Contents

1 Section 10.2

Laplace transform of the second derivative of a function

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0) \tag{1}$$

1.1 10.2.1

Solve the following differential equation by Laplace transforms. The function is subject to the given conditions.

$$y'' + 64y = 0, y(0) = 0, y'(0) = 1$$

$$\begin{aligned}
\mathcal{L}(y'') + 64\mathcal{L}(y) &= \mathcal{L}(0) \\
\left[s^2\mathcal{L}(y) - sy(0) - y'(0)\right] + 64\mathcal{L}(y) &= \mathcal{L}(0) \\
s^2\mathcal{L}(y) - 0 - 1 + 64\mathcal{L}(y) &= \mathcal{L}(0) \\
\mathcal{L}(y)(s^2 + 64) &= 1 \\
\mathcal{L}(y) &= \frac{1}{s^2 + 64} \\
y &= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 64}\right) \\
y &= \frac{1}{8}\sin(8t)
\end{aligned}$$

1.2 10.2.3

Solve the following differential equation by Laplace transforms. The function is subject to the given conditions.

$$y'' + 9y = 9, y(0) = 1, y'(0) = 3$$

$$\begin{aligned}
\mathcal{L}(y'') + 9\mathcal{L}(y) &= \mathcal{L}(9) \\
s^2\mathcal{L}(y) + sy(0) + y'(0) + 9\mathcal{L}(y) &= \frac{9}{s} \\
s^2\mathcal{L}(y) + s + 1 + 9\mathcal{L}(y) &= \frac{9}{s} \\
\mathcal{L}(y)(s^2 + 9) &= \frac{9}{s} - s - 1 \\
\mathcal{L}(y) &= \frac{-s^2 - s + 9}{s(s^2 + 9)}
\end{aligned}$$

$$
\begin{aligned}
y &= \mathcal{L}^{-1}\left(\frac{-2s-1}{s^2+9}\right) + \mathcal{L}^{-1}\left(\frac{1}{s}\right) \\
y &=
\end{aligned}
$$

$$
\begin{aligned}
mx'' + kx &= 0 \\
m(s^2\mathcal{L}(x) - sx(0) - x'(0)) + k\mathcal{L}(x) &= 0 \\
ms^2\mathcal{L}(x) - smx(0) - mx'(0) + k\mathcal{L}(x) &= 0 \\
\mathcal{L}(x)(ms^2 + k) &= smx(0) + mx'(0) \\
\mathcal{L}(x) &= \frac{smx(0) + mx'(0)}{ms^2 + k} \\
\mathcal{L}(x) &= \frac{smx(0)}{ms^2 + k} + \frac{mx'(0)}{ms^2 + k} \\
\mathcal{L}(x) &= \frac{ms}{ms^2 + k} + \frac{m}{ms^2 + k}, \quad x(0) = 1, x'(0) = 1 \\
\mathcal{L}(x) &= \frac{s}{s^2 + \frac{k}{m}} + \frac{1}{s^2 + \frac{k}{m}} \\
x &= \mathcal{L}^{-1}\left(\frac{s}{s^2 + \frac{k}{m}}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2 + \frac{k}{m}}\right) \\
x &= \cos\left(\sqrt{\frac{k}{m}}t\right) + \sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right) \\
x &= \cos(\omega t) + \omega^{-1}\sin(\omega t)
\end{aligned}
$$