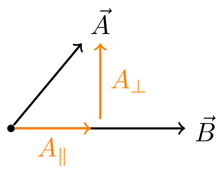
Contents

1 Energy

1.1 Review: Dot Product



$$\vec{A} \cdot \vec{B} = AB$$

if $\vec{A}$ & $\vec{B}$ are already $\parallel$

$$\vec{A} \cdot \vec{B} = AB$$

if A & B are $\perp$ then $\vec{A} \cdot \vec{B} = 0$.

Dot Product: How much of $\vec{A}$ is applied to $\vec{B}$?

Work

- by definition, is the work done by a force over a path.
- is a measure of how much a force acts on a mass for the duration of a displacement.
- 's unit is:
    1. MKS - J (Joule)
    2. CGS - erg

An amount of work is called "Energy"

$$W = \int \vec{F} \cdot d\vec{x}$$

1. W is "work"
2. $\vec{F}$ is the **applied force**
3. $d\vec{x}$ is the **displacement along a path**

1.2 Line Element

Consider force:

$$\vec{F} = (2\,\mathrm{N\,m^{-1}})x\hat{x} + (1\,\mathrm{N\,m^{-2}})xy\hat{y}$$

Recall:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

$d\vec{x}$ is called the "line element" and is tied to the coordinate system.

The line element **never** changes!

$$\begin{aligned} d\vec{x} &= dx\hat{x} + dy\hat{y} + dz\hat{z} \\ d\hat{x} &= dr\hat{r} + rd\theta\hat{\theta} + dz\hat{z} \end{aligned} \tag{1}$$

$$\begin{aligned} \hat{F} \cdot d\hat{x} &= F_x dx + F_y dy + F_z dz \\ &= (2\,\mathrm{N\,m^{-1}})xdx + (1\,\mathrm{N\,m^{-2}})xydy \end{aligned}$$

Always substitute so you are integrating one variable.

The path of integration *is* the substitution.

$$\begin{aligned} y &= 3x + 2m \\ dy &= 3dx \end{aligned}$$

Now we can solve the integral

$$\begin{aligned} W &= \int_C \vec{F} \cdot d\vec{x} \\ &= \int_C (2\,\mathrm{N\,m^{-1}})dx + \int_C (1\,\mathrm{N\,m^{-2}})xydy \quad \text{(Substitute)} \\ &= \int_0^{3m} (2\,\mathrm{N\,m^{-1}})xdx + \int_0^{3m} (1\,\mathrm{N\,m^{-2}})(x)(3x+2m)(3)dx \\ W &= \frac{2\,\mathrm{N\,m^{-1}}}{2} x^2 \Big|_0^{3m} + 3\,\mathrm{N\,m^{-2}} \int_0^{3m} 3x^2 dx + 3\,\mathrm{N\,m^{-2}} \int_0^{3m} (2\,\mathrm{m})xdx \\ &= 9\,\mathrm{N\,m} + (3\,\mathrm{N\,m^{-1}})x^3 \Big|_0^{3m} + (3\,\mathrm{N\,m^{-1}})x^2 \Big|_0^{3m} \\ &= 9\,\mathrm{N\,m} + 81\,\mathrm{N\,m} + 27\,\mathrm{N\,m} \\ W &= 117\,\mathrm{N\,m} \end{aligned}$$

$$\boxed{W = 117\,\mathrm{J}}$$

1.3 Forces & Work

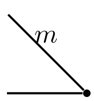
For forces that are uniform

$$\text{uniform: } \frac{df}{dx} = 0$$

$$\text{constant: } \frac{df}{dt} = 0$$

$$\begin{aligned}
W &= \int \vec{F} \cdot d\vec{x} \\
W &= \vec{F} \cdot \int d\vec{x} \\
W &= \vec{F} \cdot \Delta\vec{x}
\end{aligned}$$

Example of Finding Work



$$\begin{aligned}
m &= 1\,\mathrm{kg} \\
\theta &= 40^\circ \\
\mu_k &= 0.7 \\
\Delta x &= 19\,\mathrm{m} \\
mg &= 10\,\mathrm{N}
\end{aligned}$$

$$\begin{aligned}
\sum F_y &= 0 \\
N - mg\cos(\theta) &= 0 \\
N &= mg\cos(\theta) \\
&= (10\,\mathrm{N})\cos(40^\circ) \\
N &= 7.7\,\mathrm{N}
\end{aligned}$$

$$\begin{aligned}
f &= \mu N \\
&= (0.7)(7.7\,\mathrm{N}) \\
f &= 5.4\,\mathrm{N}
\end{aligned}$$

