Week 03 Participation Assignment

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1 Part 01

The purpose of this exercise is to prove that for any real number: $a : \sqrt{a^2} = |a|$.

First, we recall that the absolute value of any real number is defined by

$$|a| = \begin{cases} a \text{ if } a \geq 0, \text{ and} \\ -a \text{ if } a < 0. \end{cases}$$

a) Use the definition above to explain why for any real number $a : |a| \geq 0$.
Case by Case Proof:

- Case 1: $a \geq 0$.

$$\begin{aligned} &|a| = a, \quad a \geq 0 \\ &|a| = a \geq 0 \\ &|a| \geq a \end{aligned}$$

- Case 2: $a < 0$.

$$\begin{aligned} &|a| = -a, \quad a < 0 \implies -a > 0 \\ &|a| = -a > 0, \quad -a > 0 \implies -a \geq 0 \\ &|a| \geq 0 \end{aligned}$$

b) Again, using the definition, show that $|a|^2 = a^2$.
Case by Case Proof:

- Case 1: $a \geq 0$.

$$\begin{aligned} |a|^2 &= a^2 \\ |a| \cdot |a| &= a \cdot a, \quad |a| = a \geq 0 \\ a \cdot a &= a \cdot a \end{aligned}$$

- Case 2: $a < 0$.

$$\begin{aligned} |a|^2 &= a^2 \\ |a| \cdot |a| &= a \cdot a, \quad |a| = -a > 0 \implies |a| = -a \geq 0 \\ -a \cdot -a &= a \cdot a \\ a \cdot a &= a \cdot a \end{aligned}$$

c) Our next goal is to show that $\sqrt{b}$ is unique. In other words, prove that if c and d are two real numbers such that $c \geq 0$, and $d \geq 0$, and $b = c^2 = d^2$, then $c = d$.

$$\begin{aligned} c^2 &= d^2 \\ c^2 - d^2 &= 0 \\ (c+d)(c-d) &= 0 \\ c &= \pm d \\ |c| &= |d|, \quad |c| = c \geq 0, |d| = d \geq 0 \\ c &= d \end{aligned}$$

d) Rewrite the definition for $\sqrt{b}$ to define $\sqrt{a^2}$

$$\begin{aligned} b &= c^2 \\ \sqrt{b} &= c \\ \sqrt{b} &= \pm d \\ \sqrt{b} &= \sqrt{d^2}, \sqrt{(-d)^2} \\ \sqrt{b} &= \sqrt{d^2} = \sqrt{a^2}, \quad \{d \in \mathbb{R}, \in (-\infty, \infty)\} \end{aligned}$$

e) Put together all the steps above to write a complete proof that $\sqrt{a^2} = |a|$.

$$\sqrt{a^2} = |a|$$

Case by Case Proof:

- Case 1: $a \geq 0$

$$\begin{aligned} \sqrt{a^2} &= |a| \\ a^2 &= |a|^2 \\ a \cdot a &= |a| \cdot |a|, \quad |a| = a \geq 0 \\ a \cdot a &= a \cdot a \end{aligned}$$

- Case 2: $a < 0$

$$\begin{aligned} \sqrt{a^2} &= |a| \\ a^2 &= |a|^2 \\ a \cdot a &= |a| \cdot |a|, \quad |a| = a < 0 \implies -a > 0 \implies -a \geq 0 \\ a \cdot a &= -a \cdot -a \\ a \cdot a &= a \cdot a \end{aligned}$$

2 Part 02

Let's consider the powersets of a finite set. Our goal is to "calculate" how many elements are in the power set. That is the cardinality of the powerset. (We could define powerset from an infinite set).

If $X = \{x_1, x_2, \cdots, x_n\}$ is a finite set, we define $\mathcal{P}(X)$, the powerset of X, to be the set of all subsets of X. For Example, if $X = \{a, b\}$, then $\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and thus $\mathcal{P}(X)$ has 4 elements.

a) If $X = \{a, b, c\}$, list all the members of $\mathcal{P}(X)$. How many subsets does X have?