Contents

1 Section 5.5

1.1 5.5.2

Find a particular solution y_p of the following equation using the Method of Undetermined Coefficients. Primes denote the derivatives with respect to x.

$$y'' - y' - 2y = 4x + 6$$

$$\begin{aligned}
r^2 - r - 2 &= 0 \\
r &= 2, -1 \\
y(x) &= c_1 e^{2x} + c_2 e^{-x} \\
y_p(x) &= Ax + B \\
y_p'(x) &= A \\
y_p''(x) &= 0
\end{aligned}$$

$$\begin{aligned}
(0) - (A) - 2(Ax + B) &= 4x + 6 \\
-2Ax - A - 2B &= 4x + 6 \\
-2A &= 4 \\
A &= -2 \\
-A - 2B &= 6 \\
-(-2) - 2B &= 6 \\
B &= -2 \\
y(x) &= -2x - 2
\end{aligned}$$

$$\boxed{y(x) = -2x - 2}$$

1.2 5.5.3

Find a particular solution y_p of the following equation using the Method of Undetermined Coefficients. Primes denote the derivatives with respect to x.

$$y'' - y' - 6y = 20\sin(3x)$$

$$\begin{aligned}
r^2 - r - 6 = 0 \\
r &= 3, -2 \\
y(x) &= c_1e^{3x} + c_2e^{-2x} \\
y_p(x) &= A\cos(3x) + B\sin(3x) \\
y_p'(x) &= -3A\sin(3x) + 3B\cos(3x) \\
y_p''(x) &= -9A\cos(3x) + -9B\sin(3x) \\
20\sin(3x) &= (-9A\cos(3x) + -9B\sin(3x)) - (-3A\sin(3x) + 3B\cos(3x)) - 6(A\cos(3x) + B\sin(3x))
\end{aligned}$$

$$\begin{aligned}
\cos(3x)(-9A - 3B - 6A) + \sin(3x)(-9B + 3A - 6B) &= 20\sin(3x) \\
-9A - 3B - 6A &= 0 \\
-15A - 3B &= 0 \\
B &= -5A \\
-9B + 3A - 6B &= 20 \\
-15B + 3A &= 20 \\
-15(-5A) + 3A &= 20 \\
A &= \frac{10}{39} \\
B &= -5\left(\frac{10}{39}\right) \\
B &= -\frac{50}{39} \\
y(x) &= \frac{10}{39}\cos(3x) - \frac{50}{39}\sin(3x)
\end{aligned}$$

$$\boxed{y(x) = \frac{10}{39}\cos(3x) - \frac{50}{39}\sin(3x)}$$

1.3 5.5.4

Find a particular solution y_p of the following equation using the Method of Undetermined Coefficients. Primes denote the derivatives with respect to x.

$$y'' - 4y' + 5y = xe^x$$

$$
\begin{aligned}
r^2 - 4r + 5 &= 0 \\
r &= 2 \pm 1i \\
y(x) &= c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x) \\
y_p(x) &= (Ax + B)Ce^x \\
y_p(x) &= e^x(Ax + B) \\
y_p'(x) &= e^x(Ax + A + B) \\
y_p''(x) &= e^x(Ax + 2A + B) \\
xe^x &= (e^x(Ax + 2A + B)) - 4(e^x(Ax + A + B)) + 5(e^x(Ax + B))
\end{aligned}
$$

$$
\begin{aligned}
xe^x(A - 4A + 5A) + e^x(2A + B - 4A - 4B + 5B) &= xe^x \\
A - 4A + 5A &= 1 \\
A &= \frac{1}{2} \\
2A + B - 4A - 4B + 5B &= 0 \\
-2A + 2B &= 0 \\
-2\left(\frac{1}{2}\right) + 2B &= 0 \\
B &= \frac{1}{2} \\
y(x) &= e^x\left(\frac{x}{2} + \frac{1}{2}\right)
\end{aligned}
$$

$$
\boxed{y(x) = e^x\left(\frac{x}{2} + \frac{1}{2}\right)}
$$

1.4 5.5.10

Find a particular solution y_p of the following equation using the Method of Undetermined Coefficients. Primes denote the derivatives with respect to x.