Calculate the work for each force:

$$\begin{aligned}
W_N &= \vec{F}_N \cdot \Delta\vec{x} \\
&= (0\hat{x} + 7.7\,\mathrm{N}\hat{y}) \cdot (19\,\mathrm{m}\hat{x} + 0\hat{y}) \\
W_N &= 0
\end{aligned}$$

$$\begin{aligned}
W_{mg} &= m\vec{g} \cdot \Delta\hat{x} \\
&= [mg\sin(\theta)\hat{x} + (-mg\cos(\theta))\hat{y}] \cdot [\Delta x\hat{x} + 0\hat{y}] \\
W_{mg} &= mg\Delta x\sin(\theta) \\
&= (10\,\mathrm{N})(19\,\mathrm{m})\sin(40^\circ) \\
W_{mg} &= 122\,\mathrm{J}
\end{aligned}$$

$$\begin{aligned}
W_f &= \vec{f} \cdot \Delta\vec{x} \\
&= (-5.4\,\mathrm{N}\hat{x}) \cdot (19\,\mathrm{m}\hat{x}) \\
W_f &= -103\,\mathrm{J}
\end{aligned}$$

$$\boxed{W_N = 0, W_{mg} = 122\,\mathrm{J}, W_f = -103\,\mathrm{J}}$$

- $W > 0 \rightarrow$ Speeds system up
- $W < 0 \rightarrow$ Slows system up
- $W = 0 \rightarrow \Delta v = 0$
- Net work $\sum W_i$ is indicative of the overall motion

$$\begin{aligned}
W_{\text{net}} &= \sum W_i \\
&= W_{mg} + W_N + W_f \\
&= 122\,\mathrm{J} + 0 - 103\,\mathrm{J} \\
W_{\text{net}} &= 19\,\mathrm{J}
\end{aligned}$$

$$\boxed{W_{\text{net}} = 19\,\mathrm{J}}$$

How is work related to speed?

$$W = \int \vec{F} \cdot d\vec{x}$$

Consider 1D case where $\cos(\theta) \to 1$

$$\begin{aligned}
W &= \int (F)\,dx \\
&= \int (ma)\,dx \\
&= \int \left(m\frac{dv}{dt}\right) dx \\
&= \int \left(mdv\left(\frac{dx}{dt}\right)\right) \\
&= \int (mv)\,dv \\
&= \int_{v_i}^{v_f} (mv)\,dv \\
&= \frac{1}{2}mv^2\Big|_{v_i}^{v_f} \\
W &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2
\end{aligned}$$

Energy is associated with speed is called **Kinetic Energy (KE)**

$$\text{KE} = \frac{1}{2}mv^2$$

is the kinetic energy of translation.

Work is also defined as

$$W = \Delta\text{KE}$$

this is called the **Work-Kinetic Energy Theorem**

if the block started from rest:

$$\begin{aligned}
W &= \frac{1}{2}mv_f^2 - 0 \\
v_f &= \sqrt{\frac{2W_{\text{net}}}{m}} \\
&= \sqrt{\frac{2(19\,\text{J})}{1\,\text{kg}}} \\
v_f &= 6.2\,\text{m}\,\text{s}^{-1}
\end{aligned}$$

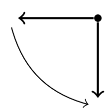
$$\boxed{v_f = 6.2\,\text{m}\,\text{s}^{-1}}$$

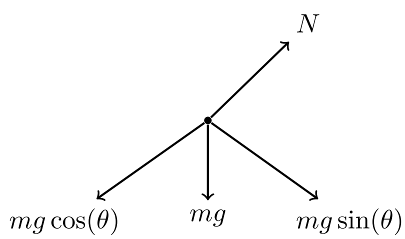
(Using the graph from above)

$$\begin{aligned}
\sin(\theta) &= \frac{h}{\Delta x} \\
h &= \Delta x \sin(\theta) \\
h &= (19\,\mathrm{m}) \sin(40^\circ) \\
h &= 12.2\,\mathrm{m}
\end{aligned}$$

How does the answer change if there is no friction?

$$\begin{aligned}
W_{\text{net}} &= W_{mg} + W_N \\
&= 122\,\mathrm{J} + 0 \\
W_{\text{net}} &= 122\,\mathrm{J}
\end{aligned}$$





$$W_N = 0$$

$$\begin{aligned}
W_{mg} &= \int \Big(mg\cos(\theta)\hat{r} + mg\sin(\theta)\hat{\theta} \Big) \cdot \Big(dr\hat{r} + rd\theta\hat{\theta} \Big) \\
&= \int \Big(mg\cos(\theta)\hat{r} + mg\sin(\theta)\hat{\theta} \Big) \cdot \Big((0)\hat{r} + rd\theta\hat{\theta} \Big) \\
&= \int_0^{90^\circ} mgr\sin(\theta)d\theta \\
W_{mg} &= mgr\cos(\theta)\Big|_0^{90^\circ} \\
&= mgr\cos(\theta)\Big|_{90^\circ}^{0} \\
W_{mg} &= mgr \\
&= (10\,\mathrm{N})(12.2\,\mathrm{m}) \\
W_{mg} &= 122\,\mathrm{J}
\end{aligned}$$

For the case of gravity, it would seem that only mg and the change in height dictate work.

$$\begin{aligned}
W &= \int \vec{F} \cdot d\hat{x} \\
&= \int mg\hat{y} \cdot (dx\hat{x} + dy\hat{y}) \\
W &= \int_0^h mgdy \\
W &= mgh
\end{aligned}$$

Gravity is completely path independent.

Conservative force - Work is path independent

If you can write a function of position called a **Potential Energy Function** whose derivative equals force, then the force is conservative.

$$\begin{aligned}
\mathrm{PE}_g &= mgy \\
-\frac{d\left(\mathrm{PE}_g\right)}{dy}\hat{y} &= -mg\hat{y} = \vec{F}_g
\end{aligned}$$