1 Section 5.1

1.1 5.1.9

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$y'' + 2y' + y = 0; y_1 = e^{-x}, y_2 = xe^{-x}; y(0) = 9, y'(0) = -10$$

(a) **Why is the function $y_1 = e^{-x}$ a solution to the differential equation?**
The function $y_1 = e^{-x}$ is a solution because when the function, its first derivative $y_1' = -e^{-x}$ and its second derivative, $y_1'' = e^{-x}$, are substituted into the equation, the result is a true statement.

(b) **Why is the function $y_2 = xe^{-x}$ a solution to the differential equation?**
The function $y_2 = xe^{-x}$ is a solution because when the function, its derivative, $y_2' = e^{-x} - xe^{-x}$, and its second derivative, $y_2'' = -e^{-x} - e^{-x} + xe^{-x}$, are substituted into the equation, the result is a true statement.

(c) The particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the initial conditions $y(0) = 9$ and $y'(0) = -10$ is $y = 9e^{-x} - xe^{-x}$.

1.2 5.1.11

A homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the given initial conditions. Primes denote derivatives with respect to x.

$$y'' - 2y' + 2y = 0; y_1 = e^x \cos(x), y_2 = e^x \sin(x); y(0) = 11, y'(0) = 14$$

(a) The function $y_1 = e^x \cos(x)$ is a solution because when the function its first derivative $y_1' = e^x \cos(x) - e^x \sin(x)$, and its second derivative, $y_1'' = e^x \cos(x) - e^x \sin(x) - e^x \sin(x) - e^x \cos(x)$, are substituted into the equation, the result is a true statement.

(b) The function $y_2 = e^x \sin(x)$ is a solution because when the function, its first derivative, $y_2' = e^x \sin(x) + e^x \cos(x)$, and its second derivative, $y_2'' = e^x \sin(x) + e^x \cos(x) + e^x \cos(x) - e^x \sin(x)$, are substituted into the equation, the result is a true statement.

(c) The particular solution of the form $y = c_1y_1 + c_2y_2$ that satisfies the initial conditions $y(0) = 11$ and $y'(0) = 14$ is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
y &= c_1 \left(e^x \cos(x)\right) + c_2 \left(e^x \sin(x)\right) \\
c_1 \left(e^0 \cos(0)\right) + c_2 \left(e^0 \sin(0)\right) &= 11 \\
c_1 &= 11 \\
y &= c_1 \left(e^x \cos(x)\right) + c_2 \left(e^x \sin(x)\right) \\
y' &= c_1 \left(e^x \cos(x) - e^x \sin(x)\right) + c_2 \left(e^x \sin(x) + e^x \cos(x)\right) \\
14 &= (11) \left(e^0 \cos(0) - e^0 \sin(0)\right) + c_2 \left(e^0 \sin(0) + e^0 \cos(0)\right) \\
14 &= (11) + c_2 \\
c_2 &= 3 \\
y &= (11)(e^x \cos(x)) + (3)(e^x \sin(x))
\end{aligned}
$$

$$\boxed{y = (11)(e^x \cos(x)) + (3)(e^x \sin(x))}$$

1.3 5.1.19

Show that $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution.

(a)

$$
\begin{aligned}
y_1 &= 1 \\
y_1' &= 0 \\
y_1'' &= 0 \\
yy'' + (y')^2 &= 0 \\
(1)(0) + (0)^2 &= 0 \\
0 &= 0
\end{aligned}
$$

(b)

$$
\begin{aligned}
y_2 &= \sqrt{x} \\
y_2' &= \frac{1}{2} x^{-1/2} \\
y_2'' &= -\frac{1}{4} x^{-3/2} \\
yy'' + (y')^2 &= 0 \\
(\sqrt{x}) \left(-\frac{1}{4} x^{-3/2}\right) + \left(\frac{1}{2} x^{-1/2}\right)^2 &= 0 \\
-\frac{1}{4x} + \frac{1}{4x} &= 0 \\
0 &= 0
\end{aligned}
$$

(c) Now consider the function $y = y_1 + y_2$.

$$
\begin{aligned}
y &= y_1 + y_2 \\
y &= (1) + (\sqrt{x}) \\
y' &= 0 + \frac{1}{2}x^{-1/2} \\
y' &= \frac{1}{2}x^{-1/2} \\
y'' &= -\frac{1}{4}x^{-3/2}
\end{aligned}
$$

(d) For the function $y = y_1 + y_2$:

$$
\begin{aligned}
yy'' &= \left(1 + \sqrt{x}\right)\left(-\frac{1}{4}x^{-3/2}\right) \\
yy'' &= -\frac{1 + \sqrt{x}}{4x^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
(y')^2 &= \left(\frac{1}{2}x^{-1/2}\right)^2 \\
(y')^2 &= \frac{1}{4x}
\end{aligned}
$$

(e) Why is the proof complete?

For the function $y = y_1 + y_2$, when the expressions above for yy'' and $(y')^2$ are substituted into the equation, the result is a false statement.

