Contents

1 Chapter 02 - Basic Structures

1.1 Sets

$\in$: belong to, is in

1.1.1 Definition 2

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x\,(x \in A \iff x \in B)$. We write $A = B$ if A and B are equal sets.

1.1.2 Definition 3

The set A is also a subset of B, and B is a superset of A, if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $B \supseteq A$. (So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.)

1.1.3 Definition 4

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

1.1.4 Countable and Uncountable Sets

- Countable
  - $\mathbb{N}$
  - $\mathbb{Z}$
  - $\mathbb{Q}$
- Uncountable
  - $\mathbb{R}$
  - $\mathbb{C}$

Let $S_0 = \{x\}$, and $S_1 = \{\{x\}\}$.

$$S_0 \neq S_1 \tag{1}$$

1.1.5 Example

1. List the members of these sets.

   a) $\left\{x \mid x \text{ is a real number such that } x^2 = 1\right\}$

   $$S = \left\{x \in \mathbb{R} \mid x^2 = 1\right\}$$

   b) $\{x \mid x \text{ is a positive integer less than } 12\}$

   $$S = \{x \in \mathbb{R} \mid 0 \leq x < 12\}$$

1.1.6 Definition 6

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

1.2 Set Operations

1.2.1 Definition 1

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\} \tag{2}$$

1.2.2 Definition 2

Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\} \tag{3}$$

1.2.3 Definition 3

Two sets are disjoint if their intersection is the empty set.

1.2.4 Definition 4

Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

1.2.5 Definition 5

Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

1.2.6 Proof

Let A, B be sets from U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

Proof:

- For " $\Longrightarrow$ "

  Given $A \subseteq B$, need to show $\overline{B} \subseteq \overline{A}$. Then $\forall x \in A$, we have $x \in B$.

  By contrapositive we have

$$\begin{aligned} \neg\,(x \in B) &\implies \neg\,(x \in A) \\ x \notin B &\implies x \notin A, \quad \overline{B} \subseteq \overline{A} \end{aligned}$$

- For " $\Longleftarrow$ "

Given $\overline{B} \subseteq \overline{A}$, we have $\forall y \in \overline{B}$, $y \in \overline{A}$ then the contrapositive is

$$\neg\left(y \in \overline{A}\right) \implies \neg\left(y \in \overline{B}\right)$$
$$y \in A \implies y \in B$$

1.2.7 Proof:

Use the identities to show that $\overline{(A \cup B)} \cap \overline{(B \cup C)} \cap \overline{(A \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}$

Proof:

$$\begin{aligned}\overline{A} \cap \overline{B} \cap \overline{C} &= \left(\overline{A} \cap \overline{B}\right) \cap \left(\overline{B} \cap \overline{C}\right) \cap \left(\overline{A} \cap \overline{C}\right) \\ &= \overline{A} \cap \left(\overline{B} \cap \overline{B}\right) \cap \left(\overline{C} \cap \overline{C}\right) \cap \overline{A} \\ &= \overline{A} \cap \overline{B} \cap \overline{C}\end{aligned}$$

1.2.8 Union

The union is a collection of sets is the set that contains those elements that are member s of at least one set in the collection.

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

1.2.9 Intersection

The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

1.3 Functions

- Sets: A, B, C, domain, codomain, range
- Relations: functions (f, g, h)
- Elements: image, preimage

1.3.1 Example

$$f_1 : \mathbb{R} \to \mathbb{R} \text{ as } y = f_1(x) = 3x - 2$$
$$f_2 : \mathbb{R} \to \mathbb{R} \text{ as } y = e^x$$
$$f_3 : \mathbb{R} \to \mathbb{R} \text{ as } y = \sqrt{x}$$

$$f : A \to B$$
$$\text{range}(f) = \{y \in B \mid \forall x \in S \subseteq A\}$$

1.3.2 Properties

1) f is injective – one-to-one

2) f is surjective –

3) f is bijective if **1)** and **2)**

Let $f : A \to B$ be a function.

