

By expanding equation (13) in powers of α, we can generate the first few meromorphic functions $g^{(0)}(z, \tau)$, $g^{(1)}(z, \tau)$ and $g^{(2)}(z, \tau)$:

$$g^{(0)}(z,\tau) = 1 \tag{15}$$

$$g^{(1)}(z,\tau) = \partial_z \ln \theta(z,\tau) \tag{16}$$

$$g^{(2)}(z,\tau) = \frac{1}{2}[(\partial_z \ln \theta(z,\tau))^2 + \partial_z^2 \ln \theta(z,\tau) - \frac{\theta''(0,\tau)}{3\theta'(0,\tau)}] \tag{17}$$

However, we are actually interested in the sphere degeneration limit of these $g^{(n)}$ functions and these are given by taking $\tau \to i\infty$:

$$g^{(0)}(z) \equiv \lim_{\tau \to i\infty} g^{(0)}(z,\tau) = 1 \tag{18}$$

$$g^{(1)}(z) = \pi \cot(\pi z) \tag{19}$$

$$g^{(2)}(z) = -\frac{\pi^2}{3} \tag{20}$$

$$g^{(3)}(z) = 0 \tag{21}$$

$$g^{(4)}(z) = -\frac{\pi^4}{45} \tag{22}$$

where we used the notation $g^{(n)}(z) := \lim_{\tau \to i\infty} g^{(n)}(z, \tau)$.

In our Mathematica code, we scale each of these $g^{(n)}$ functions by a factor of $(\frac{1}{2\pi i})^n$ and use

$$g^{(0)}(z) = 1 \tag{23}$$
$$g^{(1)}(z) = \frac{1}{2i}\cot(\pi z) \tag{24}$$
$$g^{(2)}(z) = \frac{1}{12} \tag{25}$$
$$g^{(3)}(z) = 0 \tag{26}$$
$$g^{(4)}(z) = -\frac{1}{720}. \tag{27}$$

These are all the $g^{(n)}$ functions we need for this project.

1.4 Fay Relations

The objects $g^{(n)}(z,\tau)$ are not independent of each other. They satisfy symmetry relations which can be generated by expanding the Fay relations

$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) + F(z_2,\alpha_2+\alpha_1)F(z_1-z_2,\alpha_1) \tag{28}$$

in ascending powers of α.

The first non-trivial symmetry relation is generated when we expand equation (28) at order α^0,

$$g_{ij}^{(1)}g_{jk}^{(1)} + g_{ki}^{(1)}g_{ij}^{(1)} + g_{jk}^{(1)}g_{ki}^{(1)} + g_{ik}^{(2)} + g_{kj}^{(2)} + g_{ji}^{(2)} = 0. \tag{29}$$

At order α^1, we have the relation

$$g_{ik}^{(3)} + g_{kj}^{(3)} - 2g_{ji}^{(3)} + g_{ij}^{(2)}g_{jk}^{(1)} + g_{ki}^{(1)}g_{ij}^{(2)} - g_{jk}^{(2)}g_{ki}^{(1)} - g_{jk}^{(1)}g_{ki}^{(2)} = 0. \tag{30}$$

At order α^2, we have the two relations

$$g_{ik}^{(4)} + g_{kj}^{(4)} + 3g_{ji}^{(4)} + g_{ij}^{(3)}g_{jk}^{(1)} + g_{ki}^{(1)}g_{ij}^{(3)} + g_{jk}^{(3)}g_{ki}^{(1)} + g_{jk}^{(2)}g_{ki}^{(2)} + g_{jk}^{(1)}g_{ki}^{(3)} = 0, \tag{31}$$
$$-g_{ik}^{(4)} + 3g_{kj}^{(4)} + 3g_{ji}^{(4)} - g_{ij}^{(2)}g_{jk}^{(2)} + 2g_{ki}^{(1)}g_{ij}^{(3)} + g_{ki}^{(2)}g_{ij}^{(2)} + 2g_{jk}^{(3)}g_{ki}^{(1)} + g_{jk}^{(2)}g_{ki}^{(2)} = 0. \tag{32}$$

2 4 points

2.1 Half integrand $\mathcal{I}_4$

The half integrand $\mathcal{I}_4$ has weight 0 and is given by

$$\mathcal{I}_4 = C_4(k_i,\epsilon_i) \tag{33}$$

where C_4 is a constant on the torus. From now on we drop the subscripts on k_i and ϵ_i, remembering that they are constant on the torus.

