# Contents



# 1 Section 6.1

## 1.1 6.1.1

Find the (real) eigenvalues and associated eigenvectors of the larger matrix $\boldsymbol{A}$ . Find a basis of each eigenspace of dimension 2 or larger.

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

$$\begin{aligned}
|\boldsymbol{A} - \lambda\mathbf{I}| &= \begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} \\
\det(\boldsymbol{A}) &= (4-\lambda)(1-\lambda) - (-1)(2) \\
\det(\boldsymbol{A}) &= \lambda^2 - 5\lambda + 2 = (\lambda-2)(\lambda-3) \\
\lambda &= 2, 3
\end{aligned}$$

$$\begin{aligned}
\lambda &= 2 \\
\boldsymbol{A} - \lambda\mathbf{I} &= \begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} \\
\boldsymbol{A} - \lambda\mathbf{I} &= \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
(\boldsymbol{A} - \lambda\mathbf{I})\mathbf{v} &= 0 \\
\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
2x - y &= 0 \\
x &= \frac{1}{2}y \\
2\left(\frac{1}{2}y\right) - y &= 0 \\
0 &= 0
\end{aligned}$$

$$\lambda = 2 \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}y \\ y \end{bmatrix} = y \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\lambda = 3$$
$$\boldsymbol{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$
$$(\boldsymbol{A} - \lambda\mathbf{I})\mathbf{v} = 0$$
$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x = y$$
$$2x = 2y$$
$$2(y) = 2y$$
$$0 = 0$$

$$\lambda = 3 \to \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

## 1.2 6.1.2

Find the (real) eigenvalues and associated eigenvectors of the given matrix $\boldsymbol{A}$ . Find a basis of each eigenspace of dimension 2 or larger.

$$\begin{bmatrix} 6 & -7 \\ 4 & -5 \end{bmatrix}$$

$$|\boldsymbol{A} - \lambda\mathbf{I}| = \begin{bmatrix} 6-\lambda & -7 \\ 4 & -5-\lambda \end{bmatrix}$$
$$\det(\boldsymbol{A} - \lambda\mathbf{I}) = (6-\lambda)(-5-\lambda) - (-7)(4)$$
$$\det(\boldsymbol{A} - \lambda\mathbf{I}) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$
$$\lambda_{1,2} = 2, -1$$

$$\lambda_1 = 2$$
$$[\boldsymbol{A} - \lambda_1]\,\mathbf{x} = 0$$
$$\begin{bmatrix} 6-2 & -7 \\ 4 & -5-2 \end{bmatrix} \mathbf{x} = 0$$
$$\begin{bmatrix} 4 & -7 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$
$$(4)x_1 + (-7)x_2 = 0$$
$$(4)x_1 + (-7)x_2 = 0$$

$$(4)x_1 = (7)x_2$$
$$x_1 = \left(\frac{7}{4}\right) x_2$$

$$
\begin{aligned}
(4)x_1 + (-7)x_2 &= 0 \\
(4)\left(\frac{7}{4}\right)x_2 + (-7)x_2 &= 0 \\
0 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} \left(\frac{7}{4}\right)x_2 \\ x_2 \end{bmatrix} \\
\mathbf{x} &= x_2 \begin{bmatrix} \frac{7}{4} \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_2 &= -1 \\
[\boldsymbol{A} - \lambda_2]\,\mathbf{x} &= 0 \\
\begin{bmatrix} 6-(-1) & -7 \\ 4 & -5-(-1) \end{bmatrix} \mathbf{x} &= 0 \\
\begin{bmatrix} 7 & -7 \\ 4 & -4 \end{bmatrix} \mathbf{x} &= 0 \\
\begin{bmatrix} 7 & -7 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 0 \\
(7)x_1 + (-7)x_2 &= 0 \\
(4)x_1 + (-4)x_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
(7)x_1 + (-7)x_2 &= 0 \\
x_1 &= x_2
\end{aligned}
$$

$$
\begin{aligned}
(4)x_1 + (-4)x_2 &= 0 \\
x_1 &= x_2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \\
\mathbf{x} &= x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Eigenvalues: $\lambda_{1,2} = 2, -1$

Eigenvectors: $\mathbf{x}_{1,2} = \begin{bmatrix} \frac{7}{4} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

### 1.3 6.1.13

Find the (real) eigenvalues and associated eigenvectors of the larger matrix $\boldsymbol{A}$ . Find a basis of each eigenspace of dimension 2 or larger.

$$\begin{bmatrix} 3 & 0 & 0 \\ 7 & -2 & -2 \\ -3 & 4 & 4 \end{bmatrix}$$

$$\begin{aligned} |\boldsymbol{A} - \lambda\mathbf{I}| &= \begin{bmatrix} 3-\lambda & 0 & 0 \\ 7 & -2-\lambda & -2 \\ -3 & 4 & 4-\lambda \end{bmatrix} \\ \det(\boldsymbol{A} - \lambda\mathbf{I}) &= (3-\lambda)((-2-\lambda)(4-\lambda) - (-2)(4)) - 0 - 0 \\ \det(\boldsymbol{A} - \lambda\mathbf{I}) &= -\lambda(\lambda-3)(\lambda-2) \\ \lambda_{1,2,3} &= 0, 3, 2 \end{aligned}$$

