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1 Chapter 02 - Basic Structures

1.1 Sets

$\in$: belong to, is in

1.1.1 Definition 2

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x\,(x \in A \iff x \in B)$. We write $A = B$ if A and B are equal sets.

1.1.2 Definition 3

The set A is also a subset of B, and B is a superset of A, if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $B \supseteq A$. (So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.)

1.1.3 Definition 4

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

1.1.4 Countable and Uncountable Sets

- Countable
  - $\mathbb{N}$
  - $\mathbb{Z}$
  - $\mathbb{Q}$
- Uncountable
  - $\mathbb{R}$
  - $\mathbb{C}$

Let $S_0 = \{x\}$, and $S_1 = \{\{x\}\}$.

$$S_0 \neq S_1 \tag{1}$$

1.1.5 Example

1. List the members of these sets.

   a) $\left\{x \mid x \text{ is a real number such that } x^2 = 1\right\}$

   $$S = \left\{x \in \mathbb{R} \mid x^2 = 1\right\}$$

   b) $\{x \mid x \text{ is a positive integer less than } 12\}$

   $$S = \{x \in \mathbb{R} \mid 0 \leq x < 12\}$$

1.1.6 Definition 6

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

1.2 Set Operations

1.2.1 Definition 1

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\} \tag{2}$$

1.2.2 Definition 2

Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\} \tag{3}$$

1.2.3 Definition 3

Two sets are disjoint if their intersection is the empty set.

1.2.4 Definition 4

Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

1.2.5 Definition 5

Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

1.2.6 Proof

Let A, B be sets from U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

Proof:

- For " $\Longrightarrow$ "

  Given $A \subseteq B$, need to show $\overline{B} \subseteq \overline{A}$. Then $\forall x \in A$, we have $x \in B$.

  By contrapositive we have

  $$\begin{aligned} \neg (x \in B) &\implies \neg (x \in A) \\ x \notin B &\implies x \notin A, \quad \overline{B} \subseteq \overline{A} \end{aligned}$$

- For " $\Longleftarrow$ "

  Given $\overline{B} \subseteq \overline{A}$, we have $\forall y \in \overline{B}$, $y \in \overline{A}$ then the contrapositive is

  $$\begin{aligned} \neg \left(y \in \overline{A}\right) &\implies \neg \left(y \in \overline{B}\right) \\ y \in A &\implies y \in B \end{aligned}$$

1.2.7 Proof:

Use the identities to show that $\overline{(A \cup B)} \cap \overline{(B \cup C)} \cap \overline{(A \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}$

Proof:

$$\begin{aligned}
\overline{A} \cap \overline{B} \cap \overline{C} &= \left(\overline{A} \cap \overline{B}\right) \cap \left(\overline{B} \cap \overline{C}\right) \cap \left(\overline{A} \cap \overline{C}\right) \\
&= \overline{A} \cap \left(\overline{B} \cap \overline{B}\right) \cap \left(\overline{C} \cap \overline{C}\right) \cap \overline{A} \\
&= \overline{A} \cap \overline{B} \cap \overline{C}
\end{aligned}$$

1.2.8 Union

The union is a collection of sets is the set that contains those elements that are member s of at least one set in the collection.

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

1.2.9 Intersection

The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

1.3 Functions

- Sets: A, B, C, domain, codomain, range
- Relations: functions (f, g, h)
- Elements: image, preimage

1.3.1 Example

$$\begin{aligned}
&f_1 : \mathbb{R} \to \mathbb{R} \text{ as } y = f_1(x) = 3x - 2 \\
&f_2 : \mathbb{R} \to \mathbb{R} \text{ as } y = e^x \\
&f_3 : \mathbb{R} \to \mathbb{R} \text{ as } y = \sqrt{x}
\end{aligned}$$

$$\begin{gathered}
f : A \to B \\
\text{range}(f) = \{y \in B \mid \forall x \in S \subseteq A\}
\end{gathered}$$

1.3.2 Properties

1) f is injective – one-to-one

2) f is surjective –

3) f is bijective if **1)** and **2)**

Let $f : A \to B$ be a function.

We say f is injective if

$$(x_1 \neq x_2 \implies f(x_1) \neq f(x_2)) \iff (f(x_1) = f(x_2) \implies x_1 = x_2)$$

We say f is surjective if $\forall y \in B$, $\exists x \in A$ such that $y = f(x) \iff \text{range}(f) = B$

We say f is bijective if f is injective and surjective.

Monotonic Function: We say f is increasing (strictly increasing) if $x_1 > x_2 \implies f(x_1) > f(x_2)$

1.3.3 Example 2.3.24

Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = \frac{1}{f(x)}$ is strictly decreasing.

$$\begin{aligned}
&\to \frac{1}{f(x_1)} < \frac{1}{f(x_2)} \\
&\to f(x_1) \cdot f(x_2) \left(\frac{1}{f(x_1)} \right) < f(x_1) \cdot f(x_2) \left(\frac{1}{f(x_2)} \right) \\
&\to f(x_2) < f(x_1)
\end{aligned}$$

where the first implication comes from the fact that $f(x) > 0, \forall x \in \mathbb{R}$.

1.3.4 Example 2.3.73.b

Prove or disprove each of these statements about the floor and ceiling functions.

$$\lfloor 2x \rfloor = 2 \lfloor x \rfloor$$

Consider $x = 1.6$

$$\begin{aligned}
\lfloor 2x \rfloor &= \lfloor 2 \cdot 1.6 \rfloor = \lfloor 3.2 \rfloor = 3 \\
2 \lfloor x \rfloor &= 2 \lfloor 1.6 \rfloor = 2(1) = 2
\end{aligned}$$

Hence $\lfloor 2x \rfloor = 2 \lfloor x \rfloor$ for all $x \in \mathbb{R}$ is false.

1.4 Sequences and Summations

1.5 Cardinality of Sets

1.6 Matrices