1 Section 5.1

1.1 5.1.9

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$y'' + 2y' + y = 0; y_1 = e^{-x}, y_2 = xe^{-x}; y(0) = 9, y'(0) = -10$$

(a) **Why is the function $y_1 = e^{-x}$ a solution to the differential equation?**

The function $y_1 = e^{-x}$ is a solution because when the function, its first derivative $y_1' = -e^{-x}$ and its second derivative, $y_1'' = e^{-x}$, are substituted into the equation, the result is a true statement.

(b) **Why is the function $y_2 = xe^{-x}$ a solution to the differential equation?**

The function $y_2 = xe^{-x}$ is a solution because when the function, its derivative, $y_2' = e^{-x} - xe^{-x}$, and its second derivative, $y_2'' = -e^{-x} - e^{-x} + xe^{-x}$, are substituted into the equation, the result is a true statement.

(c) The particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the initial conditions $y(0) = 9$ and $y'(0) = -10$ is $y = 9e^{-x} - xe^{-x}$.

1.2 5.1.11

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$y'' - 2y' + 2y = 0; y_1 = e^x \cos(x), y_2 = e^x \sin(x); y(0) = 11, y'(0) = 14$$

(a) The function $y_1 = e^x \cos(x)$ is a solution because when the function its first derivative $y_1' = e^x \cos(x) - e^x \sin(x)$, and its second derivative, $y_1'' = e^x \cos(x) - e^x \sin(x) - e^x \sin(x) - e^x \cos(x)$, are substituted into the equation, the result is a true statement.

(b) The function $y_2 = e^x \sin(x)$ is a solution because when the function, its first derivative, $y_2' = e^x \sin(x) + e^x \cos(x)$, and its second derivative, $y_2'' = e^x \sin(x) + e^x \cos(x) + e^x \cos(x) - e^x \sin(x)$, are substituted into the equation, the result is a true statement.

(c) The particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the initial

conditions $y(0) = 11$ and $y'(0) = 14$ is

$$y = c_1 y_1 + c_2 y_2$$
$$y = c_1 \left(e^x \cos(x)\right) + c_2 \left(e^x \sin(x)\right)$$
$$c_1 \left(e^0 \cos(0)\right) + c_2 \left(e^0 \sin(0)\right) = 11$$
$$c_1 = 11$$
$$y = c_1 \left(e^x \cos(x)\right) + c_2 \left(e^x \sin(x)\right)$$
$$y' = c_1 \left(e^x \cos(x) - e^x \sin(x)\right) + c_2 \left(e^x \sin(x) + e^x \cos(x)\right)$$
$$14 = (11) \left(e^0 \cos(0) - e^0 \sin(0)\right) + c_2 \left(e^0 \sin(0) + e^0 \cos(0)\right)$$
$$14 = (11) + c_2$$
$$c_2 = 3$$
$$y = (11)(e^x \cos(x)) + (3)(e^x \sin(x))$$

$$\boxed{y = (11)(e^x \cos(x)) + (3)(e^x \sin(x))}$$

1.3 5.1.19

Show that $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution.

(a)

$$y_1 = 1$$
$$y_1' = 0$$
$$y_1'' = 0$$
$$yy'' + (y')^2 = 0$$
$$(1)(0) + (0)^2 = 0$$
$$0 = 0$$

(b)

$$y_2 = \sqrt{x}$$
$$y_2' = \frac{1}{2} x^{-1/2}$$
$$y_2'' = -\frac{1}{4} x^{-3/2}$$
$$yy'' + (y')^2 = 0$$
$$(\sqrt{x})\left(-\frac{1}{4} x^{-3/2}\right) + \left(\frac{1}{2} x^{-1/2}\right)^2 = 0$$
$$-\frac{1}{4x} + \frac{1}{4x} = 0$$
$$0 = 0$$

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(c) Now consider the function $y = y_1 + y_2$.

$$y = y_1 + y_2$$
$$y = (1) + (\sqrt{x})$$
$$y' = 0 + \frac{1}{2}x^{-1/2}$$
$$y' = \frac{1}{2}x^{-1/2}$$
$$y'' = -\frac{1}{4}x^{-3/2}$$

(d) For the function $y = y_1 + y_2$:

$$yy'' = (1 + \sqrt{x})\left(-\frac{1}{4}x^{-3/2}\right)$$
$$yy'' = -\frac{1 + \sqrt{x}}{4x^{3/2}}$$

$$(y')^2 = \left(\frac{1}{2}x^{-1/2}\right)^2$$
$$(y')^2 = \frac{1}{4x}$$

(e) Why is the proof complete?
For the function $y = y_1 + y_2$, when the expressions above for yy'' and $(y')^2$ are substituted into the equation, the result is a false statement.

1.4   5.1.24

Determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$.

$$y_1 = \sin(t)\cos(t), y_2 = 3\sin(2t)$$

$$y_2 = 3\sin(2t) = 6\sin(t)\cos(t)$$
$$y_1 \equiv y_2$$

1.5   5.1.31

Show that $y_1 = \sin(x^2)$ and $y_2 = \cos(x^2)$ are linearly independent functions, but that their Wronskian vanishes at $x = 0$. Why does this imply that there is no differential equation of the form $y'' + p(x)y' + q(x)y = 0$, with both p and q continuous everywhere, having both y_1 and y_2 as solutions?

(a) Two functions defined on an open interval I are said to be linearly independent on I provided that <u>neither is a constant multiple of the other.</u>

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(b) The functions $y_1 = \sin(x^2)$ and $y_2 = \cos(x^2)$ are linearly independent because $\frac{y_1}{y_2} = \tan(x^2)$ is not a constant-valued function and $\frac{y_2}{y_1} = \cot(x^2)$ is not a constant-valued function.

(c) Given two function f and g, the Wronskian of f and g is the determinant defined as follows.

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

(d) The Wronskian for these functions, $W(\sin(x^2), \cos(x^2)) = -2x$, vanishes at $x = 0$ because it has a factor of x.

1.6   5.1.33

Find a general solution to the differential equation given below. Primes denote derivatives with respect to t.

$$y'' + 4y' - 21y = 0$$

$$r^2 + 4r - 21 = 0$$
$$(r - 3)(r + 7) = 0$$
$$r = 3, -7$$
$$y = c_1(e^{3t}) + c_2(e^{-7t})$$

1.7   5.1.35

Find a general solution to the differential equation given below. Primes denote derivatives with respect to x.

$$y'' + 8y' = 0$$

$$r^2 + 8r = 0$$
$$r = 0, -8$$
$$y = c_1 + c_2(e^{-8x})$$

1.8   5.1.37

Find a general solution to the differential equation given below. Primes denote derivatives with respect to x.

$$6y'' - 5y' - y = 0$$

$$6r^2 - 5r - 1 = 0$$
$$r = 1, -\frac{1}{6}$$
$$y = c_1(e^x) + c_2(e^{-x/6})$$

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1.9   5.1.38

$$12y'' + 5y' - 3y = 0$$

$$12r^2 + 5r - 3 = 0$$
$$r = \frac{1}{3}, -\frac{3}{4}$$
$$y = c_1(e^{t/3}) + c_2(e^{-3t/4})$$

1.10   5.1.39

$$9y'' + 6y' + y = 0$$

$$9r^2 + 6r + 1 = 0$$
$$r = -\frac{1}{3}$$
$$y(x) = c_1(e^{-x/3}) + xc_2(e^{-x/3})$$