Box Muller Method

Takashi Shinzato(shinzato@sp.dis.titech.ac.jp)

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1 Motivation

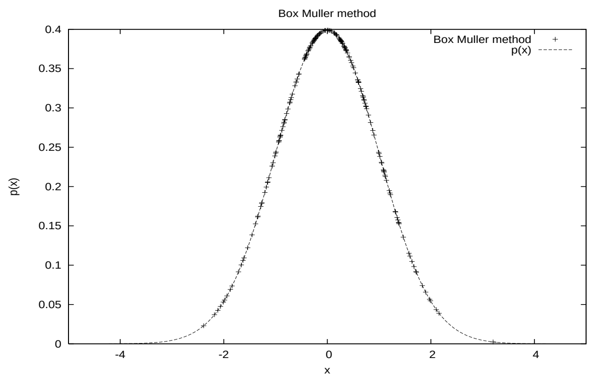

Figure 1: Box Muller method; random variable x is asymptotically distributed on $N(0,1)$.

In this informal note, we discuss normal gaussian random variable $x \sim N(0,1)$[1] using stochastic variables on uniform distribution in $[0,1]$. At first step, we indicate then the following relation as

$$\int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}} = \sqrt{2\pi}. \tag{2}$$

In order to prove eq.(2), we present via

$$\left(\int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}}\right)^2 = \int_{-\infty}^{\infty} dxdy e^{-\frac{x^2+y^2}{2}}, \tag{3}$$

and replace x and y to r and θ like[2]

$$x = r\cos\theta, \tag{4}$$
$$y = r\sin\theta. \tag{5}$$

Using eq.(4) and eq.(5), they are also held as

$$x^2+y^2 = r^2 \tag{6}$$
$$\tan^{-1}\frac{y}{x} = \theta \tag{7}$$
$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r \tag{8}$$

Eq.(8) is Jaccobian. Hence, we show so

$$\begin{aligned}\int_{-\infty}^{\infty} dxdy e^{-\frac{x^2+y^2}{2}} &= \int_0^{2\pi} d\theta \int_0^{\infty} drre^{-\frac{r^2}{2}} \\ &= 2\pi \int_0^{\infty} due^{-u} \quad \left(u=\frac{r^2}{2}\right) \\ &= 2\pi\left[-e^{-u}\right]_0^{\infty} = 2\pi. \end{aligned} \tag{9}$$

Namely, eq.(2) is true. Therefore, we give then

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} dxdy e^{-\frac{x^2+y^2}{2}}, \tag{10}$$

and in the case that x is distributed on probability measure (or mass probability) $p(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$, stochastic variable x is called gaussian random variable.

2 Box Muller Method

In this section, we present and introduce Box Muller method. Providing with the law of large numbers or center limit theorem, gaussian random variable has played an important role and one makes usefull for hypothesis and test in Data science and so on. We define a new function as

$$U(R) := \frac{1}{2\pi}\int_{x^2+y^2\le R^2} dxdy e^{-\frac{x^2+y^2}{2}}. \tag{11}$$

This integral interval is in $x^2+y^2 \le R^2$. We calcurate then briefly

$$\begin{aligned}U(R) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{R} drre^{-\frac{r^2}{2}} \\ &= \int_0^{\frac{R^2}{2}} due^{-u} = 1-e^{-\frac{R^2}{2}}\end{aligned} \tag{12}$$

$U(R)$ is nondecreasing function satisfied as

$$\lim_{R\to 0} U(R) = 0 \tag{13}$$
$$\lim_{R\to\infty} U(R) = 1 \tag{14}$$

Moreover, by eq.(11), the integral interval is replaced using $0 \le r \le R$ and eq.(4) and eq.(5). We represent that using $p \in [0,1]$ and eq.(12), variable R of $U(R) = p$ is decided as

$$U(R) = p \implies R = \sqrt{-2\log(1-p)}. \tag{15}$$

In setting $s := 1-p \in [0,1]$ and $t \in [0,1]$, it is given as

[1] Normal distribution is distributed on

$$p(x) := \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \Longleftarrow N(0,1) \tag{1}$$

with the mean 0 and the variance 1.

[2] $r > 0$ and $0 \le \theta \le 2\pi$

$$x = \begin{cases} \sqrt{-2\log(s)}\cos(2\pi t) \\ \sqrt{-2\log(s)}\sin(2\pi t)\,. \end{cases} \tag{16}$$

That is to say, we give then the scheme that we generate gaussian random variable distributed on $N(0,1)$ using uniform distribution on $[0,1]$; its scheme is called Box Muller method. More being generalized, stochastic variable $z \sim N(\mu, \sigma^2)$, with the mean μ and the variance σ^2, is given as

$$z = \mu + \sigma\sqrt{-2\log(s)}\cos(2\pi t)\,, \tag{17}$$

or

$$z = \mu + \sigma\sqrt{-2\log(s)}\sin(2\pi t)\,. \tag{18}$$

3 Conclusion

We introduce and discuss, in this informal note, the scheme of normal distribution $N(0,1)$ using unique distribution on $[0,1]$, Box Muller method.

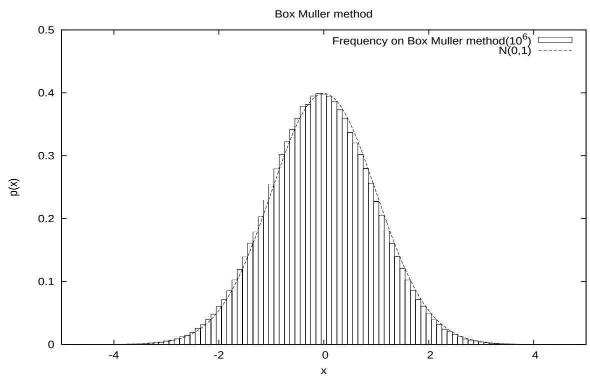

Figure 2: Frequency on Box Muller method is asymptotically distributed on $N(0,1)$.