$$
y'' + 9y = 4\cos(3x) + 6\sin(3x)
$$

$$
\begin{aligned}
r^2 + 9 &= 0 \\
r &= 0 \pm 3i \\
y(x) &= c_1\cos(3x) + c_2\sin(3x) \\
y_p(x) &= Ax\cos(3x) + Bx\sin(3x) \\
y_p'(x) &= -3Ax\sin(3x) + B\sin(3x) + 3Bx\cos(3x) + A\cos(3x) \\
y_p''(x) &= -9Bx\sin(3x) - 6A\sin(3x) - 9Ax\cos(3x) + 6B\cos(3x)
\end{aligned}
$$

$$
\begin{aligned}
&- 9Bx\sin(3x) - 6A\sin(3x) - 9Ax\cos(3x) \\
&+ 6B\cos(3x)) + 9(Ax\cos(3x) + Bx\sin(3x)) = 4\cos(3x) + 6\sin(3x) \\
6B &= 4 \\
B &= \frac{2}{3} \\
-6A &= 6 \\
A &= -1 \\
y(x) &= -x\cos(3x) + \frac{2}{3}x\sin(3x)
\end{aligned}
$$

$$
\boxed{y(x) = -x\cos(3x) + \frac{2}{3}x\sin(3x)}
$$

1.5 5.5.13

Find a particular solution y_p of the following equation using the Method of Undetermined Coefficients. Primes denote the derivatives with respect to t.

$$
y'' + 7y' + 14y = 1096e^{2t}\cos(8t)
$$

$r^2 + 7r + 14 = 0$

$$
\begin{aligned}
r &= -\frac{7}{2} \pm \frac{\sqrt{7}}{2}i \\
y(t) &= c_1 e^{-\frac{7}{2}x}\cos\left(\frac{\sqrt{7}}{2}\right) + c_2 e^{-\frac{7}{2}x}\sin\left(\frac{\sqrt{7}}{2}\right) \\
y_p(t) &= e^{2t}(A\cos(8t) + B\sin(8t)) \\
&= e^{2t}A\cos(8t) + e^{2t}B\sin(8t) \\
y_p'(t) &= 2Be^{2t}\sin(8t) - 8Ae^{2t}\sin(8t) + 8Be^{2t}\cos(8t) + 2Ae^{2t}\cos(8t) \\
y_p''(t) &= -60Be^{2t}\sin(8t) - 32Ae^{2t}\sin(8t) + 32Be^{2t}\cos(8t) - 60Ae^{2t}\cos(8t)
\end{aligned}
$$

$$
\begin{aligned}
32B - 60A + 7(8B + 2A) + 14(A) &= 1096 \\
88B - 32A &= 1096 \\
A &= -\frac{137 - 11B}{4} \\
-60B - 32A + 7(2B - 8A) + 14(B) &= 1096 \\
-32B - 88A &= 0 \\
-32B - 88\left(-\frac{137 - 11B}{4}\right) &= 0 \\
B &= 11 \\
A &= -4 \\
y(t) &= e^{2t}(-4\cos(8t) + 11\sin(8t))
\end{aligned}
$$

$$\boxed{y(t) = e^{2t}(-4\cos(8t) + 11\sin(8t))}$$

1.6 5.5.23

Using the Method of Undetermined Coefficients, determine the form of a particular solution for the differential equation. (Do not evaluate coefficients.)

$$y'' + 16y = 7t^3 \sin(4t)$$

$$\begin{aligned}
r^2 + 16r &= 0 \\
r &= 0 \pm 4i \\
y(t) &= c_1 \cos(4t) + c_2 \sin(4t) \\
y_p(t) &= t(A_3t^3 + A_2t^2 + A_1t + A_0)\cos(6t) + t(B_3t^3 + B_2t^2 + B_1t + B_0)\sin(6t)
\end{aligned}$$

1.7 5.5.33

Solve the following initial value problem.

$$y'' + 64y = \sin(2x); \quad y(0) = 1, y'(0) = 0$$

$$\begin{aligned}
r^2 + 64 &= 0 \\
r &= 0 \pm 8i \\
y_c(x) &= c_1 \cos(8x) + c_2 \sin(8x) \\
y_p(x) &= A\cos(2x) + B\sin(2x) \\
y_p'(x) &= -2A\sin(2x) + 2B\cos(2x) \\
y_p''(x) &= -4A\cos(2x) - 4B\sin(2x)
\end{aligned}$$

$$\begin{aligned}
-4B + 64(B) &= 1 \\
B &= \frac{1}{60} \\
-4A + 64(A) &= 0 \\
A &= 0 \\
y_p(x) &= \frac{1}{60}\sin(2x)
\end{aligned}$$

$$
\begin{aligned}
y(x) &= c_1\cos(8x) + c_2\sin(8x) + \frac{1}{60}\sin(2x)\\
y(0) &= c_1\cos(8(0)) + c_2\sin(8(0)) + \frac{1}{60}\sin(2(0)) = 1\\
c_1 &= 1\\
y'(x) &= -8c_1\sin(8x) + 8c_2\cos(8x) + \frac{1}{30}\cos(2x)\\
y'(0) &= -8c_1\sin(8(0)) + 8c_2\cos(8(0)) + \frac{1}{30}\cos(2(0)) = 0\\
8c_2 + \frac{1}{30} &= 0\\
c_2 &= -\frac{1}{240}\\
y(x) &= \cos(8x) - \frac{\sin(8x)}{240} + \frac{\sin(2x)}{60}
\end{aligned}
$$

$$\boxed{y(x) = \cos(8x) - \frac{\sin(8x)}{240} + \frac{\sin(2x)}{60}}$$

1.8 5.5.35

Solve the following initial value problem.

