Week 03 Participation Assignment

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1 Part 01

The purpose of this exercise is to prove that for any real number: $a : \sqrt{a^2} = |a|$.

First, we recall that the absolute value of any real number is defined by

$$|a| = \begin{cases} a \text{ if } a \geq 0, \text{ and} \\ -a \text{ if } a < 0. \end{cases}$$

a) Use the definition above to explain why for any real number $a : |a| \geq 0$.
Case by Case Proof:

- Case 1: $a \geq 0$.

$$\begin{aligned} |a| &= a, \quad a \geq 0 \\ |a| &= a \geq 0 \\ |a| &\geq a \end{aligned}$$

- Case 2: $a < 0$.

$$\begin{aligned} |a| &= -a, \quad a < 0 \implies -a > 0 \\ |a| &= -a > 0, \quad -a > 0 \implies -a \geq 0 \\ |a| &\geq 0 \end{aligned}$$

b) Again, using the definition, show that $|a|^2 = a^2$.
Case by Case Proof:

- Case 1: $a \geq 0$.

$$\begin{aligned} |a|^2 &= a^2 \\ |a| \cdot |a| &= a \cdot a, \quad |a| = a \geq 0 \\ a \cdot a &= a \cdot a \end{aligned}$$

- Case 2: $a < 0$.

$$\begin{aligned} |a|^2 &= a^2 \\ |a| \cdot |a| &= a \cdot a, \quad |a| = -a > 0 \implies |a| = -a \geq 0 \\ -a \cdot -a &= a \cdot a \\ a \cdot a &= a \cdot a \end{aligned}$$

c) Our next goal is to show that $\sqrt{b}$ is unique. In other words, prove that if c and d are two real numbers such that $c \geq 0$, and $d \geq 0$, and $b = c^2 = d^2$, then $c = d$.

$$\begin{aligned}
c^2 &= d^2 \\
c^2 - d^2 &= 0 \\
(c+d)(c-d) &= 0 \\
c &= \pm d \\
|c| &= |d|, \quad |c| = c \geq 0, |d| = d \geq 0 \\
c &= d
\end{aligned}$$

d) Rewrite the definition for $\sqrt{b}$ to define $\sqrt{a^2}$

$$\begin{aligned}
b &= c^2 \\
\sqrt{b} &= c \\
\sqrt{b} &= \pm d \\
\sqrt{b} &= \sqrt{d^2}, \sqrt{(-d)^2} \\
\sqrt{b} &= \sqrt{d^2} = \sqrt{a^2}, \quad \{d \in \mathbb{R}, \in (-\infty, \infty)\}
\end{aligned}$$

e) Put together all the steps above to write a complete proof that $\sqrt{a^2} = |a|$.

$$\sqrt{a^2} = |a|$$

Case by Case Proof:

- Case 1: $a \geq 0$

$$\begin{aligned}
\sqrt{a^2} &= |a| \\
a^2 &= |a|^2 \\
a \cdot a &= |a| \cdot |a|, \quad |a| = a \geq 0 \\
a \cdot a &= a \cdot a
\end{aligned}$$

- Case 2: $a < 0$

$$\begin{aligned}
\sqrt{a^2} &= |a| \\
a^2 &= |a|^2 \\
a \cdot a &= |a| \cdot |a|, \quad |a| = a < 0 \implies -a > 0 \implies -a \geq 0 \\
a \cdot a &= -a \cdot -a \\
a \cdot a &= a \cdot a
\end{aligned}$$

2 Part 02

Let's consider the powersets of a finite set. Our goal is to "calculate" how many elements are in the power set. That is the cardinality of the powerset. (We could define powerset from an infinite set).

If $X = \{x_1, x_2, \cdots, x_n\}$ is a finite set, we define $\mathcal{P}(X)$, the powerset of X, to be the set of all subsets of X. For Example, if $X = \{a, b\}$, then $\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and thus $\mathcal{P}(X)$ has 4 elements.

a) If $X = \{a, b, c\}$, list all the members of $\mathcal{P}(X)$. How many subsets does X have?

$$\mathcal{P}(X) = \{\emptyset, \{A, B, C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A\}, \{B\}, \{C\}\}$$
$$|\mathcal{P}(X)| = 8$$

b) Separate the list that you got in part **a)** into two columns. Place on the left column those subsets that contain c and place on the right column those that do not contain c.

