Contents

1 Section 7.1

1.1 7.1.1

Transform the given differential equation into an equivalent system of first-order differential equations.

$$x'' + 4x' - 3x = 6t$$

Let $x_1 = x$ and $x_2 = x'$. Complete the system below.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x' + 3x + 6t = -4x_2 + 3x_1 + 6t \end{aligned}$$

1.2 7.1.2

Transform the given differential equation into an equivalent system of first-order differential equations.

$$x^{(4)} + 5x'' + 2x = 5t^3 \sin(2t)$$

Let $x_1 = x$, $x_2 = x'$, $x_3 = x''$, and $x_4 = x^{(3)}$. Complete the system below.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= -5x'' - 2x + 5t^3 \sin(2t) = -5x_3 - 2x_1 + 5t^3 \sin(2t) \end{aligned}$$

1.3 7.1.5

Transform the given differential equation into an equivalent system of first-order differential equations.

$$x^{(3)} = (x'')^2 - 2\cos(x')$$

Let $x_1 = x$, $x_2 = x'$, and $x_3 = x''$. Complete the system below.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= (x_3)^2 - 2\cos(x_2) \end{aligned}$$

1.4 7.1.8

Transform the given differential equation into an equivalent system of first-order differential equations.

$$\begin{aligned} x'' + 6x' - 6x - 5y &= 0 \\ y'' - 4y' + 3x - 5y &= \sin(t) \end{aligned}$$

Let $x_1 = x$, $x_2 = x'$, $y_1 = y$, and $y_2 = y'$. Complete the system below.

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -6x' + 6x + 5y = -6x_2 + 6x_1 + 5y_1 \\ y_1' &= y_2 \\ y_2' &= 4y' - 3x + 5y + \sin(t) = 4y_2 - 3x_1 + 5y_1 + \sin(t) \end{aligned}$$

1.5 7.1.9

Transform the given differential equation into an equivalent system of first-order differential equations.

$$\begin{cases} x'' = 9x - y + 3z \\ y'' = \ x + y - 3z \\ z'' = \ 6x - y - z \end{cases}$$

$$\begin{aligned} x_1' &= x_2 \\ y_1' &= y_2 \\ z_1' &= z_2 \end{aligned}$$

$$\begin{aligned} x_2' &= 9x_1 - y_1 + 3z_1 \\ y_2' &= x_1 + y_1 - 3z_1 \\ z_2' &= 6x_1 - y_1 - z_1 \end{aligned}$$

1.6 7.1.22

(a) Beginning with the general solution of the system $x' = -2y$, $y' = 2x$, calculate $x^2 + y^2$ to show that the trajectories are circles.

(b) Show similarly that the trajectories of the system $x' = \frac{1}{2}y$, $y' = -8x$ are ellipses with equation of the form $16x^2 + y^2 = C^2$.

(a) Find the solution of the system $x' = -2y$, $y' = 2x$ below. Start with $x(t)$.

$$\begin{aligned} x' &= -2y \\ x'' &= -2y' \\ x'' &= -2(2x) \\ x'' + 4x &= 0 \end{aligned}$$

$$\begin{aligned} r^2 + 4 &= 0 \\ r &= \pm 2i \end{aligned}$$

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

Now find $y(t)$ so that $y(t)$ and the solution for $x(t)$ found in the previous step are a general solution to the system of differential equations.

$$\begin{aligned} y' &= 2x \\ y' &= 2(C_1 \cos(2t) + C_2 \sin(2t)) \\ \int y' dt &= 2 \int (C_1 \cos(2t) + C_2 \sin(2t))\, dt \\ y(t) &= C_1 \sin(2t) - C_2 \cos(2t) \end{aligned}$$

Now calculate and simplify $x^2 + y^2$.

$$\begin{aligned} x^2 + y^2 &= (C_1 \cos(2t) + C_2 \sin(2t))^2 + (C_1 \sin(2t) - C_2 \cos(2t))^2 \\ x^2 + y^2 &= C_1^2 + C_2^2 \end{aligned}$$

(b) Show that the trajectories of the system $x' = \frac{1}{2}y$, $y' = -8x$ are ellipses with equations of the form $16x^2 + y^2 = C^2$. First solve for $x(t)$.

