1 Section 7.2

1.1 7.2.1

Verify the product law for differentiation, $(\boldsymbol{AB})' = \boldsymbol{A}'\boldsymbol{B} + \boldsymbol{AB}'$ where $\boldsymbol{A}(t) = \begin{bmatrix} t & 2t-1 \\ t^3 & \frac{1}{t} \end{bmatrix}$ and $\boldsymbol{B}(t) = \begin{bmatrix} 1-t & 1+t \\ 2t^2 & 2t^3 \end{bmatrix}$.

To calculate $(\boldsymbol{AB})'$, first calculate $\boldsymbol{AB}$.

$$\boldsymbol{AB} = \begin{bmatrix} (t)(1-t) + (2t-1)(2t^2) & (t)(1+t) + (2t-1)(2t^3) \\ (t^3)(1-t) + \left(\frac{1}{t}\right)(2t^2) & (t^3)(1+t) + \left(\frac{1}{t}\right)(2t^3) \end{bmatrix}$$

$$\boldsymbol{AB} = \begin{bmatrix} 4t^3 - 3t^2 + t & 4t^4 - 2t^3 + t^2 + t \\ -t^4 + t^3 + 2t & t^4 + t^3 + 2t^2 \end{bmatrix}$$

Now take the derivative of $\boldsymbol{AB}$ to find $(\boldsymbol{AB})'$.

$$(\boldsymbol{AB})' = \begin{bmatrix} 12t^2 - 6t + 1 & 16t^3 - 6t^2 + 2t + 1 \\ -4t^3 + 3t^2 + 2 & 4t^3 + 3t^2 + 4t \end{bmatrix}$$

To calculate $\boldsymbol{A}'\boldsymbol{B} + \boldsymbol{AB}'$, first calculate $\boldsymbol{A}'$.

$$\boldsymbol{A}' = \begin{bmatrix} 1 & 2 \\ 3t^2 & -\frac{1}{t^2} \end{bmatrix}$$

Now find $\boldsymbol{A}'\boldsymbol{B}$.

$$\boldsymbol{A}'\boldsymbol{B} = \begin{bmatrix} 4t^2 - t + 1 & 4t^3 + t + 1 \\ -3t^3 + 3t^2 - 2 & 3t^3 + 3t^2 - 2t \end{bmatrix}$$

1.2 7.2.3

Write the given system in the form $\mathbf{x}' = \boldsymbol{P}(t)\mathbf{x} + \mathbf{f}(t)$.

$$x' = -9y$$
$$y' = 5x$$

$$\boldsymbol{P} = \begin{bmatrix} 0 & -9 \\ 5 & 0 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} 0 & -9 \\ 5 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

1.3 7.2.5

Write the given system in the form $\mathbf{x}' = \boldsymbol{P}(t)\mathbf{x} + \mathbf{f}(t)$.

$$x' = 9x + 4y + 6e^t$$
$$y' = 6x - y - t^3$$

$$\boldsymbol{P} = \begin{bmatrix} 9 & 4 \\ 6 & -1 \end{bmatrix}$$
$$\mathbf{f} = \begin{bmatrix} 6e^t \\ -t^3 \end{bmatrix}$$
$$\mathbf{x}' = \begin{bmatrix} 9 & 4 \\ 6 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 6e^t \\ -t^3 \end{bmatrix}$$

1.4 7.2.7

Write the given system in the form $\mathbf{x}' = \boldsymbol{P}(t)\mathbf{x} + \mathbf{f}(t)$.

$$x' = 5y + 5z$$
$$y' = 4z + 8x$$
$$z' = 8x + 2y$$

$$\boldsymbol{P} = \begin{bmatrix} 0 & 5 & 5 \\ 8 & 0 & 4 \\ 8 & 2 & 0 \end{bmatrix}$$
$$\mathbf{f} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\mathbf{x}' = \begin{bmatrix} 0 & 5 & 5 \\ 8 & 0 & 4 \\ 8 & 2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1.5 7.2.9

Write the given system in the form $\mathbf{x}' = \boldsymbol{P}(t)\mathbf{x} + \mathbf{f}(t)$.

$$x' = 8x - 9y + z + t$$
$$y' = x - 3z + t^2$$
$$z' = 3y - 9z + t^3$$

$$
\begin{aligned}
\boldsymbol{P} &= \begin{bmatrix} 7 & -9 & 1 \\ 1 & 0 & -3 \\ 0 & 3 & -9 \end{bmatrix} \\
\mathbf{x} &= \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix} \\
\mathbf{x}' &= \begin{bmatrix} 7 & -9 & 1 \\ 1 & 0 & -3 \\ 0 & 3 & -9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}
\end{aligned}
$$