Including the product of differentials, we have (note the difference between $\mathcal{I}$ and I)

$$I_4 = \left(\prod_{i=2}^{4} dz_i\right)\mathcal{I}_4 = \left(\prod_{i=2}^{4} dz_i\right)C_4(k,\epsilon) \tag{34}$$

In the sphere degeneration limit, we get

$$I_4 = \frac{\prod_A d\sigma_A}{\mathrm{vol}\,\mathrm{SL}(2,C)} \sum_{\rho\in S_4} \frac{C_4(k,\epsilon)}{(+\rho(1)\cdots\rho(4)-)} \tag{35}$$

2.2 Map to BCJ numerators

From equation (35), the map from the coefficients C_4 to the BCJ numerators is trivially

$$N(1234;\ell) = C_4(k,\epsilon) \tag{36}$$

Note that the BCJ numerators are permutation invariant at 4 points (? check).

3 5 points

3.1 Half integrand $\mathcal{I}_4$

The half integrand $\mathcal{I}_5$ has weight 1 and is given by

$$\mathcal{I}_5 = C_5^{\mu}(k,\epsilon)\, l_\mu + \sum_{\substack{i,j\\ i<j}} C_{5,ij}(k,\epsilon) g^{(1)}(z_i - z_j,\tau). \tag{37}$$

Note that $C_{5,ij}$ is antisymmetric, i.e. $C_{5,ji} = -C_{5,ij}$.

Including the product of differentials, we have

$$I_5 = \left(\prod_{i=2}^{5} dz_i\right)\mathcal{I}_4 = \left(\prod_{i=2}^{5} dz_i\right)\left(C_5^{\mu} l_\mu + \sum_{\substack{i,j\\ i<j}} C_{5,ij} g_{ij}^{(1)}\right) \tag{38}$$

where we dropped the explicit dependence on k,ϵ and used the notation $g_{ij}^{(1)} := g^{(1)}(z_i - z_j,\tau)$.

3.2 Meromorphic function $g_{ij}^{(1)}$

The meromorphic function $g^{(1)}(z,\tau)$ is generated from the Kronecker-Eisenstein series. It is given by

$$g^{(1)}(z,\tau) = \partial_z \ln\theta(z,\tau) \tag{39}$$

where $\theta(z,\tau)$ is the odd Jacobi theta function given by

$$\theta(z,\tau) = \sin(\pi z)\prod_{n=1}^{\infty}\left(1 - e^{2\pi i(n\tau+z)}\right)\left(1 - e^{2\pi i(n\tau-z)}\right). \tag{40}$$

Note that $\theta(-z,\tau) = -\theta(z,\tau)$, i.e. it is an odd in the argument z. There are in total 4 Jacobi theta functions relevant for string theory (the other 3 functions are even).

One can also check that $g^{(1)}(z,\tau)$ is also odd , i.e. $g^{(1)}(-z,\tau) = -g^{(1)}(z,\tau)$.

3.3 Sphere degeneration limit

In general, the sphere degeneration limit (or field theory limit) is given by taking $\tau \to i\infty$ or equivalently $i\tau \to -\infty$. Geometrically this is equivalent to pinching the torus at a point.

