# 1 Section 2.5

## 1.1 2.5.1

Apply the improved Euler method to approximate the solution on the interval $[0, 0.5]$ with step size $h = 0.1$. Construct a table showing values of the approximate solution and the actual solution at the points $x = 0.1, 0.2, 0.3, 0.4, 0.5$.

$$y' = -3y, y(0) = 7; y(x) = 7e^{-3x}$$

Improved Euler's method:

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ u_{n+1} &= y_n + h \cdot k_1 \\ k_2 &= f(x_{n+1}, u_{n+1}) \\ y_{n+1} &= y_n + h \cdot \frac{1}{2}(k_1 + k_2) \end{aligned}$$

$$\begin{aligned} k_1 &= -3(7) = -21 \\ u_1 &= (7) + (0.1)(-21) = 4.9 \\ k_2 &= -3(4.9) = -14.7 \\ y_1 &= (7) + (0.05)(-21 + (-14.7)) = 5.2150 \end{aligned}$$

$$\begin{aligned} k_1 &= -3(5.2150) = -15.645 \\ u_2 &= (5.2150) + (0.1)(-15.645) = 3.6505 \\ k_2 &= -3(3.6505) = -10.9515 \\ y_2 &= (5.2150) + (0.05)(-15.645 + (-10.9515)) = 3.8852 \end{aligned}$$

$$\begin{aligned} k_1 &= -3(3.8852) = -11.6555 \\ u_3 &= (3.8852) + (0.1)(-11.6555) = 2.71965 \\ k_2 &= -3(2.71965) = -8.15895 \\ y_3 &= 3.8852 + (0.05)(-11.6555 + (-8.15895)) = 2.89448 \end{aligned}$$

$$\begin{aligned} k_1 &= -3(2.89448) = -8.68343 \\ u_4 &= (2.89448) + (0.1)(-8.68343) = 2.02613 \\ k_2 &= -3(2.02613) = -6.0784 \\ y_4 &= 2.89448 + (0.05)(-8.6843 + (-6.0784)) = 2.15639 \end{aligned}$$

$$\begin{aligned} k_1 &= -3(2.15639) = -6.46916 \\ u_5 &= 2.15639 + (0.1)(-6.46916) = 1.50947 \\ k_2 &= -3(1.50947) = -4.52841 \\ k_5 &= 2.15639 + (0.05)(-6.46916 + (-4.52841)) = 1.60651 \end{aligned}$$

| $x_n$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| **Actual** $y(x_n)$ | 5.1857 | 3.8417 | 2.8460 | 2.1084 | 1.5619 |
| **Improved Euler** $y_n$ | 5.2150 | 3.8852 | 2.8945 | 2.1564 | 1.6065 |

## 1.2 2.5.5

$$y' = y - x - 1, y(0) = 1, y(x) = 2 + x - e^x$$

Improved Euler's method:

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ u_{n+1} &= y_n + h \cdot k_1 \\ k_2 &= f(x_{n+1}, u_{n+1}) \\ y_{n+1} &= y_n + h \cdot \frac{1}{2}(k_1 + k_2) \end{aligned}$$

$$\begin{aligned} k_1 &= 1 - 0 - 1 = 0 \\ u_1 &= 1 + (0.1)(0) = 1 \\ k_2 &= 1 - 0.1 - 1 = -0.1 \\ y_1 &= 1 + (0.05)(0 + (-0.1)) = 0.9950 \end{aligned}$$

$$\begin{aligned} k_1 &= 0.9950 - 0.1 - 1 = -0.105 \\ u_2 &= 0.9950 + (0.1)(-0.105) = 0.9845 \\ k_2 &= 0.9845 - 0.2 - 1 = -0.2155 \\ y_2 &= 0.9950 + (0.05)(-0.105 + (-0.2155)) = 0.978975 \end{aligned}$$

$$\begin{aligned} k_1 &= 0.978975 - 0.2 - 1 = -0.221025 \\ u_3 &= 0.978975 + (0.1)(-0.221025) = 0.956873 \\ k_2 &= 0.956873 - 0.3 - 1 = -0.343128 \\ y_3 &= 0.978975 + (0.05)(-0.221025 + (-0.343128)) = 0.950767 \end{aligned}$$

$$\begin{aligned} k_1 &= 0.950767 - 0.3 - 1 = -0.349233 \\ u_4 &= 0.950767 + (0.1)(-0.349233) = 0.915844 \\ k_2 &= 0.915844 - 0.4 - 1 = -0.484794 \\ y_4 &= 0.950767 + (0.05)(-0.349233 + (-0.484794)) = 0.909098 \end{aligned}$$

$$\begin{aligned} k_1 &= 0.909098 - 0.4 - 1 = -0.490902 \\ u_5 &= 0.909098 + (0.1)(-0.490902) = 0.860008 \\ k_2 &= 0.860008 - 0.5 - 1 = -0.639992 \\ y_5 &= 0.909098 + (0.05)(-0.490902 + (-0.639992)) = 0.852553 \end{aligned}$$

| $x_n$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| **Actual** $y(x_n)$ | 0.9948 | 0.9786 | 0.9501 | 0.9082 | 0.8513 |
| **Improved Euler** $y_n$ | 0.9950 | 0.9790 | 0.9508 | 0.9091 | 0.8526 |