Lesson 2

1 Introduction to Vectors & Scalars

1.1 Vectors & Scalars

Scalar = Magnitude
Vector = Magnitude + Direction

Vectors are represented by arrows – tail and tip

$\longrightarrow$

Triangle addition:

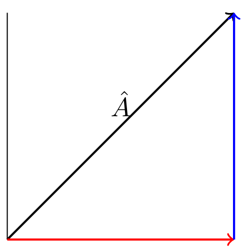
$$\hat{A} = 20 \text{ @ } 40° \text{ above} + \hat{x}$$

$\hat{x} \rightarrow$ Unit vector gives direction with Magnitude 1 "the x direction"

Vector B is 45 @ 60 ° Right of $+\hat{y}$

1.2 Algebra as an alternative to Euclidean Geometry

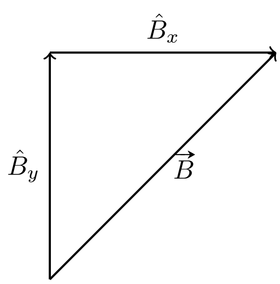
In order to add vectors, first break vectors into <u>components</u>



<u>Any vector is the sum of its components</u>

$$\boxed{\hat{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}} \tag{1}$$

$$\hat{A} = A\cos(\theta)\hat{x} + A\sin(\theta)\hat{y}\;(+0\hat{z})$$



Write $\vec{B}$ in component form:

$$\vec{B} = B\sin(\phi)\hat{x} + B\cos(\phi)\hat{y}$$

$$= (45)\sin(60°)\hat{x} + (45)\cos(60°)\hat{y}$$

$$\boxed{\vec{B} = 39\hat{x} + 23\hat{y}}$$

To add two vectors, add each direction separately:

$$15\hat{x} + 13\hat{y}$$

$$39\hat{x} + 23\hat{y}$$

$$\vec{A} + \vec{B} = 54\hat{x} + 36\hat{y}$$

To substract two vectors, subtract each direction seperately:

$$15\hat{x} + 13\hat{y}$$

$$39\hat{x} + 23\hat{y}$$

$$\vec{A} - \vec{B} = -24\hat{x} - 10\hat{y}$$

To compose (get magnitude and direction) of a vector, use:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\theta_{\vec{A}} = \arctan\left(\frac{A_i}{A_j}\right)$$

$$|\vec{A} + \vec{B}| = \sqrt{(A+B)_x^2 + (A+B)_y^2}$$

$$= \sqrt{54^2 + 36^2} = 65$$

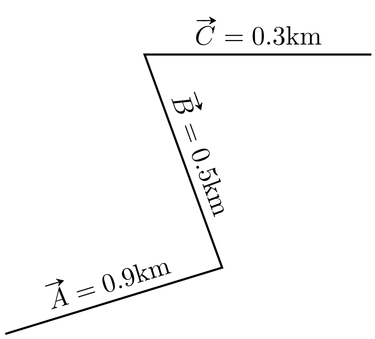
$$\theta = \arctan\left(\frac{(A+B)_y}{(A+B)_x}\right) = \arctan\left(\frac{36}{54}\right) = 34°$$

1.3 Scalar Multiplication

$$\gamma\vec{A} = \gamma A_x\hat{x} + \gamma A_y\hat{y} + \gamma A_z\hat{z}$$

$$-\vec{A} = (-A_x)\,\hat{x} + (-A_y)\,\hat{y} + (-A_z)\,\hat{z}$$

Given three vectors connected from head to tail, find the distance from the tail of $\vec{A}$ to the head of $\vec{C}$:



$$
\begin{aligned}
\vec{C} &= 0.3\text{km}\hat{x} \\
\vec{A} &= +A\cos(\theta)\hat{x} + A\sin(\theta)\hat{y} \\
&= (0.9\text{km})\cos(17°)\hat{x} + (0.9\text{km})\sin(17°)\hat{y} \\
\vec{A} &= 0.86\text{km}\hat{x} + 0.26\text{km}\hat{y} \\
\vec{B} &= -B\sin(\phi)\hat{x} + B\cos(\phi)\hat{y} \\
\vec{B} &= -0.17\text{km}\hat{x} + 0.47\text{km}\hat{y}
\end{aligned}
$$

$$
\begin{aligned}
\vec{A} &= 0.86\text{km}\hat{x} + 0.26\text{km}\hat{y} \\
\vec{B} &= -0.17\text{km}\hat{x} + 0.47\text{km}\hat{y} \\
\vec{C} &= 0.3\text{km}\hat{x} + 0\hat{y}
\end{aligned}
$$

$$\boxed{\vec{A} + \vec{B} + \vec{C} = 0.99\text{km}\hat{x} + 0.73\text{km}\hat{y}}$$