In this limit, the meromorphic function $g^{(1)}(z,\tau)$ become

$$\lim_{\tau\to i\infty} g^{(1)}(z,\tau) = \pi\cot(\pi z). \tag{41}$$

To transform the torus coordinate z to the sphere coordinate σ, we first express equation (41) in the exponential form

$$\lim_{\tau\to i\infty} g^{(1)}(z,\tau) = \frac{\pi\cos(\pi z)}{\sin(\pi z)} \tag{42}$$

$$= i\pi\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \tag{43}$$

and then subsitute the following expression

$$e^{2i\pi z} = \frac{(\sigma - \sigma_+)(\sigma_* - \sigma_-)}{(\sigma - \sigma_-)(\sigma_* - \sigma_+)} \tag{44}$$

to get

$$\lim_{\tau\to i\infty} g^{(1)}(z,\tau) \equiv i\pi \frac{(\sigma - \sigma_+)(\sigma_* - \sigma_-) + (\sigma - \sigma_-)(\sigma_* - \sigma_+)}{(\sigma - \sigma_*)(\sigma_+ - \sigma_-)}. \tag{45}$$

Correspondingly, $g^{(1)}(z_i - z_j, \tau)$ in the sphere degeneration limit becomes

$$\lim_{\tau\to i\infty} g^{(1)}(z_i - z_j,\tau) \equiv i\pi \frac{(\sigma_i - \sigma_+)(\sigma_j - \sigma_-) + (\sigma_i - \sigma_-)(\sigma_j - \sigma_+)}{(\sigma_i - \sigma_j)(\sigma_+ - \sigma_-)}. \tag{46}$$

For some reason (? check) we will scale $g^{(1)}_{ij}$ by a factor of $\frac{1}{2\pi i}$ such that from now on we use

$$\lim_{\tau\to i\infty} g^{(1)}(z_i - z_j,\tau) \equiv \frac{1}{2} \frac{(\sigma_i - \sigma_+)(\sigma_j - \sigma_-) + (\sigma_i - \sigma_-)(\sigma_j - \sigma_+)}{(\sigma_i - \sigma_j)(\sigma_+ - \sigma_-)}. \tag{47}$$

We also define

$$g^{(1)}(\sigma_i,\sigma_j) \coloneqq \frac{1}{2} \frac{(\sigma_i - \sigma_+)(\sigma_j - \sigma_-) + (\sigma_i - \sigma_-)(\sigma_j - \sigma_+)}{(\sigma_i - \sigma_j)(\sigma_+ - \sigma_-)} \tag{48}$$

for ease of notation.

We can then write down the sphere degeneration limit of I_5 given in equation (38). Using equations (8) and (47), we get

$$\lim_{\tau\to i\infty} I_5 = \frac{\prod_A d\sigma_A}{\mathrm{vol}\,\mathrm{SL}(2,C)} (-1) \sum_{\rho\in S_5} \frac{C_5^\mu l_\mu + \sum_{\substack{i,j \\ i<j}} C_{5,ij} g^{(1)}(\sigma_i,\sigma_j)}{(+\rho(1)\cdots\rho(5)-)} \tag{49}$$

3.4 Map to BCJ numerators

Using Mathematica, equation (49) can be transformed into

$$\lim_{\tau\to i\infty} I_5 = \frac{\prod_A d\sigma_A}{\mathrm{vol}\,\mathrm{SL}(2,C)} (-1) \sum_{\rho\in S_5} \frac{C_5^\mu l_\mu - \frac{1}{2}\sum_{\substack{i,j \\ i<j}} C_{5,\rho(i)\rho(j)}}{(+\rho(1)\cdots\rho(5)-)}. \tag{50}$$

Note that the σ dependence in the numerator of the permutation sum has vanished.

We can compare equation (50) with the formula involving BCJ numerators which is

$$\lim_{\tau\to i\infty} I_5 = \frac{\prod_A d\sigma_A}{\mathrm{vol}\,\mathrm{SL}(2,C)} (-1) \sum_{\rho\in S_5} \frac{N_5(\rho(1)...\rho(5);l)}{(+\rho(1)\cdots\rho(5)-)}. \tag{51}$$

We then identify the map between the coefficients and the BCJ numerators as

$$N_5(\rho(1)...\rho(5);l) = C_5^\mu l_\mu - \frac{1}{2}\sum_{\substack{i,j \\ i<j}} C_{5,\rho(i)\rho(j)}. \tag{52}$$

For example, we have

$$N_5(12345;l) = C_5^\mu l_\mu - \frac{1}{2}\sum_{\substack{i,j \\ i<j}} C_{5,ij}. \tag{53}$$