$$y'' - 2y' + 2y = x + 2; \quad y(0) = 3, y'(0) = 0$$

$$
\begin{aligned}
r^2 - 2r + 2 &= 0\\
r &= 1 \pm 1i\\
y_c(x) &= c_1e^x\cos(x) + c_2e^x\sin(x)\\
y_p(x) &= Ax + B\\
y_p'(x) &= A\\
y_p''(x) &= 0
\end{aligned}
$$

$$
\begin{aligned}
0 - 2(A) + 2(Ax + B) &= x + 2\\
2A &= 1\\
A &= \frac{1}{2}\\
-2A + 2B &= 2\\
B &= \frac{2 + 2(1/2)}{2}\\
B &= \frac{3}{2}\\
y_p(x) &= \frac{x}{2} + \frac{3}{2}
\end{aligned}
$$

$$
\begin{aligned}
y(x) &= c_1 e^x \cos(x) + c_2 e^x \sin(x) + \frac{x}{2} + \frac{3}{2} \\
y(0) &= c_1 e^0 \cos(0) + c_2 e^0 \sin(0) + \frac{0}{2} + \frac{3}{2} = 3 \\
c_1 &= \frac{3}{2} \\
y'(x) &= c_2 e^x \sin(x) - c_1 e^x \sin(x) + c_2 e^x \cos(x) + c_1 e^x \cos(x) + \frac{1}{2} \\
y'(0) &= c_2 e^0 \sin(0) - c_1 e^0 \sin(0) + c_2 e^0 \cos(0) + c_1 e^0 \cos(0) + \frac{1}{2} = 0 \\
c_2 + \frac{3}{2} + \frac{1}{2} &= 0 \\
c_2 &= -2 \\
y(x) &= \frac{3}{2} \cos(x) - 2e^x \sin(x) + \frac{x}{2} + \frac{3}{2}
\end{aligned}
$$

$$
\boxed{y(x) = \frac{3}{2} \cos(x) - 2e^x \sin(x) + \frac{x}{2} + \frac{3}{2}}
$$

1.9 5.5.49

Use the method of variation of parameters to find a particular solution of the following differential equation.

$$
y'' - 12y' + 36y = 4e^{6x}
$$

$$
\begin{aligned}
r^2 - 12r + 36 &= 0 \\
r &= 6, 6 \\
y_c(x) &= c_1 e^{6x} + c_2 x e^{6x} \\
y_1(x) &= e^{6x} \\
y_1'(x) &= 6e^{6x} \\
y_2(x) &= xe^{6x} \\
y_2'(x) &= e^{6x} + 6xe^{6x}
\end{aligned}
$$

$$W = \begin{vmatrix} e^{6x} & xe^{6x} \\ 6e^{6x} & e^{6x} + 6xe6x \end{vmatrix} = e^{12x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{6x} \\ 4e^{6x} & 6xe^{6x} \end{vmatrix} = -4e^{12x}x$$

$$W_2 = \begin{vmatrix} e^{6x} & 0 \\ 6e^{6x} & 4e^{6x} \end{vmatrix} = 4e^{12x}$$

$$c_1'(x) = \frac{W_1}{W} = -4x$$

$$\int c_1'(x) = -2x^2$$

$$c_2'(x) = \frac{W_2}{W} = 4$$

$$\int c_2'(x) = 4x$$

$$y_p(x) = -2x^2e^{6x} + 4x^2e^{6x}$$

1.10 5.5.51

Use the method of variation of parameters to find a particular solution of the following differential equation.

$$y'' + 36y = \cos(2x)$$

$$r^2 + 36r = 0$$

$$r = 0 \pm 6i$$

$$y_c(x) = c_1 \cos(6x) + c_2 \sin(6x)$$

$$y_1(x) = \cos(6x)$$

$$y_1'(x) = -6\sin(6x)$$

$$y_2(x) = \sin(6x)$$

$$y_2'(x) = 6\cos(6x)$$

$$W = \begin{vmatrix} \cos(6x) & \sin(6x) \\ -6\sin(6x) & 6\cos(6x) \end{vmatrix} = 6$$