Contents

1 Section 4.2

1.1 4.2.15

Find the solution vectors $\vec{u}$ and $\vec{v}$ such that the solution space is the set of all linear combinations of the form $s\vec{u} + t\vec{v}$.

$$\begin{cases} x_1 - 4x_2 + x_3 - 8x_4 = 0 \\ x_1 + 2x_2 + x_3 + 16x_4 = 0 \\ x_1 + x_2 + x_3 + 12x_4 = 0 \end{cases}$$

Find rref($\mathbf{A}$)

$$\mathbf{A} = \begin{bmatrix} 1 & -4 & 1 & -8 \\ 1 & 2 & 1 & 16 \\ 1 & 1 & 1 & 12 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}_2 &= \mathbf{A}_2 - \mathbf{A}_1 \\ \mathbf{A}_2 &= \frac{1}{6}\mathbf{A}_2 \\ \mathbf{A}_3 &= \mathbf{A}_3 - \mathbf{A}_1 \\ \mathbf{A} &= \begin{bmatrix} 1 & -4 & 1 & -8 \\ 0 & 1 & 0 & 4 \\ 0 & 5 & 0 & 20 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A}_3 &= \mathbf{A}_3 - 5\mathbf{A}_2 \\ \mathbf{A}_1 &= \mathbf{A}_1 + 4\mathbf{A}_2 \\ \mathbf{A} &= \begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \text{rref}(\mathbf{A}) &= \begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Find the system solution

$$\begin{cases} x_1 \qquad + x_3 + 8x_4 = 0 \\ \qquad x_2 \qquad + 4x_4 = 0 \\ \qquad\qquad\qquad 0 = 0 \end{cases}$$

From the rref($\mathbf{A}$) and system of equations, the leading variables x_1, x_2 and free variables x_3, x_4 can be determined. Solving for the leading variables:

$$\begin{aligned} x_1 + x_3 + 8x_4 &= 0 \\ x_1 &= -x_3 - 8x_4 \end{aligned}$$

$$\begin{aligned} x_2 + 4x_4 &= 0 \\ x_2 &= -4x_4 \end{aligned}$$

Thus the solution can be found as:

$$\begin{aligned} \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ \vec{x} &= \begin{bmatrix} -x_3 - 8x_4 \\ -4x_4 \\ x_3 \\ x_4 \end{bmatrix} \\ \vec{x} &= \begin{bmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -8x_4 \\ -4x_4 \\ 0 \\ x_4 \end{bmatrix} \\ \vec{x} &= x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ -4 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore the system can be described in the form $\vec{x} = s\vec{x_3} + t\vec{x_4}$:

$$\boxed{\vec{x} = s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8 \\ -4 \\ 0 \\ 1 \end{bmatrix}}$$

1.2 4.2.21

Reduce the given system to echelon form to find a single solution vector $\vec{u}$ such that the solution space is the set of all scalar multiples of $\vec{u}$.

$$\begin{cases} x_1 + 7x_2 + 3x_3 - 4x_4 = 0 \\ 2x_1 + 7x_2 + 3x_3 - x_4 = 0 \\ 3x_1 + 5x_2 + 2x_3 + 3x_4 = 0 \end{cases}$$

Begin with finding rref($\mathbf{A}$):

$$\mathbf{A} = \begin{bmatrix} 1 & 7 & 3 & -4 \\ 2 & 7 & 3 & -1 \\ 3 & 5 & 2 & 3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}_2 &= \mathbf{A}_2 - 2\mathbf{A}_1 \\ \mathbf{A}_3 &= \mathbf{A}_3 - 3\mathbf{A}_1 \\ \mathbf{A} &= \begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & -7 & -3 & 7 \\ 0 & -16 & -7 & 15 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{A}_1 + \mathbf{A}_2 \\ \mathbf{A}_2 &= -\frac{1}{7}\mathbf{A}_2 \\ \mathbf{A}_3 &= \mathbf{A}_3 + 16\mathbf{A}_2 \\ \mathbf{A} &= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & \frac{3}{7} & -1 \\ 0 & 0 & -\frac{1}{7} & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A}_2 &= \mathbf{A}_2 - \mathbf{A}_3 \\ \mathbf{A}_3 &= -7\mathbf{A}_3 \\ \mathbf{A}_2 &= \mathbf{A}_2 - \frac{4}{7}\mathbf{A}_3 \\ \mathbf{A} &= \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{bmatrix} \end{aligned}$$

The system of equations can be found as:

$$\begin{cases} x_1 & & & + 3x_4 = 0 \\ & x_2 & & - 4x_4 = 0 \\ & & x_3 & + 7x_4 = 0 \end{cases}$$

$$\begin{cases} x_1 & & & = -3x_4 \\ & x_2 & & = 4x_4 \\ & & x_3 & = -7x_4 \end{cases}$$

Aliasing $x_4 = s$, and finding $\vec{x}$ in the form $\vec{x} = s\vec{u}$:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -3x_4 \\ 4x_4 \\ -7x_4 \\ x_4 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -3s \\ 4s \\ -7s \\ s \end{bmatrix}$$

$$\vec{x} = s \begin{bmatrix} -3 \\ 4 \\ -7 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{x} = s \begin{bmatrix} -3 \\ 4 \\ -7 \\ 1 \end{bmatrix}}$$