Contents

1 Chapter 02 - Basic Structures

1.1 Sets

$\in$: belong to, is in

1.1.1 Definition 2

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x\,(x \in A \iff x \in B)$. We write $A = B$ if A and B are equal sets.

1.1.2 Definition 3

The set A is also a subset of B, and B is a superset of A, if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $B \supseteq A$. (So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.)

1.1.3 Definition 4

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

1.1.4 Countable and Uncountable Sets

- Countable
  - $\mathbb{N}$
  - $\mathbb{Z}$
  - $\mathbb{Q}$
- Uncountable
  - $\mathbb{R}$
  - $\mathbb{C}$

Let $S_0 = \{x\}$, and $S_1 = \{\{x\}\}$.

$$S_0 \neq S_1 \tag{1}$$

1.1.5 Example

1. List the members of these sets.

   a) $\big\{x \mid x \text{ is a real number such that } x^2 = 1\big\}$

   $$S = \big\{x \in \mathbb{R} \mid x^2 = 1\big\}$$

   b) $\{x \mid x \text{ is a positive integer less than } 12\}$

   $$S = \{x \in \mathbb{R} \mid 0 \leq x < 12\}$$

1.1.6 Definition 6

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

1.2 Set Operations

1.2.1 Definition 1

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\} \tag{2}$$

1.2.2 Definition 2

Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\} \tag{3}$$

1.2.3 Definition 3

Two sets are disjoint if their intersection is the empty set.

1.2.4 Definition 4

Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

1.2.5 Definition 5

Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

1.2.6 Proof

Let A, B be sets from U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

Proof:

- For " $\implies$ "

  Given $A \subseteq B$, need to show $\overline{B} \subseteq \overline{A}$. Then $\forall x \in A$, we have $x \in B$.

  By contrapositive we have

$$\begin{aligned} \neg\,(x \in B) &\implies \neg\,(x \in A) \\ x \notin B &\implies x \notin A, \quad \overline{B} \subseteq \overline{A} \end{aligned}$$

- For " $\Longleftarrow$ "

  Given $\overline{B} \subseteq \overline{A}$, we have $\forall y \in \overline{B}$, $y \in \overline{A}$ then the contrapositive is

$$\begin{aligned} \neg\left(y \in \overline{A}\right) &\implies \neg\left(y \in \overline{B}\right) \\ y \in A &\implies y \in B \end{aligned}$$

1.2.7 Proof:

Use the identities to show that $\overline{(A \cup B)} \cap \overline{(B \cup C)} \cap \overline{(A \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}$

Proof:

$$\begin{aligned} \overline{A} \cap \overline{B} \cap \overline{C} &= \left(\overline{A} \cap \overline{B}\right) \cap \left(\overline{B} \cap \overline{C}\right) \cap \left(\overline{A} \cap \overline{C}\right) \\ &= \overline{A} \cap \left(\overline{B} \cap \overline{B}\right) \cap \left(\overline{C} \cap \overline{C}\right) \cap \overline{A} \\ &= \overline{A} \cap \overline{B} \cap \overline{C} \end{aligned}$$

1.2.8 Union

The union is a collection of sets is the set that contains those elements that are member s of at least one set in the collection.

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

1.2.9 Intersection

The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

1.3 Functions

- Sets: A, B, C, domain, codomain, range
- Relations: functions (f, g, h)
- Elements: image, preimage

1.3.1 Example

$$\begin{aligned}
&f_1 : \mathbb{R} \to \mathbb{R} \text{ as } y = f_1(x) = 3x - 2\\
&f_2 : \mathbb{R} \to \mathbb{R} \text{ as } y = e^x\\
&f_3 : \mathbb{R} \to \mathbb{R} \text{ as } y = \sqrt{x}
\end{aligned}$$

$$\begin{aligned}
f : A &\to B\\
\text{range}(f) &= \{y \in B \mid \forall x \in S \subseteq A\}
\end{aligned}$$

1.3.2 Properties

1) f is injective – one-to-one

2) f is surjective –

3) f is bijective if **1)** and **2)**

Let $f : A \to B$ be a function.

We say f is injective if

$$(x_1 \neq x_2 \implies f(x_1) \neq f(x_2)) \iff (f(x_1) = f(x_2) \implies x_1 = x_2)$$

We say f is surjective if $\forall y \in B$, $\exists x \in A$ such that $y = f(x) \iff \text{range}(f) = B$

We say f is bijective if f is injective and surjective.

Monotonic Function: We say f is increasing (strictly increasing) if $x_1 > x_2 \implies f(x_1) > f(x_2)$

1.3.3 Example 2.3.24

Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in \mathbb{R}$. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = \frac{1}{f(x)}$ is strictly decreasing.

$$\begin{aligned}
&\to \frac{1}{f(x_1)} < \frac{1}{f(x_2)}\\
&\to f(x_1) \cdot f(x_2) \left(\frac{1}{f(x_1)}\right) < f(x_1) \cdot f(x_2) \left(\frac{1}{f(x_2)}\right)\\
&\to f(x_2) < f(x_1)
\end{aligned}$$

where the first implication comes from the fact that $f(x) > 0, \forall x \in \mathbb{R}$.

