Homework: Section 1.4

1 Practice Problems

1.1 Problem 1

Find the explicit general solution of the given differential equation.

$$\frac{dy}{dx} + 7xy = 0$$

$$\begin{aligned}
\frac{dy}{dx} &= -7xy \\
\int \frac{1}{y} dy &= \int -7x dx \\
ln|y| &= -\frac{7}{2}x^2 + C \\
e^{ln|y|} &= e^{-\frac{7}{2}x^2 + C} \\
y(x) &= e^{-\frac{7}{2}x^2} e^C
\end{aligned}$$

$$\boxed{y(x) = Ce^{-\frac{7}{2}x^2}}$$

1.2 Problem 2

Find the implicit general solution of the differential equation given. Assume x and y are non-zero.

$$\frac{dy}{dx} = \frac{(x-3)\,y^5}{x^2\,(6y^2 - y)}$$

$$\begin{aligned}
\int \frac{\left(6y^2 - y\right)}{y^5} dy &= \int \frac{(x-3)}{x^2} dx \\
\int \frac{6}{y^3} dy - \frac{1}{y^4} dy &= \int \frac{1}{x} dx - \frac{3}{x^2} dx \\
\frac{-3}{y^2} + \frac{1}{3y^3} &= \ln|x| + \frac{3}{x} + C
\end{aligned}$$

$$\boxed{\frac{-3}{y^2} + \frac{1}{3y^3} - \ln|x| - \frac{3}{x} = C}$$

1.3 Problem 3

Find the explicit particular solution of the initial value problem given.

$$\frac{dy}{dx} = ye^{7x},\ y(0) = 2e$$

$$\int \frac{1}{y} dy = \int e^{7x} dx$$
$$\ln|y| = \frac{1}{7} e^{7x} + C$$
$$y = e^{\frac{1}{7}e^{7x} + C}$$
$$y = Ce^{\frac{1}{7}e^{7x}}$$

$$Ce^{\frac{1}{7}e^{7x}} = 2e,\ y(0) = 2e$$
$$Ce^{\frac{1}{7}e^{7(0)}} = 2e$$
$$Ce^{\frac{1}{7}} = 2e$$
$$C = 2e^{\frac{6}{7}}$$

$$\boxed{y = \left(2e^{\frac{6}{7}}\right) e^{\frac{1}{7}e^{7x}}}$$

1.4 Problem 4

Find the explicit particular solution of the differential equation for the initial value provided.

$$\frac{dy}{dx} = 3x^3 y - y,\ y(1) = -7$$

$$\left(\frac{1}{y}\right) \frac{dy}{dx} = \left(3x^3 y - y\right) \left(\frac{1}{y}\right)$$
$$\int \left(\frac{1}{y}\right) dy = \int \left(3x^3 - 1\right) dx$$
$$\ln|y| = \frac{3}{4} x^4 - x + C$$
$$e^{\ln|y|} = e^{\frac{3}{4}x^4 - x + C}$$
$$y = Ce^{\frac{3}{4}x^4 - x}$$

$$Ce^{\frac{3}{4}(1)^4 - (1)} = -7,\ y(1) = -7$$
$$Ce^{\frac{3}{4} - 1} = -7$$
$$C = \frac{-7}{e^{\frac{-1}{4}}} = -7e^{\frac{1}{4}}$$

$$\boxed{y = \left(-7e^{\frac{1}{4}}\right) e^{\frac{3}{4}x^4 - x}}$$

1.5 Problem 5

Find the explicit particular solution of the differential equation for the initial value provided.

$$x\frac{dy}{dx} - y = 2x^2y,\ y(1) = 1$$

$$\begin{aligned}
x\frac{dy}{dx} &= 2x^2y + y \\
x\frac{dy}{dx} &= y\left(2x^2 + 1\right) \\
\int \frac{1}{y}dy &= \int \frac{2x^2+1}{x}dx \\
\ln|y| &= \int 2xdx + \int \frac{1}{x}dx \\
\ln|y| &= x^2 + \ln|x| + C \\
e^{\ln|y|} &= e^{x^2+\ln|x|+C} \\
y &= Ce^{x^2+\ln|x|}
\end{aligned}$$

$$\begin{aligned}
Ce^{x^2+\ln|x|} &= 1,\ y(1) = 1 \\
Ce &= 1 \\
C &= \frac{1}{e}
\end{aligned}$$

$$\boxed{y = \left(\frac{1}{e}\right)e^{x^2+\ln|x|}}$$

1.6 Problem 6

Find an explicit particular solution of the following initial value problem.