Contains c	Does Not Contain c
$\{A, B, C\}$	$\emptyset$
$\{A, C\}$	$\{A, B\}$
$\{B, C\}$	$\{A\}$
$\{C\}$	$\{B\}$

c) Now, cross out c from each subset on the left column. What do you notice?

Contains c	Does Not Contain c
$\{A, B\}$	$\emptyset$
$\{A\}$	$\{A, B\}$
$\{B\}$	$\{A\}$
$\{\}$ $(\emptyset)$	$\{B\}$

Let S_0 be the set of all sets that contain C

$$S_0 = \{x \mid x \in U \land C \in x\}$$

and S_1 be the power set of X that does not contain any sets containing C. It can be seen that

$$\mathcal{P}(X) - S_0 = S_1$$

or

$$\mathcal{P}(X) \cap \overline{S_0} = S_1$$

This could be understood otherwise as: The intersection of the $\mathcal{P}(X)$ (The set of all subsets of X) and the complement of S_0 (The set of all subsets of U that **does not** contain C) is the power set of X that does not contain C.

Table representation:

$\mathcal{P}(X)$	S_0	$\overline{S_0}$	$\mathcal{P}(X) \cap \overline{S_0}$
$\{A, B, C\}$	$\{A, B, C\}$	$\cdots$	$\{A, B\}$
$\{A, B\}$	$\cdots$	$\{A, B\}$	$\{A\}$
$\{A, C\}$	$\{A, C\}$	$\cdots$	$\{B\}$
$\{B, C\}$	$\{B, C\}$	$\cdots$	$\emptyset$
$\{A\}$	$\cdots$	$\{A\}$	
$\{B\}$	$\cdots$	$\{B\}$	
$\{C\}$	$\{C\}$	$\cdots$	
$\emptyset$	$\cdots$	$\emptyset$	

d) Repeat part **a)**, **b)**, and **c)** for $X = \{a, b, c, d\}$

a)

$$\begin{aligned}
\mathcal{P}(X) = \{ & \\
& \{A, B, C, D\}, \\
& \{A, B, C\}, \{A, B, D\}, \{A, C, D\}, \{B, C, D\} \\
& \{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\} \\
& \{A\}, \{B\}, \{C\}, \{D\}, \emptyset \\
& \} \\
|\mathcal{P}(X)| = 16 &
\end{aligned}$$

b)

Contains C	Does Not Contain C
$\{A, B, C, D\}$	$\{A, B, D\}$
$\{A, B, C\}$	$\{A, B\}$
$\{A, C, D\}$	$\{A, D\}$
$\{B, C, D\}$	$\{B, D\}$
$\{A, C\}$	$\{A\}$
$\{B, C\}$	$\{B\}$
$\{C, D\}$	$\{D\}$
$\{C\}$	$\emptyset$

c)

Contains C	Does Not Contain C
$\{A, B, D\}$	$\{A, B, D\}$
$\{A, B\}$	$\{A, B\}$
$\{A, D\}$	$\{A, D\}$
$\{B, D\}$	$\{B, D\}$
$\{A\}$	$\{A\}$
$\{B\}$	$\{B\}$
$\{D\}$	$\{D\}$
$\{\}$ $(\emptyset)$	$\emptyset$

e) For $X = \{x_1, x_2, \cdots, x_n\}$, guess the number of elements in the power set $\mathcal{P}(X)$.

From previous examples, it can be seen that the number of elements in a set's X power set $\mathcal{P}(X)$ is 2 raised to the cardinality of X. In other words: Let X be any set, $\mathcal{P}(X)$ be the power set of X, and $|X|$ be the cardinality or number of elements in X

$$|\mathcal{P}(X)| = 2^{|X|}$$