We say f is injective if

$$(x_1 \neq x_2 \implies f(x_1) \neq f(x_2)) \iff (f(x_1) = f(x_2) \implies x_1 = x_2)$$

We say f is surjective if $\forall y \in B$, $\exists x \in A$ such that $y = f(x) \iff \text{range}(f) = B$

We say f is bijective if f is injective and surjective.

Monotonic Function: We say f is increasing (strictly increasing) if $x_1 > x_2 \implies f(x_1) > f(x_2)$

1.3.3 Example 2.3.24

Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = \frac{1}{f(x)}$ is strictly decreasing.

$$\begin{aligned}
&\to \frac{1}{f(x_1)} < \frac{1}{f(x_2)} \\
&\to f(x_1) \cdot f(x_2) \left(\frac{1}{f(x_1)}\right) < f(x_1) \cdot f(x_2) \left(\frac{1}{f(x_2)}\right) \\
&\to f(x_2) < f(x_1)
\end{aligned}$$

where the first implication comes from the fact that $f(x) > 0, \forall x \in \mathbb{R}$.

1.3.4 Example 2.3.73.b

Prove or disprove each of these statements about the floor and ceiling functions.

$$\lfloor 2x \rfloor = 2 \lfloor x \rfloor$$

Consider $x = 1.6$

$$\lfloor 2x \rfloor = \lfloor 2 \cdot 1.6 \rfloor = \lfloor 3.2 \rfloor = 3$$
$$2 \lfloor x \rfloor = 2 \lfloor 1.6 \rfloor = 2(1) = 2$$

Hence $\lfloor 2x \rfloor = 2 \lfloor x \rfloor$ for all $x \in \mathbb{R}$ is false.

1.3.5 Inverse Function

Let $f : A \to B$ be a function. If f is injective, then there exists $g : B \to A$ such that $f \cdot g\,(y) = y, \forall y \in B$ and $g \cdot f\,(x) = x, \forall x \in A$. We call g the inverse of f. We can denote $g = f^{-1}$.

1.4 Sequences and Summations

Let $f : A \to B$ where $A = \{1, 2, 3, \cdots\}$ or $\{0, 1, 2, 3, \cdots\}$, $B = \mathbb{R}$.

Special sequences:

- Geometric Sequence
$$a_n = a_1 r^{n-1} = a_0 r^n \tag{4}$$

- Arithmetic Sequence
$$a_n = a_1 + (n-1)d = a_0 + nd \tag{5}$$

- Fibonacci Sequence: $f_0, f_1, f_2, \cdots$ is defined by the initial conditions $f_0 = 0, f_1 = 1$ and the recurrence relation
$$f_n = f_{n-1} + f_{n-2} \tag{6}$$

In order to describe a sequence:

- Closed formula
- Recurrence relation
- Verbally

1.4.1 Geometric Sequence

If $r = 1$ then $a_n = ar^n = a$ then $\sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n - 0 + 1)a$.

Let $k = j + 1$.

$$\begin{aligned}
j &= k - 1 \\
j &= 0, k = 1 \\
&= \sum_{j=0}^{j=n} ar^{j+1} \\
&= \sum_{k=1}^{k=n+1} ar^k \\
&= a + ar + ar^2 + \cdots + ar^{n+1} - a \\
&= \sum_{k=0}^{n} ar^k + ar^{n+1} - a
\end{aligned}$$

If $r \neq 1$

Notice that $(1-x)\left(1+x+x^2+\cdots+x^n\right) = 1+(x-x)+(x^2-x^2)+\cdots+(x^n-x^n)-x^{n+1} = 1-x^{n+1}$