$$\begin{aligned} x' &= \frac{1}{2}y \\ x'' &= \frac{1}{2}y' \\ x'' &= \frac{1}{2}(-8x) \\ x'' + 4x &= 0 \end{aligned}$$

$$\begin{aligned} r^2 + 4r &= 0 \\ r &= \pm 2i \end{aligned}$$

$$x(t) = A\cos(2t) + B\sin(2t)$$

Now find $y(t)$ so that $y(t)$ and the solution for $x(t)$ found in the previous step are a general solution to the system of differential equations.

$$\begin{aligned} y' &= -8x \\ \int y' dt &= -8 \int (A\cos(2t) + B\sin(2t))\, dt \\ y &= -4A\sin(2t) + 4B\cos(2t) \end{aligned}$$

Letting $C = \sqrt{A^2 + B^2}$, $A = C\cos(\alpha)$, and $B = C\sin(\alpha)$, rewrite $x(t)$ and $y(t)$ in terms of C, α, and t below.

$$\begin{aligned} x(t) &= A\cos(2t) + B\sin(2t) \\ x(t) &= (C\cos(\alpha))\cos(2t) + (C\sin(\alpha))\sin(2t) \end{aligned}$$

$$\begin{aligned} y(t) &= -4A\sin(2t) + 4B\cos(2t) \\ y(t) &= -4(C\cos(\alpha))\sin(2t) + 4(C\sin(\alpha))\cos(2t) \end{aligned}$$

Using the equations from the previous step, solve for $\cos(\alpha - 2t)$ and $\sin(\alpha - 2t)$ and rewrite $\cos^2(\alpha - 2t) + \sin^2(\alpha - 2t) = 1$ in terms of x, y and C.

$$\begin{aligned} x(t) &= (C\cos(\alpha))\cos(2t) + (C\sin(\alpha))\sin(2t) \\ x(t) &= C\cos(-2t + \alpha) \\ \cos(2t - \alpha) &= \frac{x}{C} \end{aligned}$$

$$\begin{aligned} y(t) &= -4(C\cos(\alpha))\sin(2t) + 4(C\sin(\alpha))\cos(2t) \\ y(t) &= 4C\sin(-2t + \alpha) \\ \sin(-2t + \alpha) &= \frac{y}{4C} \end{aligned}$$

$$\left(\frac{x}{C}\right)^2 + \left(\frac{y}{4C}\right)^2 = 1$$

Finally, multiply both sides of the equation found in the previous step by $16C^2$, then replace $4C$ with C, resulting in the equation $16x^2 + y^2 = C^2$.

1.7 7.1.26

Three 132 gal fermentation vats are connected as indicated in the figure, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank T_i at time $t (i = 1, 2, 3)$. Suppose that the mixture circulates between the tanks at the rate of 11 gal/min. Derive the equations.

$$
\begin{aligned}
11x_1' &= -x_1 + x_3 \\
11x_2' &= x_1 - x_2 \\
11x_3' &= x_2 - x_3
\end{aligned}
$$

$$
\begin{aligned}
11\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} &= \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \\
x_1' &= \left(-\frac{1}{11}\right) x_1 + \left(\frac{1}{11}\right) x_3 \\
x_2' &= \left(\frac{1}{11}\right) x_1 + \left(-\frac{1}{11}\right) x_2 \\
x_3' &= \left(\frac{1}{11}\right) x_2 + \left(-\frac{1}{11}\right) x_3 \\
\det(\boldsymbol{A} - \lambda\mathbf{I}) &= \begin{bmatrix} -1-\lambda & 0 & 1 \\ 1 & -1-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{bmatrix} \\
\det(\boldsymbol{A} - \lambda\mathbf{I}) &= (-1-\lambda)((-1-\lambda)(-1-\lambda) - 0) + 0 + (1)((1)(1) - 0) \\
\det(\boldsymbol{A} - \lambda\mathbf{I}) &= -\lambda^3 - 3\lambda^2 - 3\lambda = -\lambda(\lambda^2 + 3\lambda + 3) \\
\lambda &= 0, -\frac{3}{2} \pm \frac{\sqrt{3}}{2} i
\end{aligned}
$$