$$\frac{dy}{dx} = 11e^{5x-4y},\ y(0) = 0$$

$$\frac{dy}{dx} = 11e^{5x}e^{-4y}$$
$$\int \frac{1}{e^{-4y}}dy = \int 11e^{5x}dx$$
$$\int e^{4y}dy = \int 11e^{5x}dx$$
$$\frac{1}{4}e^{4y} = \frac{11}{5}e^{5x} + C$$
$$e^{4y} = \frac{44}{5}e^{5x} + 4C$$
$$4y = \ln\left(\frac{44}{5}e^{5x} + 4C\right)$$
$$y = \frac{\ln\left(\frac{44}{5}e^{5x} + 4C\right)}{4}$$

$$\frac{\ln\left(\frac{44}{5}e^{5(0)} + 4C\right)}{4} = 0,\ y(0) = 0$$
$$\frac{\ln\left(\frac{44}{5} + 4C\right)}{4} = 0$$
$$\frac{44}{5} + 4C = 1$$
$$4C = \frac{-39}{5}$$
$$C = \frac{-39}{20}$$

$$\boxed{y = \frac{\ln\left(\frac{44}{5}e^{5x} - \frac{39}{5}\right)}{4}}$$

Another example:

$$\frac{dy}{dx} = 9e^{4x-y},\ y(0) = 0$$

$$\frac{dy}{dx} = 9e^{4x}e^{-y}$$
$$\int e^{y}dy = \int 9e^{4x}dx$$
$$e^{y} = \frac{9}{4}e^{4x} + C$$
$$y = \ln\left(\frac{9}{4}e^{4x} + C\right)$$

$$\ln\left(\frac{9}{4}e^{4(0)}+C\right)=0,\ y(0)=0$$
$$\frac{9}{4}+C=1$$
$$C=\frac{-5}{4}$$

$$\boxed{y=\ln\left(\frac{9}{4}e^{4x}-\frac{5}{4}\right)}$$

1.7 Problem 7

The intensity of light I through a lake at a depth of x meters is given by the following formula.

$$\frac{dI}{dx}=(-1.59)\,I$$

(**a**) At what depth is the intensity of the light half that of the intensity at the surface, I_0 ?

(**b**) At 5 meters deep, what is the intensity of the light as a fraction of I_0 ?

(**c**) How many meters deep will the intensity of light be 9% of the intensity of the light at the surface?

(**a**)

First find the intensity at the surface I_0

$$\frac{dI}{dx}=(-1.59)\,I$$
$$\int\frac{1}{I}dI=\int-1.59dx$$
$$\ln|I|=-1.59x+C,\ (C \text{ can be ignored here})$$
$$I=Ce^{-1.59x}=e^{-1.59x}$$

Surface is when $x=0$m

$$I_0=e^{-1.59(0)}$$
$$=1$$
$$\therefore \text{ half of } I_0 \text{ would be } 0.5$$

Now finding intensity

$$e^{-1.59x}=0.5$$
$$-1.59x=\ln(0.5)$$

$$\boxed{x = 0.436\text{m, rounded to the nearest thousandth}}$$

(**b**)

Simply substitude $x = 5\text{m}$

$$\begin{aligned} I &= e^{-1.59(5)} \\ &= 0.000353 \end{aligned}$$

$$\boxed{I = 3.53 \times 10^{-4}}$$

(**c**)

First find 9% of I_0

$$I = 0.09 \cdot I_0 = 0.09$$

Now find x for when $I = 0.09$

$$\begin{aligned} e^{-1.59x} &= 0.09 \\ -1.59x &= \ln(0.09) \\ x &= \frac{\ln(0.09)}{-1.59} \end{aligned}$$

$$\boxed{x = 1.514\text{m, rounded to the nearest hundredth}}$$

1.8 Problem 8

Problem revoked by instructor