Then

$$\begin{aligned}\sum_{j=0}^{n} ar^j &= a+ar+ar^2+\cdots+ar^n \\ &= a\left(1+r+r^2+\cdots+r^n\right) \\ &= a\left(\frac{1-r^{n+1}}{1-r}\right) \\ &= a\left(\frac{r^{n+1}-1}{r-1}\right)\end{aligned}$$

1.4.2 Practice

Prove $\sum_{k=1}^{n} k = 1+2+3+\cdots+k = \frac{n(n+1)}{2}$

Method 1:

$$\begin{aligned}\sum_{k=1}^{n} k &= 1+2+3+\cdots+n \\ 2\cdot\sum_{k=1}^{n} k &= (1+2+3+\cdots+n)+(1+2+3+\cdots+n) \\ &= \sum_{k=1}^{n}(n+1) = (n+1)\cdot(n-1+n) = (n+1)\cdot n \\ \sum_{k=1}^{n} k &= \frac{n\,(n+1)}{2}\end{aligned}$$

Method 2: Telescoping

Notice that

$$\begin{aligned}\sum_{k=0}^{n-1}(a_{k+1}-a_k) &= a_1-a_0+a_2-a_1+a_3-a_2+\cdots+a_n-a_{n-1}\\ &= \sum_{i=1}^{n}(a_i-a_{i-1}) = a_n-a_0\end{aligned}$$

$$\begin{aligned}\text{and, } i^2-(i-1)^2 &= i^2-\left(i^2-2i+1\right) = 2i-1\\ \sum_{i=1}^{n}\left(i^2-(i-1)^2\right) &= \sum_{i=1}^{n}2i-1\\ n^2-0^2 &= 2\sum_{i=1}^{n}i-\sum_{i=1}^{n}1, \quad \text{where } a_i=i^2 \text{ for telescoping}\\ \sum_{i=1}^{n} &= \frac{n^2+n}{2}\end{aligned}$$

Prove $\sum_{k=1}^{n}k^2 = 1^2+2^2+3^2+\cdots+k^2 = \frac{n(n+1)(2n+1)}{6}$, Let $a_i = i^3$

$$\begin{aligned}\sum_{i=1}^{n}(a_i-a_{i-1}) &= a_n-a_0\\ i^3-(i-1)^3 &= 3i^2-3i+1\\ \sum_{i=1}^{n}\left(i^3\right) &= 3\sum_{i=1}^{n}i^2-3\sum_{i=1}^{n}i+\sum_{i=1}^{n}1\\ 3\sum_{i=1}^{n}i^2 &= \sum_{i=1}^{n}\left(i^3\right)+3\sum_{i=1}^{n}i-\sum_{i=1}^{n}1\end{aligned}$$

1.5 Cardinality of Sets

Compare numbers (including ∞)

1. Finite Elements
   - Finite: $0,1,2,3,\cdots,n$
2. Infinitely Many Elements
   - Countable: $\mathbb{N}=\mathbb{Z}^+,\mathbb{Z},\mathbb{Q}$

For $\mathbb{Z}$:

$\mathbb{N}$	$\mathbb{Z}^+$
1	0
2	1
3	-1
4	2
5	-2
6	3
7	-3

For $\mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^+, n \neq 0, \gcd(m, n) = 1 \right\}$

- Uncountable: $\mathbb{R}, \mathbb{R}^2, \cdots, \mathbb{R}^n$

1.5.1 Shroder-Bernstein Theorem Example

Let $A = [0, 1], B = (0, 1)$. Show that $|A| = |B|$.

Proof:

- Consider $f_1 : B \to A$ as $f(x) = x$. Thus f_1 is one-to-one $\implies |B| \leq |A|$.
- Next consider $f_2 : A \to B$ as $f(x) = \frac{x}{2} + a, x \in [0, 1], a \in \left(0, \frac{1}{2}\right)$. Then $f(A) = \left[a, \frac{1}{2} + a\right] \subseteq B$. Thus f_2 is one-to-one $\implies |A| \leq |B|$.

1.6 Matrices