1.3.4 Example 2.3.73.b

Prove or disprove each of these statements about the floor and ceiling functions.

$$\lfloor 2x \rfloor = 2 \lfloor x \rfloor$$

Consider $x = 1.6$

$$\lfloor 2x \rfloor = \lfloor 2 \cdot 1.6 \rfloor = \lfloor 3.2 \rfloor = 3$$
$$2 \lfloor x \rfloor = 2 \lfloor 1.6 \rfloor = 2(1) = 2$$

Hence $\lfloor 2x \rfloor = 2 \lfloor x \rfloor$ for all $x \in \mathbb{R}$ is false.

1.3.5 Inverse Function

Let $f : A \to B$ be a function. If f is injective, then there exists $g : B \to A$ such that $f \cdot g(y) = y, \forall y \in B$ and $g \cdot f(x) = x, \forall x \in A$. We call g the inverse of f. We can denote $g = f^{-1}$.

1.4 Sequences and Summations

Let $f : A \to B$ where $A = \{1, 2, 3, \cdots\}$ or $\{0, 1, 2, 3, \cdots\}$, $B = \mathbb{R}$.

Special sequences:

- Geometric Sequence
$$a_n = a_1 r^{n-1} = a_0 r^n \tag{4}$$
- Arithmetic Sequence
$$a_n = a_1 + (n-1)d = a_0 + nd \tag{5}$$
- Fibonacci Sequence: $f_0, f_1, f_2, \cdots$ is defined by the initial conditions $f_0 = 0, f_1 = 1$ and the recurrence relation
$$f_n = f_{n-1} + f_{n-2} \tag{6}$$

In order to describe a sequence:

- Closed formula
- Recurrence relation
- Verbally

1.4.1 Geometric Sequence

If $r = 1$ then $a_n = ar^n = a$ then $\sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n - 0 + 1)a$.

Let $k = j + 1$.

$$\begin{aligned}
j &= k - 1 \\
j &= 0, k = 1 \\
&= \sum_{j=0}^{j=n} ar^{j+1} \\
&= \sum_{k=1}^{k=n+1} ar^k \\
&= a + ar + ar^2 + \cdots + ar^{n+1} - a \\
&= \sum_{k=0}^{n} ar^k + ar^{n+1} - a
\end{aligned}$$

If $r \neq 1$

Notice that $(1 - x)\left(1 + x + x^2 + \cdots + x^n\right) = 1 + (x - x) + (x^2 - x^2) + \cdots + (x^n - x^n) - x^{n+1} = 1 - x^{n+1}$

Then

$$\begin{aligned}
\sum_{j=0}^{n} ar^j &= a + ar + ar^2 + \cdots + ar^n \\
&= a\left(1 + r + r^2 + \cdots + r^n\right) \\
&= a\left(\frac{1 - r^{n+1}}{1 - r}\right) \\
&= a\left(\frac{r^{n+1} - 1}{r - 1}\right)
\end{aligned}$$

1.4.2 Practice

Prove $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + k = \frac{n(n+1)}{2}$

Method 1:

$$\begin{aligned}
\sum_{k=1}^{n} k &= 1+2+3+\cdots+n \\
2\cdot\sum_{k=1}^{n} k &= (1+2+3+\cdots+n)+(1+2+3+\cdots+n) \\
&= \sum_{k=1}^{n}(n+1) = (n+1)\cdot(n-1+n) = (n+1)\cdot n \\
\sum_{k=1}^{n} k &= \frac{n\,(n+1)}{2}
\end{aligned}$$

Method 2: Telescoping

Notice that

$$\begin{aligned}
\sum_{k=0}^{n-1}(a_{k+1}-a_k) &= a_1-a_0+a_2-a_1+a_3-a_2+\cdots+a_n-a_{n-1} \\
&= \sum_{i=1}^{n}(a_i-a_{i-1}) = a_n-a_0
\end{aligned}$$

$$\begin{aligned}
\text{and, } i^2-(i-1)^2 &= i^2-\left(i^2-2i+1\right) = 2i-1 \\
\sum_{i=1}^{n}\left(i^2-(i-1)^2\right) &= \sum_{i=1}^{n} 2i-1 \\
n^2-0^2 &= 2\sum_{i=1}^{n} i-\sum_{i=1}^{n}1, \quad \text{where } a_i=i^2 \text{ for telescoping} \\
\sum_{i=1}^{n} &= \frac{n^2+n}{2}
\end{aligned}$$