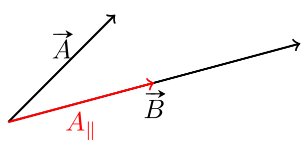
1.4 Multiplying Vectors

A dot product multiplies two vectors and returns a scalar.

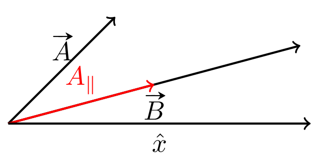
Notation:

$$\vec{A} \cdot \vec{B}$$

Essentially means: "What part of $\vec{A}$ lies in the direction of $\vec{B}$?"



Another example to show importance of angle between vectors:



$$
\begin{aligned}
A_{\parallel} &= A\cos(\theta)_{AB} \\
\vec{A} \cdot \vec{B} &= (A\cos(\theta)_{AB})\,(B) \\
\vec{A} \cdot \vec{B} &= (20\cos(10°))\,(45)
\end{aligned}
$$

$$\boxed{\vec{A} \cdot \vec{B} = 886}$$

** Where θ_{AB} is the angle between $\vec{A}$ and $\vec{B}$*

Dot product in three-dimensions:

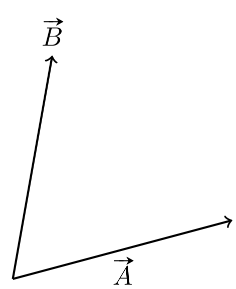
$$\begin{aligned}
\vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z \\
\vec{A} &= 15\hat{x} + 13\hat{y} \\
\vec{B} &= 39\hat{x} + 23\hat{y} \\
\vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y \\
&= (15)(39) + (13)(23)
\end{aligned}$$

$$\boxed{\vec{A} \cdot \vec{B} = 884}$$

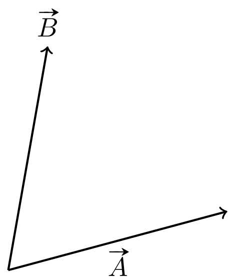
$$\begin{aligned}
\vec{A} \cdot \vec{B} &= AB\cos(\theta)_{AB} \\
\cos(\theta)_{AB} &= \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} \\
\theta_{AB} &= \arccos\left(\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}\right) \\
\theta_{AB} &= \arccos\left(\frac{884}{(20)(45)}\right) = 11^\circ
\end{aligned}$$

1.5 Cross Product

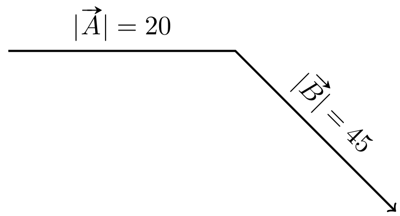
Counter-clockwise rotation - out:



Clockwise rotation - in:



$$|\vec{A} \times \vec{B}| = AB_{\perp}$$



$$\begin{aligned} |\vec{A} \times \vec{B}| &= AB_{\perp} \\ &= (20)(45)\sin(10°) \\ \vec{A} \times \vec{B} &= -156\hat{z} \to \text{(clockwise)} \end{aligned}$$

Calculating mathematically (using determinant):

$$\begin{aligned} \vec{A} \times \vec{B} &= \hat{x}A_yB_z + \hat{y}A_zB_x + \hat{z}A_xB_y - \hat{x}A_zB_y - \hat{y}A_xB_z - \hat{z}A_yB_x \\ \vec{A} \times \vec{B} &= \hat{x}\left(A_yB_z - A_zB_y\right) + \hat{y}\left(A_zB_x - A_xB_z\right) + \hat{z}\left(A_xB_y - A_yB_x\right) \end{aligned}$$

1.6 Sphere Calculation

** Three-dimensional graph omitted*

$$z = r\cos(\theta)$$
$$\rho = r\sin(\theta)$$
$$x = \rho\cos(\phi)$$
$$y = \rho\sin(\phi)$$
$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$