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1 Chapter 1

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1.7 Introduction to Proofs

We can define rational numbers as

$$\mathbb{Q} = \left\{ r | r = \frac{p}{q}, p, q, \in \mathbb{Z}, q \neq 0, gcd(p, q) = 1 \right\} \tag{1}$$

1.7.1 Basic Methods of Proofs

1. Directly
2. Case by case

3. Contradiction (unknown result, but know it is wrong) & Contrapositive (known result)
    - For contrapositive: To prove $p \implies q$, we prove $\neg q \implies \neg p$
    - For contradiction: We assume the opposite of the result, then from the assumption, we draw contradiction

1.7.2 Example

Prove that the product of two odd integers is odd.

Direct Proof: Let $a, b \in \mathbb{Z}$. Since a, b are odds, then by definition $a = 2m+1, b = 2n+1$, where $m, n \in \mathbb{Z}$.

$$\begin{aligned}
a \cdot b &= (2m+1)(2n+1) \\
a \cdot b &= 4mn + 2m + 2n + 1 \\
a \cdot b &= 2(2mn + m + n) + 1 \\
a \cdot b &= 2k + 1 \\
&\therefore a \cdot b \text{ is odd by definition}
\end{aligned}$$

1.7.3 Example

Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.

Case by Case Proof

1.7.4 Example

Prove that if $m+n$ and $n+p$ are even integers, where m, n, and p are integers, then $m+p$ is even. What kind of proof did you use?

Proof:

Case 1: When n is even.

Since $m+n$ & $n+p$ are even, then m is even & p is even. Thus, $m+p$ is even.

$$\begin{aligned}
n &= 2k_1 \\
m + n &= 2k_2 \\
m &= 2k_2 - 2k_1 \\
m &= 2(k_2 - k_1)
\end{aligned}$$

1.7.5 Example

$\forall a, b \in \mathbb{Z}$, if $a \cdot b$ is even, then either a is even or b is even.

Proof by Contrapositive: The statement is equivalent to

$$\begin{aligned}
\neg(a = 2k_1 \vee b = 2k_2) &\implies \neg(a \cdot b = 2k_3) \\
\neg(a = 2k_1) \wedge \neg(b = 2k_2) &\implies \neg(a \cdot b = 2k_3) \\
(a = 2k_1 + 1) \wedge (b = 2k_2 + 1) &\implies (a \cdot b = 2k_3 + 1)
\end{aligned}$$

1.7.6 Example

Prove that $\sqrt{2}$ is irrational.

Proof by Contadiction: We assume that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{m}{n}$, where $m, n \in \mathbb{Z}, n \neq 0, gcd(m, n) = 1$.

$$\begin{aligned}
2 &= \frac{m^2}{n^2} \\
m^2 &= 2n^2
\end{aligned}$$

Hence, m^2 is even $\implies$ m is even $\implies$ m^2 is a multiple of 4. Thus, n^2 is even $\implies$ n is even $\implies$ n^2 is a multiple of 2. $\therefore$ $gcd(m, n) \neq 1$ contradicts with $gcd(m, n) = 1$.

1.7.7 Example

Prove that there is no largest positive real number.

Proof by Contradiction: Assume that there is a largest real number. Let x be the largest positive real number.

Consider $x + 1 > x$. Then it contradicts with the assumption.

1.7.8 Example

Show that the equation has exactly one real solution.

$$2x + \cos(x) = 0$$

Start by showing there is at least one solution.

Let $a = 0, f(x) = 2x + \cos(x)$, then $f(a) = 1 > 0$. When $b = \pi$, then $f(b) = 2(-\pi) + \cos(-\pi) = -2\pi - (-1) < 0$.

Show at most one solution. By contradiction, assume we have at least two solutions: $f(x_1) = 0, f(x_2) = 0$.

By Rolle's Theorem: Show that $f'(b) = 0$. $f'(b) = 2 - \sin(x) \neq 0$.

1.7.9 Example

Prove that the arithmetic mean is greater than or equal to the geometric mean for the case when $n = 2$. That is, prove

$$\frac{x+y}{2} \geq \sqrt{xy}, x \geq, y \geq 0$$

$$\begin{aligned}
\frac{x+y}{2} &\geq \sqrt{xy} \\
\left(\frac{x+y}{2}\right)^2 &\geq xy \\
\frac{x^2+y^2+2xy}{4} &\geq xy \\
x^2+y^2+2xy &\geq 4xy \\
x^2+y^2-2xy &\geq 0 \\
(x-y)^2 &\geq 0
\end{aligned}$$

1.7.10 Existence and Uniqueness

$$\begin{aligned}
\exists! x P(x) &\implies \text{If } P(x) \text{ is } T\text{, then } P(y) \text{ is } F \text{ if } y \neq x. \\
&\iff P(x) = P(y) \text{ is true, then } x = y
\end{aligned}$$

Existence: Consider $x = \frac{c-b}{a}$ is well-defined since $a, b, c \in \mathbb{R}, a \neq 0$. Then $ax + b = a\left(\frac{c-b}{a}\right) + b = c - b + b = c \therefore x = \frac{c-b}{a}$ is a solution.

Uniqueness: Let x_1, x_2 be two real numbers that are solutions to $ax + b = c$. Then

$$\begin{cases} ax_1 + b = c \\ ax_2 + b = c \end{cases}$$

1.8 Proof Methods and Strategy