We can also reverse this mapping and obtain

$$C_5^\mu = N(\cdots;\ell)|_{\ell\mu}, \qquad C_{5,ij} = -N\left(\cdots[i,j]\cdots;\ell\right) \tag{54}$$

where we used the notation $N\left(\cdots[i,j]\cdots;\ell\right) \coloneqq N(\cdots ij\cdots;\ell) - N(\cdots ji\cdots;\ell)$. For example, we have

$$C_5^\mu = N(12345;\ell)|_{\ell\mu}, \qquad C_{5,12} = -N\left([1,2]345;\ell\right). \tag{55}$$

3.5 Monodromy Relations

To obtain monodromy relations we perform the following operation

$$D_i\mathcal{I}_n = 0 \tag{56}$$

where D_i is a linear differential operator that acts on the loop momentum ℓ and the memormophic functions $g_{jk}^{(m)}$:

$$D_i\ell_\mu = k_{i\mu}, \qquad D_i g_{jk}^{(m)} = \left(\delta_{ij} - \delta_{ik}\right) g_{jk}^{(m-1)}. \tag{57}$$

At 5 points the monodromy relations we obtain are

$$D_i\mathcal{I}_5 = C_5^\mu k_{i\mu} + \sum_{\substack{j\\ j\neq i}} C_{5,ij} = 0. \tag{58}$$

For example, we have

$$D_1\mathcal{I}_5 = C_5^\mu k_{1\mu} + \sum_{\substack{j\\ j\neq 1}} C_{5,1j} = 0. \tag{59}$$

3.6 BCJ Relations

The BCJ numerator at 5 points can be written as

$$N_5(12345;\ell) = N_5^\mu(12345)\ell_\mu + N_5^0(12345) \tag{60}$$

where we have explicitly specified the ℓ dependence.

We note that $N_5^\mu(12345)$ is permutation invariant, e.g. $N_5^\mu(12345) = N_5^\mu(21345)$ so it suffice to drop the particle numbers and only write N_5^μ.

The BCJ numerators satisfy the relation (from moving a particle around the loop and summing diagrams)

$$N_5^0(12345) - N_5^0(23451) = \sum_{\substack{j\\ j\neq 1}} N_5([1,j]) \tag{61}$$

where $N_5\left([1,j]\right) \equiv N_5\left([1,j]\cdots\right)$. We omitted the other indices in $N_5\left([1,j]\cdots\right)$ since it is permutation invariant in those indices.

We also have the relation (from choosing different loop momenta for the same diagram)

$$N_5^0(12345) - N_5^0(23451) = N_5^\mu k_{1\mu}. \tag{62}$$

Combining equations (61) and (62), we obtain the relevant BCJ relation

$$N_5^\mu k_{1\mu} - \sum_{\substack{j\\ j\neq 1}} N_5([1,j]) = 0. \tag{63}$$

We now show that the BCJ relation in equation (63) is equivalent to the monodromy relation given by equation (59). Using the map given in equation (54), the monodromy relation (59) become

$$C_5^{\mu} k_{1\mu} + \sum_{\substack{j \\ j\neq 1}} C_{5,1j} \equiv N_5^{\mu}(12345) k_{1\mu} - \sum_{\substack{j \\ j\neq 1}} N\left([1,j]\right) = 0 \tag{64}$$

which is equivalent to (63).

Using properties of BCJ numerators, we can also perform a check that our map from numerators to coefficients in equation (53) is correct. The map in (53) was

$$N_5^{\mu}(12345) = C_5^{\mu}, \qquad N^0(12345) = -\frac{1}{2}\sum_{\substack{i,j \\ i<j}} C_{5,ij}. \tag{65}$$

The first map is indeed possible since $N_5^{\mu}(12345) \equiv N^{\mu}$ is permutation invariant. The second map implies that

$$N^0(12345) = \frac{1}{2}\sum_{\substack{i,j \\ i<j}} N\left([i,j]\right) \tag{66}$$

$$= \frac{1}{2}\sum_{\substack{i,j \\ i<j}} N^0\left([i,j]\right) \tag{67}$$

where we expressed the LHS in terms of BCJ numerators. Using properties of BCJ numerators, we have checked explicitly that equation (67) true, i.e. $N^0(12345)$ cannot have components that are symmetric or antisymmetric in ≥ 3 indices. .

