Chapter 1 - First-Order Differential Equations

1 Differential Equations and Mathematical Models

1.1 What is a differential equation?

An equation relating an unknown function and one or more of its derivatives is called a **differential equation**.

The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = \frac{dx}{dt}$.

The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

inolves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y.

The study of differential equations has three principal goals:

1. To discover the differential equation that describes a specified physical situation.
2. To find – either exactly or approximately – the appropriate solution of that equation.
3. To interpret the solution that is found.

Example 1:

If C is a constant and

$$y(x) = Ce^{x^2},$$

then

$$\frac{dy}{dx} = C\left(2xe^{x^2}\right) = (2x)\left(Ce^{x^2}\right) = 2xy$$

for all x. This example defines an *infinite* family of different solutions of this differential equation; one for each choice of the arbitrary constant C.

Example 2:

Note that any function of the form

$$P(t) = Ce^{kt}$$

is a solution of the differential equation

$$\frac{dP}{dt} = kP$$

And can be verified as such

$$\begin{aligned} P'(t) &= Cke^{kt} \\ &= k\left(Ce^{kt}\right) \\ &= kP(t) \end{aligned}$$

This example exemplifies that the differential equation $\frac{dP}{dt} = kP$ has *infinitely many* different solutions of the form $P(t) = Ce^{kt}$ – one for each choice of the arbitrary constant C.

Example 3:

Suppose that $P(t) = Ce^{kt}$ is the population of a colony of bacteria at time t, that the population at time $t = 0$ (hours, h) was 1000, and that the population doubled after 1 h. This additional information about $P(t)$ yields the following equations:

$$\begin{aligned} 1000 &= P(0) = Ce^0 = C \\ 2000 &= P(1) = Ce^k \end{aligned}$$

It follows that $C = 1000$ and that $e^k = 2$, so $k = \ln(2) \approx 0.693147$. With this value of k the differential equation is

$$\frac{dP}{dt} = (\ln(2))P \approx (0.693147)P$$

Substitution of $k = \ln(2)$ and $C = 1000$ for the function P yields the particular solution

$$\begin{aligned} P(t) &= 1000e^{(\ln(2))t} \\ &= 1000\left(e^{\ln(2)}\right)^t \\ &= 1000 \cdot 2^t \rightarrow \text{because } e^{ln(2)} = 2 \end{aligned}$$

that satisfies the given conditions. This solution can be used to predict future populations of the bacteria colony – e.g.

$$P(1.5) = 1000 \cdot 2^{3/2} \approx 2828$$

The condition $P(0) = 1000$ used in **Example 3** is called an **initial condition** because differential equations are frequently written for which $t = 0$ – the ”starting time.”

1.2 Mathematical Models

Mathematical Modeling is

1. The formulation fo a real-world problem in mathematical terms; that is, the construction of a mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original real-world situation – for example, answering the question originally posed.

A satisfactory mathematical model is subject to two contradictory requirements: It must be sufficiently detailed to represent the real-world situation with relative accuracy, yet it must be sufficiently simple to make the mathematical analysis practical. In other words, if the model is so detailed that it fully represents the physical situation, then the mathematical analysis may to be too difficult to carry out. On the contrary, if the model is too simple, the results may be so inaccurate as to be useless. Finding the tradeoff is *key*.

Example 1 If C is a constant and $y(x) = \frac{1}{(C-x)^2}$, then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(C-x\right)^{-1} \\ &= \frac{-1}{\left(C-x\right)^2}\cdot\frac{d}{dx}\left(C-x\right) \\ &= \frac{1}{\left(C-x\right)^2} \to x \neq C \\ &= y^2 \end{aligned}$$

Thus

$$y(x) = \frac{1}{C-x}$$

defines a solution of the differential equation

$$\frac{dy}{dx} = y^2 \to x \neq C$$

Example 2 Verify that the function $y(x) = 2x^{1/2} - x^{1/2}\ln(x)$ satisfies the differential equation

$$4x^2y'' + y = 0$$

for all $x > 0$.

Find $y'(x)$

$$\begin{aligned} y'(x) &= x^{-1/2} - \left(\frac{1}{2}x^{-1/2} \cdot \ln(x) + \frac{1}{x} \cdot x^{1/2}\right) \\ &= x^{-1/2} - \frac{x^{-1/2}\ln(x)}{2} - x^{-1/2} \\ &= -\frac{x^{-1/2}\ln(x)}{2} \\ &= -\frac{1}{2}x^{-1/2}\ln(x) \end{aligned}$$

Find $y''(x)$

$$\begin{aligned} y''(x) &= -\frac{1}{2}\left(-\frac{1}{2}x^{-3/2} \cdot \ln(x) + \frac{1}{x} \cdot x^{-1/2}\right) \\ &= \frac{1}{4}x^{-3/2}\ln(x) - \frac{1}{2}x^{-3/2} \end{aligned}$$

Substitute for $4x^2y'' + y = 0$

$$\begin{aligned} 4x^2\left(\frac{1}{4}x^{-3/2}\ln(x) - \frac{1}{2}x^{-3/2}\right) + 2x^{1/2} - x^{1/2}\ln(x) &= 0 \\ x^{1/2}\ln(x) - 2x^{1/2} + 2x^{1/2} - x^{1/2}\ln(x) &= 0 \\ x^{1/2}\ln(x) + \left(-x^{1/2}\ln(x)\right) - 2x^{1/2} + 2x^{1/2} &= 0 \\ 0 + 0 = 0 \rightarrow 0 &= 0 \end{aligned}$$

$\therefore$ if x is positive, the differential equation is satisfied for all $x > 0$.

The **order** of a differential equation is the order of the highest derivative that appears in it. e.g.

$$y^{(4)} + x^2y^{(3)} + x^5y = \sin(x)$$

is a fourth-order equation.

The most general form of an ***n*th-order** differential equation with independent variable x and unknown function or dependent variable $y = y(x)$ is

$$F\left(x, y, y', y'', \cdots, y^{(n)}\right) = 0$$

where F is a specific real-valued function of $n + 2$ variables.

A **solution** is the continuous function $u = u(x)$ of the differential equation **on the interval** I provided that the derivatives $u', u'', \cdots, u^{(n)}$ exist on I and

$$F\left(x, u, u', u'', \cdots, u^{(n)}\right) = 0$$

for all x in I. For the sake of brevity, we may say that $u = u(x)$ **satisfies** the differential equation on I.

Example 1 If A and B are constants and

$$y(x) = A\cos(3x) + B\sin(3x)$$

then two successive differentiations yield

$$\begin{aligned} y'(x) &= -3A\sin(3x) + 3B\cos(3x) \\ y''(x) &= -9A\cos(3x) - 9B\sin(3x) = -9y(x) \end{aligned}$$

for all x.