Chapter 1

1 Introduction

1.1 Introduction to Differential Equations

Equations contain: "=" and "*solution*"

There are two types of variables: *dependent* and *independent*

Differential: contains derivative (or partial derivative)

Derivative refers to:

$$\text{ODE (ordinary differential equation):}\ \frac{\mathrm{d}}{\mathrm{d}x}$$
$$\text{PDE (partial differential equation):}\ \frac{\partial}{\partial x}$$

The solution to a differential equation is a **function**
And can be expressed in four ways:

1. Verbally
2. Table
3. Graph
4. Expression (implicitly or explicitly)

Regular equation:

$$x^3 + 3\sin(x) = 2 - 2\cos(x)$$

Claim $x = 0$ is a solution. Verify by substitution.

$$\text{When } x = 0$$

$$(0)^3 + 3\sin(0) = 2 - 2\cos(0)$$
$$0 + 3 \times 0 = 2 - 2 \times 1$$
$$0 = 0$$

Although it may be true for $x = 0$, it is not a true method of verification. See for $x = \pi$:

$$\pi^3 + 3\sin(\pi) = 2 - 2\cos(\pi)$$
$$\pi^3 + 0 = 2 - 2 \times (-1)$$
$$\pi^3 \neq 4$$

So how do we verify the equation?

Given $y'' - 2y' + 5y = 0$ (with respect to x)
Claim $y = 4e^x \cos(2x)$ is a solution

$$\begin{aligned}
y' &= 4e^x \cos(2x) + 4e^x \left(-\sin(2x)\right) \\
&= 4e^x \left(\cos(2x) - 2\sin(2x)\right) \\
y'' &= 4e^x \left(\cos(2x) - 2\sin(2x)\right) + 4e^x \left(-2\sin(2x) - 4\cos(2x)\right) \\
&= 4e^x \left(-3\cos(2x) - 4\sin(2x)\right) \\
-2y' &= 4e^x \left(-2\cos(2x) + 4\sin(2x)\right) \\
5y &= 4e^x \left(5\cos(2x)\right)
\end{aligned}$$

LHS $= y'' - 2y' + 5y = 4e^x (0 + 0) = 0 =$ RHS
$\therefore\ y = 4e^x \cos(2x)$ is a solution

1.2 Classification of Differential Equations

1. ODE v.s. PDE
   - ODE: ordinary differential equation
     $F\left(x, y, y', y'', \cdots, y^{(n)}\right) = 0$
   - PDE: partial differential equation
     $F\left(x_1, x_2, \cdots, u, u_{x_1}, u_{x_2}, \cdots, u_{x_k}, \cdots\right) = 0$

1.3 Order of Differential Equations

The order of differential equations is defined by the highest derivative.

Example:

$$y'' - 2y' + 5y = 0 \rightarrow \text{2nd order}$$

Second order ODE is generally written as:

$$F\left(x, y, y', y''\right) = 0$$

- independent variable: x
- dependent variable: y

To write the second order PDE, the independent variable must be located first

- independent variable: x, t
- dependent variable: u

$$F\left(x, t, u, u_x, u_t, u_{xt}, u_{xx}, u_{tx}, u_{tt}\right) = 0$$

1.4 Linear & Non-Linear ODE Classification

Linear equations can be solved explicitly. Non-linear equations may only sometimes be solvable. Generally analyzed or attempting to find the equation's stable point.

1. Linear Form

   $$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

   Can also be written as so:

   $$\sum_{i=0}^{n} a_i(x)y^{(i)} = f(x)$$

2. Non-Linear
   Examples include: $\sin(y), e^y, \ln(y), y^2, \sqrt{y}$

Orders covered in this course:

- 1st order: linear & some non-linear
- 2nd order: linear only
  - $\rightarrow$ $a_i(x)$ are constants
  - $\rightarrow$ $f(x)$ are "nice functions"
  - $\rightarrow$ $a_i(x)$ are polynomials
- higher order: linear with constant coefficients
  * possibly system of linear differential (using matrices & eigen-theory)