4 6 points

4.1 Half integrand I_6

The half integrand $\mathcal{I}_5$ has weight 2 and is given by

$$\mathcal{I}_6 = C_6^{\mu\nu}\ell_\mu\ell_\nu + \sum_{\substack{i,j \\ i<j}} C_{6,ij}^{\mu}\ell_\mu g_{ij}^{(1)} + \sum_{\substack{i,j,k,l\,\mathrm{dist} \\ i<j,k<l,i<k}} C_{6,ij,kl}\, g_{ij}^{(1)} g_{kl}^{(1)} + \sum_{\substack{i,j,k\,\mathrm{dist} \\ i<k}} C_{6,ijk}\, g_{ij}^{(1)} g_{kj}^{(1)} + \sum_{\substack{i,j \\ i<j}} \tilde{C}_{6,ij}\, g_{ij}^{(2)}. \tag{68}$$

where the coefficients have the following properties

- $C_{6,ij}^{\mu} = -C_{6,ji}^{\mu}$
- $C_{6,ij,kl} = -C_{6,ji,kl}, \qquad C_{6,ij,kl} = -C_{6,ij,lk}$
- $C_{6,ijk} = C_{6,kji}$
- $\tilde{C}_{6,ij} = \tilde{C}_{6,ji}$

Including the product of differentials, we have

$$I_6 = \left(\prod_{i=2}^{6} dz_i\right)\mathcal{I}_6 \tag{69}$$

$$= \left(\prod_{i=2}^{6} dz_i\right)\Bigg(C_6^{\mu\nu}\,\ell_\mu\ell_\nu + \sum_{\substack{i,j \\ i<j}} C_{6,ij}^{\mu}\,\ell_\mu\, g_{ij}^{(1)} + \sum_{\substack{i,j,k,l\,\mathrm{dist} \\ i<j,k<l,i<k}} C_{6,ij,kl}\, g_{ij}^{(1)}\, g_{kl}^{(1)} + \sum_{\substack{i,j,k\,\mathrm{dist} \\ i<k}} C_{6,ijk}\, g_{ij}^{(1)}\, g_{kj}^{(1)}$$

$$+ \sum_{\substack{i,j \\ i<j}} \tilde{C}_{6,ij}\, g_{ij}^{(2)}\Bigg) \tag{70}$$

where $g_{ij}^{(2)} \coloneqq g^{(2)}(z_i - z_j, \tau)$.

4.2 Meromorhpic function $\mathbf{g}_{ij}^{(2)}$

to be completed (scale by 2 factors of $\frac{1}{2\pi i}$)

4.3 Sphere degeneration limit

In the sphere degeneration limit, $g_{ij}^{(2)}$ is a constant given by

$$g^{(2)}(\sigma_i, \sigma_j) = \frac{1}{12}. \tag{71}$$

We can then write down the sphere degeneration limit of I_6 given in equation (70). Using equations (8), (47) and (71), we get

$$\lim_{\tau \to i\infty} I_6 = \frac{\prod_A d\sigma_A}{\mathrm{vol}\,\mathrm{SL}(2,C)} \sum_{\rho \in S_6} \frac{C_6^{\mu\nu}\,\ell_\mu \ell_\nu + \sum_{\substack{i,j\\ i<j}} C_{6,ij}^{\mu} \ell_\mu g^{(1)}(\sigma_i, \sigma_j) + \cdots + \sum_{\substack{i,j\\ i<j}} \tilde{C}_{6,ij}\, g^{(2)}(\sigma_i, \sigma_j)}{(+\rho(1)\cdots\rho(6)-)}. \tag{72}$$