$$\begin{aligned} [\boldsymbol{A} - \lambda_1]\,\mathbf{x}_1 &= 0 \\ \begin{bmatrix} 3-(0) & 0 & 0 \\ 7 & -2-(0) & -2 \\ -3 & 4 & 4-(0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 0 \\ \begin{bmatrix} 3 & 0 & 0 \\ 7 & -2 & -2 \\ -3 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 0 \end{aligned}$$

$$\begin{aligned} (3)x_1 + (0)x_2 + (0)x_3 &= 0 \\ x_1 &= 0 \end{aligned}$$

$$\begin{aligned} (7)x_1 + (-2)x_2 + (-2)x_3 &= 0 \\ x_2 &= -x_3 \end{aligned}$$

$$\begin{aligned} (-3)x_1 + (4)x_2 + (4)x_3 &= 0 \\ (4)(-x_3) + (4)x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} \\ \mathbf{x} &= x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

$$[\boldsymbol{A} - \lambda_2]\,\mathbf{x}_2 = 0$$

$$\begin{bmatrix} 3-(3) & 0 & 0 \\ 7 & -2-(3) & -2 \\ -3 & 4 & 4-(3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 7 & -5 & -2 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$(0)x_1 + (0)x_2 + (0)x_3 = 0$$
$$0 = 0$$

$$(7)x_1 + (-5)x_2 + (-2)x_3 = 0$$
$$x_1 = \left(\frac{5}{7}\right)x_2 + \left(\frac{2}{7}\right)x_3$$

$$(-3)x_1 + (4)x_2 + (1)x_3 = 0$$
$$(-3)\left(\left(\frac{5}{7}\right)x_2 + \left(\frac{2}{7}\right)x_3\right) + (4)x_2 + (1)x_3 = 0$$
$$\left(\frac{13}{7}\right)x_2 = \left(-\frac{1}{7}\right)x_3$$
$$x_2 = \left(-\frac{1}{13}\right)x_3$$

$$x_1 = \left(\frac{5}{7}\right)\left(-\frac{1}{13}\right)x_3 + \left(\frac{2}{7}\right)x_3$$
$$x_1 = \left(\frac{3}{13}\right)x_3$$

$$\mathbf{x} = \begin{bmatrix} \left(\frac{3}{13}\right)x_3 \\ \left(-\frac{1}{13}\right) \\ x_3 \end{bmatrix}$$
$$\mathbf{x} = x_3 \begin{bmatrix} \frac{3}{13} \\ -\frac{1}{13} \\ 1 \end{bmatrix}$$

$$[\boldsymbol{A} - \lambda_3]\,\mathbf{x}_3 = 0$$

$$\begin{bmatrix} 3-(2) & 0 & 0 \\ 7 & -2-(2) & -2 \\ -3 & 4 & 4-(2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 7 & -4 & -2 \\ -3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{aligned}(1)x_1 + (0)x_2 + (0)x_3 &= 0\\ x_1 &= 0\end{aligned}$$

$$\begin{aligned}(7)x_1 + (-4)x_2 + (-2)x_3 &= 0\\ x_2 &= \left(-\frac{1}{2}\right)x_3\end{aligned}$$

$$\begin{aligned}(-3)x_1 + (4)x_2 + (2)x_3 &= 0\\ (4)\left(-\frac{1}{2}\right)x_3 + (2)x_3 &= 0\\ 0 &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{x} &= \begin{bmatrix} 0 \\ \left(-\frac{1}{2}\right)x_3 \\ x_3 \end{bmatrix}\\ \mathbf{x} &= x_3\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}\end{aligned}$$

### 1.4 6.1.19

Find the (real) eigenvalues and associated eigenvectors of the larger matrix $\boldsymbol{A}$ . Find a basis of each eigenspace of dimension 2 or larger.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -6 & 3 \end{bmatrix}$$

$$\begin{aligned}|\boldsymbol{A} - \lambda\mathbf{I}| &= \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ -2 & -7 & 3-\lambda \end{bmatrix}\\ \det(\boldsymbol{A} - \lambda\mathbf{I}) &= (1-\lambda)((1-\lambda)(3-\lambda) - 0)) = -(\lambda-3)(\lambda-1)^2\\ \lambda_{1,2} &= 3, 1\end{aligned}$$

$$\begin{aligned}\lambda_2 &= 1\\ [\boldsymbol{A} - \lambda_2]\,\mathbf{x}_2 &= 0\\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 0\end{aligned}$$

$$
\begin{aligned}
(-2)x_1 + (-6)x_2 + (2)x_3 &= 0 \\
x_1 &= (-3)x_2 + x_3
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} (-3)x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} \\
\mathbf{x} &= x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_1 &= 3 \\
[\boldsymbol{A} - \lambda_1]\, \mathbf{x}_1 &= 0 \\
\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -2 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= 0
\end{aligned}
$$

$$
\begin{aligned}
(-2)x_1 + (0)x_2 + (0)x_3 &= 0 \\
x_1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
(0)x_1 + (-2)x_2 + (0)x_3 &= 0 \\
x_2 &= 0
\end{aligned}
$$

$$
\begin{aligned}
(-2)x_1 + (-6)x_2 + (0)x_3 &= 0 \\
(-2)(0) + (-6)(0) &= 0 \\
0 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x} &= \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \\
\mathbf{x} &= x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

Exactly one of the eigenspaces has dimension 2 or larger. The eigenspace associated with the eigenvalue $\lambda_2$ has basis $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.