1.6 7.2.15

Find a particular solution of the indicated linear system that satisfies the initial conditions $x_1(0) = 7$, $x_2(0) = -5$.

$$
\mathbf{x}' = \begin{bmatrix} 4 & -1 \\ 7 & -4 \end{bmatrix}; \mathbf{x}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}; \mathbf{x}_2 = \begin{bmatrix} e^{-3t} \\ 7e^{-3t} \end{bmatrix}
$$

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + C_2 \begin{bmatrix} e^{-3t} \\ 7e^{-3t} \end{bmatrix}
$$

$$
\begin{aligned}
x_1(t) &= C_1 e^{3t} + C_2 e^{-3t} \\
x_2(t) &= C_1 e^{3t} + 7C_2 e^{-3t}
\end{aligned}
$$

$$
\begin{aligned}
x_1(0) = C_1 e^{3(0)} + C_2 e^{-3(0)} &= 7 \\
C_1 + C_2 &= 7 \\
C_1 &= 7 - C_2
\end{aligned}
$$

$$
\begin{aligned}
x_2(0) = C_1 e^{3(0)} + 7C_2 e^{-3(0)} &= -5 \\
C_1 + 7C_2 &= -5 \\
(7 - C_2) + 7C_2 &= -5 \\
C_2 &= -2 \\
C_1 &= 7 - (-2) \\
C_1 &= 9
\end{aligned}
$$

$$
\begin{aligned}
x_1(t) &= 9e^{3t} - 2e^{-3t} \\
x_2(t) &= 9e^{3t} - 14e^{-3t}
\end{aligned}
$$

1.7 7.2.29

Find a particular solution of the indicated linear system that satisfies the initial conditions $x_1(0) = 5$, $x_2(0) = 4$, and $x_3(0) = 2$.

$$\mathbf{x}' = \begin{bmatrix} -14 & -17 & -1 \\ 12 & 15 & 1 \\ -12 & -12 & 2 \end{bmatrix}; \mathbf{x}_1 = e^{-2t} \begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix}, \mathbf{x}_2 = e^{2t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = e^{3t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 5 \\ -3 & -1 & -1 & 4 \\ 3 & 1 & 0 & 2 \end{array}\right]$$

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -25 \\ -6 \end{bmatrix}$$

$$x_1(t) = (C_1)4e^{-2t} + (C_2)e^{2t} + (C_3)e^{3t}$$
$$x_2(t) = (C_1) - 3e^{-2t} + (C_2) - e^{2t} + (C_3) - e^{3t}$$
$$x_3(t) = (C_1)3e^{-2t} + (C_2)e^{2t}$$

$$x_1(t) = 36e^{-2t} - 25e^{2t} - 6e^{3t}$$
$$x_2(t) = -27e^{-2t} + 25e^{2t} + 6e^{3t}$$
$$x_3(t) = 27e^{-2t} - 25e^{2t}$$

1.8 7.2.33

(a) Show that the vector functions $\mathbf{x}_1(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $\mathbf{x}_2(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$ are linearly independent on the real line.

(b) Why does it follow from the Wronskians of solutions that there is no continuous matrix $\boldsymbol{P}(t)$ such that $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions of $\mathbf{x}' = \boldsymbol{P}(t)\mathbf{x}$?

(a) The vector-valued functions $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linearly dependent on the interval $\mathbf{I}$ provided that there exist constants $c_1, c_2, \cdots, c_n$ not all zero such that $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) = 0$ for all t in $\mathbf{I}$. Otherwise they are linearly independent.
The m atrix $\mathbf{x}_2 = t \cdot \mathbf{x}_1$, so neither is a constant multiple of the other. Therefore $\mathbf{x}_1(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $\mathbf{x}_2(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$ are linearly independent.

(b) The Wronskian of solutions of $\mathbf{x}' = \boldsymbol{P}(t)\mathbf{x}$ with a continuous matrix $\boldsymbol{P}(t)$ have only two possibilities for solutions of homoegeneous systems, either $W = 0$ at every point of $\mathbf{I}$, or $W = 0$ at no point of $\mathbf{I}$. Find the Wronskian of $\mathbf{x}_1(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $\mathbf{x}_2(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$.

$$W = 0$$