1.4 5.1.24

Determine whether the functions y_1 and y_2 are linearly dependent on the interval $(0, 1)$.

$$y_1 = \sin(t)\cos(t), y_2 = 3\sin(2t)$$

$$
\begin{aligned}
&y_2 = 3\sin(2t) = 6\sin(t)\cos(t) \\
&y_1 \equiv y_2
\end{aligned}
$$

1.5 5.1.31

Show that $y_1 = \sin(x^2)$ and $y_2 = \cos(x^2)$ are linearly independent functions, but that their Wronskian vanishes at $x = 0$. Why does this imply that there is no differential equation of the form $y'' + p(x)y' + q(x)y = 0$, with both p and q continuous everywhere, having both y_1 and y_2 as solutions?

(a) Two functions defined on an open interval I are said to be linearly independent on I provided that neither is a constant multiple of the other.

(b) The functions $y_1 = \sin(x^2)$ and $y_2 = \cos(x^2)$ are linearly independent because $\frac{y_1}{y_2} = \tan(x^2)$ is not a constant-valued function and $\frac{y_2}{y_1} = \cot(x^2)$ is not a constant-valued function.

(c) Given two function f and g, the Wronskian of f and g is the determinant defined as follows.

$$W(f,g) = \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - f'g$$

(d) The Wronskian for these functions, $W(\sin(x^2), \cos(x^2)) = -2x$, vanishes at $x = 0$ because it has a factor of x.

1.6 5.1.33

Find a general solution to the differential equation given below. Primes denote derivatives with respect to t.

$$y'' + 4y' - 21y = 0$$

$$\begin{aligned} r^2 + 4r - 21 &= 0 \\ (r-3)(r+7) &= 0 \\ r &= 3, -7 \\ y &= c_1(e^{3t}) + c_2(e^{-7t}) \end{aligned}$$

1.7 5.1.35

Find a general solution to the differential equation given below. Primes denote derivatives with respect to x.

$$y'' + 8y' = 0$$

$$\begin{aligned} r^2 + 8r &= 0 \\ r &= 0, -8 \\ y &= c_1 + c_2(e^{-8x}) \end{aligned}$$

1.8 5.1.37

Find a general solution to the differential equation given below. Primes denote derivatives with respect to x.

$$6y'' - 5y' - y = 0$$

$$\begin{aligned} 6r^2 - 5r - 1 &= 0 \\ r &= 1, -\frac{1}{6} \\ y &= c_1(e^{x}) + c_2(e^{-x/6}) \end{aligned}$$

1.9 5.1.38

$$12y'' + 5y' - 3y = 0$$

$$\begin{aligned} 12r^2 + 5r - 3 &= 0 \\ r &= \frac{1}{3}, -\frac{3}{4} \\ y &= c_1(e^{t/3}) + c_2(e^{-3t/4}) \end{aligned}$$

1.10 5.1.39

$$9y'' + 6y' + y = 0$$

$$\begin{aligned} 9r^2 + 6r + 1 &= 0 \\ r &= -\frac{1}{3} \\ y(x) &= c_1(e^{-x/3}) + xc_2(e^{-x/3}) \end{aligned}$$

1.11 5.1.44

The equation below is a general solution to a homogeneous second-order differential equation $ay'' + by' + cy = 0$ with constant coefficients. Find such an equation.

$$y(x) = c_1e^{7x} + c_2e^{-7x}$$

What are the simplest integer coefficients $a > 0, b$, and c for a homogeneous second-order differential equation with the given general solution?

$$\begin{aligned} r &= 7, -7 \\ (r+7)(r-7) &= 0 \\ r^2 - 49 &= 0 \end{aligned}$$

$$\boxed{r^2 - 49 = 0}$$

1.12 5.1.45

The equation below is a general solution to a homogeneous second-order differential equation $ay'' + by' + cy = 0$ with constant coefficients. Find such an equation.

$$y(x) = c_1e^{-2x} + c_2xe^{-2x}$$

$$\begin{aligned} r &= -2 \\ (r+2)^2 &= 0 \\ r^2 + 4r + 4 &= 0 \end{aligned}$$

$$\boxed{r^2 + 4r + 4 = 0}$$

1.13 5.1.52

A second-order Euler equation is one of the form $ax^2y'' + bxy' + cy = 0$, where a, b, and c are constants. If $x > 0$, then the substitution $v = \ln(x)$ transforms the equation into the constant coefficient linear equation below, with independent variable v.

$$a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0$$

Make the substitution $v = \ln(x)$ to find the general solution of $x^2y'' + xy' - 4y = 0$, for $x > 0$.

$$\begin{aligned}
v &= \ln(x)\\
x &= e^v\\
y' &= e^{-v}\frac{dy}{dv}\\
y'' &= e^{-2v}\left(\frac{d^2y}{dv^2} - \frac{dy}{dv}\right)\\
e^{2v}\left(e^{-2v}\left(\frac{d^2y}{dv^2} - \frac{dy}{dv}\right)\right) + e^v\left(e^{-v}\frac{dy}{dv}\right) - 4y &= 0\\
\left(\frac{d^2y}{dv^2} - \frac{dy}{dv}\right) + \frac{dy}{dv} - 4y &= 0\\
\frac{d^2y}{dv^2} - 4y &= 0\\
r^2 - 4 &= 0\\
r^2 &= 4\\
r &= 2, -2\\
y &= c_1e^{2v} + c_2e^{-2v}\\
y &= c_1e^{2\ln(x)} + c_2e^{-2\ln(x)}
\end{aligned}$$

$$\boxed{y = c_1e^{2\ln(x)} + c_2e^{-2\ln(x)}}$$