Prove $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+k^2 = \frac{n(n+1)(2n+1)}{6}$, Let $a_i = i^3$

$$\begin{aligned}
\sum_{i=1}^{n}(a_i-a_{i-1}) &= a_n-a_0 \\
i^3-(i-1)^3 &= 3i^2-3i+1 \\
\sum_{i=1}^{n}\left(i^3\right) &= 3\sum_{i=1}^{n} i^2-3\sum_{i=1}^{n} i+\sum_{i=1}^{n}1 \\
3\sum_{i=1}^{n} i^2 &= \sum_{i=1}^{n}\left(i^3\right)+3\sum_{i=1}^{n} i-\sum_{i=1}^{n}1
\end{aligned}$$

1.5 Cardinality of Sets

Compare numbers (including ∞)

1. Finite Elements
    - Finite: $0, 1, 2, 3, \cdots, n$
2. Infinitely Many Elements
    - Countable: $\mathbb{N} = \mathbb{Z}^+, \mathbb{Z}, \mathbb{Q}$

      For $\mathbb{Z}$:

$\mathbb{N}$	$\mathbb{Z}^+$
1	0
2	1
3	-1
4	2
5	-2
6	3
7	-3

      For $\mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}^+, n \neq 0, \gcd(m,n) = 1 \right\}$
    - Uncountable: $\mathbb{R}, \mathbb{R}^2, \cdots, \mathbb{R}^n$

1.5.1 Shroder-Bernstein Theorem Example

Let $A = [0,1], B = (0,1)$. Show that $|A| = |B|$.

Proof:

- Consider $f_1 : B \to A$ as $f(x) = x$. Thus f_1 is one-to-one $\implies |B| \leq |A|$.
- Next consider $f_2 : A \to B$ as $f(x) = \frac{x}{2} + a, x \in [0,1], a \in \left(0, \frac{1}{2}\right)$. Then $f(A) = \left[a, \frac{1}{2} + a\right] \subseteq B$. Thus f_2 is one-to-one $\implies |A| \leq |B|$.

1.6 Matrices

1.6.1 Definition 1

A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. The plural of matrix is matrices. A matrix with the same number of rows as columns is called square. Two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

Special Matrix Sizes

- $n \times n$: Square matrix
- $1 \times n$: Row matrix
- $m \times 1$: Column matrix

1.6.2 Definition 3

Let $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{B} = [b_{ij}]$ be $m \times n$ matrices. The sum of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{A}+\boldsymbol{B}$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element. In other words, $\boldsymbol{A} + \boldsymbol{B} = [a_{ij} + b_{ij}]$.

1.6.3 Proof

Proof of matrix addition.

Let $\boldsymbol{A} = [a_{ij}]\,, \boldsymbol{B} = [b_{ij}]$.

Then $\boldsymbol{A} + \boldsymbol{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}]$

1.6.4 Definition 4

Let $\boldsymbol{A}$ be an $m \times k$ matrix and $\boldsymbol{B}$ be a $k \times n$ matrix. The product of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted by $\boldsymbol{AB}$, is the $m \times n$ matrix with its (i,j)th entry equal to the sum of the products of the corresponding elements from the ith row of $\boldsymbol{A}$ and the jth column of $\boldsymbol{B}$. In other words, if $\boldsymbol{AB} = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} \sum_{l=1}^{k} a_{il}b_{lj}$$

1.6.5 Definition 6

Let $\boldsymbol{A} = [a_{ij}]$ be an $m \times n$ matrix. The transpose of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^T$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $\boldsymbol{A}$. In other words, if $\boldsymbol{A}^T = [b_{ij}]$, then $b_{ij} = a_{ij}$ for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$.

1.6.6 Example 2.6.20

Definition: If $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$, then we say $\boldsymbol{A}$ is invertible and denote the inverse of $\boldsymbol{A}$ as $\boldsymbol{A}^{-1} = \boldsymbol{B}$.

Let $\boldsymbol{A}$ be the invertible $n \times n$ matrix. Prove that $\boldsymbol{A}^3$ is invertible and $\left(\boldsymbol{A}^3\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^3$.

Proof: Since $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^{-1}$ exists. $\boldsymbol{AA}^{-1} = \boldsymbol{I}, \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.

Notice that:

$$\begin{aligned}
\boldsymbol{A}^3 \cdot \left(\boldsymbol{A}^{-1}\right)^3 &= (\boldsymbol{AAA})\left(\boldsymbol{A}^{-1}\boldsymbol{A}^{-1}\boldsymbol{A}^{-1}\right) \\
&= (\boldsymbol{AA})\left(\boldsymbol{AA}^{-1}\right)\left(\boldsymbol{A}^{-1}\boldsymbol{A}^{-1}\right) \\
&= (\boldsymbol{AA})\,(\boldsymbol{I})\left(\boldsymbol{A}^{-1}\boldsymbol{A}^{-1}\right) \\
&= (\boldsymbol{AA})\left(\boldsymbol{A}^{-1}\boldsymbol{A}^{-1}\right), \quad etc.
\end{aligned}$$