Chapter 1

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1 Introduction

1.1 Introduction to Differential Equations

Equations contain: "=" and "solution"

There are two types of variables: dependent and independent

Differential: contains derivative (or partial derivative)

Derivative refers to:

ODE (ordinary differential equation): $\frac{\mathrm{d}}{\mathrm{d}x}$

PDE (partial differential equation): $\frac{\partial}{\partial x}$

The solution to a differential equation is a **function** And can be expressed in four ways:

- 1. Verbally
- 2. Table
- 3. Graph
- 4. Expression (implicitly or explicitly)

Regular equation:

$$x^3 + 3\sin(x) = 2 - 2\cos(x)$$

Claim x = 0 is a solution. Verify by substitution.

When
$$x = 0$$

$$(0)^3 + 3\sin(0) = 2 - 2\cos(0)$$

 $0 + 3 \times 0 = 2 - 2 \times 1$
 $0 = 0$

Although it may be true for x=0, it is not a true method of verification. See for $x=\pi$:

$$\pi^{3} + 3\sin(\pi) = 2 - 2\cos(\pi)$$
$$\pi^{3} + 0 = 2 - 2 \times (-1)$$

$$\pi^3 \neq 4$$

So how do we verify the equation?

Given y'' - 2y' + 5y = 0 (with respect to x) Claim $y = 4e^x \cos(2x)$ is a solution

$$y' = 4e^{x} \cos(2x) + 4e^{x} (-\sin(2x))$$

$$= 4e^{x} (\cos(2x) - 2\sin(2x))$$

$$y'' = 4e^{x} (\cos(2x) - 2\sin(2x)) + 4e^{x} (-2\sin(2x) - 4\cos(2x))$$

$$= 4e^{x} (-3\cos(2x) - 4\sin(2x))$$

$$-2y' = 4e^{x} (-2\cos(2x) + 4\sin(2x))$$

$$5y = 4e^{x} (5\cos(2x))$$

LHS =
$$y'' - 2y' + 5y = 4e^x (0 + 0) = 0 = \text{RHS}$$

 $\therefore y = 4e^x \cos(2x) \text{ is a solution}$

1.2 Classification of Differential Equations

- 1. ODE v.s. PDE
 - ODE: ordinary differential equation $F(x, y, y', y'', \dots, y^{(n)}) = 0$
 - PDE: partial differential equation $F\left(x_1, x_2, \cdots, u, u_{x_1}, u_{x_2}, \cdots, u_{x_k}, \cdots\right) = 0$

1.3 Order of Differential Equations

The order of differential equations is defined by the highest derivative.

Example:

$$y'' - 2y' + 5y = 0 \rightarrow 2$$
nd order

Second order ODE is generally written as:

$$F(x, y, y', y'') = 0$$

- \bullet independent variable: x
- dependent variable: y

To write the second order PDE, the independent variable must be located first

- independent variable: x, t
- \bullet dependent variable: u

$$F(x, t, u, u_x, u_t, u_{xt}, u_{xx}, u_{tx}, u_{tt}) = 0$$

1.4 Linear & Non-Linear ODE Classification

Linear equations can be solved explicitly. Non-linear equations may only sometimes be solvable. Generally analyzed or attempting to find the equation's stable point.

1. Linear Form

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

Can also be written as so:

$$\sum_{i=0}^{n} a_i(x)y^{(i)} = f(x)$$

2. Non-Linear

Examples include: $\sin(y), e^y, \ln(y), y^2, \sqrt{y}$

Orders covered in this course:

- 1st order: linear & some non-linear
- 2nd order: linear only
 - $\rightarrow a_i(x)$ are constants
 - $\rightarrow f(x)$ are "nice functions"
 - $\rightarrow a_i(x)$ are polynomials
- higher order: linear with constant coefficients
 - * possibly system of linear differential (using matrices & eigen-theory)