Week 08 Participation Assignment

Corey Mostero - 2566652

20 October 2023

1

If $X = \{x_1, x_2, \cdots, x_n\}$ is a finite set, we define $\mathcal{P}(X)$, the powerset of X, to be the set of all subsets of X.

1.1

If $X = \{a, b, c\}$, list all the members of $\mathcal{P}(X)$. How many subsets does X have?

$$\begin{aligned} \mathcal{P}(X) = \{ & \\ & \{a, b, c\} \\ & \{a, b\} \\ & \{a, c\} \\ & \{b, c\} \\ & \{a\} \\ & \{b\} \\ & \{c\} \\ & \{\emptyset\} \\ & \} \end{aligned}$$

$$|\mathcal{P}(X)| = 8$$

1.2

Separate the list that you got in part 1.1 into two columns. Place on the left column those subsets that contain c and place on the right column those that do not contain c.

Contains	**Does not contain**
$\{a, b, c\}$	$\{a, b\}$
$\{a, c\}$	$\{a\}$
$\{b, c\}$	$\{b\}$
$\{c\}$	$\{\emptyset\}$

1.3

Now, cross out c from each subset on the left column. What do you notice?

Contains	**Does not contain**
$\{a, b, \cancel{c}\}$	$\{a, b\}$
$\{a, \cancel{c}\}$	$\{a\}$
$\{b, \cancel{c}\}$	$\{b\}$
$\{\cancel{c}\}$	$\{\emptyset\}$

Contains	**Does not contain**
$\{a, b\}$	$\{a, b\}$
$\{a\}$	$\{a\}$
$\{b\}$	$\{b\}$
$\{\emptyset\}$	$\{\emptyset\}$

The set of each column ends up equivalent.

1.4

Repeat part 1.1, 1.2, and 1.3 for $X = \{a, b, c, d\}$

1.4.1

$$
\begin{aligned}
\mathcal{P}(X) = \{ & \\
& \{a, b, c, d\} \\
& \{a, b, c\} \\
& \{a, b, d\} \\
& \{a, c, d\} \\
& \{b, c, d\} \\
& \{a, b\} \\
& \{a, c\} \\
& \{a, d\} \\
& \{b, c\} \\
& \{b, d\} \\
& \{c, d\} \\
& \{a\} \\
& \{b\} \\
& \{c\} \\
& \{d\} \\
& \{\emptyset\} \\
& \}
\end{aligned}
$$

$$|\mathcal{P}(X)| = 16$$

1.4.2

Contains	**Does not contain**
$\{a, b, c, d\}$	$\{a, b, d\}$
$\{a, b, c\}$	$\{a, b\}$
$\{a, c, d\}$	$\{a, d\}$
$\{b, c, d\}$	$\{b, d\}$
$\{a, c\}$	$\{a\}$
$\{b, c\}$	$\{b\}$
$\{c, d\}$	$\{d\}$
$\{c\}$	$\{\emptyset\}$

1.4.3

Contains	**Does not contain**
$\{a, b, \cancel{c}, d\}$	$\{a, b, d\}$
$\{a, b, \cancel{c}\}$	$\{a, b\}$
$\{a, \cancel{c}, d\}$	$\{a, d\}$
$\{b, \cancel{c}, d\}$	$\{b, d\}$
$\{a, \cancel{c}\}$	$\{a\}$
$\{b, \cancel{c}\}$	$\{b\}$
$\{\cancel{c}, d\}$	$\{d\}$
$\{\cancel{c}\}$	$\{\emptyset\}$

Contains	**Does not contain**
$\{a, b, d\}$	$\{a, b, d\}$
$\{a, b\}$	$\{a, b\}$
$\{a, d\}$	$\{a, d\}$
$\{b, d\}$	$\{b, d\}$
$\{a\}$	$\{a\}$
$\{b\}$	$\{b\}$
$\{d\}$	$\{d\}$
$\{\emptyset\}$	$\{\emptyset\}$

1.5

For $X = \{x_1, x_2, \cdots, x_n\}$, guess the number of elements in the powerset $\mathcal{P}(X)$.

$$|\mathcal{P}| = 2^n$$

1.6

I hope you can guess that the number of elements in the powerset $\mathcal{P}(X)$ is 2^n. That means $|\mathcal{P}(X)| = 2^n$.

Now, use induction to prove this guess.

Proof by induction:

Where $n \in \mathbb{W}$.

- Basis step: $n = 0$

$$\begin{aligned} X &= \{\emptyset\} \\ \mathcal{P}(X) &= \{\emptyset\} \\ |\mathcal{P}(X)| &= 1 \end{aligned}$$

$$|\mathcal{P}(X)| \equiv 2^0 = 1$$

- Inductive step:

  Where $k \in \mathbb{W}$,

$$|X| = k \implies |\mathcal{P}(X)| = 2^k$$

  Let the set Y be the set with cardinality $|X| + 1$.

$$Y = \{a_1, a_2, \cdots, a_k, a_{k+1}\}$$

  Y can also be defined as

$$Y = \{a_1, a_2, \cdots, a_{k-1}, a_k\} \cup \{a_{k+1}\}$$

  The first set in the redefinition of Y can be observed as the set X; or the set not containing a_{k+1}. From our initial inductive step, we can say that the cardinality of the powerset of $\mathcal{P}\left(\{a_1, a_2, \cdots, a_{k-1}, a_k\}\right) = 2^k$. Now considering the union portion of the set $\{a_{k+1}\}$, it can be observed that it will simply be the powerset $\mathcal{P}\left(\{a_1, a_2, \cdots, a_{k-1}, a_k\}\right)$ that includes a_{k+1} within each element. It can observed that the cardinality would also be 2^k as the amount of elements in the powerset doesn't change, and only the contents of each element.

$$\begin{aligned} \therefore\ Y &= \{a_1, a_2, \cdots, a_{k-1}, a_k\} \cup \{a_{k+1}\} \\ |\mathcal{P}(Y)| &= 2^k + 2^k = 2^{k+1} \end{aligned}$$

  It is proven that a set with n elements, its powerset must have 2^n elements.