4.4 Map to BCJ numerators

Using Mathematica, equation (73) can be transformed into

$$\lim_{\tau \to i\infty} I_6 = \frac{\prod_A d\sigma_A}{\mathrm{vol}\,\mathrm{SL}(2,C)} \sum_{\rho \in S_6} \frac{N\left(\rho(1)...\rho(6);\ell\right)}{(+\rho(1)\cdots\rho(6)-)}. \tag{73}$$

where $N\left(\rho(1)...\rho(6);\ell\right)$ is the 6 point BCJ numerator given by

$$\begin{aligned} N\left(\rho(1)...\rho(6);\ell\right) = {}& C_6^{\mu\nu}\,\ell_\mu \ell_\nu - \frac{1}{2}\sum_{\substack{i,j\\ i<j}} C^{\mu}_{6,\rho(i)\rho(j)}\,\ell_\mu + \frac{1}{4}\sum_{\substack{i,j,k,l\,\mathrm{dist}\\ i<j,k<l,i<k}} C_{6,\rho(i)\rho(j),\rho(k)\rho(l)} \\ & + \frac{1}{4}\sum_{\substack{i,j,k\,\mathrm{dist}\\ i<k}} (-1)^{\mathrm{If}[i<j<k,1,0]}\, C_{6,\rho(i)\rho(j)\rho(k)} + \frac{1}{12}\sum_{\substack{i,j\\ i<j}} \tilde{C}_{6,\rho(i)\rho(j)}. \end{aligned} \tag{74}$$

For example, we have

$$\begin{aligned} N(123456;\ell) = {}& C_6^{\mu\nu}\,\ell_\mu \ell_\nu - \frac{1}{2}\sum_{\substack{i,j\\ i<j}} C^{\mu}_{6,ij}\,\ell_\mu + \frac{1}{4}\sum_{\substack{i,j,k,l\,\mathrm{dist}\\ i<j,k<l,i<k}} C_{6,ij,kl} + \frac{1}{4}\sum_{\substack{i,j,k\,\mathrm{dist}\\ i<k}} (-1)^{\mathrm{If}[i<j<k,1,0]}\, C_{6,ijk} \\ & + \frac{1}{12}\sum_{\substack{i,j\\ i<j}} \tilde{C}_{6,ij}. \end{aligned} \tag{75}$$

So equations (74) and (75) give the map from the coefficients to the BCJ numerators.

We can also reverse this mapping and obtain

$$\begin{aligned} & C_6^{\mu\nu} = N(\cdots;\ell)|_{\ell_\mu \ell_\nu}\,, \qquad C^{\mu}_{6,ij} = -N(\cdots[i,j]\cdots;\ell)|_{\ell_\mu}\,,, \qquad C_{6,ij,kl} = N(\cdots[i,j]\cdots[k,l]\cdots;\ell) \\ & C_{6,ijk} = ?\left(N(\cdots[[i,j],k]) + N(\cdots[i,[j,k]])\right) \end{aligned} \tag{76}$$

We have not found a suitable map for $C_{6,ijk}$, so look at monodromy relations and BCJ relations for some hints. We will also show that $\tilde{C}_{6,ij}$ is completely determined in terms of the other coefficients by using monodromy relations.

4.5 Monodromy Relations

Using $\mathcal{I}_6$ given in equation (70), the monodromy relation obtained at 6 points is

$$\begin{aligned} D_i\mathcal{I}_6 =& 2\ell_\mu k_{i\nu} C_6^{\mu\nu} + k_{i\mu} \sum_{\substack{j,k\\ j<k}} C^\mu_{6,jk} g^{(1)}_{jk} + \ell_\mu \sum_{j\neq i} C^\mu_{6,ij} + \sum_{\substack{jkl\neq i\\ k<l}} C_{6,ij,kl} g^{(1)}_{kl} \\ &+ \sum_{jk\neq i} \left(C_{6,ijk} g^{(1)}_{kj} + C_{6,kij} g^{(1)}_{ik} \right) + \sum_{j\neq i} \tilde{C}_{6,ij} g^{(1)}_{ij} \\ &= 0 \end{aligned} \tag{77}$$

We group the terms in equation (77) according to dependence on loop momentum ℓ_μ and rewrite it as

$$\ell_\mu A^\mu_i + B_i = 0 \tag{78}$$

where

$$A^\mu_i = 2k_{i\nu} C_6^{\mu\nu} + \sum_{j\neq i} C^\mu_{6,ij}, \tag{79}$$

$$B_i = k_{i\mu} \sum_{\substack{j,k\\ j<k}} C^\mu_{6,jk} g^{(1)}_{jk} + \sum_{\substack{jkl\neq i\\ k<l}} C_{6,ij,kl} g^{(1)}_{kl} + \sum_{jk\neq i} \left(C_{6,ijk} g^{(1)}_{kj} + C_{6,kij} g^{(1)}_{ik} \right) + \sum_{j\neq i} \tilde{C}_{6,ij} g^{(1)}_{ij}. \tag{80}$$

Since the monodromy relation in (77) is true regardless of value of the loop momentum ℓ_μ, we get

$$A^\mu_i = 0, \qquad B_i = 0. \tag{81}$$

Next we rewrite B_i in the following form

$$B_i = \sum_{j\neq i} B_{ij} g^{(1)}_{ij} + \sum_{\substack{jk\neq i\\ j<k}} B_{ijk} g^{(1)}_{jk} \tag{82}$$

where

$$B_{ij} = k_{i\mu} C^\mu_{6,ij} + \sum_{k\neq ij} C_{6,jik} + \tilde{C}_{6,ij}, \tag{83}$$

$$B_{ijk} = k_{i\mu} C^\mu_{6,jk} + \sum_{l\neq ijk} C_{6,il,jk} + C_{i[kj]}. \tag{84}$$

Note that we used the notation $C_{i[kj]} \coloneqq C_{ikj} - C_{ijk}$.

Since $B_i = 0$ is true for any point on the torus, we must have

$$B_{ij} = 0, \qquad B_{ijk} = 0. \tag{85}$$

We then write B_{ij} in symmetric and antisymmetric parts

$$B_{ij} = B_{(ij)} + B_{[ij]} \tag{86}$$

where we used the notation

$$B_{(ij)} \coloneqq \frac{1}{2}\left(B_{ij} + B_{ji}\right), \tag{87}$$

$$B_{[ij]} \coloneqq \frac{1}{2}\left(B_{ij} - B_{ji}\right). \tag{88}$$

Due to the antisymmetric property of $g_{ij}^{(1)}$ in equation (82), we must have

$$B_{[ij]} = 0 \tag{89}$$

which also implies that $B_{(ij)} = 0$.

We also note that equations (83) and (85) tell us that $\tilde{C}_{6,ij}$ is completely determined if all other coefficients are known.

In summary, the monodromy relations that we we have are

$$A_i^\mu = 0, \qquad B_{[ij]} = 0, \qquad B_{ijk} = 0. \tag{90}$$

5 7 points

5.1 Half integrand $\mathcal{I}_7$

The half integrand at 7 points has weight 3 and is given by

$$\begin{aligned}
\mathcal{I}_7 = & C^{\mu\nu\rho}\ell_\mu\ell_\nu\ell_\rho + \sum_{\substack{i,j\\ i<j}} C^{\mu\nu}_{7,ij}\ell_\mu\ell_\nu g^{(1)}_{ij} + \sum_{\substack{i,j,k,l \text{ dist}\\ i<j,k<l,i<k}} C^{\mu}_{7,ij,kl}\ell_\mu g^{(1)}_{ij} g^{(1)}_{kl} + \sum_{\substack{i,j,k \text{ dist}\\ i<k}} C^{\mu}_{7,ijk}\ell_\mu g^{(1)}_{ij} g^{(1)}_{kj} \\
& + \sum_{\substack{i,j\\ i<j}} \tilde{C}^{\mu}_{7,ij}\ell_\mu g^{(2)}_{ij} + \sum_{\substack{i,j,k,l,m,n \text{ dist}\\ i<j,k<l,m<n,i<k<m}} C_{7,ij,kl,mn} g^{(1)}_{ij} g^{(1)}_{kl} g^{(1)}_{mn} + \sum_{\substack{i,j,k,l,m \text{ dist}\\ j<k,l<m}} C_{7,i,jk,lm} g^{(1)}_{ij} g^{(1)}_{ik} g^{(1)}_{lm} \\
& + \sum_{\substack{i,j,k,l \text{ dist}\\ j<l}} C_{7,ijkl} g^{(1)}_{ji} g^{(1)}_{ik} g^{(1)}_{kl} + \sum_{\substack{i,j,k \text{ dist}\\ i<j<k}} C_{7,ijk} g^{(1)}_{ij} g^{(1)}_{jk} g^{(1)}_{ki} + \sum_{\substack{i,j,k,l \text{ dist}\\ i<j,k<l}} C_{7,ij,kl} g^{(1)}_{ij} g^{(2)}_{kl} \\
& + \sum_{i,j,k \text{ dist}} \hat{C}_{7,ijk} g^{(1)}_{ij} g^{(2)}_{ki} + \sum_{\substack{i,j\\ i<j}} \tilde{C}_{7,ij} g^{(3)}_{ij}
\end{aligned} \tag{91}$$

5.2 Monodromy Relations

The monodromy relation for $\mathcal{I}_7$ is

$$\begin{aligned}
D_n\mathcal{I}_7 = & 3C^{\mu\nu\rho}_7 k_{n\mu}\ell_\mu\ell_\rho + \sum_{i\neq p} C^{\mu\nu}_{7,ni}\ell_\mu\ell_\nu + 2\sum_{\substack{i,j\\ i<j}} C^{\mu\nu}_{7,ij} g^{(1)}_{ij} k_{n\mu}\ell_\nu + \sum_{\substack{i,j,k,l \text{ dist}\\ i<j,k<l,i<k}} C^{\mu}_{7,ij,kl} g^{(1)}_{ij} g^{(1)}_{kl} k_{n\mu} \\
& + \sum_{\substack{j,k,l \text{ dist}\\ k<l}} C^{\mu}_{7,nj,kl} g^{(1)}_{kl}\ell_\mu + \sum_{\dots} C^{\mu}_{7,ijk}(\dots) + \sum_{\substack{i,j\\ i<j}} \tilde{C}^{\mu}_{7,ij} g^{(2)}_{ij} k_{n\mu} + \sum_{i\neq n} \tilde{C}^{\mu}_{7,ni} g^{(1)}_{ni}\ell_\mu \\
& + \sum_{\substack{jklmp\\ k<l,m<p,k<m}} C_{7,nj,kl,mp} g^{(1)}_{kl} g^{(1)}_{mp} + \sum_{\substack{ijkl\\ k<l}} C_{7,n,ij,kl} g^{(1)}_{nj} g^{(1)}_{kl} - C_{7,i,nj,kl} g^{(1)}_{ij} g^{(1)}_{kl} + \sum_{\substack{ijkl\\ j<k}} C_{7,i,jk,nl} g^{(1)}_{ij} g^{(1)}_{ik} \\
& + \sum_{ijk} C_{7,injk} g^{(1)}_{ij} g^{(1)}_{jk} + C_{7,nijk}\left(g^{(1)}_{jn} g^{(1)}_{jk} + g^{(1)}_{in} g^{(1)}_{jk}\right) + \sum_{ij} C_{7,nij} g^{(1)}_{nj} g^{(1)}_{ji} \\
& + \sum_{\substack{ijk\\ j<k}} C_{7,ni,jk} g^{(2)}_{jk} + C_{7,jk,ni} g^{(1)}_{jk} g^{(1)}_{ni} + \sum_{ij} \hat{C}_{7,nij} g^{(2)}_{jp} - \hat{C}_{7,inj} g^{(2)}_{ji} + \hat{C}_{7,ijn} g^{(1)}_{ij} g^{(1)}_{ni} - \hat{C}_{7,nji} g^{(1)}_{nj} g^{(1)}_{in} \\
& + \sum_{j} \tilde{C}_{7,nj} g^{(2)}_{nj} \\
= & \ 0
\end{aligned} \tag{92}$$